Analytical solution

- $$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} - \lambda T \quad ; \quad 0 < x < L ; \ t > 0$$
- $T(x=0) = 0$, $T(0,t) = 1$, $T(L,t) = 0$

Suppose, $T(x,t) = \phi(x,t)\, e^{-\lambda t}$

$$\Rightarrow \cancel{\frac{\partial T}{\partial t} = \frac{\partial \phi}{\partial t} e^{-\lambda t} \times \lambda} \qquad \frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} \ ; \ 0 < x < 1,\ t > 0$$

Using the hint, $\phi = \int_0^t e^{\lambda t} \frac{\partial}{\partial t} [\phi_H (x, t - \tau) d\tau$

$$\phi_H (x,t) = (1 - x) + \sum_{n=1}^{\infty} c_n \sin(n\pi x)\, e^{-n^2\pi^2 t}$$

where $$c_n = \frac{\int_0^1 (x-1) \sin(n\pi x)\, dx}{\int_0^1 \sin^2 (n\pi x)\, dx} = -\frac{2}{n\pi}$$

$$\Rightarrow T(x,t) = \int_0^t e^{-\lambda(t-\tau)} \frac{\partial}{\partial t} [\phi_H (x, t - \tau) d\tau \ : \quad \underline{\text{Putting } (t-\tau) = t} \quad \Rightarrow d\tau = dt$$

$\therefore$ Integrate the above equation to get :

$$T(x,t) = (1 - x) - \sum_{n=1}^{\infty} \left[\frac{2 \sin(n\pi x)}{n\pi} \left\{ \frac{\lambda + n^2\pi^2 e^{-(n^2\pi^2 t + \lambda t)}}{\lambda + n^2\pi^2} \right\} \right]$$

$$\Rightarrow \frac{\partial T}{\partial t} = - \sum_{n=1}^{\infty} \frac{2 \sin(n\pi x)}{n\pi} \left\{ - n^2\pi^2 e^{-(n^2\pi^2 t + \lambda t)} \right\}$$

$$= \sum_{n=1}^{\infty} \left[2 \sin(n\pi x) \times (n\pi) \times e^{-(n^2\pi^2 t + \lambda t)} \right]$$

$$\Rightarrow \frac{\partial^2 T}{\partial x^2} = \sum_{n=1}^{\infty} \left[2n\pi \sin(n\pi x) \left\{ \frac{\lambda + n^2\pi^2 e^{-(n^2\pi^2 t + \lambda t)}}{n^2\pi^2 + \lambda} \right\} \right]$$

$$\Rightarrow \frac{\partial^2 T}{\partial x^2} - \lambda T = \sum_{x=1}^{\infty} \left\{ 2n\pi \sin(n\pi x)\, e^{-(n^2\pi^2 t + \lambda t} \right\} = \frac{\partial T}{\partial t}$$

$$\therefore \left.\frac{\partial T}{\partial x}\right|_{x=0} = (-1) - \sum_{n=1}^{\infty} 2(1)^n \left\{ \frac{\lambda + n^2\pi^2 e^{-(n^2\pi^2 t + \lambda t)}}{\lambda^2 + n^2\pi^2} \right\}$$

$$= \boxed{-6\,\frac{(e^{3} + e^{-3})}{e^{3} - e^{-3}}}$$