The topological theory of defects in ordered media*†

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Aspects of the theory of homotopy groups are described in a mathematical style closer to that of condensed matter physics than that of topology. The aim is to make more readily accessible to physicists the recent applications of homotopy theory to the study of defects in ordered media. Although many physical examples are woven into the development of the subject, the focus is on mathematical pedagogy rather than on a systematic review of applications.

Abhishek Dasgupta, IISER Kolkata, 22 April 2010

Why topology?

A very good description for defects and their *classification* in ordered media.

Gives some non-trivial insights, which would have been hard to get otherwise.

Outline

RECAP: Definitions, and introduction.

Examples of ordered media

Planar spins in 2D, classification of defects, ordinary spins

fundamental group, what is it? homotopies based at points, isomorphism of loops based at different points, nonabelian spaces.

Combination of line defects

Group theoretic structure of the order parameter space, with examples

Fundamental group of a continuous group

Fundamental group (revisited), how to actually find it. A theorem. Fundamental groups of the examples discussed.

Media with non-abelian fundamental groups, and what is interesting in them; combination of line defects in this case.

Higher dimensions: The second homotopy group, a quick overview (classifying point defects in 3D)

Some other applications.

Definitions, Introduction

ORDERED MEDIUM a spatial region described by $f(\mathbf{r})$. Every point is assigned an ORDER PARAMETER. The possible values of the ORDER PARAMETER make up the ORDER PARAMETER SPACE.

Medium is UNIFORM if f is a constant.

DEFECTS - regions of lower dimensionality, order parameter discontinuous.

Examples of ordered media

Planar spins

Order parameter space is the circumference of a unit circle.

$$f(\mathbf{r}) = \hat{u}\cos\phi(\mathbf{r}) + \hat{v}\sin\phi(\mathbf{r})$$

Ordinary spins

Unit vector can point in any direction in 3D.
Order parameter space is surface of unit sphere.

$$f(\mathbf{r}) = \mathbf{s}(\mathbf{r}) = s_u(\mathbf{r})\hat{u} + s_v(\mathbf{r})\hat{v} + s_w(\mathbf{r})\hat{w}, \quad s_u^2 + s_v^2 + s_w^2 = 1$$

Nematics

Nematics are like cylinders with continuous rotational symmetry and a flip symmetry.

The function f can be thought of as points on sphere with opposite points identified

or as,

$$f(\mathbf{r}) = \hat{n}(\mathbf{r})\hat{n}(\mathbf{r})$$

Practical use: liquid crystal displays

Biaxial nematics

Three distinct axes of symmetry. The molecule symmetry is of a cuboid (point group D_2).

The order parameter space here is complicated, parameter space for SO(3) with the sets of 4 points identified.

Superfluid He-3

Exotic order parameter, different for phases.

Dipole locked A phase

Order parameter pair of orthonormal axes, $\hat{\phi}^{(1)}, \hat{\phi}^{(2)}$

$$\hat{\phi}^{(1)} \cdot \hat{\phi}^{(1)} = \hat{\phi}^{(2)} \cdot \hat{\phi}^{(2)} = 1, \quad \hat{\phi}^{(1)} \cdot \hat{\phi}^{(2)} = 0$$

or a single complex vector $\mathbf{e} = \hat{\phi}^{(1)} + i\hat{\phi}^{(2)}$
with $\mathbf{e} \cdot \mathbf{e}^* = 1$ and $\mathbf{e} \cdot \mathbf{e} = 0$.

Like biaxial nematic, but no proper symmetry. Order parameter space is like 4D sphere with diametrically opposite points identified.

Dipole-free A phase

Order parameter like $A(\mathbf{r}) = \hat{n}(\mathbf{r})e(\mathbf{r})$, locally five dimensional.

Order parameter pitfalls

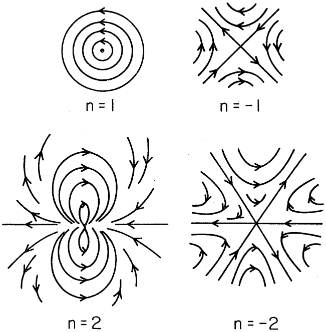
Usually we start with some objects (vectors, projection operators) and then specify the order parameter space.

Points to note:

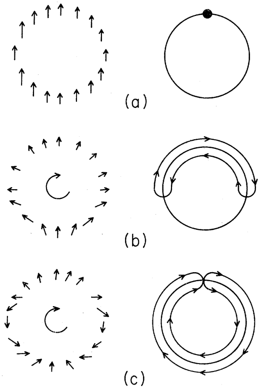
- Points in O.P. space in 1-1 correspondence with the “objects”.
- Representation is continuous

Planar spins 2D

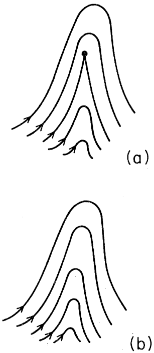
Concluding that the vector field is singular somewhere in an internal region by examining a contour around that region.



Winding number, removal of 0 w.n. singularity



spin configurations on circular contours



removal of a zero winding number singularity

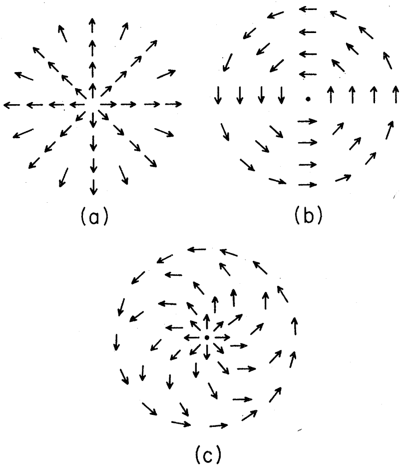
Specifying order parameter along real contour $\Rightarrow$ mapping of contour into order parameter space.

In the planar spin case, two *mappings* can be transformed into each other nonviolently iff their winding number is same. Thus winding number of the contour around the defect **characterises** the defect.

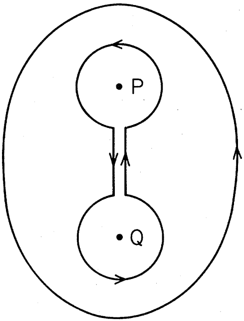
Winding number zero $\Rightarrow$ topologically unstable / removable defect.

Topologically unstable need not imply physically unstable.

Patching and combining



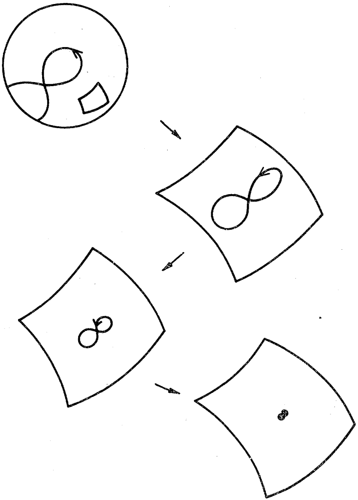
two planar spin configurations, w.n. 1



combining two defects, winding numbers add

Ordinary spins in the plane

All defects are unstable.



Two loops are HOMOTOPIC if they are continuously deformable into each other. Corresponding singularities are TOPOLOGICALLY EQUIVALENT.

Explicit construction of the deformation is a HOMOTOPY. So if $f_0(\mathbf{r})$ and $f_1(\mathbf{r})$ are two homotopic mappings, then the homotopy is a function $h(t,\mathbf{r})$, where $h(0,\mathbf{r}) = f_0(\mathbf{r})$ and $h(1,\mathbf{r}) = f_1(\mathbf{r})$.

Fundamentals of fundamental group

Classifying maps of loops (contours) into order parameter space gives us a way of classifying defects. Each distinct class of loops characterises a defect.

Classes of homotopic maps can be given a group structure (the FUNDAMENTAL GROUP or FIRST HOMOTOPY GROUP of the order parameter space R).

Combination law for elements of fundamental group related to that of combination of point defects (2D) or line defects (3D).

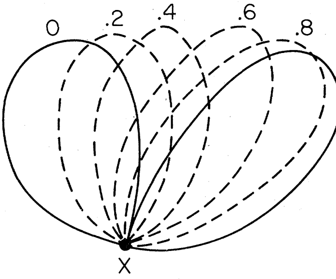
Maps of real space contours into order parameter space give loops in order parameter space.

1. Impose a group structure on set of homotopy classes of loops *based at a point*.
2. Fundamental groups at different base points are isomorphic. So we talk about fundamental group of R (denoted $\pi_1(R)$)

Fundamental group at a point $\pi_1(R, x)$

(Loops at x) $f : [0,1] \to R$,
$f(0) = f(1) = x$.
Homotopies at x:
$h_t(z) : [0,1] \to R$

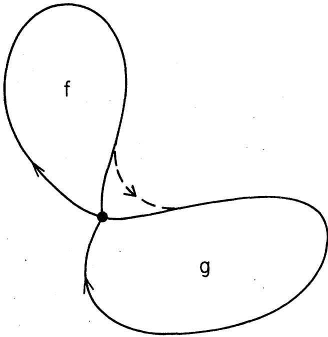
1. $h_0 = f$
2. $h_1 = g$
3. $h_t(0) = h_t(1) = x, \quad \forall t$



Product of two loops

$$
\begin{aligned}
f \circ g(z) &= f(2z), \quad 0 \leq z \leq \frac{1}{2}; \\
&= g(2z-1), \frac{1}{2} \leq z \leq 1
\end{aligned}
$$

Problem with using this as
group operation:
parametrisation, comes into
play when proving
associativity:
$f \circ (g \circ k) = (f \circ g) \circ k$.
Way out?



Product of homotopy classes of loops

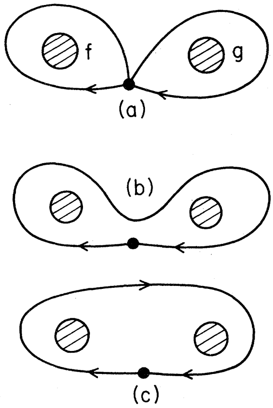
$[f]$ set of loops homotopic (at x) to f.
Loops at x separated into distinct *classes of mutually homotopic* loops.

$$[f] \circ [g] = [f \circ g]$$

Product homotopy class does not depend upon particular representative, as

$$f \sim f',\ g \sim g' \Rightarrow f \circ g \sim f' \circ g'$$

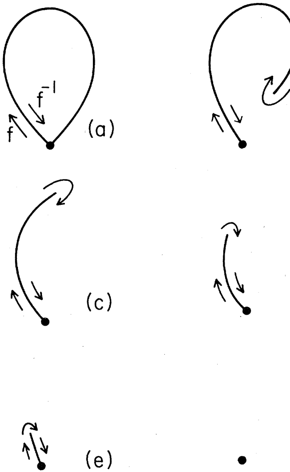
Freed from parametrisation problem!



Fundamental group at x

1. associativity, parametrisation problem solved.
2. identity, $e(z) = x \forall z$. $[e]$ all loops which can be shrunk to point.
3. inverse, $f^{-1}(z) = f(1-z),\ 0 \le z \le 1$, $[f]^{-1} = [f^{-1}]$

Just to show that $\alpha \circ \alpha^{-1} = e$. To show $f \circ f^{-1} \sim e$:



$$\begin{aligned} h_t(z) &= f(2zt),\ 0 \le z \le 1/2; \\ &= f(2t(1-z)), 1/2 \le z \le 1 \end{aligned}$$

Simple examples

Circle: $\pi_1(S_1) = Z$

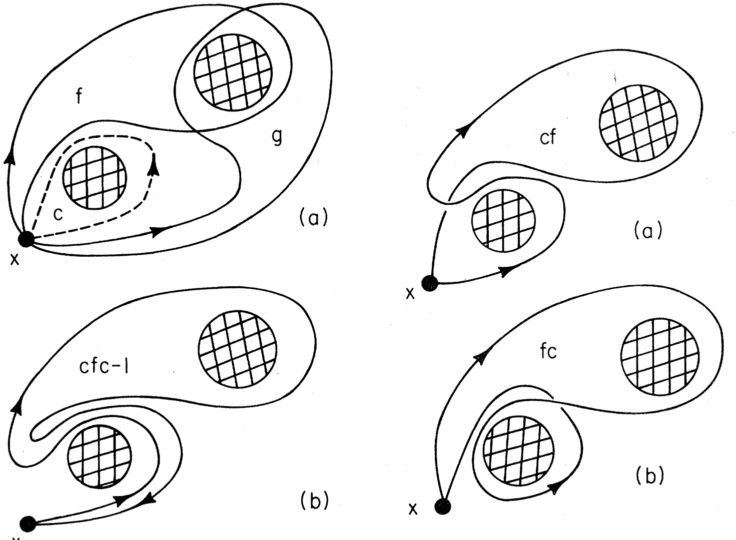
That is why winding numbers are additive in the same way as integers.

Sphere: $\pi_1(S_2) = 0$

In both the above cases, it is obvious that the base point does not matter, that will be proved later.

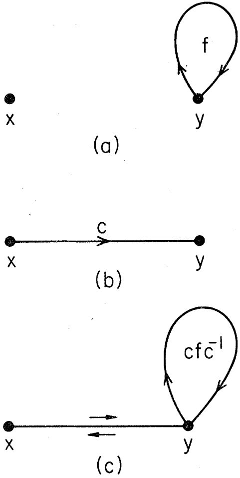
The figure 8 space

Simplest example of case with non-abelian fundamental group, $c \circ f \circ c^{-1} \sim g$. Note that $c \circ f$ and $f \circ f$ are freely homotopic.



Isomorphism of f.g. at different base points

Construction of a natural *path isomorphism* between $\pi_1(R,x)$ and $\pi_1(R,y)$.
What's a path? $c : [0,1] \to R$, $c(0) = x$ and $c(1) = y$.
$c([f]) = [cfc^{-1}]$,
$c[f_1] = c[f_2] \Rightarrow f_1 = f_2$.



It also follows from

$$(cfc^{-1})(cgc^{-1}) \sim c(fg)c^{-1}$$

that $c(\alpha)c(\beta) = c(\alpha\beta)$.

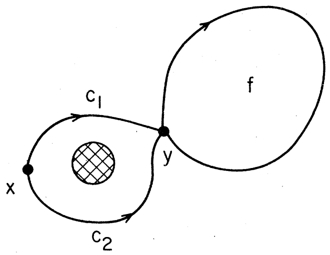
Thus $\alpha \to c(\alpha)$ is isomorphism between $\pi_1(R,x)$ and $\pi_1(R,y)$.

Fundamental group abelian $\Rightarrow$ 1-1 between classes of freely homotopic loops and group elements, otherwise 1-1 with conjugacy classes.

Uniqueness (or lack thereof) of the path isomorphism

Proposition: Path isomorphism independent of path iff f.g. abelian.

Suppose path isomorphism is path dependent. Two paths from x to y, c_1 and c_2 and class of loops $\alpha \in \pi_1(R, y)$ s.t. $c_1[\alpha] \neq c_2[\alpha]$.

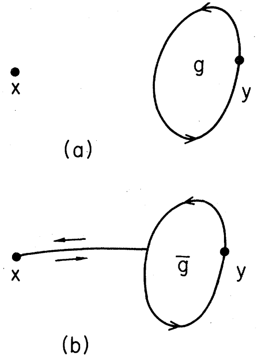


Assertions:

1. nonhomotopic at x: $c_1 f c_1^{-1}$ and $c_2 f c_2^{-1}$
2. nonhomotopic at y: f and $(c_1^{-1}c_2) f (c_1^{-1}c_2)^{-1}$.
3. $c_1^{-1}c_2$ is y-based loop, g. Then $f \not\simeq gfg^{-1}$
4. $fg \not\simeq gf$ at y. $\pi_1(R,y)$ thus $\pi_1(R)$ nonabelian

Converse, suppose $\pi_1(R)$ and thus $\pi_1(R,y)$ non-abelian. Thus:

1. There are 2 y based loops, f and g with $f \not\sim gfg^{-1}$
2. Replace g by homotopic (at y) loop $\bar{g}$, as shown
3. Decompose $\bar{g}$ into c_1, c_2 with $\bar{g} = c_2^{-1}c_1$.
4. $c_1fc_1^{-1} \not\sim c_2fc_2^{-1}$



If $\pi_1(R)$ abelian, unique isomorphism between the different based f.g. (the path isomorphism which is independent of path).

$\pi_1(R)$ nonabelian, various path iso can differ, but only by inner automorphism of $\pi_1(R,x)$. e.g. $k = c_1 c_2^{-1}$ loop at x, two mappings of $\pi_1(R,y)$ to $\pi_1(R,x)$ related by $c_1 f c_1^{-1} \sim k(c_2 f c_2^{-1})k^{-1}$.

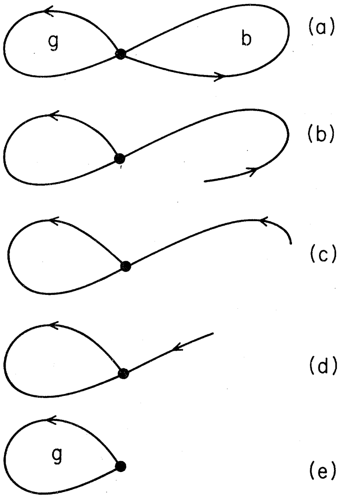
Isomorphic images differ by i.a. $\alpha \to [k]\circ\alpha\circ[k]^{-1}$.

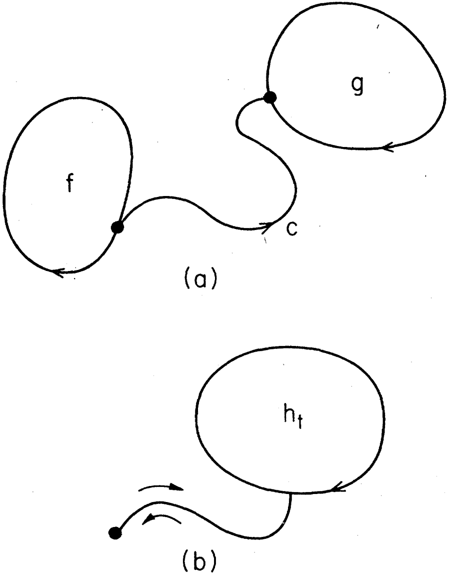
Given any i.a. of $\pi_1(R,x)$ we can take loop from $[k]$, break it into two, and use those two generate path iso between the two based f.g.

Conjugacy classes of group invariant under inner automorphisms.

Path iso: unique correspondence between conjugacy classes of based f.g. (elements for abelian f.g.)

Line defect characterised by set of loops **equivalent under free homotopy** in order parameter space. Two distinct elements in f.g. may be freely homotopic even though elements are distinct (when they belong to same conjugacy class), $f \sim bgb^{-1}$. Converse true (proof).



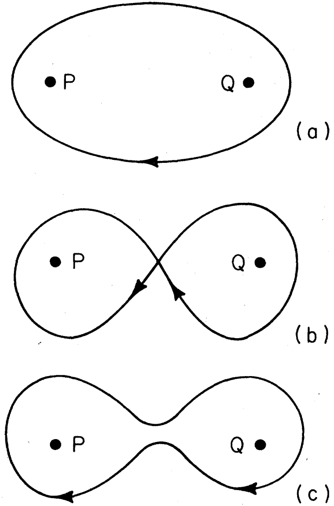


Constructing homotopy between two based f.g., labeling a loop

Fundamental group abelian implies elements of f.g. characterise defects.

Non-abelian case more intricate, discussed later.

Single defect equivalent to combined line defects characterised by conjugacy classes of f.g. (members products of members of classes of original defects). Again abelian f.g makes life simpler ⇒ combination law simply group multiplication.



Some more definitions

A connected space R is SIMPLY CONNECTED if its f.g. contains only identity, example S_2. All loop shrinkable to point.

Theorem

R is product of two spaces R_1, R_2. Usual extension of continuity.

$$\pi_1(R_1 \times R_2) = \pi_1(R_1) \times \pi_1(R_2)$$

Group theoretic structure of order parameter space

Till now, an intuitive notion of the order parameter space (cubes, spheres, etc.)

Formalising the notion in terms of group structure helps in computation of the fundamental group.

Continuous groups

Kind of like Lie groups but without the need for an explicit parametrisation. A group with the notion of continuity and convergence.

Group operations are **continuous**.

1. G is a topological group, G_0 is the subset connected to identity, then G_0 is a normal subgroup of G.
2. Disjoint connected components of G are cosets of the subgroup G_0
   - 2.1 each coset is connected
   - 2.2 any elements in G that can be joined by a continuous path of elements are in the same coset of G_o.

Crucial: g_t sequence of group elements, ag_t continuous.

$\pi_0(G)$ is G/G_0. The zeroth homotopy group (maps of points into R).

Group theoretic description of o.p. space

Essence: We fix a *reference* order parameter, say f and consider the group G of continuous transformations which act on it to give the entire order parameter space.

The group G need not be the smallest that is necessary.

The ISOTROPY SUBGROUP H_f: operations which leave f unchanged: $gf = f$. In general not normal ($f_2 = gf_1$, $H_{f_2} = gH_{f_1}f^{-1}$.

The structure we shall get is independent of the particular choice of reference order parameter. If we change the ref. o.p from f to f', then $H \to gHg^{-1}$ $(f' = gf)$.

Structures from $G,\ H_f$ converted to $G,\ H_{f'}$ by i.a. $G \to gGg^{-1}$. Continuous transformation.

The result (shall be proved):

Order parameter space $R = G/H$.

Why $R = G/H$

What needs to be shown:

- 1-1 correspondence ($af = bf = f'$, then $aH = bH$)
- Continuity (If g_n is convergent sequence in G then g_nH is convergent in $G/H = R$.

Symmetry completely broken $\Rightarrow$ H is only identity and G is the order parameter space.

Summary: G symmetry group of a disordered phase, H subgroup giving symmetry of ordered phase.

Examples

Planar spins: $G = SO(2)$, $H = 0$. What if we took $G = O(2)$?
Same thing.
Or we could take the translation group $T(1)$.

Spins in the form $\mathbf{s} = \hat{x}\cos\theta + \hat{y}\sin\theta$. Group operations:
$\theta \to \theta - \phi$. Isotropy subgroup is translations through 2π.
$G/H = SO(2)$ (H normal subgroup, G abelian).

Ordinary spins $R = G/H$ = SO(3)/SO(2).

Or we can take the larger group SU(2), then H has matrices of from $\exp(i\theta\sigma_z)$ iso. to U(1). So $R =$SU(2)/U(1)

Nematics: $R =$ SO(3)/D_∞.

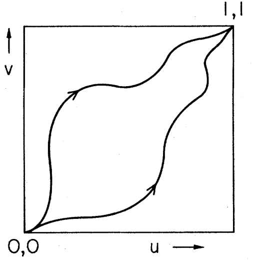
Biaxial nematics: $R =$SU(2)/Q, where Q is the quaternion group.

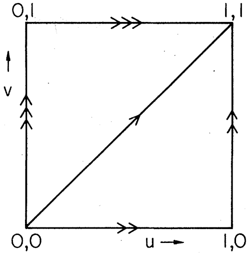
Superfluid He-3: dipole locked A phase $R =$ SO(3).

Dipole free A phase: $G =$SO(3) $\times$ SO(3).

Properties of the f.g. of topological group

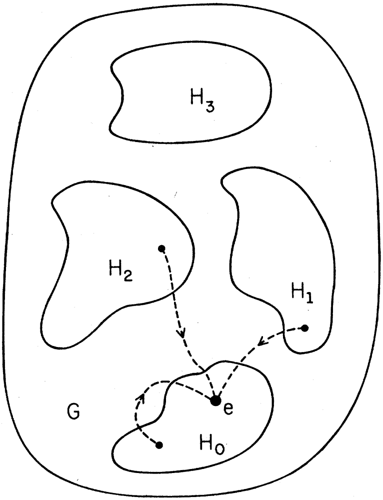
Fundamental group of a continuous group is abelian.

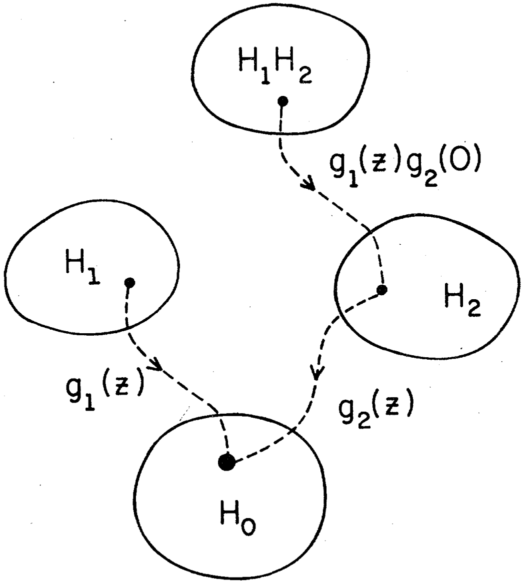




Fundamental group of the o.p. space

Main result:
$\pi_1(G/H) = H/H_0$ where H_0 is the set of points in H that are connected to identity by continuous paths lying entirely in H.
Correspondence between loops in coset space and connected components of the isotropy subgroup.




H_1H_2
$g_1(z)g_2(0)$
H_1
H_2
$g_1(z)$
$g_2(z)$
H_0

Examples

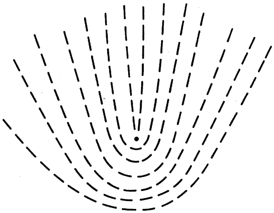
Planar spins: $H = \{T_{2\pi n}, n = 0, 1, 2, \ldots\}$, $\pi_1(R) = Z$.

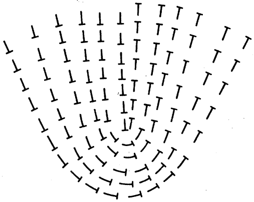
Ordinary spins: $\pi_1(R) = 0$

Nematics: $\pi_1(R) = Z_2$.

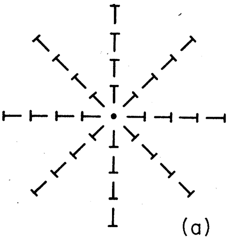
Biaxial nematics: $\pi_1(R) = Q$ (nonabelian)

$$H = \{\pm 1, \pm i\sigma_x, \pm i\sigma_y, \pm i\sigma_z\}$$

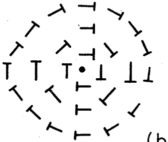




Escape in 3D



(a)



(b)

F.G. corresponding to Superfluid He-3

Dipole locked A phase: $\pi_1(R) = Z_2$.

Dipole free A phase: $\pi_1(R) = Z_4$.

Media with nonabelian f.g.

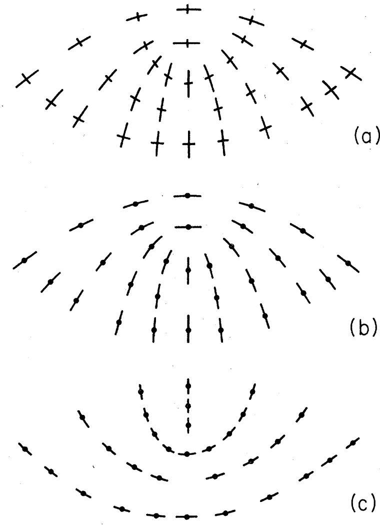
Here the elements of the f.g. do not correspond to classes of freely homotopic loops, instead the conjugacy classes do.

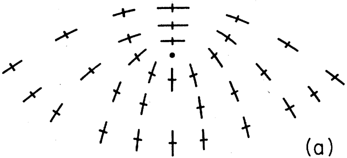
Biaxial nematics, conjugacy classes of Q

$C_0 = \{1\}$, $\bar{C}_0 = \{-1\}$,

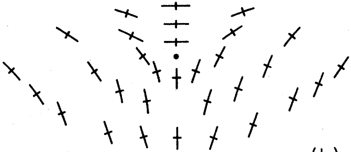
$C_x = \{\pm i\sigma_x\}$, $C_y = \{\pm i\sigma_y\}$,

$C_z = \{\pm i\sigma_z\}$.





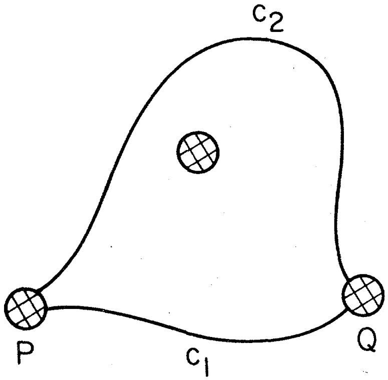
(a)



(b)

	C_0	$\overline{C}_0$	C_x	C_y	
C_0	C_0	$\overline{C}_0$	C_x	C_y	
$\overline{C}_0$	$\overline{C}_0$	C_0	C_x	C_y	
C_x	C_x	C_x	$2C_0+2\overline{C}_0$	$2C_z$	
C_y	C_y	C_y	$2C_z$	$2C_0+2\overline{C}_0$	
C_z	C_z	C_z	$2C_y$	$2C_x$	2

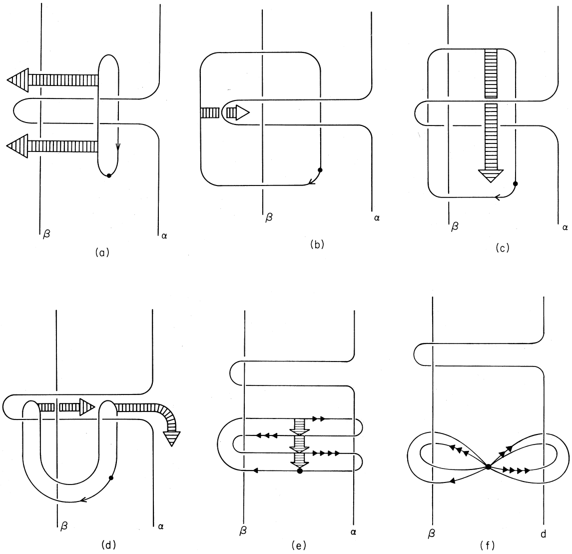
Combination of defects in n.a. case



The second homotopy group

Group of maps of spheres into the order parameter space, to classify point defects in 3D.

Fundamental theorem: $\pi_2(G/H) = \pi_1(H_0)$. G should be simply connected and $\pi_2(G) = 0$.



β
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β
α
(e)
β
d
(f)

Some other applications

To crystal dislocations

The third homotopy group can be applied to solitons

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