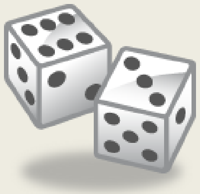
A VERY QUICK INTRODUCTION TO DISCRETE PROBABILITY

DISCRETE PROBABILITY (KR 7.1)

- Finite Probability
- Probability of complements and unions



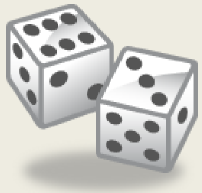
Probability of an Event

Pierre-Simon Laplace (1749-1827)

In the 18th century, Pierre-Simon Laplace developed the classical theory of probability when he analyzed games of chance.

Key terms:

- An *experiment* is a procedure that yields one of a given set of possible outcomes.
- The *sample space* of the experiment is the set of possible outcomes.
- An *event* is a subset of the sample space.



Probability of an Event

Definition: S is a finite sample space of equally likely outcomes and E is an event, $E \subseteq S$, then the *probability* of E is $p(E) = |E| / |S|$.

For every event E, we have $0 \leq p(E) \leq 1$.
This follows directly from the definition as

$$0 \leq p(E) = |E|/|S| \leq |S|/|S| \leq 1$$

and $0 \leq |E| \leq |S|$.

Applying Laplace's Definition

Example: An urn contains four blue balls and five red balls. What is the probability that a ball chosen from the urn is blue?

Solution: The probability that the ball is chosen is $4/9$ since there are nine possible outcomes, and four of these produce a blue ball.

Example: What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7?

Solution: By the product rule there are $6^2 = 36$ possible outcomes. Six of these sum to 7. Hence, the probability of obtaining a 7 is $6/36 = 1/6$.

Applying Laplace’s Definition

Example: In a lottery, a player wins a large prize when they pick four digits that match, in correct order, four digits selected by a random mechanical process.

What is the probability that a player wins the prize?

1822

Solution:

By the product rule there are $10^4 = 10{,}000$ ways to pick four digits.

There is only 1 way to pick the correct digits.
Hence, the probability of winning the large prize is $1/10{,}000 = 0.0001$.

1822

A smaller prize is won if only three digits are matched. What is the probability that a player wins the small prize?

Solution:

If exactly three digits are matched, one of the four digits must be incorrect and the other three digits must be correct. For the digit that is incorrect, there are 9 possible choices.

By the sum rule, there a total of $9\times4=36$ possible ways to choose four digits that match exactly three of the winning four digits.

The probability of winning the small price is $36/10{,}000 = 9/2500 = 0.0036$.

Applying Laplace’s Definition

Example: There are many lotteries that award prizes to people who correctly choose a set of six numbers out of the first *n* positive integers, where *n* is usually between 30 and 60.

What is the probability that a person picks the correct six numbers out of 40?

Solution:

The number of ways to choose six numbers out of 40 is

$$C(40,6) = 40!/(34!6!) = 3,838,380.$$

Hence, the probability of picking a winning combination is $1/\ 3,838,380 \approx 0.00000026$.

Poker hands and probability

Example: What is the probability that a hand of five cards in poker contains four cards of one kind (.e., four 9's, 4 Queens, etc)?

Solution:

The number of ways to choose 5 cards out of 52 is C(52,5) = 2,598,960

Let be the event E that a hand has four cards of one kind.

The number of ways to choose a hand with four cards of a kind is 13×48: 13 ways for the kind and 48 ways for the 5-th card.

Hence $P(E) = 13 \times 48/2{,}598{,}960 \approx 0.00024$

Applying Laplace's Definition

Example:

What is the probability that the numbers 11, 4, 17, 39, and 23 are drawn in that order from a bin with 50 balls labeled with the numbers 1,2, …, 50 if the ball selected is not returned to the bin.

Solution: Use the product rule.

Sampling without replacement:
The probability is $1/(50 \cdot 49 \cdot 48 \cdot 47 \cdot 46)$ since there are $50 \cdot 49 \cdot 48 \cdot 47 \cdot 46$ ways to choose the five balls.

Applying Laplace's Definition

Example:

What is the probability that the numbers 11, 4, 17, 39, and 23 are drawn in that order from a bin with 50 balls labeled with the numbers 1,2, …, 50 if the ball selected is returned to the bin before the next ball is selected.

Solution: Use the product rule.

Sampling with replacement:

The probability is $1/50^5$.

The Probability of Complements and Unions of Events

Theorem 1: Let *E* be an event in sample space *S*. The probability of the event $\bar{E} = S - E$, the complementary event of *E*, is given by

$$p(\overline{E}) = 1 - p(E).$$

Proof: Using the fact that $|\bar{E}| = |S| - |E|$,

$$p(\overline{E}) = \frac{|S| - |E|}{|S|} = 1 - \frac{|E|}{|S|} = 1 - p(E).$$

The Probability of Complements and Unions of Events

Example:

A sequence of 10 bits is chosen randomly. What is the probability that at least one of these bits is 0?

Solution: Let *E* be the event that at least one of the 10 bits is 0. Then, $\bar{E}$ is the event that all of the bits are 1s. The size of the sample space *S* is 2^{10}.

Hence, $p(E) = 1 - p(\overline{E}) = 1 - \frac{|\overline{E}|}{|S|} = 1 - \frac{1}{2^{10}} = 1 - \frac{1}{1024} = \frac{1023}{1024}$.

The Probability of Complements and Unions of Events

Theorem 2: Let E_1 and E_2 be events in the sample space S. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

Proof: Given the inclusion-exclusion formula $|A \cup B| = |A| + |B| - |A \cap B|$, it follows that

$$p(E_1 \cup E_2) = \frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|}$$

$$= \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|}$$

$$= p(E_1) + p(E_2) - p(E_1 \cap E_2).$$

The Probability of Complements and Unions of Events

Example: What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

Solution: Let E_1 be the event that the integer is divisible by 2 and E_2 be the event that it is divisible 5.

Then, the event that the integer is divisible by 2 or 5 is $E_1 \cup E_2$ and $E_1 \cap E_2$ is the event that it is divisible by 2 and 5.

It follows that:

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

$$= 50/100 + 20/100 - 10/100 = 3/5.$$

PROBABILITY THEORY (KR 7.2)

- Assigning Probabilities
- Conditional Probability
- Independence

Assigning Probabilities

Laplace's definition assumes that all outcomes are equally likely.

A general definition of probabilities avoids this restriction.

Let S be a sample space of an experiment with a finite number of outcomes.
We assign a probability p(s) to each outcome s, so that:

i. $0 \leq p(s) \leq 1$ for each $s \in S$
ii. $\sum_{s \in S} p(s) = 1$

The function p from the set of all outcomes of the sample space S is called a **probability distribution**.

Assigning Probabilities

Example:

What probabilities should we assign to the outcomes *H* (heads) and *T* (tails) when a fair coin is flipped?

For a fair coin, we have $p(H) = p(T) = ½$.

What probabilities should be assigned to these outcomes when the coin is biased so that heads comes up twice as often as tails?

For such a biased coin, we have $p(H) = 2p(T)$.

Because $p(H)+p(T)=1$, it follows that $2p(T)+p(T)=3p(T)=1$.

Hence, $p(T) = 1/3$ and $p(H) = 2/3$.

Uniform Distribution

Definition: Suppose that S is a set with n elements. The *uniform distribution* assigns the probability $1/n$ to each element of S. (Note that we could have used Laplace's definition here.)

Example: Consider again the coin flipping example, but with a fair coin. Now $p(H) = p(T) = 1/2$.

Probability of an Event

Definition: The probability of the event *E* is the sum of the probabilities of the outcomes in *E*.

$$p(E) = \sum_{s \in S} p(s)$$

Note: no assumption is being made about the distribution.

Example:

Suppose that a die is biased so that 3 appears twice as often as each other number, but that the other five outcomes are equally likely.

What is the probability that an odd number appears when we roll this die?

Solution: We want the probability of the event $E = \{1,3,5\}$.

We have $p(3) = 2/7$ and
$p(1) = p(2) = p(4) = p(5) = p(6) = 1/7$.

Hence, $p(E) = p(1) + p(3) + p(5) =$

$$1/7 + 2/7 + 1/7 = 4/7.$$

Probabilities of Complements and Unions of Events

Complements: $p(\overline{E}) = 1 - p(E)$ still holds.

Since each outcome is in either E or $\overline{E}$, but not both,

$$\sum_{s \in S} p(s) = 1 = p(E) + p(\overline{E}).$$

Unions: $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$

still holds under the new definition.

Conditional Probability

Definition: Let E and F be events with $p(F) > 0$. The conditional probability of E given F, denoted by $P(E|F)$, is defined as:

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

Example: A bit string of length four is generated at random so that each of the 16 bit strings of length 4 is equally likely.

What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0?

Conditional Probability

Example: A bit string of length four is generated at random so that each of the 16 bit strings of length 4 is equally likely.
What is the probability that it contains at least two consecutive 0's, given that its first bit is a 0?

Solution: Let E be the event that the bit string contains at least two consecutive 0s, and F be the event that the first bit is a 0.

- Since $E \cap F = \{0000, 0001, 0010, 0011, 0100\}$, $p(E \cap F) = 5/16$.
- Because 8 bit strings of length 4 start with a 0, $p(F) = 8/16 = ½$.

Hence, $$p(E|F) = \frac{p(E \cap F)}{p(F)} = \frac{5/16}{1/2} = \frac{5}{8}.$$

Conditional Probability

Example: What is the conditional probability that a family with two children has two boys, given that they have at least one boy.
Assume that each of the possibilities BB, BG, GB, and GG is equally likely where B represents a boy and G represents a girl.

Solution: Let E be the event that the family has two boys and let F be the event that the family has at least one boy. Then $E = \{BB\}$, $F = \{BB, BG, GB\}$, and $E \cap F = \{BB\}$.

– It follows that $p(F) = 3/4$ and $p(E \cap F) = 1/4$.

Hence,

$$p(E|F) = \frac{p(E \cap F)}{p(F)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

Independence

Definition:

Events *E* and *F* are independent if and only if

$$p(E \cap F) = p(E)p(F).$$

Which events are independent?

1. H H on two consecutive coin tosses.
2. 4 2 on two consecutive rolls of a 6-sided die.
3. Choosing a card from a 52-card deck. Put back and choose a second card. Choosing a jack and then an 8.
4. Choosing a card from a 52-card deck. Not put back and choose a second card. Choosing a jack and then an 8.

3. Choosing a card from a 52-card deck.
Put back and choose a second card.
Choosing a jack and then an 8.

$$P(\text{jack}) = \frac{4}{52}$$

$$P(8) = \frac{4}{52}$$

$$P(\text{jack and } 8) = \frac{4}{52}\,\frac{4}{52}$$

4. Choosing a card from a 52-card deck.
<u>Not</u> put back and choose a second card.
Choosing a jack and then an 8.

$$P(\text{jack and } 8) = \frac{4}{52}\,\frac{4}{51}$$

Independence

Example: E is the event that a randomly generated bit string of length four begins with a 1. F is the event that the bit string contains an even number of 1s.

Are E and F independent if the 16 bit strings of length four are equally likely?

Solution: There are 8 bit strings of length 4 that begin with a 1, and 8 bit strings of length 4 that contain an even number of 1s. Since the number of bit strings of length 4 is 16,

$$p(E) = p(F) = 8/16 = ½.$$

Since $E \cap F = \{1111, 1100, 1010, 1001\}$, $p(E \cap F) = 4/16 = 1/4$.

Events E and F are independent, because

$$p(E \cap F) = 1/4 = (½)\,(½) = p(E)\,p(F)$$

Independence

Example: Assume that each of the four ways a family can have two children (*BB*, *GG*, *BG*,*GB*) is equally likely. Are the events *E*, that a family with two children has two boys, and *F*, that a family with two children has at least one boy, independent?

Solution: Because $E = \{BB\}$, $p(E) = 1/4$. We saw previously that that $p(F) = 3/4$ and $p(E \cap F) = 1/4$. The events *E* and *F* are not independent since

$$p(E)\, p(F) = 3/16 \neq 1/4 = p(E \cap F) .$$

Bernoulli Trials

Definition: Suppose an experiment can have only two possible outcomes, *e.g.*, the flipping of a coin or the random generation of a bit.

- Each performance of the experiment is called a *Bernoulli trial*.
- One outcome is called a *success* and the other a *failure*.
- If p is the probability of success and q the probability of failure, then $p + q = 1$.

Many problems involve determining the probability of k successes when an experiment consists of n mutually independent Bernoulli trials.

Bernoulli Trials

Example: A coin is biased so that the probability of heads is $2/3$.
What is the probability that exactly four heads occur when the coin is flipped seven times?

Solution: There are $2^7 = 128$ possible outcomes.
The number of ways four of the seven flips can be heads is $C(7,4)$.
The probability of each of the outcomes is $(2/3)^4(1/3)^3$ since the seven flips are independent.
Hence, the probability that exactly four heads occur is

$$C(7,4)\,(2/3)^4(1/3)^3 = (35 \cdot 16)/\,2^7 = 560/\,2187.$$

Probability of *k* Successes in *n* Independent Bernoulli Trials.

Theorem 2: The probability of exactly k successes in n independent Bernoulli trials, with probability of success p and probability of failure $q = 1 - p$, is $C(n,k)p^kq^{n-k}$.

Proof: The outcome of n Bernoulli trials is an n-tuple $(t_1,t_2,\ldots,t_n)$, where each is t_i either S (success) or F (failure).

The probability of each outcome of n trials consisting of k successes and $n-k$ failures (in any order) is p^kq^{n-k}.

There are $C(n,k)$ n-tuples of S's and F's that contain exactly k S's.

Hence, the probability of k successes is $C(n,k)p^kq^{n-k}$.

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STAT 350 (and 416) will build on counting and the probability basics we covered

Probability and statistics are important in many computer science applications.

Example: Bayes' theorem allows answers questions such as the following:

- Given that someone tests positive for having a particular disease, what is the probability that they actually do have the disease?
- Given that someone tests negative for the disease, what is the probability, that in fact they do have the disease?

Bayes' theorem has applications to machine learning, spam filtering, medicine, law, engineering, and many other areas and real life applications.

The Famous Birthday Problem

How many people are needed in a room to ensure that the probability of at least two of them having the same birthday is more than ½?

Solution: Assume all birthdays are equally likely and a year has 366 days.

Imagine people entering the room one by one. The probability that at least two have the same birthday is $1 - p_n$.

- The probability that the birthday of the second person is different from that of the first is 365/366.
- The probability that the birthday of the third person is different from the other two, when these have two different birthdays, is 364/366.

In general, the probability that the jth person has a birthday different from those already in the room, assuming that these people all have different birthdays, is $(366 - (j - 1))/366 = (367 - j)/366$.

The Famous Birthday Problem (2)

How many people are needed in a room to ensure that the probability of at least two of them having the same birthday is more than ½?

Hence, $p_n = (365/366)(364/366)\cdots(367 - n)/366$.

Therefore , $1 - p_n = 1 - (365/366)(364/366)\cdots(367 - n)/366$.

Checking various values for n with computation help tells us that for $n = 22$, $1 - p_n \approx 0.457$, and for $n = 23$, $1 - p_n \approx 0.506$.

Consequently, a minimum number of 23 people are needed so that that the probability that at least two of them have the same birthday is greater than 1/2.