Introduction to Game Theory
Nash Equilibrium Computation and Applications

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Agenda

- Examples: How to compute NE?
- Iterated Dominance
- Two Player non-zero sum games and LCP
- Complexity of computing a NE

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Meet at the Cafe Game

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A \ B	L	G
L	2,1	0.5,0.5
G	0,0	1,2

× No dominant strategy equilibibrium

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- (L,L) and (G,G) PSNE
- Does it have MSNE?
- Say Player A plays L with prob p and Player B plays with q (Note this is enough to specify mixed strategy completely though more precise way is $\sigma_1 = (p, 1-p), \sigma_2 = (q, 1-q)$ and $\sigma = (\sigma_1, \sigma_2)$)

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- What is $U_A(L, q), U_A(G, q)$?

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- $U_A(L,q) = 2q + 0.5(1-q)$ and $U_A(G,q) = 1 - q \Rightarrow q = \frac{1}{5}$

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- Similarly what can we say about p?
- And $p = \frac{1}{5}$ or $p = \frac{4}{5}$?

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Hawk-Dove Game

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- Say Player 1 plays H with prob p and Player 2 with q
- What can we say about p and q?
- $5(1-q) = q + 3(1-q)$ and $5(1-p) = p + 3(1-p) \Rightarrow \; p = \frac{2}{3} = q$

Consider the following game:

	1	2	3	4	5	6
1	(0,0)	(5,1)	(3,4)	(1,0)	(3,7)	(0,0)
2	(8,0)	(2,2)	(-4,4)	(-1,0)	(8,7)	(6,4)
3	(2,3)	(3,2)	(3,-4)	(4,0)	(2,7)	(0,0)
4	(1,1)	(6,4)	(2,1)	(2,6)	(3,7)	(0,0)
5	(5,6)	(7,5)	(1,2)	(3,4)	(4,7)	(0,0)
6	(-1,2)	(9,6)	(8,5)	(1,6)	(5,2)	(0,5)
7	(1,4)	(0,7)	(7,9)	(1,0)	(1,7)	(0,0)

- What are PSNE?

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- What are PSNE?(2,5) and (6,2). May be many more...
- How about MSNE?
- Let $\sigma_1 = (p_1, p_2, \ldots, p_7)$ and $\sigma_2 = (q_1, \ldots, q_6)$
- What we can say about these first? $\sum p_i = 1$ $\sum q_j = 1$.

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- What we can say about these first? $\sum p_i = 1$ $\sum q_j = 1$. What more?
- Can we say $U_1(1, \sigma_2) = U_1(2, \sigma_2) = \ldots = U_1(7, \sigma_2)$?

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Nash Equilibrium Computation

Recall Prisoner's Dilemma

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- Where does the assumption of this equality go wrong?

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- Where does the assumption of this equality go wrong?
- Row player is indifferent **among** the actions for which she assigns **non-zero probability** given a mixed strategy of column player.

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  $U_i(s_i, \sigma{-}i) \geq U_i(s_i', \sigma{-}i)\ \forall s_i \in \Omega_i, s_i' \in S_i \setminus \Omega_i$

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Nash Equilibrium Computation

$$w_i = U_i(s_i, \sigma_{-i}) \qquad \forall s_i \in \Omega_i \ \forall i \tag{1}$$

$$w_i \geq U_i(s_i', \sigma_{-i}) \qquad \forall s_i' \in S_i \setminus \Omega_i \ \forall i \tag{2}$$

$$\sigma_i(s_i) > 0 \qquad \forall s_i \in \Omega_i \ \forall i \tag{3}$$

$$\sigma_i(s_i') = 0 \qquad \forall s_i' \in S_i \setminus \Omega_i \ \forall i \tag{4}$$

$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1 \qquad \forall i \tag{5}$$

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$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1 \qquad \forall i \tag{5}$$

Convince yourself that there are $n + 2\sum_i |S_i|$ equations

If we have found $w_1, \ldots, w_n, \sigma_1, \ldots, \sigma_n$ satisfying the above, we have found a NE.

Two Player non-Zero Sum Games

Two Player non-Zero sum games, also called **Bi-matrix** Games

$$U_1(s_i, \sigma_2) = \sum_{s_2 \in S_2} \sigma_2(s_2) \times u_1(s_i, s_2)$$

- $U_1(), U_2()$ are linear equations

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- $U_1(), U_2()$ are linear equations
- Hence, equations (1) - (5) are linear
- Can we solve it in polynomial time?Why or How?

Bi-matrix Games and LCP

- Solving (1) - (5) for two players is called **Linear Complementarity Problem** (LCP)
- LCPs are well studied and useful in LP, Quadratic Programming, computational mechanics
- LCP: **No objective function**, more about feasibility

LCP in Standard Form
Given $M(\in R^{n\times n}), q(\in R^n)$ find $w, z \in R^n$ s.t.

$$\begin{aligned} w^T z &= 0 \\ w &= Mz + q \\ w, z &\geq 0 \end{aligned}$$

- It is shown that Bi-matrix game is equivalent to the above

Complexity of Nash Equilibrium Computation

- Lemke-Howson[1]: Used LCP to solve bi-matrix games
- Time complexity: Worst case exponential
- Nash (1951): NASH reduces to BROWER
- PPAD: Polynomial Parity Arguments on Directed graphs (Papadimtriou 1994)
- Daskalakis, Goldberg, Papadimitriou[2], Chen and Deng[3]: NASH is PPAD complete

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[1] Lemke, Carlton E., and Joseph T. Howson, Jr. "Equilibrium points of bimatrix games." Journal of the Society for Industrial and Applied Mathematics 12.2 (1964): 413-423.

[2] Daskalakis, Constantinos, Paul W. Goldberg, and Christos H. Papadimitriou. "The complexity of computing a Nash equilibrium." SIAM Journal on Computing 39.1 (2009): 195-259.

[3] Chen, Xi, and Xiaotie Deng. "Settling the Complexity of Two-Player Nash Equilibrium." FOCS. Vol. 6. 2006.

- Packet Forwarding Game 1
- Packet Forwarding Game 2
- In Networks:
  Papadimitriou, Christos. "Algorithms, games, and the internet." Proceedings of the thirty-third annual ACM symposium on Theory of computing. ACM, 2001.
- Security and Game Theory
  - where to locate strong antivirus firewalls in network to make it secure
  - Patrolling at air-ports
  - **Patrolling for fare-invasion**
  - ....

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Better Patrolling with Game Theory

- Indian Railways caught a racket in 2012: Travel ticketless in trains; if caught, touts pay for you
  `http://www.thehindu.com/news/national/travel-ticketless-in-trains-if-caught-touts-pay-for-you/article5252855.ece`
- Prof Millind Tambe, University of Southern California: POineer in using Game Theory for Security
- **TRUSTS: Scheduling Randomized Patrols for Fare Inspection in Transit Systems**, Conference on Innovative Applications of Artificial Intelligence (IAAI)

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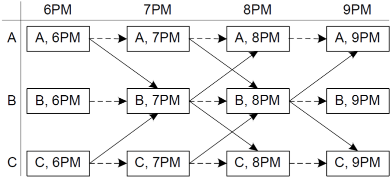
TRUSTS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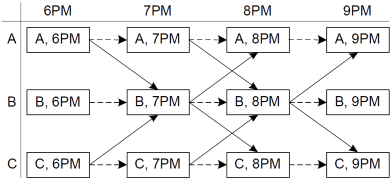
TRUSTS



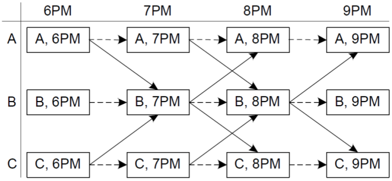
$$\max_{\mathbf{x},\mathbf{u}} \sum_{\lambda \in \Lambda} p_\lambda u_\lambda$$

$$\text{s.t. } u_\lambda \leq \min\{\rho,\ \tau \sum_{e \in \lambda} x_e f_e\}, \text{ for all } \lambda \in \Lambda$$

$$\sum_{v \in V^+} x_{(v^+,v)} = \sum_{v \in V^-} x_{(v,v^-)} \leq \gamma$$

$$\sum_{(v',v) \in E} x_{(v',v)} = \sum_{(v,v^\dagger) \in E} x_{(v,v^\dagger)}, \text{ for all } v \in V$$

$$\sum_{e \in E} l_e \cdot x_e \leq \gamma \cdot \kappa, 0 \leq x_e \leq \alpha, \forall e \in E$$



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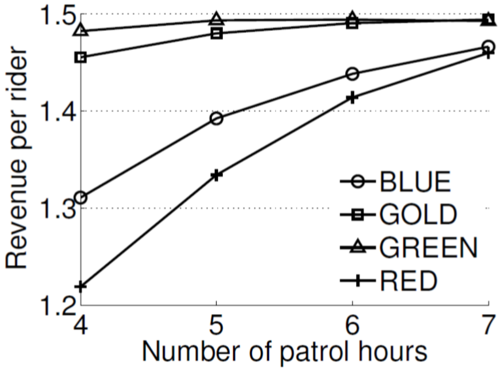
$$\sum_{e\in E} l_e\cdot x_e \leq \gamma\cdot\kappa, 0\leq x_e\leq\alpha, \forall e\in E$$

Λ : set of possible types ($\lambda\in\Lambda$)
x_e: Probability of patrolling on edge e (marginal representation of patrolling strategy)
f_e: Proabablity of getting caught on e if patrolling happens
ρ: Fare per ride
τ : Fine if caught traveling without ticket
γ: No of patrol units available
K: duration for which each patrol unit can work

Performance



- Fare ρ: \$1.5, Fine τ: \$100
- $\gamma = 1$

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Further Reading

- **Game Theory and Mechanism Design**, Y Narahari. World Scientific Publishing Company, 2014.
- **Multiagent systems: Algorithmic, game-theoretic, and logical foundations**, Shoham, Yoav, and Kevin Leyton-Brown. Cambridge University Press, 2008. (Free download).
- **Game Theory** by Roger Myerson. Harvard University press, 2013.
- **Algorithmic Game Theory**, edited by Noam Nisan, Tim Roughgarden, Eva Tardos and Vijay Vazerani. (Non-printable version available online).

`http://gametheory.net/`

`http://lcm.csa.iisc.ernet.in/gametheory/lecture.html`

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