CS182 – Foundations of Computer Science

Homework 8 (100 points)

Due: Thursday, April 23, 2020, 11:59pm

Upload the homework to Gradescope. DO NOT SUBMIT TO BLACKBOARD LEARN. No late submissions accepted. Only typed solutions will be graded.

Remark: Provide a brief justification for each of your answers (no more than five lines), explaining which counting rules you used and what your thought process was. Feel free to have expressions of the form $3 \cdot 5^{26}$, etc. in your final answers; no need to use calculators to compute such powers.

Problem 1. (30=3*10 points)
Suppose you have a set S of 40 objects, split into four classes (C_1, C_2, C_3, and C_4) of 10 objects each. Objects within a class are identical.

- You pick one object from S at random. What is the probability that the object belongs to the class C_4?
- You pick three objects at random, one at a time, *with replacement*. What is the probability that all three belong to the class C_4?
- You pick three objects at random, one at a time, *without replacement*. What is the probability that all three belong to the class C_1?

Solution-Problem 1

1. $(\frac{10}{40}) = (\frac{1}{4})$
2. $(\frac{1}{4}) * (\frac{1}{4}) * (\frac{1}{4}) = (\frac{1}{64})$
3. $(\frac{10}{40}) * (\frac{9}{39}) * (\frac{8}{38}) = (\frac{3}{247})$

Problem 2. (40=5*8 points)
What is the probability of each of the following events:

- What is the probability that a card chosen from an ordinary deck of 52 cards belongs to a red suit?
- What is the probability that the sum of two randomly selected integers chosen from the first 200 positive integers $1, 2, 3, \ldots, 200$ is even?
- What is the probability that the sum of the numbers on four dice rolled is even?
- What is the probability that a fair coin lands heads exactly 15 times out of 18 flips?
- If a biased coin has a probability of heads that is twice the probability of tails and is flipped 100 times, what is the probability of exactly ten heads?

Solution-Problem 2

1. $(\frac{26 RedCards}{52 TotalCards}) = (\frac{1}{2})$

2. Note:
   Even + Even = Even✓
   Even + Odd = Odd×
   Odd + Odd = Even✓
   Odd + Even = Odd×

   Let EO denote the event of choosing an even number first and then a odd number, with the following symbols denoting analogous events: OE is odd even, EE is even even, OO is odd odd, EO is even odd. The two numbers are selected with replacement, so the same number can be selected twice. It is easy to verify that
   $$P(EO) = P(OE) = P(EE) = P(OO) = 1/4$$
   by direct computations. Thus, the probability that we seek to compute is simply $P(EE) + P(OO) = 1/2$.

3. The result of every dice roll can be classified in two different ways: an even number or a odd number; both events are equiprobable. The probability of getting an even number in a single roll is the exact same as a getting an odd number. Here there are four dice being rolled, which would give us 4^2 possible outcomes (with respect to even/odd numbers), all equally likely to occur. Half of them (EEEE,EOEO,EEOO,OOOO,OEOE,OOEE,EOOE,OEEO - we use the same notation as in the question above) result in an even sum and the other half result in an odd sum. Thus, the probability that the sum of four dice rolls is equal to 1/2.

4. $\binom{n}{k} * (P)^k * (1-P)^{n-k}$, where $P = 1 - P = \frac{1}{2}$ Which Implies,
   $$\binom{n}{k} * (P)^n = \binom{18}{15} * (\frac{1}{2})^{18}$$
   $$C(18, 15) * \frac{1}{2^{18}} = (\frac{51}{16384})$$

5. $\binom{n}{k} * (P)^k * (1-P)^{n-k}$, where $P = 2(1-P)$, also to note Heads is P and Tails is (1-P).
   $$P + (1-P) = 1$$
   $$3(1-p) = 1$$
   $$(1-p) = \frac{1}{3}$$
   $$p = \frac{2}{3}$$
   Which implies,
   $$\binom{100}{10} * (\frac{2}{3})^{10} * (\frac{1}{3})^{90}$$

Problem 3. (30=3*10 points)

1. A group of 15 undergraduate and 15 graduate students are in a room. If 10 of the 30 students are selected at random and called sequentially by the instructor (order matters), what is the probability that all 10 are graduate students or all 10 are undergraduate students?

2. Select uniformly at random a permutation of $\{1, 2, ..., 100\}$. What is the probability that in the selected permutation all numbers are in ascending order?

3. A group of 40 women and 40 men are in a room. A committee of 15 is chosen at random. Find the probability that the committee consists only of women.

Solution-Problem 3

1. We have $P(15, 10)$ ways of getting 10 graduate students (We use permutation here since the order matters). Similarly, we have $P(15, 10)$ ways of getting 10 undergraduate students. In total, we have $P(30, 10)$ ways in which the instructor calls students. Thus, our answer would be,

$$prob(\text{only graduate or only undergraduate}) = \frac{(P(15, 10) + P(15, 10)))}{P(30, 10)}$$

$$= \frac{2 * (P(15, 10))}{P(30, 10)}$$

2. There are 100! permutations of the numbers $1 \ldots 100$. Of those, only one has all numbers in ascending order, namely the permutation $1, 2, 3, \ldots, n$. Thus the probability of choosing it is $1/(100!)$.

3. We have, $C(40, 15)$ ways of choosing a committee made up of only women. We have $C(80, 15)$ ways of choosing the committee in total. Thus,

$$prob(\text{Committee made up of only women}) = \frac{C(40, 15)}{C(80, 15)}$$