Finding eigenvectors.

Recall: The eigenvalues of a matrix A are the roots of the characteristic polynomial

$$P_A(\lambda) = \det(A - \lambda I)$$

If A is $n \times n$, there are n of these, counting with multiplicity & including complex eigenvalues.

Suppose X is an eigenvector of A with eigenvalue λ. Then

$$AX = \lambda X = (\lambda I)X$$

$$(A - \lambda I)X = 0.$$

This is just a system of equations that we can solve for X.

ex.

$$A = \begin{bmatrix} -2 & 0 & 1 \\ -2 & -1 & 2 \\ 2 & -1 & 0 \end{bmatrix} \qquad P_A(\lambda) = \begin{vmatrix} -2-\lambda & 0 & 1 \\ -2 & -1-\lambda & 2 \\ 2 & -1 & -\lambda \end{vmatrix}.$$

$$= (-2-\lambda)\begin{vmatrix} -1-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} + 1\begin{vmatrix} -2 & -1-\lambda \\ 2 & -1 \end{vmatrix}$$

$$= (-2-\lambda)\left[(-1-\lambda)(-\lambda) + 2\right] + \left[2 - 2(-1-\lambda)\right]$$

$$= (-2-\lambda)(\lambda^2 + \lambda + 2) + (2\lambda + 4)$$

$$= -\lambda^3 - 3\lambda^2 - 2\lambda = -\lambda(\lambda^2 + 3\lambda + 2)$$

$$= -\lambda(\lambda + 2)(\lambda + 1)$$

The eigenvalues: $\lambda = 0, -2,$ and -1.

$\lambda = -2$: $(A+2I)X = 0$.

$$\begin{bmatrix} 0 & 0 & 1 \\ -2 & 1 & 2 \\ 2 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Rightarrow \quad z = 0.$$

$$\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2x + y = 0$, so $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, for any $c \neq 0$.

$\lambda = -1$. $A + I = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ 2 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 3 \end{bmatrix}$

$$\to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} x - z &= 0 \\ y - 3z &= 0, \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

$\lambda = 0$: $(A + 0I)X = 0$, i.e. $AX = 0$.

$$\begin{bmatrix} -2 & 0 & 1 \\ -2 & -1 & 2 \\ 2 & -1 & 0 \end{bmatrix} \xrightarrow[R_1+R_3]{-R_1+R_2} \begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[-R_3+R_2]{} \begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} -2x + z &= 0 \\ -y + z &= 0, \end{aligned} \quad \text{so} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = c\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

ex. $A=\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\qquad P_A(\lambda) = (\lambda-2)^2$.

eigenvalues: $\lambda = 2$ (double eigenvalue).

$(A-2I)X = 0$

$A-2I = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

So any X is a solution.

ex. $S=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\qquad P_S(\lambda) = (\lambda-1)^2$

eigenvalues: $\lambda = 1$ (double eigenvalue)

$(S - I)X = 0$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X = 0. \Rightarrow X = c\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

In this case, 1 is a double eigenvalue, but has only a 1-dim'l space of eigenvectors.

In general: a k-fold eigenvalue can have a space of eigenvectors of dimension anywhere between 1 and k.

Exercise: Find 3x3 matrices with triple eigenvalues and (a) a 1-dim'l space of eigenvectors,

(b) a 2-dim'l // //

(c) a 3-dim'l // //

ex. $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad P_R(\lambda) = \lambda^2+1$

$\lambda = \pm i$.

$\lambda = i$: $(R - iI)X = 0$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -ix - y = 0 \\ x - iy = 0. \end{cases}$$

$x = iy \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = c\begin{bmatrix} i \\ 1 \end{bmatrix}, \quad c \in \mathbb{C}.$

$\lambda = -i$: $\begin{bmatrix} x \\ y \end{bmatrix} = c\begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad c \in \mathbb{C}.$