Abhishek Gunasekar

CS 182: Homework 5

Problem 1 (20 Points)

Prove or disprove that for any positive integer n ≥ 0, $n^3 + 2n + 3$ is divisible by 3.

$P(n)$: $n^3 + 2n + 3 = 3x \qquad n \geq 0, x \geq 1$
1. $\boldsymbol{Basis\ step}$: $P(0) = 0^3 + 2(0) + 3 = 3 = 3(1), \quad DONE$
2. $\boldsymbol{Inductive\ Hypothesis}$: $Assume\ P(n)\ holds\ for\ any\ arbitary\ integer\ n \geq 0$
3. $\boldsymbol{Will\ Prove}$: $P(n) \rightarrow P(n+1)$
$P(n+1)$: $(n+1)^3 + 2(n+1) + 3$
$= (n+1)(n^2 + 2n + 1) + 2(n+1) + 3$
$= n^3 + 3n^2 + 3n + 1 + 2n + 2 + 3$
$= (n^3 + 2n + 3) + (3n^2 + 3n + 3)$
$= 3(x) + 3(n^2 + n + 1) \qquad Inductive\ Hypothesis$
$= 3(x + n^2 + n + 1) = 3l\ where\ l\ is\ any\ positive\ integer \qquad DONE$
4. $\boldsymbol{Hence}, \boldsymbol{proven}\ that\ for\ any\ positive\ integer\ n \geq 0, n^3 + 2n + 3\ is\ divisible\ by\ 3.$

Problem 2 (20 Points)

Prove or disprove that for all n ≥ 1

$$\sum_{i=1}^{n} \boldsymbol{i2^i \leq 2^{n+1}n}.$$

$$P(n): \sum_{i=1}^{n} i2^i \leq 2^{n+1}n. for\ n \geq 1$$

1. $\boldsymbol{Basis\ Step}: P(1) is\ true\ since \sum_{i=1}^{1}(1)2^1 = 2 \leq 2^2 1 = 4,$ DONE

2. $\boldsymbol{Inductive\ Hypothesis}: Assume\ P(n)\ \ holds\ for\ any\ arbitrary\ integer\ n \geq 1.$

3. $\boldsymbol{Will\ Prove}: P(n) \rightarrow P(n+1)$

$P(n): (1)(2)^1 + (2)(2)^2 + (3)(2)^3 + \cdots + n(2)^n \leq 2^{n+1}n$

$$P(n+1): \sum_{i=1}^{n+1} i2^i = \sum_{i=1}^{n} i2^i + (n+1)2^{n+1} \leq 2^{n+2}(n+1)$$

$= (n-1)(2^{n+1}) + 2 + (n+1)2^{n+1} \leq 2^{n+2}(n+1)$

$= 2^{n+1}(n - 1 + n + 1) + 2 \leq 2^{n+2}(n+1)$

$= 2^{n+1}(2n) + 2 \leq 2^{n+2}(n+1)$

$= (2^{n+2})n + 2 \leq (2^{n+2})n +\ 2^{n+2}$
$= (2^{n+2})n + 2 \leq (2^{n+2})n + 4(2^n)$

4. $\boldsymbol{Therefore} \sum_{i=1}^{n} \boldsymbol{i2^i \leq 2^{n+1}n\ for\ n \geq 1}$

Problem 3. (20 Points)

$$\boldsymbol{Let\ \ H_n = \sum_{i=1}^{n} \frac{1}{i}}$$

$\boldsymbol{for\ any\ n \geq 1. Prove\ or\ disprove\ that\ for\ any\ n \geq 1\ H_{2n} \geq \frac{n}{4}}$

1. $\boldsymbol{Basis\ Step}: n = 1\ \left(\frac{1}{2} + \frac{1}{1}\right) \geq \frac{1}{4}$, then P(1) holds, DONE

2. $\boldsymbol{Inductive\ Hypothesis}: Assume\ H_{2k} \geq \frac{k}{4}$ holds for any arbitrary integer $\boldsymbol{k \geq 1}$

3. $\boldsymbol{Will\ Prove}: \boldsymbol{P(k) \to P(k+1)}$

$$H_{2(k+1)} \geq \frac{k+1}{4}$$

$$\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} \geq \frac{k+1}{4} = \frac{k}{4} + \frac{1}{4}$$

$$H_{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} \geq \frac{k}{4} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}}$$

$$\frac{k}{4} + \left(2^k * \left(\frac{1}{2^k+1}\right)\right) \geq \frac{k}{4} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}}$$

$$\frac{k}{4} + \frac{1}{2} \geq \frac{k}{4} + \frac{1}{4}$$

4. $\boldsymbol{Therefore, by\ induction, we\ have\ proven}\ that\ for\ any\ n \geq 1\ H_{2n} \geq \frac{n}{4}$

Problem 4. (20 Points)

$\boldsymbol{Let\ a_1 = 2, a_2 = 9, and\ a_n = 2_{a_{n-1}} + 3_{a_{n-2}} for\ n \geq 3.}$

$\boldsymbol{Prove\ or\ disprove\ that\ a_n \leq 3^n\ for\ all\ positive\ integers\ n.}$

$P(n)$: $a_n \leq 3^n$ for all positive integers n

1. $\boldsymbol{Basis\ Step}$: $P(1)\ holds\ since\ 2 \leq 3$
   $P(2)\ holds\ since\ 9 \leq 9$
   $P(3)\ holds\ since\ 2(9) + 3(2) = 24 \leq 27, DONE$

2. $\boldsymbol{Inductive\ Hypothesis}$: $Assume\ P(n)\ holds\ for\ any\ arbitrary\ positive\ integer\ n$

3. $\boldsymbol{Will\ Prove}$: $P(n) \rightarrow P(n+1)$

$P(n+1)$: $a_{n+1} \leq 3^{n+1}$

$a_{n+1} = 2a_n + 3a_{n-1}$

$Suppose\ a_n = 3^n$

$$\begin{aligned} a_{n+1} &= 2(3^n) + (3(3^{n-1})) \\ &= 2(3^n) + 3^n \end{aligned}$$

$a_{n+1} = 3^n(2+1) = 3^{n+1}$

$If\ a_n \leq 3^n$ $\quad$ $Inductive\ Hypothesis$

$a_{n+1} \leq 2(3^n) + 3(3^{n-1})$

$a_{n+1} \leq 2(3^n) + 3^n$

$a_{n+1} \leq 3^n(2+1)$

$a_{n+1} \leq 3^{n+1}$

4. $\boldsymbol{Therefore, by\ induction, we\ have\ proven}$ $that\ a_n \leq 3^n$

Problem 5. (20 Points)

Describe the most efficient algorithm to compute $5^{\log_2 m}$ where m is a nonnegative integer. Prove its correctness.

Algorithm:

```
Author: Abhishek Gunasekar
procedure power(x, m) {
    int n = log2(m)
    if (n == 0) {
        return 1;
    }
    else if (n == 1) {
        return x;
    }
    else if (n is even) {
        return power(x * x, n/2);
    }
    else if (n is odd) {
        return x * power(x * x, (n -1) /2);
    }
}
```

Proof of correctness:

$P(n)$: *for all nonnegative integers* m, $Power(5, m)$ *correctly computes* $5^{\log_2 m}$

Procedure as a Piecewise Function.

$$x^n = \begin{cases} (x^2)^{\frac{n}{2}} & n \text{ is even} \\ x(x^2)^{(\frac{n-1}{2})} & n \text{ is odd} \end{cases}$$

The relationship between n *and* $\log_2 m$ *is as follows*:

If $n = \log_2 m$,

then, $n + 1 = \log_2(m + m)$

1. ***Basis Step***: *if* $m = 2 \rightarrow n = 1$, *then* $(x^2)^{\frac{1}{2}} = x$, *which is true because* $5^1 = 5$

2. ***Inductive Hypothesis***: $x = 5$ *in this case, so* $power(x, n) = x^n = (x^2)^{\frac{n}{2}}$ *if* n *is even*;

$$x^n = x(x^2)^{\frac{n+1}{2}} \text{ if n is odd}$$

3. ***Will Prove***: $P(n) \rightarrow P(n + 1)$

Case 1: *If* n *is even, then* $(n + 1)$ *is odd*

$P(n + 1)$: $x(x^2)^{\frac{(n+1)-1}{2}} = x(x^2)^{\frac{n}{2}}$ *conclusion to arrive at*

$$x^{n+1} = x * x^n = x(x^2)^{\frac{n}{2}} \qquad By\ inductive\ hypothesis$$

$Therefore,\ proved\ case\ 1\ using\ induction$

$Case\ 2: If\ n\ is\ odd, then\ (n+1)\ is\ even$

$P(n+1): (x^2)^{\frac{(n+1)}{2}} \quad conclusion\ to\ arrive\ at$

$x^{n+1} = x * x^n$

$= x * (x(x^2)^{\frac{n-1}{2}})$

$= x^2((x^2)^{\frac{n-1}{2}})$

$= (x^2)^{\frac{n-1+2}{2}}$

$= (x^2)^{\frac{n+1}{2}}$

$Therefore, proved\ case\ 2\ using\ induction$

4. $\boldsymbol{Therefore}, \boldsymbol{by\ strong\ induction}, we\ have\ proven\ the\ correctness\ of\ the\ algorithm\ to\ get\ 5^{\log_2 m}$