MA 351 Homework 9

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I General Problem:

Write down a 4×4 matrix with no zeroes. Compute its determinant, any way you like. Figure out how to check your answer by calculator, or computer, and do so.

Ans

Let A be a 4×4 matrix, such that

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \xrightarrow[\substack{R_3 \to -9R_1+R_3 \\ R_4 \to -13R_1+R_4}]{R_2 \to -5R_1+R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \\ 0 & -12 & -24 & -36 \end{bmatrix} \xrightarrow{R_2 \to -\frac{1}{4}R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -8 & -16 & -24 \\ 0 & -12 & -24 & -36 \end{bmatrix}$$

$$\xrightarrow[R_4 \to 12R_2+R_4]{R_3 \to 8R_2+R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\det(A)\cdot(-1/4) = \det(B)$

Since $\det(B)=0$, $\boxed{\det(A) = 0}$

II Exercise.

4.4] Use expansion along the third row to express each of the following determinants as a sum of determinants of 4×4 matrices. Without expanding further, explain why $3\alpha = \beta$. What theorem from the text does this exercise demonstrate?

$$\alpha = \begin{vmatrix} 24 & -13 & 7 & 9 & 5 \\ 11 & 16 & -37 & 99 & 64 \\ 1 & 4 & 2 & 2 & -3 \\ 31 & -42 & 78 & 55 & -3 \\ 62 & 47 & 29 & -14 & -8 \end{vmatrix}, \quad \beta = \begin{vmatrix} 24 & -13 & 7 & 9 & 5 \\ 11 & 16 & -37 & 99 & 64 \\ 3 & 12 & 6 & 6 & -9 \\ 31 & -42 & 78 & 55 & -3 \\ 62 & 47 & 29 & -14 & -8 \end{vmatrix}$$

$$\alpha = (1)\begin{vmatrix} -13 & 7 & 9 & 5 \\ 16 & -37 & 99 & 64 \\ -42 & 78 & 55 & -3 \\ 47 & 29 & -14 & -8 \end{vmatrix} - (4)\begin{vmatrix} 24 & 7 & 9 & 5 \\ 11 & -37 & 99 & 64 \\ 31 & 78 & 55 & -3 \\ 62 & 29 & -14 & -8 \end{vmatrix} + (2)\begin{vmatrix} 24 & -13 & 9 & 5 \\ 11 & 16 & 99 & 64 \\ 31 & -42 & 55 & -3 \\ 62 & 47 & -14 & -8 \end{vmatrix} - (2)\begin{vmatrix} 24 & -13 & 7 & 5 \\ 11 & 16 & -37 & 64 \\ 31 & -42 & 78 & -3 \\ 62 & 47 & 29 & -8 \end{vmatrix}$$

$$+ (-3)\begin{vmatrix} 24 & -13 & 7 & 9 \\ 11 & 16 & -37 & 99 \\ 31 & -42 & 78 & 55 \\ 62 & 47 & 29 & -14 \end{vmatrix}$$

$$\beta = (3)\begin{vmatrix} -13 & 7 & 9 & 5 \\ 16 & -37 & 99 & 64 \\ -42 & 78 & 55 & -3 \\ 47 & 29 & -14 & -8 \end{vmatrix} - (12)\begin{vmatrix} 24 & 7 & 9 & 5 \\ 11 & -37 & 99 & 64 \\ 31 & 78 & 55 & -3 \\ 62 & 29 & -14 & -8 \end{vmatrix} + (6)\begin{vmatrix} 24 & -13 & 9 & 5 \\ 11 & 16 & 99 & 69 \\ 31 & -42 & 55 & -3 \\ 62 & 47 & -14 & -8 \end{vmatrix}$$

$$- (6)\begin{vmatrix} 24 & -13 & 7 & 3 \\ 11 & 16 & -37 & 64 \\ 31 & -42 & 78 & -3 \\ 62 & 47 & 29 & -8 \end{vmatrix} + (-9)\begin{vmatrix} 24 & -13 & 7 & 9 \\ 11 & 16 & -37 & 99 \\ 31 & -42 & 78 & 55 \\ 62 & 47 & 29 & -14 \end{vmatrix}$$

∴ As it can be observed each of a_{3j} elements of β is just $3\cdot a_{3j}$ elements of α. This example demonstrates the $\boxed{\text{row scalar theorem}}$ such that

$$\det\left(\begin{bmatrix} A \\ cX \\ B \end{bmatrix}\right) = c\cdot\det\left(\begin{bmatrix} A \\ X \\ B \end{bmatrix}\right)$$

where A, B are some rows of a matrix and c is some scalar $\in \mathbb{R}$.

4.6] This exercise discusses the proof of the statement that a matrix with integral [entries] has integral determinant.

(a) Prove that if all the entries of a 2×2 matrix are integers, then its determinant [must] be an integer.

ANS

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $a, b, c, d \in \mathbb{Z}$, then $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \in \mathbb{Z}$

$\therefore$ Hence proved.

(b) Use part (a) and formula (4.2) on page 239 to prove the statement in part (a) for 3×3 matrices.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, with all entries a_{ij} to $a_{3j} \in \mathbb{Z}$, then,

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \in \mathbb{Z}$$

because all other determinants are with entries in $\mathbb{Z}$.

4.9] Let $X = [a_1, a_2, a_3]^t$, $Y = [b_1, b_2, b_3]^t$, and $Z = [c_1, c_2, c_3]^t$ be vectors in $\mathbb{R}^3$. Let $X \times Y = [d_1, d_2, d_3]^t$, where

$d_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$, $d_2 = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}$, $d_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$; Prove that $Z^t(X \times Y) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

ANS Now, $X \times Y = [a_1, a_2, a_3]^t \times [b_1, b_2, b_3]^t = \left[\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right]$

$= [a_2 b_3 - a_3 b_2,\ b_1 a_3 - a_1 b_3,\ a_1 b_2 - a_2 b_1]^t \longrightarrow ①$

$= [d_1, d_2, d_3]^t$

Now, $Z^t(X \times Y) = [c_1, c_2, c_3]^t [d_1, d_2, d_3]^t$

$= [c_1 d_1, c_2 d_2, c_3 d_3]^t \longrightarrow ②$

$L.H.S = Z^t(X \times Y) = [c_1(a_2 b_3 - a_3 b_2), c_2(b_1 a_3 - a_1 b_3), c_3(a_1 b_2 - a_2 b_1)] = c_1(a_2 b_3 - a_3 b_2)$
$- c_2(a_1 b_3 - a_3 b_1)$
$+ c_3(a_1 b_2 - a_2 b_1) \longrightarrow ③$

From L.H.S eq ②

$Z^t(X \times Y) = c_1(a_2 b_3 - a_3 b_2) - c_2(a_1 b_3 - b_1 a_3) + c_3(a_1 b_2 - a_2 b_1) \longrightarrow ④$

Hence, from eq ③ and ④, we get

$$Z^t(X \times Y) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$\boxed{\therefore \text{QED}}$

...ancouyhnett's Problem

...ow reduction to prove that $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y-x)(z-x)(z-y)$. What has to be true about x, y, z for the row vectors $[1, x, x^2]$, $[1, y, y^2]$, and $[1, z, z^2]$ to be linearly dependent?

ANS

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} = (1)\left[(y-x)(z^2-x^2) - (z-x)(y^2-x^2)\right]$$

$$= (1)(y-x)(z-x)\,(z+x-y-x)$$

$$= (y-x)(z-x)(z-y)$$

Let $\alpha, \beta, \gamma \in \mathbb{R}$, then

$\alpha(1, x, x^2) + \beta(1, y, y^2) + \gamma(1, z, z^2) = (0, 0, 0)$

$\alpha + \beta + \gamma = 0 \longrightarrow$ (i)

$\alpha x + \beta x + \gamma x = 0 \longrightarrow$ (ii)

$\alpha x^2 + \beta x^2 + \gamma x^2 = 0 \longrightarrow$ (iii)

(i) $\times x$ − (ii)

$\alpha x + \beta x + \gamma x = 0$

$\alpha x + \beta y + \gamma z = 0$

$\beta(x-y) + \gamma(x-z) = 0 \longrightarrow$ (iv)

(ii) $\times x$ − (iii)

$\alpha x^2 + \beta xy + \gamma xz = 0$

$\alpha x^2 + \beta y^2 + \gamma z^2 = 0$

$\beta y(x-y) + \gamma z(x-z) = 0 \longrightarrow$ (v)

(iv) $\times y$ − (v)

$\beta y(x-y) + \gamma y(x-z) = 0$

$-\beta y(x-y) + \gamma z(x-z) = 0$

$\gamma(x-z)(y-z) = 0$

$\therefore (x-z) \neq 0 \quad (y-z) \neq 0 \implies \boxed{\gamma = 0}$

By substituting $\gamma = 0$ in (iv), we get $\beta(x-y) = 0$ since $(x-y) \neq 0$

$\boxed{\beta = 0}$, and from (i), we get $\boxed{\alpha = 0}$

$\therefore$ If $x \neq y \neq z$, then $(1, x, x^2)$, $(1, y, y^2)$ & $(1, z, z^2)$ are linearly independent.

4.12] Use Row-Reduction to compute the following determinants:

(a) $\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} \xrightarrow[R_3 \to -3R_1 + R_3]{R_2 \to -2R_1 + R_2} \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{vmatrix} \xrightarrow{R_3 \to -2R_2 + R_3} \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix}$ $\therefore$ $\boxed{\det(A) = \det(RREF(A)) = 0}$

Since 3rd row is zero.

(b) $\begin{vmatrix} -3 & 2 & 2 \\ 1 & 4 & 1 \\ 7 & 6 & -2 \end{vmatrix} \xrightarrow{R_1 \to R_1/-3} \begin{vmatrix} 1 & -2/3 & -2/3 \\ 1 & 4 & 1 \\ 7 & 6 & -2 \end{vmatrix} \xrightarrow[R_3 \to -7R_1 + R_3]{R_2 \to -R_1 + R_2} \begin{vmatrix} 1 & -2/3 & -2/3 \\ 0 & 14/3 & 5/3 \\ 0 & 32/3 & 8/3 \end{vmatrix} \xrightarrow{R_2 \to R_2 \cdot 14/3} \begin{vmatrix} 1 & -2/3 & -2/3 \\ 0 & 1 & 5/14 \\ 0 & 32/3 & 8/3 \end{vmatrix}$

$\xrightarrow{R_3 \to -\frac{32}{3}R_2 + R_3} \begin{vmatrix} 1 & -2/3 & -2/3 \\ 0 & 1 & 5/14 \\ 0 & 0 & -8/7 \end{vmatrix} \xrightarrow{R_3 \to R_3 \cdot -7/8} \begin{vmatrix} 1 & -2/3 & -2/3 \\ 0 & 1 & 5/14 \\ 0 & 0 & 1 \end{vmatrix}$

$\det(RREF(A)) = \det(A)\,(-1/3)(3/14)(-7/8)$

Since $\det(RREF(A)) = 1 \cdot 1 \cdot 1 = 1$ b/c RREF(A) is also triangular matrix.

$\boxed{\det(A) = 16}$

(c) $\begin{vmatrix} 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 2 \\ 1 & 0 & 0 & 5 \end{vmatrix} \xrightarrow[R_2 \to -R_1 + R_2]{R_1 \to 2R_1} \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 1 & 0 & 0 & 5 \end{vmatrix} \xrightarrow{R_4 \to -R_1 + R_4} \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -1 & 5 \end{vmatrix} \xrightarrow{R_3 \to \frac{1}{3}R_3} \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & -1 & 5 \end{vmatrix}$

$\xrightarrow{R_4 \to R_3 + R_4} \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 17/3 \end{vmatrix} \xrightarrow{R_4 \to \frac{3}{17}R_4} \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 1 \end{vmatrix}$

$\det(RREF(A)) = \det(A)\,(1/2)(1/3)(3/17)$

Since $\det(RREF(A)) = 1$, $\boxed{\det(A) = 34}$

4.17] Let $A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$, represent a 3X3 matrix and suppose that $A_3 = 2A_1 + A_2$. Use the row scalar and additive properties to prove that $\det A = 0$.

ANS

$A = \begin{bmatrix} A_1 \\ A_2 \\ 2A_1 + A_2 \end{bmatrix}$, given

$\therefore A = \begin{bmatrix} A_1 \\ A_2 \\ 2A_1 \end{bmatrix} + \begin{bmatrix} A_1 \\ A_2 \\ A_2 \end{bmatrix}$, Row Additivity Law

$\Rightarrow A = 2 \cdot \begin{bmatrix} A_1 \\ A_2 \\ A_1 \end{bmatrix} + \begin{bmatrix} A_1 \\ A_2 \\ A_2 \end{bmatrix}$

$= (2 \cdot 0) + 0$

$= \boxed{0}$ $\therefore$ $\det(A) = 0$, Hence proved.

4.6(c) Since I know induction, let me prove it for $n \times n$ instead of 4×4 matrices.] → optional

→ The statement is true for $n=2$ → basis case.

→ suppose that the statement is true for some $n-1$

→ ∴ Now we know that

$$\det\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = a_{11} \det\begin{bmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n2} & \cdots & a_{nn} \end{bmatrix} - a_{12}\begin{bmatrix} a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$+ \cdots \pm a_n \begin{vmatrix} a_{21} & \cdots & a_{2(n-1)} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{n(n-1)} \end{vmatrix}$$

Since all the determinants are integer from the inductive hypothesis, they are also multiplied by elements $\in \mathbb{Z}$. Therefore, the determinants are also integer.