CS182 – Foundations of Computer Science

PSO sessions 1 and 2, week of Apr 6, 2020

PSO 1

Problem 1. Consider all bit strings (i.e., strings of 1s and 0s) of length 12.

- How many begin with 110?
- How many begin with 11 and end in 10?
- How many begin with 11 or end in 10?

Solution 1.

We'll start by finding the number of bit strings of length 12, and then use similar reasoning to do the problem.

For the first bit, there are 2 choices. For the next, there are also 2. The choice at each bit has nothing to do with the choices for the others. Therefore, in total, there are

$$2 \cdot 2 \cdot 2 \cdot ... \cdot 2 = 2^{12}$$

The number beginning with 110 is then a question of what happens when some of the choices are made for us. For the first 3 bits, each has 1 possibility. For the remaining 9, each has 2 as before. Therefore, there are

$$1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 \cdot ... \cdot 2 = 1^3 \cdot 2^9 = 2^9$$

The same logic applies for those beginning with 11 and ending with 10. For the first and last two, each has 1 choice. For the remaining in the middle, each has 2.

$$1^2 \cdot 2^8 \cdot 1^2 = 2^8$$

Those beginning with 11 or ending in 10 is not simply the number beginning with 11 plus the number ending in 10. This is because some bit strings both start with 11 and end with 10. Such bit strings would be counted twice. Therefore, we can still add up the number starting with 11 and the number ending in 10, but we must make sure to remove the double counted ones via subtraction.

$$(1^2 \cdot 2^{10}) + (2^{10} \cdot 1^2) - (1^2 \cdot 2^8 \cdot 1^2) = 2^{10} + 2^{10} - 2^8 = 1792$$

Problem 2. Let A and B be sets.

- Let $|A| = 4$ and $|B| = 10$. Find the number of functions $f : A \to B$.
- Let $|A| = 4$ and $|B| = 10$. Find the number of 1-1 functions $f : A \to B$.
- Let $|A| = 10$ and $|B| = 4$. Find the number of 1-1 functions $f : A \to B$.

Solution 2.

Here we are tested on the definition of the term *function*. Remember, though the range need not equal the codomain, a function must be defined for its entire domain. In this problem, then, not all elements of B need be used, but every element from A has to be.

In the first part we have no restrictions on f. Each element of A may be assigned to any element of B freely. Thus, the first element has 10 choices for assignment, the second has 10 also, etc. And since these choices have no effect on each other, we can compute the number of functions by the Product Rule.

$$10 \cdot 10 \cdot 10 \cdot 10 = 10^4$$

The second part restricts f to be an injection. Thus, no two distinct elements in A may be assigned the same element in B. One way to ensure this is as follows.

- Pick 1 of the 10 choices for the first element in A.
- Pick 1 of the remaining 9 choices for the second.
- Pick 1 of the remaining 8 for the third.
- Pick 1 of the remaining 7 for the fourth.

In total we have $10 \cdot 9 \cdot 8 \cdot 7 = 5040$ possible functions. This is also the same as $P(10, 4)$ since we are drawing 4 choices from the 10 in B, and we care about order.

The last part makes A larger in size. Since every element in A must be used for a function f to be well defined, and since, by the Pigeonhole Principle, at least $\lceil \frac{10}{4} \rceil = 3$ elements in A must share the same element in B, no injections exist. So the number is 0.

Problem 3. DNA sequences are sequences of bases, where each base can take one of the four "values" A, C, T, and G. Two examples of DNA sequence of length eight are $GACCATTT$ and $GTAATTAC$.

- How many length eight DNA sequences start with C and end with C?
- How many length eight DNA sequences do not contain C?
- How many length eight DNA sequences do not contain all four bases A, C, T, and G?
- How many length eight DNA sequences contain exactly four C's?

Solution 3.

This is much like the bit strings problem, except each place value has 4 choices. For the first part, 2 choices have been made for us, yielding

$$1^1 \cdot 4^6 \cdot 1^1 = 4^6$$

Not containing C simply means we have 3 choices for each base.

$$3^8$$

For part 3, we must deal with the great potential to over count. Suppose we simply add up all the ways to exclude A, exclude C, exclude T, and exclude G. We would over-count all the ways to exclude both A and C, as well as the other pairs. So suppose we subtract out those over-counted pair-wise exclusive sequences. Well, then we'd be over-removing all sequences excluding A, C and T, and the other triples. Well, we could add those back in. But then we'd over-count those excluding all 4 bases. Ah, well that's no problem since no such sequences exist.

The logic above is the basis for the inclusion-exclusion principle, which we shall apply below. Let the symbols below represent what's being excluded.

$|\mathrm{A}| + |\mathrm{C}| + |\mathrm{T}| + |\mathrm{G}| = \binom{4}{1}|\mathrm{A}| = \binom{4}{1}3^8$

$|\mathrm{A,\ C}| + |\mathrm{A,\ T}| + |\mathrm{A,\ G}| + |\mathrm{C,\ T}| + |\mathrm{C,\ G}| + |\mathrm{T,\ G}| = \binom{4}{2}|\mathrm{A,\ C}| = \binom{4}{2}2^8$

$|\mathrm{C,\ T,\ G}| + |\mathrm{A,\ T,\ G}| + |\mathrm{A,\ C,\ G}| + |\mathrm{A,\ T,\ C}| = \binom{4}{3}|\mathrm{A,\ T,\ C}| = \binom{4}{3}1^8$

So, in total, we have $\binom{4}{1}3^8 - \binom{4}{2}2^8 + \binom{4}{3}1^8 = 24712$

For part 4, let's first analyze a given such sequence. If we started with 4 C's, we'd have $1^4 \cdot 3^4 = 3^4$ possibilities since the remaining 4 bases can't contain a C. Order doesn't change anything, since ultimately there will always be 4 remaining bases, each of which will only have 3 options. So if we knew the number of ways to place the C's in the sequence, we could multiply by 3^4, the number of ways to fill the remaining bases. There are 8 positions, and so $\binom{8}{4}$ placements. Note that since C's are indistinguishable, order doesn't matter. Thus there are:

$$\binom{8}{4} \cdot 3^4$$

PSO 2

Problem 1. You pick cards one at a time without replacement from an ordinary deck of 52 playing cards. What is the minimum number of cards you must pick in order to guarantee that you get:
(a) a pair (for example, two kings or two 5s);
(b) three of a kind (for example, three 7s).

Solution 1a. Consider the adversarial case, wherein we select one of each rank before finally repeating one. There are 13 ranks (A, 2, 3, ..., J, Q, K). Thus, at most 14 cards must be drawn before a pair is formed.

Notice we can also use the Pigeonhole Principle for a hand of h cards.

$$2 = \lceil \frac{h}{13} \rceil$$

which is true for the minimal $h = 14$. Thus, 14 is the answer.

Solution 1b. The Pigeonhole Principle may be applied for a hand of h cards.

$$3 = \lceil \frac{h}{13} \rceil$$

which is true for the minimal $h = 27$. Thus, 27 is the answer. This is equivalent to considering the adversarial scenario wherein you draw 2 of each rank before finally being forced on the 27^{th} card to draw a three of a kind.

Problem 2.

- Find the number of subsets of $S = \{1, 2, 3, ..., 10\}$ that contain the number 5.
- Find the number of subsets of $S = \{1, 2, 3, ..., 10\}$ that contain neither 5 nor 6.
- Find the number of subsets of $S = \{1, 2, 3, ..., 10\}$ that contain both 5 and 6.
- Find the number of subsets of $S = \{1, 2, 3, ..., 10\}$ that contain no odd numbers.

Solution 2

Recall that, for a set S, the powerset $\mathcal{P}(S)$ is constructed by taking all subsets of S. Formally, this is expressed in set builder notation as

$$\mathcal{P}(S) \equiv \{A \mid A \subseteq S\}$$

Some examples are below

$$\begin{aligned}
\mathcal{P}(\{1\}) &= \{\emptyset, \{1\}\} \\
\mathcal{P}(\{1,2\}) &= \{\emptyset, \{1\}, \{2\}, \{1,2\}\} \\
\mathcal{P}(\{1,2,3\}) &= \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}
\end{aligned}$$

Look above at the sizes of the sets. Notice that they are powers of 2. This is because the power set's construction involves creating all subsets. Each subset essentially makes the following choice for each element: include or exclude. So there's 2 choices for each element, meaning if a set has size n, it's powerset has size 2^n. This gives us a mechanism to compute the sizes in this problem.

- Being forced to including 5 means only having the choice to include or exclude for the remaining 9, so there are $1^1 \cdot 2^9 = 2^9$ such sets.
- Being forced to exclude 5 and 6 means having the 2 choices for the remaining 8, so we there are $1^2 \cdot 2^8 = 2^8$ such sets.
- Being forced to include 5 and 6 is the same as the above, with the only difference being the choice we are forced to make for 5 and 6. Thus, there are 2^8 such sets.
- Containing no odd numbers is the same as excluding the odd numbers. Thus, there are $1^5 \cdot 2^5 = 2^5$ such sets.