CS182 – Foundations of Computer Science

PSO sessions 1 and 2, week of April 13, 2020

PSO 1

Problem 1. How many permutations of 12345 are there that leave 3 in the third position but leave no other integer in its own position?

Solution 1. Since 3 is always in the same position, we can effectively ignore it in this analysis. One may simply insert the 3 into the middle of any permutation of the remaining four integers.

There are 4! ways to permute four numbers. Thus, 4! strings exist that have 3 in the middle.

But some of those permutations leave the other four digits in their own positions, thus violating the criteria. Let's count how many are in violation, and then subtract them out.

We'll use inclusion-exclusion, since simply adding up the number of ways to keep each integer in the same position overcounts, as some permutations keep more than one integer in its own place.

$\binom{4}{1}3!$ keep at least 1 digit in its place, as the other 3 may freely permute.

$\binom{4}{2}2!$ keep at least 2 digits in their places, as the other 2 may freely permute.

$\binom{4}{3}1!$ keep at least 3 digits in their places, as the last 1 may freely permute.

$\binom{4}{4}0!$ keep at least 4 digits in their places, as none are left to permute.

So the number in violation are

$$\binom{4}{1}3! - \binom{4}{2}2! + \binom{4}{3}1! - \binom{4}{4}0! = 15$$

Meaning $4! - 15 = 9$ strings match the criteria.

Problem 2. In how many ways can five distinct Martians and five distinct Jovians be seated at a circular table if no two Martians sit together.

Solution 2. Since no two martians can sit together, the order of the aliens about the table must be pairwise. That is, for the species denoted A and B, we have the ordering

A B A B A B A B A B

Observe that, since the table is circular, rotations are not distinct. So – in terms of species – there's only one way to do it. But each alien is distinct, so order within each species matters. Let's think about each species individually, and then consider the table's circularity.

There are 2 ways of choosing which species is represented by A. Once the choice is made, each species is free to permute about its 5 positions. So there are 5! ways of placing a given species. Notice too that after the choice for A, the placement of individuals in one species has absolutely no bearing on the placement of individuals in the other.

Therefore, temporarily disregarding the table's circularity, there are $2 \cdot 5! \cdot 5!$ ways of arranging the aliens. Now let's consider the table.

Formally, we have computed the number of ways to arrange 5 Martians and 5 Jovians in a line. But in a circle, we have 10 equivalent rotations. Since the start of the line could represent any one of the 10 seats of the table, each true arrangement has been counted 10 times. Thus, there are actually

$$\frac{2 \cdot 5! \cdot 5!}{10} = \frac{2 \cdot 5! \cdot 5!}{2 \cdot 5} = 4! \cdot 5!$$

distinct arrangements of the aliens.

Problem 3. Show that in a group of ten people (where any two people are either friends or enemies) there are either three mutual friends or four mutual enemies.

Solution 3. Let A be one of the people. Either A has at least 4 friends, or A has at least six enemies.

Case 1: A has at least 4 friends. Consider 4 of A's friends.

- If at least two are friends with each other, then there are 3 mutual friends.
- Otherwise, all 4 are pairwise enemies, so there are four mutual enemies.

Case 2: A has at least 6 enemies. Consider 6 of A's enemies. Let B denote one of them. B will have at least 3 friends or at least 3 enemies.

- If B has at least 3 friends and at least two are friends with each other, then there are three mutual friends. Otherwise, all three are pairwise enemies, and since each is also an enemy of A, we have 4 mutual enemies.
- If B has at least 3 enemies and at least two are enemies with each other, then them also being enemies with A and B means we have 4 mutual enemies. Otherwise, all three are pairwise friends.

PSO 2

Problem 1. Prove **Vandermonde's Identity**:

$$C(m+n,r) = \sum_{k=0}^{r} C(m,r-k)C(n,k).$$

Solution 1. Although a complicated, purely algebraic proof exists, the vast majority of the work can be skipped if we understand what the identity is saying.

Suppose we have two disjoint sets, A and B, of sizes $n \geq r$ and $m \geq r$, respectively. How many ways are there to draw r elements from $A \cup B$? That's what the left side is asking by $C(m+n,r)$.

We shall show that the identity's right side answers the question using a counting construction. Notice, one way to choose r elements from $A \cup B$ is to add up the following:

- The number of ways to take 0 elements from A and r from B
- The number of ways to take 1 element from A and $r-1$ from B
- The number of ways to take 2 elements from A and $r-2$ from B
- ...
- The number of ways to take $r-2$ elements from A and 2 from B
- The number of ways to take $r-1$ elements from A and 1 from B
- The number of ways to take r elements from A and 0 from B

Clearly, the scenarios above don't overlap since A and B are disjoint. For any given scenario, say, k elements from A and $r-k$ elements from B, we have

$$C(n,k)C(m,r-k)$$

since, after making the decision that k shall be taken from A, the individual choices you make for A have no bearing as to the individual choices you make for B. In other words, the Product Rule applies.

Therefore, it's the simple sum of the above pairwise products. Done.

$$\sum_{k=0}^{r} C(m,r-k)C(n,k).$$

Problem 2. A doughnut shop sells 30 kinds of doughnuts. In how many ways can you *(a)* get a bag of 12 doughnuts? *(b)* get a bag of 12 doughnuts if you want at least 3 glazed doughnuts and at least 4 raspberry doughnuts?

Solution 2. This is a version of the famous "Stars and Bars" question, which we'll need to understand before we can answer this problem.

Consider, how many ways are there to permute the word "Missisippi"?

There are 10! ways to permute 10 letters, but since some letters are repeated, this over-counts. Notice that, if we just focus on the letter s, the number of strings not over-counting s get counted 3! times. Likewise, just focusing on i, we have each counted 4! times. Likewise, for p we have each counted 2! times. Lastly, for M we have each counted 1! time. Since each relies on permuting the indistinguishable copies of each letter, they have no effect on one another once places for the letter types are chosen. Therefore, there are really

$$\frac{10!}{3! \cdot 4! \cdot 2! \cdot 1!}$$

ways to do it. Another way to prove the above is by construction. First, we place the 3 s's. There are 10 locations and we need 3 of them, yielding $\binom{10}{3}$ possibilities. Now we place the 4 i's. There are 7 remaining locations, and we need 4 of them. So we have $\binom{7}{4}$ possibilities for each placement of the s's. Next we place the 2 p's. There are 3 remaining locations, and we need 2 of them. So we have $\binom{3}{2}$ possibilities for each placement of the s's and p's. Lastly, the M takes the singular remaining position. So, overall, there are

$$\binom{10}{3}\binom{7}{4}\binom{3}{2}\binom{1}{1} = \frac{10!}{3! \cdot 7!} \cdot \frac{7!}{4! \cdot 3!} \cdot \frac{3!}{2! \cdot 1!} \cdot \frac{1!}{1! \cdot 0!} = \frac{10!}{3! \cdot 4! \cdot 2! \cdot 1!}$$

Now for stars and bars.

How many ways are there to place 5 indistinguishable objects into 2 distinct groups. We're going to create a notation that will let us solve the problem.

For 3 in the first group and 2 in the second, we write	$* * * \mid * *$
For 1 in the first group and 4 in the second, we write	$* \mid * * * *$
For 5 in the first group and 0 in the second, we write	$* * * * * \mid$

Suppose instead we have 3 groups.

$* \mid * * \mid * *$	means 1 in the first, 2 in the second, and 2 in the third.
$* * * \mid \mid * *$	means 3 in the first, 0 in the second, and 2 in the third.

In other words, placing k objects into m groups is isomorphic to permuting the string of k stars and $m-1$ bars. From the 'Mississippi' example, we know that there are

$$\frac{(k+m-1)!}{k! \cdot (m-1)!} = \binom{k+m-1}{k}$$

ways to do it. So the question is, can we turn our doughnut question into a stars and bars question.

Part a. We have 12 doughnuts to pick, and 30 flavors. The picks can be the stars, and the flavors can be the bars! We can imagine buying a specific doughnut flavor as a pick being an indistinguishable object we can place into one of the 30 flavor groups. Thus, there are 12 stars and $30 - 1$ bars.

$$\binom{12+30-1}{12} = \binom{41}{12}$$

Part b. 3 glazed and 4 raspberries simply means some of the choices have been made for us. So we have 5 remaining stars, but still $30 - 1$ bars.

$$\binom{5+30-1}{5} = \binom{34}{5}$$

Problem 3. Assume that you have 50 pennies and three jars, labeled A, B, and C. In how many ways can you put the pennies in the jars, assuming that the pennies are distinguishable?

Solution 3. If they were indistinguishable, we'd have to use Stars and Bars. But since each is unique, we may do the following.

For each penny, place it in one of the 3 jars. Each choice has no effect on any other, thus, there are 3^{50} ways to do it by the Product Rule.