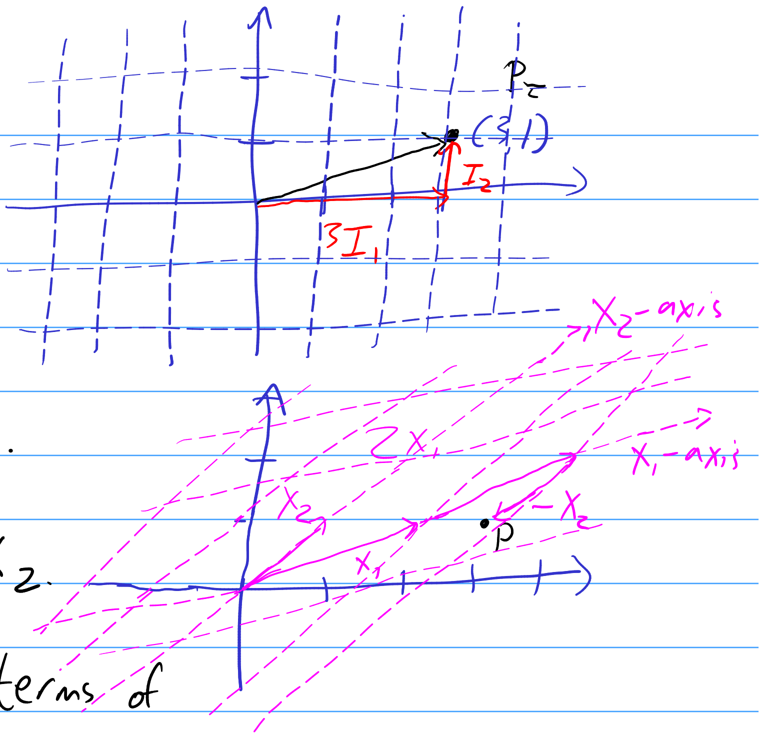
Bases & coordinates



The coordinates (3, 1) mean that $P = 3I_1 + I_2$.

ex. $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.
$\quad\quad X_1 \quad\quad X_2$

solve system $a_1X_1 + a_2X_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Then $\begin{bmatrix} 3 \\ 1 \end{bmatrix} = 2X_1 - X_2$.

The coordinates of P in terms of $\mathcal{B}$ are (2, -1).

Can think about different bases as giving:

- different axes / grids
- different languages
- different systems of units.

ex. There's a 1-dimensional vector space of lengths

One basis is $\mathcal{B}_1 = \{1 \text{ inch}\}$

Another basis is $\mathcal{B}_2 = \{1 \text{ cm}\}$.

If X is the vector "2 inches",
in terms of $\mathcal{B}_2$, X = "5.08 cm".

How to translate between coordinate systems?

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = 2X_1 - X_2 = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

$$\underbrace{\begin{bmatrix}3\\1\end{bmatrix}}_{\text{point}} = \underbrace{\begin{bmatrix}| & |\\ X_1 & X_2\\ | & |\end{bmatrix}}_{\substack{\text{matrix whose}\\ \text{columns are } \mathcal{B}.\\ = \text{"point matrix" for } \mathcal{B}. = P_{\mathcal{B}}}} \underbrace{\begin{bmatrix}2\\-1\end{bmatrix}}_{\substack{\text{coordinates}\\ \text{in terms of } \mathcal{B}}}$$

The point matrix has maximal rank, so it's invertible.

Say $C_{\mathcal{B}} = P_{\mathcal{B}}^{-1}$, the "coordinate matrix".

$$\underbrace{\begin{bmatrix}2\\-1\end{bmatrix}}_{\substack{\text{coordinates}\\ \text{in terms of } \mathcal{B}}} = \underbrace{C_{\mathcal{B}}}_{\substack{\text{coord.}\\ \text{matrix}}} \underbrace{\begin{bmatrix}3\\1\end{bmatrix}}_{\text{point}}.$$

In this case $C_{\mathcal{B}} = \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix}1 & -1\\-1 & 2\end{bmatrix}$

$$= \frac{1}{2-1}\begin{bmatrix}1 & -1\\-1 & 2\end{bmatrix} = \begin{bmatrix}1 & -1\\-1 & 2\end{bmatrix}.$$

So $\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}1 & -1\\-1 & 2\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}.$

<u>ex</u>. Say $\mathcal{B} = \left\{ \overset{Y_1}{\begin{bmatrix}1\\1\end{bmatrix}}, \overset{Y_2}{\begin{bmatrix}2\\-3\end{bmatrix}} \right\}$, $P = \begin{bmatrix}0\\2\end{bmatrix}$.

The point matrix is $P_{\mathcal{B}} = \begin{bmatrix}1 & 2\\1 & -3\end{bmatrix}$.

The coordinate matrix is $C_{\mathcal{B}} = P_{\mathcal{B}}^{-1}$.

$$\left[\begin{array}{cc|cc}1 & 2 & 1 & 0\\1 & -3 & 0 & 1\end{array}\right] \to \left[\begin{array}{cc|cc}1 & 2 & 1 & 0\\0 & -5 & -1 & 1\end{array}\right] \to \left[\begin{array}{cc|cc}1 & 2 & 1 & 0\\0 & 1 & 1/5 & -1/5\end{array}\right]$$

$$\qquad P_{\mathcal{B}} \qquad\quad I$$

$$\to \left[\begin{array}{cc|cc}1 & 0 & 3/5 & 2/5\\0 & 1 & 1/5 & -1/5\end{array}\right]$$

$$\qquad I \qquad\quad C_{\mathcal{B}}$$

So the coordinates of P in terms of $\mathcal{B}$ are

$$C_{\mathcal{B}} \cdot P = \begin{bmatrix} 3/5 & 2/5 \\ 1/5 & -1/5 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4/5 \\ -2/5 \end{bmatrix}.$$

So $P = \frac{4}{5} \underset{\sim}{v}_1 - \frac{2}{5} \underset{\sim}{v}_2$.

Note that: - if $\mathcal{B}$ doesn't span the vector space, not every point will have coordinates in terms of $\mathcal{B}$.

- if B isn't linearly independent, a given point may have more than one set of coordinates in terms of $\mathcal{B}$.
- if $\mathcal{B}$ is a basis, coordinates in terms of $\mathcal{B}$ exist + are unique for every point.

Also, we need to order the elements of $\mathcal{B}$.

The coordinates of P in terms of $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2\}$ are $(4/5, -2/5)$.

The coordinates of P in terms of $\{\underset{\sim}{v}_2, \underset{\sim}{v}_1\}$ are $(-2/5, 4/5)$.