Row linearity.

Fix some $1 \le i \le n$ and: a $(i-1) \times n$ matrix A (can be 0)

$(n-i) \times n$ matrix B.

Think about matrices

$$\underbrace{\begin{bmatrix} A \\ \hline X \\ \hline B \end{bmatrix}}_{n} \begin{array}{l} \}\, i-1 \\ \leftarrow i^{th} \text{ row} \\ \}\, n-i \end{array}$$

Row additivity: $\det\left(\begin{bmatrix} A \\ \hline X+Y \\ \hline B \end{bmatrix}\right) = \det\left(\begin{bmatrix} A \\ \hline X \\ \hline B \end{bmatrix}\right) + \det\left(\begin{bmatrix} A \\ \hline Y \\ \hline B \end{bmatrix}\right).$

Row scalar: $\det\left(\begin{bmatrix} A \\ \hline cX \\ \hline B \end{bmatrix}\right) = c \cdot \det\left(\begin{bmatrix} A \\ \hline X \\ \hline B \end{bmatrix}\right)$

Zero row: $\det\left(\begin{bmatrix} A \\ \hline 0 \\ \hline B \end{bmatrix}\right) = 0.$

All together, these say: the function $X \longmapsto \det\left(\begin{bmatrix} A \\ \hline X \\ \hline B \end{bmatrix}\right)$

is a linear transformation $M(1,n) \longrightarrow \mathbb{R}$.

Slogan: "det is linear in each row separately"

"det is multilinear."

ex. Say that A is 3×3. What's $\det(2A)$?

$$\begin{vmatrix} 2a_{11} & 2a_{12} & \\ 2a_{21} & \ddots & \\ 2a_{31} & & \ddots \end{vmatrix} = 2\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} & \ddots & \\ 2a_{31} & & \ddots \end{vmatrix} = 4\begin{vmatrix} a_{11} & a_{12} & \\ a_{21} & \ddots & \\ 2a_{31} & & \ddots \end{vmatrix} = 8\begin{vmatrix} a_{11} & & \\ & \ddots & \\ & & \ddots \end{vmatrix}$$

$= 8\det(A).$

More generally: if A is $n\times n$, $\det(cA) = c^n \cdot \det(A)$.

ex.
$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{vmatrix} = 6$$

$1 \cdot 0 \cdot 3 = 0$ $\quad$ $1 \cdot 2 \cdot 3 = 6$ $\quad$ $1 \cdot 0 \cdot 3 = 0$

Proofs by induction

Write (n) for "the statement is true for $n \times n$ matrices".
We'll prove (1) and (n-1) implies (n).
To get (4), ... you know (1) ... (1) $\Rightarrow$ (2), so you know (2) ... (2) $\Rightarrow$ (3), so you know (3), ...

Proof of row scalar property.

(1) $A = [a]$.
$\det([ca]) = ca = c \cdot \det([a])$.

Assume (n-1), i.e. it's true for $(n-1) \times (n-1)$ matrices.
Let M be an $n \times n$ matrix, $M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$

Case 1: Scaling first row of M,
$$\begin{vmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ a_{21} & \cdots & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{vmatrix} = ca_{11}\det(M_{11}) - ca_{12}\det(M_{12}) + \cdots + (-1)^{1+n} ca_{1n}\det(M_{1n}),$$
$$= c\left(a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + \cdots + (-1)^{1+n} a_{1n}\det(M_{1n})\right)$$
$$= c\det(M).$$

Case 2: Scaling second row of M,
$$M' = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ ca_{21} & \cdots & ca_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}.$$
Then $\det(M') =$
$a_{11}\det(M'_{11}) - a_{12}\det(M'_{12}) + \cdots + (-1)^{1+n} a_{1n}\det(M'_{1n})$.

$M'_{11} = \begin{bmatrix} ca_{22} & \cdots & ca_{2n} \\ \vdots & & \vdots \\ a_{n2} & \cdots & a_{nn} \end{bmatrix} = M_{11}$ with its first row scaled by c.

By $(n-1)$, $\det(M'_{11}) = c\det(M_{11})$.
Likewise, $\det(M'_{1j}) = c\det(M_{1j})$.

$$\det(M') = a_{11}\det(M'_{11}) - a_{12}\det(M'_{12}) + \cdots + (-1)^{1+n} a_{1n}\det(M'_{1n})$$
$$= a_{11}\cdot c\det(M_{11}) - a_{12}\cdot c\cdot\det(M_{12}) + \cdots + (-1)^{1+n} a_{1n}\cdot c\cdot\det(M_{1n})$$
$$= c\left[a_{11}\det(M_{11}) - \cdots + (-1)^{1+n} a_{1n}\det(M_{1n})\right]$$
$$= c\cdot\det(M).$$

Case 3: Scaling other rows — essentially the same as Case 2. $\blacksquare$

Proof of row additivity

(1) For 1×1 matrices: $\det([x]+[y]) = \det([x+y]) = x+y$
$$= \det([x]) + \det([y]).$$

Assume $(n-1)$, i.e. we know row additivity for $(n-1)\times(n-1)$ matrices.

We want to prove row additivity for the i^{th} row of an $n\times n$ matrix.

Case 1: $i=1$.
$$\det\left(\begin{bmatrix} X+Y \\ \hline B \end{bmatrix}\right) = \begin{vmatrix} x_1+y_1 & x_2+y_2 & \cdots & x_n+y_n \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \vdots \\ b_{n1} & \cdots & \cdots & b_{nn} \end{vmatrix}$$
$$= (x_1+y_1)\det(B_{11}) - (x_2+y_2)\det(B_{12}) + \cdots + (-1)^{1+n}(x_n+y_n)\det(B_{1n}).$$
$$= x_1\det(B_{11}) - x_2\det(B_{12}) + \cdots + (-1)^{1+n} x_n\det(B_{1n})$$
$$+ y_1\det(B_{11}) - y_2\det(B_{12}) + \cdots + (-1)^{1+n} y_n\det(B_{1n}).$$
$$= \det\left(\begin{bmatrix} X \\ \hline B \end{bmatrix}\right) + \det\left(\begin{bmatrix} Y \\ \hline B \end{bmatrix}\right).$$

Case 2: $i=2$. $\begin{bmatrix} A \\ \hline X+Y \\ \hline B \end{bmatrix}$

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ x_1+y_1 & x_2+y_2 & \cdots & x_n+y_n \\ b_{31} & \cdots & \cdots & b_{3n} \\ \vdots & & & \vdots \\ b_{n1} & \cdots & \cdots & b_{nn} \end{vmatrix} = a_{11}\begin{vmatrix} x_2+y_2 & \cdots & x_n+y_n \\ b_{32} & & b_{3n} \\ \vdots & \ddots & \\ b_{n1} & & b_{nn} \end{vmatrix} - \cdots + (-1)^{1+n} a_{1n}\begin{vmatrix} x_1+y_1 & \cdots & x_{n-1}+y_{n-1} \\ b_{31} & \cdots & b_{3,n-1} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{n,n-1} \end{vmatrix}$$
$$= a_{11}\left(\begin{vmatrix} x_2 & \cdots & x_n \\ b_{32} & \cdots & b_{3n} \\ & \ddots & \end{vmatrix} + \begin{vmatrix} y_2 & \cdots & y_n \\ b_{32} & & b_{3n} \\ & \ddots & \end{vmatrix}\right) + \cdots$$

$$= a_{11}\begin{vmatrix} x_2 & \cdots & x_n \\ b_{32} & \cdots & b_{3n} \\ & \ddots & \\ & & b_{nn} \end{vmatrix} - a_{12}\begin{vmatrix} x_1 & x_3 & \cdots & x_n \\ b_{31} & b_{33} & & b_{3n} \\ & & \ddots & \\ & & & b_{nn} \end{vmatrix} + \cdots \Bigg] = \det\left(\begin{bmatrix} A \\ \hline x \\ \hline B \end{bmatrix}\right)$$

$$+ a_{11}\begin{vmatrix} y_2 & \cdots & y_n \\ b_{32} & \cdots & b_{3n} \\ & \ddots & \\ & & b_{nn} \end{vmatrix} - a_{12}\begin{vmatrix} y_1 & y_3 & \cdots & y_n \\ b_{31} & b_{33} & \cdots & b_{3n} \\ & & \ddots & \\ & & & b_{nn} \end{vmatrix} + \cdots \Bigg] = \det\left(\begin{bmatrix} A \\ \hline y \\ \hline B \end{bmatrix}\right)$$

Other cases very similar to case 2. $\blacksquare$