

Coordinates in abstract vector spaces.

ex. Vector spaces of functions, $M(m,n)$ of $m \times n$ matrices, ...

ex. $P_3 = \{ \text{cubic functions } ax^3 + bx^2 + cx + d \}$

There's a linear transformation

$$D: P_3 \rightarrow P_2 = \{ \text{quadratic functions} \}$$

$$D(f) = f'$$

$$D(ax^3 + bx^2 + cx + d) = 3ax^2 + 2bx + c$$

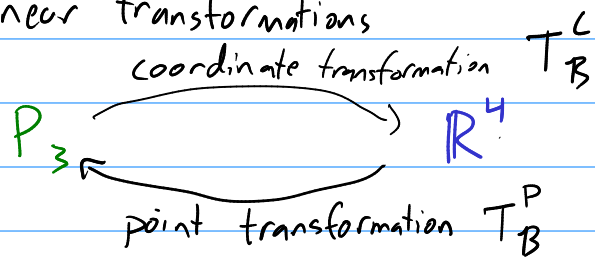
D is linear because $D(f+g) = D(f) + D(g)$,
 $D(cf) = cD(f)$ for $c \in \mathbb{R}$.

Pick bases $B = \{1, x, x^2, x^3\}$ for P_3
and $\bar{B} = \{1, x, x^2\}$ for P_2 .

A function $f \in P_3$ has coordinates in terms of B
 $ax^3 + bx^2 + cx + d$
"point" in P_3 , i.e.,
a polynomial

$[d, c, b, a]^T$
a 4×1 column vector, i.e.,
an element of \mathbb{R}^4 .

There are linear transformations



$$T_B^P \left(\begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix} \right) = ax^3 + bx^2 + cx + d$$

$$T_B^C(ax^3 + bx^2 + cx + d) = \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix}$$

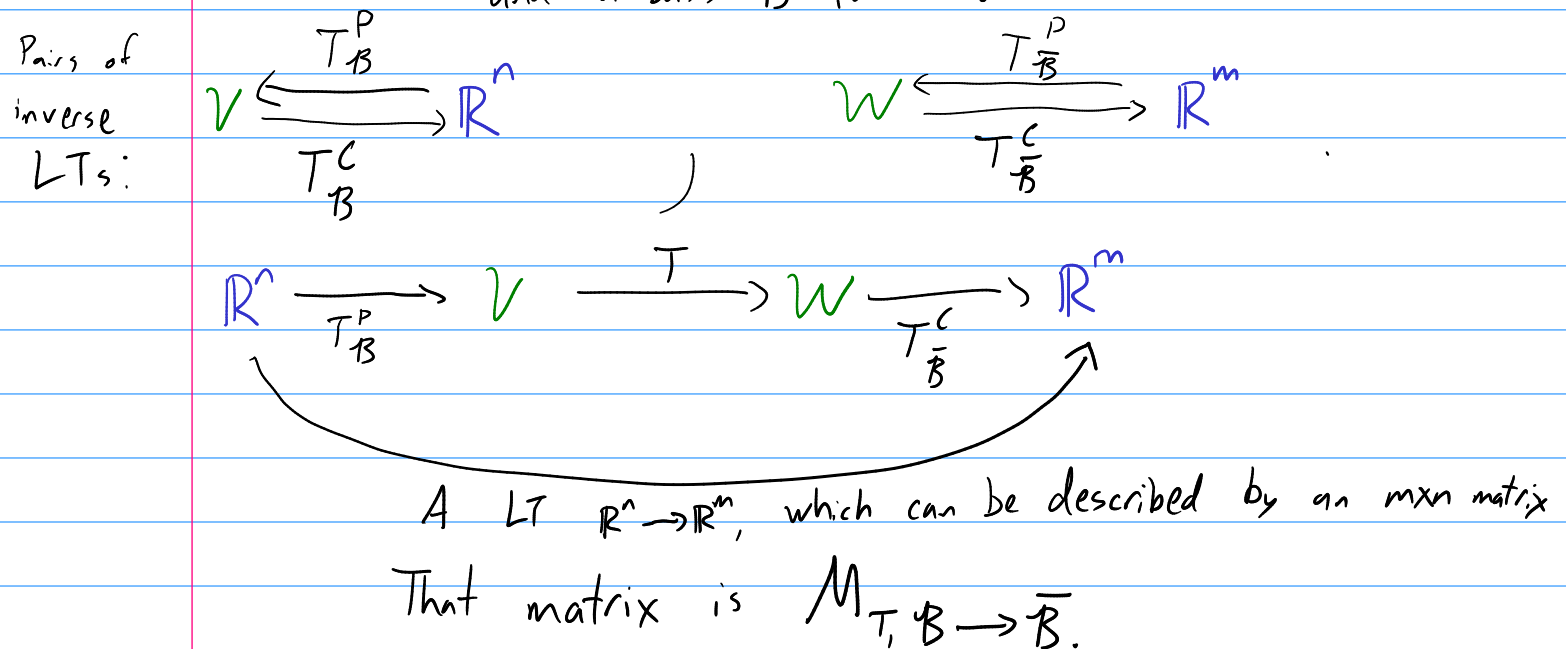
There's a matrix $M_{D, B \rightarrow \bar{B}}$ such that, if Q is the B -coordinates for $f \in P_3$, then $M_{D, B \rightarrow \bar{B}} Q$ is the \bar{B} -coordinates for $D(f) \in P_2$

$$M \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = D(1) = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$M \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = D(x) = 1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

General framework: Say we have $T: V \rightarrow W$,
 $\dim n$ $\dim m$
 a basis B for V ,
 and a basis \bar{B} for W .



ex. Let V be the set of functions $f(x)$ such that $f''(x) = -f(x)$.

This is a subspace of $F(\mathbb{R})$:

$0 \in V$ ✓

If $f, g \in V$, then $(f+g)'' = f'' + g'' = -f - g = -(f+g)$. ✓

If $f \in V$, $c \in \mathbb{R}$, then $(cf)'' = c \cdot f'' = c(-f) = -cf$. ✓

This vector space is 2-dimensional and has a basis $\{\sin(x), \cos(x)\} = B$

Any $f \in V$ is of the form $a \sin(x) + b \cos(x)$.

In this case, f has coordinates $\begin{bmatrix} a \\ b \end{bmatrix}$

Let's consider the LT $D: V \rightarrow V$

given by $D(f) = f'$

(If $f \in V$, then $D(f) = f'$

$$D(f)' = (f')' = f'' = (f'')' = (-f)' = -f' = -D(f).)$$

$$\text{Then } M_{D, B \rightarrow B} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = D(\sin(x)) = \cos(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$M_{D, B \rightarrow B} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = D(\cos(x)) = -\sin(x) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\text{So } M_{D, B \rightarrow B} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

$$\mathbb{R}^2 \xrightarrow{T_B^P} V \xrightarrow{D} V \xrightarrow{T_B^C} \mathbb{R}^2$$
