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CS 182: Homework 7

Problem 1. (45=5*9 points)

Suppose you have 18 objects (10 of type A, 5 of type B, and 3 of type C). Objects of type A are indistinguishable from each other; objects of type B are indistinguishable from each other; and objects of type C are indistinguishable from each other. In how many ways can you:

1. **Put the 18 objects in a row?**

   Total No. Of Objects = 18

   No. Type A object = 10; No. Type B Object = 5; No. Type C Object = 3;

   Ways to put 18 objects in a row: By **Division Rule.**

$$\frac{\mathbf{18}!}{\mathbf{10}! * \ \mathbf{5}! * \ \mathbf{3}!}, \boldsymbol{total\ ways}$$

2. **Pick 3 of the 18 objects (order does not matter)?**

$\boldsymbol{Method\ 1}$: $Coefficient\ of\ x^3 in\ Binomial\ Theorem\ represents\ the\ number\ of\ ways\ that\ you\ can\ enumerate$

$3\ objects\ from\ n\ objects, which\ can\ be\ represented\ by\ the\ below\ equation.$

$$= \ (1 + x + x^2 + \cdots + x^{10}) * (1 + x + x^2 \ldots + x^5) * (1 + x + x^2 + x^3)$$

$$= \left(\frac{(1-x^{11})}{1-x}\right) * \left(\frac{1-x^6}{1-x}\right) * \left(\frac{1-x^4}{1-x}\right)$$

$$= (1 - x^{11}) * (1 - x^6) * (1 - x^4) * (1 - x)^{-3}$$

$$= \ (1 - x^{11}) * (1 - x^6) * (1 - x^4) * (1 + 3c_1 x + 4c_2 x^2 + 5c_3 x^3 + \cdots)$$

$Therefore, Coefficient\ of\ x^3 = 5C_3 = \mathbf{10}\ \boldsymbol{Total\ Ways}$

$\boldsymbol{Method\ 2}$: $\boldsymbol{Repetition\ Allowed\ and\ Order\ Does\ not\ matter, n = 3, r = 3}$

$$\binom{n + r - 1}{r} = \binom{5}{3} = \mathbf{10}\ \boldsymbol{Total\ Ways}$$

$\boldsymbol{Method}\ \mathbf{3}\ (\boldsymbol{Brute\ Force})$: $\boldsymbol{C} = \{\mathbf{300, 030, 003, 120, 210, 102, 201, 021, 012, 111}\} = |\boldsymbol{c}| = \mathbf{10}.$

3. **Pick 4 of the 18 objects (order does not matter)?**

$Coefficient\ of\ x^4 in\ Binomial\ Theorem\ represents\ the\ number\ of\ ways\ that\ you\ can\ enumerate$

$4\ objects\ from\ n\ objects, which\ can\ be\ represented\ by\ the\ equation\ continued\ from\ last\ problem.$

$$= (1-x^{11})(1-x^6)(1-x^4)*(1+3c_1x+4c_2x^2+5c_3x^3+6c_4x^4+\cdots)$$

$$= -1+6c_4$$

$$= -1+15$$

$$= \mathbf{14\ \boldsymbol{total\ ways}}.$$

$\boldsymbol{Method\ 2: Repetition\ Allowed\ and\ Order\ Does\ not\ matter, n = 3, r = 4}$

$$\binom{n+r-1}{r} = \binom{6}{4} - 1\ (\#impossible\ ways) = \mathbf{15-1=14}\ \boldsymbol{Total\ Ways}$$

$\boldsymbol{Brute\ Force\ Verification}$:

$= \{\mathbf{400, 040, 310, 301, 013, 103, 031, 130, 202, 220, 211, 022, 121, 112}\}$

⇨ $|\boldsymbol{c}| = \mathbf{14}$

4. **Pick 5 of the 18 objects (order does not matter)?**

$Coefficient\ of\ x^5 in\ Binomial\ Theorem\ represents\ the\ number\ of\ ways\ that\ you\ can\ enumerate$

$5\ objects\ from\ n\ objects, which\ can\ be\ represented\ by\ the\ equation\ continued\ from\ last\ problem.$

$$= (1-x^{11})(1-x^6)(1-x^4)*(1+3c_1x+4c_2x^2+5c_3x^3+6c_4x^4+2c_3x^5+\cdots$$

$$= -3c_1+7c_5$$

$$= 21-3$$

$$= \mathbf{18}\ \boldsymbol{Total\ Ways}$$

$\boldsymbol{Method\ 2: Repetition\ Allowed\ and\ Order\ Does\ not\ matter, n = 3, r = 5}$

$$\binom{n+r-1}{r} = \binom{7}{5} - 3\ (\#impossible\ ways) = \mathbf{21-3=18}\ \boldsymbol{Total\ Ways}$$

Brute Force: c = {500,050,401,410,041,140,320,302,023,203,032,230,122,131,113,211,112,312}

- ⇨ **|c| = 18**

5. **Pick nine objects out of the 18 objects so that exactly three objects are of type A and exactly two objects are of type B (order does not matter)?**

$There\ are\ \mathbf{0}\ \boldsymbol{ways}\ to\ select\ nine\ objects\ out\ of\ the\ 18\ objects\ because\ if\ there\ are\ three\ objects\ of\ type\ A\ and\ exactly\ two\ objects\ of\ type\ B\ would\ mean\ that\ you\ would\ need\ \mathbf{4}\ \boldsymbol{of\ type\ C}, which\ is\ clearly\ not\ possible\ since\ only\ 3\ objects\ of\ type\ C\ exists.$

Problem 2. (27 = 3*9 points)

A movie theater can play 30 westerns, 15 science fiction movies, and 10 horror movies (all movies are distinct from each other). Its standard daily program typically consists of a western followed by a science fiction movie, and then a horror movie.

1. **How many different programs can it play?**

   $Number\ of\ Different\ Programs$

   $= P(Westerns) * P(Scifi) * P(Horror)$ $\qquad$ $\boldsymbol{Product\ Rule}$

   $= 30 * 15 * 10$

   $= \mathbf{4500}\ \boldsymbol{different\ programs\ can\ be\ played}.$

2. **How many different programs are there if the three movies can be played in any order? How does this number compare to the previous number and why?**

   $\#\ Total\ Programs = 3! * P(Standard\ Daily\ Program)$

   $= \mathbf{3}! * (\mathbf{30} * \mathbf{15} * \mathbf{10})\ \boldsymbol{Total\ Programs}$

   $This\ answer\ is\ just\ 3!\ times\ the\ first\ answer\ because\ the\ number\ of\ possibilities$

   $for\ 3\ such\ movies\ would\ be\ P(3,3) = 3!$

3. **How many different three-movie programs are there if there are absolutely no restrictions (e.g., the same movie can be played twice, movies can be played in any order, categories do not matter, etc.)? How does this number compare to the previous numbers and why?**

   $\boldsymbol{Repetitions}: Allowed;\ \boldsymbol{Order}: Matter$

   $= 55\ Total\ movies\ are\ there$

   $= n^r = \mathbf{55^3}\ \boldsymbol{different\ three\ movie - programs\ are\ there}.$

$This\ number\ is\ much\ different\ because\ repetitions\ are\ allowed\ and\ the\ order\ does\ matter.$

$This\ is\ different\ than\ the\ previous\ numbers\ because\ in\ the\ previous\ cases\ the\ movies\ were\ distinct.$

$But\ in\ this\ scenario\ the\ movies\ are\ basically\ all\ the\ same\ and\ they\ are\ indistinguishable\ from\ each\ other.$

Problem 3. (16 = 2*8 points)

You have 10 of each of the following type of objects: A, B, C, and D. The objects of each type are distin- guishable (e.g., the 10 objects of type A are different from each other, think of them as $\boldsymbol{A_1} \dots \boldsymbol{A_{10}}$; same for the other three types).

1. **In how many ways can you arrange all objects in a row?**

   $\boldsymbol{Order}: Matters; \quad \boldsymbol{Repetition}: No\ repetition$

   $Number\ of\ distinct\ objects\ (n) = 40, and\ Number\ of\ to\ be\ selected\ objects\ (r) = 40$

   $\boldsymbol{Therefore, total\ possibilities}: \boldsymbol{P(n,r) = P(40,40) = 40!}$

2. **In how many ways can you choose a set S of 10 objects?**

   $\boldsymbol{Order}: \boldsymbol{DOES\ NOT}\ Matter; \quad \boldsymbol{Repetition}: No\ Repetition$

   $n = 40, and\ r = 10$

   $$= \boldsymbol{C(40,10)} = \frac{\boldsymbol{40!}}{\boldsymbol{10!\,(40-10)!}} = \frac{\boldsymbol{40!}}{\boldsymbol{10!\,30!}}$$

Problem 4. (12 = 2*6 points)

1. $\boldsymbol{What\ is\ the\ coefficient\ of\ x^8y^4\ in\ the\ expansion\ of\ (4x-4y)^{28}?}$

$$Binomial\ Theorem\text{:}\ (a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j}b^j$$

$$(4x-4y)^{28} \coloneqq \binom{28}{0} a^{28}b^0 + \cdots + \binom{28}{k} a^{28-k}b^k + \binom{28}{28} b^{28}, where\ a = 4x, and\ b = -4y$$

$Let\ z\ be\ n-j\ and\ q\ be\ j, then\ z+q = 28\ or\ z = 28-q$

$\boldsymbol{Since\ there's\ no\ possibility\ of\ x^8y^4\ occuring\ in\ the\ binomial\ expansion,}$
$\boldsymbol{the\ Coefficient\ of\ x^8y^4 = 0.}$

2. $\boldsymbol{Prove\ that\ \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}}$

R.H.S:

$$\frac{n!}{(n-k)!\,k!} + \frac{n!}{\big(n-(k-1)\big)!\,(k-1)!} = \frac{n!}{(n-k)!\,k!} + \frac{n!}{(n-k+1)!\,(k-1)!}$$

$$= \frac{n!}{(k-1)!\,(n-k)!}\left(\frac{1}{k} + \frac{1}{n+1-k}\right)$$

$$= \frac{n!}{(k-1)!\,(n-k)!}\left(\frac{(n+1-k)+k}{k\,(n+1-k)}\right)$$

$$= \frac{n!\,(n+1)}{(k-1)!\,(n-k)!\,k\,(n+1-k)}$$

$$= \frac{(n+1)\,!}{k!\,(n+1-k)} = L.H.S$$

$\boldsymbol{Since\ L.H.S = R.H.S, \qquad Therefore\ Proved.}$