---

1. (a)
PDF of Rayleigh Distribution is given by:
$f(r) = \xi\, r\, exp\{-\xi\, r^2/2\}$

Since, we know that the above distribution's PDF depends on parameter ξ, we can say that:
$f(r|\xi) = f(r) = \xi\, r\, exp\{-\xi\, r^2/2\}$

Also, we know that prior on ξ is exponential with the rate parameter λ, therefore:

Prior: $\pi(\xi) = \lambda\, exp\{-\lambda\, \xi\}$

Therefore, using Bayes theorem:

Posterior: $\pi(\xi/r) \propto f(r|\xi) * \pi(\xi)$
$= C\, \xi\, r\, exp\{-\xi\, r^2/2\} * \lambda\, exp\{-\lambda\, \xi\}$ [where C is some constant]
$= C\, \xi\, \lambda\, r\, exp\{-\xi\, r^2/2 - \lambda\, \xi\}$
$= C\, \xi\, \lambda\, r\, exp\{-\xi\, r^2/2 - \lambda\, \xi\}$
$= C\, \xi\, \lambda\, r\, exp\{-\xi\, (\lambda + r^2/2)\}$
We didn't compute marginal here, because our likelihood $f(r|\xi)$ and prior $\pi(\xi)$ belong to the same family of distributions and differ by their degrees of freedom. More details:
(https://en.wikipedia.org/wiki/Relationships_among_probability_distributions#Multiple_of_a_random_variable)

PDF of Gamma distribution is given by:
$f(\xi, \alpha, \beta) = \beta^{\alpha}\xi^{\alpha-1}e^{-\beta\xi}/\Gamma(\alpha)$

Comparing the kernels, clearly we can see that $\pi(\xi/r)$ is a Gamma distribution with:
$\alpha - 1 = 1 \Rightarrow \alpha = 2$
$\beta = \lambda + r^2/2$
Therefore, $\pi(\xi/r)$ is a Gamma distribution with $\alpha = 2$ and $\beta = \lambda + r^2/2$
$\Rightarrow \pi(\xi/r) = Ga(\alpha, \beta) = Ga(2, \lambda + r^2/2)$

---

1. (b)
Assuming $r_1, r_2, \ldots, r_n$ are observed:

Likelihood = $f(r_1|\xi) * f(r_2|\xi) * \ldots\ldots * f(r_n|\xi)$
$= \xi\, r_1\, exp\{-\xi\, r_1{}^2/2\} * \xi\, r_2\, exp\{-\xi\, r_2{}^2/2\} * \ldots\ldots\ldots * \xi\, r_n\, exp\{-\xi\, r_n{}^2/2\}$

$= \xi^n\, r_1\, r_2 \ldots r_n\, exp\{-\xi \sum_{i=1}^{n} r_i^2/2\}$

We know that prior on ξ is exponential with the rate parameter λ, therefore:

Prior: $\pi(\xi) = \lambda\, exp\{-\lambda\, \xi\}$

Using Bayes theorem:
Posterior: $\pi(\xi/r_1, r_2, \ldots, r_n) \propto Likelihood * Prior$
$= C\, \xi^n\, r_1\, r_2 \ldots r_n\, exp\{-\xi \sum_{i=1}^{n} r_i^2\} * \lambda\, exp\{-\lambda\, \xi\}$
$= C\, \xi^n\, \lambda\, r_1\, r_2 \ldots r_n\, exp\{-\xi \sum_{i=1}^{n} r_i^2/2 - \lambda\, \xi\}$
$= C\, \xi^n\, \lambda\, r_1\, r_2 \ldots r_n\, exp\{-\xi\, (\lambda + 1/2 * \sum_{i=1}^{n} r_i^2)\}$
We didn't compute marginal here, because our likelihood $_{i=1}{}^{n} f(r_i|\xi)$ and prior $\pi(\xi)$ belong to the same family of distributions and differ by their degrees of freedom. More details: (https://en.wikipedia.org/wiki/Relationships_among_probability_distributions#Multiple_of_a_random_variable)

PDF of Gamma distribution is given by:
$f(\xi, \alpha, \beta) = \beta^{\alpha} \xi^{\alpha-1} e^{-\beta\xi} / \Gamma(\alpha)$
Comparing the kernels, clearly we can see that $\pi(\xi/r_1, r_2, \ldots, r_n)$ is a Gamma distribution with:
$\alpha - 1 = n \Rightarrow \alpha = n + 1$
$\beta = \lambda + 1/2 * \sum_{i=1}^{n} r_i^2$
Therefore, $\pi(\xi/r_1, r_2, \ldots, r_n)$ is a Gamma distribution with $\alpha = n + 1$ and $\beta = \lambda + 1/2 * \sum_{i=1}^{n} r_i^2$
=> $\pi(\xi/r_1, r_2, \ldots, r_n) = Ga(\alpha, \beta) = Ga(n + 1, \lambda + 1/2 * \sum_{i=1}^{n} r_i^2)$

As given in the question, we know that $R_1 = 3$, $R_2 = 4$, $R_3 = 2$, and $R_4 = 5$

Therefore, in the distribution above: $\alpha = n + 1 = 4 + 1 = 5$ (given n = 4)
$\beta = \lambda + 1/2 * \sum_{i=1}^{n} r_i^2$
$= \lambda + 1/2 * (3^2 + 4^2 + 2^2 + 5^2)$
$= \lambda + 1/2 * (3^2 + 4^2 + 2^2 + 5^2)$
$= \lambda + 1/2 * (9 + 16 + 4 + 25)$
$= \lambda + 1/2 * (54)$
$= \lambda + 27$
=> This distribution will be Gamma $(5, \lambda + 27)$
E(ξ) of Gamma (ξ, α, β) is given by α/β
=> Bayes estimator of ξ is given by E(ξ)
E(ξ) for Gamma $(5, \lambda + 27) = 5/(\lambda + 27)$
Therefore, here the Bayes estimator of ξ is $5/(\lambda + 27)$

1.(c)

For $\lambda = 1$ in Gamma distribution obtained in part (b), we get distribution: Gamma(5, 28)

To find 95% (equal-tailed) credible set for ξ, we run the following two commands in R:
(R code attached in file gammaCredibleSet.r)

a) qgamma(0.025, shape = 5, rate = 28)
   0.05798166
b) qgamma(0.975, shape = 5, rate = 28)
   0.365771

Therefore, our 95% (equal-tailed) credible set for ξ is (0.05798166, 0.365771)

---

2. (a)

Elicitation of Beta prior on proportion that models the oncologist's beliefs:
$\mu = 0.9$
$\mu - 2\sigma = 0.8 \Rightarrow \sigma = 0.05$
For a Beta(p, α, β) distribution:
Mean = E (p) = $\alpha/(\alpha + \beta) = \mu = 0.9$
=> $\alpha = 0.9\alpha + 0.9\beta$
=> $0.1\alpha = 0.9\beta$
=> $\alpha = 9\beta$

Variance = $\alpha\beta/(\alpha + \beta)^2(\alpha + \beta + 1) = \sigma^2 = 0.05^2$
=> $9\beta^2/((10\beta)^2 * (10\beta + 1)) = 0.0025$
=> $10\beta + 1 = 36$
=> $\beta = 3.5$
=> $\alpha = 9\beta = 9 * 3.5 = 31.5$

=> $\pi(p) = Beta(\alpha, \beta) = Beta(31.5, 3.5)$

Likelihood:
Since there is a trial in which 30 patients treated and 22 responded, this distribution can be represented by Bin(n,p).
=> X|p ∼ Bin(n,p)
=> $f(x|p) = {}^{n}C_x p^x (1-p)^{n-x}$
=> $f(x|p) = {}^{30}C_{22}\, p^{22}(1-p)^8$ {as n = 30, x = 22}

Finding Posterior distribution:
Given prior: $\pi(p)$ = Beta(α, β) = $p^{\alpha-1}(1-p)^{\beta-1}/B(\alpha,\beta)$

Given likelihood: $f(x|p) = {}^{n}C_x p^x (1-p)^{n-x}$

=> Posterior $\pi(p|x) \propto f(x|p) * \pi(p)$

$= c * {}^{n}C_x\, p^x\, (1-p)^{n-x} * p^{\alpha-1}(1-p)^{\beta-1}/B(\alpha,\beta)$ {Where c is some constant}

$= c * {}^{n}C_x\, p^{x+\alpha-1}(1-p)^{n+\beta-x-1}/B(\alpha,\beta)$

The above distribution models Beta(x + α, n - x + β) = Beta(22 + 31.5, 30 - 22 + 3.5) = Beta(53.5, 11.5)

Therefore our posterior distribution is: Beta(53.5, 11.5)

Bayes estimator of p in Beta(x + α, n - x + β) = (x + α)/(n + α + β)
= 53.5/(53.5 + 11.5) = 53.5/65 = 0.8231

---

2. (b)
To find 95% (equal-tailed) credible set for p, we run the following two commands in R:
(R code attached in file betaCredibleSet.r)

1. qbeta(0.025, 53.5, 11.5, 0)

   0.7222732
2. qbeta(0.975, 53.5, 11.5, 0)

   0.9051004

Therefore, our 95% (equal-tailed) credible set for p is (0.7222732, 0.9051004)

---

2. (c)
(MATLAB code attached in file Hypothesis_test.m)
Hypothesis test:
H_0 : p >= ⅘ vs H_1 : p < ⅘

From (a), we know that posterior is given by Beta(53.5, 11.5)
Evaluating Posteriors integrals :

$p_1 = \int_{-\infty}^{4/5} Beta(53.5,\ 11.5)dp$

Used MATLAB command given below to evaluate above integral:
betacdf(4/5,53.5, 11.5)
0.29457

=> p_1 = 0.29457

=> p_0 = 1 - p_1 = 1 - 0.29457 = 0.70543

From (a), we know that prior is given by Beta(53.5, 11.5)
Evaluating Prior integrals:

$\pi_1 = {}_{-\infty}\int^{4/5} Beta(31.5,\ 3.5)dp$

Used MATLAB command given below to evaluate above integral:

```
betacdf(4/5,31.5, 3.5)
0.041495
```

=> π_1 = 0.041495

$\pi_0 = 1 - \pi_1 = 1 - 0.041495 = 0.958505$

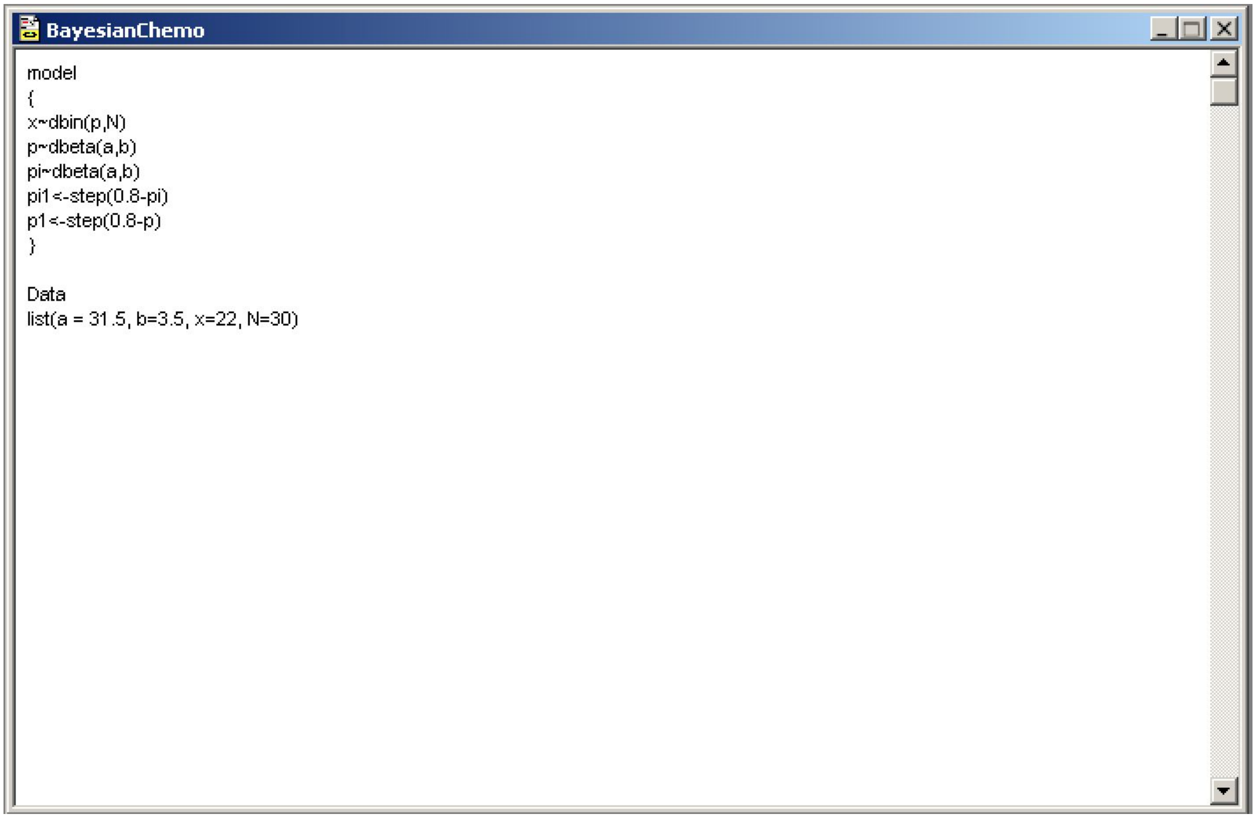
Therefore,
$B_{10} = (p_1/p_0)/(\pi_1/\pi_0) = (0.29457/0.70543)/(0.041495/0.958505) = 9.64568785228$

$\log(B_{10}) = 0.98433320344$

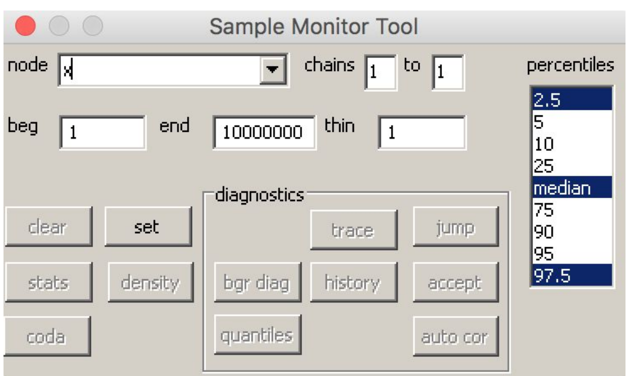
Based on the Bayesian calibration table discussed in the lectures, here $0.5 < \log(B_{10}) = 0.98433320344 <= 1$ means substantial evidence against H_0.

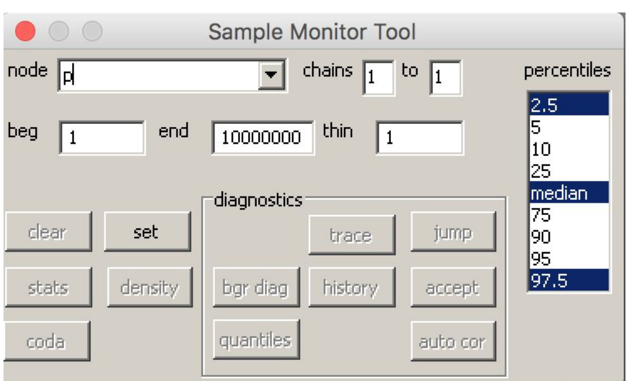
---

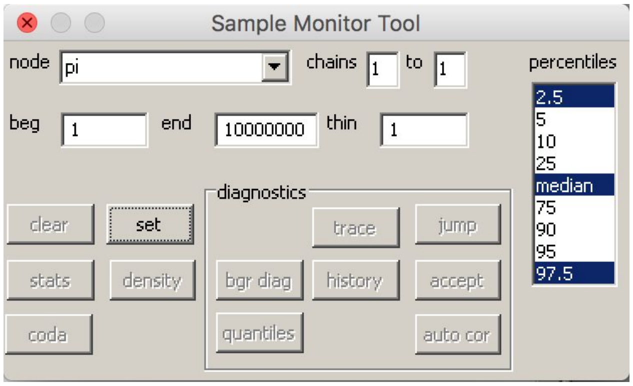
2. (d)
(OpenBUGS code attached in file BayesianChemo.odc)
Using below code in OpenBUGS:

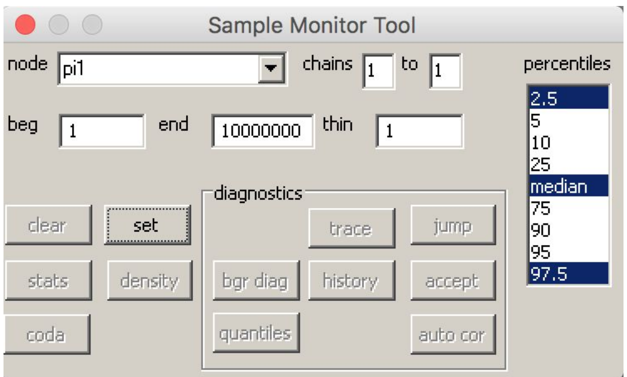


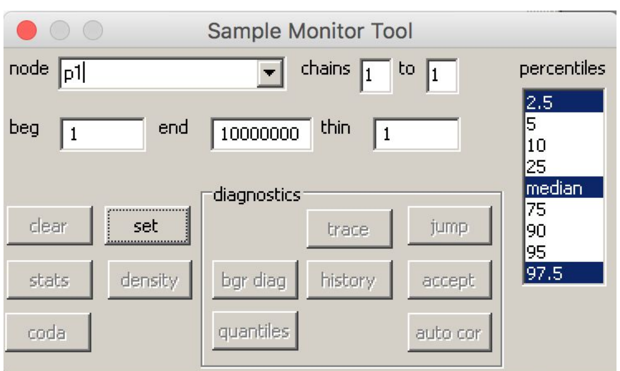
```
BayesianChemo
model
{
x~dbin(p,N)
p~dbeta(a,b)
pi~dbeta(a,b)
pi1<-step(0.8-pi)
p1<-step(0.8-p)
}

Data
list(a = 31.5, b=3.5, x=22, N=30)
```

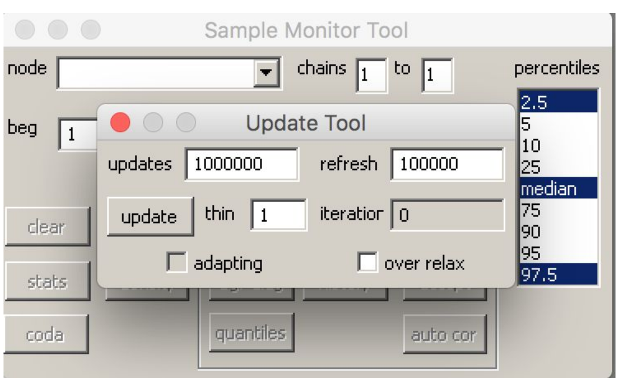


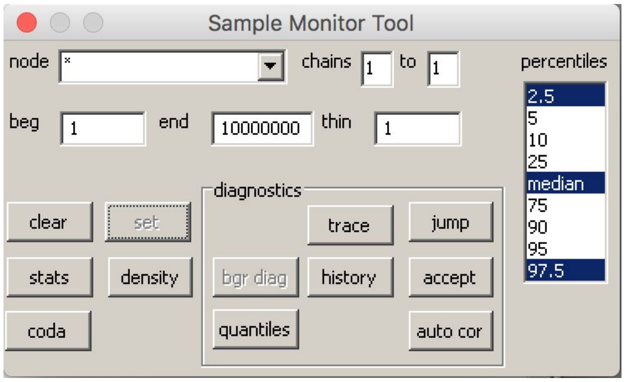


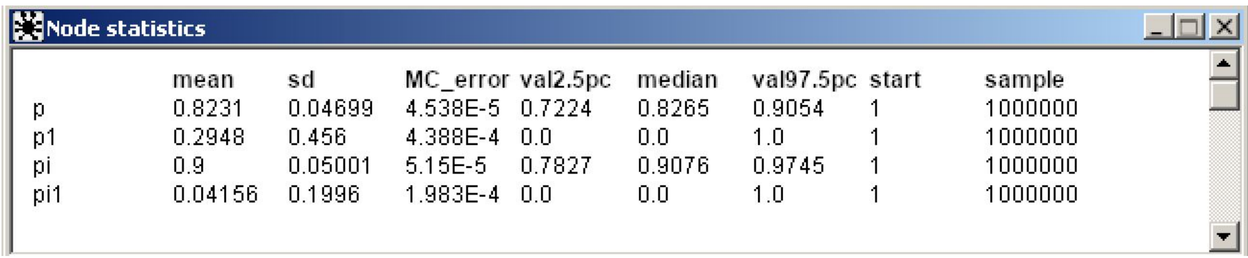

Sample Monitor Tool
node pi
chains 1 to 1
percentiles
beg 1
end 10000000
thin 1
diagnostics
clear
set
trace
jump
stats
density
bgr diag
history
accept
coda
quantiles
auto cor
2.5
5
10
25
median
75
90
95
97.5


Sample Monitor Tool
node pi1
chains 1 to 1
percentiles
beg 1
end 10000000
thin 1
diagnostics
clear
set
trace
jump
stats
density
bgr diag
history
accept
coda
quantiles
auto cor
2.5
5
10
25
median
75
90
95
97.5


Sample Monitor Tool
node p1
chains 1 to 1
percentiles
beg 1
end 10000000
thin 1
diagnostics
clear
set
trace
jump
stats
density
bgr diag
history
accept
coda
quantiles
auto cor
2.5
5
10
25
median
75
90
95
97.5


Sample Monitor Tool
node
chains 1 to 1
percentiles
beg 1
Update Tool
updates 1000000
refresh 100000
update
thin 1
iteratior 0
adapting
over relax
clear
stats
coda
quantiles
auto cor
2.5
5
10
25
median
75
90
95
97.5




Node statistics

	mean	sd	MC_error	val2.5pc	median	val97.5pc	start	sample
p	0.8231	0.04699	4.538E-5	0.7224	0.8265	0.9054	1	1000000
p1	0.2948	0.456	4.388E-4	0.0	0.0	1.0	1	1000000
pi	0.9	0.05001	5.15E-5	0.7827	0.9076	0.9745	1	1000000
pi1	0.04156	0.1996	1.983E-4	0.0	0.0	1.0	1	1000000

As obtained above we can see that mean of p = 0.8231 which is exactly the same as the Bayesian estimate of p that we got in part (a).
As obtained above, the 95% credible set of p is given by (0.7224, 0.9054) which is almost the same to the credible set for p given by (0.7222732, 0.9051004) that we obtained in part (b).

For hypothesis testing:
Using step function in OpenBUGS, I have calculated:
p_1 = mean of p1 in the node statistics obtained in the last screenshot = 0.2948
(this is very similar to p_1 = 0.29457 we got in part (c))

=> $p_0 = 1 - p_1 = 1 - 0.2948 = 0.7052$
(this is very similar to p_0 = 0.70543 we got in part (c))

π_1 = mean of pi1 in the node statistics obtained in the last screenshot = 0.04156
(this is very similar to π_1 = 0.041495 we got in part (c))

=> $\pi_0 = 1 - \pi_1 = 1 - 0.04156 = 0.95844$
(this is very similar to π_0 = 0.958505 we got in part (c))

$B_{10} = (p_1/p_0)/(\pi_1/\pi_0) = (0.2948/0.7052)/(0.04156/0.95844) = 9.64061117277$
(this is very similar to B_{10} = 9.64568785228 we got in part (c))

$\log(B_{10}) = 0.98410456715$

(this is very similar to $\log(B_{10})= 0.98433320344$ we got in part (c))

Based on the Bayesian calibration table discussed in the lectures, $0.5 < \log(B_{10}) = 0.98410456715 <= 1$ means substantial evidence against H_0. This result is the exact same as what we got in part (c).
