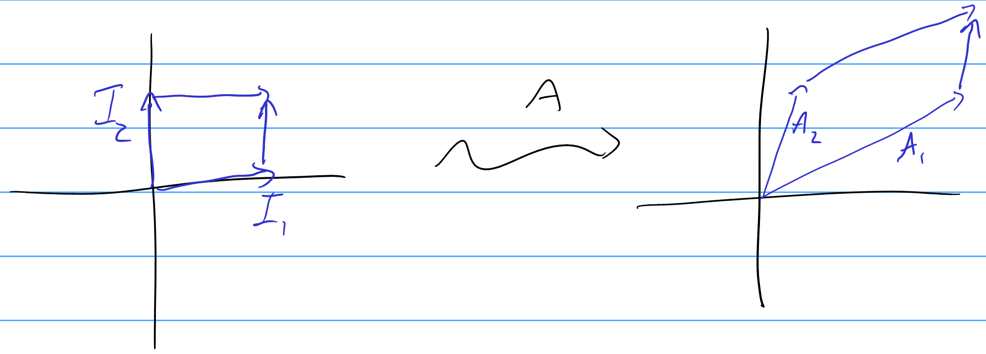
# Determinant and volume.

$A \in M(n,n) \Longleftrightarrow T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n.$

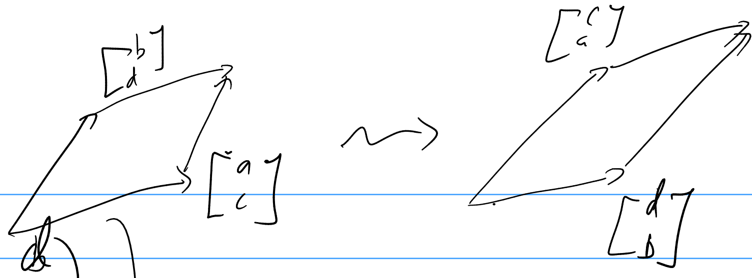



Main idea. If $n=2$, $|\det A| =$ area of the parallelogram generated by $A_1$ and $A_2$.
More generally, if $S \subseteq \mathbb{R}^2$ has area $x$, then $T_A(S)$ has area $x \cdot |\det A|$.
If $n=3$, $|\det A| =$ volume of the parallelepiped generated by $A_1$, $A_2$, and $A_3$.
If $S \subseteq \mathbb{R}^3$ has volume $x$, then $T_A(S)$ has volume $x \cdot |\det A|$.
Likewise in $\mathbb{R}^4$, $\mathbb{R}^5$, ... .

Proof. Slight variant of uniqueness thm.
$A \mapsto |\det(A)|$ is the unique function $D$ such that:
- $D(I) = 1$.
- Interchanging two rows of $A$ doesn't change $D(A)$,
- Scaling a row of $A$ by $c$ scales $D(A)$ by $|c|$.
- Doing a row operation $cR_i + R_j \longrightarrow R_j'$, $i \neq j$, doesn't change $D(A)$.

in $\mathbb{R}^2$ Now define $D(A) =$ area of parallelogram generated by $A_1$ and $A_2$.
- $D(I) =$ area of unit square $= 1$.

→ Interchange rows:

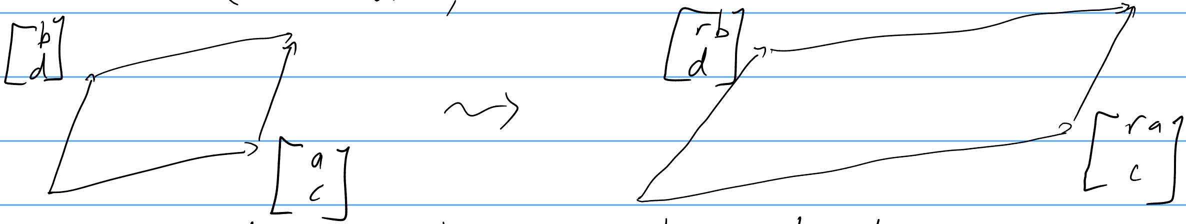



$$\left(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow \begin{pmatrix} c & d \\ a & b \end{pmatrix}\right)$$

This is just reflection across $y=x$, which doesn't change area.

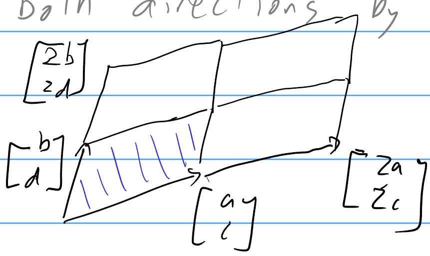
→ Scale row:

$$\left(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow \begin{pmatrix} ra & rb \\ c & d \end{pmatrix}, \text{ for some } r \in \mathbb{R}.\right)$$

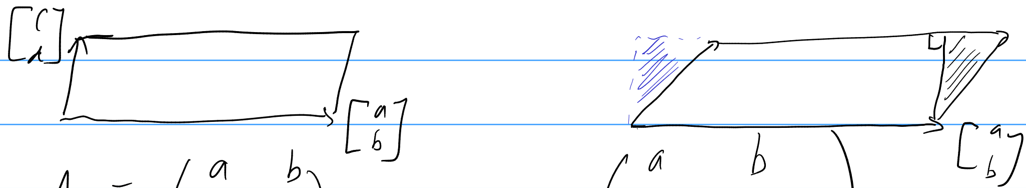



Scaling in one direction by $r$ also scales the area by $r$.

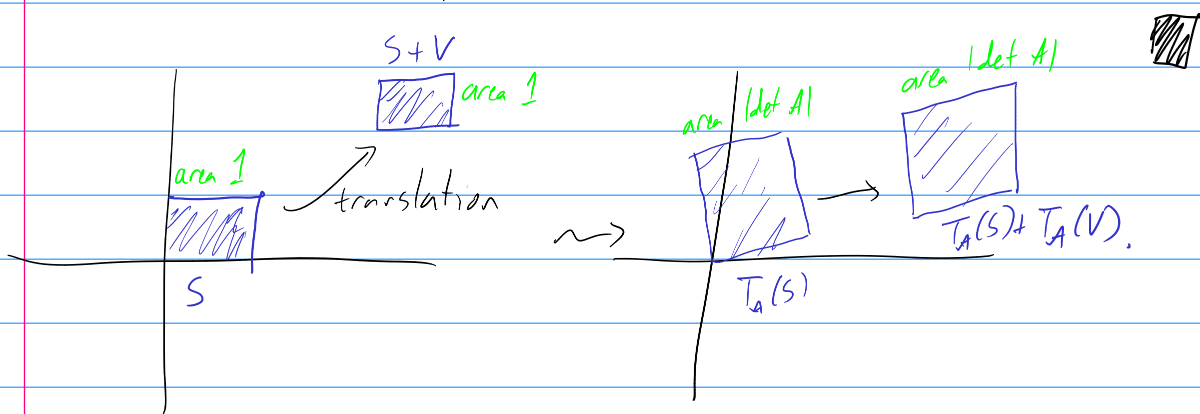
// Scaling in both directions by $r$ scales the area by $r^2$.

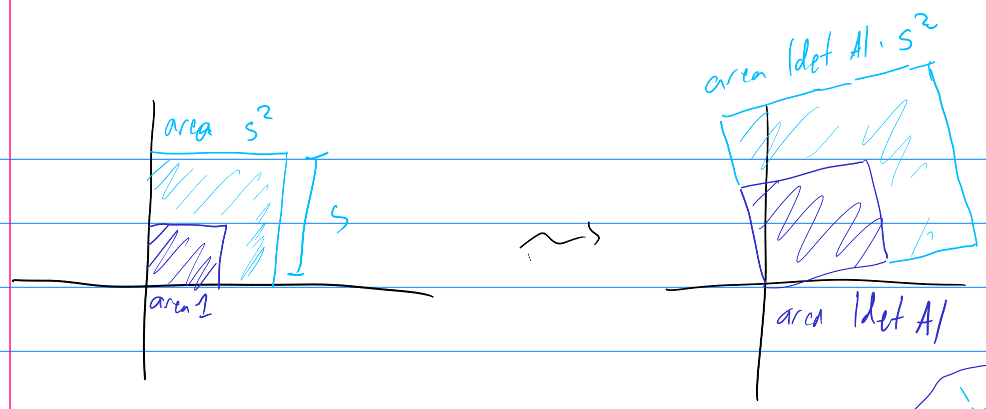


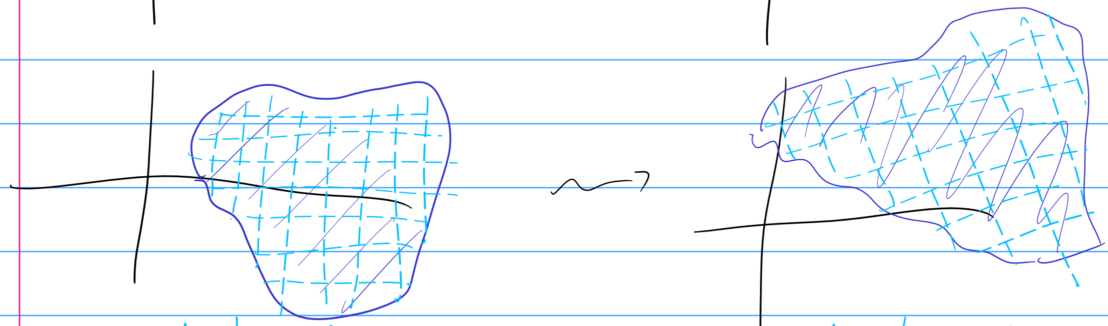

→ Type II row operations.




$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow \begin{pmatrix} a & b \\ c+ra & d+rb \end{pmatrix}$$





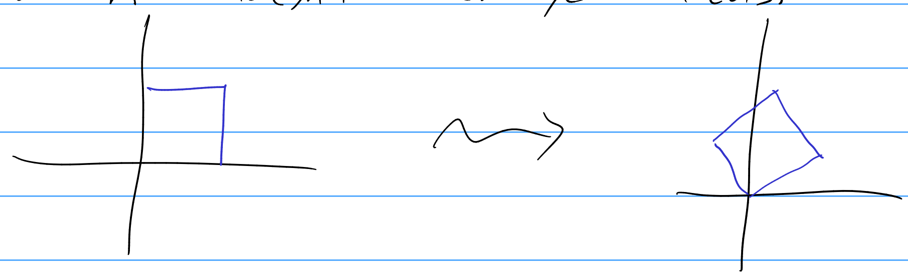
covered by $N$ squares of area $\varepsilon$. covered by $N$ parallelograms of size $|\det A|\cdot\varepsilon$.

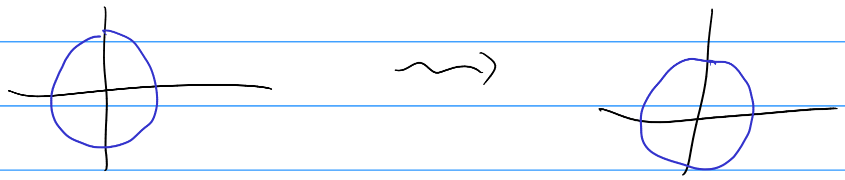
Conclusion: $T_A$ scales all areas by $|\det A|$!

ex. Rotation matrices: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = A.$

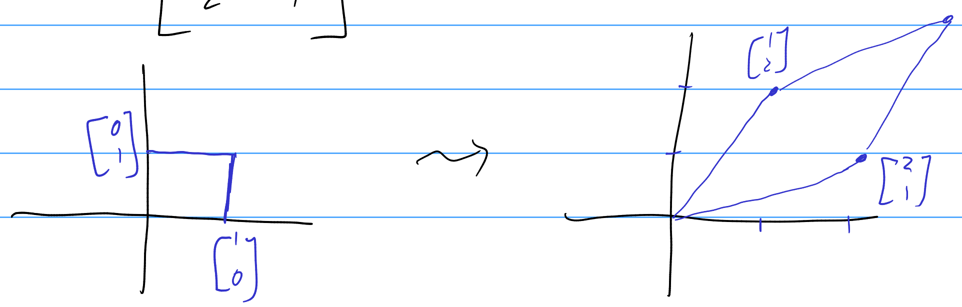
$\det A = \cos^2\theta + \sin^2\theta = 1.$

So $A$ doesn't change areas.

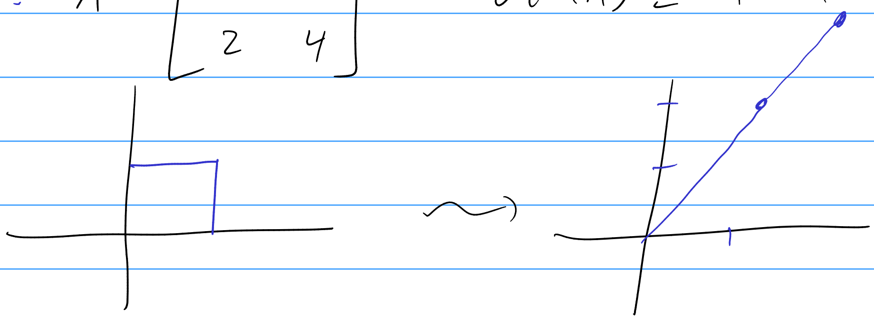




ex. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $\qquad \det A = 1 - 4 = -3.$



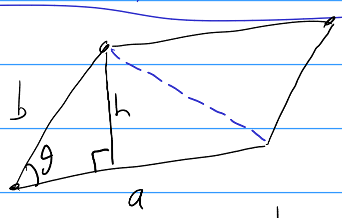
ex. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $\det(A) = 4 - 4 = 0.$



$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has length 1 $\rightsquigarrow$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ has length $\sqrt{1^2+2^2} = \sqrt{5}.$
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has length 1 $\rightsquigarrow$ $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ has length $\sqrt{2^2+4^2} = 2\sqrt{5}.$
$\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ has length $\sqrt{5}$ $\rightsquigarrow$ $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has length 0.

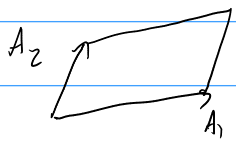
Conclusion: Nothing nice can be said about what a noninvertible matrix does to lengths.

Classical way of thinking about parallelograms.



area of $\triangle$ = $\frac{1}{2} a h$
area of $\square$ = $a h$.
$h = b \sin\theta$, so area of $\square$ $= ab \sin\theta$.

Cross product in $\mathbb{R}^3$: $V \times W$ = the vector $\perp$ to $V$ and $W$ with length $= |V| \cdot |W| \cdot \sin\theta$.



area $= |A_1 \times A_2|$.

Formula for the cross product:

$$V \times W = \begin{vmatrix} I_1 & I_2 & I_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{vmatrix}$$

← row of standard basis vectors
← rows of numbers.

$$= [V_2 W_3 - W_2 V_3,\ V_3 W_1 - W_3 V_1,\ V_1 W_2 - W_1 V_2]^t.$$

ex. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ area of $T_A(\square) = |A_1 \times A_2|. = \left| \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right|$

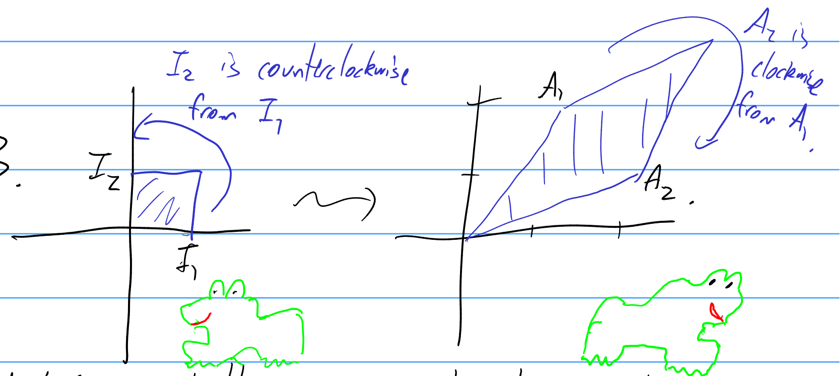
$$= \text{length} \begin{vmatrix} I_1 & I_2 & I_3 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \end{vmatrix} = \text{length}\left( \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} I_3. \right) = \text{abs}\left( \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \right).$$

## In $\mathbb{R}^3$:

volume of a parallelepiped with sides $U$, $V$, and $W$, is $|U \cdot (V \times W)|$. $= \begin{vmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{vmatrix}$.

## Orientation.

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\det(A) = -3$.




Conclusion: The sign of $\det(A)$ tells you whether it preserves or reverses orientation.