CS182 – Foundations of Computer Science

PSO sessions 1 and 2, week of April 20, 2020

PSO 1

Problem 1. Find the probability that a hand of five cards in poker contains four cards of one kind (i.e., four cards of the same rank).

Solution 1.
Recall that the probability of a condition C may be expressed as

$$P(C) = \frac{|S|}{|\Omega|}$$

where Ω is the set of all possibilities and $S = \{s \in \Omega \mid C(s) \equiv T\}$. Our goal, then, is to determine the sizes of the sets Ω and S.

Ω in our problem is the set of all possible hands. There are 52 cards, and we want 5 of them. Order doesn't matter, so it's simply

$$|\Omega| = \binom{52}{5}$$

S equals all hands containing four of one kind. We'll determine it's size by the counting construction below.

Select a rank.	$\binom{13}{1}$
Take all four cards from that rank.	$\binom{4}{4}$
Choose a remaining card so we have 5 in total.	$\binom{48}{1}$

Hence there are $\binom{13}{1}\binom{4}{4}\binom{48}{1}$ hands, a member of which is drawn with probability

$$\frac{\binom{13}{1}\binom{4}{4}\binom{48}{1}}{\binom{52}{5}}$$

Problem 2. What is the probability of generating the binary string 0011010 among all bit strings of length seven provided that each spot is generated independently and the probability of zero at any spot is equal to twice the probability of one?

Solution 2.

First, let's find the probability of generating a 0 vs a 1.

$$\begin{aligned} P(0) &= 2P(1) && \text{is something we are given.} \\ P(0) + P(1) &= 1 && \text{since all probabilities add up to 1.} \end{aligned}$$

Solving the system, we find that $P(0) = \frac{2}{3}$ and $P(1) = \frac{1}{3}$.

We are told that the probabilities among the bits are independent. Analogous to the Product Rule from counting, we may take the simple product

$$P(0)\ P(0)\ P(1)\ P(1)\ P(0)\ P(1)\ P(0)$$

$$\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{1^3\ 2^4}{3^7} = \frac{2^4}{3^7}$$

Problem 3. Find the probability that a randomly selected k-digit decimal number is also a valid octal number (a number whose digits are between zero and eight).

Solution 3.

As with Problem 1, we may count the total number of octal numbers and divide by the total number of decimal numbers.

How many octal numbers are k-long? There are 7 choices for the first digit, since, besides as in the trivial case where $k = 1$, the first digit cannot be a 0. That's because we are working with octal *numbers* as opposed to octal *strings*. The remaining digits may freely be any digit 0 through 7. Hence, there are then

$$7 \cdot 8^{k-1}$$

For decimals it's much the same, except we have 2 more digits available. Hence there are 9 choices for the first digit, and 10 for the remaining. Overall, there are then

$$9 \cdot 10^{k-1}$$

Hence, the probability is the ratio

$$\frac{7 \cdot 8^{k-1}}{9 \cdot 10^{k-1}}$$

And, for completeness, when $k = 1$ we have $\frac{8}{10}$.

As a final note, observe that we could have solved this using the independent probabilities for each digit, just as we did in Problem 2. This yields the equivalent

$$\left(\frac{7}{9}\right)\left(\frac{8}{10}\right)\left(\frac{8}{10}\right)\cdots\left(\frac{8}{10}\right) = \left(\frac{7}{9}\right)\left(\frac{8}{10}\right)^{k-1}$$

Problem 4. Suppose you pick two cards, one at a time, at random, from an ordinary deck of 52 cards. Find (a) the probability that both cards are diamonds; and (b) the probability that the cards form a pair (they are the same).

Solution 4.

(a) Recall that there are 4 suites, so $\frac{52}{4} = 13$ cards are diamonds. Our construction is then as follows.

Select a diamond from the 52 cards.	$\frac{13}{52}$
Select a diamond from the remaining 51 cards.	$\frac{12}{51}$

$$\frac{13}{52} \cdot \frac{12}{51}$$

(b) Recall that there are 13 ranks, so $\frac{52}{13} = 4$ cards exist for each rank. Our construction is then as follows.

Select any card from the deck.	$\frac{52}{52}$
Select another of the same rank from the remaining 51 cards.	$\frac{3}{51}$

$$\frac{52}{52} \cdot \frac{3}{51}$$

PSO 2

Problem 1: What is more likely: rolling a total of 8 when two dice are rolled or rolling a total of 8 when three dice are rolled?

Solution 1: To prove this by exhaustion, we'd need to go through all $6 \cdot 6 + 6 \cdot 6 \cdot 6 = 252$ possibilities and manually determine which add up to 8.

So, instead, we'll use Stars and Bars. Suppose we have 2 dice.

Rolling a total of 8 means having 8 total pips show up on the 2 dice. The number of ways to do this is can be thought of as the number of ways to place 8 pips onto the 2 die faces. So the pips are the stars and the faces are the bars.

There are 2 faces, meaning we have $2 - 1 = 1$ bar.

There are 8 pips but, since these are normal dice, each face has to have at least one pip. So there's 6 remaining pips to place among the 2 faces. Furthermore, no face can have more than 6 pips, so we'll have to subtract out the 2 scenarios where all 6 go to one or the other. Hence, we have 6 stars, yielding

$$\binom{6+2-1}{6} - 2 = \binom{7}{6} - 2 = 7 - 2 = 5$$

ways to place the pips. There are $6^2 = 36$ ways to roll 2 dice, and so the probability of rolling a total of 8 is $\frac{5}{36}$

What about 3 dice? Now we have 1 less star, since the third die needs a pip. As a result, there's no way to accidentally have more than 6 pips on a die, which means that now there's nothing to subtract out. There's also one more bar. Hence, the number of ways of rolling a total of 8 is

$$\binom{5+3-1}{5} = \binom{7}{5} = 21$$

with a probability of $\frac{21}{6^3} = \frac{21}{216}$.

Since $\frac{5}{36} = \frac{30}{216} > \frac{21}{216}$, it's more likely to happen when you have 2 dice.

Problem 2: Urn 1 contains 2 blue tokens and 8 red tokens; urn 2 contains 12 blue tokens and 3 red tokens. You pick an urn at random and draw out a token at random from that urn. Given that the token is blue, what is the probability that the token came from urn 1?

Solution 2: First, observe the following.

For two events, A and B, we define the conditional probability A given B as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Equivalently, we have

$$P(A \cap B) = P(A|B)P(B)$$

And since $A \cap B = B \cap A$, we also have

$$P(A \cap B) = P(B|A)P(A)$$

So, all together, we can rewrite $P(A|B)$ as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$

The above is a proof of Bayes' Theorem, which we shall use to solve the problem.

Let B denote the event that the token is blue. Let U_i denote the event that the token came from urn i. We are asked to find $P(U_1|B)$ but are indirectly told $P(B|U_1)$ and $P(B|U_2)$. Applying Bayes' Theorem, we have

$$P(U_1|B) = \frac{P(B|U_1)P(U_1)}{P(B)}$$

Observe that $P(B)$, the probability of drawing a blue token in general, equals the total probability of the two scenarios wherein a blue token is drawn. That is, the probability of selecting urn 1 and then drawing a blue token, plus the probability of selecting urn 2 and then drawing a blue token. Hence, we have

$$\begin{aligned} P(U_1|B) &= \frac{P(B|U_1)P(U_1)}{P(B|U_1)P(U_1) + P(B|U_2)P(U_2)} \\ &= \frac{(2/10)(1/2)}{(2/10)(1/2) + (12/15)(1/2)} \\ &= \frac{1}{5} \end{aligned}$$

Note that $P(U_1) = P(U_2) = \frac{1}{2}$ since the urn is chosen without bias.

$P(B|U_1) = \frac{2}{10}$ since 2 of urn 1's 10 tokens are blue.

$P(B|U_2) = \frac{12}{15}$ since 12 of urn 2's 15 tokens are blue.

Problem 3: Urn 1 contains 2 blue tokens and 8 red tokens; urn 2 contains 12 blue tokens and 3 red tokens. You roll a die to determine which urn to choose: if you roll a 1 or 2 you choose urn 1; if you roll a 3, 4, 5, or 6 you choose urn 2. Once the urn is chosen, you draw out a token at random from that urn. Given that the token is blue, what is the probability that the token came from urn 1?

Solution 3: This question is exactly like the previous, except the urn is chosen with bias. Hence, we'll still use Bayes' Theorem, but with the each urn's new selection probability. Specifically, the facts that

$$P(U_1) = \frac{2}{6} = \frac{1}{3} \qquad P(U_2) = \frac{4}{6} = \frac{2}{3}$$

By Bayes' Theorem we see then that

$$\begin{aligned} P(U_1|B) &= \frac{P(B|U_1)P(U_1)}{P(B)} \\ &= \frac{P(B|U_1)P(U_1)}{P(B|U_1)P(U_1) + P(B|U_2)P(U_2)} \\ &= \frac{(2/10)(1/3)}{(2/10)(1/3) + (12/15)(2/3)} \\ &= \frac{1}{9} \end{aligned}$$

Problem 4: Flip a biased coin, where the probability of heads is 3/4 and the probability of tails is 1/4, ten times. Find the probability of getting exactly nine heads; exactly seven heads; at least seven heads.

Solution 4: Let H denote the event that heads is flipped, and let T denote the event that tails is flipped. From the information given, we know that

$$P(H) = \frac{3}{4} \qquad P(T) = \frac{1}{4}$$

We'll take a hybrid approach. No matter which of the 10 flips is a tail, the probability of any specific sequence of 9 heads and 1 tail is

$$P(H)^9 P(T)$$

There are $\binom{10}{1}$ ways to select the flip that lands tails. Thus, the probability of flipping exactly 9 heads is

$$\binom{10}{1} P(H)^9 P(T) = 10 \left(\frac{3}{4}\right)^9 \left(\frac{1}{4}\right)$$

Getting exactly 7 heads means picking 3 flips to be tails, yielding

$$\binom{10}{3} P(H)^7 P(T)^3 = \binom{10}{3} \left(\frac{3}{4}\right)^7 \left(\frac{1}{4}\right)^3$$

Getting 7 or more heads is the disjoint union of getting exactly 7, 8, 9, or 10 heads. Hence, we'll sum the probabilities of the 4 scenarios.

$$\binom{10}{3} P(H)^7 P(T)^3 + \binom{10}{2} P(H)^8 P(T)^2 + \binom{10}{1} P(H)^9 P(T)^1 + \binom{10}{0} P(H)^{10} P(T)^0$$

$$\binom{10}{3} \left(\frac{3}{4}\right)^7 \left(\frac{1}{4}\right)^3 + \binom{10}{2} \left(\frac{3}{4}\right)^8 \left(\frac{1}{4}\right)^2 + \binom{10}{1} \left(\frac{3}{4}\right)^9 \left(\frac{1}{4}\right)^1 + \binom{10}{0} \left(\frac{3}{4}\right)^{10} \left(\frac{1}{4}\right)^0$$