MATHEMATICS 3 20171089

ASSIGNMENT 1 ABHIGYAN GHOSH

Ans 1. $f(x) = \begin{cases} cx^2 & |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$

i) $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} cx^2 dx + \int_{1}^{\infty} 0\,dx = 1$

$\Rightarrow c\left[\frac{x^3}{3}\right]_{-1}^{1} = 1$

$\Rightarrow c \cdot \frac{2}{3} = 1 \Rightarrow \boxed{c = \frac{3}{2}}$

ii) $E[x] = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx = \int_{-\infty}^{-1} x \cdot 0 \cdot dx + \int_{-1}^{1} x \cdot cx^2 \cdot dx + \int_{1}^{\infty} x \cdot 0 \cdot dx$

$= \int_{-1}^{1} cx^3 dx = \left[\frac{cx^4}{4}\right]_{-1}^{1} = c\left[\frac{1}{4} - \frac{1}{4}\right] = 0$

$\therefore \boxed{E[x] = 0}$

$Var(x) = E(x^2) - (E(x))^2$

$= \int_{-\infty}^{\infty} x^2 \cdot f(x) \cdot dx - (0)^2$

$= \int_{-\infty}^{\infty} cx^4 dx = \int_{-1}^{1} cx^4 dx = \frac{c}{5}\left[x^5\right]_{-1}^{1}$

$= \frac{2c}{5} = \frac{2 \times 3}{5 \cdot 2} = \frac{3}{5}$

$\therefore \boxed{Var(x) = \frac{3}{5}}$

iii) $P(x \ge 1/2) = \int_{1/2}^{\infty} f(x) \cdot dx = \int_{1/2}^{1} cx^2 dx = \frac{c}{3}\left[x^3\right]_{1/2}^{1}$

$= \frac{7c}{24} = \frac{7 \times 3}{24 \cdot 2} = \frac{7}{16}.$

$\therefore \boxed{P(x \ge 1/2) = 7/16}$

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2. a) $F(x) = \frac{x^3+k}{40}$

$F(1) = \frac{k+1}{40}$ $F(2) = \frac{k+8}{40}$ $F(3) = \frac{k+27}{40}$.

$P(1) = F(1) = \frac{k+1}{40}$ $P(2) = F(2) - F(1) = \frac{7}{40}$

$P(3) = F(3) - F(2) = \frac{19}{40}$

$\therefore \sum_i P(x_i) = 1$

$\Rightarrow \frac{k+1}{40} + \frac{7}{40} + \frac{19}{40} = 1$

$\Rightarrow \boxed{k = 13}$

b) $P(x=1) = \frac{k+1}{40} = \frac{14}{40}$, $P(x=2) = \frac{7}{40}$ $P(x=3) = \frac{19}{40}$.

c) $var(ax+b) = a^2 \, var(x)$

Given $var(x) = \frac{259}{320}$

$\Rightarrow Var(4x-5) = 16 \, var(x)$

$= \cancel{16} \times \frac{259}{\cancel{320}_{20}} = \frac{259}{20}$

$\therefore \boxed{var(4x-5) = \frac{259}{20}}$

3. Let ~~be the~~ $P(x=i)$ be the probability of getting 6 on the i^{th} throw.

$\therefore P(x=2) = \frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$

$P(x=4) = \left(\frac{5}{6}\right)^3 \times \frac{1}{6}$

$P(x=6) = \left(\frac{5}{6}\right)^5 \times \frac{1}{6}$

$\therefore$ probability of getting 6 in an even throw

$$= \frac{5}{6}\times\frac{1}{6} + \left(\frac{5}{6}\right)^3 \times \frac{1}{6} + \left(\frac{5}{6}\right)^5 \times \frac{1}{6} + \cdots\cdots \infty$$

$$= \frac{\frac{5}{6}\times\frac{1}{6}}{1-\frac{25}{36}} = \frac{5}{11}$$

$\therefore$ Probability of getting first six on second throw given that ~~six~~ six occurs on even throws $= \frac{P(x=2)}{\sum_{i=1}^{\infty} P(x=i)} = \frac{5/36}{5/11} = \frac{11}{36}$ Ans

4.

A. Let A be the event that 0 was sent and B be the event that 0 was recieved.

$$P(A\cap B) = \frac{2}{3}\times\left(1-\frac{1}{5}\right) = \frac{2}{3}\times\frac{4}{5} = \frac{8}{15}$$

$$P(B) = \frac{2}{3}\left(1-\frac{1}{5}\right) + \frac{1}{3}\times\frac{1}{6} = \frac{8}{15}+\frac{1}{18} =$$

$$\therefore P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{8/15}{8/15+1/18} = \frac{8}{8+\frac{5}{6}} = \frac{48}{53}.$$

$$\therefore \boxed{P(A/B) = 48/53}$$

5. $f(x) = \begin{cases} cx^2 & 0<x<3 \\ 0 & \text{otherwise.} \end{cases}$

i) $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow$

$$\Rightarrow \int_{-\infty}^{0} 0\,dx + \int_{0}^{3} cx^2\,dx + \int_{3}^{\infty} 0\,dx = 1$$

$$\Rightarrow \left[\frac{cx^3}{3}\right]_0^3 = 1 \Rightarrow \boxed{c = \frac{1}{9}}.$$

ii) $P(1<x<2) = \int_1^2 f(x)\,dx = \int_1^2 cx^2\,dx = \frac{c}{3}\left[x^3\right]_1^2 = \frac{7c}{3}$

$$\therefore \boxed{P(1<x<2) = \frac{7c}{3} = \frac{7}{27}}$$

6. $f(x) = \begin{cases} \dfrac{c}{x^2+1} & -\infty < x < \infty. \end{cases}$

a. $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{c\,dx}{x^2+1} = c\left[\tan^{-1}x\right]_{-\infty}^{\infty}$

$= c\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = c\pi = 1 \Rightarrow c = \frac{1}{\pi}.$

b. : $\frac{1}{3} < x^2 < 1$

$\Rightarrow x \in \left(-1, \frac{-1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, 1\right)$

$P(1/3 < x^2 < 1) = P(-1 < x < -1/\sqrt{3}) + P(1/\sqrt{3} < x < 1).$

$= \int_{-1}^{-1/\sqrt{3}} f(x)\,dx + \int_{1/\sqrt{3}}^{1} f(x)\,dx$

$= c\left[\int_{-1}^{-1/\sqrt{3}} \frac{dx}{x^2+1} + \int_{1/\sqrt{3}}^{1} \frac{dx}{x^2+1}\right] = c\left(\left[\tan^{-1}x\right]_{-1}^{-1/\sqrt{3}} + \left[\tan^{-1}x\right]_{1/\sqrt{3}}^{1}\right)$

$= c\left[\left(-\frac{\pi}{6} + \frac{\pi}{4}\right) + \left(\frac{\pi}{4} - \frac{\pi}{6}\right)\right]$

$= c\left(\frac{\pi}{6}\right) = \frac{1}{6}$

$\therefore \boxed{P(1/3 < x^2 < 1) = \frac{1}{6}}$

7. i) CDF $= F(x) = \begin{cases} 1 - e^{-2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

PDF $= \frac{d}{dx}F(x) = f(x) = \begin{cases} 2e^{-2x} & x \geq 0. \\ 0 & x < 0. \end{cases}$

ii) $\therefore P(x > 2) =$ ~~$\int$~~ $F(\infty) - F(2)$

$= \left[1 - \frac{1}{e^{2x}}\right]_{2}^{\infty}$

$= 1 - 1 + \frac{1}{e^4} = 1/e^4 = e^{-4}.$

$\therefore P(x > 2) = e^{-4}.$

iii) $P(-3 < x \leq 4) = F(4) - F(-3)$

$= \left(1 - \frac{1}{e^8}\right) - 0$

$= 1 - e^{-8}$.

$\therefore \boxed{P(-3 < x \leq 4) = 1 - e^{-8}}$

8a) $F1(x) = \begin{cases} 0 & x \leq 0 \\ 0.5x & 0 < x \leq 1 \\ 0.25(1+x) & 1 < x \leq 3 \\ 1 & x > 3 \end{cases}$

For a function to be a CDF, it should be

1. non-decreasing 2. Right continuous

3. $\lim_{x \to -\infty} F(x) = 0$ 4. $\lim_{x \to +\infty} F(x) = 1$.

$F1(0^-) = 0$; $F1(0^+) = 0$

$F1(1^-) = 0.5 \times 1 = 0.5$; $F1(1^+) = 0.25(1+1) = 0.5$

$F1(3^-) = 0.25(1+3) = 1$ $F1(3^+) = 1$.

$\therefore$ F1(x) is ~~non-decreasing, right co~~ satisfying all the above conditions. Hence it is a valid CDF.

8b) $F2(x) = \begin{cases} 0.5 & x < 1 \\ 0.75 & 1 \leq x < 3 \\ 1 & x > 3. \end{cases}$

$F2(1^-) = 0.5$ $F2(1^+) = 0.75$.

$\therefore$ Clearly F2 is discontinuous.

Hence F2 cannot be a valid CDF

9. 8 white (W), 4 black (B), 2 orange (O)

$\therefore$ All possible outcomes are

WW	WB	WO	BB	BO	OO	
−2	1	−1	4	2	0	← Values of X

$P(X = -2) = \frac{8 \times 7}{14 \times 13} = \frac{4}{13}$.

$\cancel{P(x)}$ $P(x=-1) = \dfrac{\dfrac{8}{14} \times \dfrac{2}{13}}{2} = \dfrac{16}{91}$

$P(x=0) = \dfrac{1}{14 \times \frac{13}{2}} = \dfrac{1}{91}$

$P(x=1) = \dfrac{8 \times 4}{14 \times 13/2} = \dfrac{32}{91}$

$P(x=2) = \dfrac{4 \times 2}{14 \times 13/2} = \dfrac{8}{91}$

$P(x=4) = \dfrac{4 \times 3}{14 \times 13} = \dfrac{6}{91}$

10. $P(x=0) = {}^5C_2 \times \dfrac{3!}{5!} = \dfrac{1}{2}$ $\quad P_1 < P_2$

$P(x=1) = {}^5C_3 \times \dfrac{2!}{5!} = \dfrac{1}{6}$ $\quad P_2 < P_1 < P_3$.

$P(x=2) = {}^5C_4 \times \dfrac{2!}{5!} = \dfrac{1}{12}$ $\quad P_{2,3} < P_1 < P_4$

$P(x=3) = {}^5C_5 \cdot \dfrac{3!}{5!} = \dfrac{1}{20}$ $\quad P_2, P_3, P_4 < P_1 < P_5$

$P(x=4) = \dfrac{4!}{5!} = \dfrac{1}{5}$ $\quad P_2, P_3, P_4, P_5 < P_1$.

11. $P(x=0) = \dfrac{16}{20} \times \dfrac{15}{19} \times \dfrac{14}{18} = \dfrac{28}{57}$

$P(x=1) = \dfrac{16}{20} \times \dfrac{15}{19} \times \dfrac{4}{18} = \dfrac{24}{57}$

$P(x=2) = \dfrac{16}{20} \times \dfrac{4}{19} \times \dfrac{3}{18} = \dfrac{8}{95}$

$P(x=3) = \dfrac{4}{20} \times \dfrac{3}{19} \times \dfrac{2}{18} = \dfrac{1}{285}$

$\therefore E[x] = \sum x P(x)$

$= 0 \times \dfrac{28}{57} + 1 \times \dfrac{24}{57} + 2 \times \dfrac{8}{95} + 3 \times \dfrac{1}{285}$

$\Rightarrow \boxed{E(x) = \dfrac{171}{285}}$

12. If 0, 1, 2 bits are correct, then the correct message will be recieved.

Let P(i) denote the number of incorrectly recieved digits in a message

$P(0) = {}^5C_5 \times (0.8)^5 = 0.32768$

$P(1) = {}^5C_1 \times 0.2 \times (0.8)^4 = 0.4096.$

$P(2) = {}^5C_2 \times (0.2)^2 \times (0.8)^3 = 0.2048.$

$\therefore$ Probability that the message is correctly deciphered $= P(0) + P(1) + P(2) = 0.94208.$

$\therefore$ Probability that the message is incorrectly decoded $= 1 - 0.94208 = 0.05792.$

The assumption here is that the transfer of each bit is independent of the transfer of other bits in the same message.

13. Clearly the lowest rank that can be achieved by the highest ranking woman is 6.

$\therefore P(x=7) = P(x=8) = P(x=9) = P(x=10) = 0.$

$$P(x=1) = \frac{{}^5C_1 \times 9!}{10!} = \frac{1}{2}$$

$$P(x=2) = \frac{{}^5C_1 \times {}^5C_1 \times 8!}{10!} = \frac{5}{18}$$

$$P(x=3) = \frac{{}^5C_2 \times {}^5C_1 \times 7!\, 2!}{10!} = \frac{5}{36}.$$

$$P(x=4) = \frac{{}^5C_3 \times 3! \times {}^5C_1 \times 6!}{10!} = \frac{5}{84}$$

$$P(x=5) = \frac{{}^5C_4 \times 4! \times {}^5C_1 \times 5!}{10!} = \frac{5}{252}$$

$$P(x=6) = \frac{{}^5C_5 \times {}^5C_1 \times 4! \times 5!}{10!} = \frac{1}{252}$$

14. Number of ways to create 2 groups out of 12 members $= {}^{12}C_6 = 924.$

Number of ways to create groups with 3 memo in each group $= {}^6C_3 \times {}^6C_3 = 20 \times 20 = 400$

$\therefore \boxed{\text{Probability} = \frac{400}{924} = \frac{100}{231}}$

15. The problem can be broken down into 2 cases.

Case I: They were fraternal twins.

$P = \frac{92}{100} \times \frac{1}{2} =$

Case II: They were identical twins

$P = \frac{8}{100}$

$\therefore$ Probability that they were identical twins

$$= \frac{\frac{8}{100}}{\frac{46}{100} + \frac{8}{100}} = \frac{8}{54}$$

16. $f(x,y) = \begin{cases} 2 & x+y \le 1 \quad x>0, y>0 \\ 0 & \text{otherwise} \end{cases}$

$$E(xy) = \int_0^x \int_0^y xy\, f(x,y)\, dy\, dx$$

$$= \int_0^x \left(\int_0^{1-x} 2x\, y\, dy \right) dx$$

$$= \int_0^x x \left[\frac{2y^2}{2} \right]_0^{1-x} dx$$

$$= \int_0^x x(1-x)^2\, dx$$

$$= \int_0^1 (x^3 - 2x^2 + x)\, dx$$

$$= \left[\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} \right]_0^1 = \frac{1}{4} - \frac{2}{3} + \frac{1}{2} = \frac{1}{12}$$

$$E(x) = \int_0^x \int_0^y 2x \, dy \, dx = \int_0^1 2x \left(\int_0^{1-x} dy \right) dx$$

$$= \int_0^1 2x \, [y]_0^{1-x} \, dx$$

$$= \int_0^1 2x(1-x) \, dx$$

$$= \int_0^1 2x - 2x^2 \, dx$$

$$= \left[x^2 - \frac{2x^3}{3} \right]_0^1 = 1 - \frac{2}{3} = \frac{1}{3}.$$

$$E(y) = \int_0^x \int_0^y 2y \, dy \, dx = \int_0^1 \left(\int_0^{1-x} 2y \, dy \right) dx$$

$$= \int_0^1 (1-x)^2 \, dx$$

$$= \left[\frac{(1-x)^3}{-3} \right]_0^1 = \frac{1}{3}$$

$$\therefore Cov(x,y) = E(xy) - E(x)E(y)$$

$$= \frac{1}{12} - \frac{1}{3} \times \frac{1}{3} = -\frac{1}{36}.$$

17. The given statement x cannot be verified to be true. ~~Becau~~ But it is assumed to be true. Therefore ~~the p although we cannot~~ the probability of truth of the statement is either $P(x) = 1$ or $P(x) = 0$. Experimentally, $P(x) = 0$ but according to the assumptions that go into determining the probability of getting a head or tail, we can say that $P(x) = 1$.