Determinants

$\det : M(n,n) \longrightarrow \mathbb{R}$.

⚠ Not a linear map.
⚠ Only defined for square matrices.

What is it for?

① Test for invertibility: A is invertible $\iff \det(A) \neq 0$.
② Quickly compute inverses.
③ If A represents a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$, then $\det(A)$ measures how A scales volumes.

Def ① If $A = [a]$, then $\det(A) = a$.

② If $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$, then:

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) - \cdots + (-1)^{1+n} a_{1n} \det(A_{1n}).$$

where A_{ij} is the matrix obtained by removing the i^{th} row and j^{th} column of A.

A_{ij} are called minors of A, and $\pm\det(A_{ij})$ are called cofactors.

cofactor expansion along the first row of A.

$$\det \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

⚠ Not a matrix! A number!

ex. 2×2 matrices

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

Then $\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12})$

$A_{11} = \begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} \\ \cancel{a_{21}} & a_{22} \end{bmatrix} = [a_{22}]$. $A_{12} = \begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} \\ a_{21} & \cancel{a_{22}} \end{bmatrix} = [a_{21}]$

So $\det(A) = a_{11}a_{22} - a_{12}a_{21}$
$= \boxed{ad - bc.}$

$ad - bc \neq 0 \iff A$ is invertible.

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

ex. 3×3 matrices. $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & -1 \\ 3 & 5 & -5 \end{bmatrix}$.

Method 1: Cofactor expansion along the first row. (Same as in definition)

$\det(A) = 2\det(A_{11}) - 0\cdot\det(A_{12}) + 1\cdot\det(A_{13})$.

$A_{11} = \begin{bmatrix} \cancel{2} & \cancel{0} & \cancel{1} \\ \cancel{1} & 0 & -1 \\ \cancel{3} & 5 & -5 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 5 & -5 \end{bmatrix}$ $\begin{vmatrix} 0 & -1 \\ 5 & -5 \end{vmatrix} = 0\cdot(-5) - (-1)\cdot 5 = 5$.

$A_{13} = \begin{bmatrix} \cancel{2} & \cancel{0} & \cancel{1} \\ 1 & 0 & \cancel{-1} \\ 3 & 5 & \cancel{-5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 5 \end{bmatrix}$ $\begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 1\cdot 5 - 0\cdot 3 = 5$.

$\det(A) = 2\cdot 5 + 1\cdot 5 = 15$.

Remark: $A^{-1} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & -1 \\ 3 & 5 & -5 \end{bmatrix}^{-1} = \begin{bmatrix} 1/3 & 1/3 & 0 \\ 2/15 & -13/15 & 1/5 \\ 1/3 & -2/3 & 0 \end{bmatrix}$

Method 2: "Diagonal trick" ⚠ Only for 3×3!

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & -1 \\ 3 & 5 & -5 \end{bmatrix} \begin{matrix} 2 & 0 \\ 1 & 0 \\ 3 & 5 \end{matrix}$$

forward diagonals

backward diagonals.

$\det(A) = \sum$ products along forward diagonals
$\quad - \sum$ products along backward diagonals.

$= 2\cdot 0\cdot(-5) + 0\cdot(-1)\cdot 3 + 1\cdot 1\cdot 5$
$\quad - (1\cdot 0\cdot 3 + 2\cdot(-1)\cdot 5 + 0\cdot 1\cdot(-5))$
$= 15.$

Method 3: Cofactor expansion along second column.

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & -1 \\ 3 & 5 & -5 \end{bmatrix}$$

$-a_{21}\det(A_{21}) + a_{22}\det(A_{22}) - a_{32}\det(A_{32})$
$= -0\cdot\det(A_{21}) + 0\cdot\det(A_{22}) - 5\cdot\det(A_{32})$
$= -0 + 0 - 5\cdot\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$
$= -5\cdot(-3) = 15.$

Cofactor expansion thm. "Determinants can be computed using any row or column."

Let $1 \le i \le n$. Then $\det(A) = (-1)^{i+1}a_{i1}\det(A_{i1}) + (-1)^{i+2}a_{i2}\det(A_{i2}) + \cdots + (-1)^{i+n}a_{in}\det(A_{in})$.

Let $1 \le j \le n$. Then $\det(A) = (-1)^{1+j}a_{1j}\det(A_{1j}) + (-1)^{2+j}a_{2j}\det(A_{2j}) + \cdots + (-1)^{n+j}a_{nj}\det(A_{nj})$.

Signs: $\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$

ex. Lower triangular matrices.

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 5 & 3 & 3 & 0 \\ \pi & e & \sqrt{7} & 4 \end{vmatrix} = 1\begin{vmatrix} 2 & 0 & 0 \\ 3 & 3 & 0 \\ e & \sqrt{7} & 4 \end{vmatrix} = 1\cdot 2\cdot\begin{vmatrix} 3 & 0 \\ \sqrt{7} & 4 \end{vmatrix} = 1\cdot 2\cdot 3\cdot 4$$

More generally: the determinant of a lower triangular matrix is the product of the entries on the diagonal.
(Same is true for upper triangular matrices — why?)

Row linearity.

can be 0

Fix some $1 \le i \le n$ and: a $(i-1) \times n$ matrix A
$(n-i) \times n$ matrix B.

Think about matrices
$$\underbrace{\left[\begin{array}{c} A \\ \hline X \\ \hline B \end{array}\right]}_{n} \begin{array}{l} \}\, i-1 \\ \leftarrow i^{\text{th}} \text{ row} \\ \}\, n-i \end{array} \Bigg\}\, n$$

Row additivity: $\det\left(\left[\begin{array}{c} A \\ \hline X+Y \\ \hline B \end{array}\right]\right) = \det\left(\left[\begin{array}{c} A \\ \hline X \\ \hline B \end{array}\right]\right) + \det\left(\left[\begin{array}{c} A \\ \hline Y \\ \hline B \end{array}\right]\right).$

Row scalar: $\det\left(\left[\begin{array}{c} A \\ \hline cX \\ \hline B \end{array}\right]\right) = c \cdot \det\left(\left[\begin{array}{c} A \\ \hline X \\ \hline B \end{array}\right]\right)$

Zero row: $\det\left(\left[\begin{array}{c} A \\ \hline 0 \\ \hline B \end{array}\right]\right) = 0.$

All together, these say: the function $X \longmapsto \det\left(\left[\begin{array}{c} A \\ \hline X \\ \hline B \end{array}\right]\right)$

is a linear transformation $M(1,n) \longrightarrow \mathbb{R}$.

Slogan: "det is linear in each row separately"
"det is multilinear."

ex. Say that A is 3×3. What's $\det(2A)$?

$$\begin{vmatrix} 2a_{11} & 2a_{12} & \\ 2a_{21} & \ddots & \\ 2a_{31} & & \ddots \end{vmatrix} = 2\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} & \ddots & \\ 2a_{31} & & \ddots \end{vmatrix} = 4\begin{vmatrix} a_{11} & a_{12} & \\ a_{21} & \ddots & \\ 2a_{31} & & \ddots \end{vmatrix} = 8\begin{vmatrix} a_{11} & & \\ & \ddots & \\ & & \ddots \end{vmatrix}$$
$= 8\det(A).$

More generally: if A is $n\times n$, $\det(cA) = c^n \cdot \det(A)$.

ex. $\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{vmatrix} = 6$

$$\begin{array}{ccc} \| & \| & \| \\ 1\cdot 0\cdot 3 & 1\cdot 2\cdot 3 & 1\cdot 0\cdot 3 \\ \| & \| & \| \\ 0 & 6 & 0 \end{array}$$

Proofs by induction

Write (n) for "the statement is true for $n \times n$ matrices".
We'll prove (1) and (n-1) implies (n).
To get (4), ... you know (1) ... (1) $\Rightarrow$ (2), so you know (2) ... (2) $\Rightarrow$ (3), so you know (3), ...

Proof of row scalar property.

(1) $A = [a]$.

$\det([ca]) = ca = c \cdot \det([a])$.

Assume (n-1), i.e. it's true for $(n-1)\times(n-1)$ matrices.

Let M be an $n \times n$ matrix, $M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$

Case 1: Scaling first row of M,

$$\begin{vmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ a_{21} & \cdots & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{vmatrix} = ca_{11}\det(M_{11}) - ca_{12}\det(M_{12}) + \cdots + (-1)^{1+n} ca_{1n}\det(M_{1n})$$
$$= c\left(a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + \cdots + (-1)^{1+n} a_{1n}\det(M_{1n})\right)$$
$$= c\det(M).$$

Case 2: Scaling second row of M,

$$M' = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ ca_{21} & \cdots & ca_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}.$$

Then $\det(M') =$
$a_{11}\det(M'_{11}) - a_{12}\det(M'_{12}) + \cdots + (-1)^{1+n} a_{1n}\det(M'_{1n})$.

$M'_{11} = \begin{bmatrix} ca_{22} & \cdots & ca_{2n} \\ \vdots & & \vdots \\ a_{n2} & \cdots & a_{nn} \end{bmatrix} = M_{11}$ with its first row scaled by c.

By (n-1), $\det(M'_{11}) = c\det(M_{11})$.
Likewise, $\det(M'_{1j}) = c\det(M_{1j})$.

$$\det(M') = a_{11}\det(M'_{11}) - a_{12}\det(M'_{12}) + \cdots + (-1)^{1+n} a_{1n}\det(M'_{1n})$$
$$= a_{11}\cdot c\det(M_{11}) - a_{12}\cdot c\cdot\det(M_{12}) + \cdots + (-1)^{1+n} a_{1n}\cdot c\cdot\det(M_{1n})$$
$$= c\left[a_{11}\det(M_{11}) - \cdots + (-1)^{1+n} a_{1n}\det(M_{1n})\right]$$
$$= c\cdot\det(M).$$

Case 3: Scaling other rows — essentially the same as Case 2.

Proof of row additivity

(1) For 1×1 matrices: $\det([x]+[y]) = \det([x+y]) = x+y$
$= \det([x]) + \det([y])$.

Assume (n-1), i.e. we know row additivity for $(n-1)\times(n-1)$ matrices.

We want to prove row additivity for the i^{th} row of an $n\times n$ matrix.

Case 1: $i=1$.

$$\det\left(\begin{bmatrix} X+Y \\ \hline B \end{bmatrix}\right) = \begin{vmatrix} x_1+y_1 & x_2+y_2 & \cdots & x_n+y_n \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \vdots \\ b_{n1} & \cdots & & b_{nn} \end{vmatrix}$$

$$= (x_1+y_1)\det(B_{11}) - (x_2+y_2)\det(B_{12}) + \cdots + (-1)^{1+n}(x_n+y_n)\det(B_{1n}).$$
$$= x_1\det(B_{11}) - x_2\det(B_{12}) + \cdots + (-1)^{1+n} x_n\det(B_{1n})$$
$$+ y_1\det(B_{11}) - y_2\det(B_{12}) + \cdots + (-1)^{1+n} y_n\det(B_{1n}).$$
$$= \det\left(\begin{bmatrix} X \\ \hline B \end{bmatrix}\right) + \det\left(\begin{bmatrix} Y \\ \hline B \end{bmatrix}\right).$$

Case 2: $i=2$.

$$\begin{bmatrix} A \\ \hline X+Y \\ \hline B \end{bmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ x_1+y_1 & x_2+y_2 & \cdots & x_n+y_n \\ b_{31} & \cdots & \cdots & b_{3n} \\ \vdots & & & \vdots \\ b_{n1} & \cdots & \cdots & b_{nn} \end{vmatrix} = a_{11}\begin{vmatrix} x_2+y_2 & \cdots & x_n+y_n \\ b_{32} & & b_{3n} \\ \vdots & \ddots & \\ b_{n1} & & b_{nn} \end{vmatrix} - \cdots + (-1)^{1+n} a_{1n}\begin{vmatrix} x_1+y_1 & \cdots & x_{n-1}+y_{n-1} \\ b_{31} & \cdots & b_{3,n-1} \\ \vdots & & \\ b_{n1} & \cdots & b_{n,n-1} \end{vmatrix}$$

$$= a_{11}\left(\begin{vmatrix} x_2 & \cdots & x_n \\ b_{32} & \cdots & b_{3n} \\ & \ddots & \end{vmatrix} + \begin{vmatrix} y_2 & \cdots & y_n \\ b_{32} & & b_{3n} \\ & \ddots & \end{vmatrix}\right) + \cdots$$

$$= a_{11} \begin{vmatrix} x_2 & \cdots & x_n \\ b_{32} & \cdots & b_{3n} \\ & \ddots & \\ & & b_{nn} \end{vmatrix} - a_{12} \begin{vmatrix} x_1 & x_3 & \cdots & x_n \\ b_{31} & b_{33} & & b_{3n} \\ & & \ddots & \\ & & & b_{nn} \end{vmatrix} + \cdots \Bigg] = \det\left(\begin{bmatrix} A \\ \hline x \\ \hline B \end{bmatrix}\right)$$

$$+ a_{11} \begin{vmatrix} y_2 & \cdots & y_n \\ b_{32} & \cdots & b_{3n} \\ & \ddots & \\ & & b_{nn} \end{vmatrix} - a_{12} \begin{vmatrix} y_1 & y_3 & \cdots & y_n \\ b_{31} & b_{33} & \cdots & b_{3n} \\ & & \ddots & \\ & & & b_{nn} \end{vmatrix} + \cdots \Bigg] = \det\left(\begin{bmatrix} A \\ \hline y \\ \hline B \end{bmatrix}\right)$$

Other cases very similar to case 2. $\blacksquare$

Row interchange property

Interchanging two rows of a matrix changes the sign of the determinant.

i.e. $$\det\left(\begin{bmatrix} A \\ \hline X \\ \hline B \\ \hline Y \\ \hline C \end{bmatrix}\right) = -\det\left(\begin{bmatrix} A \\ \hline Y \\ \hline B \\ \hline X \\ \hline C \end{bmatrix}\right)$$

(if X and Y are rows)

ex. $$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -(ad-bc) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

ex. $$\begin{vmatrix} 0 & 0 & 3 \\ 1 & 5 & -1 \\ 0 & 2 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 5 & -1 \\ 0 & 0 & 3 \\ 0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

Equal rows property. If A has two equal rows, then $\det(A) = 0$.

Proof. Let B be the matrix obtained from A by swapping the two equal rows.
Then $\det(A) = -\det(B) = -\det(A) \Rightarrow \det(A) = 0$. ∎

Idea of proof for row interchange property

True for 2×2 matrices ✓

Assume we know it for $(n-1)\times(n-1)$ matrices.

Case 1: Swapping two rows, neither of which is the first row.

$$A = \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline A_3 \\ \hline \vdots \\ \hline A_n \end{bmatrix} \qquad A' = \begin{bmatrix} A_1 \\ \hline A_3 \\ \hline A_2 \\ \hline \vdots \\ \hline A_n \end{bmatrix}$$

(for example; other row swaps are similar)

$$\det(A') = a_{11}\det(A'_{11}) - a_{12}\det(A'_{12}) + \cdots + (-1)^{1+n} a_{1n}\det(A'_{1n}).$$

$$A'_{11} = \begin{bmatrix} a_{32} & a_{33} & \cdots & a_{3n} \\ a_{22} & a_{23} & \cdots & a_{2n} \\ a_{42} & a_{43} & \cdots & a_{4n} \\ \vdots & & & \vdots \\ a_{n2} & \cdots & & a_{nn} \end{bmatrix} \quad \text{so} \quad \det(A'_{11}) = -\det(A_{11}).$$

Likewise, $\det(A'_{1j}) = -\det(A_{1j})$, so $\det(A') = -\det(A)$.

Case 2: Swapping 1st row and ith row, $i > 2$.

This is equivalent to: Swap rows 1 and 2
Swap rows 2 and i (changes sign by case 1)
Swap rows 1 and 2
(changes sign by case 3)

Case 3: Swapping 1st row and 2nd row.

$$\begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{31} & \cdots & & a_{3n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{vmatrix} = a_{21}\begin{vmatrix} a_{12} & \cdots & a_{1n} \\ a_{32} & \cdots & a_{3n} \\ \vdots & & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} - a_{22}\begin{vmatrix} a_{11} & a_{13} & \cdots & a_{1n} \\ a_{31} & a_{33} & \cdots & a_{3n} \\ \vdots & & & \vdots \\ a_{n1} & & \cdots & a_{nn} \end{vmatrix} + \cdots$$

$$= a_{21}a_{12}\begin{vmatrix} a_{33} & \cdots & a_{3n} \\ \vdots & & \vdots \\ a_{n3} & \cdots & a_{nn} \end{vmatrix} - a_{21}a_{13}\begin{vmatrix} a_{32} & a_{34} & \cdots & a_{3n} \\ \vdots & & & \vdots \\ a_{n2} & a_{n4} & \cdots & a_{nn} \end{vmatrix} + \cdots$$

A with first 2 rows & first 2 columns deleted.

$$- a_{22}a_{11}\begin{vmatrix} a_{33} & \cdots & a_{3n} \\ \vdots & & \\ a_{n3} & \cdots & a_{nn} \end{vmatrix} + a_{22}a_{13}\begin{vmatrix} & & \end{vmatrix} - \cdots$$

$$+ \cdots$$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & & a_{2n} \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & & \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} - \cdots = a_{11}a_{22}\begin{vmatrix} a_{33} & \cdots & a_{3n} \\ \vdots & & \vdots \\ a_{n3} & \cdots & a_{nn} \end{vmatrix} \pm \cdots$$

Hard part: show every sign changes!

Determinants & row operations.

I : Interchange two rows $\Rightarrow$ determinant gets multiplied by (-1).

III : Multiply a row by c $\Rightarrow$ determinant gets multiplied by c.

II : Add a multiple of row i to row j $(i \neq j)$. $\Rightarrow$ determinant doesn't change.

Why?

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 4 & 5 \\ 1 & 6 & 11 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ \hline 1+0 & 4+2 & 5+6 \\ \hline 0 & 0 & 3 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 4 & 5 \\ 1 & 6 & 11 \\ 0 & 0 & 3 \end{vmatrix} = \underbrace{\begin{vmatrix} 1 & 4 & 5 \\ 1 & 4 & 5 \\ 0 & 0 & 3 \end{vmatrix}}_{= 0} + \underbrace{\begin{vmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{vmatrix}}_{= \det(A)} \quad \text{by row additivity,}$$

$$= \det(A).$$

$$A = \begin{bmatrix} B_1 \\ \hline R_j \\ \hline B_2 \end{bmatrix} \begin{matrix} \} \ j-1 \text{ rows} \\ \leftarrow j^{th} \text{ row} \\ \} \ n-j \text{ rows} \end{matrix}$$

$$\begin{vmatrix} B_1 \\ \hline cR_i + R_j \\ \hline B_2 \end{vmatrix} = \begin{vmatrix} B_1 \\ \hline cR_i \\ \hline B_2 \end{vmatrix} + \begin{vmatrix} B_1 \\ \hline R_j \\ \hline B_2 \end{vmatrix} \qquad \text{(row additivity)}$$

$$= c \begin{vmatrix} B_1 \\ \hline R_i \\ \hline B_2 \end{vmatrix} + \det(A) \qquad \text{(row scalar)}$$

$$= 0 + \det(A) \qquad \text{(equal rows)}$$

$$= \det(A).$$

Suppose that A is invertible.
Then $RREF(A) = I$.
We can get from A to $RREF(A)$ by applying row operations, and each row operation multiplies the determinant by a <u>nonzero number</u>.
So: $\det(I) = c \cdot \det(A)$, $c \neq 0$
$1 = c \cdot \det(A)$, $c \neq 0$.
Thus, $\det(A) \neq 0$.

Suppose that A is not invertible. (but still $n \times n$)
Then $RREF(A)$ has a zero row,
So $\det(RREF(A)) = 0$.
But $\det(RREF(A)) = c \cdot \det(A)$, for some $c \neq 0$.
So $\det(A) = 0$.

Conclusion: $\det(A) \neq 0 \iff A$ is invertible.

Uniqueness of the determinant.

Review properties of the determinant:

- $\Rightarrow$ Row interchange: If we swap two rows of a matrix, the determinant gets multiplied by (-1).
- $\Rightarrow$ Row scalar: If we multiply a row of a matrix by c, the determinant gets multiplied by c.
- $\Rightarrow$ Row additivity: $\det\left(\begin{bmatrix} A \\ \hline X+Y \\ \hline B \end{bmatrix}\right) = \det\left(\begin{bmatrix} A \\ \hline X \\ \hline B \end{bmatrix}\right) + \det\left(\begin{bmatrix} A \\ \hline Y \\ \hline B \end{bmatrix}\right)$.
- $\Rightarrow$ Row operations: If B is obtained from A by a row operation $cR_i + R_j \to R_j$ (where $i \neq j$), then $\det(B) = \det(A)$.
  - ↳ This follows from the first 3 properties.

Uniqueness theorem. Let $D : M(n,n) \to \mathbb{R}$ be a function such that:

1. $D(I) = 1$,
2. D satisfies the row interchange, row scalar, and row additivity properties.

Then: $D(A) = \det(A)$ for all $A \in M(n,n)$.

$D(I)=1$ row interchange

(Fancy words: det is the unique normalized alternating multilinear form on n-tuples of $1 \times n$ row vectors.)

refers to row additivity + scalar properties

Proof. If D satisfies the properties in ②, we know what row operations do to it.

I. Swap two rows $\Rightarrow$ change the sign of D

II. Add a multiple of a row to another row $\Rightarrow$ D doesn't change

III. Scale a row by c $\Rightarrow$ D gets multiplied by c.

Given A, put A in RREF.

If A is invertible: $RREF(A) = I$. $\Rightarrow D(RREF(A)) = \det(RREF(A)) \overset{"}{=} 1$.

If A isn't invertible: $RREF(A)$ has a zero row.

$$\Downarrow$$

$$D(RREF(A)) = \det(RREF(A)) = 0.$$

There's some sequence of row ops which turns $RREF(A)$ back into A.

These row operations change D in the same way as they do det. So $\det(A) = D(A)$. ☒

Consequences of uniqueness

Product theorem If A and B are $n \times n$ matrices,

$$\det(AB) = \det(A)\det(B).$$

Proof. Case 1: $\det(B) = 0$ Then $\operatorname{rank}(B) < n$.

So $\operatorname{rank}(AB) < n$. So $\det(AB) = 0$.

Case 2: $\det(B) \neq 0$

Define $D(A) = \dfrac{\det(AB)}{\det(B)}$. Want to show $D = \det$.

Check uniqueness thm:

$\Rightarrow$ $D(I) = \det(IB)/\det(B) = \det(B)/\det(B) = 1$.

$\Rightarrow$ Row interchange property:

$$\begin{bmatrix} - A_1 - \\ - A_2 - \\ \vdots \\ - A_n - \end{bmatrix} B = \begin{bmatrix} - A_1 B - \\ - A_2 B - \\ \vdots \\ - A_n B - \end{bmatrix}$$

The rows of AB are $A_1 B, \ldots, A_n B$.

So interchanging two rows of A $\Rightarrow$ interchanges the same two rows of AB $\Rightarrow$ changes the sign of $\det(AB)$ while leaving $\det(B)$ unchanged.

$\Rightarrow$ changes the sign of $D(A) = \det(AB)/\det(B)$.

→ Row scalar property: if we scale A_i by c, we also scale $(AB)_i = A_i B$ by c.
So we scale $\det(AB)$ by c, so we scale $D(AB)$ by c.

→ Row additivity property:

$$A = \begin{bmatrix} - & A_1 & - \\ & \vdots & \\ - & X+Y & - \\ & \vdots & \\ - & A_n & - \end{bmatrix} \quad i^{th} \text{ row}$$

Let $A_X = \begin{bmatrix} - & A_1 & - \\ & \vdots & \\ - & X & - \\ & \vdots & \\ - & A_n & - \end{bmatrix}$, $\quad A_Y = \begin{bmatrix} - & A_1 & - \\ & \vdots & \\ - & Y & - \\ & \vdots & \\ - & A_n & - \end{bmatrix}$

Then $AB = \begin{bmatrix} - & A_1 B & - \\ & \vdots & \\ - & (X+Y)B & - \\ & \vdots & \\ - & A_n B & - \end{bmatrix}$, $\quad A_X B = \begin{bmatrix} - & A_1 B & - \\ & \vdots & \\ - & XB & - \\ & \vdots & \\ - & A_n B & - \end{bmatrix}$, $\quad A_Y B = \begin{bmatrix} - & A_1 B & - \\ & \vdots & \\ - & YB & - \\ & \vdots & \\ - & A_n B & - \end{bmatrix}$

Since $(X+Y)B = XB + YB$,

$$\det(AB) = \det(A_X B) + \det(A_Y B)$$
$$D(A) = D(A_X) + D(A_Y).$$

divide by $\det(B)$.

■

Cor. $\det(AB) = \det(BA)$.

Last part of cofactor expansion thm:

$$\det(A) = (-1)^{1+j} a_{1j} \det(A_{1j}) + (-1)^{2+j} a_{2j} \det(A_{2j}) + \cdots + (-1)^{n+j} a_{nj} \det(A_{nj}).$$

Proof. Let $D(A)$ be the expression on the right-hand side.
(I'll just do $j=1$, so

$$D(A) = a_{11}\det(A_{11}) - a_{21}\det(A_{21}) + \cdots + (-1)^{n+1} a_{n1}\det(A_{n1}).$$
$$= \det(A^t)$$

→ $D(I) = \det(I^t) = \det(I) = 1$.

$$A = \begin{bmatrix} a_{11} & & \\ a_{21} & & \\ \vdots & & A_{11} \\ a_{n1} & & \end{bmatrix} \qquad A' = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \\ a_{n1} & & & \ddots \end{bmatrix}$$

→ Row interchange:

Let's say we swap rows 1 and 2, and get A',

$$D(A') = a'_{11}\det(A'_{11}) - a'_{21}\det(A'_{21}) + \cdots + (-1)^{n+1} a'_{n1}\det(A'_{n1}),$$

$$= a_{21}\det(A'_{11}) - a_{11}\det(A'_{21}) + a_{31}\det(A'_{31}) - \cdots + (-1)^{n+1} a_{n1}\det(A'_{n1}).$$

$$= a_{21}\det(A_{21}) - a_{11}\det(A_{11})$$
$$- a_{31}\det(A_{31}) + \cdots$$
$$+ (-1)^n a_{n1}\det(A_{n1}).$$
$$= -D(A).$$

A'_{31}: A_{31} with 1st 2 rows switched

A'_{n1}: A_{n1} with 1st 2 rows switched.

→ Row scalar:
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} \end{bmatrix} \to A' = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ a_{21} & & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} \end{bmatrix}$$

$$D(A) = a_{11}\det(A_{11}) - a_{21}\det(A_{21}) + \cdots + (-1)^{n+1} a_{n1}\det(A_{n1}),$$
$$D(A') = ca_{11}\det(A_{11}) - a_{21}\det(A'_{21}) + \cdots + (-1)^{n+1} a_{n1}\det(A'_{n1})$$
$$= ca_{11}\det(A_{11}) - ca_{21}\det(A_{21}) + \cdots + (-1)^{n+1} c\, a_{n1}\det(A_{n1})$$
$$= c\, D(A).$$

→ Row additivity:
$$A_x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & & & \ddots \end{bmatrix} \qquad A_y = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ a_{21} & & & \\ \vdots & & \ddots & \\ a_{n1} & & & \ddots \end{bmatrix}$$

$$A = \begin{bmatrix} x_1+y_1 & x_2+y_2 & \cdots & x_n+y_n \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & & & \ddots \end{bmatrix}$$

$$D(A) = (x_1+y_1)\det(A_{11}) - a_{21}\det(A_{21}) + \cdots + (-1)^{n+1} a_{n1}\det(A_{n1}).$$
$$A_{11} = (A_x)_{11} = (A_y)_{11}$$

For $i > 1$, $\det(A_{i1}) = \det((A_x)_{i1}) + \det((A_y)_{i1})$

So
$$D(A) = x_1\det((A_x)_{11}) + y_1\det((A_y)_{11})$$
$$- a_{21}\det((A_x)_{21}) - a_{21}\det((A_y)_{21}) + \cdots$$
$$= D(A_x) + D(A_y).$$

Conclusion: $\det(A)$ can be defined by cofactor expansion along columns.

Also: $\det(A) = \det(A^t)$

det behaves well with respect to column operations. It satisfies column interchange, column scalar, and column additivity properties.

Cramer $\rightarrow$

Cramer's Rule.

The rule: Consider a system $A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = Y$ where A is $n\times n$, $\det(A) \neq 0$,

Then $$x_j = \frac{\det\left(\begin{bmatrix} | & & | & y_1 & | & & | \\ A_1 & \cdots & A_{j-1} & \vdots & A_{j+1} & \cdots & A_n \\ | & & | & y_n & | & & | \end{bmatrix}\right)}{\det(A)} = \frac{\det([A_1, \cdots, A_{j-1}, Y, \cdots, A_n])}{\det(A)}$$

example: $$\underbrace{\begin{bmatrix} 2 & 2 & 3 \\ -1 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} \underbrace{=}_{=} \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{Y}.$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & 2 & 3 \\ -1 & 0 & 0 \\ 0 & 5 & 1 \end{vmatrix} = (-1)(-1)\begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} = -13.$$

$$\det(Y, A_2, A_3) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \\ 3 & 5 & 1 \end{vmatrix} = (-1)2\begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} = 26.$$

$$\det(A_1, Y, A_3) = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \\ 0 & 3 & 1 \end{vmatrix} = (-1)(-1)\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix}$$
$$= -8 + 4 = -4$$

$$\det(A_1, A_2, Y) = \begin{vmatrix} 2 & 2 & 1 \\ -1 & 0 & 2 \\ 0 & 5 & 3 \end{vmatrix} = (-1)(-1)\begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 0 & 5 \end{vmatrix}$$
$$= 1 - 20 = -19$$

So $x_1 = \frac{26}{-13} = -2$

$x_2 = -4/-13 = 4/13$

$x_3 = -19/-13 = 19/13.$

Check: $$\begin{bmatrix} 2 & 2 & 3 \\ -1 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 4/13 \\ 19/13 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

example:
$$\begin{bmatrix} 2 & 2 & 3 \\ -1 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$
$$A \quad X = Y$$

$\det(A) = -13.$

$$\det([Y, A_2, A_3]) = \begin{vmatrix} y_1 & 2 & 3 \\ y_2 & 0 & 0 \\ y_3 & 5 & 1 \end{vmatrix} = -y_2 \begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} = 13y_2.$$

$$\det([A_1, Y, A_3]) = \begin{vmatrix} 2 & y_1 & 3 \\ -1 & y_2 & 0 \\ 0 & y_3 & 1 \end{vmatrix} = (-1)(-1)\begin{vmatrix} y_1 & 3 \\ y_3 & 1 \end{vmatrix} + y_2 \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix}$$
$$= y_1 - 3y_3 + 2y_2$$
$$= y_1 + 2y_2 - 3y_3$$

$$\det([A_1, A_2, Y]) = \begin{vmatrix} 2 & 2 & y_1 \\ -1 & 0 & y_2 \\ 0 & 5 & y_3 \end{vmatrix} = (-1)(-1)\begin{vmatrix} 2 & y_1 \\ 5 & y_3 \end{vmatrix} - y_2 \begin{vmatrix} 2 & 2 \\ 0 & 5 \end{vmatrix}$$
$$= 2y_3 - 5y_1 - 10y_2$$
$$= -5y_1 - 10y_2 + 2y_3.$$

So $$\begin{bmatrix} 2 & 2 & 3 \\ -1 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -\frac{1}{13} \begin{bmatrix} 13y_2 \\ y_1 + 2y_2 - 3y_3 \\ -5y_1 - 10y_2 + 2y_3 \end{bmatrix}$$

$$A^{-1} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = -\frac{1}{13} \begin{bmatrix} 13y_2 \\ y_1 + 2y_2 - 3y_3 \\ -5y_1 - 10y_2 + 2y_3 \end{bmatrix}$$

So, $$A^{-1} = -\frac{1}{13} \begin{bmatrix} 0 & 13 & 0 \\ 1 & 2 & -3 \\ -5 & -10 & 2 \end{bmatrix}$$

Formula for A^{-1}, in general.

The j^{th} column of A^{-1} is $A^{-1} I_j$.
In other words, it's the solution to
$$AX = I_j.$$
Using Cramer's rule, the j^{th} column of A^{-1} is
$$\frac{\det([A_1, \dots, A_{j-1}, I_j, A_{j+1}, \dots, A_n])}{\det(A)}.$$

Proof of Cramer's rule (A = fixed $n \times n$ matrix)

Consider the following $T: \mathbb{R}^n \to \mathbb{R}^n$.
$T(X)$ has j^{th} coordinate $\dfrac{\det([A_1, \dots, A_{j-1}, X, A_{j+1}, \dots, A_n])}{\det(A)}$.

T is a linear transformation
(because det is linear in each column separately!)

$T(A_j)$ has j^{th} coordinate $\dfrac{\det([A_1, \dots, A_{j-1}, A_j, A_{j+1}, \dots, A_n])}{\det(A)}$
$$= 1$$
$T(A_j)$ has k^{th} coordinate $(k \neq j)$ $\dfrac{\det([A_1, \dots, A_{k-1}, A_j, A_{k+1}, \dots, A_n])}{\det(A)}$
$$= 0.$$
So $T(A_j) = I_j$.

If B = the matrix of T, then $BA_j = I_j$.
So $BA = I$
So $B = A^{-1}$. ∎

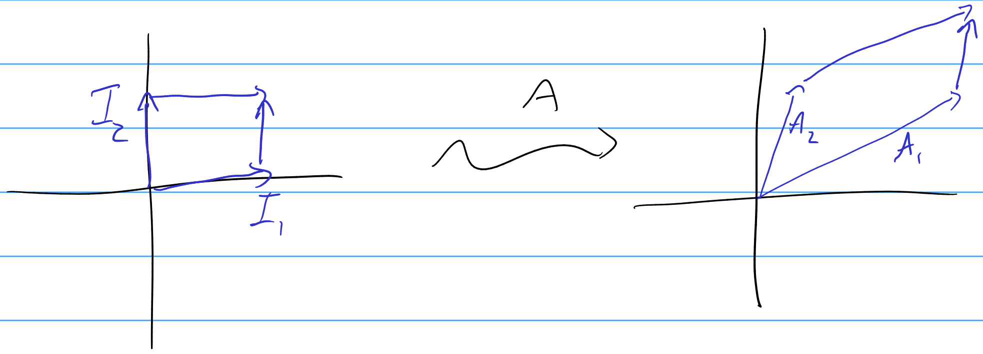
Why do we care?

This is slower than computing A^{-1} via row reduction.

1. Sometimes nice to have explicit formulas (especially for small-dimensional inverses).
2. Given $AX = Y$, you can just solve for X_1.

Determinant and volume

$A \in M(n,n) \Longleftrightarrow T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n.$

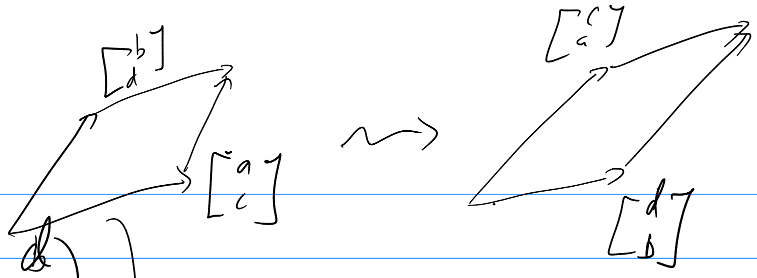


Main idea. If $n=2$, $|\det A| =$ area of the parallelogram generated by A_1 and A_2.
More generally, if $S \subseteq \mathbb{R}^2$ has area x, then $T_A(S)$ has area $x \cdot |\det A|$.
If $n=3$, $|\det A| =$ volume of the parallelepiped generated by A_1, A_2, and A_3.
If $S \subseteq \mathbb{R}^3$ has volume x, then $T_A(S)$ has volume $x \cdot |\det A|$.
Likewise in $\mathbb{R}^4$, $\mathbb{R}^5$, ….

Proof. Slight variant of uniqueness thm.
$A \mapsto |\det(A)|$ is the unique function D such that:
→ $D(I) = 1$.
→ Interchanging two rows of A doesn't change $D(A)$,
→ Scaling a row of A by c scales $D(A)$ by $|c|$.
→ Doing a row operation $cR_i + R_j \longrightarrow R_j$, $i \neq j$, doesn't change $D(A)$.

in $\mathbb{R}^2$ Now define $D(A) =$ area of parallelogram generated by A_1 and A_2.
→ $D(I) =$ area of unit square $= 1$.

→ Interchange rows:

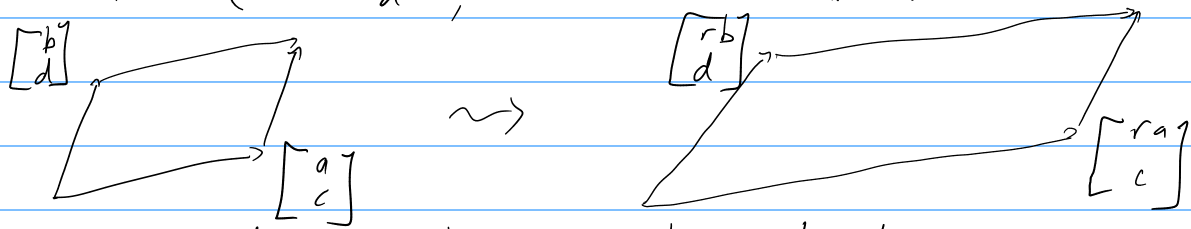


$\left(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow \begin{pmatrix} c & d \\ a & b \end{pmatrix}\right)$

This is just reflection across $y=x$, which doesn't change area.

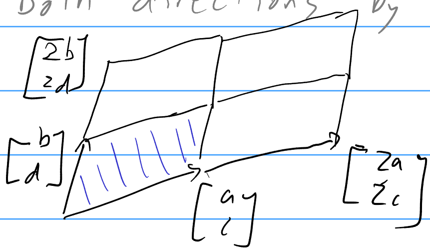
→ Scale row:

$\left(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow \begin{pmatrix} ra & rb \\ c & d \end{pmatrix}, \text{ for some } r \in \mathbb{R}.\right)$

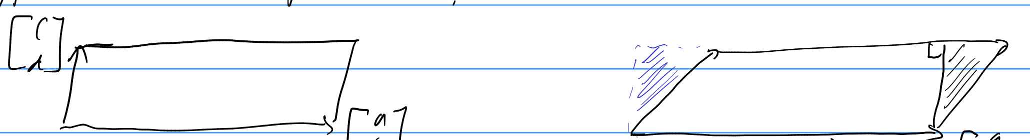


Scaling in one direction by r also scales the area by r.

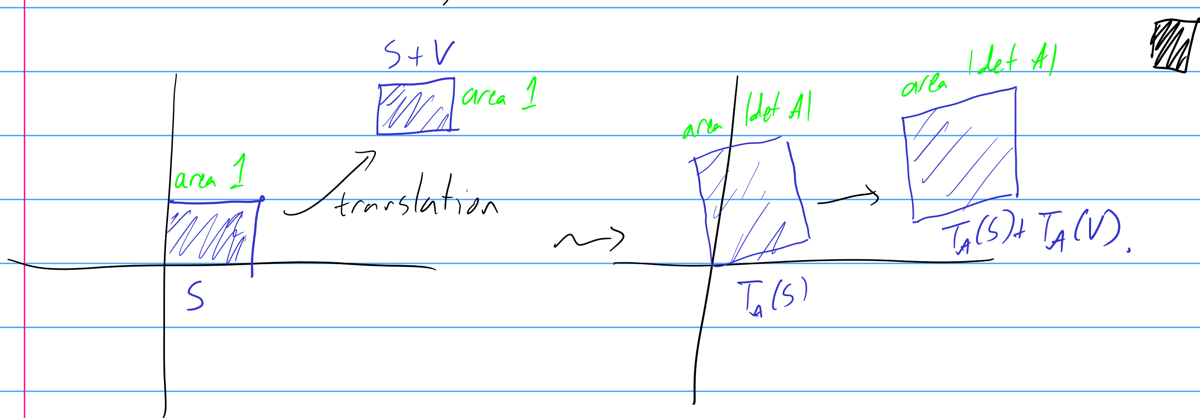
// Scaling in both directions by r scales the area by r^2.

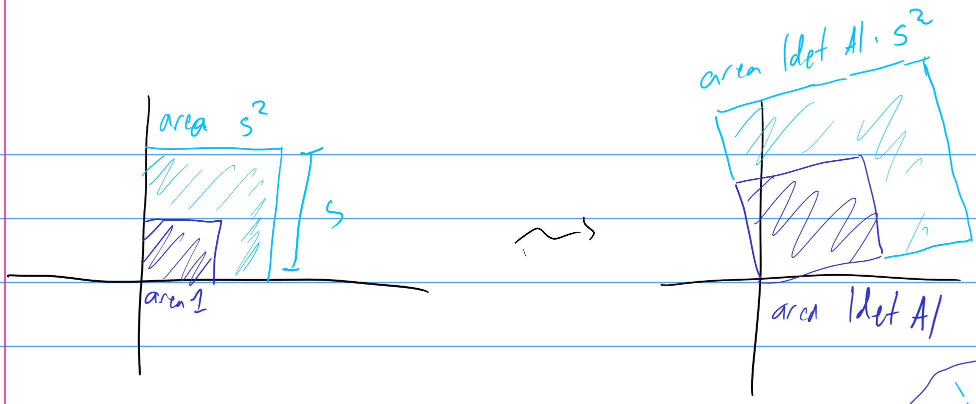


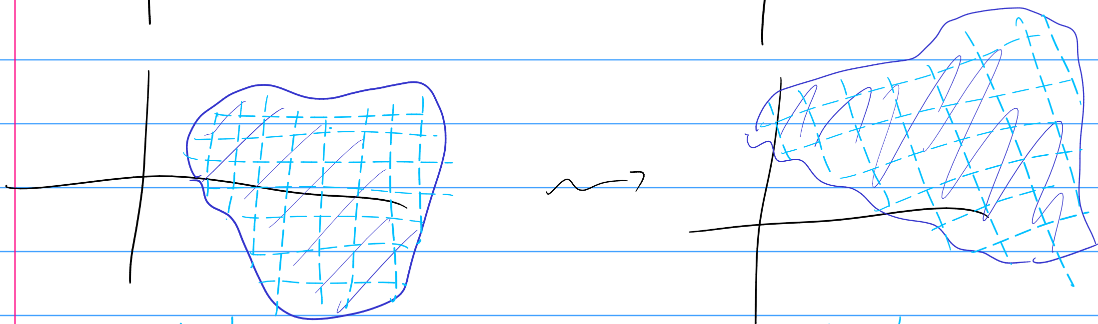
→ Type III row operations.



$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow \begin{pmatrix} a & b \\ c+ra & d+rb \end{pmatrix}$







covered by N squares of area ε.

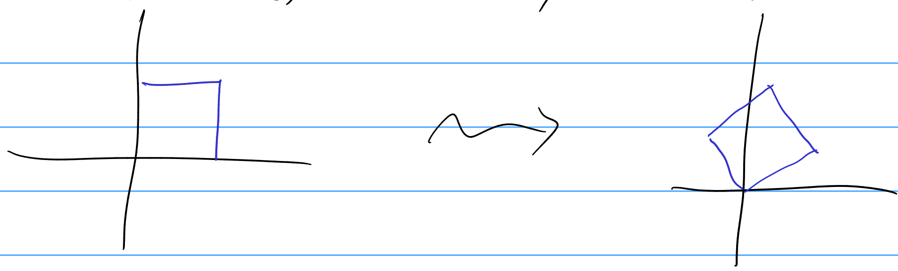
covered by N parallelograms of size $|\det A| \cdot \varepsilon$.

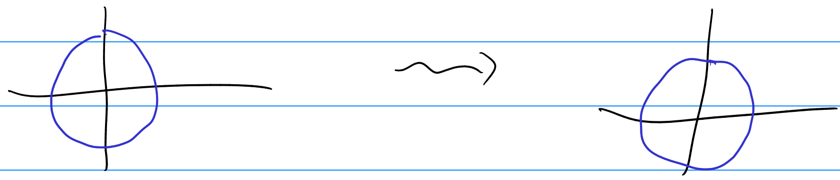
Conclusion: T_A scales all areas by $|\det A|$.

ex. Rotation matrices: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = A$.

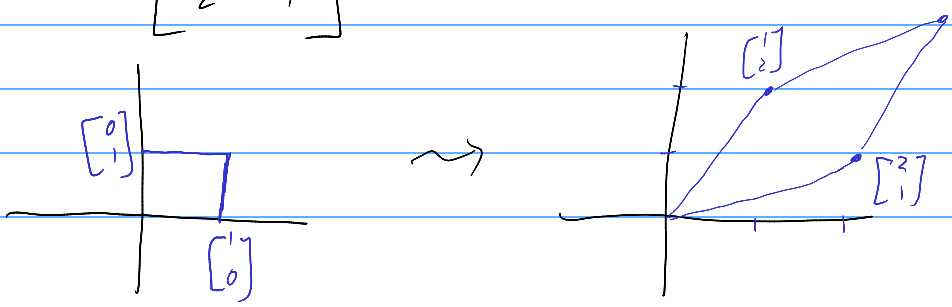
$\det A = \cos^2\theta + \sin^2\theta = 1.$

So A doesn't change areas.

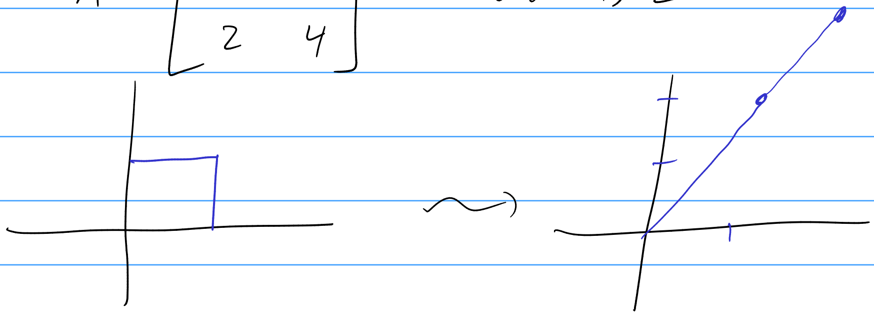




ex. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $\quad \det A = 1 - 4 = -3.$



ex. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $\quad \det(A) = 4 - 4 = 0.$



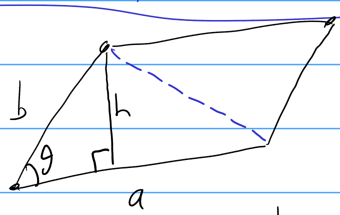
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has length 1 $\rightsquigarrow$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ has length $\sqrt{1^2+2^2} = \sqrt{5}$.

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has length 1 $\rightsquigarrow$ $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ has length $\sqrt{2^2+4^2} = 2\sqrt{5}$.

$\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ has length $\sqrt{5}$ $\rightsquigarrow$ $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has length 0.

Conclusion: Nothing nice can be said about what a noninvertible matrix does to lengths.

Classical way of thinking about parallelograms.

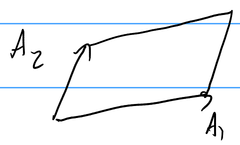


area of $\triangle = \frac{1}{2} a h$

area of $\square = a h$.

$h = b \sin\theta$, so area of $\square = ab \sin\theta$.

Cross product in $\mathbb{R}^3$: $V \times W$ = the vector $\perp$ to V and W with length $= |V| \cdot |W| \cdot \sin\theta$.



area $= |A_1 \times A_2|$.

Formula for the cross product:

$$V \times W = \begin{vmatrix} I_1 & I_2 & I_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{vmatrix}$$

← row of standard basis vectors

← rows of numbers

$$= [V_2 W_3 - W_2 V_3,\ V_3 W_1 - W_3 V_1,\ V_1 W_2 - W_1 V_2]^t.$$

ex. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $\quad$ area of $T_A(\square) = |A_1 \times A_2| = \left| \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right|$

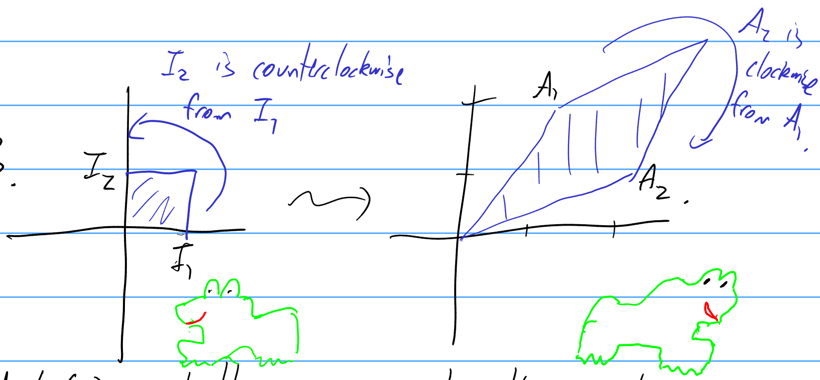
$$= \text{length} \begin{vmatrix} I_1 & I_2 & I_3 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \end{vmatrix} = \text{length}\left(\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} I_3 \right) = \text{abs}\left(\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \right).$$

In $\mathbb{R}^3$:

volume of a parallelepiped with sides u, v, and w, is $|u \cdot (v \times w)|$ $= \left| \begin{matrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{matrix} \right|$.

Orientation.

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\det(A) = -3$.



Conclusion: The sign of $\det(A)$ tells you whether it preserves or reverses orientation.