Determinants

$\det : M(n,n) \longrightarrow \mathbb{R}.$

⚠ Not a linear map.
⚠ Only defined for square matrices.

What is it for?

① Test for invertibility: A is invertible $\Leftrightarrow \det(A) \neq 0$.
② Quickly compute inverses.
③ If A represents a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$, then $\det(A)$ measures how A scales volumes.

Def ① If $A = [a]$, then $\det(A) = a$.

② If $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$, then:

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) - \cdots + (-1)^{1+n} a_{1n}\det(A_{1n}).$$

where A_{ij} is the matrix obtained by removing the i^{th} row and j^{th} column of A.

A_{ij} are called minors of A, and $\pm\det(A_{ij})$ are called cofactors.

cofactor expansion along the first row of A.

$$\det \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

⚠ Not a matrix! A number!

ex. 2×2 matrices

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

Then $\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12})$

$A_{11} = \begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} \\ \cancel{a_{21}} & a_{22} \end{bmatrix} = [a_{22}]$. $\quad A_{12} = \begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} \\ a_{21} & \cancel{a_{22}} \end{bmatrix} = [a_{21}]$

So $\det(A) = a_{11}a_{22} - a_{12}a_{21}$

$= \boxed{ad - bc.}$

$ad - bc \neq 0 \iff A$ is invertible.

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

ex. 3×3 matrices. $\quad A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & -1 \\ 3 & 5 & -5 \end{bmatrix}$.

Method 1: Cofactor expansion along the first row. (Same as in definition)

$\det(A) = 2\det(A_{11}) - 0\cdot\det(A_{12}) + 1\cdot\det(A_{13})$.

$A_{11} = \begin{bmatrix} \cancel{2} & \cancel{0} & \cancel{1} \\ \cancel{1} & 0 & -1 \\ \cancel{3} & 5 & -5 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 5 & -5 \end{bmatrix}$ $\qquad \begin{vmatrix} 0 & -1 \\ 5 & -5 \end{vmatrix} = 0\cdot(-5) - (-1)\cdot 5 = 5.$

$A_{13} = \begin{bmatrix} \cancel{2} & \cancel{0} & \cancel{1} \\ 1 & 0 & \cancel{-1} \\ 3 & 5 & \cancel{-5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 5 \end{bmatrix}$ $\qquad \begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 1\cdot 5 - 0\cdot 3 = 5.$

$\det(A) = 2\cdot 5 + 1\cdot 5 = 15.$

Remark: $A^{-1} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & -1 \\ 3 & 5 & -5 \end{bmatrix}^{-1} = \begin{bmatrix} 1/3 & 1/3 & 0 \\ 2/15 & -13/15 & 1/5 \\ 1/3 & -2/3 & 0 \end{bmatrix}$

Method 2: "Diagonal trick" ⚠ Only for 3×3!

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & -1 \\ 3 & 5 & -5 \end{bmatrix} \begin{matrix} 2 & 0 \\ 1 & 0 \\ 3 & 5 \end{matrix}$$

forward diagonals

backward diagonals.

$$\det(A) = \sum \text{ products along forward diagonals} - \sum \text{ products along backward diagonals.}$$

$$= 2\cdot 0\cdot(-5) + 0\cdot(-1)\cdot 3 + 1\cdot 1\cdot 5 - \left(1\cdot 0\cdot 3 + 2\cdot(-1)\cdot 5 + 0\cdot 1\cdot(-5)\right)$$

$$= 15.$$

Method 3: Cofactor expansion along second column.

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & -1 \\ 3 & 5 & -5 \end{bmatrix}$$

$$-a_{21}\det(A_{21}) + a_{22}\det(A_{22}) - a_{32}\det(A_{32})$$

$$= -0\cdot\det(A_{21}) + 0\cdot\det(A_{22}) - 5\cdot\det(A_{32})$$

$$= -0 + 0 - 5\cdot\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= -5\cdot(-3) = 15.$$

Cofactor expansion thm. "Determinants can be computed using any row or column."

Let $1 \le i \le n$. Then $\det(A) = (-1)^{i+1} a_{i1}\det(A_{i1}) + (-1)^{i+2} a_{i2}\det(A_{i2}) + \cdots + (-1)^{i+n} a_{in}\det(A_{in})$.

Let $1 \le j \le n$. Then $\det(A) = (-1)^{1+j} a_{1j}\det(A_{1j}) + (-1)^{2+j} a_{2j}\det(A_{2j}) + \cdots + (-1)^{n+j} a_{nj}\det(A_{nj})$.

Signs:
$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

ex. Lower triangular matrices.

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 5 & 3 & 3 & 0 \\ \pi & e & \sqrt{7} & 4 \end{vmatrix} = 1\begin{vmatrix} 2 & 0 & 0 \\ 3 & 3 & 0 \\ e & \sqrt{7} & 4 \end{vmatrix} = 1\cdot 2\cdot\begin{vmatrix} 3 & 0 \\ \sqrt{7} & 4 \end{vmatrix} = 1\cdot 2\cdot 3\cdot 4$$

More generally: the determinant of a lower triangular matrix is the product of the entries on the diagonal.
(Same is true for upper triangular matrices — why?)