Introduction to Game Theory
Nash Equilibrium

Sujit Prakash Gujar

sujit.gujar@iiit.ac.in

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Agenda

- Recap

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- General n person Games
  - Iterated Dominance

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  - Notion of Nash Equilibrium

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  - Meaning and Implication of Nash Equilibrium

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  - Notion of Nash Equilibrium
  - Meaning and Implication of Nash Equilibrium
  - Nash's Theorem and Existence

Meet at the Cafe Game

Recall meet at the cafe game[1]:

A \ B	L	G
L	2,1	0.5,0.5
G	0,0	1,2

- Does it has any dominant strategy equilibirium?

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- Can we apply minimax theorem?
- we need another notion of **equilibrium**?

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n-player Games

Recall, n player game in strategic form is represented as
$\Gamma = < N, (S_i)_{i \in N}, (u_i)_{i \in N} >$

N : Set of players
$N = \{1, 2, \ldots, n\}$

S_1 : Strategies available to player 1
S_2 : Strategies available to player 2
$\vdots$
S_n : Strategies available to player n

$S = S_1 \times S_2 \times \ldots \times S_n$
Strategy space of all the players

$u_1 : S \to \mathbb{R}$
$u_2 : S \to \mathbb{R}$
$\vdots$
$u_n : S \to \mathbb{R}$

Utility Functions

- Note that we denote the space of strategies include randomization, that is, it is set of **mixed strategies** by $\Delta(S_i)$ and represent a mixed strategy for a player i as σ_i.

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- After deducing the above, what should Player 2 play?

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- (T,L) is equilibrium

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- In general, if s_i is weakly dominated by some strategy, will not be played by rational agents
- So we can restrict our attention to $S_i \setminus s_i$

  and it may lead to newer dominated strategies!! (Assumption of rationality and common knowledge)

Iterated Dominance...

Algorithm 1: Iterated Dominance

Input: $\Gamma =< N, (S_i)_{i \in N}, (u_i)_{i \in N} >$
Output: $\Gamma' =< N, (Z_i)_{i \in N}, (u_i)_{i \in N} >$
for $i = 1 \rightarrow n$ **do**
  $Z_i^0 = S_i$
$k = 1$
flag=true
while *flag* **do**
  flag=false
  for $i = 1 \rightarrow n$ **do**
    $Z_i^k = Z_i^{k-1}$
    if $\exists s_i$ **dominated** *by some other strategy* **then**
      $Z_i^k = Z_i^k \setminus s_i$
      flag=true;
    $k = k + 1$
for $i = 1 \rightarrow n$ **do**
  $Z_i = Z_i^{k-1}$

Example 2

Consider the following game:

	L	R
T	(0,0)	(4,2)
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- What are strongly/weakly dominated strategies for Player 1? Player 2?

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- What are strongly/weakly dominated strategies for Player 1? Player 2?
- Can the above algorithm give any result even if we consider dominance in mixed strategy?

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What Next

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- Solution concept Dominant strategy equilibrium hardly exists and Iterated dominance may not always be helpful
- Most celebrated solution concept in Game Theory: **Nash Equilibrium**

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Nash Equilibrium

What should be desirable properties of an equilibrium?

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Definition (Pure Strategy Nash Equilibrium)

A strategy profile $(s_1^*, s_2^*, \ldots, s_n^*)$ is called as Pure Strategy Nash Equilibrium (**PSNE**), if for each player i, s_i^* is a best response strategy to s_{-i}*.

That is, $\forall i$

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \; \forall \, s_i \; \in \; S_i$$

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Definition (Mixed Strategy Nash Equilibrium)

A strategy profile $(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ is called as Mixed Strategy Nash Equilibrium, if for each player i, σ_i^* is a best response strategy to σ_{-i}*.

That is, $\forall i$

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Notice difference between u_i and U_i

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John Nash: Nobel Memorial Prize in Economic Sciences (1994)

Image Credits: Elke Wetzig (Elya) - Own work.

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  - Evolution and Steady State

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Does Nash Equilibrium Exists?

- Matching coins game

A \ B	H	T
H	10,-10	-10,10
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- What about minimax theorem? How does the equilibrium we defined for two player zero sum games related to NE?

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Nash Theorem

Theorem (Nash)

Every finite game has at least one Nash Equilibrium (NE).

- What is finite game?

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- What is finite game? NE: PSNE or MSNE?
- In NE player plays his/her best response to the strategy played by the remaining agents

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- Let $b_i(\sigma_{-i})$ denote a set of strategies for i that are best response to σ_{-i}

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- Let $b_i(\sigma_{-i})$ denote a set of strategies for i that are best response to σ_{-i}
- $b : \prod_{i \in N} \Delta(S_i) \rightarrow 2^{\prod \Delta(S_i)}$ as
  $b(\sigma) = (b_1(\sigma_{-1}), b_2(\sigma_{-2}), \ldots, b_n(\sigma_{-n},))$

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  $b(\sigma) = (b_1(\sigma_{-1}), b_2(\sigma_{-2}), \ldots, b_n(\sigma_{-n},))$
- b is called **best response correspondence** of a game.

- Fixed point theorems

Proof: Nash Theorem

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  - Continuity,

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- Fixed point theorems
  - Continuity, Closed

Proof: Nash Theorem

- Fixed point theorems
  - Continuity, Closed and Bounded (Compact)

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Proof: Nash Theorem

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Theorem

Let A be a **non-empty, compact** *and* **convex** *subset of some Euclidean space R^n. Let $\phi : A \rightarrow 2^A$ be a* **set-valued function** *(***correspondence***) on A with a closed graph and the property that $\phi(x)$ is non-empty and convex for all $x \in A$. Then ϕ has a fixed point.*

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- Also true, if (i) $\prod_i \Delta(S_i)$ is compact, convex, and nonempty. (ii) $b(\sigma)$ non-empty. (iii) $b(\sigma)$ is convex. (iv) $b(\sigma)$ is upper hemi-continuous (follows from continuity of $U_i's$)

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- Convince yourself that $\sigma^* \in b(\sigma^*) \Rightarrow \sigma^*$ is NE

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Examples

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- (L,L) and (G,G) are PSNE
- $\sigma_1 = ()$ and $\sigma_2 = ()$ is MSNE..Next Class.

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Hawk-Dove Game

	Hawk	Dove
Hawk	(0,0)	(5,1)
Dove	(1,5)	(3,3)

- (Hawk,Dove) and (Dove,Hawk) are PSNE
- $\sigma_1 = ()$ and $\sigma_2 = ()$ is MSNE..Next Class.

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Further Reading

- **Game Theory and Mechanism Design**, Y Narahari. World Scientific Publishing Company, 2014.
- **Multiagent systems: Algorithmic, game-theoretic, and logical foundations**, Shoham, Yoav, and Kevin Leyton-Brown. Cambridge University Press, 2008. (Free download).
- **Game Theory** by Roger Myerson. Harvard University press, 2013.
- **Algorithmic Game Theory**, edited by Noam Nisan, Tim Roughgarden, Eva Tardos and Vijay Vazerani. (Non-printable version available online).

`http://gametheory.net/`

`http://lcm.csa.iisc.ernet.in/gametheory/lecture.html`

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