Solutions to Tutorial sheet - 7 IEC102

(Q1) The switch in the circuit in Fig. Q1 moves from position-1 to position-2 at $t=0$. Compute $i_o(t)$ for $t>0$ and use this current to determine $v_o(t)$ for $t>0$.
(Assume that the circuit is in steady state at $t=0-$).

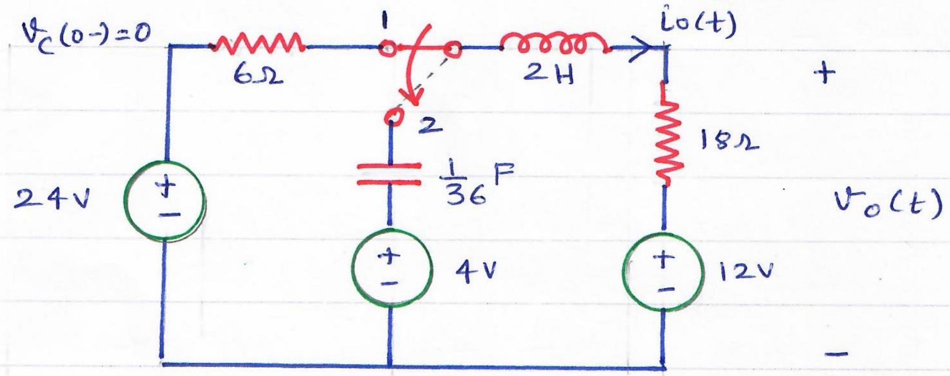
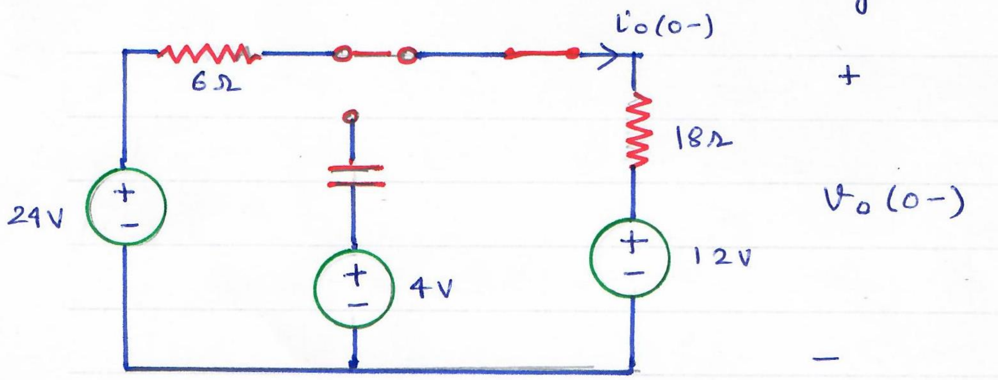


Fig. Q1

Sol.

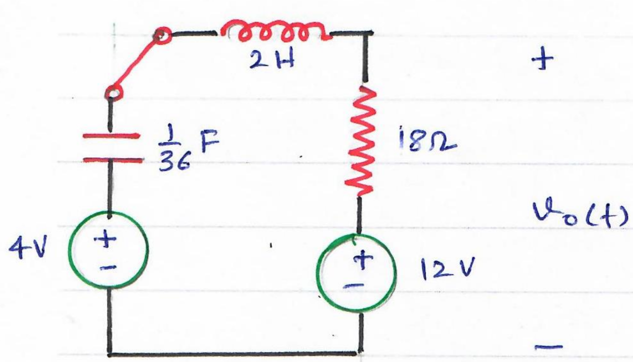
Circuit at $t=0-$ (It is in steady state) (The switch is in position-1)



$$i(0-) = \frac{24-12}{6+18} = \frac{12}{24} = 0.5\,A = i(0) = i(0+)$$

$$v_o(0-) = 18\,i_o(0-) + 12 = 9 + 12 = 21\,V$$

At $t=0$ (the switch is in position-2)



The circuit is a forced RLC series circuit.

$$\alpha = \frac{R}{2L} = \frac{18}{2\times 2} = \frac{18}{4} = 4.5 \text{ rad/s}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{2\times\frac{1}{36}}} = \sqrt{18} = 3\sqrt{2} \text{ rad/s}$$

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$$S_1, S_2 = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2}$$

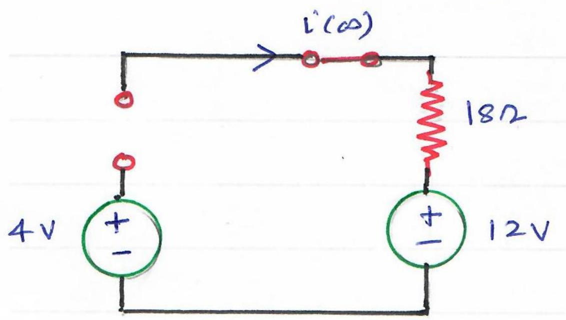
$$= -\frac{9}{2} \mp \sqrt{\left(\frac{9}{2}\right)^2 - (3\sqrt{2})^2}$$

$$= -\frac{9}{2} \mp \frac{3}{2} = -6, -3$$

The solution can be of the form

$$i_o(t) = K + A_1 e^{S_1 t} + A_2 e^{S_2 t}$$
$$= K + A_1 e^{-6t} + A_2 e^{-3t}$$

At $t = \infty$, the circuit will be in steady state. The equivalent circuit at $t = \infty$ is



$i_o(\infty) = 0$

$+$

$v_o(\infty)$

$-$

$$\therefore i_o(t) = K + A_1 e^{-6t} + A_2 e^{-3t}$$
$$i_o(\infty) = 0 = K + 0 + 0$$
$$\Rightarrow \boxed{K = 0}$$

$$\therefore i_o(t) = A_1 e^{-6t} + A_2 e^{-3t}$$

$$i_o(0) = 0.5 = A_1 + A_2 \Rightarrow A_1 + A_2 = 0.5 \quad \cdots (A)$$

$$0 = v_c(0-) = v_c(0+) = v_c(0)$$

At any time $t \geq 0$

$$-4 - v_c(t) + L\frac{di_o(t)}{dt} + 18\, i_o(t) + 12 = 0$$

$$\Rightarrow -4 - v_c(0) + L\frac{di_o(0)}{dt} + 18\, i_o(0) + 12 = 0$$

$$\Rightarrow L\frac{di_o(0)}{dt} = 4 + 0 - 18 \times 0.5 - 12$$
$$= 4 - 9 - 12 = -17$$

$$\therefore \frac{di_o(0)}{dt} = \frac{-17}{L} = -\frac{17}{2}\ A/s$$

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$i_o(0) = 0.5\ A$

$\frac{di_o(0)}{dt} = -\frac{17}{2}\ A/s$

$i_o(t) = A_1 e^{-6t} + A_2 e^{-3t}$

$i_o(0) = A_1 + A_2 = 0.5 \quad \cdots (A)$

$\frac{di_o(t)}{dt} = -6A_1 e^{-6t} - 3A_2 e^{-3t}$

$\frac{di_o(0)}{dt} = -6A_1 - 3A_2 = -\frac{17}{2}$

$\Rightarrow \quad 6A_1 + 3A_2 = \frac{17}{2} \quad \cdots (B)$

solving (A) and (B)

$A_1 = \frac{14}{6}\ ; \quad A_2 = -\frac{11}{6}$

$$\boxed{\therefore\ i_o(t) = \frac{14}{6} e^{-6t} - \frac{11}{6} e^{-3t}}$$

$v_o(t) = 18\, i_o(t) + 12$

$= 12 + 18\left[\frac{14}{6} e^{-6t} - \frac{11}{6} e^{-3t}\right]$

$$\Rightarrow \boxed{v_o(t) = 12 + 42 e^{-6t} - 33 e^{-3t}\ V}$$

(Q2) Find $v_c(t)$ for $t>0$ in the circuit shown in Fig. Q2. The circuit is in steady state at $t=0-$

The switch is open at $t=0$.

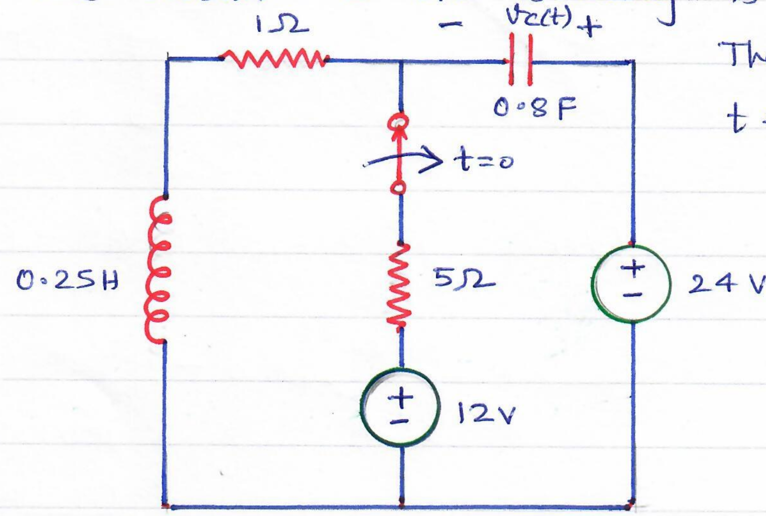
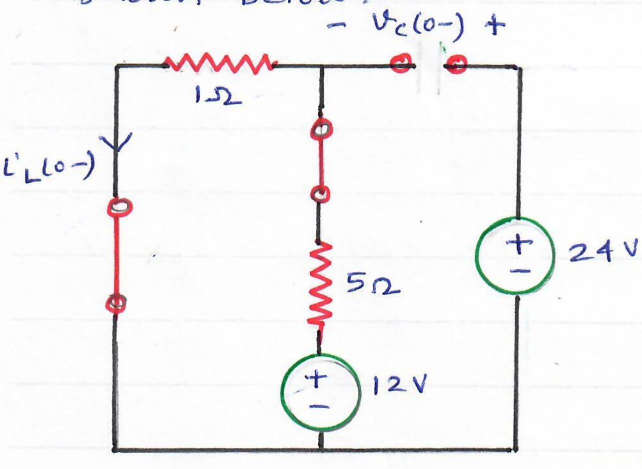


Fig. Q2

Sol. At $t=0-$ the circuit is in steady state and it is as shown below.



$$v_c(0-) = 24 - \frac{12\times 1}{1+5} = v_c(0) = v_c(0+)$$

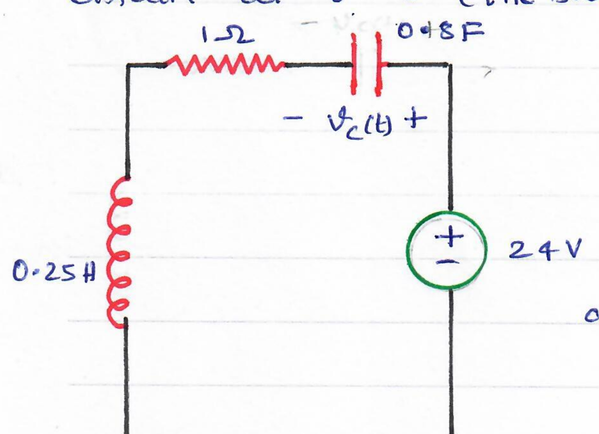
$$= 24 - \frac{12}{6}$$

$$= 24 - 2 = 22V$$

$$i_L(0-) = \frac{12}{1+5} = \frac{12}{6} = 2A$$

$$= i_L(0) = i_L(0+)$$

Circuit at $t=0$ (the switch is open)



This is forced series RLC circuit

with $\alpha = \frac{R}{2L} = \frac{1}{2\times 0.25} = \frac{1}{0.5} = 2$ rad/s

and $w_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.25\times 0.8}} = \frac{1}{\sqrt{0.2}} = \sqrt{5}$ rad/s since $\alpha < \omega_0$, the circuit is underdamped forced :

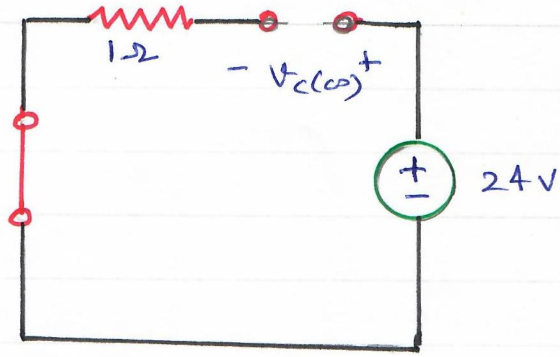
$\therefore$ The solution can be of the form

$$v_c(t) = K + e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

where $\alpha = 2$; $\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{5-4} = 1$ rad/s

$$\therefore v_c(t) = K + e^{-2t}(B_1 \cos t + B_2 \sin t)$$

Circuit at $t = \infty$ (It will be in steady state and capacitor acts as open circuit and inductor acts as short circuit).



$v_c(\infty) = 24$

$$v_c(t) = K + e^{-2t}(B_1 \cos t + B_2 \sin t)$$

$$\Rightarrow v_c(\infty) = K + 0 = 24$$

$$\therefore K = 24$$

$$\therefore v_c(t) = 24 + e^{-2t}(B_1 \cos t + B_2 \sin t)$$

$v_c(0) = 22V$; $i_L(0) = 2A$

At any time $t \geq 0$

$$C\frac{dv_c}{dt} = i_L$$

$$\Rightarrow \frac{dv_c(0)}{dt} = \frac{i_L(0)}{C} = \frac{2}{0.8} = \frac{20}{8} = 2.5 \text{ V/s}$$

$$v_c(t) = 24 + e^{-2t}(B_1 \cos t + B_2 \sin t)$$

$$v_c(0) = 24 + (1)(B_1) = 22$$

$$\Rightarrow B_1 = 22 - 24 = -2$$

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$$v_c(t) = 24 + e^{-2t}(B_1 \cos t + B_2 \sin t)$$

$$\frac{dv_c(t)}{dt} = e^{-2t}(-B_1 \sin t + B_2 \cos t) - 2e^{-2t}(B_1 \cos t + B_2 \sin t)$$

$$\frac{dv_c(0)}{dt} = (1)(B_2) - 2B_1 = 2.5$$

$$\Rightarrow B_2 = 2.5 + 2B_1$$

$$= 2.5 - 2\times 2 = 2.5 - 4 = -1.5$$

$$\therefore \quad v_c(t) = 24 + e^{-2t}(-2\cos t - 1.5 \sin t)$$

$$\Rightarrow \quad \boxed{v_c(t) = 24 - e^{-2t}(2\cos t + 1.5 \sin t) \quad V}$$

Q. The switch in the circuit shown in Fig. Q3 has been closed for a long time and is opened at $t=0$. Find $i_L(t)$ for $t>0$.

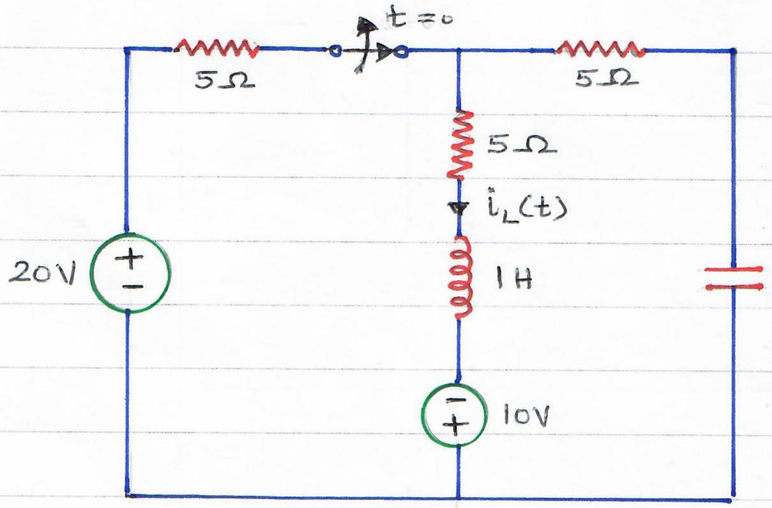
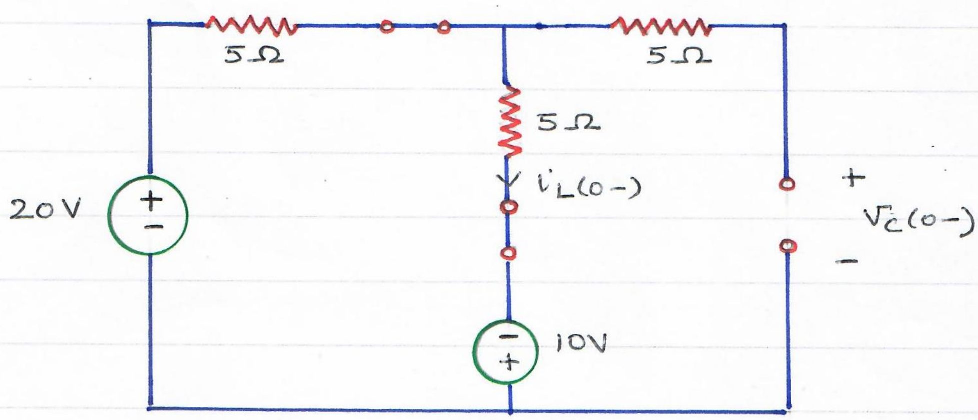


Fig. Q3

Given that the circuit is in steady state at $t=0-$.

Sol.

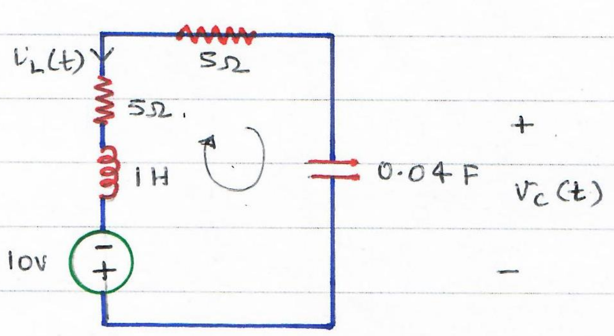
Circuit at $t=0-$



$$i_L(0-) = \frac{20+10}{5+5} = \frac{30}{10} = 3\,A$$

$$V_c(0-) = 20 - 5\,i_L(0-) = 20 - 5\times 3 = 20-15 = 5\,V$$

Circuit at $t>0$



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It is a forced series RLC circuit

$$\alpha = \frac{R}{2L} = \frac{10}{2} = 5$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.04}} = \frac{1}{0.2} = 5$$

$\alpha = \omega_0$, therefore the circuit is critically damped.

$\therefore$ Natural response $= i_{Ln}(t) = (A_1 t + A_2)e^{-\alpha t}$

Forced response $= K$

Complete response $= i_L(t) = K + (A_1 t + A_2)e^{-\alpha t}$

$i_L(0-) = i_L(0) = 3\,A$; $v_c(0-) = v_c(0) = 5\,V$

Applying KVL around the loop

$$10 - \frac{di_L(t)}{dt} - 10\,i_L(t) + v_c(t) = 0$$

$$\Rightarrow \frac{di_L(t)}{dt} = 10 - 10\,i_L(t) + v_c(t)$$

$$\left.\frac{di_L(t)}{dt}\right|_{t=0} = 10 - 10\,i_L(0) + v_c(0)$$

$$= 10 - 10 \times 3 + 5$$

$$= 10 - 30 + 5 = -15\ A/s$$

$$i_L(t) = K + (A_1 t + A_2)e^{-\alpha t}$$

$$\boxed{i_L(0) = K + A_2 = 3} \quad \cdots (I)$$

$$\frac{di_L(t)}{dt} = -\alpha(A_1 t + A_2)e^{-\alpha t} + A_1 e^{-\alpha t}$$

$$\left.\frac{di_L}{dt}\right|_{t=0} = -\alpha A_2 + A_1 = -15$$

$$\boxed{-5A_2 + A_1 = 15} \quad \cdots (II)$$

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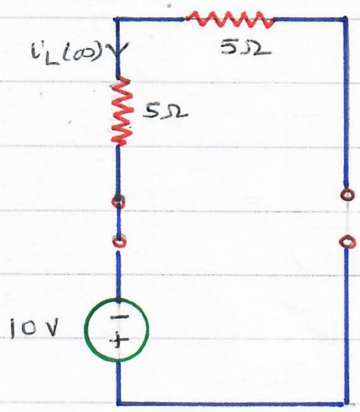
Circuit at $t = \infty$



$i_L(\infty) = 0$

$i_L(t) = K + (A_1 t + A_2) e^{-\alpha t}$

$\Rightarrow i_L(\infty) = 0 = K$

$\Rightarrow \boxed{K = 0}$

(I) $\therefore$ $K + A_2 = 3$

$\Rightarrow \boxed{A_2 = 3}$

(II) $-5A_2 + A_1 = -15$

$\Rightarrow \boxed{A_1 = -15 + 5A_2 = -15 + 5 \times 3 = 0}$

$i_L(t) = K + (A_1 t + A_2) e^{-5t}$

$\Rightarrow \boxed{i_L(t) = 3e^{-5t} \ A}$

Q) The response of a series RLC circuit is

$$v_c(t) = 50 - 56.25e^{-t} + 6.25e^{-9t} \quad \text{and}$$

$$i_L(t) = 506.25e^{-t} - 506.25e^{-9t}$$

where $v_c(t)$ and $i_L(t)$ are the capacitor voltage and inductor currents respectively. Determine the values of R, L, and C.

Sol. The response is that of a overdamped RLC series circuit.

$$s_1, s_2 = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2}$$

$$-\alpha - \sqrt{\alpha^2 - \omega_0^2} = -9 \quad \dots (A)$$

$$-\alpha + \sqrt{\alpha^2 - \omega_0^2} = -1 \quad \dots (B)$$

Adding eqns (A) and (B)

$$-2\alpha = -10$$

$$\Rightarrow \alpha = 5$$

For a series RLC circuit $\frac{R}{2L} = \alpha$

$$\Rightarrow \boxed{\frac{R}{2L} = 5}$$

Subtracting eq. (A) from (B) gives

$$2\sqrt{\alpha^2 - \omega_0^2} = 8$$

$$\Rightarrow \alpha^2 - \omega_0^2 = 4^2$$

$$\Rightarrow \omega_0^2 = 5^2 - 4^2 = 3^2$$

$$\Rightarrow \omega_0 = 3 \text{ rad/s}$$

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}} = 3} \Rightarrow \boxed{\frac{1}{LC} = 9}$$

$$i_L = C\frac{dv_c}{dt}$$

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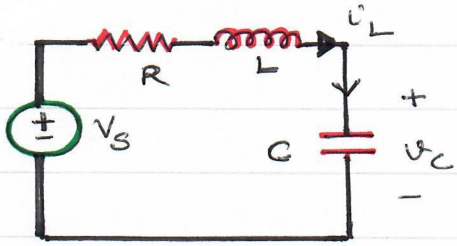
$v_c(0) = 50 - 56.25 + 6.25 = 0\,V$

$i_L(0) = 506.25 - 506.25 = 0\,A$

$v_c(t) = 50 - 56.25e^{-t} + 6.25e^{-9t}$

$\frac{dv_c(t)}{dt} = 56.25e^{-t} - 56.25e^{-9t}$

$C\frac{dv_c(t)}{dt} = i_L(t)$ (since, both L and C are in series



$C(56.25e^{-t} - 56.25e^{-9t}) = 506.25e^{-t} - 506.25e^{-9t}$

$56.25C(e^{-t} - e^{-9t}) = 506.25(e^{-t} - e^{-9t})$

$\boxed{C = \frac{506.25}{56.25} = 9F}$

$\frac{1}{LC} = 9 \quad \Rightarrow L = \frac{1}{9C} = \frac{1}{81}\,H$

$\frac{R}{2L} = 5 \quad \Rightarrow R = \frac{10}{81}\,\Omega$

$\therefore \boxed{R = \frac{10}{81}\,\Omega \;;\quad L = \frac{1}{81}\,H \;;\quad C = 9F}$