Programming assignment III
ME685A – APPLIED NUMERICAL METHODS

6th April 2022

Discretize the unsteady 1D diffusion equation for species concentration in a slab with radioactive decay on a grid of *N* nodes. Adopt the FTCS implicit method. The applicable governing equation with IC/BC is:

$$\frac{\partial T}{\partial t}=\alpha\frac{\partial^2 T}{\partial x^2}-\lambda T \quad 0<x<L;\ \ t>0$$
$$t=0,\ T(x,0)=0$$
$$x=0,\ T(0,t)=1;\ x=L,\ T(L,t)=0$$

Solve the system of algebraic equations arising from FTCS using the successive substitution formula (Gauss-Seidel method). Also derive the analytical steady state solution.

Weightage of numerical solution 90% and analytical 10%

Show results for the number of points *N*=101. For definiteness, set $\alpha=1;\ \lambda=9;\ L=1.$ Choose a time step of 0.005 and perform time integration to at least *t*=1. The program output should comprise the left wall concentration gradient as well as concentration at the mid-point of the slab, as a function of time. Compare the numerical solution with the analytical (steady state).

Summarize your results in the form of a report. The zipped file should contain the pdf of your report and the computer program.

Hint: The full analytical solution is derived by first introducing a new dependent variable $T(x,t)=\phi(x,t)\exp(-\lambda t)$. Hence, with $\alpha=1;\ \lambda=9;\ L=1$ we get

$$\frac{\partial \phi}{\partial t}=\frac{\partial^2 \phi}{\partial x^2} \quad 0<x<1;\ \ t>0$$
$$t=0,\ \phi(x,0)=0$$
$$x=0,\ \phi(0,t)=\exp(\lambda t);\ x=1,\ \phi(1,t)=0$$

The analytical solution is obtained using Duhamel's theorem.

Hence we get the following results: {ϕ and ϕ_H are intermediate solutions.}

$$\phi_H(x,t)=(1-x)+\sum_{n=1}^{\infty}c_n\sin(n\pi x)\exp(-n^2\pi^2 t) \text{ with } c_n=\frac{\int_0^1 (x-1)\sin n\pi x dx}{\int_0^1 \sin^2 n\pi x dx}$$

Consider 100 terms in $\sum_{n=1}^{\infty} C_n$.

$$\phi(x,t)=\int_0^t \exp(\lambda\tau)\frac{\partial}{\partial t}\left[\phi_H(x,t-\tau)\right]d\tau \text{ and}$$

$$T(x,t)=\int_0^t \exp(-\lambda(t-\tau))\frac{\partial}{\partial t}\left[\phi_H(x,t-\tau)\right]d\tau \longleftarrow$$

$\frac{\partial}{\partial t}[\phi_H(x,t-\tau)]$ is evaluated as $\frac{\partial \phi_H(x,t)}{\partial t}$ followed by 't' replaced by '$t-\tau$'.

This is the required analytical solution for *T*(*x*,*t*). The steady state solution is much simpler and is obtained by solving the ODE

$$0=\frac{d^2 T}{dx^2}-\lambda T \quad 0<x<L;\ \ t>0$$
$$x=0,\ T(0,t)=1;\ x=L,\ T(L,t)=0$$