Determinants & row operations.

I : Interchange two rows $\Rightarrow$ determinant gets multiplied by (-1).
III : Multiply a row by c $\Rightarrow$ determinant gets multiplied by c.
II : Add a multiple of row i $\Rightarrow$ determinant doesn't change.
to row j $(i \neq j)$.

Why?

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} 1 & 4 & 5 \\ 1 & 6 & 11 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ \hline 1+0 & 4+2 & 5+6 \\ \hline 0 & 0 & 3 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 4 & 5 \\ 1 & 6 & 11 \\ 0 & 0 & 3 \end{vmatrix} = \underbrace{\begin{vmatrix} 1 & 4 & 5 \\ 1 & 4 & 5 \\ 0 & 0 & 3 \end{vmatrix}}_{= 0} + \underbrace{\begin{vmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{vmatrix}}_{= \det(A)}$$ by row additivity.

$$= \det(A).$$

$$A = \begin{bmatrix} B_1 \\ \hline R_j \\ \hline B_2 \end{bmatrix} \begin{array}{l} \} \; j-1 \text{ rows} \\ \leftarrow j^{th} \text{ row} \\ \} \; n-j \text{ rows} \end{array}$$

$$\begin{vmatrix} B_1 \\ \hline cR_i + R_j \\ \hline B_2 \end{vmatrix} = \begin{vmatrix} B_1 \\ \hline cR_i \\ \hline B_2 \end{vmatrix} + \begin{vmatrix} B_1 \\ \hline R_j \\ \hline B_2 \end{vmatrix}$$ (row additivity)

$$= c \begin{vmatrix} B_1 \\ \hline R_i \\ \hline B_2 \end{vmatrix} + \det(A)$$ (row scalar)

$$= 0 + \det(A)$$ (equal rows)

$$= \det(A).$$

Suppose that A is invertible.
Then $RREF(A) = I$.
We can get from A to $RREF(A)$ by applying row operations, and each row operation multiplies the determinant by a <u>nonzero number</u>.

So: $\det(I) = c \cdot \det(A)$, $c \neq 0$
$$1 = c \cdot \det(A), \quad c \neq 0.$$
Thus, $\det(A) \neq 0$.

Suppose that A is not invertible. (but still $n \times n$)
Then $RREF(A)$ has a zero row,
So $\det(RREF(A)) = 0$.
But $\det(RREF(A)) = c \cdot \det(A)$, for some $c \neq 0$.
So $\det(A) = 0$.

Conclusion: $\det(A) \neq 0 \iff A$ is invertible.