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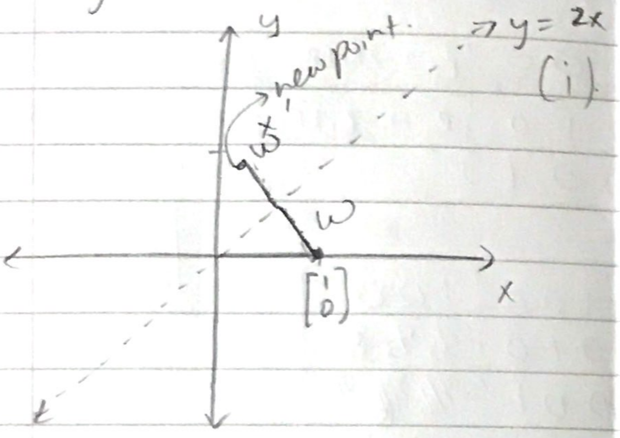
MA 351 Homework 8

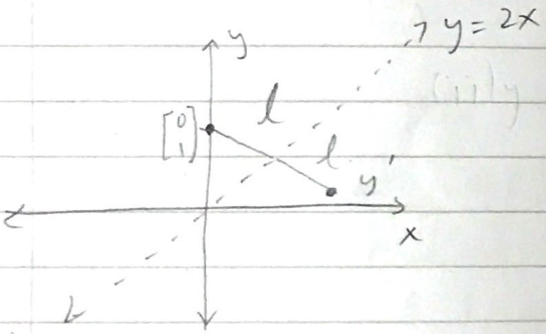
I Professor Vancoughnett's Quiz Question.

Q] Find the matrix for the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by the reflection across the line $y = 2x$



- First, we are looking for a 2×2 transformation matrix because $T: \mathbb{R}^2 \to \mathbb{R}^2$.

- To find the matrix, we need to use the standard basis vectors $[1, 0]$ and $[0, 1]$.



(i) $y = -\frac{1}{2}x + \frac{1}{2}$; $(0) = -\frac{1}{2}(1) + b$

$b = \frac{1}{2}$

intersection between $\begin{bmatrix}1\\0\end{bmatrix}$ & $y = 2x$ is:

$-\frac{1}{2}x + \frac{1}{2} = 2x$ $\quad \boxed{\left(\frac{1}{5}, \frac{2}{5}\right)}$

$\frac{1}{2} = \frac{5}{2}x$

$x = \frac{1}{5}$ $\quad y = 2x = \frac{2}{5}$

$w = \begin{bmatrix}-4/5\\2/5\end{bmatrix}$

(ii) $y = -\frac{1}{2}x + 1$

$1 = -\frac{1}{2}(0) + b \Rightarrow b = 1$

intersection: $2x = -\frac{1}{2}x + 1$ $\quad \boxed{\left(\frac{2}{5}, \frac{4}{5}\right)}$

$\frac{5}{2}x = 1$; $x = \frac{2}{5}$ $\quad y = 2x$

$l = \begin{bmatrix}2/5\\-1/5\end{bmatrix}$ $\quad y' = \begin{bmatrix}0\\1\end{bmatrix} + 2\begin{bmatrix}2/5\\-1/5\end{bmatrix} = \begin{bmatrix}4/5\\3/5\end{bmatrix}$ $\quad x' = \begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}-4/5\\2/5\end{bmatrix} = \begin{bmatrix}-3/5\\4/5\end{bmatrix}$

$$\boxed{\text{Transformation Matrix} \Big\} : \begin{bmatrix}-3/5 & 4/5\\4/5 & 3/5\end{bmatrix}}$$

II

3.3 Inverses Exercises

3.64] Use the method of Example 3-8 on page 185 to invert the following matrices (if possible)

(p)

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -1 & 2 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to -R_1 + R_3]{R_2 \to -2R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & -2 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3/2]{R_2 \to R_2/3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2/3 & 1/3 & 0 \\ 0 & -1/2 & 1 & -1/2 & 0 & 1/2 \end{array}\right] \xrightarrow{R_3 \to \frac{1}{2}R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2/3 & 1/3 & 0 \\ 0 & 0 & 1 & -5/6 & 1/6 & 1/2 \end{array}\right]$$

$I \qquad A^{-1}$

$$\left(\frac{1}{2}\right)\left(-\frac{2}{3}\right) + \left(-\frac{1}{2}\right)$$
$$-\frac{1}{3} + -\frac{1}{2}$$
$$\frac{-2 + -3}{6} = -5/6$$

$$\boxed{A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1/3 & 0 \\ -5/6 & 1/6 & 1/2 \end{bmatrix}}$$

3.75] A certain matrix A has inverse

$$B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} = A^{-1}$$

Find a matrix X such that $XA = C$, where

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad X = CA^{-1} = CB$$

$$CB = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 9 & 8 \\ 6 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

3.81] Using properties of inverses and matrix multiplication, prove that if A and B are invertible $n \times n$ matrices, then

$$(B^{-1}A^{-1})(AB) = I$$

How does this prove that $(AB)^{-1} = B^{-1}A^{-1}$?

L.H.S

$$\begin{aligned}
&(B^{-1}A^{-1})(AB) &&\longrightarrow \text{given} \\
&= (B^{-1}(A^{-1}A))B &&\longrightarrow \text{Associative law} \\
&= (B^{-1}(I))B &&\longrightarrow \text{Inverse law} \\
&= B^{-1}B &&\longrightarrow \text{Identity law} \\
&= I &&\longrightarrow \text{Inverse law.}
\end{aligned}$$

Since L.H.S = R.H.S we have proven that $(B^{-1}A^{-1})(AB) = I$

$$\begin{aligned}
(B^{-1}A^{-1})(AB) &= I \\
(AB)^{-1}(B^{-1}A^{-1})(AB) &= I(AB^{-1}) &&\longrightarrow \text{Multiply both sides by } (AB)^{-1} \\
I(B^{-1}A^{-1}) &= (AB)^{-1} &&\longrightarrow \text{Inverse law} \\
B^{-1}A^{-1} &= (AB)^{-1} &&\longrightarrow \text{Identity law.}
\end{aligned}$$

$\therefore$ QED

3.100] For the following matrices L, U, and Y, solve the system $AX = Y$ for X where $A = LU$ by first finding a Z such that $LZ = Y$ and then finding an X such that $UX = Z$

(b)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad y = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

(i) $LZ = Y$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 2 & 1 & 0 & 2 \\ 2 & 2 & 1 & 1 \end{array}\right] \xrightarrow[R_3 \to -2R_1 + R_3]{R_2 \to -2R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 2 & 1 & -5 \end{array}\right] \xrightarrow{R_3 \to -2R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$$

(ii) $UX = Z$

$$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} \Longrightarrow \left[\begin{array}{cccc|c} 2 & 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 & -4 \\ 0 & 0 & 1 & 1 & 5 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 2 & 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 & -4 \\ 0 & 0 & 1 & 1 & 5 \end{array}\right] \xrightarrow{R_1 \to R_1/2} \left[\begin{array}{cccc|c} 1 & 1/2 & 0 & 1/2 & 3/2 \\ 0 & 1 & 1 & 2 & -4 \\ 0 & 0 & 1 & 1 & 5 \end{array}\right] \xrightarrow{R_2 \to -R_3 + R_2} \left[\begin{array}{cccc|c} 1 & 1/2 & 0 & 1/2 & 3/2 \\ 0 & 1 & 0 & 1 & -9 \\ 0 & 0 & 1 & 1 & 5 \end{array}\right]$$

$$\xrightarrow{R_1 \to -\frac{1}{2}R_2 + R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 1 & -9 \\ 0 & 0 & 1 & 1 & 5 \end{array}\right]$$

$x_1 = 6$

$x_2 = -9 - x_4$

$x_3 = 5 - x_4$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ -9 \\ 5 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}}$$

3.101] Compute the LU Factorization for the following matrices

(c) $\begin{bmatrix} 2 & 4 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

$-\frac{3}{2}+3$

$-\frac{3}{2}+1$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$

$\downarrow \begin{matrix} R_2 \to -\frac{1}{2}R_1 + R_2 \\ R_3 \to -\frac{1}{2}R_1 + R_3 \end{matrix}$

$\begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & 3/2 \\ 0 & 0 & -1/2 \end{bmatrix} = U$

$A = LU$

$$\boxed{\begin{bmatrix} 2 & 4 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & 3/2 \\ 0 & 0 & -1/2 \end{bmatrix}}$$

3.113] These exercises refer to the permutation transformation $T: \mathbb{R}^5 \to \mathbb{R}^5$ defined by

$$T([x_1, x_2, x_3, x_4, x_5]^t) = [x_5, x_1, x_4, x_2, x_3]^t$$

(a) Find a 5×5 permutation matrix P such that, for all $X \in \mathbb{R}^5$, $T(x) = PX$

P_x (Permutation Matrix):

$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

5×5 Identity matrix:

$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = I$

b) The inverse transformation T^{-1} satisfies
$$T^{-1}([x_5, x_1, x_4, x_2, x_3]) = [x_1, x_2, x_3, x_4, x_5]^t.$$

Give a formula for
$$T^{-1}([y_1, y_2, y_3, y_4, y_5]^t)$$

Then find the 5×5 matrix Q such that, for all $y \in \mathbb{R}^5$, $Qy = T^{-1}(y)$. Check that $QP = I$

let $x, y \in \mathbb{R}^5$

$x = [x_1, x_2, x_3, x_4, x_5]^t \qquad y = [y_1, y_2, y_3, y_4, y_5]^t$

let $Tx = y$

$$\Rightarrow \begin{pmatrix} x_5 \\ x_1 \\ x_4 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} \Rightarrow \begin{matrix} x_5 = y_1 \\ x_1 = y_2 \\ x_4 = y_3 \\ x_2 = y_4 \\ x_3 = y_5 \end{matrix}$$

$$Tx = y \Rightarrow T^{-1}[y] = x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} y_2 \\ y_4 \\ y_5 \\ y_3 \\ y_1 \end{bmatrix}$$

$T^{-1}(l_1) = l_5 \quad T^{-1}(l_3) = l_4$

$T^{-1}(l_2) = l_1 \quad T^{-1}(l_4) = l_2$

$T^{-1}(l_5) = l_3$

$$Q = [T^{-1}] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$T^{-1}y = Qy$

$$QP = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \boxed{I}$$

$\therefore$ Hence, verified

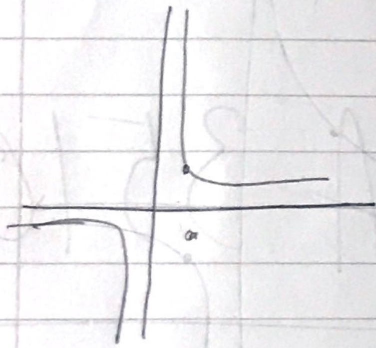
$\square$

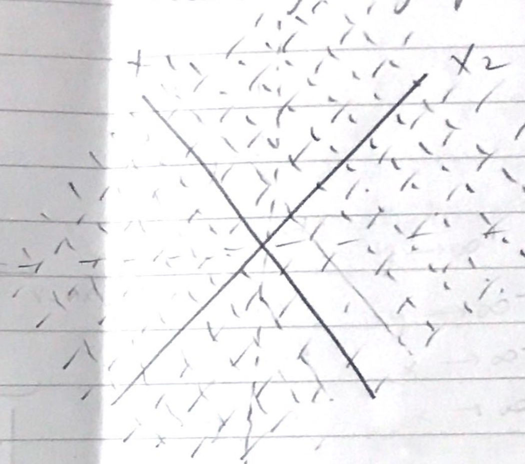
worked with
ASG[illegible]

3.114 Sketch the graph of the curve $xy=1$. Use the ordered basis $X_1 = [1,1]^t$ and $X_2 = [1,-1]^t$ to define new coordinates for $\mathbb{R}^2$

(a) Sketch the new coordinate axes on your graph.

$xy = 1$





(b) Show that in these coordinates the curve is described by the equation
$$(x')^2 - (y')^2 = 1$$
This proves that $y = \frac{1}{x}$ represents a hyperbola. [HINT: show that if $[x',y']^t$ is the coordinate vector for $[x,y]^t$, then $x = x'+y'$ and $y = [x-y']$

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$xy = x'\begin{bmatrix} 1 \\ 1 \end{bmatrix} + y'\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x' + y' \\ x' - y' \end{bmatrix}$$

$xy = 1$, by substitution, we get
$$(x'+y')(x'-y') = 1$$
$$(x')^2 - (y')^2 = 1 \implies$$ which we know is a hyperbola

$\boxed{\therefore \text{Hence, proved}}$

MA 35100

$\boxed{\text{HOMEWORK 8}}$

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