Coordinates in abstract vector spaces.

ex. Vector spaces of functions, $M(m,n)$ of $m \times n$ matrices, ...

ex. $P_3 = \{$cubic functions $ax^3+bx^2+cx+d\}$

There's a linear transformation

$$D : P_3 \longrightarrow P_2 = \{\text{quadratic functions}\}$$

$$D(f) = f'$$

$$D(ax^3+bx^2+cx+d) = 3ax^2+2bx+c$$

D is linear because $D(f+g) = D(f)+D(g)$,

$D(cf) = c\,D(f)$ for $c \in \mathbb{R}$.

Pick bases $\mathcal{B} = \{1, x, x^2, x^3\}$ for P_3

and $\bar{\mathcal{B}} = \{1, x, x^2\}$ for P_2.

A function $f \in P_3$ has coordinates in terms of $\mathcal{B}$

ax^3+bx^2+cx+d $[d, c, b, a]^T$

"point" in P_3, i.e., a polynomial   a 4×1 column vector, i.e., an element of $\mathbb{R}^4$.

There are linear transformations

$$\begin{tikzcd} P_3 \arrow[r, bend left, "\text{coordinate transformation } T_{\mathcal{B}}^{C}"] & \mathbb{R}^4 \arrow[l, bend left, "\text{point transformation } T_{\mathcal{B}}^{P}"] \end{tikzcd}$$

$$T_{\mathcal{B}}^{P}\left(\begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix}\right) = ax^3+bx^2+cx+d$$

$$T_{\mathcal{B}}^{C}(ax^3+bx^2+cx+d) = \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix}$$

M

There's a matrix $M_{D,\mathcal{B}\to\bar{\mathcal{B}}}$ such that, if Q is the $\mathcal{B}$-coordinates for $f\in P_3$, then $M_{D,\mathcal{B}\to\bar{\mathcal{B}}}Q$ is the $\bar{\mathcal{B}}$-coordinates for $D(f)\in P_2$

$$M\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = D(1) = 0 = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$M\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = D(x) = 1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \Rightarrow M = \begin{bmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\end{bmatrix}$$

$\vdots$

General framework: Say we have $T: \underset{\dim n}{V} \longrightarrow \underset{\dim m}{W}$,
a basis $\mathcal{B}$ for V,
and a basis $\bar{\mathcal{B}}$ for W.

Pairs of inverse LTs:

$$V \underset{T^C_{\mathcal{B}}}{\overset{T^P_{\mathcal{B}}}{\rightleftarrows}} \mathbb{R}^n \qquad\qquad W \underset{T^C_{\bar{\mathcal{B}}}}{\overset{T^P_{\bar{\mathcal{B}}}}{\rightleftarrows}} \mathbb{R}^m$$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "T^P_{\mathcal{B}}"'] \arrow[rrr, bend right] & V \arrow[r, "T"] & W \arrow[r, "T^C_{\bar{\mathcal{B}}}"'] & \mathbb{R}^m \end{tikzcd}$$

A LT $\mathbb{R}^n\to\mathbb{R}^m$, which can be described by an $m\times n$ matrix

That matrix is $M_{T,\mathcal{B}\to\bar{\mathcal{B}}}$.

ex. Let V be the set of functions $f(x)$ such that
$f''(x) = -f(x)$.

This is a subspace of $F(\mathbb{R})$:

$0\in V$ ✓

If $f, g\in V$, then $(f+g)'' = f''+g'' = -f-g = -(f+g)$. ✓

If $f\in V$, $c\in\mathbb{R}$, then $(cf)'' = c\cdot f'' = c(-f) = -cf$. ✓

This vector space is 2-dimensional and has a basis $\{\sin(x), \cos(x)\} = \mathcal{B}$

Any $f \in V$ is of the form $a\sin(x) + b\cos(x)$.

In this case, f has coordinates $\begin{bmatrix} a \\ b \end{bmatrix}$

Let's consider the LT $D: V \to V$

given by $D(f) = f'$.

(If $f \in V$, then $D(f) = f'$

$D(f)'' = (f')'' = f''' = (f'')' = (-f)' = -f' = -D(f)$.)

Then $M_{D, \mathcal{B} \to \mathcal{B}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = D(\sin(x)) = \cos(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$M_{D, \mathcal{B} \to \mathcal{B}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = D(\cos(x)) = -\sin(x) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

So $M_{D, \mathcal{B} \to \mathcal{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "T_{\mathcal{B}}^{P}"'] \arrow[rrr, bend right] & V \arrow[r, "D"'] & V \arrow[r, "T_{\mathcal{B}}^{C}"'] & \mathbb{R}^2 \end{tikzcd}$$