Linear transformations in terms of coordinates.

Let $\mathcal{B} = \{X_1, \dots, X_n\}$ be an ordered basis for $\mathbb{R}^n$.
If $Y = a_1X_1 + \cdots a_nX_n$, then we say
Y has **coordinates** $[a_1, \dots, a_n]^T$ **with respect to $\mathcal{B}$**

We have the **point matrix** $P_{\mathcal{B}} = \begin{bmatrix} | & & | \\ X_1 & \cdots & X_n \\ | & & | \end{bmatrix}$

and the **coordinate matrix** $C_{\mathcal{B}} = P_{\mathcal{B}}^{-1}$.

$$\mathcal{B}\text{-coordinates} \underset{C_{\mathcal{B}}}{\overset{P_{\mathcal{B}}}{\rightleftarrows}} \text{point}$$

<u>ex</u>. Say $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Then $P_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

So, the point with $\mathcal{B}$-coordinates $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+1-2 \\ 1-1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Another example: $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ = the 2nd basis vector of $\mathcal{B}$.

$$C_{\mathcal{B}} = P_{\mathcal{B}}^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_3+R_1]{-R_3+R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]. \qquad \text{So } C_{\mathcal{B}} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

So the coordinates of $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ in terms of $\mathcal{B}$ are:

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0-2-3 \\ 2-3 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix}$$

Say we have a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$.
Write M_T for the matrix of T (with respect to the standard basis)

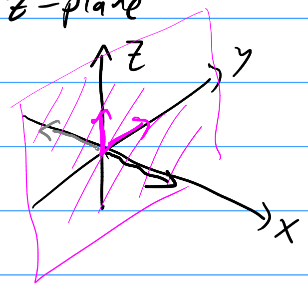
Say $\mathcal{B}$ is an ordered basis for $\mathbb{R}^n$.
There's a matrix $M_{T,\mathcal{B}}$ such that, if $Q = [a_1, \dots, a_n]^T$ is the $\mathcal{B}$-coordinates of a point $P \in \mathbb{R}^n$, then $M_{T,\mathcal{B}} Q$ is the $\mathcal{B}$-coordinates of the point $T(P)$.

How do we find the matrix $M_{T,\mathcal{B}}$?

$$\underset{(\mathcal{B}\text{-coords})}{Q} \longrightarrow \underset{(\text{point in } \mathbb{R}^n)}{P = P_{\mathcal{B}} Q} \longrightarrow \underset{(\text{point in } \mathbb{R}^n)}{M_T P_{\mathcal{B}} Q} \longrightarrow \underset{(\mathcal{B}\text{-coords})}{C_{\mathcal{B}} M_T P_{\mathcal{B}} Q}$$

So: $M_{T,\mathcal{B}} = C_{\mathcal{B}} M_T P_{\mathcal{B}} = P_{\mathcal{B}}^{-1} M_T P_{\mathcal{B}}.$

ex. In standard coords, reflection across the yz-plane has the matrix $M_T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$



How about in the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$?

The formula: $M_{T,\mathcal{B}} = C_{\mathcal{B}} M_T P_{\mathcal{B}}$

$$= \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So, for example, if we take the point with $\mathcal{B}$-coordinates $[1, 1, 1]^T$, its reflection across the yz-plane has $\mathcal{B}$-coords

$$M_{T,\mathcal{B}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}.$$