Uniqueness of the determinant.

Review properties of the determinant:

- ⇒ **Row interchange**: If we swap two rows of a matrix, the determinant gets multiplied by (-1).
- ⇒ **Row scalar**: If we multiply a row of a matrix by c, the determinant gets multiplied by c.
- ⇒ **Row additivity**: $\det\left(\begin{bmatrix} A \\ \hline X+Y \\ \hline B\end{bmatrix}\right) = \det\left(\begin{bmatrix} A \\ \hline X \\ \hline B\end{bmatrix}\right) + \det\left(\begin{bmatrix} A \\ \hline Y \\ \hline B\end{bmatrix}\right)$.
- ⇒ **Row operations**: If B is obtained from A by a row operation $cR_i + R_j \rightarrow R_j$ (where $i \neq j$), then $\det(B) = \det(A)$.
  - ↳ This follows from the first 3 properties.

Uniqueness theorem. Let $D : M(n,n) \rightarrow \mathbb{R}$ be a function such that:

① $D(I) = 1$,

② D satisfies the row interchange, row scalar, and row additivity properties.

Then: $D(A) = \det(A)$ for all $A \in M(n,n)$.

$D(I)=1$ row interchange

(Fancy words: det is the unique (normalized alternating multilinear form on n-tuples of $1 \times n$ row vectors.)

refers to row additivity + scalar properties

Proof. If D satisfies the properties in ②, we know what row operations do to it.

I. Swap two rows ⇒ change the sign of D

II. Add a multiple of a row to another row ⇒ D doesn't change

III. Scale a row by c ⇒ D gets multiplied by c.

Given A, put A in RREF.

If A is invertible: $\text{RREF}(A) = I$. $\Rightarrow D(\text{RREF}(A)) = \det(\text{RREF}(A)) = 1$.

If A isn't invertible: $\text{RREF}(A)$ has a zero row.

$$\Downarrow$$

$$D(\text{RREF}(A)) = \det(\text{RREF}(A)) = 0.$$

There's some sequence of row ops which turns $\text{RREF}(A)$ back into A.

These row operations change D in the same way as they do det. So $\det(A) = D(A)$. ∎

Consequences of uniqueness

Product theorem If A and B are $n \times n$ matrices,

$$\det(AB) = \det(A)\det(B).$$

Proof. Case 1: $\det(B) = 0$. Then $\text{rank}(B) < n$.

So $\text{rank}(AB) < n$. So $\det(AB) = 0$.

Case 2: $\det(B) \neq 0$

Define $D(A) = \dfrac{\det(AB)}{\det(B)}$. Want to show $D = \det$.

Check uniqueness thm:

$\Rightarrow$ $D(I) = \det(IB)/\det(B) = \det(B)/\det(B) = 1$.

$\Rightarrow$ Row interchange property:

$$\begin{bmatrix} - & A_1 & - \\ - & A_2 & - \\ & \vdots & \\ - & A_n & - \end{bmatrix} B = \begin{bmatrix} - & A_1 B & - \\ - & A_2 B & - \\ & \vdots & \\ - & A_n B & - \end{bmatrix}$$

The rows of AB are $A_1 B, \ldots, A_n B$.

So interchanging two rows of $A \Rightarrow$ interchanges the same two rows of $AB \Rightarrow$ changes the sign of $\det(AB)$ while leaving $\det(B)$ unchanged.

$\Rightarrow$ changes the sign of $D(A) = \det(AB)/\det(B)$.

→ Row scalar property: if we scale A_i by c, we also scale $(AB)_i = A_i B$ by c.
So we scale $\det(AB)$ by c, so we scale $D(AB)$ by c.

→ Row additivity property: $A = \begin{bmatrix} - & A_1 & - \\ & \vdots & \\ - & X+Y & - \\ & \vdots & \\ - & A_n & - \end{bmatrix}$ i^{th} row

Let $A_X = \begin{bmatrix} - & A_1 & - \\ & \vdots & \\ - & X & - \\ & \vdots & \\ - & A_n & - \end{bmatrix}$, $A_Y = \begin{bmatrix} - & A_1 & - \\ & \vdots & \\ - & Y & - \\ & \vdots & \\ - & A_n & - \end{bmatrix}$

Then $AB = \begin{bmatrix} - & A_1 B & - \\ & \vdots & \\ - & (X+Y)B & - \\ & \vdots & \\ - & A_n B & - \end{bmatrix}$, $A_X B = \begin{bmatrix} - & A_1 B & - \\ & \vdots & \\ - & XB & - \\ & \vdots & \\ - & A_n B & - \end{bmatrix}$, $A_Y B = \begin{bmatrix} - & A_1 B & - \\ & \vdots & \\ - & YB & - \\ & \vdots & \\ - & A_n B & - \end{bmatrix}$

Since $(X+Y)B = XB + YB$,

$$\det(AB) = \det(A_X B) + \det(A_Y B)$$
$$D(A) = D(A_X) + D(A_Y).$$

divide by $\det(B)$.

▨

Cor. $\det(AB) = \det(BA)$.

Last part of cofactor expansion thm:

$$\det(A) = (-1)^{1+j} a_{1j} \det(A_{1j}) + (-1)^{2+j} a_{2j} \det(A_{2j}) + \cdots + (-1)^{n+j} a_{nj} \det(A_{nj}).$$

Proof. Let $D(A)$ be the expression on the right-hand side.
(I'll just do $j=1$, so

$$D(A) = a_{11}\det(A_{11}) - a_{21}\det(A_{21}) + \cdots + (-1)^{n+1} a_{n1} \det(A_{n1}),$$
$$= \det(A^t)$$

→ $D(I) = \det(I^t) = \det(I) = 1.$

$$A = \begin{bmatrix} a_{11} & & \\ a_{21} & & \\ \vdots & A_{11} & \\ a_{n1} & & \end{bmatrix} \qquad A' = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \\ a_{n1} & & & \end{bmatrix}$$

→ Row interchange:

Let's say we swap rows 1 and 2, and get A',

$$D(A') = a'_{11}\det(A'_{11}) - a'_{21}\det(A'_{21}) + \cdots + (-1)^{n+1}a'_{n1}\det(A'_{n1}),$$

$$= a_{21}\det(A'_{11}) - a_{11}\det(A'_{21}) + a_{31}\det(A'_{31}) - \cdots + (-1)^{n+1}a_{n1}\det(A'_{n1}).$$

A_{31} with 1st 2 rows switched

A_{n1} with 1st 2 rows switched.

$$= a_{21}\det(A_{21}) - a_{11}\det(A_{11})$$
$$- a_{31}\det(A_{31}) + \cdots$$
$$+ (-1)^n a_{n1}\det(A_{n1}).$$
$$= -D(A).$$

→ Row scalar:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} \end{bmatrix} \to A' = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ a_{21} & & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} \end{bmatrix}$$

$$D(A) = a_{11}\det(A_{11}) - a_{21}\det(A_{21}) + \cdots + (-1)^{n+1}a_{n1}\det(A_{n1}),$$
$$D(A') = ca_{11}\det(A_{11}) - a_{21}\det(A'_{21}) + \cdots + (-1)^{n+1}a_{n1}\det(A'_{n1})$$
$$= ca_{11}\det(A_{11}) - ca_{21}\det(A_{21}) + \cdots + (-1)^{n+1}c\,a_{n1}\det(A_{n1})$$
$$= c\,D(A).$$

→ Row additivity:

$$A_x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & & & \end{bmatrix} \qquad A_y = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ a_{21} & & & \\ \vdots & & \ddots & \\ a_{n1} & & & \end{bmatrix}$$

$$A = \begin{bmatrix} x_1+y_1 & x_2+y_2 & \cdots & x_n+y_n \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & & & \end{bmatrix}$$

$$D(A) = (x_1+y_1)\det(A_{11}) - a_{21}\det(A_{21}) + \cdots + (-1)^{n+1}a_{n1}\det(A_{n1}).$$

$$A_{11} = (A_x)_{11} = (A_y)_{11}$$

For $i > 1$, $\det(A_{i1}) = \det((A_x)_{i1}) + \det((A_y)_{i1})$

So
$$D(A) = x_1\det((A_x)_{11}) + y_1\det((A_y)_{11})$$
$$- a_{21}\det((A_x)_{21}) - a_{21}\det((A_y)_{21}) + \cdots$$
$$= D(A_x) + D(A_y).$$

▨

Conclusion: $\det(A)$ can be defined by cofactor expansion along columns.

Also: $\det(A) = \det(A^t)$

det behaves well with respect to column operations. It satisfies column interchange, column scalar, and column additivity properties.