Midterm 2 MA 35100 – solutions

Q3 Find the PLU factorization of the following matrix:

$$\begin{pmatrix} 2 & -1 & 3 & 1 \\ -2 & 2 & -1 & -1 \\ 6 & -3 & 8 & 4 \end{pmatrix}$$

Solution: Reduce the matrix A to REF, using only row operations of the form

$$cR_i + R_j \to R_j, \quad i < j.$$

In this case, we can use the row operations $R_1 + R_2 \to R_2$ and $-3R_1 + R_3 \to R_3$ to get

$$\begin{pmatrix} 2 & -1 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} = U.$$

L is obtained from the negatives of the coefficients in the row operations:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

Since we didn't rearrange the rows at all, P is the identity matrix, and the PLU factorization is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

Note: I took a point off for not giving P. Also, this isn't the only solution: you could apply a nontrivial permutation of the rows and then do an LU factorization of the result, giving a different PLU factorization. These answers were also fine.

Q4 Let $\mathcal{V}$ be the vector space of functions spanned by e^x and e^{-x}. Let $D : \mathcal{V} \to \mathcal{V}$ be the linear transformation that sends $f(x)$ to its derivative $f'(x)$. Find a basis for $\mathcal{V}$, such that the matrix for D with respect to this basis is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Solution: I'll give the *most straightforward* way to solve this – there are more complicated ways using point & coordinate matrices, but no need to work harder than you have to. Let's say that the basis we're looking for is $\{ae^x+be^{-x}, ce^x+de^{-x}\}$. The operator D is supposed to switch the two elements of the basis, so

$$\begin{aligned} D(ae^x + be^{-x}) &= ce^x + de^{-x}, \\ D(ce^x + de^{-x}) &= ae^x + be^{-x}. \end{aligned}$$

However,

$$\begin{aligned} D(ae^x + be^{-x}) &= ae^x - be^{-x}, \\ D(ce^x + de^{-x}) &= ce^x - de^{-x}. \end{aligned}$$

Comparing the two sets of equations, we see that $a = c$ and $b = -d$. So one solution is

$$\mathcal{B} = \left\{ \frac{e^x + e^{-x}}{2}, \frac{e^x - e^{-x}}{2} \right\}.$$

Other correct answers could be obtained by multiplying both basis vectors in my $\mathcal{B}$ by the same scalar.

Note: The elements of $\mathcal{B}$ are the **hyperbolic trig functions**

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

We just proved that

$$\frac{d}{dx} \cosh(x) = \sinh(x), \quad \frac{d}{dx} \sinh(x) = \cosh(x).$$

These functions also satisfy "twisted" versions of many other familiar trig identities, which I encourage you to explore for yourself!

Q5 Let T be the linear transformation such that $T\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$ and $T\begin{pmatrix}-2\\3\end{pmatrix} = \begin{pmatrix}4\\6\end{pmatrix}$. What is $T\begin{pmatrix}2\\0\end{pmatrix}$?

(a) $\begin{pmatrix}2\\2\end{pmatrix}$

(b) $\begin{pmatrix}2\\4\end{pmatrix}$

(c) $\begin{pmatrix}2\\6\end{pmatrix}$

(d) $\begin{pmatrix}-1/2\\1/2\end{pmatrix}$

(e) $\begin{pmatrix}2\\0\end{pmatrix}$

Solution: (c). Since the vectors $\begin{pmatrix}-1\\1\end{pmatrix}$ and $\begin{pmatrix}-2\\3\end{pmatrix}$ are a basis for $\mathbb{R}^2$, it's possible to explicitly figure out what T is. Indeed, if the matrix of T is $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$, then

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}-a+b\\-c+d\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix},$$

and

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}-2\\3\end{pmatrix} = \begin{pmatrix}-2a+3b\\-2c+3d\end{pmatrix} = \begin{pmatrix}4\\6\end{pmatrix}.$$

Solving these systems of equations shows that

$$\begin{pmatrix}a & b\\c & d\end{pmatrix} = \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}.$$

Thus,

$$T\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}2\\6\end{pmatrix}.$$

Q6 Which one of the following statements is true? In all statements, A and B are $n \times n$ matrices.

(a) For all A and B, $(A+B)(A-B) = A^2 - B^2$.

(b) For all A, there exists B such that $AB = BA$.

(c) If A is invertible, then $AB = BA^{-1}$.

(d) If A and B are invertible, then $(AB)^{-1} = A^{-1}B^{-1}$.

(e) There exist A and B such that $AB = BA$ but $A^2B \neq BA^2$.

Solution: (b). For example, if B is the identity matrix, then $AB = BA = A$. Or if $B = A$, then $AB = BA = A^2$.

(a) is true if and only if $AB = BA$ (so not for all A and B). There's no reason to believe (c). For (d), what actually holds is $(AB)^{-1} = B^{-1}A^{-1}$, which is only equal to $A^{-1}B^{-1}$ if $AB = BA$. For (e), note that if $AB = BA$ then

$$A^2B = A(AB) = A(BA) = (AB)A = (BA)A = BA^2.$$

Q7 Which one of the following statements is false? All the matrices in the statements are $n \times n$.

(a) The inverse of a permutation matrix is a permutation matrix.

(b) The inverse of a lower triangular matrix is a lower triangular matrix.

(c) The inverse of an upper triangular matrix is an upper triangular matrix.

(d) The inverse of the identity matrix is the identity matrix.

(e) The inverse of the zero matrix is the zero matrix.

Solution: (e). The zero matrix is not invertible! For a matrix A to be invertible, there needs to be a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. But the zero matrix times any other matrix gives zero – never the identity matrix.

Q8 Consider the following sequence of row operations: Add row 1 to row 2, then swap row 2 and row 3.

Which of the following matrices encodes this sequence of row operations (in the sense that left multiplication by that matrix is the same as performing the sequence of row operations)?

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$

Solution: (a). Adding row 1 to row 2 is encoded in the lower triangular matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Swapping row 1 and row 3 is encoded in the permutation matrix

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

To do the whole sequence of row operations, we multiply the matrices:

$$BA = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The order matters here. We calculate BA and not AB because we're going to be *left* multiplying other matrices by this matrix, and we want A to do its thing *first*. To check that the answer's right, multiply BA by any other $3 \times n$ matrix, and observe that it does the right thing.

Q9 Let T be the linear transformation given by reflecting across the line $y = x$, then reflecting across the line $y = 0$. What is the matrix for T?

(a) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(e) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Solution: **(b)** The matrix for reflecting across the line $y = x$ is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Multiplying A by any vector switches its x and y coordinate. The matrix for reflecting across $y = 0$ is

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Multiplying B by a vector keeps its x coordinate the same, and multiplies its y coordinate by -1. To compose the transformations, we multiply the matrices.

$$BA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We choose this order because we want A to get multiplied by the vectors first.

Remark: You may recognize the answer as a rotation matrix. Is there a general rule for what happens when you reflect across two different lines in succession?

Q10 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an invertible linear transformation. Which of the following is not always true?

(a) T sends lines to lines.

(b) T sends ellipses to ellipses.

(c) T sends rectangles to rectangles.

(d) T sends triangles to triangles.

(e) T sends parallelograms to parallelograms.

Solution: (c) For example, the shearing transformation given by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ sends the unit square to a parallelogram, which is not a rectangle.

If we weren't necessarily talking about an *invertible* linear transformation, it would be possible for a linear transformation to send a line to a point, or some sort of 2-dimensional shape to a line.

Q11 Which of the following vector spaces is not isomorphic to $\mathbb{R}^6$?

(a) The vector space $M(2, 3)$ of 2×3 matrices.

(b) The vector space $\mathbb{C}^3$ of ordered triples (z_1, z_2, z_3) of complex numbers.

(c) The vector space $\mathcal{P}_5$ of polynomials of degree at most 5.

(d) The nullspace of a 7×3 matrix of rank 1.

(e) The vector space $\mathcal{P}_6$ of polynomials of degree at most 6.

Solution: (d) or (e) Recall that two vector spaces are isomorphic if and only if there's an invertible linear transformation from one to the other. If both vector spaces are finite-dimensional, this is true if and only if they have the same dimension. So the question is really asking, which vector space doesn't have dimension 6? In other words, which one doesn't have a basis of size 6?

The correct answers are (e), which has dimension 7 (take the basis $\{1, x, x^2, x^3, x^4, x^5, x^6\}$), and (d), which has dimension 2 by the rank-nullity theorem. (I meant to write

3×7 here, which would've made this vector space dimension 6! Oh well!) The others all have dimension 6. For (a), we can pick the basis

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For (b), we can pick the basis

$$(1,0,0), \quad (i,0,0), \quad (0,1,0), \ldots, (0,0,i).$$

(In other words, $\mathbb{C}$ is a two-dimensional vector space, so three distinct complex coordinates give six dimensions.) For (c), we can use the basis $\{1, x, \ldots, x^5\}$.

Note about (b): As a *complex* vector space – meaning we allow ourselves to think about multiplication by *complex* scalars – this has dimension 3. However, real vector spaces are all we've talked about or are going to talk about in this class.

Q12 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix $\begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}$. Which of the following is the matrix of T in terms of the ordered basis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$?

(a) $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} -5 & 5 \\ -5 & 6 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 5 \\ 1 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 5 & 10 \\ 8 & 15 \end{pmatrix}$

(e) $\begin{pmatrix} 0 & 1 \\ 5 & 1 \end{pmatrix}$

Solution: (c). The point matrix for the basis $\mathcal{B}$ is

$$P_{\mathcal{B}} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

The coordinate matrix is

$$C_{\mathcal{B}} = P_{\mathcal{B}}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

To get the matrix of T in terms of $\mathcal{B}$, we need to change $\mathcal{B}$-coordinates to a point (using $P_{\mathcal{B}}$), apply T, and change the result into $\mathcal{B}$-coordinates (using $C_{\mathcal{B}}$). So the answer is

$$C_{\mathcal{B}} T P_{\mathcal{B}} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 1 & 0 \end{pmatrix}.$$