Homework-2

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1 Cell Clusters in 3D Petri Dishes.

Poisson distribution always has $mean = \lambda$. From question, we know that this Poisson distribution has $mean = 5 \Rightarrow \lambda = 5$. PMF of Poisson distribution is given by:

$$f(x) = p(X = x) = \frac{\lambda^x * e^{-\lambda}}{x!}$$

The distribution we should get for $\lambda = 5$ is given by:

$$f(x) = p(X = x) = \frac{5^x * e^{-5}}{x!}$$

However, the CDF (cumulative distribution function) is given by

$$F(x) = p(X \leq x) = e^{-\lambda} \sum_{m=0}^{x} \frac{\lambda^m}{m!}$$

1. The percentage of Petri dishes that have (a) 0 clusters

$$\begin{aligned} f(0) = p(X = 0) &= \frac{5^0 * e^{-5}}{0!} \\ &= e^{-5} \\ &= 0.0067 \\ &= 0.67\% \end{aligned}$$

2. The percentage of Petri dishes that have (b) at least one cluster

$$\begin{aligned} P(X \geq 1) &= 1 - P(X \leq 0) \\ &= 1 - P(X = 0) \\ &= 1 - 0.0067 \\ &= 0.9933 \\ &= 99.33\% \end{aligned}$$

3. The percentage of Petri dishes that have(c) more than 8 clusters

$$
\begin{aligned}
P(X > 8) &= 1 - F(8) = 1 - P(X \leq 8) \\
&= 1 - \sum_{m=0}^{8} \frac{5^m * e^{-5}}{m!} \\
&= 1 - (\frac{5^0 * e^{-5}}{0!} + \frac{5^1 * e^{-5}}{1!} + \frac{5^2 * e^{-5}}{2!} + \frac{5^3 * e^{-5}}{3!} + \frac{5^4 * e^{-5}}{4!} + \frac{5^5 * e^{-5}}{5!} + \frac{5^6 * e^{-5}}{6!} + \\
&\frac{5^7 * e^{-5}}{7!} + \frac{5^8 * e^{-5}}{8!}) \\
&= 1 - e^{-5}(1 + 5^1 + \frac{5^2}{2} + \frac{5^3}{6} + \frac{5^4}{24} + \frac{5^5}{120} + \frac{5^6}{720} + \frac{5^7}{5040} + \frac{5^8}{40320}) \\
&= 0.0681 \\
&= 6.81\%
\end{aligned}
$$

4. The percentage of Petri dishes that have (d) between 4 and 6 clusters inclusive.

$$
\begin{aligned}
P(4 \leq X \leq 6) &= f(4) + f(5) + f(6) \\
&= \frac{5^4 * e^{-5}}{4!} + \frac{5^5 * e^{-5}}{5!} + \frac{5^6 * e^{-5}}{6!} \\
&= 0.4972 \\
&= 49.72\%
\end{aligned}
$$

2 Silver-Coated Nylon Fiber.

1. Find the probabilities that (a) a run continues for at least 10 hours. Given T follows exponential distribution with rate parameter $\frac{1}{10}$. A run T continues for at least 10 hours is

$$
\begin{aligned}
P((T \geq 10) &= \int_{10}^{\infty} \frac{1}{10} e^{-\frac{1}{10}x} dx \\
&= \frac{1}{10} \int_{10}^{\infty} e^{-\frac{1}{10}x} dx \\
&= -[0 - (e^{-\frac{1}{10}10})] \\
&= -[0 - e^{-1}] \\
&= \frac{1}{e} \\
&= 0.368
\end{aligned}
$$

2. Find the probabilities that (b) a run lasts less than 15 hours,

$$
\begin{aligned}
p(T < 15) &= \int_0^{15} \frac{1}{10} e^{-\frac{1}{10}x} dx \\
&= \frac{1}{10} \int_0^{15} e^{-\frac{1}{10}x} dx \\
&= -[(e^{-\frac{1}{10}*15}) - (e^{-\frac{1}{10}*0})] \\
&= -[e^{-1.5} - 1] \\
&= 1 - e^{-1.5} \\
&= 0.777
\end{aligned}
$$

3. Find the probabilities that (c) a run continues for at least 20 hours, given that it has lasted 10 hours.

$$
\begin{aligned}
P(T \geq 20 | T \geq 10) &= \frac{P(T \geq 20) \cap P(T \geq 10))}{P(T \geq 10)} \\
&= \frac{P(T \geq 20)}{P(T \geq 10)} \\
&= \frac{\int_{20}^{\infty} \frac{1}{10} e^{-\frac{1}{10}x dx}}{\int_{10}^{\infty} \frac{1}{10} e^{-\frac{1}{10}x} dx} \\
&= \frac{e^{-2}}{e^{-1}} \\
&= \frac{1}{e} \\
&= 0.368
\end{aligned}
$$

3 2-D Density Tasks.

(a) Show that: (a) marginal distribution $f_x(x)$.

$$
\begin{aligned}
f_x(x) &= \int f(x, y) dy \\
&= \int_x^{\infty} \lambda^2 * e^{-\lambda * y} dy \\
&= -\lambda * e^{-\lambda * y} \\
&= -(0 - \lambda * e^{-\lambda * x}) \\
&= \lambda * e^{-\lambda * x}
\end{aligned}
$$

Therefore, marginal distribution $f_x(x)$ is exponential (λ).

(b) Show that (b) marginal distribution $f_y(y)$. is Gamma $Ga(2, \lambda)$.

$$\begin{aligned} f_y(y) &= \int f(x,y)dx \\ &= \int_0^y \lambda^2 * e^{-\lambda * y} dx \\ &= \lambda^2 * e^{-\lambda * y} \int_0^y dx \\ &= \lambda^2 * e^{-\lambda * y} * y \\ &= \lambda^2 * y^{2-1} * e^{-\lambda * y} \end{aligned}$$

We know that Gamma function is given by: $f(y, \alpha, \beta) = \frac{\beta^{\alpha} * y^{\alpha-1} * e^{-\beta * y}}{\tau(\alpha)}$
Clearly from distribution $f_y(y)$, $\alpha = 2$ and $\beta = \lambda$.
Therefore, marginal distribution $f_y(y)$. is given by $Ga(2, \lambda)$.

(c) conditional distribution $f(y|x)$ is shifted exponential,

$$\begin{aligned} f(y|x) &= \frac{f(x,y)}{f(x)} \\ &= \frac{\lambda^2 * e^{-\lambda * y}}{\lambda * e^{-\lambda * x}} \\ &= \lambda * e^{-\lambda(y-x)} \end{aligned}$$

(d) conditional distribution $f(x|y)$ is uniform $U(0, y)$.

$$\begin{aligned} f(x|y) &= \frac{f(x,y)}{f(y)} \\ &= \frac{\lambda^2 * e^{-\lambda * y}}{\lambda^2 * y * e^{-\lambda * y}} \\ &= \frac{1}{y} \\ &= \frac{1}{y-0} \text{ for } 0 \le x \le y \end{aligned}$$

For all other values for x, $f(x, y) = 0 \Rightarrow f(x|y) = 0$.

The PDF of uniform distribution is given as $\frac{1}{b-a}$ from a to b.
By comparing, we get a = 0, and b = y. Therefore, $f(x|y)$ is uniform with $U(0, y)$.

4 Nylon Fiber Continued.

(a)

$$
\begin{aligned}
Likelihood &= (\lambda * e^{-3*\lambda})(\lambda * e^{-13*\lambda})(\lambda * e^{-8*\lambda}) \\
&= \lambda^3 * e^{-24*\lambda}
\end{aligned}
$$

To maximize this likelihood using classical statistics, we differentiate w.r.t λ and equate it to 0.

$$
\begin{aligned}
\frac{d}{d\lambda}(\lambda^3 * e^{-24\lambda})) &= 0 \\
3 * \lambda^2 * e^{-24\lambda} - 24\lambda^3 * e^{-24\lambda} &= 0 \\
e^{-24\lambda} * 3(1 - 8\lambda) * \lambda^2 &= 0 \\
\lambda &= \frac{1}{8}, 0
\end{aligned}
$$

Here, we got two values of λ. However, we neglect $\lambda = 0$ as for $\lambda = 0$ distribution won't be exponential. Therefore, the value at which likelihood will be maximized is $\frac{1}{8}$ and this value will be the classical statistician estimate of λ.

(b) Prior $= \pi(\lambda) = \frac{1}{\sqrt{\lambda}}$
Likelihood $= \lambda^3 * e^{-24\lambda}$
posterior $\propto prior * Likelihood$
$posterior = C * prior * Likelihood$
(where C is some constant).
$posterior = C\lambda^{\frac{5}{2}} e^{-24\lambda}$
$Gamma(x, \alpha, \beta) = \frac{\beta^\alpha}{\tau(\alpha)} * x^{\alpha-1} * e^{-\beta*x}$

Clearly posterior is a gamma distribution with $\beta = 24, \alpha = \frac{7}{2}$.
Also, Bayes estimator of an unknown parameter is given by it's mean of posterior distribution. Here, the posterior distribution is Gamma and the E(x) of a $Gamma(x, \alpha, \beta)$ is given by $\frac{\alpha}{\beta}$. Similarly,

$$
\begin{aligned}
E(\lambda) &= \frac{\alpha}{\beta} \\
&= \frac{7}{2 * 24} \\
&= \frac{7}{48}
\end{aligned}
$$

Therefore, the Bayes estimator of $\lambda = \hat{\lambda} = \frac{7}{48}$