COUNTING

$\left\lceil \frac{N}{k} \right\rceil$

Problems we want to solve

How many ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left and right neighbor?

$\frac{4!}{4} = 3!$

$\left\lceil \frac{52}{4} \right\rceil = 9$ cards

How many cards must Bob select from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

Alice likes 4 types of bagels. How many ways are there to select a dozen bagels when she chooses only among the 4 types she likes?

$\binom{12+4-1}{4} = \binom{15}{4}$

BASICS OF COUNTING (6.1)

- Basic Counting Principles and Counting Rules

Counting Rules

- Product Rule
- Sum Rule
- Subtraction Rule
- Division Rule

The Product Rule

Example: How many bit strings of length seven are there?

Solution: Since each of the seven bits is either a 0 or a 1, the answer is $2^7 = 128$.’

The Product Rule: A procedure can be broken down into a sequence of two tasks. There are

- n_1 ways to do the first task and
- n_2 ways to do the second task.

Then, there are $n_1 \cdot n_2$ ways to do the procedure.

Example: License Plates

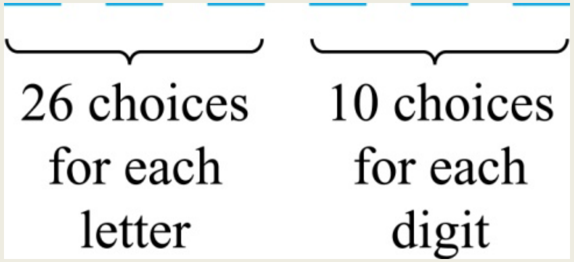
How many different license plates can be made if each plate is a sequence of three uppercase English letters followed by three digits?

e.g., AGF349

Solution: By the product rule, there are

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$$

different possible license plates.



Example: Telephone Numbering Plan

The *North American Numbering Plan* (*NANP*) specifies that a telephone number consists of 10 digits:
3-digit area code, 3-digit office code, a 4-digit station code.

There are some restrictions on the digits.

- Let *X* denote a digit from 0 through 9.
- Let *N* denote a digit from 2 through 9.
- Let *Y* denote a digit that is 0 or 1.
- Format in old plan (used in 1960s): *NYX-NNX-XXXX*.
- Format in new plan: *NXX-NXX-XXXX*.

How many different telephone numbers are possible under the old plan and the new plan?

$X = \{0, \ldots, 9\}$, $N = \{2, \ldots, 9\}$, $Y = \{0, 1\}$

old plan format: *NYX-NNX-XXXX*

new plan format: *NXX-NXX-XXXX*

How many different telephone numbers are possible?

$X = \{0, \ldots, 9\}$, $N = \{2, \ldots, 9\}$, $Y = \{0, 1\}$

old plan format: *NYX-NNX-XXXX*

new plan format: *NXX-NXX-XXXX*

How many different telephone numbers are possible?

Solution: Use the Product Rule.

- $8 \cdot 2 \cdot 10 = 160$ area codes with the format *NYX.*
- $8 \cdot 10 \cdot 10 = 800$ area codes with the format *NXX.*
- $8 \cdot 8 \cdot 10 = 640$ office codes with the format *NNX.*
- $10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ station codes with the format *XXXX.*

Number of old plan telephone numbers:

$$160 \cdot 640 \cdot 10{,}000 = 1{,}024{,}000{,}000.$$

Number of new plan telephone numbers:

$$800 \cdot 800 \cdot 10{,}000 = 6{,}400{,}000{,}000.$$

Example: Counting Subsets of a Finite Set

Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$.

Solution: List the elements of S, |S|=k, in an arbitrary order.

There is a one-to-one correspondence between subsets of *S* and bit strings of length k.

- When the *i*-th element is in the subset, the bit string has a 1 in the *i*-th position and a 0 otherwise.

By the product rule, there are 2^k such bit strings, and therefore $2^{|S|}$ subsets.

Product Rule in Terms of Sets

- If A_1, A_2, … , A_m are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set.
- The task of choosing an element in the Cartesian product $A_1 \times A_2 \times \cdots \times A_m$ is done by choosing an element in A_1, an element in A_2 , …, and an element in A_m.
- By the product rule, it follows that:

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|.$$

The Sum Rule

Example:

Friday night you can see one of five movies, go to one of two concerts, or stay home. How many choices do you have for spending Friday night?

There are 5 + 2 + 1 = 8 choices

Sum Rule:

If there are n_1 ways for one task and n_2 ways for another task and the two tasks cannot be done at the same time, then there are $n_1 + n_2$ ways to select one of these tasks.

Sum Rule Example

The CS department chooses either a student or a faculty as a representative for a committee.

How many choices are there for this representative if there are 47 CS faculty and 558 CS majors and no one is both a faculty member and a student.

There are 47 + 558 = 605 possible ways

Sum Rule in terms of sets

The sum rule can be phrased in terms of sets.

$|A \cup B| = |A| + |B|$ as long as A and B are disjoint sets.

Or more generally,

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$
$$\text{when } A_i \cap A_j = \emptyset \text{ for all } i, j.$$

Combining Sum and Product Rule

Example:

Suppose an ID can be either a two letters or a letter followed by a digit.

Find the number of possible IDs.

Solution: $(26 \cdot 26) + (26 \cdot 10) = 936$

Example: Counting Passwords

A password must be 6-8 characters long; each character is an uppercase letter or a digit. Each password must contain at least one digit.

How many possible passwords are there?

Solution:

Let P be the total number of passwords.

Let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.

By the sum rule, $P = P_6 + P_7 + P_8$.

Example: Counting Passwords

A password must be 6-8 characters long; each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution: Let P be the total number of passwords. Let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8. By the sum rule, $P = P_6 + P_7 + P_8$.

To find each of P_6, P_7, and P_8, find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters.

$$P_6 = 36^6 - 26^6 = 2{,}176{,}782{,}336 - 308{,}915{,}776 = 1{,}867{,}866{,}560.$$

$$P_7 = 36^7 - 26^7 = 78{,}364{,}164{,}096 - 8{,}031{,}810{,}176 = 70{,}332{,}353{,}920.$$

$$P_8 = 36^8 - 26^8 = 2{,}821{,}109{,}907{,}456 - 208{,}827{,}064{,}576 = 2{,}612{,}282{,}842{,}880.$$

Consequently, $P = P_6 + P_7 + P_8 = 2{,}684{,}483{,}063{,}360.$

The Subtraction Rule

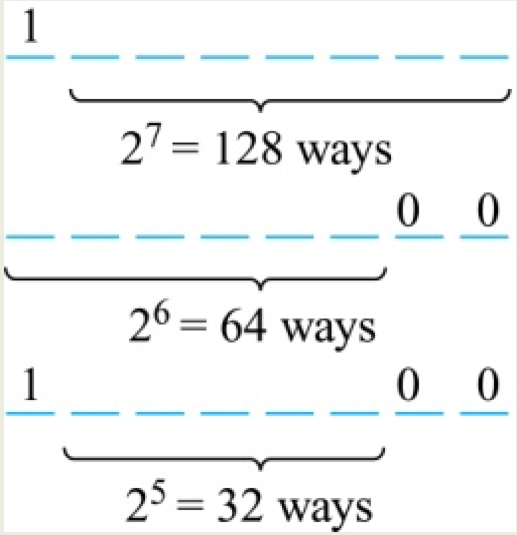
Subtraction Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

Also known as, the *principle of inclusion-exclusion*:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example: Counting Bit Strings

How many bit strings of length eight either start with a 1 or end with 00?



Solution: Use the subtraction rule.

- Number of bit strings of length eight that start with a 1: $2^7 = 128$
- Number of bit strings of length eight that end with 00: $2^6 = 64$
- Number of bit strings of length eight that start with a 1 <u>and</u> end with 00: $2^5 = 32$

Hence, the number is $128 + 64 - 32 = 160$.

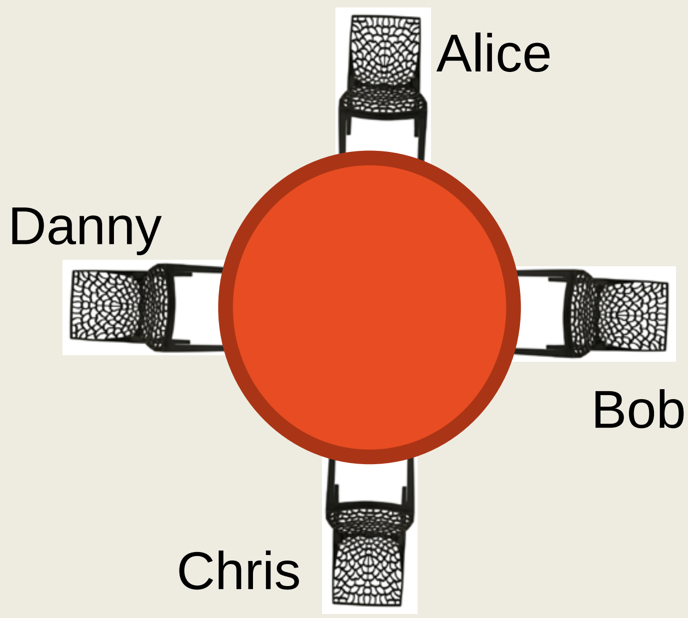
The Division Rule

Bob counts the number of people in the room by courting the number of ears (assume every person has two ears). To get the number of people, he needs to divide the number of ears by 2.

Division Rule: If a task can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w, then there n/d ways to do the task.

The Division Rule

Example: How many ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left and right neighbor?



Solution:

Number the seats around the table from 1 to 4 proceeding clockwise.

There are four ways to select the person for seat 1, 3 ways for seat 2, 2 ways for seat 3, and one way for seat 4. Thus there are $4! = 24$ ways to order the four people.

But, two seatings are the same when each person has the same left and right neighbor. Each of the four choices for seat 1 leads to the same seating arrangement.

Therefore, by the division rule, there are $24/4 = 6$ different seating arrangements.

Tree Diagrams

We can solve many counting problems through the use of ***tree diagrams***

- a branch represents a possible choice
- the leaves of the tree represent possible outcomes.

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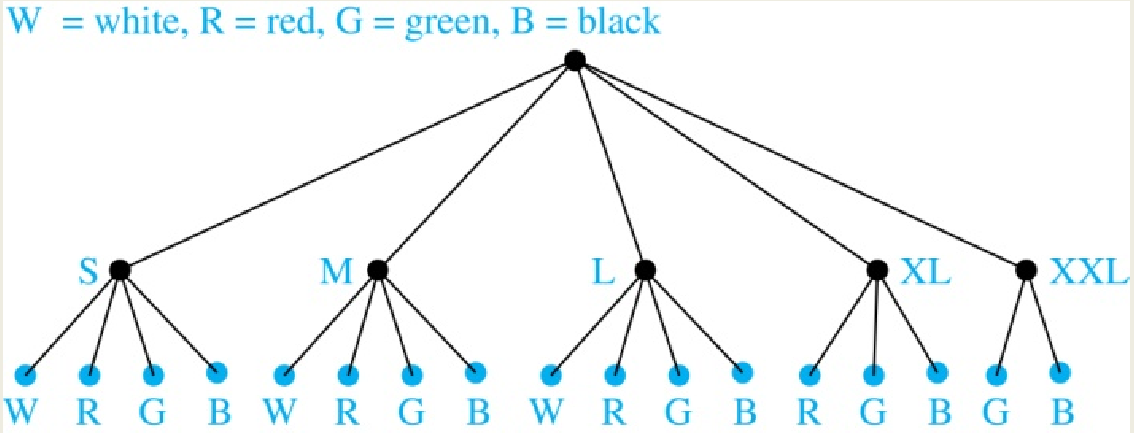


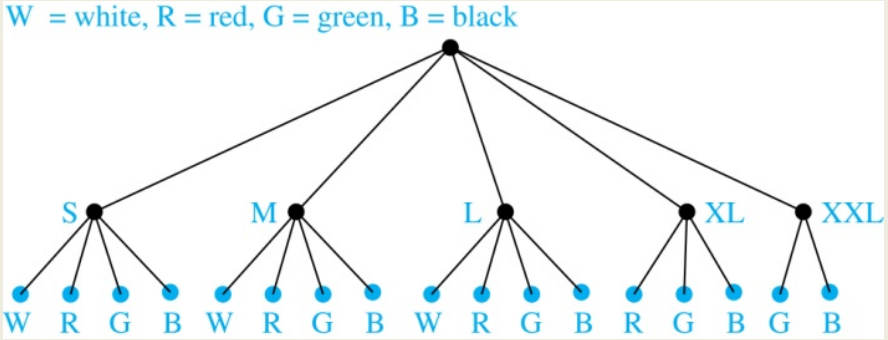
Example: Tree Diagrams

A T-shirt comes in five different sizes: S, M, L, XL, and XXL. Each size comes in four colors: white, red, green, and black, except

- XL comes only in red, green, and black
- XXL comes only in green and black.

What is the minimum number of T-shirts that a store needs to stock to have one of each size and color available?

Solution:
Draw the tree diagram.
17 T-shirts must be stocked.



PIGEONHOLE PRINCIPLE (6.2)

- Basic principle
- Applications

The Pigeonhole Principle

If a flock of 26 pigeons roosts in a set of 25 pigeonholes, one of the pigeonholes must have more than 1 pigeon.

https://en.wikipedia.org/wiki/Pigeonhole_principle

$\left\lceil \frac{26}{25} \right\rceil =$

1K containers

1K-1 1 obj each

1 at least 2 obj each

The Pigeonhole Principle

Pigeonhole Principle:

If *k* is a positive integer and *k* + 1 objects are placed into *k* boxes, then at least one box contains two or more objects.

Proof: We use a proof by contradiction.

Suppose none of the *k* boxes has more than one object. Then the total number of objects would be at most *k*. This contradicts the statement that we have *k* + 1 objects.

Example: Among any group of 367 people, there must be at least two with the same birthday because there are only 366 possible birthdays.

Example: Pigeonhole Principle

Every positive integer *n* has a multiple that has only 0’s and 1’s in its decimal expansion.
For example, for n=6, 1110 = 185×6.

Solution: Let *n* be a positive integer.

Every positive integer n has a multiple that has only 0's and 1's in its decimal expansion; e.g., for n=6, 1110 = 185×6.

Solution:

Let n be a positive integer.

Consider the n + 1 integers 1, 11, 111, …., 11…1 (where the last integer has (n + 1) 1's).

There are n possible remainders when an integer is divided by n.

Divide each of the n + 1 integers by n. By the pigeonhole principle, at least two integers must have the same remainder (i.e., s = kn+r, t = jn+r)

Subtract the smaller from the larger.

The result is a multiple of n that has only 0's and 1's in its decimal expansion.

The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Prove by contraposition: If all boxes contain at most $\lceil N/k \rceil - 1$ objects, the total number of objects cannot be N.

Example: Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

Example: Generalized Pigeonhole Principle

How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

Solution:

$$\lceil N/4 \rceil \geq 3$$

$$N = 9$$

Example: Generalized Pigeonhole Principle

How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

Solution:

Assume there are four boxes, one for each suit.
We place cards in the box reserved for its suit.
After N cards have been placed into boxes, at least one box contains at least $\lceil N/4 \rceil$ cards.

At least three cards of one suit have been selected if $\lceil N/4 \rceil \geq 3$.
The smallest integer N such that $\lceil N/4 \rceil \geq 3$ is
$N = 2 \cdot 4 + 1 = 9$.
Hence, **select 9 cards**.

Pigeonhole Principle Example

42 13 13 13 13

How many cards must be selected from a standard deck of 52 cards to guarantee that at least three hearts are selected?

Solution:

A deck contains 52 cards and 13 hearts.
Hence, 39 cards are not hearts.

If we select 41 cards, we may have 39 cards which are not hearts along with 2 hearts.
However, when we select **42 cards**, we must have at least three hearts.

PERMUTATIONS AND COMBINATIONS (6.3)

- Permutations and r-permutations
- Combinations and r-combinations
- Binomial coefficients

Permutations

Definition: A *permutation* of a set of distinct objects is an ordered arrangement of these objects.
An ordered arrangement of r elements of a set is called an *r-permutation*.

Example: Let $S = \{1,2,3\}$.

The ordered arrangement 3,1, 2 is a **permutation** of S.

The ordered arrangement 3, 2 is a **2-permutation** of S.

Permutations

The number of ***r*-permutations** of a set with n elements is denoted by $P(n,r)$.

The 2-permutations of $S = \{1,2,3\}$ are

1,2; 1,3; 2,1; 2,3; 3,1; 3,2

Hence, $P(3,2) = 6$.

$P(n,r) = n(n-1)(n-2) \ldots (n-r+1)$ with $1 \leq r \leq n$

Solving Counting Problems by Counting Permutations

Example: How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution:

$$P(100,3) = 100 \cdot 99 \cdot 98 = 970{,}200$$

Solving Counting Problems by Counting Permutations

Example: Suppose a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order.

How many possible orders exist?

Solution: The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

If you need to find the tour with the shortest path that visits all the cities, do you need to consider all 5040 paths?

Theorem: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$\boldsymbol{P}(\boldsymbol{n},\boldsymbol{r}) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

r-permutations of a set with n distinct elements.

Proof: Use the product rule.

- The first element can be chosen in n ways.
- The second element can be chosen in $n-1$ ways,

  .

  .
- until there are $(n-(r-1))$ ways to choose the last element.

Note: $P(n,0) = 1$. There is only one way to order zero elements.

Solving Counting Problems by Counting Permutations

Example: How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

Solution: We solve this problem by counting the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

Combinations

Definition: An *r-combination* of elements of a set is an unordered selection of *r* elements from the set.

An *r*-combination is a subset with *r* elements.

The number of *r*-combinations of a set with n distinct elements is denoted by $C(n, r)$.

Notation: $C(n,r) = \binom{n}{r}$ is called a *binomial coefficient*.

Example: Combinations

$S = \{a, b, c, d\}$

$\{a, c, d\}$ is a 3-combination from S.
It is the same as $\{d, c, a\}$ since the order does not matter.

$C(4,2) = 6$

The 2-combinations of set $\{a, b, c, d\}$ are six subsets: $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.

Combinations

Theorem: The number of *r*-combinations of a set with *n* elements, $n \geq r \geq 0$, is $C(n,r) = \frac{n!}{(n-r)!r!}$.

Proof:

The P(n,r) r-permutations of the set can be obtained by

- forming the $C(n,r)$ r-combinations and then
- ordering the elements in each which can be done in r! ways

By the product rule $P(n, r) = C(n,r) \cdot r!$ The result follows.

$$C(n,r) = \frac{n!}{(n-r)!r!}.$$

Useful identities

$$C(n,r) = \frac{P(n,r)}{r!}$$

$$P(n,r) = C(n,r) \cdot r!$$

$$C(n,r) = C(n,n-r)$$

Example: Combinations

How many poker hands of five cards can be dealt from a standard deck of 52 cards?

Solution: Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$C(52,5) = \frac{52!}{5!47!}$$

$$= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960$$

Example: Combinations

How many ways are there to select 47 cards from a deck of 52 cards?

The different ways to select 47 cards from 52 is

$$C(52,47) = \frac{52!}{47!5!} = C(52,5) = 2,598,960.$$

Combinations

Corollary: Let n and r be nonnegative integers, $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Proof: Since $C(n,r) = \frac{n!}{(n-r)!r!}$

and $C(n, n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!}$.

$C(n, r) = C(n, n - r)$ follows. ◀

Combinations

Example: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.

Solution: The number of 5-combinations of a set with 10 elements is $C(10,5) = \frac{10!}{5!5!} = 252.$

Combinations

Example: How many bit strings of length n contain exactly r ones?

Solution: A bit string of length n has the bit position {1, 2, 3, …, n}.

We need r positions set to 1.
The number of r-combinations of a set with n elements is $C(n,r) = \frac{n!}{(n-r)!r!}$.

BINOMIAL COEFFICIENTS AND IDENTITIES (6.4, 6.5)

- Binomial Theorem
- Permutations with repetition and indistinguishable objects

Expand $(x + y)^3 = (x + y)(x + y)(x + y)$

Terms of the form x^3, x^2y, $x y^2$, y^3 arise.

What are the coefficients?

- To obtain x^3 , an x must be chosen from each of the sums. There is only one way to do this. So, the coefficient of x^3 is 1.
- To obtain x^2y, an x must be chosen from two of the sums and a y from the other. There are $\binom{3}{2}$ ways to do this. Hence, the coefficient of x^2y is 3.
- To obtain xy^2, an x must be chosen from one of the sums and a y from the other two . There are $\binom{3}{2}$ ways to do this. Hence, the coefficient of xy^2 is 3.
- To obtain y^3, a y must be chosen from each of the sums. The coefficient of y^3 is 1.

We have used a counting argument to show that

$$(x + y)^3 = x^3 + 3x^2y + 3x y^2 + y^3.$$

Binomial Theorem

- We have used a counting argument to show that $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.
- The binomial theorem gives the coefficients of the terms in the expansion of $(x + y)^n$.
- A *binomial* expression is the sum of two terms, such as $x + y$.

Binomial Theorem.

Let *x* and *y* be variables, and let be *n* a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j =$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Proof: We use combinatorial reasoning. The terms in the expansion of $(x + y)^n$ are of the form $x^{n-j}y^j$, $0 \leq j \leq n$. To form $x^{n-j}y^j$, it is necessary to choose $n-j$ x's from the n sums. Therefore, the coefficient of $x^{n-j}y^j$ *is* $\binom{n}{j}$ which equals $\binom{n}{n-j}$. ◀

$$(1+X)^0 = 1$$

$$(1+X)^1 = 1 + 1X$$

$$(1+X)^2 = 1 + 2X + 1X^2$$

$$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$$

$$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$$

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1
1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1

By Pascal's identity, adding two adjacent binomial coefficients results is the binomial coefficient in the next row between these two coefficients.

Using the Binomial Theorem

Example: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution: We view the expression as $(2x + (-3y))^{25}$. By the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^j.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$.

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!12!}2^{12}3^{13}.$$

Useful Identities

Corollary: With *n* ≥ 0, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

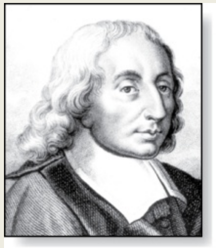
Proof (*using the Binomial Theorem*):
With *x* = 1 and *y* = 1, we get:

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{(n-k)} = \sum_{k=0}^{n} \binom{n}{k}.$$

Two other identities:

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0 \quad and \quad \sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n$$

Blaise Pascal
(1623-1662)



Pascal's Identity

Pascal's Identity:

If n and k are integers with $n \geq k \geq 0$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Pascal's Triangle

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6} \\
\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7} \\
\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8} \\
\dots
\end{array}
$$

(a)

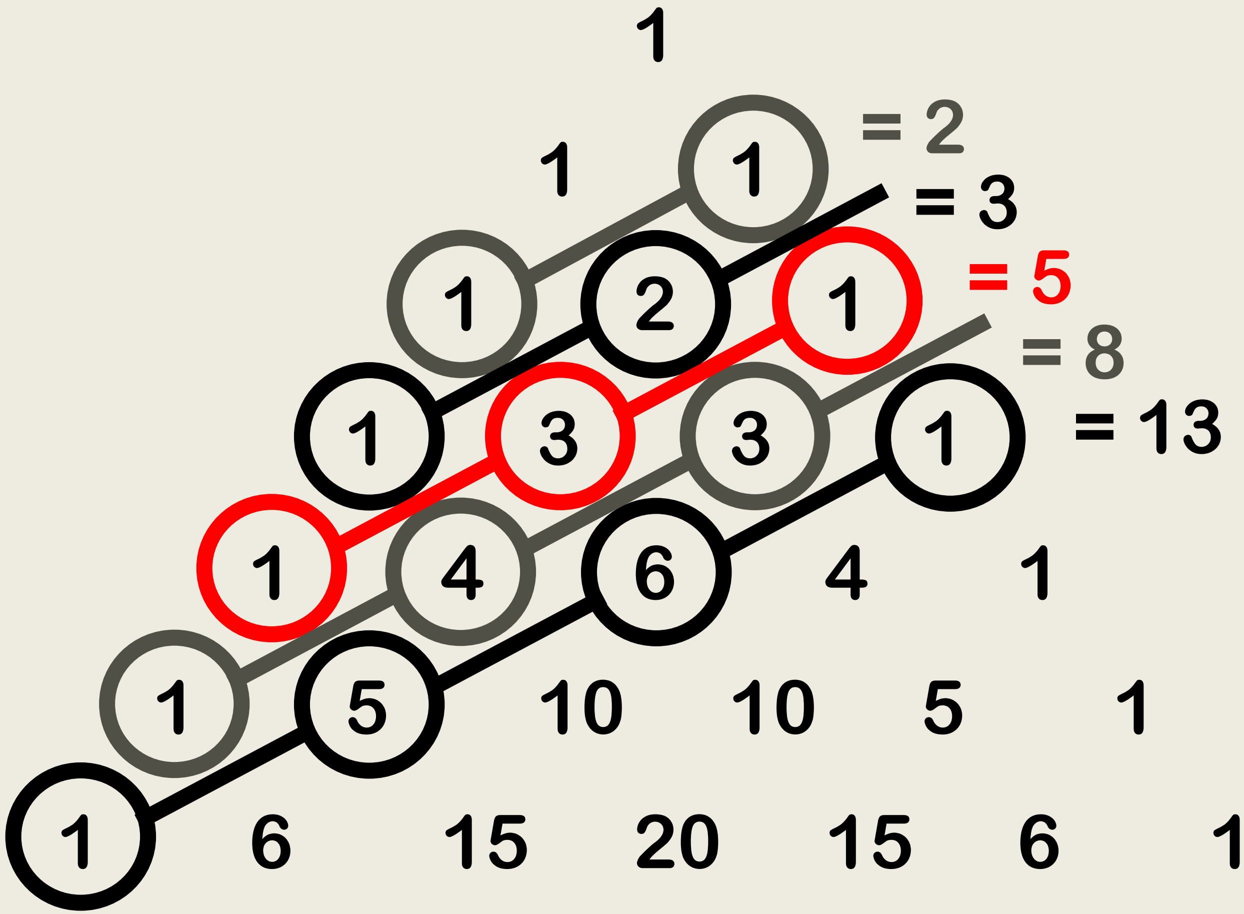
By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
\dots
\end{array}
$$

(b)

Fibonacci Numbers in Pascal's Triangle



Sums of Squares

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1^2 & & 2^2 & & 1^2 & & & & \\
 & & & 1^2 & & 3^2 & & 3^2 & & 1^2 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Permutations with Repetition

Theorem: The number of *r*-permutations of a set of *n* objects with repetition allowed is n^r.

Proof: There are *n* ways to select an element of the set for each of the *r* positions in the *r*-permutation when repetition is allowed. By the product rule, there are n^r *r*-permutations with repetition. ◀

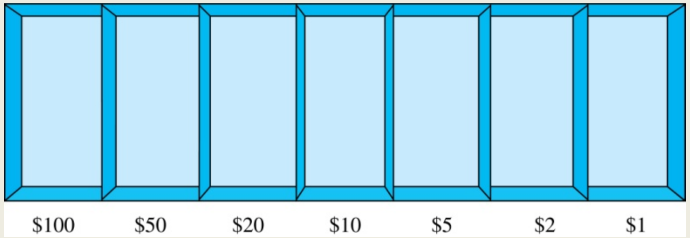
Example: How many strings of length *r* can be formed from the uppercase letters of the English alphabet?

Solution: The number of such strings is 26^r, which is the number of *r*-permutations of a set with 26 elements.

Combinations with Repetition

Example: How many ways are there to select five bills from a box containing at least five of each of the denominations: \$1, \$2, \$5, \$10, \$20, \$50, and \$100?

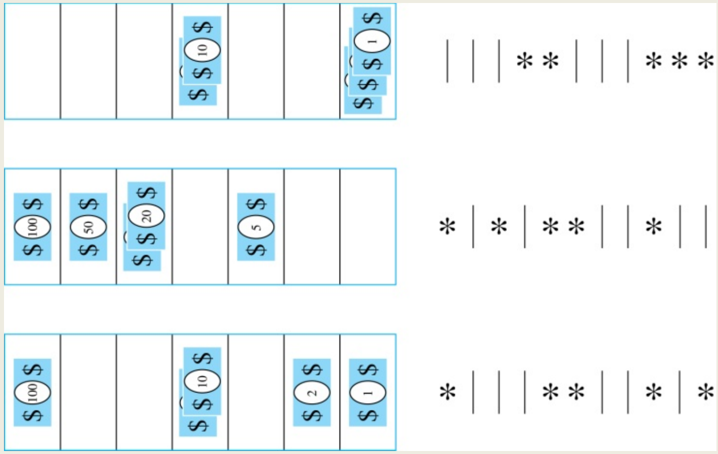
Solution: Place the selected bills in the appropriate position of a cash box illustrated below:



continued →

Combinations with Repetition

- Some possible ways of placing the five bills:



- The number of ways to select five bills corresponds to the number of ways to arrange six bars and five stars in a row.

- This is the number of unordered selections of 5 objects from a set of 11. Hence, there are

$$C(11,5) = \frac{11!}{5!6!} = 462$$

ways to choose five bills with seven types of bills.

Combinations with Repetition

Theorem: The number of r-combinations from a set with n elements when repetition of elements is allowed is

$$C(n + r - 1, r) = C(n + r - 1, n - 1).$$

Proof: Each r-combination of a set with n elements with repetition allowed can be represented by a list of $n - 1$ bars and r stars. The bars mark the n cells containing a star for each time the i-th element of the set occurs in the combination.

The number of such lists is $C(n + r - 1, r)$, because each list is a choice of the r positions to place the stars, from the total of $n + r - 1$ positions.
This is also equal to $C(n + r - 1, n - 1)$, which is the number of ways to place the $n - 1$ bars. ◀

Combinations with Repetition

Example: How many solutions does the equation

$x_1 + x_2 + x_3 = 11$

have, where x_1 , x_2 and x_3 are nonnegative integers?

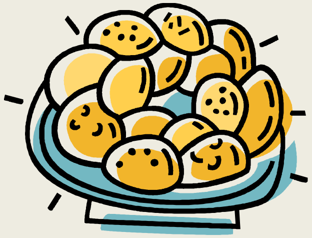
Solution: Each solution corresponds to a way to select 11 items from a set with three elements; x_1 elements of type one, x_2 of type two, and x_3 of type three.

It follows that there are

$$C(3+11-1,11) = C(13,11) = C(13,2) = \frac{13\cdot 12}{1\cdot 2} = 78$$

solutions.

Combinations with Repetition



Example: Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?

Solution: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. By Theorem 2

$$C(9,6) = C(9,3) = \frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3} = 84$$

is the number of ways to choose six cookies from the four kinds.

Summary for Counting Permutations and Combinations with and without Repetition

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

Permutations with Indistinguishable Objects

Example: How many different strings can be made by reordering the letters of the word *SUCCESS*.

Solution: There are seven possible positions for the three S's, two C's, one U, and one E.

- The three S's can be placed in $C(7,3)$ different ways, leaving four positions free.
- The two C's can be placed in $C(4,2)$ different ways, leaving two positions free.
- The U can be placed in $C(2,1)$ different ways, leaving one position free.
- The E can be placed in $C(1,1)$ way.

By the product rule, the number of different strings is:

$$C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!} = \frac{7!}{3!2!1!1!} = 420.$$

Permutations with Indistinguishable Objects

Theorem: The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, …., and n_k indistinguishable objects of type k, is:

$$\frac{n!}{n_1! n_2! \cdots n_k!} .$$