



CSE251

Basics of Computer Graphics

Module: Geometry

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Spring 2019

Overview

Preliminary Concepts

Translations and Rotations

Other Transforms

Composite Transformations

Transformations About A Point

Points and Frames

Points and Frames

Rolling Wheel

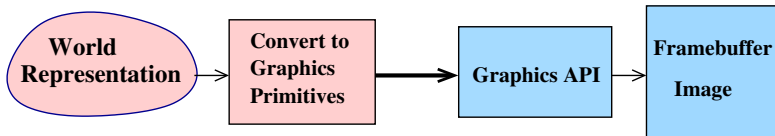
Rotations

3D Rotations about an Axis

Arbitrary Axis, Point

Transforming Lines and Planes

Graphics Process



- ▶ **Model** the desired world in your head.
- ▶ **Represent** it using natural structures in the program. Convert to standard primitives supported by the API
- ▶ **Processing** is done by the API. Converts the primitives in stages and forms an image in the framebuffer
- ▶ The image is displayed automatically on the device

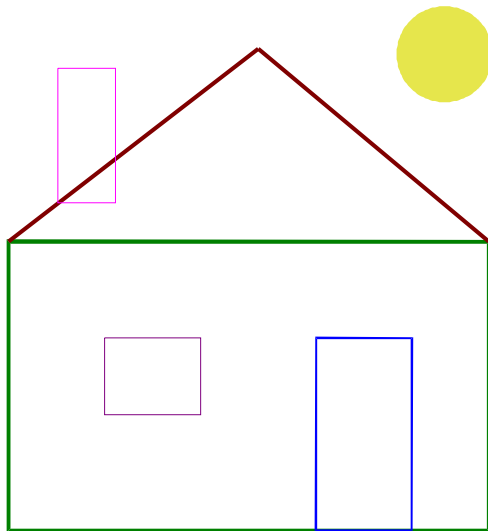
How to Draw A House?

- ▶ Compose out of basic shapes

<code>drawRectangle(v1, v2, v3, v4);</code>	<code>// Main part</code>
<code>drawTriangle(v2, v3, v5);</code>	<code>// Roof</code>
<code>drawRectangle(...);</code>	<code>// Door</code>
<code>drawRectangle(...);</code>	<code>// Window</code>
<code>drawRectangle(...);</code>	<code>// Chimney</code>
<code>drawCircle(...);</code>	<code>// Sun</code>

- ▶ That's all, really!

Resulting House



Graphics Primitives

- ▶ Points: 2D or 3D. (x, y) or (x, y, z) .
- ▶ Lines: specified using end-points
- ▶ Triangles/Polygons: specified using vertices
- ▶ Why not **circles, ellipses, hyperbolas**?

Graphics Attributes

- ▶ Colour, Point width.
- ▶ Line width, Line style.
- ▶ Fill, Fill Pattern.

Point Representation

- ▶ A point is represented using 2 or 3 numbers $(x, y, [z])$ that are the projections on to the respective coordinate axes.
- ▶ Fundamental shape-defining primitive in most Graphics APIs. Everything else is built from it!
- ▶ Represented using **byte, short, int, float, double**, etc.
- ▶ The scale and unit are application dependent. Could be angstroms or lightyears!
- ▶ Points undergo transformations:
Translations, Rotations, Scaling, Shearing.

3D Coordinates

► Cartesian: (x, y, z) .

► Polar: (ρ, θ, ϕ)

► $z =$

$y =$

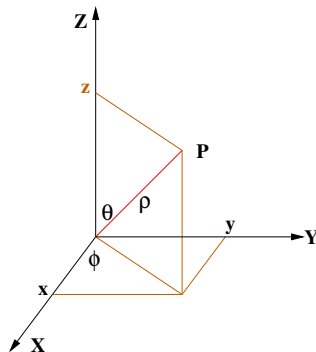
$x =$

► $\rho =$

$\phi =$

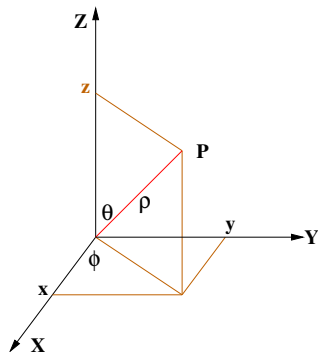
$\theta =$

► Elevation: θ , Azimuthal: ϕ



3D Coordinates

- ▶ Cartesian: (x, y, z) .
- ▶ Polar: (ρ, θ, ϕ)
- ▶ $z = \rho \cos \theta$
 $y = \rho \sin \theta \sin \phi$
 $x = \rho \sin \theta \cos \phi$
- ▶ $\rho^2 = x^2 + y^2 + z^2$
 $\phi = \tan^{-1}(y/x)$
 $\theta = \tan^{-1}(\sqrt{x^2 + y^2}/z)$
- ▶ Elevation: θ , Azimuthal: ϕ



Translation

- ▶ Translate a point $P = (x, y, [z])$ by $(a, b, [c])$.
- ▶ Points coordinates become $P' = (?, ?, ?)$.
- ▶ In vector form, $P' = ?$.

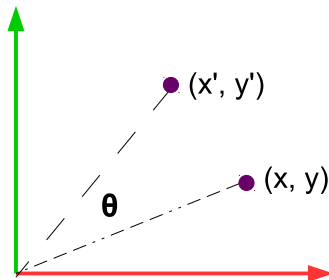
Translation

- ▶ Translate a point $P = (x, y, [z])$ by $(a, b, [c])$.
- ▶ Points coordinates become $P' = (x + a, y + b, [z + c])$.
- ▶ In vector form, $P' = P + T$, where $T = (a, b, [c])$.
- ▶ Distances, angles, parallelism are all maintained.

2D Rotation

- ▶ Rotate about origin CCW by θ .
- ▶ $x' = ?$, $y' = ?$
- ▶ Matrix notation: $P' = R P$

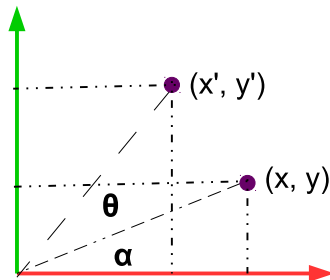
$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



2D Rotation

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$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

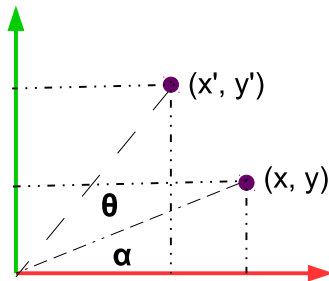


2D Rotation

- ▶ Rotate about origin CCW by θ .
- ▶ $x' = x \cos \theta - y \sin \theta$,
 $y' = x \sin \theta + y \cos \theta$.

- ▶ Matrix notation: $P' = R P$

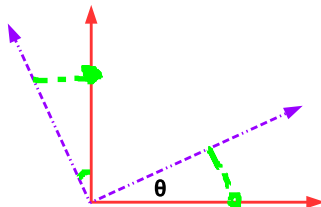
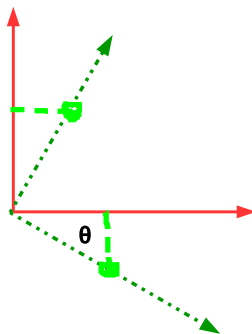
$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



2D Rotation: Observations

$$R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

- ▶ Orthonormal: $R^{-1} = R^T$
- ▶ Rows: vectors that **rotate to** coordinate axes
- ▶ Cols: vectors coordinate axes **rotate to**
- ▶ Invariants: distances, angles, parallelism.



3D Rotations

- ▶ Rotation could be about any axis in 3D!
- ▶ About Z-axis: Just like 2D rotation case. Z-coordinates don't change anyway.
- ▶ X-Y coordinates change exactly the same way as in 2D.
- ▶ CCW +ve, when looking into the **arrowhead**

$$R_z(\theta) = ??$$

3D Rotations

- ▶ Rotation could be about any axis in 3D!
- ▶ About Z-axis: Z-coordinates don't change anyway

$$R_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- ▶ CCW +ve; orthonormal; length preserving
- ▶ Rows: vectors that rotate onto axes; columns: vectors that axes rotate into....

3D Rotations

$$R_y = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

- ▶ CCW +ve; orthonormal
- ▶ Rows: vectors that rotate onto axes; columns: vectors that axes rotate into....
- ▶ Rotation about an arbitrary axis, for example, $[1, 1, 1]^T$??
Coming soon

Non-uniform Scaling

- ▶ Scale along X, Y, Z directions by s , u , and t .
- ▶ $x' = s x$, $y' = u y$, $z' = t z$.
- ▶ We are more comfortable with $P' = S P$ or

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} s & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & t \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- ▶ Invariants: parallelism, ratios of lengths in any direction (Angles also for uniform scaling.)

Shearing

- Add a little bit of x to y or other combinations

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} 1 & x_y & x_z \\ y_x & 1 & y_z \\ z_x & z_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- One of $x_y, x_z, y_x, y_z, z_x, z_y \neq 0$. Rectangles can become parallelograms.
- Invariants: parallelism, ratios of lengths in any direction.

Reflection

- ▶ Negative entries in a matrix indicate reflection.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- ▶ Reflection needn't be about a coordinate axis alone

General Transformation

- ▶ Rotation, scaling, shearing, and reflection: **Matrix-vector** product. Vectors get transformed into other vectors
- ▶ Translation alone is a **vector-vector** addition
- ▶ Sequence of: Translation, rotation, scaling, translation and rotation: $\mathbf{P}' = \mathbf{R}_2 [\mathbf{S} \mathbf{R}_1 (\mathbf{P} + \mathbf{t}_1) + \mathbf{t}_2]$
- ▶ If translation is also a matrix-vector product, we can combine above transformations into a single matrix:
 $\mathbf{P}' = \mathbf{R}_2 \mathbf{T}_2 \mathbf{S} \mathbf{R}_1 \mathbf{T}_1 \mathbf{P} = \mathbf{M} \mathbf{P}$
- ▶ How?

General Transformation

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- ▶ How? Answer: **homogeneous coordinates**!

Homogeneous Coordinates

- ▶ Add a *non-zero scale factor* w to each coordinate.
A 2D point is represented by a vector $[x \ y \ w]^T$
- ▶ $[x \ y \ w]^T \equiv (x/w, y/w)$.
- ▶ Translate $[x \ y]^T$ by $[a \ b]^T$ to get $[x + a \ y + b]^T$

$$\begin{bmatrix} x + a \\ y + b \\ 1 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

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$$\begin{bmatrix} x + a \\ y + b \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- ▶ Now, translation is also: $\mathbf{P}' = \mathbf{T} \mathbf{P}$, a matrix-vector product and a linear operation.

Homogeneous Coordinates

- ▶ Add a *non-zero scale factor* w to each coordinate.
A 2D point is represented by a vector $[x \ y \ w]^T$
- ▶ $[x \ y \ w]^T \equiv (x/w, y/w)$.
- ▶ Now, translation is also: $\mathbf{P}' = \mathbf{T} \mathbf{P}$
- ▶ For a point: Rotation followed by translation followed by scaling, followed by another rotation: $\mathbf{P}' = \mathbf{R}_2 \mathbf{S} \mathbf{T} \mathbf{R}_1 \mathbf{P}$.
- ▶ Similarly for 3D. Points represented by: $[x \ y \ z \ w]^T$.
- ▶ All matrices are 3×3 in 2D. Last row is $[0 \ 0 \ 1]$.
- ▶ All matrices are 4×4 in 3D. Last row is $[0 \ 0 \ 0 \ 1]$.

Homogeneous Representation

- ▶ Convert to a 4-vector with a scale factor as fourth.
 $(x, y, z) \equiv [kx \ ky \ kz \ k]^T$ for any $k \neq 0$.
- ▶ Translation, rotation, scaling, shearing, etc. become linear operations represented by 4×4 matrices.
- ▶ What does $[x \ y \ z \ 0]^T$ mean?
- ▶ $[a \ b \ c \ d]^T$ could be a point or a plane. Lines are specified using two such vectors, either as join of two points or intersection of two planes!

Transformation Matrices: Rotations

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$R_y = \begin{bmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad R_z = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- CCW +ve; orthonormal; length preserving; rows give direction vectors that rotate onto each axis; columns

Translation, Scaling, Composite

$$T(a, b, c) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad S(a, b, c) = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- ▶ A sequence of transforms can be represented using a composite matrix: $\mathbf{M} = \mathbf{R}_x \mathbf{T} \mathbf{R}_y \mathbf{S} \mathbf{T} \dots$
- ▶ Operations are not commutative, but are associative.
- ▶ \mathbf{RT} and \mathbf{TR} ??

Rotation and Translation

► $T_{4 \times 4} = \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$ and

$$R_{4 \times 4} = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$$

► $T R = \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} = ?$

► $R T = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} = ?$

Rotation and Translation

$$\blacktriangleright T_{4 \times 4} = \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \quad \text{and} \quad R_{4 \times 4} = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$\blacktriangleright T R = \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$\blacktriangleright R T = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{Rt} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$\blacktriangleright TR = RT \text{ if: (a) } \mathbf{R} = \mathbf{I} \text{ or (b) } \mathbf{t} = \mathbf{0} \text{ or (c) } \mathbf{Rt} = ?$$

Rotation and Translation

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$$\blacktriangleright TR = RT \text{ if: (a) } \mathbf{R} = \mathbf{I} \text{ or (b) } \mathbf{t} = \mathbf{0} \text{ or (c) } \mathbf{Rt} = \mathbf{t}$$

$$\blacktriangleright \text{When is } \mathbf{Rt} = \mathbf{t} \text{? eigenvector of } \mathbf{R}$$

Rotation and Translation

- ▶ $T_{4 \times 4} = \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$ and $R_{4 \times 4} = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$
- ▶ $TR = \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$
- ▶ $RT = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{Rt} \\ \mathbf{0} & 1 \end{bmatrix}$
- ▶ $TR = RT$ if: (a) $\mathbf{R} = \mathbf{I}$ or (b) $\mathbf{t} = \mathbf{0}$ or (c) $\mathbf{Rt} = \mathbf{t}$
- ▶ When is $\mathbf{Rt} = \mathbf{t}$? when \mathbf{t} is an eigenvector of \mathbf{R}

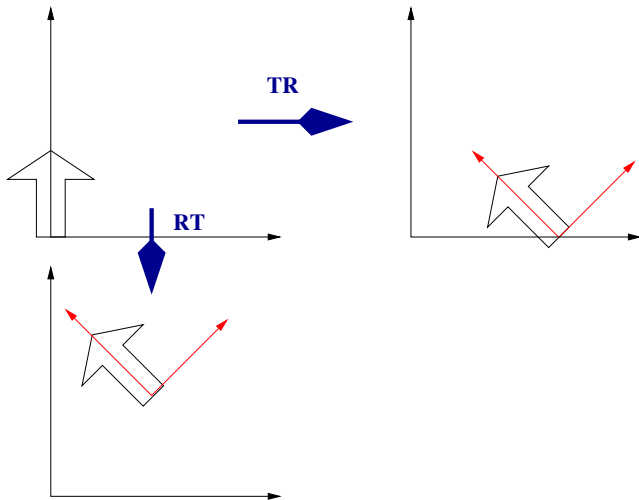
Commutativity

- ▶ Translations are commutative: $\mathbf{T}_1\mathbf{T}_2 = \mathbf{T}_2\mathbf{T}_1$
- ▶ Scaling is commutative: $\mathbf{S}_1\mathbf{S}_2 = \mathbf{S}_2\mathbf{S}_1$
- ▶ Are rotations commutative? $\mathbf{R}_1\mathbf{R}_2 \stackrel{?}{=} \mathbf{R}_2\mathbf{R}_1$
- ▶ What would be an example?
Consider the effect on Z-axis of:

Commutativity

- ▶ Translations are commutative: $T_1T_2 = T_2T_1$
- ▶ Scaling is commutative: $S_1S_2 = S_2S_1$
- ▶ Are rotations commutative? $R_1R_2 \neq R_2R_1$
- ▶ Consider the effect on Z-axis of $R_x(90)R_y(90)$ and $R_y(90)R_x(90)$
- ▶ $RT \neq TR$. (If translation is not parallel to rotation axis)
- ▶ Consider: $R(\pi/4)$ and $T(5, 0)$.
Where does the origin $(0, 0)$ go in TR and RT ?

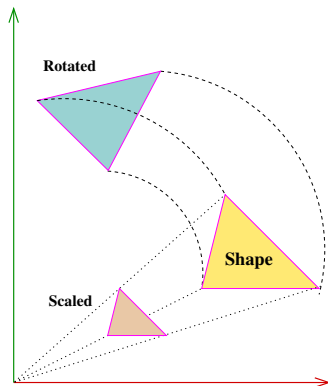
TR and RT



TR keeps it on X axis to $(5, 0)$. **RT** takes it to $(\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}})$.

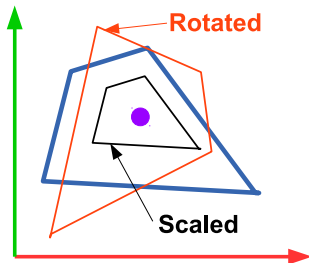
Objects Away from Origin

- ▶ Object “**translates**” when rotated or scaled!!
- ▶ Default: Perform these **about the origin**
- ▶ How do we transform points “**in place**”?
- ▶ Rotate or scale about the centroid of the object. Or about an arbitrary point
- ▶ How?



Transformations About A Point

- ▶ Rotating about point P
 - Align P with origin
 - Rotate/scale about origin
 - Translate back
- ▶ Rotation:
 $\mathbf{R}_C(\theta) = \mathbf{T}(C) \mathbf{R} \mathbf{T}(-C)$
- ▶ Scaling:
 $\mathbf{S}_C() = \mathbf{T}(C) \mathbf{S}() \mathbf{T}(-C)$
- ▶ A transformation \mathbf{M} :
 $\mathbf{M}_C = \mathbf{T}(C) \mathbf{M} \mathbf{T}(-C)$



R, T Operations on Points

- ▶ **T(5,0) R($\pi/4$)**: Impact on a point:

- ▶ R($\pi/4$):

(Point stays at **(0, 0)**)

- ▶ T(5, 0) :

(Point goes to **(5, 0)**)

- ▶ **R($\pi/4$) T(5,0)**: Impact on the point:

- ▶ T(5, 0):

(Point moves to **(5, 0)**)

- ▶ R($\pi/4$).

(Point rotates about origin)

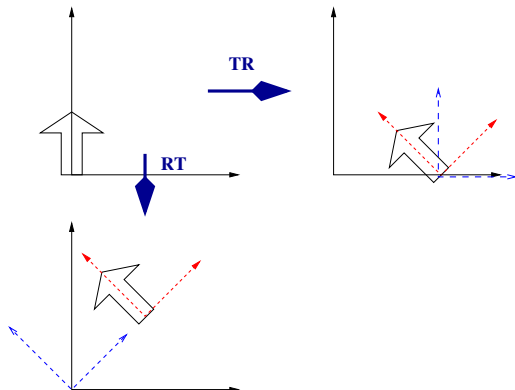
- ▶ All points on the shape undergo the same motions and get new coordinates

R, T Operations on Frames

- ▶ **T(5,0) R($\pi/4$)**: Impact on coordinate frame:
 - ▶ T(5, 0): (Origin shifted to (5, 0))
 - ▶ R($\pi/4$). (Axes rotated at new origin)
- ▶ **R($\pi/4$) T(5,0)**: Impact on coordinate frame:
 - ▶ R($\pi/4$): (Axes rotate by 45 degrees))
 - ▶ T(5, 0). (Point moves to (5, 0) in new axes)
- ▶ Frames move around, giving new coordinates to points on objects!!

Relating Coordinate Frames

- ▶ $T(5, 0)$ and $R(\pi/4)$
- ▶ Start: Black axes
Next: Blue axes
Last: Red axes



Points and Frames in General

- ▶ Points go through changes in a common coordinate frame when a sequence of transformations is **viewed from right to left**
- ▶ Coordinate system goes through the same transformations when the sequence is **viewed from left to right**
- ▶ Composite transformations $P' = M_1 M_2 M_3 P$ relates the coordinates in successive coordinate frames as we go from left to right, starting with $X'Y'$ coordinate frame to finally the XY frame.

Transforming the World Reference

- ▶ Consider $P_4 = \mathbf{M}_4\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1 P_0$
- ▶ Point P_0 undergoes 4 operations and get coordinates P_4
- ▶ Imagine the point having coordinates P_1, P_2, P_3 after operations $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3$
- ▶ We can also visualize coordinate frames $\Pi_4, \Pi_3, \Pi_2, \Pi_1, \Pi_0$ in which a point has coordinates P_4 to P_0 respectively
- ▶ Operation \mathbf{M}_i represents a change in coordinates from Π_i to Π_{i-1} , resulting in new labels for the point.

Let us look at Ourselves

- ▶ Model IIIT Campus as a whole. Campus is our “world”
- ▶ Global coordinate frame Π_G for the campus: at the Gate
- ▶ Buildings: Himalaya, Vindhya, Bakul, Parul, ..., Palash. Each has a natural coordinate frame. Π_H is Himalaya's
- ▶ Himalaya has several rooms: H105, H204, H205, H304, etc., with own coordinate frames. Π_C is of H205 (our class)
- ▶ H205 has 55 desks, with coord frames Π_{Di} for desk i
- ▶ Desks are identical in geometry; the coord frame Π_{Di} places each in its location.

Consider a Desk

- ▶ Consider a corner point P of desk 37, with coordinates $(200, 30, 100)$ in Π_{D37} . That is: $P_{D37} = (200, 30, 100)$
- ▶ Since our world is the campus, we have to ultimately describe everything in the coordinate frame Π_G

$$P_G = M_{GH} M_{HC} M_{CD37} P_{D37}$$

- ▶ M_{GH} aligns Π_G to Π_H . M_{HC} aligns Π_H to Π_C .
 M_{CD37} aligns Π_C to Π_{D37}

- ▶ $P_G = M_{GH} \overset{P_H}{\mid} M_{HC} \overset{P_C}{\mid} M_{CD37} \overset{P_{D37}}{\mid} P$ (for any point P on Desk37)

- ▶ We can place a given desk in any **building, room, place!**

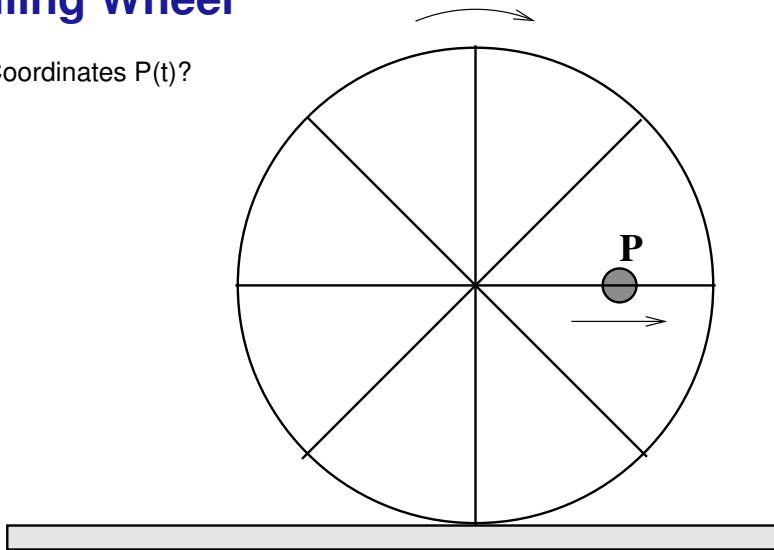
Walking on Stage

- ▶ Person walking horizontally on stage, with swinging arms
- ▶ How does the hand-tip move w.r.t each student? **How?**
- ▶ Student knows own position in room's reference frame
- ▶ Start at a student's eye. (That provides the viewpoint!)
- ▶ Align to room's reference frame using \mathbf{M}_1 . Different matrix for each student, but everyone same now....
- ▶ Walk: pure translation. \mathbf{M}_2 aligns to person coord frame
- ▶ Arm swing: Simple harmonic motion with angle $\theta(t)$

Simpler viewpoints in newer coord frames.

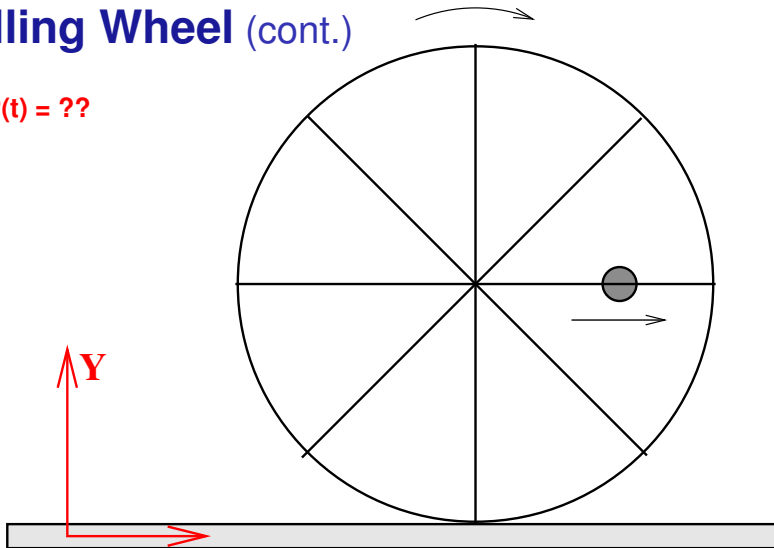
Rolling Wheel

Coordinates $P(t)$?



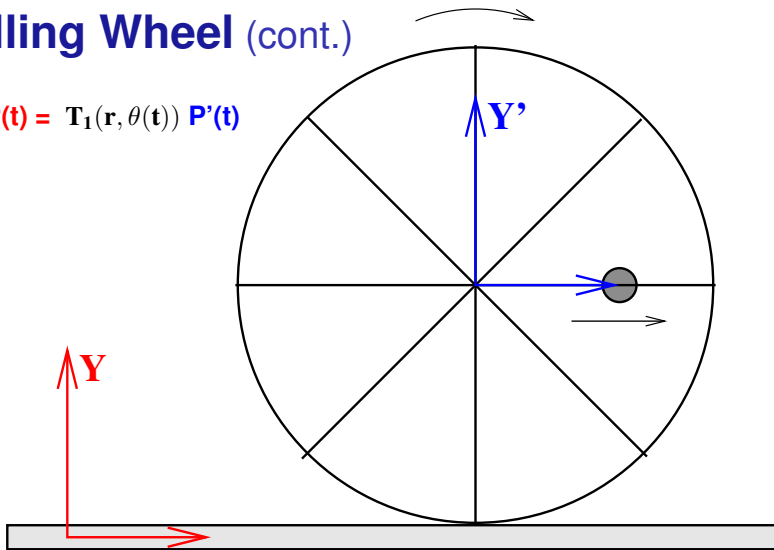
Rolling Wheel (cont.)

$P(t) = ??$



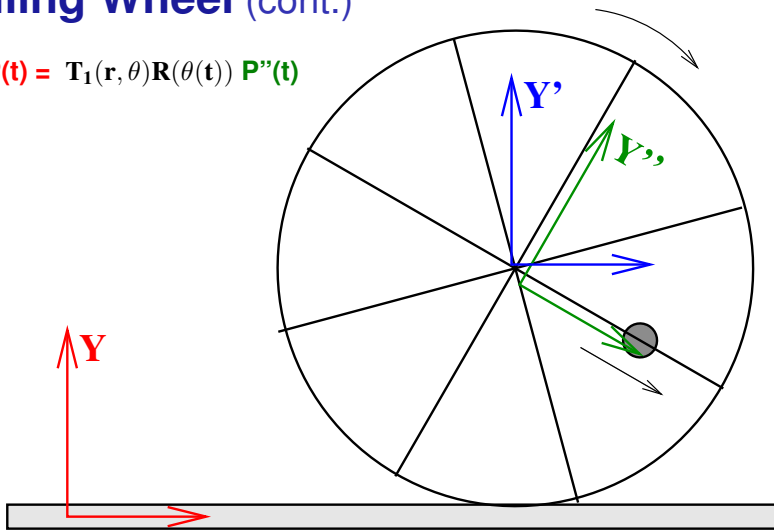
Rolling Wheel (cont.)

$$\mathbf{P}(t) = \mathbf{T}_1(\mathbf{r}, \theta(t)) \mathbf{P}'(t)$$



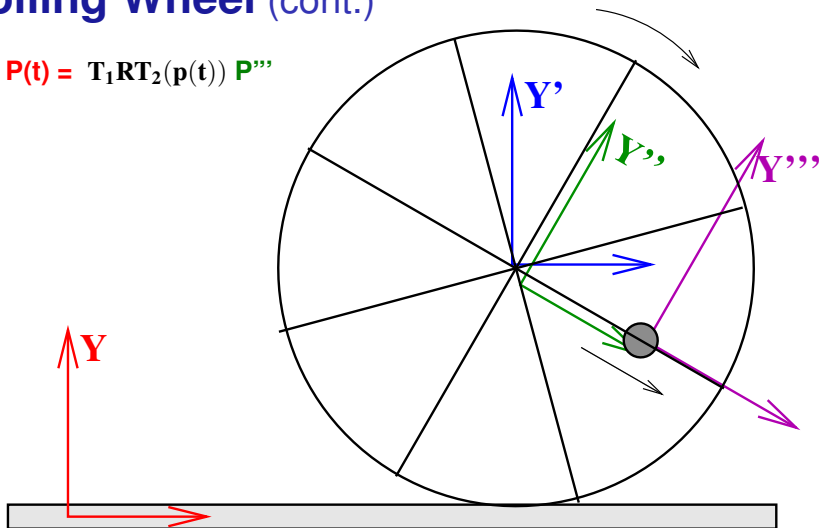
Rolling Wheel (cont.)

$$\mathbf{P}(t) = \mathbf{T}_1(\mathbf{r}, \theta) \mathbf{R}(\theta(t)) \mathbf{P}''(t)$$



Rolling Wheel (cont.)

$$\mathbf{P}(t) = \mathbf{T}_1 \mathbf{R} \mathbf{T}_2(\mathbf{p}(t)) \mathbf{P}'''$$

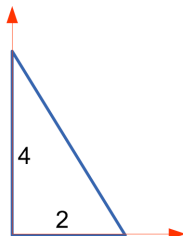


Final Transformation

- ▶ $\mathbf{P}(\mathbf{t}) = \mathbf{T}_1(\mathbf{t}) \mathbf{R}(\theta(\mathbf{t})) \mathbf{T}_2(\mathbf{p}(\mathbf{t})) \mathbf{P}'''$
- ▶ $\mathbf{T}_1(\mathbf{t}) = \mathbf{T}(\mathbf{r} \theta(\mathbf{t}), \mathbf{r}) = \mathbf{T}(\mathbf{r} \omega \mathbf{t}, \mathbf{r})$ (A translation matrix)
- ▶ $\mathbf{R}(\theta(\mathbf{t})) = \mathbf{R}_Z(\omega \mathbf{t})$ (A normal rotation matrix)
- ▶ $\mathbf{T}_2(\mathbf{t}) = \mathbf{T}(\mathbf{p}(\mathbf{t}), \mathbf{0}) = \mathbf{T}(\mathbf{v} \mathbf{t}, \mathbf{0})$ (A translation matrix)
- ▶ $\mathbf{P}''' = [0, 0, 1]^T$ (Origin of the bead)
- ▶ Lot simpler than thinking about it all together.
- ▶ What if we have a pendulum swinging freely on the bead?

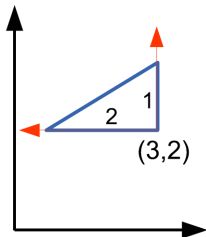
Given an object

- ▶ An object `triangleObj` is given. Can be drawn using `drawObject (triangleObj)`



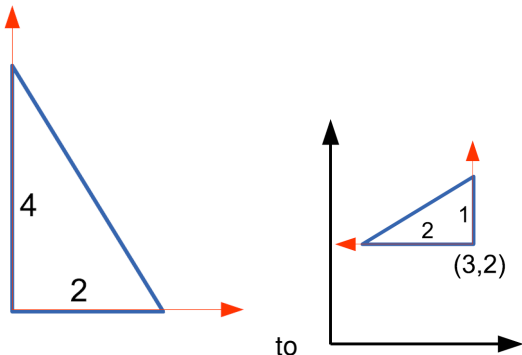
- ▶ `drawObject (triangleObj)` draws the object at (current) origin

Draw it in a different configuration



- Use `drawObject (triangleObj)`, with right transformations

Transformations

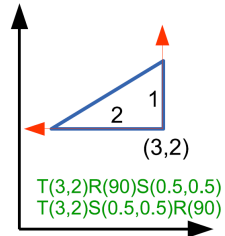
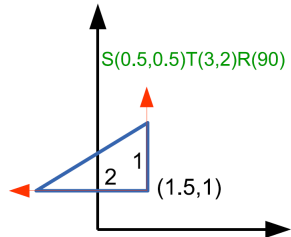
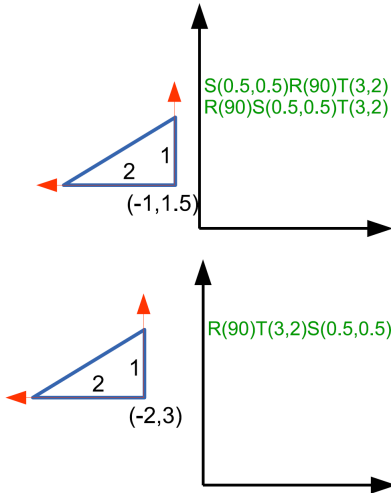


- ▶ What are the transformations? Combination of Translation, Rotation, Scaling!!
- ▶ Operations involved: $S(0.5, 0.5)$, $T(3, 2)$, $R(90)$

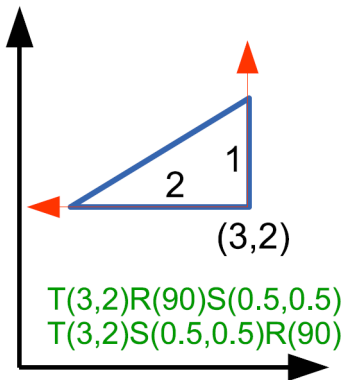
Which combination ?

1. **S(0.5, 0.5), R(90), T(3, 2)**
2. **S(0.5, 0.5), T(3, 2), R(90)**
3. **T(3, 2), R(90), S(0.5, 0.5)**
4. **T(3, 2), S(0.5, 0.5), R(90)**
5. **R(90), S(0.5, 0.5), T(3, 2)**
6. **R(90), T(3, 2), S(0.5, 0.5)**

Which combination ? (cont.)



Several Correct Situations

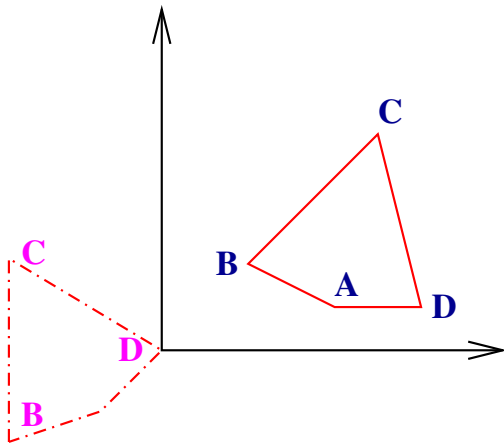


Another Similar Scenario

- ▶ A clock is hanging from a nail fixed to a flat plate. The plate is being translated with a velocity \vec{v} and acceleration \vec{a} . The pendulum of the clock swings back and forth with a time period of 5 seconds and a max angle of $\pm\theta$. An ant travels from the bottom tip of the pendulum up to the centre.
- ▶ How do we compute the ant's position with respect to a fixed coordinate system coplanar with the plate?

Please sketch the situation and work it out for yourself

A Transformation Problem

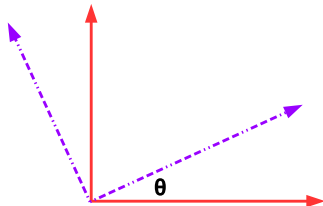
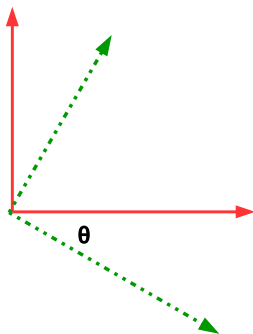


Bring **D** to origin and **BC** parallel to the Y axis as shown

2D Rotation: Observations

$$R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

- ▶ Orthonormal: $R^{-1} = R^T$
- ▶ Rows: vectors that **rotate to** coordinate axes
- ▶ Cols: vectors coordinate axes **rotate to**
- ▶ Invariants: distances, angles, parallelism.



Transformation Computation

- ▶ Step 1: Translate by $-\mathbf{D}$. What is the orientation of BC?
- ▶ Step 2: Rotate to have unit vector $\vec{\mathbf{u}} = [u_x \ u_y]^T$ from \mathbf{B} to \mathbf{C} on the Y axis. That is the second row of \mathbf{R} matrix
- ▶ The matrix for the total operation: $\mathbf{M} = \mathbf{T}(-\mathbf{D})\mathbf{R}$
- ▶ Two options for first row. $[u_y \ -u_x]^T$ and $[-u_y \ u_x]^T$
- ▶ \mathbf{R} matrix: (a) $\begin{bmatrix} u_y & -u_x \\ u_x & u_y \end{bmatrix}$ or (b) $\begin{bmatrix} -u_y & u_x \\ u_x & u_y \end{bmatrix}$?
- ▶ Difference? The direction aligned to the X-axis!
- ▶ Option (a) is correct. **Why?** Draw Option (b)!

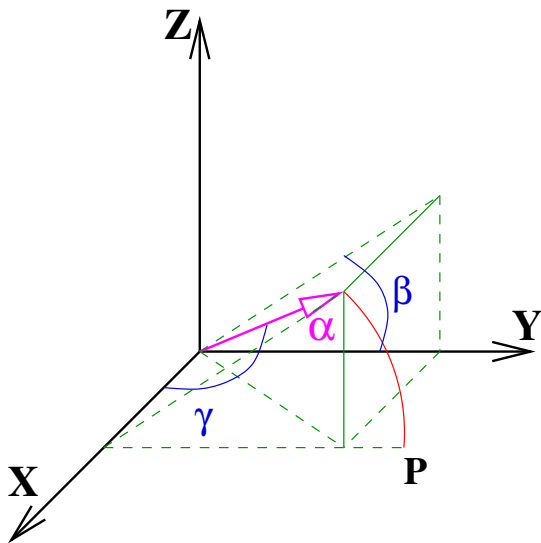
Rotation about an axis parallel to Z

- ▶ An axis parallel to Z axis, passing through point $(x, y, 0)$.
- ▶ Translate so that the axis passes through the origin: $\mathbf{T}(-x, -y, k)$ for any k !!
- ▶ Overall: $\mathbf{M} = \mathbf{T}(x, y, -k) \mathbf{R}_Z(\theta) \mathbf{T}(-x, -y, k)$
- ▶ Why shouldn't k matter? \mathbf{R}_Z doesn't affect the z coordinate. So, whatever k is added first will be subtracted later

3D Rotation about an axis α

- ▶ What is $\mathbf{R}_\alpha(\theta)$?
- ▶ 3-step process:
 1. Apply $\mathbf{R}_{\alpha\mathbf{x}}$ to align α with the X axis.
 2. Rotate about X by angle θ .
 3. Undo the first rotation using $\mathbf{R}_{\alpha\mathbf{x}}^{-1}$
- ▶ Net result: $\mathbf{R}_\alpha(\theta) = \mathbf{R}_{\alpha\mathbf{x}}^{-1} \mathbf{R}_{\mathbf{x}}(\theta) \mathbf{R}_{\alpha\mathbf{x}}$
- ▶ Quite simple!? What is $\mathbf{R}_{\alpha\mathbf{x}}(\theta)$?
- ▶ **(We can align α with Y or Z axis also)**

3D Rotation about an axis α (cont.)



Computing \mathbf{R}_α

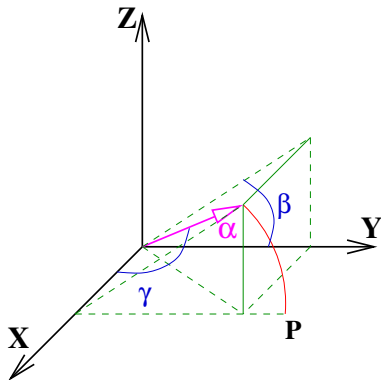
- ▶ First rotate by $-\beta$ about X axis. Vector α would lie in the XY plane, with tip at point \mathbf{P} .
- ▶ $\beta = ?$, $\tan \beta = ?$
- ▶ Next rotate by $-\gamma$ about Z axis. Vector α would coincide with the X axis.
- ▶ $\gamma = ?$, $\tan \gamma = ?$

Computing \mathbf{R}_α

- ▶ Rotate by $-\beta$ about X axis to bring α to XY plane
- ▶ $\tan \beta = \frac{\alpha_z}{\alpha_y}$
- ▶ Rotate by $-\gamma$ about Z axis to bring α to X axis
- ▶ $\tan \gamma = \frac{\sqrt{\alpha_y^2 + \alpha_z^2}}{\alpha_x} = \frac{\sqrt{1 - \alpha_x^2}}{\alpha_x}$ if $|\alpha| = 1$.
- ▶ $\mathbf{R}_{\alpha\mathbf{x}} = \mathbf{R}_z(-\gamma)\mathbf{R}_x(-\beta)$ and $\mathbf{R}_{\alpha\mathbf{x}}^{-1} = \mathbf{R}_x(\beta)\mathbf{R}_z(\gamma)$
- ▶ Alternative: Don't we know about **rotation matrices** and direction cosines that go **to/from coordinate axes**?

Final

► $\mathbf{R}_\alpha(\theta) = \mathbf{R}_x(\beta)\mathbf{R}_z(\gamma) \quad \mathbf{R}_x(\theta) \quad \mathbf{R}_z(-\gamma)\mathbf{R}_x(-\beta)$



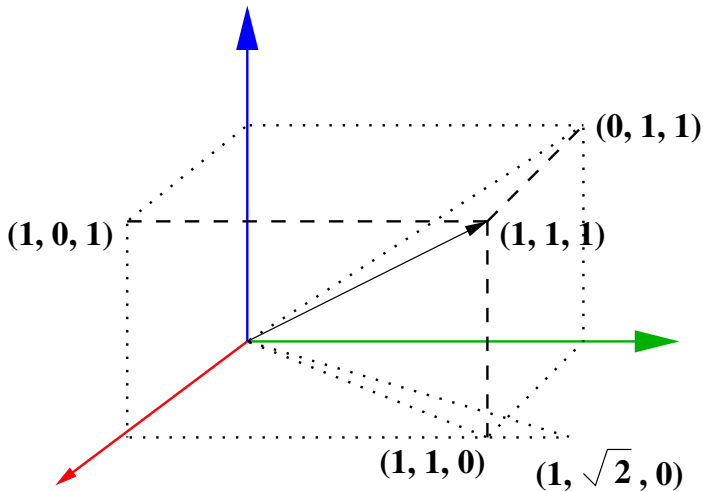
Alternate $\mathbf{R}_{\alpha\mathbf{x}}$

- ▶ After rotation, α will align with X-axis. Hence that is the first row \mathbf{r}_1 of the rotation matrix
- ▶ Find a direction orthogonal to α to be row \mathbf{r}_2 . How?
- ▶ Take any vector \mathbf{v} not parallel to α . $\mathbf{r}_2 = \alpha \times \mathbf{v}$ will work!!

▶ Lastly, $\mathbf{r}_3 = \mathbf{r}_1 \times \mathbf{r}_2$ and $\mathbf{R}_{\alpha\mathbf{x}} = \begin{bmatrix} \alpha & 0 \\ \alpha \times \mathbf{v} & 0 \\ \mathbf{r}_1 \times \mathbf{r}_2 & 0 \\ \mathbf{0} & 1 \end{bmatrix}$

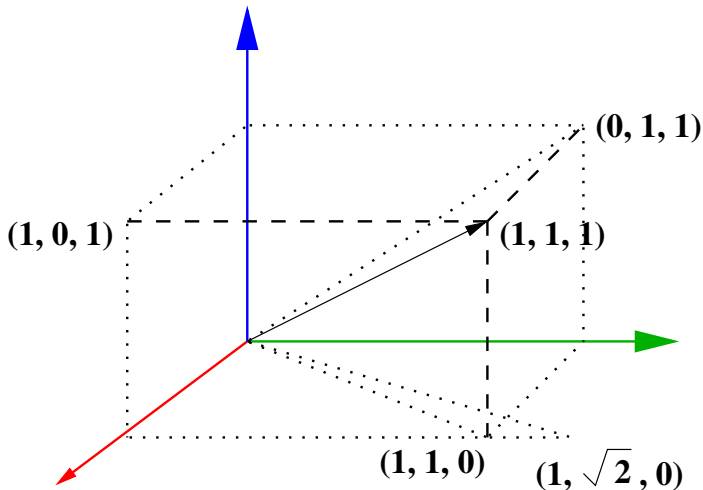
- ▶ Many possibilities, all with the same result (hopefully...)

Example: Rotation about $[1\ 1\ 1]^T$



$$\beta = ?, \quad \gamma = ?$$

Example: Rotation about $[1\ 1\ 1]^T$



$$\tan \beta = 1, \quad \tan \gamma = \sqrt{2}$$

Computing $\mathbf{R}_{\alpha\mathbf{X}}$: Method 1

► Rotate by $-\pi/4$ about X. $\mathbf{R}_{\mathbf{X}}(-\frac{\pi}{4}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

► $\mathbf{R}_{\mathbf{Z}}(-\arctan(\sqrt{2})) = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

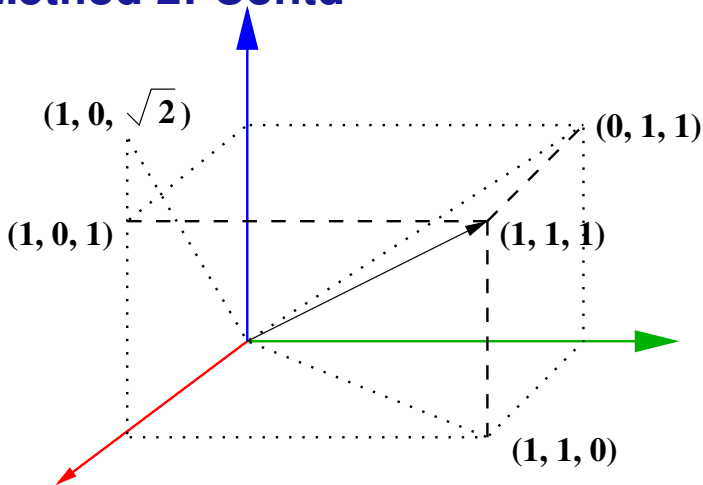
► $\mathbf{R}_{\alpha\mathbf{X}}^{\mathbf{I}} = \mathbf{R}_{\mathbf{Z}}(-\tan^{-1}(\sqrt{2})) \mathbf{R}_{\mathbf{X}}(-\frac{\pi}{4}) = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Computing $\mathbf{R}_{\alpha\mathbf{X}}$: Method 2

- ▶ $[1\ 1\ 1]^T$ will lie on X-axis. First row $\mathbf{r}_1 = [\frac{1}{\sqrt{3}}\ \frac{1}{\sqrt{3}}\ \frac{1}{\sqrt{3}}]^T$.
- ▶ Second row: $\mathbf{r}_2 = \alpha \times [1\ 0\ 0]^T = [0\ \frac{1}{\sqrt{2}}\ \frac{-1}{\sqrt{2}}]^T$
- ▶ Third row: $\mathbf{r}_3 = \alpha \times [0\ \frac{1}{\sqrt{2}}\ \frac{-1}{\sqrt{2}}]^T = [\frac{2}{\sqrt{6}}\ \frac{-1}{\sqrt{6}}\ \frac{-1}{\sqrt{6}}]^T$

$$\text{▶ } \mathbf{R}_{\alpha\mathbf{X}}^{\Pi} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{R}_Y(\tan^{-1}(\sqrt{2})) \mathbf{R}_X(\frac{\pi}{4})$$

$R_{\alpha X}$ Method 2: Contd



Question: Which vector v yields the matrix of Method 1?

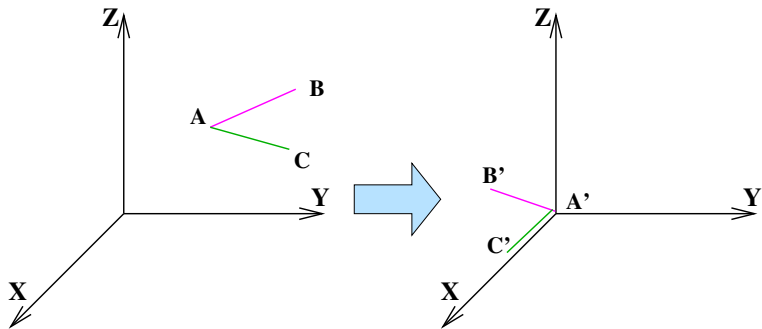
Rotation: Arbitrary Axis, Point

- ▶ An arbitrary axis may not pass through the origin.
- ▶ Translate by \mathbf{T} so that it passes through the origin.
- ▶ Apply \mathbf{R}_α .
- ▶ Translate back using \mathbf{T}^{-1} .
- ▶ Composite transformation: $\mathbf{T}^{-1} \mathbf{R}_\alpha \mathbf{T}$.

3D Transformations

- ▶ Many ways to *think about* a given transform.
- ▶ Ultimately, there is only one transform given the starting and ending configurations.
- ▶ Different methods let us analyze the problem from different perspectives.

Another Example



Transforming Lines

- ▶ A composite transformation can be seen as changing points in the coordinate system.
- ▶ These transformations preserve collinearity. Thus, points on a line remain on a (transformed) line.
- ▶ Take two points on the line, transform them, and compute the line between new points.
- ▶ Lines are defined as a join of 2 points or intersection of 2 planes in 3D. The same holds for transformed lines using transformed points or planes!

Transforming Planes

- ▶ A plane is defined by a 4-vector \mathbf{n} (called the **normal** vector) in homogeneous coordinates.
- ▶ The plane consists of points \mathbf{p} such that $\mathbf{n}^T \mathbf{p} = 0$.
- ▶ Let \mathbf{Q} transform \mathbf{n} when points are transformed by \mathbf{M} .
- ▶ Coplanarity is preserved: $(\mathbf{Qn})^T \mathbf{Mp} = 0 = \mathbf{n}^T \mathbf{Q}^T \mathbf{Mp}$.
- ▶ True when $\mathbf{Q}^T \mathbf{M} = \mathbf{I}$, or $\mathbf{Q} = \mathbf{M}^{-T}$.
- ▶ \mathbf{Q} is the Matrix of cofactors of \mathbf{M} in the general case when \mathbf{M}^{-1} doesn't exist.

Understanding Geometric Transformations

- ▶ Geometry transformation of objects is very important to compose graphics environments
- ▶ Understand what you want to be achieved, visualize it in your mind and compose the series of transformations
- ▶ Needs getting used to the ideas. Think about getting into a simpler situation from the current one.