LU factorization – What is it? How do you find it?

LU factorization theorem

Some matrices A can be written

$$A = L \cdot U$$

L: lower triangular with 1s on the diagonal. U: REF

A: $m \times n$, L: $m \times m$, U: $m \times n$.

ex. $A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ -5 & 7 & -8 & 3 \\ 0 & 4 & 4 & 2 \end{bmatrix}$

Basic Row ops:
1. Swap two rows
2. Add a multiple of one row to a lower row.
3. Multiply a row by a scalar.

Want to row-reduce A using only operations of type 2.

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ -5 & 7 & -8 & 3 \\ 0 & 4 & 4 & 2 \end{bmatrix} \xrightarrow{5R_1 + R_2} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 2 & 2 & 3 \\ 0 & 4 & 4 & 2 \end{bmatrix} \xrightarrow{-2R_2 + R_3} \underbrace{\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & 0 & -4 \end{bmatrix}}_{U}$$

If we used a row operation $aR_j + R_i$ $(j < i)$

then $L_{ij} = -a$.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ -5 & 7 & -8 & 3 \\ 0 & 4 & 4 & 2 \end{bmatrix} = \overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & 0 & -4 \end{bmatrix}}$$

Q: For which A can we do this?

Q: Why does this procedure let us find L?

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ -5 & 7 & -8 & 3 \\ 0 & 4 & 4 & 2 \end{bmatrix} \underset{\xleftarrow{-5R_1+R_2}}{\xrightarrow{5R_1+R_2}} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 2 & 2 & 3 \\ 0 & 4 & 4 & 2 \end{bmatrix} \underset{\xleftarrow{2R_2+R_3}}{\xrightarrow{-2R_2+R_3}} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

Left-multiplying any matrix by the matrix

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

does the row operations $2R_2 + R_3$, then $-5R_1 + R_2$.