Cramer →

Cramer's Rule.

The rule: Consider a system $A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = Y$ where A is $n \times n$, $\det(A) \neq 0$,

Then $x_j = \dfrac{\det\left(\begin{bmatrix} | & & | & y_1 & | & & | \\ A_1 & \cdots & A_{j-1} & \vdots & A_{j+1} & \cdots & A_n \\ | & & | & y_n & | & & | \end{bmatrix}\right)}{\det(A)}$, $= \dfrac{\det([A_1, \ldots, A_{j-1}, Y, \ldots, A_n])}{\det(A)}$

example:

$$\underbrace{\begin{bmatrix} 2 & 2 & 3 \\ -1 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} \underbrace{=}_{=} \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{Y}.$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & 2 & 3 \\ -1 & 0 & 0 \\ 0 & 5 & 1 \end{vmatrix} = (-1)(-1)\begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} = -13.$$

$$\det(Y, A_2, A_3) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \\ 3 & 5 & 1 \end{vmatrix} = (-1)2\begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} = 26,$$

$$\det(A_1, Y, A_3) = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \\ 0 & 3 & 1 \end{vmatrix} = (-1)(-1)\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix}$$
$$= -8 + 4 = -4$$

$$\det(A_1, A_2, Y) = \begin{vmatrix} 2 & 2 & 1 \\ -1 & 0 & 2 \\ 0 & 5 & 3 \end{vmatrix} = (-1)(-1)\begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 0 & 5 \end{vmatrix}$$
$$= 1 - 20 = -19$$

So $x_1 = \dfrac{26}{-13} = -2$

$x_2 = -4/-13 = 4/13$

$x_3 = -19/-13 = 19/13.$

Check: $\begin{bmatrix} 2 & 2 & 3 \\ -1 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 4/13 \\ 19/13 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$

example:

$$\begin{bmatrix} 2 & 2 & 3 \\ -1 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$A \quad X = Y$$

$\det(A) = -13.$

$$\det([Y, A_2, A_3]) = \begin{vmatrix} y_1 & 2 & 3 \\ y_2 & 0 & 0 \\ y_3 & 5 & 1 \end{vmatrix} = -y_2 \begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} = 13y_2.$$

$$\det([A_1, Y, A_3]) = \begin{vmatrix} 2 & y_1 & 3 \\ -1 & y_2 & 0 \\ 0 & y_3 & 1 \end{vmatrix} = (-1)(-1)\begin{vmatrix} y_1 & 3 \\ y_3 & 1 \end{vmatrix} + y_2 \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix}$$

$$= y_1 - 3y_3 + 2y_2$$

$$= y_1 + 2y_2 - 3y_3$$

$$\det([A_1, A_2, Y]) = \begin{vmatrix} 2 & 2 & y_1 \\ -1 & 0 & y_2 \\ 0 & 5 & y_3 \end{vmatrix} = (-1)(-1)\begin{vmatrix} 2 & y_1 \\ 5 & y_3 \end{vmatrix} - y_2 \begin{vmatrix} 2 & 2 \\ 0 & 5 \end{vmatrix}$$

$$= 2y_3 - 5y_1 - 10y_2$$

$$= -5y_1 - 10y_2 + 2y_3.$$

So
$$\begin{bmatrix} 2 & 2 & 3 \\ -1 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -\frac{1}{13} \begin{bmatrix} 13y_2 \\ y_1 + 2y_2 - 3y_3 \\ -5y_1 - 10y_2 + 2y_3 \end{bmatrix}$$

$$A^{-1} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = -\frac{1}{13} \begin{bmatrix} 13y_2 \\ y_1 + 2y_2 - 3y_3 \\ -5y_1 - 10y_2 + 2y_3 \end{bmatrix}$$

So,
$$A^{-1} = -\frac{1}{13} \begin{bmatrix} 0 & 13 & 0 \\ 1 & 2 & -3 \\ -5 & -10 & 2 \end{bmatrix}$$

Formula for A^{-1}, in general.

The j^{th} column of A^{-1} is $A^{-1} I_j$.
In other words, it's the solution to
$$AX = I_j.$$
Using Cramer's rule, the j^{th} column of A^{-1} is
$$\frac{\det([A_1, \ldots, A_{j-1}, I_j, A_{j+1}, \ldots, A_n])}{\det(A)}.$$

Proof of Cramer's rule

(A = fixed $n \times n$ matrix)

Consider the following $T: \mathbb{R}^n \to \mathbb{R}^n$.
$T(X)$ has j^{th} coordinate $\dfrac{\det([A_1, \ldots, A_{j-1}, X, A_{j+1}, \ldots, A_n])}{\det(A)}$.

T is a linear transformation
(because det is linear in each column separately!)

$T(A_j)$ has j^{th} coordinate $\dfrac{\det([A_1, \ldots, A_{j-1}, A_j, A_{j+1}, \ldots, A_n])}{\det(A)}$
$$= 1$$
$T(A_j)$ has k^{th} coordinate $(k \neq j)$ $\dfrac{\det([A_1, \ldots, A_{k-1}, A_j, A_{k+1}, \ldots, A_n])}{\det(A)}$
$$= 0.$$
So $T(A_j) = I_j$.

If B = the matrix of T, then $BA_j = I_j$.
So $BA = I$
So $B = A^{-1}$. $\blacksquare$

Why do we care?

This is slower than computing A^{-1} via row reduction.

1. Sometimes nice to have explicit formulas (especially for small-dimensional inverses)
2. Given $AX = Y$, you can just solve for x_1.