Finding eigenvalues.

Suppose λ is an eigenvalue of A. (with X an eigenvector)

$$AX = \lambda X = (\lambda I)X.$$

$$AX - (\lambda I)X = 0$$

$$(A - \lambda I)X = 0.$$

So $\mathrm{null}(A - \lambda I)$ is bigger than $\{0\}$.

i.e. $A - \lambda I$ is not invertible.

i.e. $\det(A - \lambda I) = 0$.

Def. The characteristic polynomial of A is

$$P_A(\lambda) = \det(A - \lambda I).$$

If A is $n \times n$, this is a degree n polynomial.

ex. $A = \begin{bmatrix} 1/3 & 4/3 \\ 8/3 & 5/3 \end{bmatrix}$

$$P_A(\lambda) = \left| \begin{bmatrix} 1/3 & 4/3 \\ 8/3 & 5/3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} 1/3 - \lambda & 4/3 \\ 8/3 & 5/3 - \lambda \end{vmatrix}$$

$$= (1/3 - \lambda)(5/3 - \lambda) - (4/3) \cdot (8/3)$$

$$= \lambda^2 - 2\lambda - 3. = (\lambda - 3)(\lambda + 1)$$

Zeros are $\lambda = 3$, $\lambda = -1$ These are the eigenvalues of A.

ex. $D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. $D - \lambda I = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}$

$$P_D(\lambda) = \begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = (\lambda - 2)^2.$$

Only eigenvalue is $\lambda = 2$. This is a double eigenvalue, because $(\lambda - 2)^2$ appears in $P_D(\lambda)$.

More generally, we say that the multiplicity of an eigenvalue λ_0 is r if the characteristic polynomial is divisible by $(\lambda - \lambda_0)^r$ (and not $(\lambda-\lambda_0)^{r+1}$).

What we know about factorizing polynomials.

Fundamental Thm of Algebra.
A degree n polynomial $p(\lambda)$ has exactly n roots.
... if you count with multiplicity
... and you include complex roots.

In other words,
$$p(\lambda) = c(\lambda-\lambda_1)^{n_1}(\lambda-\lambda_2)^{n_2}\cdots(\lambda-\lambda_k)^{n_k}$$
where $\lambda_1, \ldots, \lambda_k$ may be complex, and
$$n_1 + n_2 + \cdots + n_k = n.$$

Cor. An $n \times n$ matrix has exactly n eigenvalues
... if you count with multiplicity
... and you include complex eigenvalues.

A few more useful reminders:

→ A quadratic polynomial can always be factorized, using the quadratic formula if necessary.
→ Higher-degree polynomials can be factorized numerically by computer, but by hand it's pretty hard/impossible.
→ if the polynomial has real coefficients, then any non-real roots occur in conjugate pairs.
(if $a+bi$ is a root w/ multiplicity k, then $a-bi$ is also a root w/ multiplicity k.)

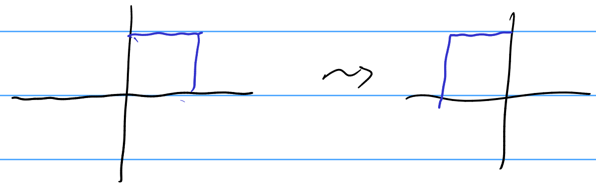
ex. $U = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$. $P_U(\lambda) = \begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{vmatrix}$

$= (2-\lambda)(2-\lambda)(3-\lambda)$.

So the eigenvalues of U are 2 (double) & 3.

Conclusion: eigenvalues of an upper triangular matrix are the entries on the diagonal.
(Same for lower triangular matrices.)

ex. $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.



$P_R(\lambda) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$.

So $P_R(\lambda) = 0 \iff \lambda^2 + 1 = 0$

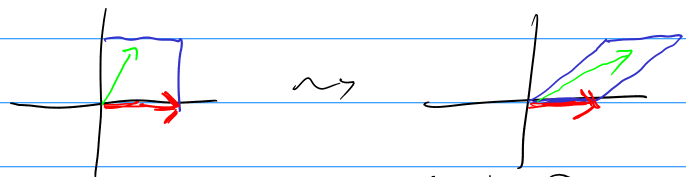
$\lambda^2 = -1$

$\lambda = \pm i$.

$R \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \end{bmatrix}$.

So, $\begin{bmatrix} 1 \\ i \end{bmatrix}$ is an eigenvector of R with eigenvalue $-i$.

ex. $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$



Only eigenvalue of S is $\lambda = 1$ (double).
Only eigenvectors are $\begin{bmatrix} x \\ 0 \end{bmatrix}$.