Eigenvalues & eigenvectors

eigen = "own"

eigenvalues = "matrix's own values."

Def. Let A be an $n \times n$ matrix
A vector $X \neq 0$ is an eigenvector of A with eigenvalue λ if

$$A \cdot X = \lambda \cdot X.$$

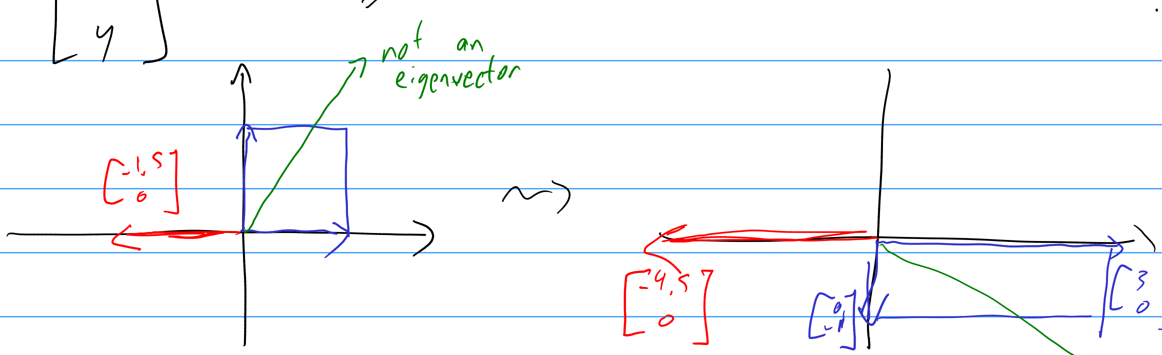
ex. $A = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$. Then $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

So $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of A with eigenvalue 3.

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue -1.

$\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} x \\ 0 \end{bmatrix}$ " of A with eigenvalue 3

$\begin{bmatrix} 0 \\ y \end{bmatrix}$ " of A " -1.



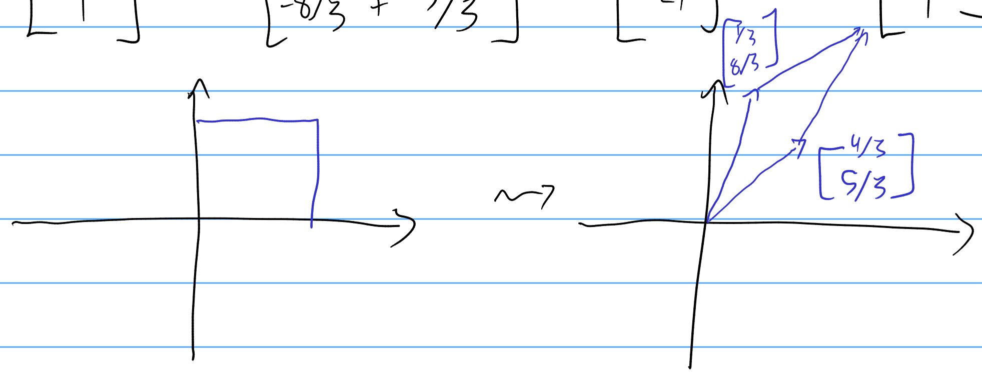
The x-axis & y-axis consist of eigenvectors.

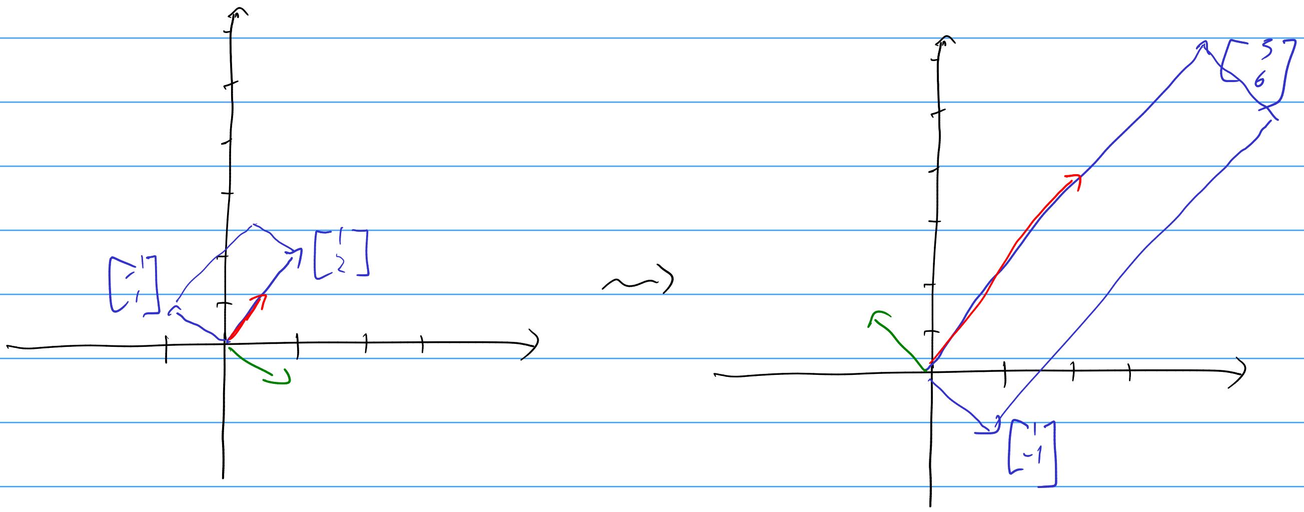
In fact, these are the only eigenvectors.

ex. $B = \begin{bmatrix} 1/3 & 4/3 \\ 8/3 & 5/3 \end{bmatrix}$

$$B \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 + 2 \cdot 4/3 \\ 8/3 + 2 \cdot 5/3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$B \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/3 + 4/3 \\ -8/3 + 5/3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$





ex. $C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Every vector X is an eigenvector of C, with eigenvalue 2.

ex. X is an eigenvector with eigenvalue 0 for some A

$\iff A \cdot X = 0 \cdot X = 0$

$\iff X \in null(A)$.

A has at least one eigenvector with eigenvalue 0

$\iff A$ has a nontrivial null space

$\iff A$ is not invertible.

Prop. The set of eigenvectors of A with eigenvalue λ is a subspace of $\mathbb{R}^n$.

Proof. Suppose X, Y have eigenvalue λ.

Then $A\cdot(X+Y) = A\cdot X + A\cdot Y = \lambda\cdot X + \lambda\cdot Y = \lambda(X+Y)$

Suppose X has eigenvalue λ and $c \in \mathbb{R}$

Then $A\cdot(cX) = c\cdot AX = c\cdot\lambda \cancel{X} = \lambda\cdot(cX)$. ∎

⚠ Eigenvectors with distinct eigenvalues generally don't form a subspace.

If X has eigenvalue λ, Y has eigenvalue μ, for some A,

then $A\cdot(X+Y) = AX + AY = \lambda X + \mu Y$

This is probably not a scalar multiple of $X+Y$.

Prop. Eigenvectors with different eigenvalues are linearly independent.

Proof. Say X has eigenvalue λ, Y has eigenvalue μ.

If X and Y are linearly dependent, then let's say $X = cY$. But then X would also have eigenvalue μ. $\rightarrow\leftarrow$ so X, Y are linearly independent. ∎.

Cor. At most n possible eigenvalues for an $n\times n$ matrix.

Proof. Any eigenvalue has at least one eigenvector.

... so has at least a 1-dim'l space of eigenvectors.

The eigenvectors for distinct eigenvalues are LI from each other, so $\leq n$ of these spaces of eigenvectors. ∎

Finding eigenvalues.

Suppose λ is an eigenvalue of A. (with X an eigenvector)

$$AX = \lambda X = (\lambda I)X.$$

$$AX - (\lambda I)X = 0$$

$$(A - \lambda I)X = 0.$$

So $\mathrm{null}(A - \lambda I)$ is bigger than $\{0\}$.

i.e. $A - \lambda I$ is not invertible.

i.e. $\det(A - \lambda I) = 0$.

Def. The characteristic polynomial of A is

$$P_A(\lambda) = \det(A - \lambda I).$$

If A is $n \times n$, this is a degree n polynomial.

ex. $A = \begin{bmatrix} 1/3 & 4/3 \\ 8/3 & 5/3 \end{bmatrix}$

$$P_A(\lambda) = \left| \begin{bmatrix} 1/3 & 4/3 \\ 8/3 & 5/3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} 1/3 - \lambda & 4/3 \\ 8/3 & 5/3 - \lambda \end{vmatrix}$$

$$= (1/3 - \lambda)(5/3 - \lambda) - (4/3) \cdot (8/3)$$

$$= \lambda^2 - 2\lambda - 3. = (\lambda - 3)(\lambda + 1)$$

Zeros are $\lambda = 3$, $\lambda = -1$ These are the eigenvalues of A.

ex. $D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. $\quad D - \lambda I = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}$

$$P_D(\lambda) = \begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = (\lambda - 2)^2.$$

Only eigenvalue is $\lambda = 2$. This is a double eigenvalue, because $(\lambda - 2)^2$ appears in $P_D(\lambda)$.

More generally, we say that the multiplicity of an eigenvalue λ_0 is r if the characteristic polynomial is divisible by $(\lambda-\lambda_0)^r$ (and not $(\lambda-\lambda_0)^{r+1}$).

What we know about factorizing polynomials.

Fundamental Thm of Algebra. A degree n polynomial $p(\lambda)$ has exactly n roots.
... if you count with multiplicity
... and you include complex roots.

In other words,

$$p(\lambda) = c(\lambda-\lambda_1)^{n_1}(\lambda-\lambda_2)^{n_2}\cdots(\lambda-\lambda_k)^{n_k}$$

where $\lambda_1, \ldots, \lambda_k$ may be complex, and

$$n_1+n_2+\cdots+n_k = n.$$

Cor. An $n\times n$ matrix has exactly n eigenvalues
... if you count with multiplicity
... and you include complex eigenvalues.

A few more useful reminders:

- → A quadratic polynomial can always be factorized, using the quadratic formula if necessary.
- → Higher-degree polynomials can be factorized numerically by computer, but by hand it's pretty hard/impossible.
- → if the polynomial has real coefficients, then any non-real roots occur in conjugate pairs.
  (if $a+bi$ is a root w/ multiplicity k, then $a-bi$ is also a root w/ multiplicity k.)

ex. $U = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$. $\quad P_U(\lambda) = \begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{vmatrix}$

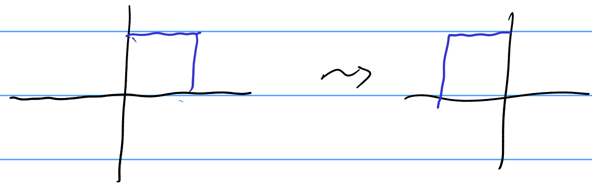
$$= (2-\lambda)(2-\lambda)(3-\lambda).$$

So the eigenvalues of U are 2 (double) & 3.

Conclusion: eigenvalues of an upper triangular matrix are the entries on the diagonal.

(Same for lower triangular matrices.)

ex. $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.



$$P_R(\lambda) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1.$$

So $P_R(\lambda) = 0 \iff \lambda^2 + 1 = 0$

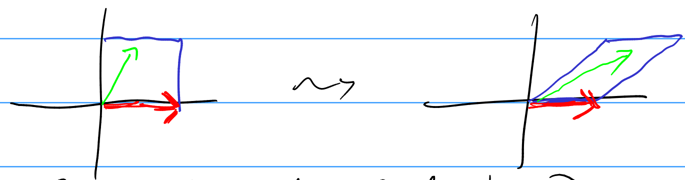
$$\lambda^2 = -1$$

$$\lambda = \pm i.$$

$$R\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i\begin{bmatrix} 1 \\ i \end{bmatrix}.$$

So, $\begin{bmatrix} 1 \\ i \end{bmatrix}$ is an eigenvector of R with eigenvalue $-i$.

ex. $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$



Only eigenvalue of S is $\lambda = 1$ (double).

Only eigenvectors are $\begin{bmatrix} x \\ 0 \end{bmatrix}$.

Finding eigenvectors.

Recall: The eigenvalues of a matrix A are the roots of the characteristic polynomial

$$P_A(\lambda) = \det(A - \lambda I).$$

If A is $n \times n$, there are n of these, counting with multiplicity & including complex eigenvalues.

Suppose X is an eigenvector of A with eigenvalue λ. Then

$$AX = \lambda X = (\lambda I)X$$

$$(A - \lambda I)X = 0.$$

This is just a system of equations that we can solve for X.

ex.

$$A = \begin{bmatrix} -2 & 0 & 1 \\ -2 & -1 & 2 \\ 2 & -1 & 0 \end{bmatrix} \qquad P_A(\lambda) = \begin{vmatrix} -2-\lambda & 0 & 1 \\ -2 & -1-\lambda & 2 \\ 2 & -1 & -\lambda \end{vmatrix}.$$

$$= (-2-\lambda)\begin{vmatrix} -1-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} + 1\begin{vmatrix} -2 & -1-\lambda \\ 2 & -1 \end{vmatrix}$$

$$= (-2-\lambda)\left[(-1-\lambda)(-\lambda) + 2\right] + \left[2 - 2(-1-\lambda)\right]$$

$$= (-2-\lambda)(\lambda^2 + \lambda + 2) + (2\lambda + 4)$$

$$= -\lambda^3 - 3\lambda^2 - 2\lambda = -\lambda(\lambda^2 + 3\lambda + 2)$$

$$= -\lambda(\lambda + 2)(\lambda + 1)$$

The eigenvalues: $\lambda = 0, -2,$ and -1.

$\lambda = -2$: $(A+2I)X = 0.$

$$\begin{bmatrix} 0 & 0 & 1 \\ -2 & 1 & 2 \\ 2 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Rightarrow \quad z = 0.$$

$$\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2x + y = 0$, so $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, for any $c \neq 0$.

$\lambda = -1$. $A + I = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ 2 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 3 \end{bmatrix}$

$$\to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} x - z &= 0 \\ y - 3z &= 0, \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

$\lambda = 0$: $(A + 0I)X = 0$, i.e. $AX = 0$.

$$\begin{bmatrix} -2 & 0 & 1 \\ -2 & -1 & 2 \\ 2 & -1 & 0 \end{bmatrix} \xrightarrow[R_1+R_3]{-R_1+R_2} \begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[-R_3+R_2]{} \begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} -2x + z &= 0 \\ -y + z &= 0 \end{aligned}, \text{ so } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = c\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

ex. $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\qquad P_A(\lambda) = (\lambda - 2)^2$.

eigenvalues: $\lambda = 2$ (double eigenvalue).

$(A - 2I)X = 0$

$A - 2I = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

So any X is a solution.

ex. $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\qquad P_S(\lambda) = (\lambda - 1)^2$

eigenvalues: $\lambda = 1$ (double eigenvalue)

$(S - I)X = 0$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X = 0. \implies X = c\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

In this case, 1 is a double eigenvalue, but has only a 1-dim'l space of eigenvectors.

In general: a k-fold eigenvalue can have a space of eigenvectors of dimension anywhere between 1 and k.

Exercise: Find 3×3 matrices with triple eigenvalues and (a) a 1-dim'l space of eigenvectors,

(b) a 2-dim'l space of eigenvectors

(c) a 3-dim'l space of eigenvectors

ex. $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad P_R(\lambda) = \lambda^2 + 1$

$\lambda = \pm i.$

$\lambda = i$: $(R - iI)X = 0$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -ix - y = 0 \\ x - iy = 0. \end{cases}$$

$$x = iy \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = c \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad c \in \mathbb{C}.$$

$\lambda = -i$: $\begin{bmatrix} x \\ y \end{bmatrix} = c \begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad c \in \mathbb{C}.$

Diagonalization

Say that A is an $n \times n$ matrix representing $T_A : \mathbb{R}^n \to \mathbb{R}^n$. Suppose that there's a basis $\mathcal{B} = \{X_1, \dots, X_n\}$ for $\mathbb{R}^n$ such that each X_i is an eigenvector for A, with eigenvalue λ_i.

What's the matrix for T_A in terms of the basis $\mathcal{B}$?

$$T_A(X_i) = A \cdot X_i = \lambda_i X_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix} \text{ in } \mathcal{B}\text{-coords.}$$

So the matrix for T_A in terms of $\mathcal{B}$ is

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}, \quad \text{a diagonal matrix.}$$

$$A = P_{\mathcal{B}} \cdot D \cdot C_{\mathcal{B}}$$

$$\boxed{A = P_{\mathcal{B}} \cdot D \cdot P_{\mathcal{B}}^{-1}} \quad \text{where } P_{\mathcal{B}} = \begin{bmatrix} | & & | \\ X_1 & \cdots & X_n \\ | & & | \end{bmatrix}.$$

Def. A is **diagonalizable** if there's a basis for $\mathbb{R}^n$ consisting of eigenvectors for A.

ex.

$$A = \begin{bmatrix} -2 & 0 & 1 \\ -2 & -1 & 2 \\ 2 & -1 & 0 \end{bmatrix} \qquad \lambda_1 = -2 \quad X_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad \lambda_2 = -1 \quad X_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \qquad \lambda_3 = 0 \quad X_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

$$P_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-R_2+R_3]{-R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 2 & 2 & -1 & 1 \end{array}\right] \xrightarrow[R_3/2]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1/2 & 1/2 \end{array}\right]$$

$$\xrightarrow{-R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1/2 & -1/2 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1/2 & 1/2 \end{array}\right] \qquad \text{So } C_B = \begin{bmatrix} 2 & -1/2 & -1/2 \\ -2 & 1 & 0 \\ 1 & -1/2 & 1/2 \end{bmatrix}$$

Conclusion: $A = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix}}_{P_B} \underbrace{\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 2 & -1/2 & -1/2 \\ -2 & 1 & 0 \\ 1 & -1/2 & 1/2 \end{bmatrix}}_{P_B^{-1}}$.

1 Application:
$$\begin{aligned} A^{500} &= (P_B D P_B^{-1})^{500} \\ &= P_B D \cancel{P_B^{-1}} \cdot \cancel{P_B} D \cancel{P_B^{-1}} \cdot \ldots \cdot \cancel{P_B} D P_B^{-1} \\ &= P_B D^{500} P_B^{-1} \\ &= P_B \begin{bmatrix} (-2)^{500} & 0 & 0 \\ 0 & (-1)^{500} & 0 \\ 0 & 0 & 0 \end{bmatrix} P_B^{-1}. \end{aligned}$$

When is a matrix diagonalizable?

Prop. If A has n distinct eigenvalues, then it's diagonalizable.

Proof. Each eigenvalue has at least one eigenvector, and these are LI for distinct eigenvalues. So there's a basis for $\mathbb{R}^n$ consisting of one eigenvector for each eigenvalue. ▨

ex $S=\begin{bmatrix}1&1\\0&1\end{bmatrix}$. eigenvalues: $\lambda=1$ (double)

eigenvectors: $x=c\begin{bmatrix}1\\0\end{bmatrix}$.

There's no basis of $\mathbb{R}^2$ consisting of eigenvectors for S. So S is not diagonalizable.

ex. $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$ $\quad P_A(\lambda)=\begin{vmatrix}1-\lambda&2\\3&4-\lambda\end{vmatrix}$

$$=(1-\lambda)(4-\lambda)-6=\lambda^2-5\lambda-2$$

$$\lambda=\frac{5\pm\sqrt{25+8}}{2}=\frac{5}{2}\pm\frac{\sqrt{33}}{2}$$

$\lambda=\frac{5}{2}+\frac{\sqrt{33}}{2}$: $A-\lambda I=\begin{bmatrix}-\frac{3}{2}-\frac{\sqrt{33}}{2}&2\\3&\frac{3}{2}-\frac{\sqrt{33}}{2}\end{bmatrix}$.

Solutions are solutions to $\left(-\frac{3}{2}-\frac{\sqrt{33}}{2}\right)x+2y=0$.

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2\\\frac{3}{2}+\frac{\sqrt{33}}{2}\end{bmatrix}$$

Check: $\begin{bmatrix}-\frac{3}{2}-\frac{\sqrt{33}}{2}&2\\3&\frac{3}{2}-\frac{\sqrt{33}}{2}\end{bmatrix}\begin{bmatrix}2\\\frac{3}{2}+\frac{\sqrt{33}}{2}\end{bmatrix}=\begin{bmatrix}\cancel{2\left(-\frac{3}{2}-\frac{\sqrt{33}}{2}\right)+2\left(\frac{3}{2}+\frac{\sqrt{33}}{2}\right)}\\6+\frac{9}{4}+\frac{3\sqrt{33}}{4}-\frac{3\sqrt{33}}{4}-\frac{33}{4}\end{bmatrix}$

$$=\begin{bmatrix}0\\0\end{bmatrix}.$$

$\lambda=\frac{5}{2}-\frac{\sqrt{33}}{2}$ $\quad A-\lambda I=\begin{bmatrix}-\frac{3}{2}+\frac{\sqrt{33}}{2}&2\\3&\frac{3}{2}+\frac{\sqrt{33}}{2}\end{bmatrix}$

$$\left(-\frac{3}{2}+\frac{\sqrt{33}}{2}\right)x+2y=0$$

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2\\\frac{3}{2}-\frac{\sqrt{33}}{2}\end{bmatrix}.$$

$$B = \left\{ \begin{bmatrix} 2 \\ \frac{3}{2} + \frac{\sqrt{33}}{2} \end{bmatrix}, \begin{bmatrix} 2 \\ \frac{3}{2} - \frac{\sqrt{33}}{2} \end{bmatrix} \right\}$$

$\lambda = \frac{5}{2} + \frac{\sqrt{33}}{2}$ $\qquad$ $\lambda = \frac{5}{2} - \frac{\sqrt{33}}{2}$

$$P_B = \begin{bmatrix} 2 & 2 \\ \frac{3}{2} + \frac{\sqrt{33}}{2} & \frac{3}{2} - \frac{\sqrt{33}}{2} \end{bmatrix}$$

$$\det(P_B) = 2\left(\frac{3}{2} - \frac{\sqrt{33}}{2}\right) - 2\left(\frac{3}{2} + \frac{\sqrt{33}}{2}\right) = -2\sqrt{33}.$$

$$C_B = P_B^{-1} = \frac{1}{-2\sqrt{33}} \begin{bmatrix} \frac{3}{2} - \frac{\sqrt{33}}{2} & -2 \\ -\frac{3}{2} - \frac{\sqrt{33}}{2} & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 2 \\ \frac{3}{2} + \frac{\sqrt{33}}{2} & \frac{3}{2} - \frac{\sqrt{33}}{2} \end{bmatrix} \begin{bmatrix} \frac{5}{2} + \frac{\sqrt{33}}{2} & 0 \\ 0 & \frac{5}{2} - \frac{\sqrt{33}}{2} \end{bmatrix} \left(\frac{1}{-2\sqrt{33}} \begin{bmatrix} \frac{3}{2} - \frac{\sqrt{33}}{2} & -2 \\ -\frac{3}{2} - \frac{\sqrt{33}}{2} & 2 \end{bmatrix} \right)$$