# Row interchange property

Interchanging two rows of a matrix changes the sign of the determinant.

i.e. $$\det\left(\begin{bmatrix} A \\ \hline X \\ \hline B \\ \hline Y \\ \hline C \end{bmatrix}\right) = -\det\left(\begin{bmatrix} A \\ \hline Y \\ \hline B \\ \hline X \\ \hline C \end{bmatrix}\right)$$

(if $X$ and $Y$ are rows)

ex. $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -(ad - bc) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

ex. $\begin{vmatrix} 0 & 0 & 3 \\ 1 & 5 & -1 \\ 0 & 2 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 5 & -1 \\ 0 & 0 & 3 \\ 0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{vmatrix} = 6$.

**Equal rows property.** If $A$ has two equal rows, then $\det(A) = 0$.

*Proof.* Let $B$ be the matrix obtained from $A$ by swapping the two equal rows.
Then $\det(A) = -\det(B) = -\det(A) = 0$. ∎

## Idea of proof for row interchange property

True for $2\times 2$ matrices ✓

Assume we know it for $(n-1)\times(n-1)$ matrices.

Case 1: Swapping two rows, neither of which is the first row.

$$A = \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline A_3 \\ \vdots \\ \hline A_n \end{bmatrix} \qquad A' = \begin{bmatrix} A_1 \\ \hline A_3 \\ \hline A_2 \\ \vdots \\ \hline A_n \end{bmatrix}$$

(for example; other row swaps are similar)

$$\det(A') = a_{11}\det(A'_{11}) - a_{12}\det(A'_{12}) + \cdots + (-1)^{1+n} a_{1n}\det(A'_{1n}).$$

$$A'_{11} = \begin{bmatrix} a_{32} & a_{33} & \cdots & a_{3n} \\ a_{22} & a_{23} & \cdots & a_{2n} \\ a_{42} & a_{43} & \cdots & a_{4n} \\ \vdots & & & \vdots \\ a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix} \quad \text{so} \quad \det(A'_{11}) = -\det(A_{11}).$$

Likewise, $\det(A'_{1j}) = -\det(A_{1j})$, so $\det(A') = -\det(A)$.

## Case 2: Swapping 1st row and $i$th row, $i > 2$.

This is equivalent to:

Swap rows 1 and 2

Swap rows 2 and $i$ (changes sign by case 1)

Swap rows 1 and 2

(changes sign by case 3)

## Case 3: Swapping 1st row and 2nd row.

$$\begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{31} & \cdots & \cdots & a_{3n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{vmatrix} = a_{21}\begin{vmatrix} a_{12} & \cdots & a_{1n} \\ a_{32} & \cdots & a_{3n} \\ \vdots & & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} - a_{22}\begin{vmatrix} a_{11} & a_{13} & \cdots & a_{1n} \\ a_{31} & a_{33} & \cdots & a_{3n} \\ \vdots & & & \vdots \\ a_{n1} & & \cdots & a_{nn} \end{vmatrix} + \cdots$$

$$= a_{21}a_{12}\begin{vmatrix} a_{33} & \cdots & a_{3n} \\ \vdots & & \vdots \\ a_{n3} & \cdots & a_{nn} \end{vmatrix} - a_{21}a_{13}\begin{vmatrix} a_{32} & a_{34} & \cdots & a_{3n} \\ \vdots & & & \vdots \\ a_{n2} & a_{n4} & \cdots & a_{nn} \end{vmatrix} + \cdots$$

A with first 2 rows & first 2 columns deleted.

$$- a_{22}a_{11}\begin{vmatrix} a_{33} & \cdots & a_{3n} \\ \vdots & & \vdots \\ a_{n3} & \cdots & a_{nn} \end{vmatrix} + a_{22}a_{13}\begin{vmatrix} & \\ & \end{vmatrix} - \cdots$$

$$+ \cdots$$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & a_{2n} \\ \vdots & & & \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & & \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} - \cdots = a_{11}a_{22}\begin{vmatrix} a_{33} & \cdots & a_{3n} \\ \vdots & & \vdots \\ a_{n3} & \cdots & a_{nn} \end{vmatrix} + \cdots$$

Hard part: show every sign changes!