Introduction to Game Theory
Two Player Zero Sum Games

Sujit Prakash Gujar

sujit.gujar@iiit.ac.in

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Agenda

- Recap

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- Two Player Zero Sum Games
  - Mini-max Strategy
  - Saddle Points
  - Mixed Strategies
  - von Neumann - Morgenstern Utility Theory
  - von Neumann minimax Theorem

- Gain of one player = Loss to the Other
  Total sum of utilities = 0.

[1] This term is different than Matrix Form Games

Two Player Zero Sum Games

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- $\Gamma =< \{1,2\}, S_1, S_2, u_1, -u_1 >$. Also called as **Matrix Games**.[1]

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- We can represent the game by a single $m \times n$ matrix. For Example, game $\Gamma^Z =$:

	L	M	R
T	1	2	2
M	-3	-1	-2
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- Consider another game $\Gamma^Z =$:

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- How to analyze this?

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- Any dominant strategy for Player 1?Player 2?
- We need different notion of equilibrium

Saddle Points

- **Saddle Point**: Given a matrix A, (i^*, j^*) is a saddle point if

$$a_{i^*j^*} \leq a_{i^*l} \ \forall l = 1, 2, \ldots, n;$$
$$a_{i^*j^*} \geq a_{kj*} \ \forall k = 1, 2, \ldots, m;$$

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- Any relation between u_R, u_C and saddle points?

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- Exercise: $u_R \leq u_C$

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- Let $u_R = \max_i \min_j a_{ij}$ and $u_C = \min_j \max_i a_{ij}$.
- Any relation between u_R, u_C and saddle points?
- Exercise: $u_R \leq u_C$
- If saddle point exists show that $u_R = u_C$. If $u_R = u_C$, show that saddle point exists.

- If saddle point exists, then for row player: she is maximizing her min assured gain.
  For column player: she is minimizing her worst loss (same as maximizing her min assured gain).

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- Let (i^*, j^*) be a saddle point: What can we say if row player is playing i^*?

Equilibrium In Two Player Zero-Sum Games

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  For column player: she is minimizing her worst loss (same as maximizing her min assured gain).
- Let (i^*, j^*) be a saddle point: What can we say if row player is playing i^*?The column player cannot reduce her loss by deviating from j^*. Convince yourself that same holds true for row player
- If such saddle point exists, it is called an equilibrium.
  The strategy that achieves this is **mini-max** strategy.

Equilibrium Continued

Game Γ^Z:

A \ B	L	M	R
T	1	2	1
M	0	-1	2
B	-1	0	-2

What is u_R? u_C?What is equilibrium?

Equilibrium Continued

Game Γ^Z:

A \ B	L	M	R
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$u_R = 1, u_C = 1.$ Hence (T,L) is an equilibrium.

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Equilibrium Continued

Game Γ^Z:

B / A	L	M	R
T	1	2	1
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$u_R = 1, u_C = 1$. Hence (T,L) is an equilibrium.

	L	M	R
T	0	2	0
M	-1	5	-2
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What is u_R? u_C?What is equilibrium?

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Equilibrium Continued

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$u_R = 0 = u_C$ (T,L) and (B,R) are equilibrium

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Did we list all saddle points?

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	L	M	R
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$u_R = 0 = u_C$ (T,L) and (B,R) are equilibrium
Have you observed that all equilibrium utilities are the same

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Equilibrium Continued

Game Γ^Z:

B / A	L	M	R
T	1	2	1
M	0	-1	2
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$u_R = 1, u_C = 1.$ Hence (T,L) is an equilibrium.

	L	M	R
T	0	2	0
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$u_R = 0 = u_C$ (T,L) and (B,R) are equilibrium
(Exercise: Show that all the saddle of a matrix have same value)

Matching Coins without Observations

A \ B	H	T
H	10,-10	-10,10
T	-10,10	10,-10

$u_R = -10, u_C = 10.$

Matching Coins without Observations

A \ B	H	T
H	10,-10	-10,10
T	-10,10	10,-10

$u_R = -10, u_C = 10.$

No pure strategy equilibrium.

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Mixed Strategies

- In matching pennies game, row player tosses a coin and if H, then play H, else T.
- Similarly column player plays her action.
- Row player expected payoff = Pr(H,H) - Pr(H,T) - Pr(T,H) + Pr(T,T) = 0
- Column Player expected utility = -Pr(H,H) + Pr(H,T) + Pr(T,H) - Pr(T,T)
- Such randomization over across is called as mixed strategy

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Mixed Strategies

- Say for player i, there are i_k actions, $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$.
- She decides to play these actions with probabilities $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$ with $p_{i_1} + p_{i_2} + \ldots + p_{i_k} = 1$
- $\sigma_i = (p_{i_1}, p_{i_2}, \ldots, p_{i_k})$ is mixed strategy of the player i.
- Mixed strategy space $\Delta S_i = i_k - 1$ dimensional simplex
- Examples of simplex: 1-Δ, 2-Δ.

Expected Utility

- For Player i, expected payoff
$$U_i(\sigma_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} p_{i_1} * p(s_{-i}) * U(a_{i_1}, s_{-i}) + p_{i_2} * p(s_{-i}) * U(a_{i_2}, s_{-i}) + \ldots + p_{i_k} * p(s_{-i}) * U(a_{i_k}, s_{-i})$$

- For two-player zero sum games, we refer mixed strategies as $p = (p_1, \ldots, p_m), q = (q_1, \ldots, q_n)^T$
- Mixed strategies leads to **Utility Theory**

Utility Theory (1)

Let X be the set of outcomes. $\succ$ be the preference of a player over the set of outcomes.
Axioms

- **Completeness**: every pair of outcomes is ranked
- **Transitivity**: If $x_1 \succ x_2$ and $x_2 \succ x_3$ then $x_1 \succ x_3$.
- **Substitutability**: If $x_1 \sim x_2$ then any lottery in which x_1 is substituted by x_2 is equally preferred.
- **Decomposability**: two different lotteries assign same probability to each outcome, then player is indifferent between these two lotteries
- **Monotonicity**: If $x_1 \succ x_2$ and $p > q$ then $[x_1 : p,\ x_2 : 1 - p] \succ [x_1 : q,\ x_2 : 1 - q]$
- **Continuity**: If $x_1 \succ x_2 \succ x_3$, $\exists p \ni x_2 \sim [x_1 : p,\ x_3 : 1 - p]$

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Utility Theory (2)

Von Neumann and Morgenstern

Theorem

Given a set of outcomes X and a preference relation on X that satisfies above six axioms, there exists a utility function $u : X \rightarrow [0,1]$ with the following properties:

1. $u(x_1) \geq u(x_2) iff \;\; x_1 \succ x_2$
2. $U([x_1 : p_1; x_2 : p_2; \ldots; x_m : p_m]) = \sum_{j=1}^{m} p_j u(x_j)$

Zero Sum Games

- Recall: Zero sum games where one player's gain = other player's loss.
- We studied saddle points and pure strategy equilibrium

X Matching coins without observation: no pure strategy equilibrium

- Can it have a mixed strategy equilibrium?

$\surd$ **Yes**.

- Let p and q be the mixed strategies of row and column player respectively.

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Equilibrium in Zero Sum Games

Von Neumann and Morgenstern showed:

Theorem (Mini-Max Theorem)

For every $(m \times n)$ *matrix* A, *there is a stochastic row vector* $p^* = (p_1^*, \ldots, p_n^*)$ *and a stochastic column vector* ${q^*}^T = (q_1^*, \ldots, q_n^*)$ *such that*

$$\min_{q \in \Delta(S_2)} p^* A q = \max_{p \in \Delta(S_1)} p A q^*$$

(p^*, q^*) is equilibrium.

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What is equilibrium of Matching Coins without Observation game?

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$p^* = (0.5, 0.5) = {q^*}^T$

Important Lemmas

Lemma 1

For any matrix A,

$$\min_{q \in \Delta S_2} pAq = \min_j \sum_i a_{ij} p_i$$

Lemma 2

For any matrix A,

$$\max_{p \in \Delta S_1} pAq = \max_i \sum_j a_{ij} q_j$$

Row Player's Objective:

$$\max_p \min_q pAq$$

s.t.

$$\sum_i p_i = 1$$

$$p_i \geq 0 \ \forall i = 1, 2, \ldots, m$$

Column Player's Objective:

$$\min_q \max_p pAq$$

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Proof: Mini-Max Theorem Cntd...

Row Player's Objective:

$$\max z$$

s.t.

$$z - \sum_i a_{ij} p_i \leq 0 \; \forall j = 1, 2, \ldots, n$$

$$\sum_i p_i = 1$$

$$p_i \geq 0 \; \forall i = 1, 2, \ldots, m$$

Column Player's Objective:

$$\min w$$

s.t.

$$w - \sum_j a_{ij} q_j \geq 0 \; \forall j = 1, 2, \ldots, m$$

$$\sum_j q_j = 1$$

$$q_j \geq 0 \; \forall j = 1, 2, \ldots, n$$

Mini-max Theorem then follows from strong duality of **LP**

The above problems can be solved in polynomial time

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Further Reading

- **Game Theory and Mechanism Design**, Y Narahari. World Scientific Publishing Company, 2014.
- **Multiagent systems: Algorithmic, game-theoretic, and logical foundations**, Shoham, Yoav, and Kevin Leyton-Brown. Cambridge University Press, 2008. (Free download).
- **Game Theory** by Roger Myerson. Harvard University press, 2013.
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http://gametheory.net/

http://lcm.csa.iisc.ernet.in/gametheory/lecture.html

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