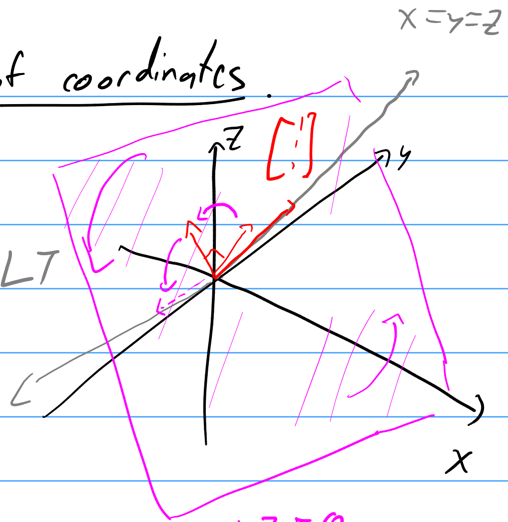
More linear transformations in terms of coordinates

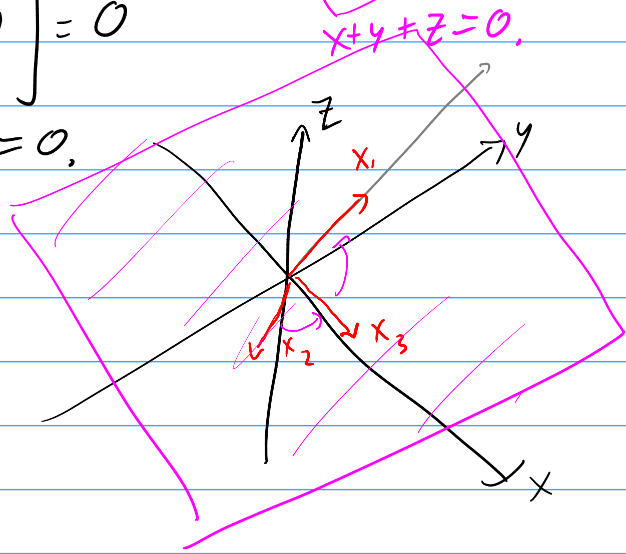


<u>ex</u>. Rotation by $90°$ around $x=y=z$.

Strategy: First find the matrix for this LT with respect to a <u>different</u> basis, then convert to standard coordinates.

$$\begin{bmatrix}x\\y\\z\end{bmatrix} \perp \begin{bmatrix}1\\1\\1\end{bmatrix} \text{ iff } \begin{bmatrix}x\\y\\z\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}=0$$

$$x+y+z=0.$$



$$\underset{X_1}{\begin{bmatrix}1\\1\\1\end{bmatrix}},\ \underset{X_2}{\begin{bmatrix}1\\-1\\0\end{bmatrix}},\ \underset{X_3}{\begin{bmatrix}1\\1\\-2\end{bmatrix}}$$

The linear transformation T sends

$$X_1 \longrightarrow X_1$$
$$X_2 \longrightarrow X_3$$
$$X_3 \longrightarrow -X_2.$$

If the B-coordinates of a point are $[a_1, a_2, a_3]^T$, then its image under T has B-coords $[a_1, -a_3, a_2]^T$

So $M_{T,B} = \begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}$

$$\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}=\begin{bmatrix}a_1\\-a_3\\a_2\end{bmatrix}.$$

Note: The i^{th} column of the matrix says where the i^{th} B-basis vector goes, in B-coordinates.

$$\underline{M_{T,\mathcal{B}}} = P_{\mathcal{B}}^{-1}\, \underline{M_T}\, P_{\mathcal{B}}.$$

$$\begin{aligned} P_{\mathcal{B}} M_{T,\mathcal{B}} &= P_{\mathcal{B}} P_{\mathcal{B}}^{-1} M_T P_{\mathcal{B}} \\ &= I\, M_T P_{\mathcal{B}} \\ &= M_T P_{\mathcal{B}} \end{aligned}$$

$$\begin{aligned} P_{\mathcal{B}} M_{T,\mathcal{B}} P_{\mathcal{B}}^{-1} &= M_T P_{\mathcal{B}} P_{\mathcal{B}}^{-1} \\ &= M_T I \\ &= M_T \end{aligned}$$

Conclusion: $P_{\mathcal{B}} M_{T,\mathcal{B}} P_{\mathcal{B}}^{-1} = M_T$

(Compare: $P_{\mathcal{B}}^{-1} M_T P_{\mathcal{B}} = M_{T,\mathcal{B}}$)

Our $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}$. So $P_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -1 & -3 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2/(-2)} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -1 & -3 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_2+R_3]{-R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -\frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right]$$

$$\xrightarrow{R_3/(-3)} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \end{array}\right] \xrightarrow{-R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \end{array}\right] = \left[\, I \mid P_{\mathcal{B}}^{-1} \right]$$

$$M_T = P_{\mathcal{B}} M_{T,\mathcal{B}} P_{\mathcal{B}}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \end{bmatrix} = \boxed{\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ 1 & 0 & 0 \\ -\frac{2}{3} & \frac{4}{3} & \frac{1}{3} \end{bmatrix}}.$$