CS 182 – Foundations of Computer Science

PSO sessions 1 and 2, week of April 27, 2020

PSO 1 and 2

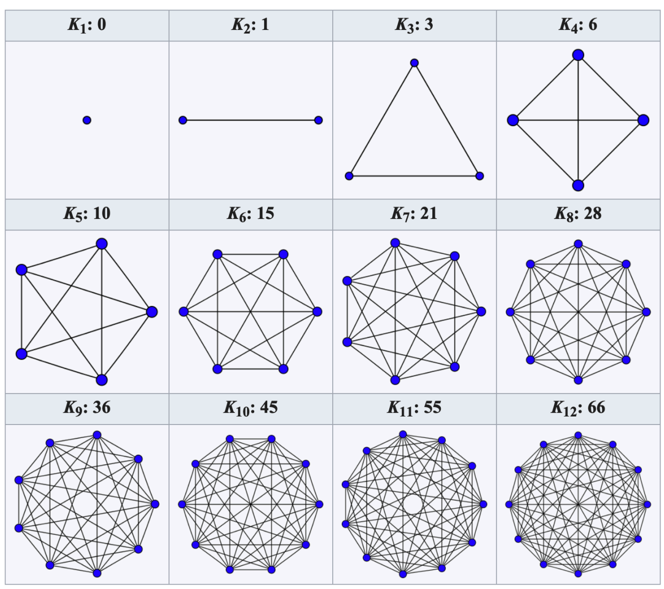
Problem 1. Graph K_n is the complete, undirected graph on n vertices (see 10.3 in KR for more details).

1. For what values of n does K_n contain an Euler cycle? Explain your answer.

2. For what values of n is K_n a bipartite graph.

Solution 1.1.

Recall that an Euler cycle is a path that goes through *every edge exactly once*, starting and ending at the same vertex. Since it must start and end at the same vertex, every vertex must have even degree. Otherwise, the path would have to enter and exit any given odd-degree vertex an unequal number of times, which is impossible.



The first few orders (Wikipedia)

Observe that for K_n every vertex is connected to every other. Hence, each has degree $n - 1$.

Since $2 \nmid n - 1$ for n even, K_n does not have an Euler cycle when n is even.

But do we know that there actually *is* a path for n odd?

Well, we could use an inductive construction like you saw in class. But since complete graphs have so much symmetry, there's a pleasing direct proof.

Arrange the vertices in a circle as in the Wikipedia drawings. For any given vertex *v*, define the vertex "*k* vertices away" as the vertex you get by going around the circle *k* vertices clockwise.

Notice what happens if we only take edges some fixed $k \leq n/2$ vertices away. For *k* coprime with *n*, you'll go through all the vertices, looping back to where you started.

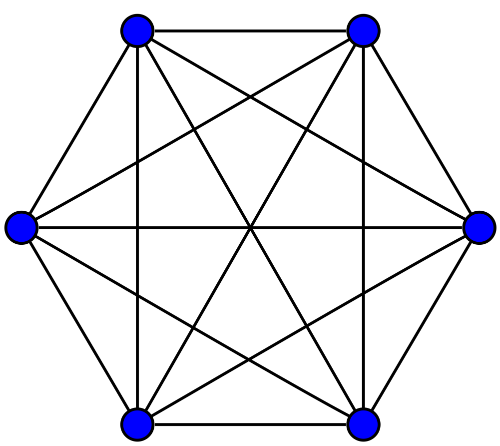
For *k* not coprime with *n*, we'll still get back to where we started, but we'll only go through n / k vertices. By symmetry, we have *k* unique ways to start the cycle, each of which goes through a unique set of n / k vertices as well.

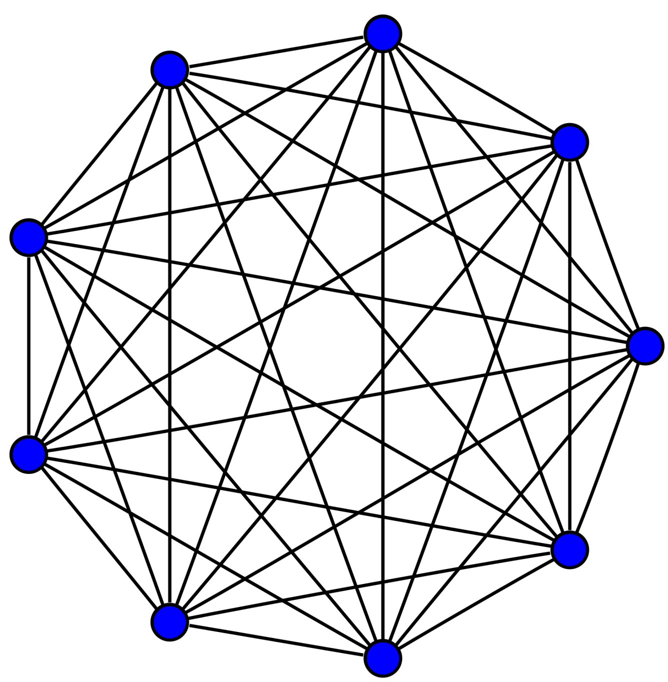
One way, then, we could try to construct an Euler cycle is by taking the union of all such paths involving edges successive values of *k* vertices away. The union cycle may integrate the coprime paths as a modified walk through the first path $k = 1$, wherein before going to each neighbor 1 vertex away, we redirect to any novel coprime paths starting at the given vertex.

Notice that, in each sub-path, the edges used are unique to the sub-path. That's because any given edge connects 2 vertices a fixed distance away. After all, 2 vertices cannot simultaneously be *k* and unequal *k'* away from each other Hence, if there's any edge walked twice, it's *within* a given sub-path, not *between* 2 or more.

When does this happen? Only when *n* is even, since, in that case, the sub-path walking $n/2$ distance edges consists of degenerate pair-wise paths, walking the same antipodal edge across the graph and back.

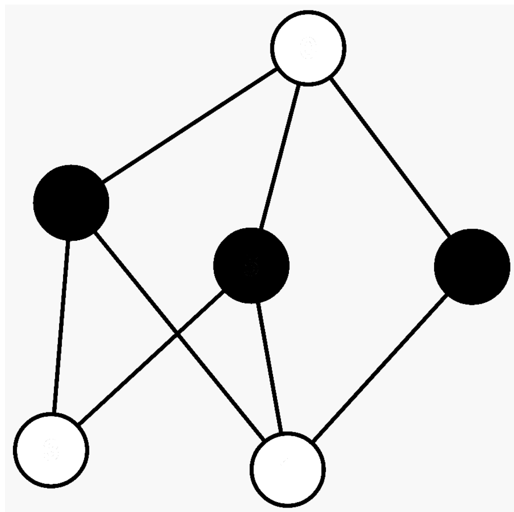
If you have trouble seeing this, try this process on the two graphs below and see what happens.





Solution 1.2

Recall that a bipartite graph is one such that the vertices can be partitioned into 2 disjoint sets such that all the edges are *between* and not *within* the sets. An isomorphic way to look at this is as a coloring problem: is there a way to color some of the vertices black and some white such that no edge connects a black and white vertex?



A bipartite graph

For complete graphs, it's only possible when $n < 3$.

Every vertex must have a different color than each of its neighbors. For 1 or 2 vertices, the coloring is trivial.

Suppose K_3 has vertices A, B, and C. A must have a different color than B's and C's. But B and C must have different colors from each other too. So A can't have B's color, nor C's different color. Ergo, 2 colors aren't enough, and so it's impossible to color some black and some white. Every K_n afterward has K_3 as a subgraph, so it's impossible for them too.

Problem 2.

Define the n-cube graph Q_n, using the recursive definition given in 10.2, Example 8.

i. Q_n has 2^n vertices. How many edges does Q_n have?

ii. Express the number of vertices and edges of Q_n in terms of the number of vertices of Q_{n-1}, respectively.

iii. Is Q_n a bipartite graph?

Solution 2.

We'll solve these out of order. First, read the alternative construction on the next page. Clearly Q_n is bipartite, as each step involves connecting a white vertex to a black vertex.

At each step, we add a chiral copy. Hence, the number of vertices doubles each time.

$$V_n = 2\ V_{n-1} \qquad\qquad V_0 = 1$$

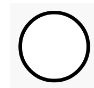
Observe too that the number of edges a specific vertex has increases by one each time a chiral copy is connected in. Hence, the total degree of all vertices is nV_n. But since the total degree double counts each edge, we divide by 2, yielding the following number of edges.

$$E_n = (n/2)\ V_n = (n/2)\ 2^n = n\ 2^{n-1}$$

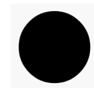
Recursively, this amounts to saying

$$E_n = (n/2)\ V_n = (n/2)\ (2V_{n-1}) = n\ V_{n-1} \qquad\qquad V_0 = 1$$

Start with a single white vertex, Q_0.



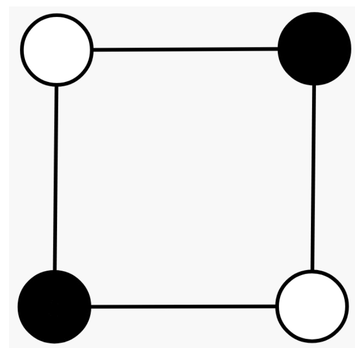
Create it's chiral equivalent.



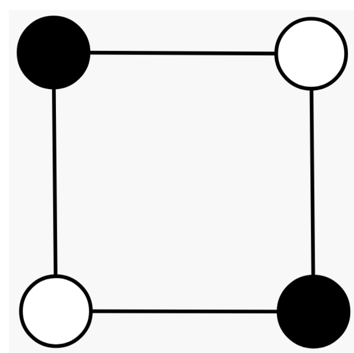
Connect them together to form Q_1.

Create it's chiral equivalent.

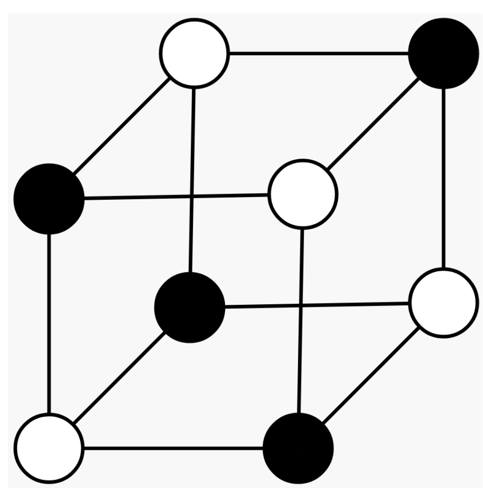
Connect them together to form Q_2.



Create it's chiral equivalent.



Connect them together to form Q_3.



Problem 3.

A simple graph is regular if every vertex has the same degree.

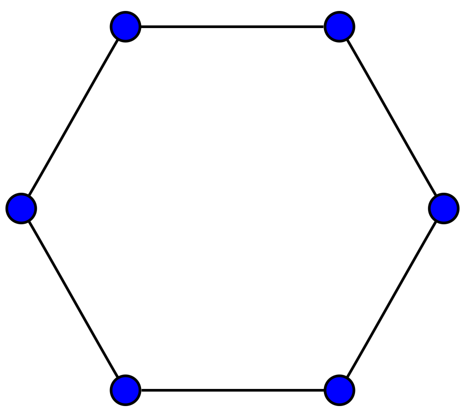
i. For which positive integers n are the following graphs regular: C_n, W_n, K_n?

ii. For which positive integers m and n is the complete bipartite graph $K_{m,n}$ regular?

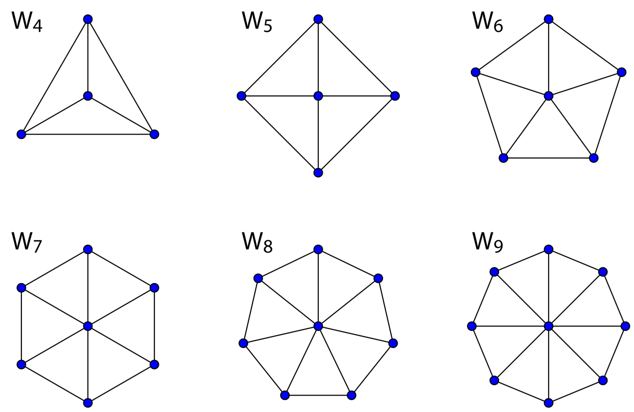
Solution 3.1

We know from earlier that each vertex in K_n has $n - 1$ edges, since all the vertices are connected together. Hence, all complete graphs are regular.

For the other two, recall their definitions.



The cycle graph C_6 (Wikipedia)



A few wheel graphs (Wikipedia)

In non-degenerate a cycle graph, each vertex has 2 edges. Hence, they are all regular.

A Wheel is like a cycle graph, but additionally each vertex on the rim is connected to one in the center. Each rim vertex has 3 edges, but the central vertex has $n - 1$ edges. Ergo, the only regular Wheel is W_4, since

$$n - 1 = 3$$
$$n = 4$$

Solution 3.2

The complete bipartite graph $K_{n,m}$ is a bipartite graph wherein each black vertex is connected to every white vertex. If there are n white vertices and m black vertices, then all white vertices have degree m and all black vertices have degree n. Thus, the only regular ones have n and m equal. Depending on definitions, we do also have the degenerate $K_{0,1}$ and $K_{1,0}$. For other degenerates like $K_{0,m}$ where $m > 1$, the graph is not simple, and therefore not regular.