Problem 1.

Prove that $n^3 + 3n$ is a multiple of two for all $n \geq 1$.

We shall use induction.

Base Case
Base case where $n = 1$

$n^2 + 3n \rightarrow (1)^2 + 3(1) = 4$

$4 = 2(2) \rightarrow 2 \mid n^2 + 3n$

Hence the base case holds!

Inductive Hypothesis
Let's assume that, for $k \geq 1$

$$2 \mid k^2 + 3k$$

Inductive Step
Consider the claim for $k + 1$

$(k + 1)^2 + 3(k + 1)$ expands to

$(k^2 + 2k + 1) + (3k + 3)$ which is equal to

$(k^2 + 3k) + (2k + 4)$ which is, by the Inductive Hypothesis,

$2a + (2k + 4)$ for some integer a. This can be rewritten as

$2(a + k + 2)$

And since $a + k + 2$ is an integer, $2 \mid (k + 1)^2 + 3(k + 1)$

So the inductive step holds!

Ok, so if the statement is true for k, we've proven it must be true for k+1.
The statement is true for 1. So it's true for all integers greater than 1.

Done.

Problem 2.
Prove that $7^n - 1$ is a multiple of six for all $n \geq 1$.

We shall use induction.

Base Case
Suppose $n = 1$

$$7^n - 1 = 7^1 - 1 = 6 = 6\,(1) \rightarrow 6 \mid 7^n - 1$$

Inductive Hypothesis
Assume that $6 \mid 7^k - 1$ for some $k \geq 1$.

Inductive Step
Consider the claim for $k + 1$

$7^{k+1} - 1$ is equal to

$7(7^k) - 1$ which is

$6(7^k) + 1(7^k) - 1$ and simplifies to

$6(7^k) + (7^k - 1)$.

Notice, $7^k - 1$ is, by the Inductive Hypothesis, a multiple of 6. Ergo we may write:

$6(7^k) + 6m$ for some integer m.

This is equal to $6\,(7^k + m)$, and, since $7^k + m$ is an integer,

$$6 \mid 7^{k+1} - 1$$

So the inductive step holds!
Thus, we have proven the claim by induction.

Problem 3

What is wrong with the following proof by induction?

> Theorem: For every non-negative integer n, 5n = 0.
> Basis Step: 5 0 = 0.
> Induction hypothesis: 5 * j = 0 for all non-negative integers j = 0, 1, …, k.
> Will prove: 5 * (k + 1) = 0. Write k + 1 = i + j, where i and j are natural numbers strictly less than k + 1. By the inductive hypothesis, 5 * i = 5 * j = 0 and thus
>
> $$5 * (k + 1) = 5 * (i + j) = 5i + 5j = 0.$$

This is an (erroneous) proof by Strong Induction.

Yes, if it really is the case that, for a number like 6, you knew for all $h < 6$, that $5h = 0$, then

> 5(6) = 5(5) + 5(1) which is, by the Inductive Hypothesis,
>
> 0 + 0 = 0

But are we actually able to reach numbers like 6?

Let's try the number 1.

> We have, by assumption, that for all $h < 1$, $5h = 0$
>
> But that's just the number 0, and we need two numbers, *i* and *j* that add up to 1.
>
> So we only have 0 to work with, and there's no way 0 + 0 = 1.
>
> So for 5(1) = 5(i) + 5(j), there's no *i* and *j* that we could use.
>
> Ergo, the proof fails.