1. X represents the no. of heads when a coin is tossed three times

i) for $X = 0$, $S = \{TTT\}$

$P(X) = 1/8$

ii) for $X = 1$, $S = \{HTT, THT, TTH\}$

$P(X) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$

iii) for $X = 2$ $S = \{HHT, HTH, THH\}$

$P(X) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$

iv) for $X = 3$ $S = \{HHH\}$

$P(X) = \frac{1}{8}$

Req. Probability distribution table,

X	0	1	2	3
P(X)	1/8	3/8	3/8	1/8

2. No. of bad apples = 4

No. of good apples = 16

Total no. of apples = 20

Let X represent no. of bad apples in a draw of two apples

i) For $X=0$, both apples drawn are good apples

$\therefore P(X=0) = \dfrac{^{16}C_2}{^{20}C_2} = \dfrac{16\times15}{2!}\times\dfrac{2!}{20\times19} = \dfrac{12}{19}$

ii) For $X=1$, one of the apples drawn is good while the other one is a bad apple.

$\therefore P(X=1) = \dfrac{^{16}C_1 \times {^4C_1}}{^{20}C_2} = \dfrac{16}{1!}\times\dfrac{4}{1!}\times\dfrac{2!}{20\times19} = \dfrac{32}{95}$

iii) For $X=2$, both apples drawn are bad apples

$\therefore P(X=2) = \dfrac{^{4}C_2}{^{20}C_2} = \dfrac{4\times3}{2!}\times\dfrac{2!}{20\times19} = \dfrac{3}{95}$

Req. probability distribution is,

x	0	1	2
$P(x)$	$12/19$	$32/95$	$3/95$

3. $P(x) = \begin{cases} \dfrac{x}{15}, & x = 1,2,3,4,5 \\ 0 & \text{otherwise} \end{cases}$

a) $P(X=1 \text{ or } X=2) = \dfrac{1}{15} + \dfrac{2}{15} = \dfrac{3}{15} = \dfrac{1}{5}$

b) $P\left\{\dfrac{1/2 < X < 5/2}{X > 1}\right\} = \dfrac{P(5/2) - P(1/2)}{1 - P(X \leq 1)}$

$= \dfrac{5/30 - 1/30}{1 - 1/15} = \dfrac{2/15}{14/15} = \dfrac{1}{7}$

4. Given, $f(x) = 3x^2;\ 0 \le x \le 1$

i) $P(X \le a) = P(X > a)$

$\Rightarrow P(X \le a) = 1 - P(X \le a)$

$\Rightarrow 2P(X \le a) = 1$

$\Rightarrow 2\int_0^a f(x)dx = 1 \Rightarrow 2\int_0^a 3x^2 dx = 1$

$\Rightarrow 6\left[\frac{x^3}{3}\right]_0^a = 1 \Rightarrow a^3 = \frac{1}{2} \Rightarrow a = \left(\frac{1}{2}\right)^{1/3}$

ii) $P(X > b) = 0.05$

$\Rightarrow 1 - P(X \le b) = 0.05 \Rightarrow 0.95 = P(X \le b)$

$\Rightarrow \int_0^b 3x^2 dx = \frac{19}{20} \Rightarrow \left[3\left(\frac{x^3}{3}\right)\right]_0^b = \frac{19}{20}$

$\Rightarrow b^3 = \left(\frac{19}{20}\right) \Rightarrow b = \left(\frac{19}{20}\right)^{1/3}$

5. Moment Generating Function $M_X(t) = E[e^{tx}]$

$= \int_0^2 \frac{x}{2} e^{tx} dx = \left[\frac{x}{2}\int e^{tx}dx - \int\left\{\frac{d}{dx}\left(\frac{x}{2}\right)\cdot\int e^{tx}dx\right\}\right]_0^2$

$= \left[\frac{xe^{tx}}{2t} - \int\frac{e^{tx}}{2t}\right]_0^2 = \left[\frac{xe^{tx}}{2t} - \frac{e^{tx}}{2t^2}\right]_0^2$

$= \left[\frac{2e^{2t}}{2t} - \frac{e^{2t}}{2t^2} - 0 + \frac{1}{2t^2}\right] = \frac{1}{2t^2}\left(1 + 2te^{2t} - e^{2t}\right)$

6. Given, $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$

Let $P(X=3) = x$

$\Rightarrow P(X=1) = x/2$,

$P(X=2) = x/3$

and $P(X=4) = x/5$

Since, $\sum_{i=1}^{n} P(X=i) = 1$ [$\because$ Sum of probabilities in a probability distribution is 1]

$\Rightarrow P(X=1) + P(X=2) + P(X=3) + P(X=4) = 1$

$\Rightarrow \frac{x}{2} + \frac{x}{3} + x + \frac{x}{5} = 1 \Rightarrow \frac{15x + 10x + 30x + 6x}{30} = 1$

$\Rightarrow x = \frac{30}{61}$

So, the req. probability distribution is:

X	1	2	3	4
P(X)	15/61	10/61	30/61	6/61

7. $F_x(x) = ke^{-t}(1-e^{-t})^{x-1}$ (given)

We know, $\sum_{x=1}^{\infty} f_X(x) = 1$ ——— (1)

$\Rightarrow \sum_{x=1}^{\infty} ke^{-t}(1-e^{-t})^{x-1} = \frac{ke^{-t}}{1-(1-e^{-t})} = \frac{ke^{-t}}{e^{-t}} = k$

From (1), $k = 1$

Mean $\mu_x = \sum_{x=1}^{\infty} x \cdot f_x(x) = \sum_{x=1}^{\infty} x \cdot e^{-t}(1-e^{-t})^{x-1}$

$= \sum_{x=1}^{\infty} x \cdot e^{-t}(1-e^{-t})^{x-1} = e^{-t}\sum_{x=0}^{\infty}(x+1)(1-e^{-t})^{x}$

Any summation of the form

$$\sum_{u=0}^{\infty}(x+1)x^u = (1-x)^{-2}$$

So, $\mu_x = e^{-t}(1-(1-e^{-t}))^{-2} = e^{-t}(e^{-t})^{-2} = e^{t}$

$Var(X), \sigma_x^2 = E(X^2) - [E(X)]^2$

$$E(X^2) = \sum_{x=1}^{\infty} x^2 \cdot e^{-t}(1-e^{-t})^{x-1}$$

Let $x-1 = u$

$\Rightarrow E(X^2) = \sum_{u=0}^{\infty}(u+1)^2 e^{-t}(1-e^{-t})^u$

$= \underset{(i)}{\sum_{u=0}^{\infty} u^2 e^{-t}(1-e^{-t})^u} + \underset{(ii)}{\sum_{u=0}^{\infty} e^{-t}(1-e^{-t})^u}$

$+ \underset{(iii)}{\sum_{u=0}^{\infty} 2ue^{-t}(1-e^{-t})^u}$

Part (i) $\sum_{u=0}^{\infty} u^2 e^{-t}(1-e^{-t})^u = f(t) = \sum_{u=0}^{\infty} u^2 e^{-t}(1-e^{-t})^u$

$$\Rightarrow \frac{f(t)}{1-e^{-t}} = \sum_{u=0}^{\infty} u^2 e^{-t}(1-e^{-t})^{u-1}$$

Integrating

$$\int \frac{f(t)}{1-e^{-t}}\,dt = \int \sum_{u=0}^{\infty} u^2 \cdot e^{-t}(1-e^{-t})^{u-1}\,dt$$

$$= \sum_{u=0}^{\infty} u(1-e^{-t})^u + C = \frac{1-e^{-t}}{(1-(1-e^{-t}))^2} + C$$

$$= \frac{1-e^{-t}}{(e^{-t})^2}$$

$$\int \frac{f(t)}{1-e^{-t}}\,dt = e^{2t} - e^{t} + C$$

$$\Rightarrow \frac{f(t)}{1-e^{-t}} = 2e^{2t} - e^{t} \quad \Rightarrow f(t) = 2e^{2t} - 3e^{t} + 1$$

Part (ii) $\sum_{u=0}^{\infty} e^{-t}(1-e^{-t})^u = e^{-t}\sum_{u=0}^{\infty}(1-e^{-t})^u$

$$= e^{-t} \cdot \frac{1}{1-(1-e^{-t})} = e^{-t} \cdot \frac{1}{e^{-t}} = 1$$

Part (iii) $\sum_{u=0}^{\infty} 2u e^{-t}(1-e^{-t})^u = 2e^{-t}\sum_{u=0}^{\infty} u \cdot (1-e^{-t})^u$

$$= 2e^{-t} \cdot \frac{1-e^{-t}}{(1-(1-e^{-t}))^2} = 2e^{-t} \cdot \frac{1-e^{-t}}{(e^{-t})^2} = 2e^{t} - 2$$

$E(X^2) = 2e^{2t} - 3e^{t} + 1 + 1 + 2e^{t} - 2 = 2e^{2t} - e^{t}$

$Var(X), \sigma_X^2 = E(X^2) - [E(X)]^2 = 2e^{2t} - e^{t} - (e^{t})^2$

$$= e^{2t} - e^{t}$$

8. Given, $F(x) = \begin{cases} 1 - \frac{4}{x^2}, & \text{for } x > 2 \\ 0 & \text{for } x \leq 2 \end{cases}$

$$P(4 < X < 5) = F(5) - F(4)$$
$$= \left[\cancel{1} - \frac{4}{25} - \cancel{1} + \frac{4}{16}\right]$$
$$= 0.25 - 0.16 = 0.09$$

9. Given, $f(x) = [2(1+x)]/27$ and the continuous random variable can assume any value between $X = 2$ and $X = 5$

Then, $P(X < 4) = P(2 \leq X < 4) = P(2 \leq X \leq 4)$
$$= \int_2^4 f(x)\,dx = \int_2^4 \frac{2(1+x)}{27}\,dx$$
$$= \frac{2}{27}\left[x + \frac{x^2}{2}\right]_2^4$$
$$= \frac{2}{27}\left[4 + \frac{16}{2} - 2 - \frac{4}{2}\right] = \frac{2}{27} \times 8 = \frac{16}{27}$$

10. Given, $f(x) = x^2$ for $1 < x < 2$

$$E(\log x) = \int_1^2 x^2 \log x\,dx = \left[\frac{x^3 \log x}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3}\right]$$
$$= \left[\frac{x^3 \log x}{3} - \frac{x^3}{9}\right]_1^2 = \left[\frac{8 \log 2}{3} - \frac{8}{9} - 0 + \frac{1}{9}\right]$$
$$= \frac{8}{3}\log 2 - \frac{7}{9}$$

11. When a fair dice is tossed

$$P(X=i)=\frac{1}{6} \text{ for } i=1,2,3,4,5,6$$

MGf, $$M_x(t)=\sum_{x=1}^{6} e^{tx} P(X=i)$$

$$=\frac{1}{6}\sum_{i=1}^{6} e^{tx} = \frac{1}{6}(e^{t}+e^{2t}+e^{3t}+e^{4t}+e^{5t}+e^{6t})$$

Mean $$= E(x) = \left[\frac{d}{dt} M_x(t)\right]_{t=0} \cancel{= \frac{1}{6}(e^{t}+2e^{2t}+3e^{3t}+4e^{4t}+5e^{5t}+6e^{6t})}$$

$$= \frac{1}{6}\left[e^{t}+2e^{2t}+3e^{3t}+4e^{4t}+5e^{5t}+6e^{6t}\right]_{t=0}$$

$$= \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = \frac{7}{2}$$

Now, $$E(x^2)_{t=0} = \frac{d^2}{dt^2} M_x(t)_{t=0} = \frac{d}{dt} E(x)_{t=0}$$

$$= \frac{1}{6}\left[e^{t}+4e^{2t}+9e^{3t}+16e^{4t}+25e^{5t}+36e^{6t}\right]_{t=0}$$

$$= \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6}$$

Variance, $$Var(x) = E(x^2) - [E(x)]^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

12. Given, $E(x)=1$ and $E[x(x-1)]=4$

$\Rightarrow E[x^2-x]=4$

$\Rightarrow E(x^2)-E(x)=4$

Then, $E(x^2)-1=4 \Rightarrow E(x^2)=5$

We know $Var(X) = E(X^2) - [E(X)]^2$
$$= 5 - 1^2 = 4$$

By the property. $Var(X/2) = \frac{1}{4} Var(X) = \frac{1}{4} \times 4 = 1$

$$Var(2-3X) = Var(2) + 3^2 Var(X)$$
$$= 0 + 9 \times 4 = 36$$

13. $M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r' = \sum_{r=0}^{\infty} \frac{t^r}{r!} (r+1)! \, 2^r$
$$= \sum_{r=0}^{\infty} (r+1)(2t)^r = 1 + 2(2t) + 3(2t)^2 + \cdots$$
$$= \frac{1}{(1-2t)^2}$$

14. $M_x(t) = \frac{3}{3-t}$ & $E(X) = \left[\frac{d}{dt} M_x(t)\right]_{t=0} = \left[\frac{d}{dt} \frac{3}{(3-t)}\right]_{t=0}$
$$\& \; E(X) = \left[\frac{3}{(3-t)^2}\right]_{t=0} = \frac{1}{3}$$

Now,
$$E(X^2) = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0} = \left[\frac{6(3-t)}{(3-t)^4}\right]_{t=0}$$
$$= 2/9$$

$Var(X) = E(X^2) - [E(X)]^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$

15. Given, PDF $f(x) = y_0(x - x^2)$ for $0 \leq x \leq 1$

$$\int_0^1 f(x) \cdot dx = 1 \Rightarrow y_0 \int_0^1 (x - x^2) dx = 1$$

$$\Rightarrow \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{y_0} \Rightarrow \frac{1}{2} - \frac{1}{3} = \frac{1}{y_0}$$

$$\Rightarrow \frac{1}{6} = \frac{1}{y_0} \Rightarrow y_0 = 6$$

$$E(X) = \text{Mean} = \mu_x = \int x \cdot f(x) dx$$

$$= \int_0^1 x \cdot 6 \cdot (x - x^2) \cdot dx = 6\int_0^1 (x^2 - x^3) dx$$

$$= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6 \cdot \left[\frac{1}{3} - \frac{1}{4}\right] = 6 \cdot \frac{1}{12} = \frac{1}{2}$$