Audio Filter

EE23BTECH11023 - Abhignya *

I. Software Installation

I.1 Run the following commands(for laptop)

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1
    python3−scipy python3−numpy python3−
    matplotlib
sudo pip install cffi pysoundfile
```

I.2 Run the following commands(for termex(mobile))

```
apt−get update
apt−get install libffi−dev libsndfile1 python3−
    scipy python3−numpy python3−
    matplotlib
apt install python3−cffi python3−soundfile
```

II. Digital Filter

II.1 The sound file used for this code is obtained from the below link

```
https://github.com/abhignya08/Audio_Filter/
    blob/main/codes/abhi.wav
```

II.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem in the spectrogram and play. Observe the spectrogram. What do you find?

Solution: The audio file is analyzed using spectrogram using the online platform https://academo.org/demos/spectrum-analyzer. There are a lot of yellow lines between 440 Hz to 1KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

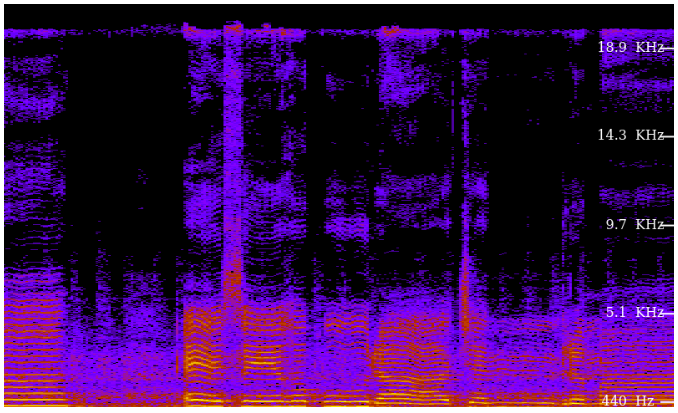

Fig. 1. Spectrogram of the audio file before Filtering

II.3 A Python Code is written to achieve Audio Noise Filtering

```
import soundfile as sf
from scipy import signal

#read .wav file
input_signal,fs = sf.read('abhi.wav')

#sampling frequency of Input signal
sampl_freq=fs

#order of the filter
order=4

#cutoff frquency 1kHz
cutoff_freq=6000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order,Wn, 'low')
print(a)
print(b)
#filter the input signal with butterworth filter
#output_signal = signal.filtfilt(b, a,
    input_signal)
output_signal = signal.lfilter(b, a,
input_signal)

#write the output signal into .wav file
sf.write('abhi_filtered.wav', output_signal, fs
    )
```

II.4 The output of the python script in Problem II.3 is the audio file Sound-with-Reduced-noise.wav. Play the file in the spectrogram in Problem II.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 6 kHz.

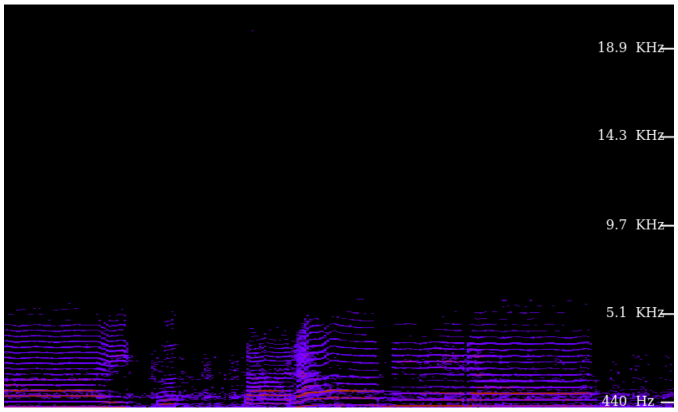

Fig. 2. Spectrogram of the audio file after Filtering

III. Difference Equation

III.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{1}$$

Sketch $x(n)$.

III.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2), \qquad y(n) = 0, n < 0 \tag{2}$$

Sketch $y(n)$.
Solve
Solution: The C code calculates $y(n)$ and generates values in a text file.

```
https://github.com/abhignya08/Audio_Filter/
    blob/main/codes/3.c
```

The following code plots (1) and (2)

```
https://github.com/abhignya08/Audio_Filter/
    blob/main/codes/3.py
```

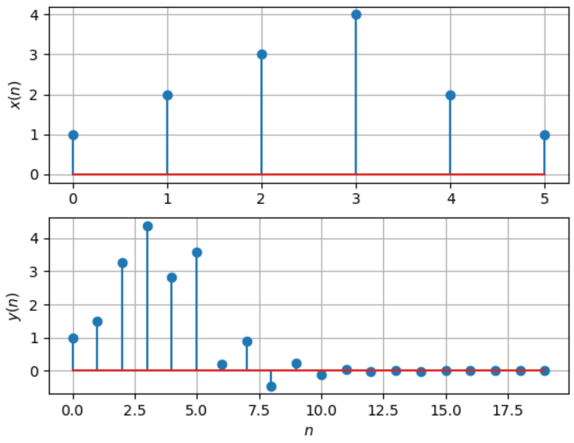

Fig. 3. Plot of $x(n)$ and $y(n)$

IV. Z-Transform

IV.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{3}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{5}$$

Solution: From (3),

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{6}$$

$$\text{Lets, take } n-1 = k \tag{7}$$

$$= \sum_{k=-\infty}^{\infty} x(k)z^{-(k+1)} \tag{8}$$

$$= z^{-1}\sum_{k=-\infty}^{\infty} x(k)z^{-k} \tag{9}$$

$$= z^{-1}X(z) \tag{10}$$

resulting in (4). Similarly, it can be shown that

$$\implies \mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{11}$$

IV.2 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{12}$$

from (2) assuming that the Z-transform is a linear operation.

Solution: Applying (11) in (2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{13}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{14}$$

IV.3 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{17}$$

Solution: It is easy to show that

$$\delta(n) \overset{Z}{\leftrightarrow} 1 \tag{18}$$

and from (16),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{19}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{20}$$

using the formula for the sum of an infinite geometric progression.

IV.4 Show that

$$a^n u(n) \overset{Z}{\leftrightarrow} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{21}$$

Solution:

$$a^n u(n) \overset{Z}{\leftrightarrow} \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \tag{22}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} \tag{23}$$

$$= 1 + az^{-1} + a^2 z^{-2} + ... \tag{24}$$

$$= \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{25}$$

IV.5 Let

$$H\left(e^{j\omega}\right) = H\left(z = e^{j\omega}\right). \tag{26}$$

Plot $\left|H\left(e^{j\omega}\right)\right|$. Comment. $H(e^{j\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$.

Solution: The following code plots the magnitude of transfer function.

```
https://github.com/abhignya08/Audio_Filter/
    blob/main/codes/4.5.py
```

Substituting $z = e^{j\omega}$ in (14), we get

$$\left|H\left(e^{j\omega}\right)\right| = \left|\frac{1 + e^{-2j\omega}}{1 + \frac{1}{2}e^{-j\omega}}\right| \tag{27}$$

$$= \sqrt{\frac{(1 + \cos 2\omega)^2 + (\sin 2\omega)^2}{\left(1 + \frac{1}{2}\cos\omega\right)^2 + \left(\frac{1}{2}\sin\omega\right)^2}} \tag{28}$$

$$= \frac{4|\cos\omega|}{\sqrt{5 + 4\cos\omega}} \tag{29}$$

$$\left|H\left(e^{j(\omega+2\pi)}\right)\right| = \frac{4|\cos(\omega + 2\pi)|}{\sqrt{5 + 4\cos(\omega + 2\pi)}} \tag{30}$$

$$= \frac{4|\cos\omega|}{\sqrt{5 + 4\cos\omega}} \tag{31}$$

$$= \left|H\left(e^{j\omega}\right)\right| \tag{32}$$

Therefore its fundamental period is 2π , which verifies that DTFT of a signal is always periodic.

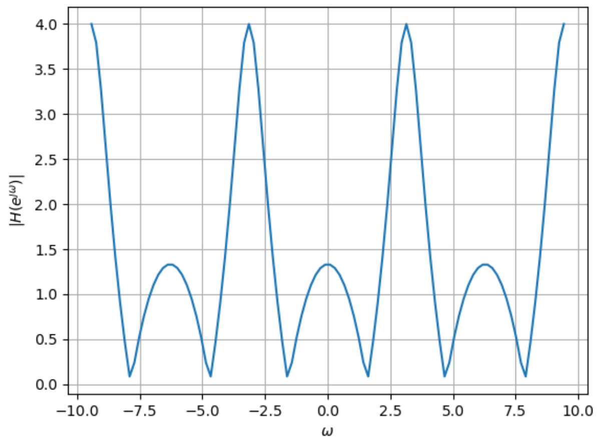

Fig. 4. $\left|H\left(e^{j\omega}\right)\right|$

V. Impulse Response

V.1 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{Z}{\leftrightarrow} H(z) \tag{33}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse*

response of the system defined by (2).
Solution: From (14),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{34}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{35}$$

using (21) and (11).

V.2 Sketch $h(n)$. Is it bounded? Convergent?
Solution:

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{36}$$

$$h(n) \text{ is convergent equation} \tag{37}$$

$$\left(\frac{-1}{2}\right)^n \to 0, \text{ when } n \to \infty \text{ So,} \tag{38}$$

$$h(n) \to 0, \text{ when } n \to \infty \tag{39}$$

The following code plots $h(n)$

```
https://github.com/abhignya08/Audio_Filter/
    blob/main/codes/5.2.py
```

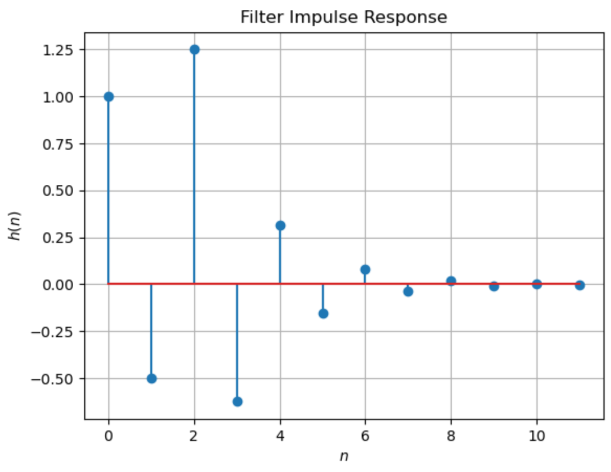

Fig. 5. $h(n)$ as the inverse of $H(z)$

V.3 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{40}$$

Is the system defined by (2) stable for the impulse response in (33)?

Solution: For stable system (40) should converge.
By using ratio test for convergence:

$$\lim_{n\to\infty} \left| \frac{h(n+1)}{h(n)} \right| < 1 \tag{41}$$

$$\tag{42}$$

For large n

$$u(n) = u(n-2) = 1 \tag{43}$$

$$\lim_{n\to\infty} \left(\frac{h(n+1)}{h(n)} \right) = 1/2 < 1 \tag{44}$$

Hence it is stable.

V.4 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{45}$$

This is the definition of $h(n)$.
Solution:
Definition of $h(n)$: The output of the system when $\delta(n)$ is given as input.

The following code plots Fig. 6. Note that this is the same as Fig. 5.

```
https://github.com/abhignya08/Audio_Filter/
    blob/main/codes/hndef.py
```

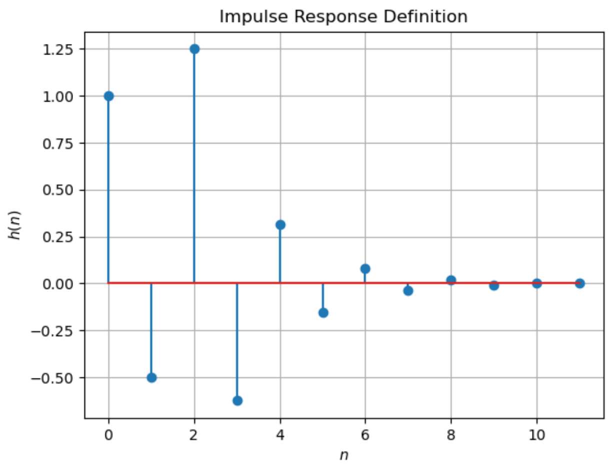

Fig. 6. $h(n)$ from the definition is same as Fig. 5

V.5 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \tag{46}$$

Comment. The operation in (46) is known as *convolution*.
Solution: The following code plots Fig. 7. Note that this is the same as $y(n)$ in Fig. 3.

```
https://github.com/abhignya08/Audio_Filter/
    blob/main/codes/5.5.py
```

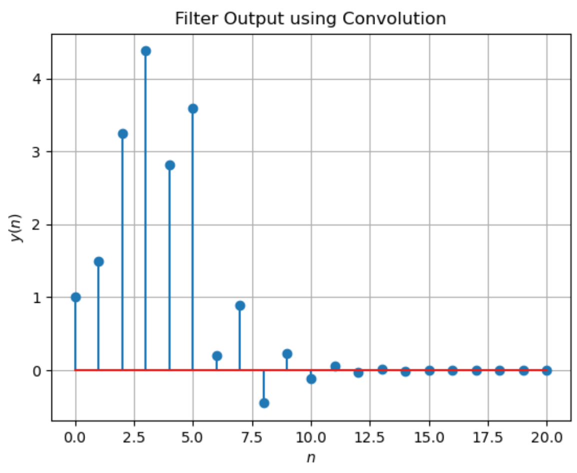

Fig. 7. $y(n)$ from the definition of convolution

V.6 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{47}$$

Solution: In (46), we substitute $k = n - k$ to get

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \tag{48}$$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)\, h(k) \tag{49}$$

$$\implies y(n) = \sum_{k=-\infty}^{\infty} x(n-k)\, h(k) \tag{50}$$

Hence,proved

VI. DFT and FFT

VI.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{51}$$

and $H(k)$ using $h(n)$.
Solution:

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{52}$$

$$H(k) = \sum_{n=0}^{N-1} h(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{53}$$

VI.2 Compute

$$Y(k) = X(k)H(k) \tag{54}$$

VI.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{j2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \tag{55}$$

Solution: The above three questions are solved using the code below.

```
https://github.com/abhignya08/Audio_Filter/
    blob/main/codes/6.123.py
```

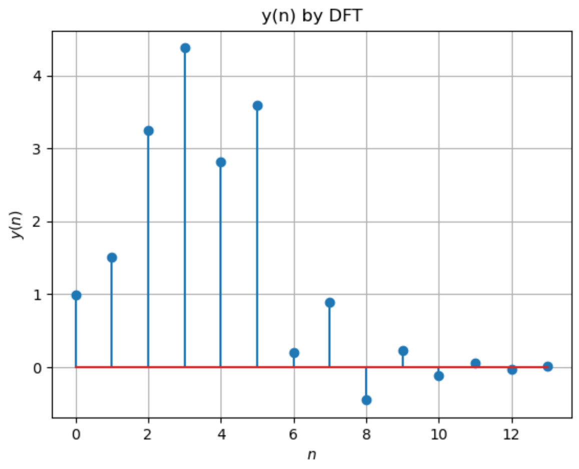

Fig. 8. $y(n)$ obtained from IDFT is plotted

VI.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution: The solution of this question can be found in the code below.

```
https://github.com/abhignya08/Audio_Filter/
    blob/main/codes/6.4.py
```

This code verifies the result by plotting the obtained result with the result obtained by DFT.

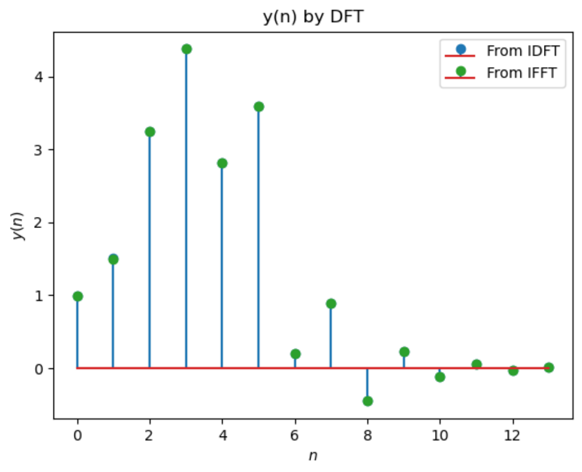

Fig. 9. $y(n)$ obtained from IDFT and IFFT is plotted and verified

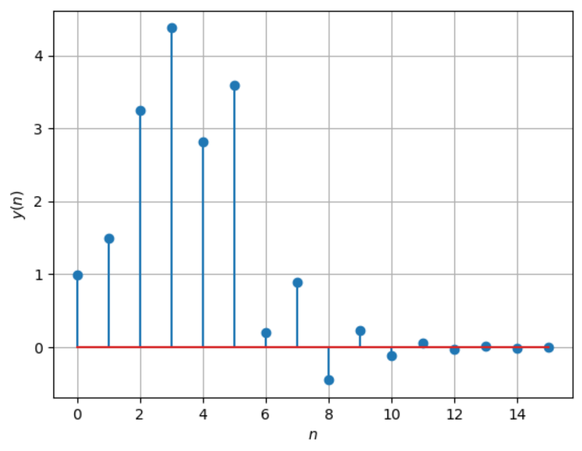

Fig. 10. $y(n)$ obtained from DFT Matrix

VI.5 Wherever possible, express all the above equations as matrix equations.
Solution: The DFT matrix is defined as :

$$\mathbf{W} = \begin{pmatrix} \omega^0 & \omega^0 & \dots & \omega^0 \\ \omega^0 & \omega^1 & \dots & \omega^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^0 & \omega^{N-1} & \dots & \omega^{(N-1)(N-1)} \end{pmatrix} \tag{56}$$

where $\omega = e^{-\frac{j2\pi}{N}}$. Now any DFT equation can be written as

$$\mathbf{X} = \mathbf{W}\mathbf{x} \tag{57}$$

where

$$\mathbf{x} = \begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(n-1) \end{pmatrix} \tag{58}$$

$$\mathbf{X} = \begin{pmatrix} X(0) \\ X(1) \\ \vdots \\ X(n-1) \end{pmatrix} \tag{59}$$

Thus we can rewrite (54) as:

$$\mathbf{Y} = \mathbf{X} \odot \mathbf{H} = (\mathbf{W}\mathbf{x}) \odot (\mathbf{W}\mathbf{h}) \tag{60}$$

where the $\odot$ represents the Hadamard product which performs element-wise multiplication.

```
https://github.com/abhignya08/Audio_Filter/
    blob/main/codes/6.5.py
```

VII. Exercises

Answer the following questions by looking at the python code in Problem II.3.

VII.1 **The command**

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem II.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{61}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal. filtfilt** with your own routine and verify.

Solution: The below code gives the output of an Audio Filter without using the built in function signal.lfilter.

```
https://github.com/abhignya08/Audio_Filter/
    blob/main/codes/7.1.py
```

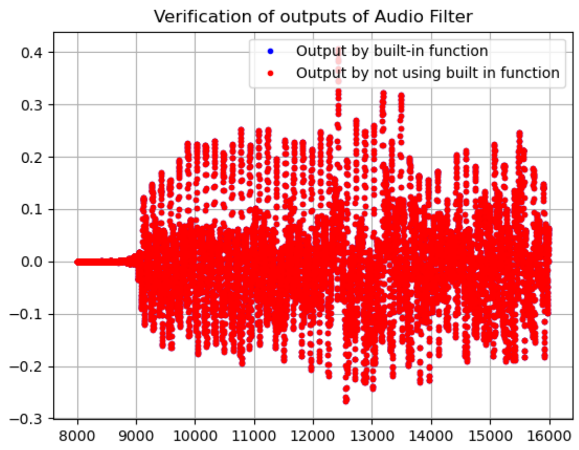

Fig. 11. Both the outputs using and without using function overlap

VII.2 Repeat all the exercises in the previous sections for the above a and b.
Solution: The code in II.3 generates the values of a and b which can be used to generate a difference equation.
And,

$$M = 5 \tag{62}$$
$$N = 5 \tag{63}$$

From 61

$$a(0)\,y(n) + a(1)\,y(n-1) + a(2)\,y(n-2) + a(3) \tag{64}$$
$$y(n-3) + a(4)\,y(n-4) = b(0)\,x(n) + b(1)\,x(n-$$
$$+\,b(2)\,x(n-2) + b(3)\,x(n-3) + b(4)\,x(n-4)$$

Difference Equation is given by :

$$\begin{aligned} &y(n) - (1.97)\,y(n-1) + (1.73)\,y(n-2) \\ &- (0.724)\,y(n-3) + (0.12)\,y(n-4) \\ &= (0.01)\,x(n) + (0.04)\,x(n-1) \\ &+ (0.06)\,x(n-2) + (0.04)\,x(n-3) \\ &+ (0.01)\,x(n-4) \end{aligned} \tag{65}$$

From (61)

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-N}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-M}} \tag{66}$$

$$H(z) = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{\sum_{k=0}^{M} a(k) z^{-k}} \tag{67}$$

Partial fraction on (67) can be generalised as:

$$H(z) = \sum_i \frac{r(i)}{1 - p(i) z^{-1}} + \sum_j k(j) z^{-j} \tag{68}$$

Now,

$$a^n u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}} \tag{69}$$

$$\delta(n-k) \overset{\mathcal{Z}}{\longleftrightarrow} z^{-k} \tag{70}$$

Taking inverse z transform of (68) by using (69) and (70)

$$h(n) = \sum_i r(i)[p(i)]^n u(n) + \sum_j k(j)\delta(n-j) \tag{71}$$

The below code computes the values of $r(i)$, $p(i)$, $k(i)$ and plots $h(n)$

```
https://github.com/abhignya08/Audio_Filter/
    blob/main/codes/7.2.py
```

$r(i)$	$p(i)$	$k(i)$
$0.06029142 - 0.14682007j$	0.88475217+0.0445749j	$2.19006287e-05$
$0.06029142 + 0.14682007j$	0.88475217-0.0445749j	–
$-0.06029459 + 0.02518904j$	0.94427798+0.11485352j	–
$-0.06029459 - 0.02518904j$	0.94427798-0.11485352j	–

TABLE 1
VALUES OF $r(i), p(i), k(i)$

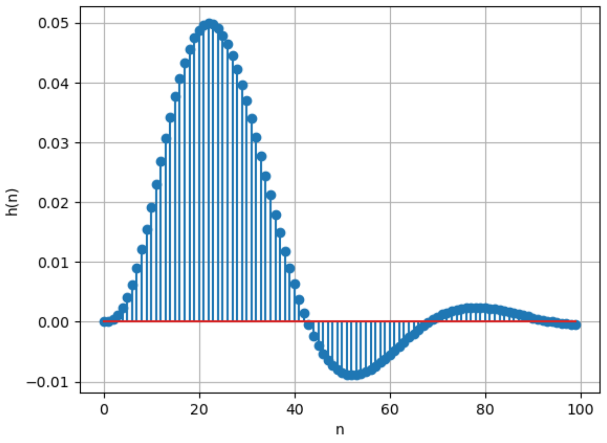

Fig. 12. h(n) of Audio Filter

Stability of h(n):
According to (40)

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} \tag{72}$$

$$H(1) = \sum_{n=0}^{\infty} h(n) = \frac{\sum_{k=0}^{N} b(k)}{\sum_{k=0}^{M} a(k)} < \infty \tag{73}$$

As both $a(k)$ and $b(k)$ are finite length sequences they converge.
The below code plots Filter frequency response

```
https://github.com/abhignya08/Audio_Filter/
    blob/main/codes/filter_response.py
```

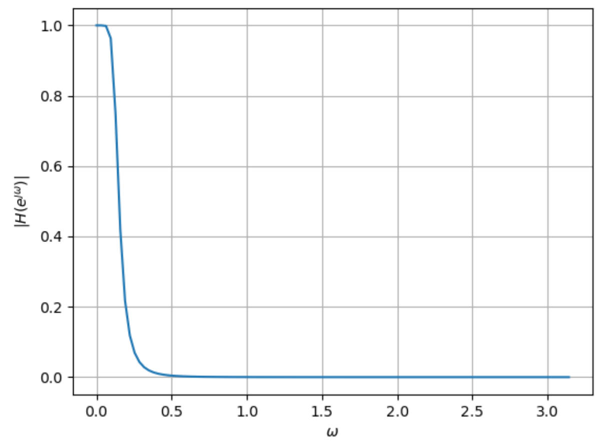

Fig. 13. Frequency Response of Audio Filter

VII.3 What is the sampling frequency of the input signal?
Solution: The Sampling Frequency is 44.1KHz

VII.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low-pass with order=4 and cutoff-frequency=6kHz.

VII.5 Modify the code with different input parameters and get the best possible output.
Solution: A better filtering was found on setting the order of the filter to be 5.