CS182 – Foundations of Computer Science

Homework 7 (100 points)

Due: Thursday, April 16, 2020, 11:59pm

Upload the homework to Gradescope. DO NOT SUBMIT TO BLACKBOARD LEARN. No late submissions accepted. Only typed solutions will be graded.

<u>Remark:</u> Provide a brief justification for each of your answers (no more than five lines), explaining which counting rules you used and what your thought process was. Feel free to have expressions of the form $3 \cdot 6^{10}$, etc. in your final answers; no need to use calculators to compute such powers.

Problem 1. (45=5*9 points)
Suppose you have 18 objects (10 of type A, 5 of type B, and 3 of type C). Objects of type A are indistinguishable from each other; objects of type B are indistinguishable from each other; and objects of type C are indistinguishable from each other. In how many ways can you:

1. Put the 18 objects in a row?
2. Pick 3 of the 18 objects (order does not matter)?
3. Pick 4 of the 18 objects (order does not matter)?
4. Pick 5 of the 18 objects (order does not matter)?
5. Pick nine objects out of the 18 objects so that exactly three objects are of type A and exactly two objects are of type B (order does not matter)?

Solution-Problem 1
Show you answer in the form of $P(n,r)$ or $C(n,r)$.

1. Step 1: choose where the Type A objects go. There are $C(18,10)$ ways of arranging the Type A Objects and not Type A. Step 2: of the spaces labeled for use by not-Type A, choose which of those spaces will be occupied by Type B: There are $C(18-10,5)$ number of ways to do this. Step 3: of the spaces labeled for use by not-Type A and not-Type B, choose which are occupied by Type C: There are $C(18-10-5,3)$ number of ways. Taking what we have done we can come up with there are $C(18,10) * C(8,5) * C(3,3)$ which is $(\frac{(18!8!3!)}{10!8!5!3!3!0!}) = (\frac{(18!)}{10!5!3!}) = 2450448$ number of ways to accomplish this.
2. Combinations with Repetition. Assume that x_1, x_2 and x_3 is the number of objects we take from A,B and C respectively. We have
$$x_1 + x_2 + x_3 = 3$$
$$C(3+3-1,3)$$
$$C(5,3) = 10$$
is the solution.
3. Combinations with Repetition. An easy approach is to break the problem into different cases where we will solve individually and then sum the combinations. We need to ensure that $x_1 + x_2 + x_3 = 4$, but objects of type C (corresponding to x_3) cannot exceed the number 3. Case 1:
$$x_3 = 0$$
$$x_1 + x_2 = 4$$
$$C(5,4)$$

Case 2:

$$x_3 = 1$$
$$x_1 + x_2 = 3$$
$$C(4, 3)$$

Case 3:

$$x_3 = 2$$
$$x_1 + x_2 = 2$$
$$C(3, 2)$$

Case 4:

$$x_3 = 3$$
$$x_1 + x_2 = 1$$
$$C(2, 1)$$

Finally,
Sum:

$$C(5, 4) + C(4, 3) + C(3, 2) + C(2, 1) = 14$$

is the solution.

4. Combinations with Repetition. Again, the easiest approach is to break the problem into different cases which we will solve individually and then sum the combinations. We need to ensure that $x_1+x_2+x_3 = 5$ but objects of type C (corresponding to x_3) cannot exceed the number 3.
Case 1:

$$x_3 = 0$$
$$x_1 + x_2 = 5$$
$$C(6, 5)$$

Case 2:

$$x_3 = 1$$
$$x_1 + x_2 = 4$$
$$C(5, 4)$$

Case 3:

$$x_3 = 2$$
$$x_1 + x_2 = 3$$
$$C(4, 3)$$

Case 4:

$$x_3 = 3$$
$$x_1 + x_2 = 2$$
$$C(3, 2)$$

Finally,
Sum:

$$C(6, 5) + C(5, 4) + C(4, 3) + C(3, 2) = 18.$$

is the solution

5. Three objects are of $Type - A$, need to find $(x_1 + x_2 + x_3 = 9)$.

$$x_1 = 3$$
$$x_2 = 2$$
$$x_3 = 4$$

The Solution is 0 because x_3 can not be bigger than 3.

Problem 2. (27 = 3*9 points)
A movie theater can play 30 westerns, 15 science fiction movies, and 10 horror movies (all movies are distinct from each other). Its standard daily program typically consists of a western followed by a science fiction movie, and then a horror movie.

1. How many different programs can it play?
2. How many different programs are there if the three movies can be played in any order? How does this number compare to the previous number and why?
3. How many different three-movie programs are there if there are absolutely no restrictions (e.g., the same movie can be played twice, movies can be played in any order, categories do not matter, etc.)? How does this number compare to the previous numbers and why?

Solution-Problem 2

1. We would simple just multiply each total genre of movie with one another as followed: $30 Westerns * 15 ScienceFiction * 10 Horror = 4500$ total different programs.
2. If we assume that we need one movie from each genre: $(30 * 15 * 10) * 3! = 27,000$. We have more options due to freedom of choice.
3. We would begin by summing up all movie selections to get $30 + 15 + 10 = 55$. Because the question asks for how many different three-movie programs we would take the sum and multiply it by itself three times: $(55 * 55 * 55) = 166375$. Again, like the previous problem there are more options due to freedom of choice.

Problem 3. (16 = 2*8 points)
You have 10 of each of the following type of objects: A, B, C, and D. The objects of each type are distinguishable (e.g., the 10 objects of type A are different from each other, think of them as $A_1 \dots A_{10}$; same for the other three types).

1. In how many ways can you arrange all objects in a row?
2. In how many ways can you choose a set S of 10 objects?

Solution-Problem 3

1. Since all objects of A (or B,C,D) can be distinguished from one another, there is no point in having separate categories. Therefore we have 40 objects in total and the solution is (40!).
2. The solution would be $C(40, 10)$.

Problem 4. (12 = 2*6 points)

1. What is the coefficient of x^8y^4 in the expansion of $(4x - 4y)^{28}$?
2. Prove that
$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

Solution-Problem 4

1. The key is the use of the Binomial Theorem.

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} * x^{n-j} * y^j$$

We are interested in the term x^8y^4 in $(4x-4y)^{28}$, where $n = 28$ and $j = 4$

$$(4x-4y)^{28} = 4^{28}(x-y)^{28}$$

We can see that the coefficient is actually 0. This is due to the fact that in $(x+y)^n$ every term has degree n, but in $x^r y^n$ the degree is equal to $8+4 = 12 \neq 28$.

Reducing the problem this far suffices.

2. Need to Prove:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Focusing our work on the right side.(Proof Consists of Maintaining Good Algebra Skills)

$$= \binom{n}{k} + \binom{n}{k-1} = \frac{(n!)}{k!(n-k)!} + \frac{(n!)}{(k-1)!(n-k+1)!}$$

$$= \frac{(n!)(n-k+1) + n!(k)}{(k!(n-k+1)!}$$

$$= \frac{(n+1)!}{(k!(n-k+1)!}$$

$$= \binom{n+1}{k}$$