Practice Midterm 2 MA 35100 – solutions

1. Let $\mathcal{P}_3$ be the vector space of cubic polynomials, with the ordered basis

$$\mathcal{B} = \{(x-1)^3, (x-1)^2, (x-1), 1\}.$$

Let $\mathcal{P}_2$ be the vector space of quadratic polynomials, with the ordered basis

$$\overline{\mathcal{B}} = \{x^2, x, 1\}.$$

Let $D : \mathcal{P}_3 \to \mathcal{P}_2$ be the linear transformation $D(f) = f'$. Write the matrix for D in terms of the two bases given.

Solution: We have

$$\begin{aligned} D(x-1)^3 &= 3(x-1)^2 = 3x^2 - 6x + 3 \\ D(x-1)^2 &= 2(x-1) = 2x - 2 \\ D(x-1) &= 1 \\ D(1) &= 0 \end{aligned}$$

These equations express the images of the basis vectors in $\mathcal{B}$ under D as linear combinations of the basis vectors of $\overline{\mathcal{B}}$. The corresponding matrix is

$$M_{D,\mathcal{B}\to\overline{\mathcal{B}}} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ -6 & 2 & 0 & 0 \\ 3 & -2 & 1 & 0 \end{pmatrix}.$$

(Note: a previous version of this document had this matrix completely wrong!) Note that multiplying this matrix by a vector such as $[1,0,0,0]^T$ – the coordinate vector of $(x-1)^3$ in the basis $\mathcal{B}$ – gives the coordinates for D applied to this function.

2. Suppose that A has the LU factorization

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

Solve the system of equations

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Solution: We don't need to find A to solve this. Instead, write the system as

$$AX = LUX = Y.$$

Define $W = UX = [w_1, w_2, w_3]^T$, and first solve

$$LW = Y.$$

It's easy to do this by substitution, since L is lower triangular. I got

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ 3y_1 + y_2 \\ -6y_1 - 2y_2 + y_3. \end{pmatrix}.$$

Then solve

$$UX = W.$$

I got

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}w_1 + \frac{1}{2}w_2 - \frac{3}{4}w_3 \\ w_2 - w_3 \\ \frac{1}{2}w_3 \end{pmatrix} = \begin{pmatrix} -11y_1 - \frac{5}{2}y_2 - \frac{1}{2}y_3 \\ 9y_1 + 3y_2 - y_3 \\ -3y_1 - y_2 + \frac{1}{2}y_3 \end{pmatrix}.$$

3. Let A be an $n \times n$ square matrix. Which of the following *does not* imply that A is invertible?

   (a) A has n linearly independent rows.

   (b) The system of equations $AX = 0$ has a unique solution.

   (c) For any B in $\mathbb{R}^n$, the system of equations $AX = B$ has at least one solution.

   (d) A is in reduced row echelon form.

   (e) A is a permutation matrix.

   Solution: (d).

   There are RREF matrices which are not invertible – for example, the zero matrix is RREF. (a) implies A is rank n, so invertible. (b) implies the nullity of A is zero, so its rank is n, so it's invertible. (c) implies the column space of A is equal to $\mathbb{R}^n$, which implies that the rank is n, so A is invertible. (e) is a little trickier: you could observe that any permutation matrix has n linearly independent rows; or that they're row equivalent to the identity matrix and thus rank n; or that any rearrangement of rows can be reversed, giving an inverse to the corresponding permutation matrix.

4. Suppose that A is a 3×4 matrix of rank 2, and B is a 4×5 matrix of rank 3. Which of the following *could be* true about AB?

   (a) AB has rank 1.

   (b) AB has rank 3.

   (c) AB is invertible.

   (d) AB is a permutation matrix.

   (e) AB is not defined.

   Solution: (a).

   The key fact here is that $\text{rank}(AB) \leq \text{rank}(A)$ and $\text{rank}(B)$. Thus, AB has rank at most 2. It's possible to make it rank 1 (can you find an example?). Note that AB is always defined since the number of columns of A equals the number of rows of B, and that it's a 3×5 matrix, which means it can't be invertible or a permutation matrix.

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by rotating $\pi/4$ radians counterclockwise about the origin, and then applying the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Which of the following is a basis for the space of vectors X such that $T(X) = 0$?

   (a) $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$

   (b) $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$

   (c) $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$

   (d) $\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$

   (e) $\left(\begin{smallmatrix} 1 \\ \sqrt{3} \end{smallmatrix}\right)$

   Solution: (c).

   The matrix for the rotation is $\left(\begin{smallmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{smallmatrix}\right)$. So the composition of the two operations has matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 \end{pmatrix}.$$

   This is the matrix of T, and the subspace referred to is the nullspace of this matrix. You can check that only vector (c) is actually in the nullspace.

   You can also think about this geometrically. The matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ projects everything onto the x-axis, so it sends only those vectors with zero x-coordinate to 0. What vectors have zero x-coordinate after you rotate by $\pi/4$? The ones on the line $y = x$. So the subspace is the line $y = x$, i.e., the span of $[1, 1]^T$.

6. Which of the following matrices *does not* have an LU factorization?

(a) $\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & 3 \\ 2 & 0 & 1 \\ 1 & 1 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 1 & 3 \\ -5 & 1 & 4 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$

(e) All of the above matrices have an LU factorization.

Solution: (b).

The way to do this is to try to find the LU factorization, by row-reducing the matrices only using type-2 operations. That is, only use row operations that involve adding a multiple of a higher row to a lower row. In the case of (b), we'd start with the row operations

$$-2R_1 + R_2 \to R_2,$$

$$-R_1 + R_3 \to R_3,$$

giving

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & -5 \\ 0 & 1 & 1. \end{pmatrix}$$

The matrix can't be reduced any further without rearranging rows.

Likewise, you can check that (c) *does* have an LU factorization. Also, (a) is already in REF (so it has the LU factorization where L is the identity and the given matrix is U), and (b) has the LU factorization where the given matrix is L and U is the identity.

7. A certain matrix A has

$$A^{-1} = \begin{pmatrix} 3 & 0 & 3 \\ 1 & 2 & -1 \\ 2 & 3 & 0 \end{pmatrix}$$

Which of the following is the solution to

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}?$$

(a) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(b) $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

(c) $\begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix}$

(d) $\begin{pmatrix} 1 \\ -1/3 \\ -2/3 \end{pmatrix}$

(e) No solution exists.

Solution: (c).

You don't have to find the matrix A! The equation

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

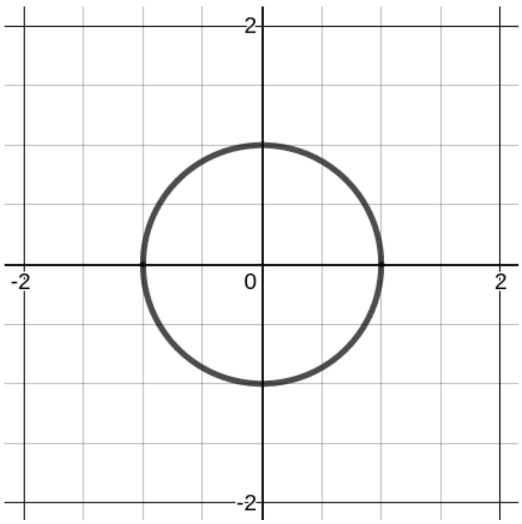
is equivalent to

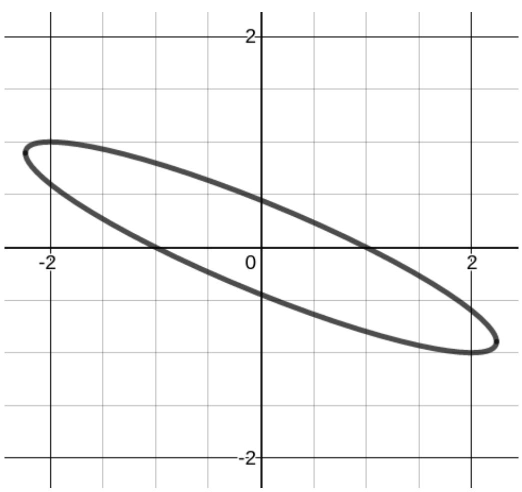
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

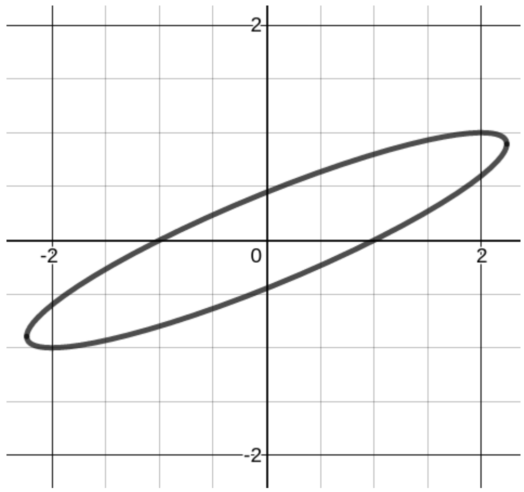
Just left-multiply both equations by A^{-1}, and use properties of inverses. You can then compute $A^{-1}[1, 1, 1]^T$ using the definition of matrix multiplication.

8. Which of the following is the image of the unit circle, $x^2 + y^2 = 1$, under the linear transformation given by the matrix

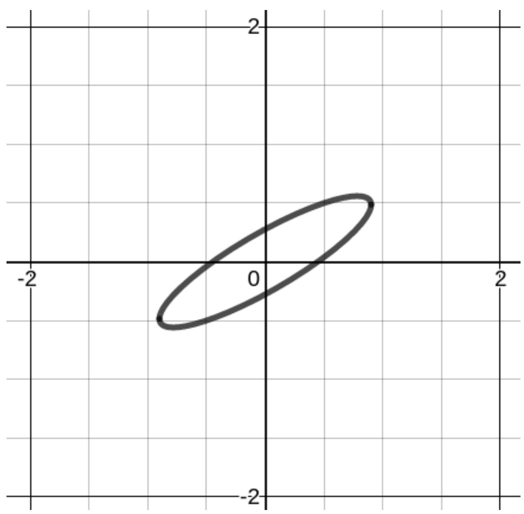
$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}?$$

(a) 

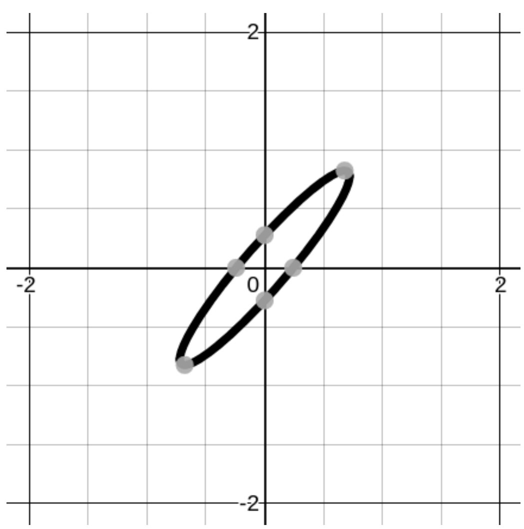
(b) 



(c)



(d)



(e)

Solution: (b).

The standard basis vectors $[1, 0]^T$ and $[0, 1]^T$ are on the unit circle, and get sent to $[1, 0]^T$ and $[-2, 1]^T$. So these points have to be on the image of the unit circle. The only graph that passes through these points is (b). (If you want, you can check even more points on the unit circle, like $[1/\sqrt{2}, 1/\sqrt{2}]^T$, by multiplying the given matrix by them.)

9. Suppose that A is a 3×4 matrix. Let A' be the matrix by doing the following row operation to A:

add (-2) times row 2 to row 1.

Which of the following is true about the matrix B such that $BA = A'$?

(a) B is lower triangular.

(b) B is upper triangular.

(c) B is a permutation matrix.

(d) B is a diagonal matrix.

(e) No such B exists.

Solution: (b).

This is probably the trickiest problem on the test, since it uses ideas I've never talked about explicitly. The matrix B is

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

You can check that multiplying B by *any* 3×4 matrix – in fact, by any matrix with 3 rows – does the prescribed row operation.

How would you find B? Well, you could write down some simple matrices A, figure out what A' is, and try to solve for the entries of B using the equation $BA = A'$. For example, the equation

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

has enough information to solve for the entries of B. You could even make the entries of A arbitrary – a_{11} and so on – and solve for the entries of B.

You may also remember that row operations of the form "add a multiple of a higher row to a lower row" are encoded in left-multiplication by *lower* triangular matrices. Well, row operations of the form "add a multiple of a lower row to a higher row" are encoded left-multiplication by *upper* triangular matrices. Can you see why?

10. Which of the following matrices, viewed as a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$, sends some rectangle to a line segment?

(a) $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 3 \\ 3 & 6 \end{pmatrix}$ (**Note:** This originally said $\begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}$.)

Solution: (d) ((e) was also correct in the original version.)

Here's how I think about this. If a linear transformation sends a rectangle to a line segment, then it can't be invertible. Since all these linear transformations are given by 2×2 matrices, not being invertible is equivalent to having rank 0 or 1. Rank 0 would mean the zero matrix, and rank 1 means that one of the rows is a scalar multiple of the other one. Now we can observe that only (d) (and (e) in the original version) have this property. If you have another way of doing this problem, write something about it in the forum!