Multiple bases at once.

Say that we have $T: \mathbb{R}^n \to \mathbb{R}^m$, with "standard" matrix M_T.

$\mathcal{B}$ - a basis for $\mathbb{R}^n$

$\bar{\mathcal{B}}$ - a basis for $\mathbb{R}^m$.

We want to find the matrix $M_{T,\mathcal{B}\to\bar{\mathcal{B}}}$ which takes the $\mathcal{B}$-coords for $P \in \mathbb{R}^n$ and outputs the $\bar{\mathcal{B}}$-coords for $T(P) \in \mathbb{R}^m$.

$$\underbrace{Q}_{(\mathcal{B}\text{-coords for } P \in \mathbb{R}^n)} \longrightarrow \underbrace{P = P_{\mathcal{B}} Q}_{(\text{point in } \mathbb{R}^n)} \longrightarrow \underbrace{M_T P_{\mathcal{B}} Q}_{(\text{point in } \mathbb{R}^m)} \longrightarrow C_{\bar{\mathcal{B}}} M_T P_{\mathcal{B}} Q$$

$$C_{\bar{\mathcal{B}}} M_T P_{\mathcal{B}} Q = \underbrace{P_{\bar{\mathcal{B}}}^{-1} M_T P_{\mathcal{B}} Q}_{(\bar{\mathcal{B}}\text{-coords for a point in } \mathbb{R}^m)},$$

$$\underbrace{M_{T,\mathcal{B}\to\bar{\mathcal{B}}}}_{m\times n} = \underbrace{P_{\bar{\mathcal{B}}}^{-1}}_{(m\times m)} \underbrace{M_T}_{(m\times n)} \underbrace{P_{\mathcal{B}}}_{(n\times n)}$$

(<u>exercise</u>: $M_{T,\mathcal{B}\to\bar{\mathcal{B}}}$ has the same rank as M_T.).

<u>ex</u>. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^3$

$\bar{\mathcal{B}} = \left\{ \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ for $\mathbb{R}^2$

$M_T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Then $M_{T,\mathcal{B}\to\bar{\mathcal{B}}} = P_{\bar{\mathcal{B}}}^{-1} M_T P_{\mathcal{B}}$.

$$P_{\bar{\mathcal{B}}}^{-1} = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}^{-1} = \frac{1}{2\cdot 2 - 5\cdot 1} \begin{bmatrix} 2 & -1 \\ -5 & 2 \end{bmatrix} = \frac{1}{(-1)} \begin{bmatrix} 2 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 5 & -2 \end{bmatrix}$$

So $$M_{T,\mathcal{B}\to\bar{\mathcal{B}}} = \begin{bmatrix} -2 & 1 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 9 & 19 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 5 \\ -3 & -3 & -3 \end{bmatrix}$$