Diagonalization

Say that A is an $n \times n$ matrix representing $T_A : \mathbb{R}^n \to \mathbb{R}^n$.
Suppose that there's a basis $\mathcal{B} = \{X_1, \ldots, X_n\}$ for $\mathbb{R}^n$ such that each X_i is an eigenvector for A, with eigenvalue λ_i.

What's the matrix for T_A in terms of the basis $\mathcal{B}$?

$$T_A(X_i) = A \cdot X_i = \lambda_i X_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix} \text{ in } \mathcal{B}\text{-coords,}$$

So the matrix for T_A in terms of $\mathcal{B}$ is

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}, \quad \text{a diagonal matrix.}$$

$$A = P_{\mathcal{B}} \cdot D \cdot C_{\mathcal{B}}$$

$$\boxed{A = P_{\mathcal{B}} \cdot D \cdot P_{\mathcal{B}}^{-1}} \quad \text{where } P_{\mathcal{B}} = \begin{bmatrix} | & & | \\ X_1 & \cdots & X_n \\ | & & | \end{bmatrix}.$$

Def. A is *diagonalizable* if there's a basis for $\mathbb{R}^n$ consisting of eigenvectors for A.

ex.

$$A = \begin{bmatrix} -2 & 0 & 1 \\ -2 & -1 & 2 \\ 2 & -1 & 0 \end{bmatrix} \qquad \lambda_1 = -2 \quad X_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad \lambda_2 = -1 \quad X_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \qquad \lambda_3 = 0 \quad X_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

$$P_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-R_2+R_3]{-R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 2 & 2 & -1 & 1 \end{array}\right] \xrightarrow{R_3/2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1/2 & 1/2 \end{array}\right]$$

$$\xrightarrow{-R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1/2 & -1/2 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1/2 & 1/2 \end{array}\right]$$

So $C_B = \begin{bmatrix} 2 & -1/2 & -1/2 \\ -2 & 1 & 0 \\ 1 & -1/2 & 1/2 \end{bmatrix}$

<u>Conclusion:</u> $A = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix}}_{P_B} \underbrace{\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 2 & -1/2 & -1/2 \\ -2 & 1 & 0 \\ 1 & -1/2 & 1/2 \end{bmatrix}}_{P_B^{-1}}$.

<u>1 Application:</u>
$$\begin{aligned} A^{500} &= (P_B D P_B^{-1})^{500} \\ &= P_B D \cancel{P_B^{-1}} \cdot \cancel{P_B} D \cancel{P_B^{-1}} \cdot \ldots \cdot \cancel{P_B} D P_B^{-1} \\ &= P_B D^{500} P_B^{-1} \\ &= P_B \begin{bmatrix} (-2)^{500} & 0 & 0 \\ 0 & (-1)^{500} & 0 \\ 0 & 0 & 0 \end{bmatrix} P_B^{-1}. \end{aligned}$$

When is a matrix diagonalizable?

<u>Prop.</u> If A has n distinct eigenvalues, then it's diagonalizable.

<u>Proof</u>. Each eigenvalue has at least one eigenvector, and these are LI for distinct eigenvalues. So there's a basis for $\mathbb{R}^n$ consisting of one eigenvector for each eigenvalue. $\blacksquare$

ex $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. eigenvalues: $\lambda = 1$ (double)

eigenvectors: $X = c\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

There's no basis of $\mathbb{R}^2$ consisting of eigenvectors for S. So S is not diagonalizable.

ex. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad P_A(\lambda) = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix}$

$= (1-\lambda)(4-\lambda) - 6 = \lambda^2 - 5\lambda - 2$

$\lambda = \dfrac{5 \pm \sqrt{25+8}}{2} = \dfrac{5}{2} \pm \dfrac{\sqrt{33}}{2}$

$\lambda = \frac{5}{2} + \frac{\sqrt{33}}{2}$: $A - \lambda I = \begin{bmatrix} -\frac{3}{2} - \frac{\sqrt{33}}{2} & 2 \\ 3 & \frac{3}{2} - \frac{\sqrt{33}}{2} \end{bmatrix}$.

Solutions are solutions to $\left(-\frac{3}{2} - \frac{\sqrt{33}}{2}\right)x + 2y = 0$.

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{3}{2} + \frac{\sqrt{33}}{2} \end{bmatrix}$

Check: $\begin{bmatrix} -\frac{3}{2} - \frac{\sqrt{33}}{2} & 2 \\ 3 & \frac{3}{2} - \frac{\sqrt{33}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ \frac{3}{2} + \frac{\sqrt{33}}{2} \end{bmatrix} = \begin{bmatrix} \cancel{2\left(-\frac{3}{2} - \frac{\sqrt{33}}{2}\right) + 2\left(\frac{3}{2} + \frac{\sqrt{33}}{2}\right)} \\ 6 + \frac{9}{4} + \frac{3\sqrt{33}}{4} - \frac{3\sqrt{33}}{4} - \frac{33}{4} \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\lambda = \frac{5}{2} - \frac{\sqrt{33}}{2}$ $\quad A - \lambda I = \begin{bmatrix} -\frac{3}{2} + \frac{\sqrt{33}}{2} & 2 \\ 3 & \frac{3}{2} + \frac{\sqrt{33}}{2} \end{bmatrix}$

$\left(-\frac{3}{2} + \frac{\sqrt{33}}{2}\right)x + 2y = 0$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{3}{2} - \frac{\sqrt{33}}{2} \end{bmatrix}$.

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ \frac{3}{2} + \frac{\sqrt{33}}{2} \end{bmatrix}, \begin{bmatrix} 2 \\ \frac{3}{2} - \frac{\sqrt{33}}{2} \end{bmatrix} \right\}$$

$\lambda = \frac{5}{2} + \frac{\sqrt{33}}{2}$ $\quad$ $\lambda = \frac{5}{2} - \frac{\sqrt{33}}{2}$

$$P_{\mathcal{B}} = \begin{bmatrix} 2 & 2 \\ \frac{3}{2} + \frac{\sqrt{33}}{2} & \frac{3}{2} - \frac{\sqrt{33}}{2} \end{bmatrix}$$

$$\det(P_{\mathcal{B}}) = 2\left(\frac{3}{2} - \frac{\sqrt{33}}{2}\right) - 2\left(\frac{3}{2} + \frac{\sqrt{33}}{2}\right) = -2\sqrt{33}$$

$$C_{\mathcal{B}} = P_{\mathcal{B}}^{-1} = \frac{1}{-2\sqrt{33}} \begin{bmatrix} \frac{3}{2} - \frac{\sqrt{33}}{2} & -2 \\ -\frac{3}{2} - \frac{\sqrt{33}}{2} & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 2 \\ \frac{3}{2} + \frac{\sqrt{33}}{2} & \frac{3}{2} - \frac{\sqrt{33}}{2} \end{bmatrix} \begin{bmatrix} \frac{5}{2} + \frac{\sqrt{33}}{2} & 0 \\ 0 & \frac{5}{2} - \frac{\sqrt{33}}{2} \end{bmatrix} \left(\frac{1}{-2\sqrt{33}} \begin{bmatrix} \frac{3}{2} - \frac{\sqrt{33}}{2} & -2 \\ -\frac{3}{2} - \frac{\sqrt{33}}{2} & 2 \end{bmatrix} \right)$$