Eigenvalues & eigenvectors

eigen = "own"
eigenvalues = "matrix's own values"

Def. Let A be an $n \times n$ matrix.
A vector $X \neq 0$ is an eigenvector of A with eigenvalue λ if
$$A \cdot X = \lambda \cdot X.$$

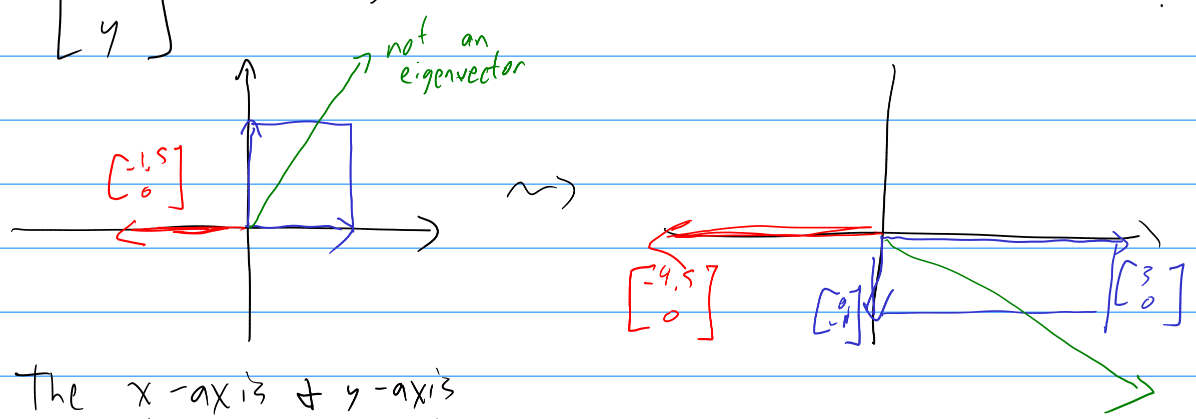
ex. $A = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$. Then $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

So $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of A with eigenvalue 3.

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue -1.

$\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} x \\ 0 \end{bmatrix}$ " of A with eigenvalue 3

$\begin{bmatrix} 0 \\ 4 \end{bmatrix}$ " of A " -1.



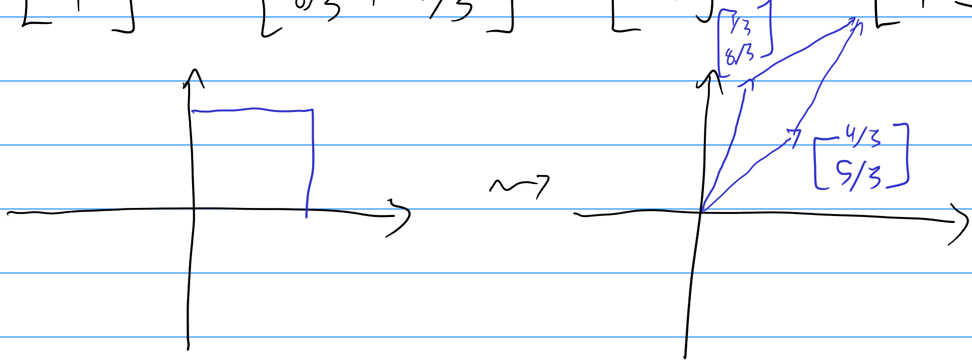
The x-axis & y-axis consist of eigenvectors.

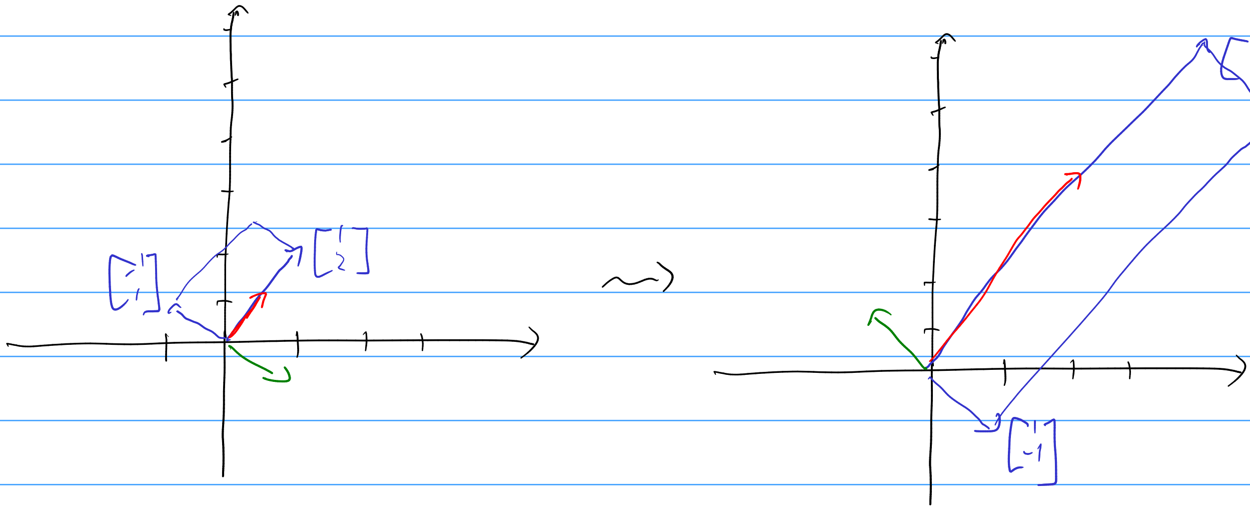
In fact, these are the only eigenvectors.

ex. $B = \begin{bmatrix} 1/3 & 4/3 \\ 8/3 & 5/3 \end{bmatrix}$

$$B\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix} 1/3 + 2\cdot 4/3 \\ 8/3 + 2\cdot 5/3 \end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix} = 3\begin{bmatrix}1\\2\end{bmatrix}$$

$$B\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix} -1/3 + 4/3 \\ -8/3 + 5/3 \end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix} = -1\cdot\begin{bmatrix}-1\\1\end{bmatrix}$$





ex. $C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Every vector X is an eigenvector of C, with eigenvalue 2.

ex. X is an eigenvector with eigenvalue 0 for some A

$\iff A\cdot X = 0\cdot X = 0$

$\iff X \in \text{null}(A)$.

A has at least one eigenvector with eigenvalue 0

$\iff A$ has a nontrivial null space

$\iff A$ is not invertible.

Prop. The set of eigenvectors of A with eigenvalue λ is a subspace of $\mathbb{R}^n$.

Proof. Suppose X, Y have eigenvalue λ.

Then $A \cdot (X+Y) = A \cdot X + A \cdot Y = \lambda \cdot X + \lambda \cdot Y = \lambda (X+Y)$

Suppose X has eigenvalue λ and $c \in \mathbb{R}$

Then $A \cdot (cX) = c \cdot AX = c \cdot \lambda X = \lambda \cdot (cX)$. ∎

⚠ Eigenvectors with distinct eigenvalues generally don't form a subspace.

If X has eigenvalue λ, Y has eigenvalue μ, for some A, then $A \cdot (X+Y) = AX + AY = \lambda X + \mu Y$

This is probably not a scalar multiple of $X+Y$.

Prop. Eigenvectors with different eigenvalues are linearly independent.

Proof. Say X has eigenvalue λ, Y has eigenvalue μ.

If X and Y are linearly dependent, then let's say $X = cY$. But then X would also have eigenvalue μ. $\rightarrow\leftarrow$ so X, Y are linearly independent. ∎.

Cor. At most n possible eigenvalues for an $n \times n$ matrix.

Proof. Any eigenvalue has at least one eigenvector.

... so has at least a 1-dim'l space of eigenvectors.

The eigenvectors for distinct eigenvalues are LI from each other, so $\leq n$ of these spaces of eigenvectors. ∎