Problem 1.

Find the best (i.e. asymptotically smallest) big-Oh function for $\frac{x^3+7x}{3x+1}$.

Let $f(x) = \frac{x^3+7x}{3x+1}$. We seek a function $g(x)$ such that, for some cut-off value *k* and constant *c*,

$$\forall x \quad x \geq k \;\Rightarrow\; f(x) \leq c\,g(x)$$

$\frac{x^3+7x}{3x+1} \leq \frac{x^3+7x^3}{3x+1} = \frac{8x^3}{3x+1}$ when $x \geq 1$. So let *k* tentatively be 1.

$\frac{8x^3}{3x+1} \leq \frac{8x^3}{3x}$ when $x > 0$. So $k = 1$ still works.

Since $\frac{8x^3}{3x} = \frac{8}{3}x^2$, if we let $c = \frac{8}{3}$, then we have

$$\forall x \quad x \geq 1 \;\Rightarrow\; \frac{x^3+7x}{3x+1} \leq \frac{8}{3}x^2 \quad .$$

So we have such a $g(x)$, the function x^2 . So $\frac{x^3+7x}{3x+1} \in O(x^2)$.

We have found our answer. But how do we know that our choice of *g* is asymptotically smallest? If we can show *g* also under-bounds *f*, then there can be nothing tighter. Let's therefore prove that, for some potentially different cutoff *k*' and constant *c*', that

$$\forall x \quad x \geq k' \;\Rightarrow\; f(x) \geq c'x^2 \quad .$$

$\frac{x^3+7x}{3x+1} \geq \frac{x^3+7x}{3x+x}$ when $x \geq 1$. So let *k*' tentatively be 1.

$\frac{x^3+7x}{3x+x} = \frac{x^3+7x}{4x} = \frac{1}{4}x^2+\frac{7}{4} \geq \frac{1}{4}x^2$

So $\frac{x^3+7x}{3x+1} \in \Omega(x^2)$ too. So *g* is asymptotically smallest.

Problem 2

Prove that $n^3 + 3n^2 + 2n$ is a multiple of three for all $n \geq 1$.

We shall use induction.

Base case where $n = 1$
$n^3 + 3n^2 + 2n = (1)^3 + 3(1)^2 + 2(1) = 6$
$6 = 3(2) \rightarrow 3 \mid n^3 + 3n^2 + 2n$
$\rightarrow$ the base case holds

Let's assume the Inductive Hypothesis that, $k \geq 1$

$$3 \mid k^3 + 3k^2 + 2k$$

Consider the claim for $k + 1$

$(k+1)^3 + 3(k+1)^2 + 2(k+1)$ expands to

$(k^3 + 3k^2 + 3k + 1) + (3k^2 + 6k + 3) + (2k + 2)$ which simplifies to

$k^3 + 6k^2 + 11k + 6$ which is equal to

$k^3 + 3k^2 + 2k + 3k^2 + 9k + 6$ which is also

$(k^3 + 3k^2 + 2k) + 3k^2 + 9k + 6$

Ah, $(k^3 + 3k^2 + 2k)$ was assumed to be a multiple of 3 (that's our inductive hypothesis).

So we can write

$(k^3 + 3k^2 + 2k) + 3k^2 + 9k + 6$ as

$3m + 3k^2 + 9k + 6$ for some integer m

$3(m + k^2 + 3k + 2)$

and since $m + k^2 + 3k + 2$ is an integer, $3 \mid (k+1)^3 + 3(k+1)^2 + 2(k+1)$

Ok, so if the statement is true for k, we've proven it must be true for k+1.
The statement is true for 1. So it's true for all integers greater than 1.

Done.