How to use (P)LU factorizations to solve systems.

ex. $A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ -5 & 7 & -8 & 3 \\ 0 & 4 & 4 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & 0 & -4 \end{bmatrix}}_{U}$

Solve $A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. $(AX = B)$

$LUX = B.$

Let $UX = W = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$.

Then the system becomes $LW = B$.

$$\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \begin{aligned} w_1 &= 1 \\ -5w_1 + w_2 &= 2 \\ 2w_2 + w_3 &= 3. \end{aligned}$$

$-5 + w_2 = 2 \Rightarrow w_2 = 7$

$14 + w_3 = 3 \Rightarrow w_3 = -11.$

$W = \begin{bmatrix} 1 \\ 7 \\ -11 \end{bmatrix}$

$$UX = W \Rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ -11 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ 2x_2 + 2x_3 + 3x_4 &= 7 \\ -4x_4 &= -11. \end{aligned}$$

$$x_1 - x_2 + 2x_3 = 1$$
$$2x_2 + 2x_3 + 3x_4 = 7$$
$$-4x_4 = -11.$$

$$x_4 = \frac{11}{4}$$

$$\Rightarrow 2x_2 + 2x_3 + \frac{33}{4} = 7 = \frac{28}{4}$$

$$2x_2 + 2x_3 = -\frac{5}{4}$$

$$x_2 = -x_3 - \frac{5}{8}$$

$$x_1 + x_3 + \frac{5}{8} + 2x_3 = 1$$

$$x_1 = \frac{3}{8} - 3x_3.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3/8 \\ -5/8 \\ 0 \\ 11/4 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} x_3$$

ex.

$$A = \begin{bmatrix} 2 & 1 & -3 & 4 \\ -4 & -2 & 1 & 3 \\ 7 & 1 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 7/2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 & -3 & 4 \\ 0 & -\frac{5}{2} & \frac{21}{2} & -13 \\ 0 & 0 & -5 & 11 \end{bmatrix}}_{U}$$

Solve

$$PLUX = AX = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Let $V = LUX$, $W = UX$.

Step 1: Solve $PV = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$

$$v_1 = 1$$
$$v_3 = 2$$
$$v_2 = 3$$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

Step 2: Solve $LW = V \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 7/2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$

$$w_1 = 1$$
$$\frac{7}{2}w_1 + 3w_2 = 3 \Rightarrow \frac{7}{2} + 3w_2 = 3 \Rightarrow w_2 = -\frac{1}{6}$$
$$-2w_1 + w_3 = 2 \Rightarrow -2 + w_3 = 2 \Rightarrow w_3 = 4.$$

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/6 \\ 4 \end{bmatrix}.$$

Step 3: Solve $UX = W$.

$$\begin{bmatrix} 2 & 1 & -3 & 4 \\ 0 & -\frac{5}{2} & \frac{21}{2} & -13 \\ 0 & 0 & -5 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/6 \\ 4 \end{bmatrix}$$

$$2x_1 + x_2 - 3x_3 + 4x_4 = 1$$
$$-\frac{5}{2}x_2 + \frac{21}{2}x_3 - 13x_4 = -\frac{1}{6}$$
$$-5x_3 + 11x_4 = 4$$

$$-5x_3 = 4 - 11x_4$$
$$x_3 = -\frac{4}{5} + \frac{11}{5}x_4$$

$$-\frac{5}{2}x_2 - \frac{42}{5} + \frac{231}{10}x_4 - 13x_4 = -\frac{1}{6}$$
$$-\frac{5}{2}x_2 + \frac{101}{10}x_4 = -\frac{1}{6} + \frac{42}{5} = -\frac{5}{30} + \frac{252}{30} = \frac{247}{30}$$
$$-\frac{5}{2}x_2 = -\frac{101}{10}x_4 + \frac{247}{30}$$
$$x_2 = \frac{101}{25}x_4 - \frac{247}{75}$$

$$2x_1 + x_2 - 3x_3 + 4x_4 = 1$$
$$2x_1 + \frac{101}{25}x_4 - \frac{247}{75} + \frac{12}{5} - \frac{33}{5}x_4 + 4x_4 = 1$$
$$2x_1 + \frac{101 - 165 + 100}{25}x_4 = 1 + \frac{247}{75} - \frac{180}{75} = \frac{75 + 247 - 180}{75}$$
$$= \frac{142}{75}$$

$$2x_1 + \frac{36}{25}x_4 = \frac{142}{75}$$
$$x_1 = -\frac{36}{50}x_4 + \frac{71}{75}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 71/75 \\ -247/75 \\ -4/5 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 71/75 \\ 101/25 \\ 11/5 \\ 1 \end{bmatrix}$$

1
12
x15
60
12
180