CS182 - Foundations of Computer Science (FoCS)
Final Exam

Name: CS 182 - SPRING17

Purdue ID: Solutions

Problem	Maximum	Score
1	12	
2	16	
3	16	
4	14	
5	10	
6	10	
7	12	
8	10	
Total	100	

Table 1: **This table is for TA use only; do not modify.**

Instructions:

- Make sure that you are writing your first and last name as it is on your myPurdue Career Account (no nickname, short name, etc.).
- You have **2 hours** to complete this test. The test is worth a total of **100 points** and has **14 pages**, including four **scratch pages** at the end. Check your copy and exchange it immediately if it is defective.
- You **are not allowed** to use laptops, cell phones, calculators or any other type of electronic device.
- You can have expressions of the form $\binom{m}{n}$, m^n, $m!$, etc. in your final answers.
- **If you do not know the answer to a particular question leave it blank!** We will, by default, give one point to a blank answer.
- Please read each question carefully several times before beginning to work and especially before asking questions. We generally will not answer questions except when there is a glaring mistake or ambiguity in the statement of a question.

PROBLEM 1 (12 points)

1. **(8 points)** Prove or disprove: For all integers a, b, c, if $a|c$ and $b|c$, then $ab|c^2$.
2. **(4 points)** Prove or disprove: The sum of two primes is a prime.

Your answers:

1.

$$\left.\begin{array}{l} a|c \rightarrow c = a \cdot c' \\ b|c \rightarrow c = b \cdot c'' \end{array}\right\} \rightarrow c^2 = (ac')(bc'') = (ab)(c'c'') = (ab)c'''$$

$$\Rightarrow ab \mid c^2$$

2. Disprove by counter example:

Primes $\begin{cases} P_1 = 3 \\ P_2 = 5 \end{cases}$ However $P_1 + P_2 = 8$ not a prime

PROBLEM 2 (16 points)
For each of the following counting problems, state the answer in the space below. Provide a *brief* (no more than 3-5 lines) justification of your answer.

1. **(8 points)** You have 20 pennies, 30 nickels, and 40 dimes. Assume that the pennies are identical, the nickels are identical, and the dimes are identical. In how many ways can you put all the coins in a row?

2. **(8 points)** Find the number of subsets of $S = \{1, 2, 3, \ldots, 10\}$ that contain exactly five elements, all of them even.

Your answers:

1. $$\frac{(20+30+40)!}{20! \times 30! \times 40!} = \frac{90!}{20!\,30!\,40!}$$

2. Any subset of interest may contain only one of the following elements: $S' = \{2,4,6,8,10\}$
Hence, it is enough to count number of 5-subsets of S':

$$\binom{|S'|}{5} = \binom{5}{5} = 1$$

PROBLEM 3 (16 points)
For each of the following counting problems, state the answer in the space below. Provide a *brief* (no more than 3-5 lines) justification of your answer.

1. **(3 points)** Find the number of solutions to $x + y = 29$, where x and y are integers.
2. **(5 points)** Find the number of solutions to $x + y = 29$, where x and y are nonnegative integers.
3. **(8 points)** Find the number of solutions to $x + y + z = 29$, where $x \geq 7$, $y \geq 7$, and $z \geq 0$ are integers.

Your answers:

1. Since x and y can take any value (i.e. negative, zero, positive integers), there are (countably) infinite pairs $\langle x, y \rangle$ s.t. $x + y = 29$

2. Using "stars and bars" approach, we get:

$$\binom{29+2-1}{2-1} = \binom{30}{1} = 30$$

3. Set $\begin{cases} x' = x - 7 \\ y' = y - 7 \\ z' = z - 0 \end{cases}$ and solve $x' + y' + z' = 29 - (7+7+0)$

$\Leftrightarrow x' + y' + z' = 15$

Again, we use the "stars and bars" method:

$$\binom{15+3-1}{3-1} = \binom{17}{2} = \frac{17 \times 16}{2} = 136$$

PROBLEM 4 (14 points)
For each of the following problems, state the answer in the space below. Provide a *brief* (no more than 3-5 lines) justification of your answer.

1. **(7 points)** Find the coefficient of x^5 in $(-1+x^2+3)^{12}$.
2. **(7 points)** How many ways are there to arrange the letters of the word NONSENSE that start with the letter O?

Your answers:

1. $$(-1+x^2+3)^{12} = (x^2+2)^{12} = \sum_{i=0}^{12} \binom{12}{i} (x^2)^i \, 2^{12-i}$$
$$= \sum_{i=0}^{12} \binom{12}{i} x^{2i} \, 2^{12-i}$$

As we can, see, for $0 \leq i \leq 12$ only x^{2i} has non-zero coefficient; i.e. any odd power of x, including 5, has coefficient zero!

2. Count of letters: $\begin{cases} N : 3 \\ O : 1 \\ S : 2 \\ E : 2 \end{cases}$

O _ _ _ _ _ _ _

fixed

Now, we need to count permutations of other 7 letter (Notice, these MUST be distinct):

$$\frac{(3+2+2)!}{3!\,2!\,2!} = \frac{7!}{3!\,2!\,2!} = 210$$

PROBLEM 5 (10 points)
A computer is programmed to print subsets of $\{1, 2, 3, 4, 5\}$ at random. If the computer prints 40 subsets, prove that some subset must have been printed at least twice.

Your answer:

$A = \{1, 2, 3, 4, 5\}$

We use pigeon-hole principle:

- Total number of subsets: $2^{|A|} = 2^5 = 32$

[H] - Pigeon-holes $\equiv$ # of subsets $= 32$

[P] - Pigeons $\equiv$ # of printed subsets $= 40$

- Since $P > H$ $(P = 1 \times H + 8)$, then by pigeon-hole principle, there is at least $\underline{1}$ pigeon-hole with more than $\underline{1}$ pigeon in it; i.e., there is at least one subset printed more than once.

PROBLEM 6 (10 points)
Pick a bit string (i.e., a string of zeros and ones) from the set of all bit strings of length ten. What is the probability that the bit string has the sum of its digits equal to seven? Explain your answer.

Your answer:

\# of all bit-strings of length 10 : 2^{10}

\# of all bit-strings of length 10 with exactly seven 1s and three 0s :

$$\binom{10}{7} = \frac{10!}{7!\,3!} = \frac{10 \times 9 \times 8}{3 \times 2} = 120$$

Hence,

$$\Pr_{\text{len}=10}[\text{Sum of bits} = 7] = \frac{120}{2^{10}} = \frac{120}{1024} \simeq 0.117$$

PROBLEM 7 (12 points)

1. (**4 points**) Let A and B be two events defined over a sample space S. What is the probability $P(A|B)$ if A is the empty set and B is non-empty? Can you answer the same question in the case where A is non-empty but B is the empty set? Finally, compute the probabilities $P(A|S)$ and $P(S|A)$ where A is non-empty.

2. (**8 points**) A red and a green die (both fair) are rolled. What is the probability of getting a sum of six, given that the number on the green die is odd? Explain your answer in 3-5 lines.

Your answer:

1. $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$, $P(B) > 0$

- $A = \phi$, then $P(A|B) = 0$ since $P(B) > 0$.
- $A \neq \phi$, $B = \phi$ $P(A|B)$ is undefined since $P(B) = 0$
- $P(A|S) = \dfrac{P(A \cap S)}{P(S)} = \dfrac{P(A)}{P(S)} = P(A)$
- $P(S|A) = P(S \cap A)/P(A) = P(A)/P(A) = 1$

2. Since the number on the green die has been rolled and is odd, our sample space consists of only 18 entries (e.g. $\{(1,1), (1,3), (1,5), \dots (6,1), (6,3), (6,5)\}$)
Of those 18 entries, 3 sum up to 6 $((1,5), (3,3), (5,1))$ and thus the probability is $3/18 = 1/6$.

PROBLEM 8 (10 points)

1. (**4 points**) Consider the language $L_1 = \{a^n b^n, 0 \leq n \leq 2017\}$ over the alphabet $\Sigma = \{a, b\}$. Is L_1 a regular language (explain why)?

2. (**6 points**) Design a deterministic or non-deterministic finite automaton for the language

$$L_2 = \{w \in \Sigma^* : \text{strings in } L \text{ do not contain 001 or 111}\},$$

where $\Sigma = \{0, 1\}$.

Your answer:

1. Yes, because L_1 is a finite language and as proved in the lectures, we know that all finite languages are regular.

2.