How the LU factorization can fail

ex. $A = \begin{bmatrix} 2 & 1 & -3 & 4 \\ -4 & -2 & 1 & 3 \\ 7 & 1 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 & -3 & 4 \\ -4 & -2 & 1 & 3 \\ 7 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{2R_1 + R_2} \begin{bmatrix} 2 & 1 & -3 & 4 \\ 0 & 0 & -5 & 11 \\ 7 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{-\frac{7}{2}R_1 + R_3} \begin{bmatrix} 2 & 1 & -3 & 4 \\ 0 & 0 & -5 & 11 \\ 0 & -\frac{5}{2} & \frac{21}{2} & -13 \end{bmatrix}$$

Problem: can't finish row-reducing without swapping rows

Say that a matrix is "type 2" if it can be row-reduced using only operations of the form:

add a multiple of a higher row to a lower row.

If A is type 2 $\Rightarrow$ A has an LU factorization

Type 2 row operations are encoded in lower triangular matrices with 1s on diagonal.

ex. $L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

Then $LB = \begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 5 & 6 \end{bmatrix}$ = the result of the row operation $-3R_1 + R_2$ applied to B.

Def. A permutation matrix is a square matrix with the property that every row and column has a single 1 and every other entry is 0.

ex. $P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$

Then $PC = \begin{bmatrix} 5 & 6 \\ 7 & 8 \\ 3 & 4 \\ 1 & 2 \end{bmatrix}$

Exercise: Think of a possible rearrangement of the rows of a matrix, and then find a permutation matrix P such that left-multiplying by P performs that rearrangement.

PLU factorization. Every matrix A can be written

$$\underset{}{A} = \underbrace{P}_{\substack{\text{permutation} \\ \text{matrix}}} \cdot \underbrace{L}_{\substack{\text{lower triangular,} \\ \text{with 1s on} \\ \text{diagonal}}} \cdot \underbrace{U}_{\text{REF.}}$$

ex. $A = \begin{bmatrix} 2 & 1 & -3 & 4 \\ -4 & -2 & 1 & 3 \\ 7 & 1 & 0 & 1 \end{bmatrix}$ $A' = \begin{bmatrix} 2 & 1 & -3 & 4 \\ 7 & 1 & 0 & 1 \\ -4 & -2 & 1 & 3 \end{bmatrix}$

$$A' = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P^{-1}} \underbrace{\begin{bmatrix} 2 & 1 & -3 & 4 \\ -4 & -2 & 1 & 3 \\ 7 & 1 & 0 & 1 \end{bmatrix}}_{A} \Rightarrow A = PA'$$

P^{-1} is the permutation matrix that swaps R_2 + R_3

P // // // that swaps R_2 + R_3.

(reversing P^{-1}).

So $P = P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

$$A' = \begin{bmatrix} 2 & 1 & -3 & 4 \\ 7 & 1 & 0 & 1 \\ -4 & -2 & 1 & 3 \end{bmatrix} \xrightarrow{-\frac{7}{2}R_1 + R_2} \begin{bmatrix} 2 & 1 & -3 & 4 \\ 0 & -\frac{5}{2} & \frac{21}{2} & -13 \\ -4 & -2 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{2R_1 + R_3} \begin{bmatrix} 2 & 1 & -3 & 4 \\ 0 & -\frac{5}{2} & \frac{21}{2} & -13 \\ 0 & 0 & -5 & 11 \end{bmatrix} = U \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 7/2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$A' = LU$

$$A = PA' = PLU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 7/2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 & 4 \\ 0 & -\frac{5}{2} & \frac{21}{2} & -13 \\ 0 & 0 & -5 & 11 \end{bmatrix}$$