CS 228 : Logic in Computer Science

Krishna. S

GNBA

- Generalized NBA, a variant of NBA
- Only difference is in acceptance condition
- Acceptance condition in GNBA is a set $\mathcal{F} = \{F_1, \ldots, F_k\}$, each $F_i \subseteq Q$
- An infinite run ρ is accepting in a GNBA iff

$$\forall F_i \in \mathcal{F}, \mathit{Inf}(\rho) \cap F_i \neq \emptyset$$

- Note that when $\mathcal{F} = \emptyset$, all infinite runs are accepting
- GNBA and NBA are equivalent in expressive power.

Word View (On the board)

- $w = \{a\}\{a, b\}\{\} \ldots,$
- $\varphi = a \,\mathsf{U}(\neg a \wedge b)$
- Subformulae of $\varphi = \{a, \neg a, b, \neg a \wedge b, \varphi\}$
- Parse trees to compute all subformulae

Closure of φ, $cl(\varphi)$

- $cl(\varphi)$=all subformulae of φ and their negations, identifying $\neg\neg\psi$ to be ψ.
- Example for $\varphi = a\,\mathsf{U}(\neg a \wedge b)$
- $cl(\varphi) = \{a, \neg a, b, \neg b, \neg a \wedge b, \neg(\neg a \wedge b), \varphi, \neg\varphi\}$

Elementary Sets

Let φ be an LTL formula. Then $B \subseteq cl(\varphi)$ is elementary provided:

- B is propositionally and maximally consistent : for all $\varphi_1 \wedge \varphi_2, \psi \in cl(\varphi)$,
  - $\varphi_1 \wedge \varphi_2 \in B \Leftrightarrow \varphi_1 \in B \wedge \varphi_2 \in B$
  - $\psi \in B \Leftrightarrow \neg\psi \notin B$
  - $true \in cl(\varphi) \Rightarrow true \in B$
- B is locally consistent wrt U. That is, for all $\varphi_1 \mathsf{U} \varphi_2 \in cl(\varphi)$,
  - $\varphi_2 \in B \Rightarrow \varphi_1 \mathsf{U} \varphi_2 \in B$
  - $\varphi_1 \mathsf{U} \varphi_2 \in B, \varphi_2 \notin B \Rightarrow \varphi_1 \in B$
- B is elementary : B is propositionally, maximally and locally consistent
- Given a $B \subseteq cl(\varphi)$, how can you check if B is elementary?

Check Elementary

Let $\varphi = a\,\mathrm{U}(\neg a \wedge b)$

- $B_1 = \{a, b, \neg a \wedge b, \varphi\}$

Check Elementary

Let $\varphi = a \, \mathsf{U} (\neg a \wedge b)$

- $B_1 = \{a, b, \neg a \wedge b, \varphi\}$ No, propositionally inconsistent
- $B_2 = \{\neg a, b, \varphi\}$

Check Elementary

Let $\varphi = a \, \mathsf{U} (\neg a \wedge b)$

- $B_1 = \{a, b, \neg a \wedge b, \varphi\}$ No, propositionally inconsistent
- $B_2 = \{\neg a, b, \varphi\}$ No, not maximal as $\neg a \wedge b \notin B_2$, $\neg(\neg a \wedge b) \notin B_2$
- $B_3 = \{\neg a, b, \neg a \wedge b, \neg \varphi\}$

Check Elementary

Let $\varphi = a\,\mathsf{U}(\neg a \wedge b)$

- $B_1 = \{a, b, \neg a \wedge b, \varphi\}$ No, propositionally inconsistent
- $B_2 = \{\neg a, b, \varphi\}$ No, not maximal as $\neg a \wedge b \notin B_2$, $\neg(\neg a \wedge b) \notin B_2$
- $B_3 = \{\neg a, b, \neg a \wedge b, \neg\varphi\}$ No, not locally consistent for U
- $B_4 = \{\neg a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$

Check Elementary

Let $\varphi = a \, \mathrm{U}(\neg a \wedge b)$

- $B_1 = \{a, b, \neg a \wedge b, \varphi\}$ No, propositionally inconsistent
- $B_2 = \{\neg a, b, \varphi\}$ No, not maximal as $\neg a \wedge b \notin B_2$, $\neg(\neg a \wedge b) \notin B_2$
- $B_3 = \{\neg a, b, \neg a \wedge b, \neg\varphi\}$ No, not locally consistent for U
- $B_4 = \{\neg a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$ Yes, elementary

LTL φ to GNBA G_φ

- States of G_φ are elementary sets B_i
- For a word $w = A_0 A_1 A_2 \ldots$ the sequence of states $\sigma = B_0 B_1 B_2 \ldots$ will be a run for w
- σ will be accepting iff $w \models \varphi$ iff $\varphi \in B_0$
- In general, a run $B_i B_{i+1} \ldots$ for $A_i A_{i+1} \ldots$ is accepting iff $A_i A_{i+1} \ldots \models \psi$ for all $\psi \in B_i$.

LTL to GNBA

- Let $\varphi = \bigcirc a$.
- Subformulae of φ : $\{a, \bigcirc a\}$. Let $A = \{a, \bigcirc a, \neg a, \neg \bigcirc a\}$.
- Possibilities at each state
  - $\{a, \bigcirc a\}$
  - $\{\neg a, \bigcirc a\}$
  - $\{a, \neg \bigcirc a\}$
  - $\{\neg a, \neg \bigcirc a\}$
- Our initial state(s) must guarantee truth of $\bigcirc a$. Thus, initial states: $\{a, \bigcirc a\}$ and $\{\neg a, \bigcirc a\}$

LTL to GNBA

$\{a, \bigcirc a\}$

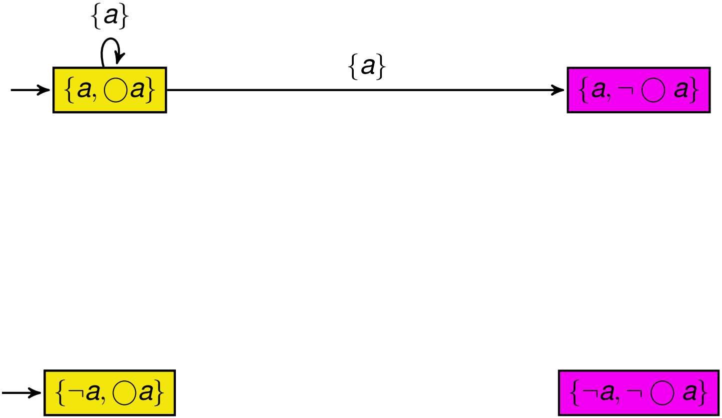
$\{a, \neg \bigcirc a\}$

$\{\neg a, \bigcirc a\}$

$\{\neg a, \neg \bigcirc a\}$

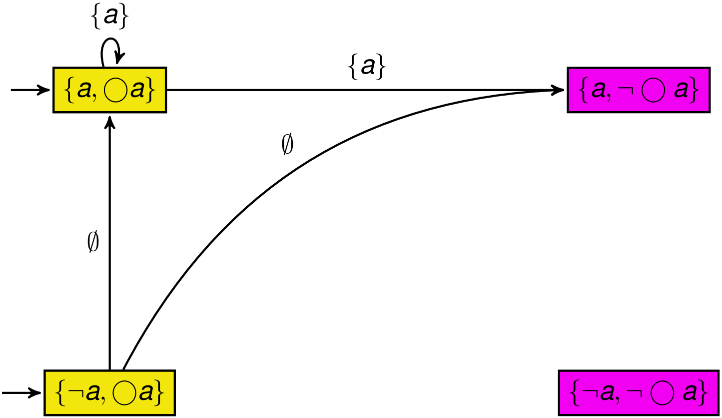
LTL to GNBA

LTL to GNBA



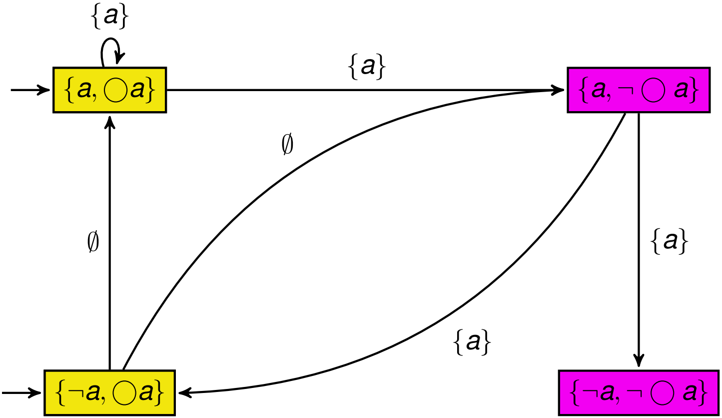
$\{a\}$
$\{a, \bigcirc a\}$
$\{a\}$
$\{a, \neg \bigcirc a\}$
$\{\neg a, \bigcirc a\}$
$\{\neg a, \neg \bigcirc a\}$

LTL to GNBA



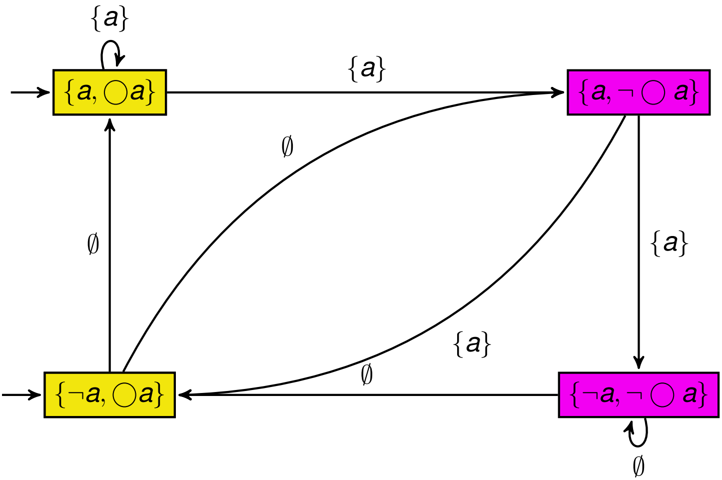
$\{a\}$
$\{a, \bigcirc a\}$
$\{a\}$
$\{a, \neg \bigcirc a\}$
$\emptyset$
$\emptyset$
$\{\neg a, \bigcirc a\}$
$\{\neg a, \neg \bigcirc a\}$

LTL to GNBA



$\{a\}$
$\{a, \bigcirc a\}$
$\{a\}$
$\{a, \neg \bigcirc a\}$
$\emptyset$
$\emptyset$
$\{a\}$
$\{a\}$
$\{\neg a, \bigcirc a\}$
$\{\neg a, \neg \bigcirc a\}$

LTL to GNBA


$\{a\}$
$\{a, \bigcirc a\}$
$\{a\}$
$\{a, \neg \bigcirc a\}$
$\emptyset$
$\emptyset$
$\{a\}$
$\{a\}$
$\emptyset$
$\{\neg a, \bigcirc a\}$
$\{\neg a, \neg \bigcirc a\}$
$\emptyset$

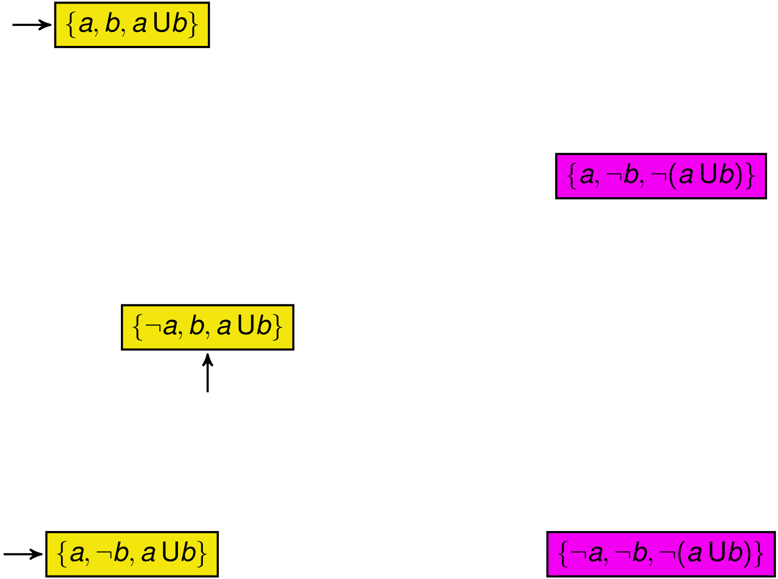
LTL to GNBA

- Claim : Runs from a state labelled set B indeed satisfy B
- No good states. All words having a run from a start state are accepted.
- Automaton for $\neg \bigcirc a$ same, except for the start states.

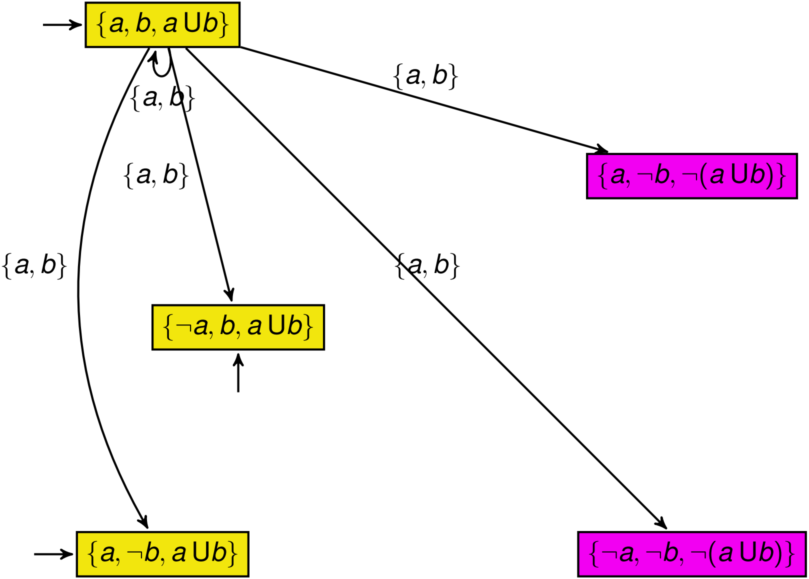
LTL to GNBA

- Let $\varphi = a \, \mathsf{U} b$.
- Subformulae of φ : $\{a, b, a \, \mathsf{U} b\}$. Let $B = \{a, \neg a, b, \neg b, a \, \mathsf{U} b, \neg(a \, \mathsf{U} b)\}$.
- Possibilities at each state
  - $\{a, \neg b, a \, \mathsf{U} b\}$
  - $\{\neg a, b, a \, \mathsf{U} b\}$
  - $\{a, b, a \, \mathsf{U} b\}$
  - $\{a, \neg b, \neg(a \, \mathsf{U} b)\}$
  - $\{\neg a, \neg b, \neg(a \, \mathsf{U} b)\}$
- Our initial state(s) must guarantee truth of $a \, \mathsf{U} b$. Thus, initial states: $\{a, b, a \, \mathsf{U} b\}$ and $\{\neg a, b, a \, \mathsf{U} b\}$ and $\{a, \neg b, a \, \mathsf{U} b\}$.

LTL to GNBA

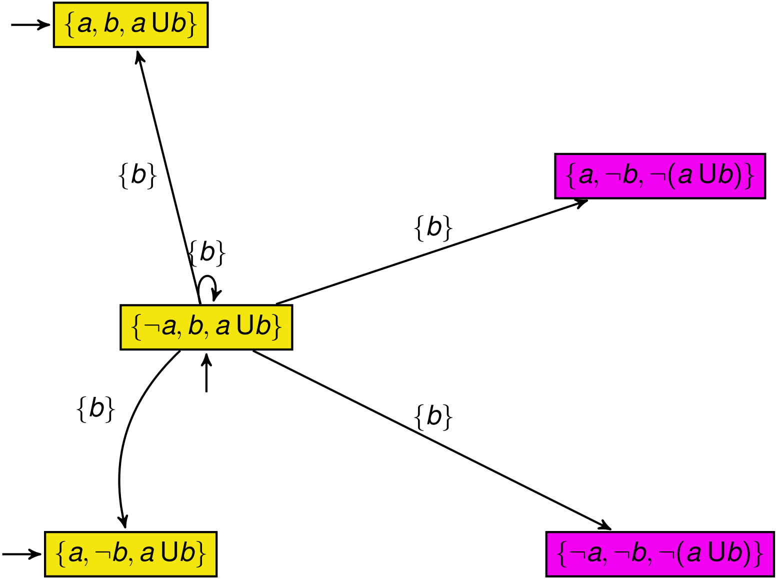

$\{a, b, a \,\mathsf{U}\, b\}$
$\{a, \neg b, \neg(a \,\mathsf{U}\, b)\}$
$\{\neg a, b, a \,\mathsf{U}\, b\}$
$\{a, \neg b, a \,\mathsf{U}\, b\}$
$\{\neg a, \neg b, \neg(a \,\mathsf{U}\, b)\}$

LTL to GNBA



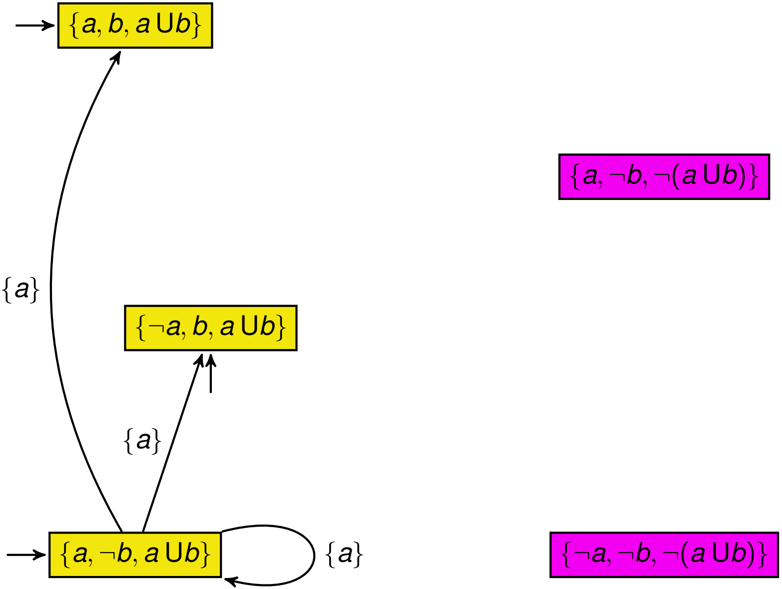
$\{a, b, a\,\mathsf{U}\,b\}$
$\{a, b\}$
$\{a, b\}$
$\{a, b\}$
$\{a, b\}$
$\{a, b\}$
$\{a, \neg b, \neg(a\,\mathsf{U}\,b)\}$
$\{\neg a, b, a\,\mathsf{U}\,b\}$
$\{a, \neg b, a\,\mathsf{U}\,b\}$
$\{\neg a, \neg b, \neg(a\,\mathsf{U}\,b)\}$

LTL to GNBA



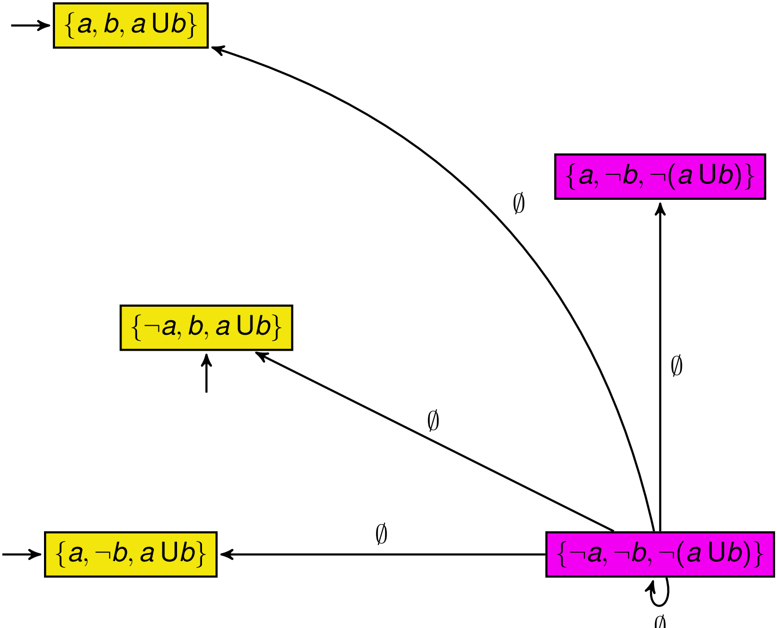
$\{a, b, a \,\mathsf{U}\, b\}$
$\{b\}$
$\{a, \neg b, \neg(a \,\mathsf{U}\, b)\}$
$\{b\}$
$\{b\}$
$\{\neg a, b, a \,\mathsf{U}\, b\}$
$\{b\}$
$\{b\}$
$\{a, \neg b, a \,\mathsf{U}\, b\}$
$\{\neg a, \neg b, \neg(a \,\mathsf{U}\, b)\}$

LTL to GNBA

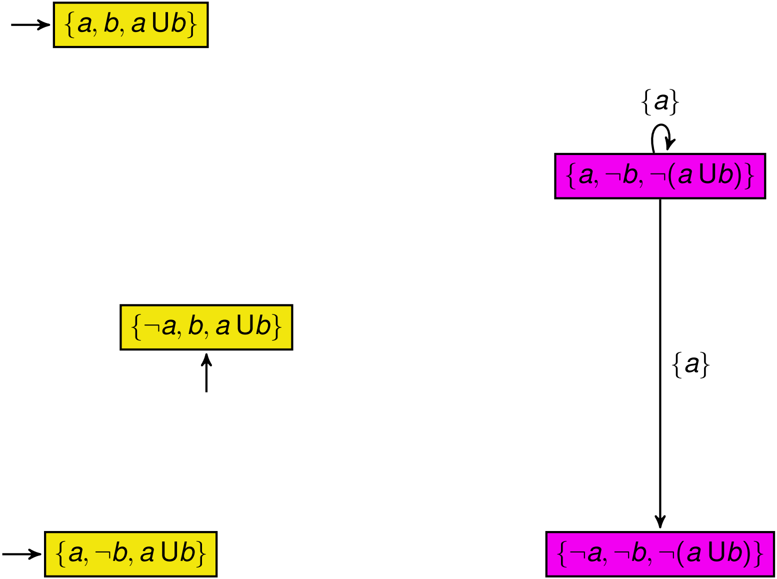


$\{a, b, a \,\mathsf{U}\, b\}$
$\{a, \neg b, \neg(a \,\mathsf{U}\, b)\}$
$\{a\}$
$\{\neg a, b, a \,\mathsf{U}\, b\}$
$\{a\}$
$\{a, \neg b, a \,\mathsf{U}\, b\}$
$\{a\}$
$\{\neg a, \neg b, \neg(a \,\mathsf{U}\, b)\}$

LTL to GNBA

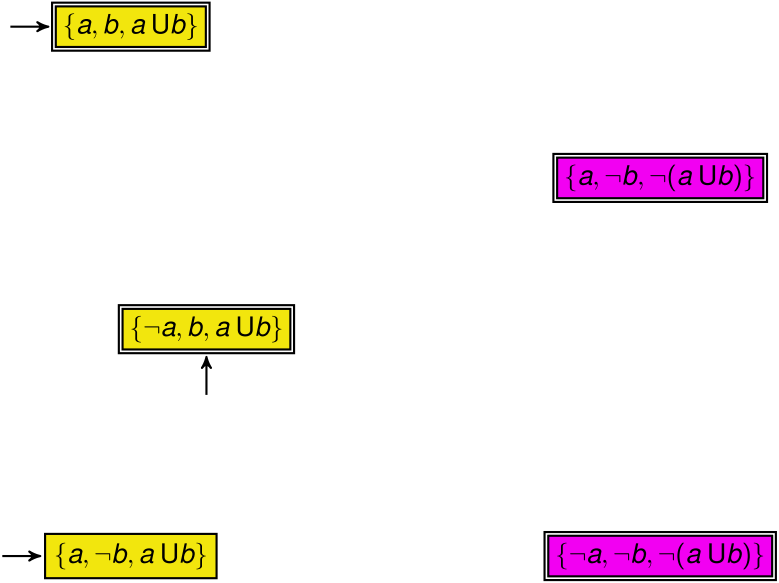

$\{a, b, a\,\mathsf{U}\,b\}$
$\{a, \neg b, \neg(a\,\mathsf{U}\,b)\}$
$\emptyset$
$\{\neg a, b, a\,\mathsf{U}\,b\}$
$\emptyset$
$\emptyset$
$\emptyset$
$\{a, \neg b, a\,\mathsf{U}\,b\}$
$\{\neg a, \neg b, \neg(a\,\mathsf{U}\,b)\}$
$\emptyset$

LTL to GNBA



$\{a, b, a\,\mathsf{U}\,b\}$
$\{a\}$
$\{a, \neg b, \neg(a\,\mathsf{U}\,b)\}$
$\{\neg a, b, a\,\mathsf{U}\,b\}$
$\{a\}$
$\{a, \neg b, a\,\mathsf{U}\,b\}$
$\{\neg a, \neg b, \neg(a\,\mathsf{U}\,b)\}$

LTL to GNBA : Accepting States



LTL to GNBA

Construct GNBA for $\neg(a \, \mathsf{U} b)$.

LTL to GNBA

- Let $\varphi = a\,\mathsf{U}(\neg a\,\mathsf{U} c)$. Let $\psi = \neg a\,\mathsf{U} c$
- Subformulae of φ : $\{a, \neg a, c, \psi, \varphi\}$. Let $B = \{a, \neg a, c, \neg c, \psi, \neg\psi, \varphi, \neg\varphi\}$.
- Possibilities at each state
    - $\{a, c, \psi, \varphi\}$
    - $\{\neg a, c, \psi, \varphi\}$
    - $\{a, \neg c, \neg\psi, \varphi\}$
    - $\{a, \neg c, \neg\psi, \neg\varphi\}$
    - $\{\neg a, \neg c, \psi, \varphi\}$
    - $\{\neg a, \neg c, \neg\psi, \neg\varphi\}$

LTL to GNBA

$\rightarrow \{a, c, \psi, \varphi\}$

$\{\neg a, \neg c, \psi, \varphi\} \leftarrow$

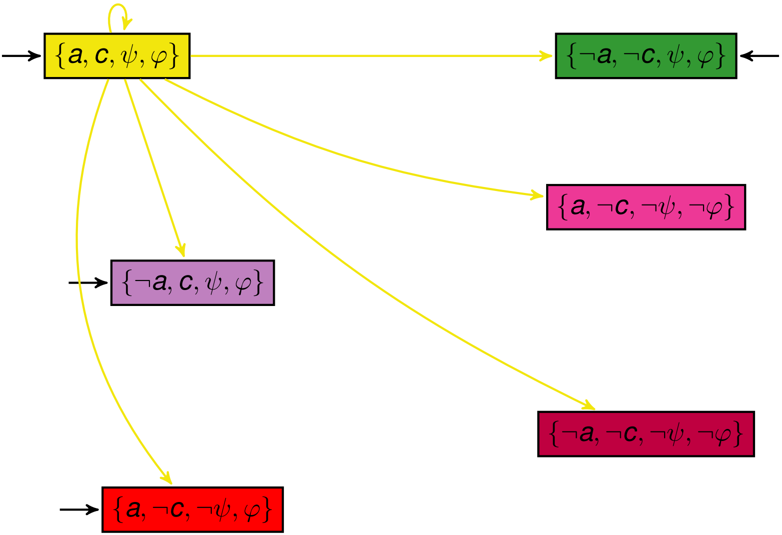
$\{a, \neg c, \neg\psi, \neg\varphi\}$

$\rightarrow \{\neg a, c, \psi, \varphi\}$

$\{\neg a, \neg c, \neg\psi, \neg\varphi\}$

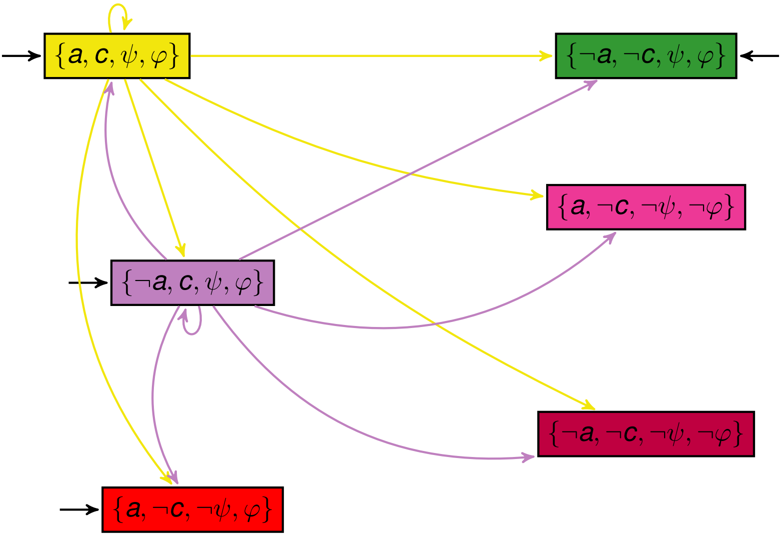
$\rightarrow \{a, \neg c, \neg\psi, \varphi\}$

LTL to GNBA



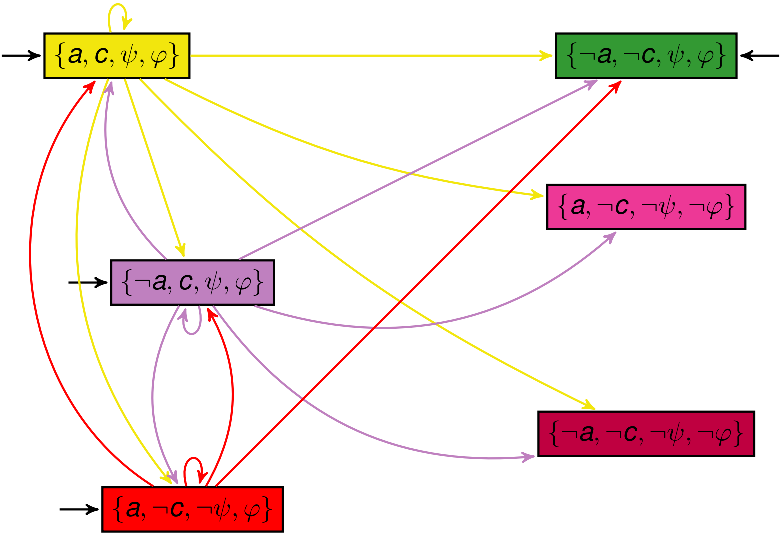
$\{a, c, \psi, \varphi\}$
$\{\neg a, \neg c, \psi, \varphi\}$
$\{a, \neg c, \neg\psi, \neg\varphi\}$
$\{\neg a, c, \psi, \varphi\}$
$\{\neg a, \neg c, \neg\psi, \neg\varphi\}$
$\{a, \neg c, \neg\psi, \varphi\}$

LTL to GNBA



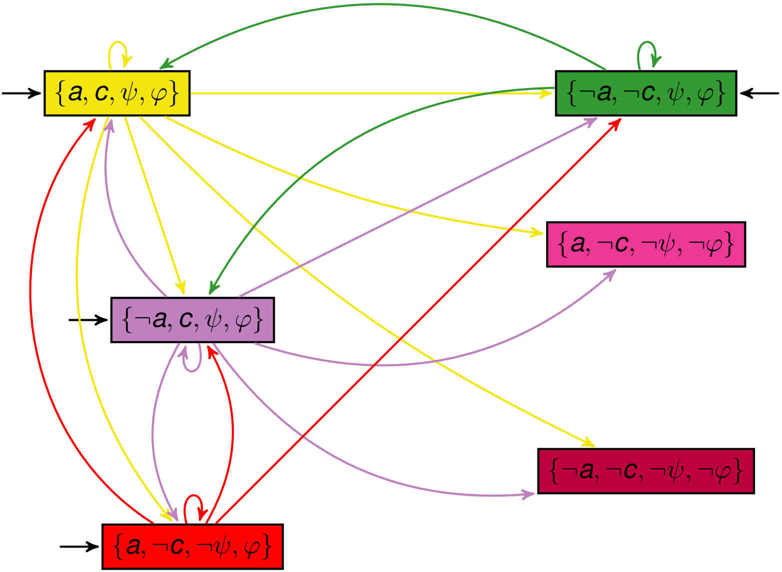
$\{a, c, \psi, \varphi\}$
$\{\neg a, \neg c, \psi, \varphi\}$
$\{a, \neg c, \neg\psi, \neg\varphi\}$
$\{\neg a, c, \psi, \varphi\}$
$\{\neg a, \neg c, \neg\psi, \neg\varphi\}$
$\{a, \neg c, \neg\psi, \varphi\}$

LTL to GNBA



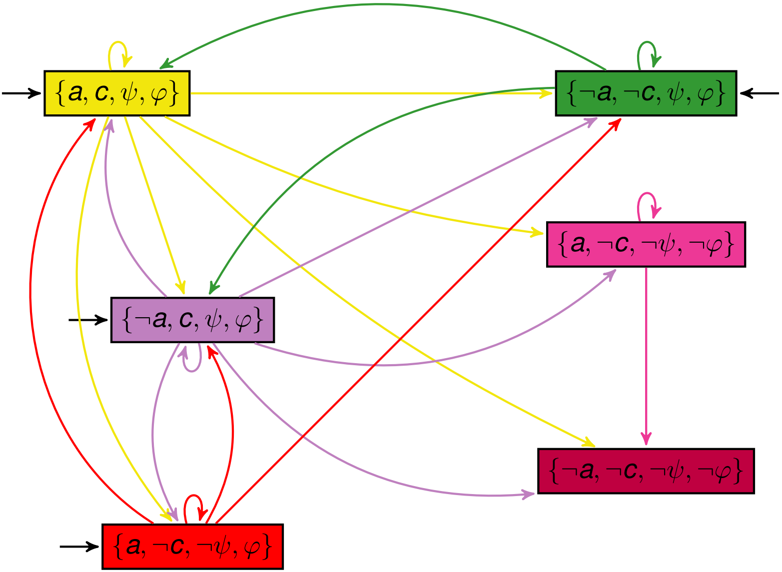
$\{a, c, \psi, \varphi\}$
$\{\neg a, \neg c, \psi, \varphi\}$
$\{a, \neg c, \neg \psi, \neg \varphi\}$
$\{\neg a, c, \psi, \varphi\}$
$\{\neg a, \neg c, \neg \psi, \neg \varphi\}$
$\{a, \neg c, \neg \psi, \varphi\}$

LTL to GNBA

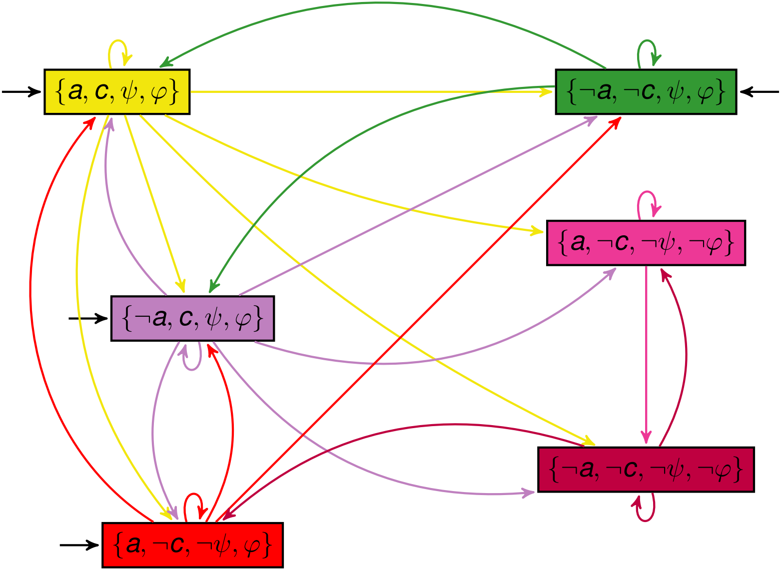


$\{a, c, \psi, \varphi\}$
$\{\neg a, \neg c, \psi, \varphi\}$
$\{a, \neg c, \neg\psi, \neg\varphi\}$
$\{\neg a, c, \psi, \varphi\}$
$\{\neg a, \neg c, \neg\psi, \neg\varphi\}$
$\{a, \neg c, \neg\psi, \varphi\}$

LTL to GNBA


$\{a, c, \psi, \varphi\}$
$\{\neg a, \neg c, \psi, \varphi\}$
$\{a, \neg c, \neg\psi, \neg\varphi\}$
$\{\neg a, c, \psi, \varphi\}$
$\{\neg a, \neg c, \neg\psi, \neg\varphi\}$
$\{a, \neg c, \neg\psi, \varphi\}$

LTL to GNBA


$\{a, c, \psi, \varphi\}$
$\{\neg a, \neg c, \psi, \varphi\}$
$\{a, \neg c, \neg\psi, \neg\varphi\}$
$\{\neg a, c, \psi, \varphi\}$
$\{\neg a, \neg c, \neg\psi, \neg\varphi\}$
$\{a, \neg c, \neg\psi, \varphi\}$

GNBA Acceptance Condition

- $\psi = \neg a \, \mathsf{U} c$
- $\varphi = a \, \mathsf{U} \psi$
- $F_1 = \{B \mid \psi \in B \to c \in B\}$
- $F_2 = \{B \mid \varphi \in B \to \psi \in B\}$
- $\mathcal{F} = \{F_1, F_2\}$

Final States

$\{a, c, \psi, \varphi\} \in F_1, F_2$

$\{\neg a, \neg c, \psi, \varphi\} \in F_2$

$\{a, \neg c, \neg\psi, \neg\varphi\} \in F_1, F_2$

$\{\neg a, c, \psi, \varphi\} \in F_1, F_2$

$\{\neg a, \neg c, \neg\psi, \neg\varphi\} \in F_1, F_2$

$\{a, \neg c, \neg\psi, \varphi\} \in F_1$

Putting Together

- Given φ, build $Cl(\varphi)$, the set of all subformulae of φ and their negations

Putting Together

- Given φ, build $Cl(\varphi)$, the set of all subformulae of φ and their negations
- Consider those $B \subseteq Cl(\varphi)$ which are consistent
  - $\varphi_1 \wedge \varphi_2 \in B \leftrightarrow \varphi_1 \in B$ and $\varphi_2 \in B$

Putting Together

- Given φ, build $Cl(\varphi)$, the set of all subformulae of φ and their negations
- Consider those $B \subseteq Cl(\varphi)$ which are consistent
  - $\varphi_1 \wedge \varphi_2 \in B \leftrightarrow \varphi_1 \in B$ and $\varphi_2 \in B$
  - $\psi \in B \rightarrow \neg\psi \notin B$ and $\psi \notin B \rightarrow \neg\psi \in B$

Putting Together

- Given φ, build $Cl(\varphi)$, the set of all subformulae of φ and their negations
- Consider those $B \subseteq Cl(\varphi)$ which are consistent
  - $\varphi_1 \wedge \varphi_2 \in B \leftrightarrow \varphi_1 \in B$ and $\varphi_2 \in B$
  - $\psi \in B \rightarrow \neg\psi \notin B$ and $\psi \notin B \rightarrow \neg\psi \in B$
  - Whenever $\psi_1 \, \mathsf{U} \psi_2 \in Cl(\varphi)$,
    - $\psi_2 \in B \rightarrow \psi_1 \, \mathsf{U} \psi_2 \in B$
    - $\psi_1 \, \mathsf{U} \psi_2 \in B$ and $\psi_2 \notin B \rightarrow \psi_1 \in B$

Putting Together

Given φ over AP, construct $A_\varphi = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$,

- $Q = \{B \mid B \subseteq Cl(\varphi) \text{ is consistent }\}$
- $Q_0 = \{B \mid \varphi \in B\}$
- $\delta : Q \times 2^{AP} \to 2^Q$ is such that

Putting Together

Given φ over AP, construct $A_\varphi = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$,

- $Q = \{B \mid B \subseteq Cl(\varphi)$ is consistent $\}$
- $Q_0 = \{B \mid \varphi \in B\}$
- $\delta : Q \times 2^{AP} \rightarrow 2^{Q}$ is such that
  - For $C = B \cap AP$, $\delta(B, C)$ is enabled and is defined as :
  - If $\bigcirc\psi \in Cl(\varphi)$, $\bigcirc\psi \in B$ iff $\psi \in \delta(B, C)$
  - If $\varphi_1 \,\mathsf{U} \varphi_2 \in Cl(\varphi)$,
    $\varphi_1 \,\mathsf{U} \varphi_2 \in B$ iff $(\varphi_2 \in B \vee (\varphi_1 \in B \wedge \varphi_1 \,\mathsf{U} \varphi_2 \in \delta(B, C)))$

Putting Together

Given φ over AP, construct $A_\varphi = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$,

- $Q = \{B \mid B \subseteq Cl(\varphi) \text{ is consistent } \}$
- $Q_0 = \{B \mid \varphi \in B\}$
- $\delta : Q \times 2^{AP} \to 2^Q$ is such that
  - For $C = B \cap AP$, $\delta(B, C)$ is enabled and is defined as :
  - If $\bigcirc\psi \in Cl(\varphi)$, $\bigcirc\psi \in B$ iff $\psi \in \delta(B, C)$
  - If $\varphi_1 \mathsf{U} \varphi_2 \in Cl(\varphi)$,
    $\varphi_1 \mathsf{U} \varphi_2 \in B$ iff $(\varphi_2 \in B \vee (\varphi_1 \in B \wedge \varphi_1 \mathsf{U} \varphi_2 \in \delta(B, C)))$
- $\mathcal{F} = \{F_{\varphi_1 \mathsf{U} \varphi_2} \mid \varphi_1 \mathsf{U} \varphi_2 \in Cl(\varphi)\}$, with
  $F_{\varphi_1 \mathsf{U} \varphi_2} = \{B \in Q \mid \varphi_1 \mathsf{U} \varphi_2 \in B \to \varphi_2 \in B\}$

Putting Together

Given φ over AP, construct $A_\varphi = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$,

- $Q = \{B \mid B \subseteq Cl(\varphi) \text{ is consistent }\}$
- $Q_0 = \{B \mid \varphi \in B\}$
- $\delta : Q \times 2^{AP} \to 2^Q$ is such that
  - For $C = B \cap AP$, $\delta(B, C)$ is enabled and is defined as :
  - If $\bigcirc\psi \in Cl(\varphi)$, $\bigcirc\psi \in B$ iff $\psi \in \delta(B, C)$
  - If $\varphi_1 \mathsf{U} \varphi_2 \in Cl(\varphi)$,
    $\varphi_1 \mathsf{U} \varphi_2 \in B$ iff $(\varphi_2 \in B \vee (\varphi_1 \in B \wedge \varphi_1 \mathsf{U} \varphi_2 \in \delta(B, C)))$
- $\mathcal{F} = \{F_{\varphi_1 \mathsf{U} \varphi_2} \mid \varphi_1 \mathsf{U} \varphi_2 \in Cl(\varphi)\}$, with
  $F_{\varphi_1 \mathsf{U} \varphi_2} = \{B \in Q \mid \varphi_1 \mathsf{U} \varphi_2 \in B \to \varphi_2 \in B\}$
- Prove that $L(\varphi) = L(A_\varphi)$

$L(\varphi) \subseteq L(A_\varphi)$

Let $\sigma = A_0 A_1 A_2 \cdots \in L(\varphi)$. Show that there is an accepting run $B_0 A_0 B_1 A_1 B_2 A_2 \ldots$ in A_φ for σ, B_i are the states, such that $B_i = \{\psi \mid A_i A_{i+1} \ldots \models \psi\}$.

Structural induction on φ

- $\varphi = a$. All starting states contain a, and can go to all successor states with all combinations of propositions.
- If $a \in B_i$, every run starting at B_i starts with a. Hence, $A_i A_{i+1} \ldots \models a$

$L(\varphi) \subseteq L(A_\varphi)$

Let $\sigma = A_0 A_1 A_2 \cdots \in L(\varphi)$. Show that there is an accepting run $B_0 A_0 B_1 A_1 B_2 A_2 \ldots$ in A_φ for σ, B_i are the states, such that $B_i = \{\psi \mid A_i A_{i+1} \ldots \models \psi\}$.

- $\varphi = \bigcirc a$, then all initial states contain $\bigcirc a$, and all successor states contain a by construction.
- If $\bigcirc a \in B_i$, then by construction, $B_{i+1} \in \delta(B_i, B_i \cap AP)$ iff $a \in B_{i+1}$, for every successor B_{i+1}.
  Then $A_{i+1} \ldots \models a$, and hence $A_i A_{i+1} \ldots \models \bigcirc a$.

$L(\varphi) \subseteq L(A_\varphi)$

Let $\sigma = A_0 A_1 A_2 \cdots \in L(\varphi)$. Show that there is an accepting run $B_0 A_0 B_1 A_1 B_2 A_2 \ldots$ in A_φ for σ, B_i are the states, such that $B_i = \{\psi \mid A_i A_{i+1} \ldots \models \psi\}$.

- If $\varphi_1 \, \mathsf{U} \varphi_2 \in B_i$, then by construction, either $\varphi_2 \in B_i$ or $\varphi_1, \varphi_1 \, \mathsf{U} \varphi_2 \in B_i$. If $\varphi_2 \in B_i$ then $A_i A_{i+1} \ldots \models \varphi_2$ by induction hypothesis, and hence, $A_i A_{i+1} \ldots \models \varphi_1 \, \mathsf{U} \varphi_2$.

$L(\varphi) \subseteq L(A_\varphi)$

Let $\sigma = A_0 A_1 A_2 \cdots \in L(\varphi)$. Show that there is an accepting run $B_0 A_0 B_1 A_1 B_2 A_2 \ldots$ in A_φ for σ, B_i are the states, such that $B_i = \{\psi \mid A_i A_{i+1} \ldots \models \psi\}$.

- If $\varphi_1 \mathsf{U} \varphi_2 \in B_i$, then by construction, either $\varphi_2 \in B_i$ or $\varphi_1, \varphi_1 \mathsf{U} \varphi_2 \in B_i$. If $\varphi_2 \in B_i$ then $A_i A_{i+1} \ldots \models \varphi_2$ by induction hypothesis, and hence, $A_i A_{i+1} \ldots \models \varphi_1 \mathsf{U} \varphi_2$.
- If $\varphi_1, \varphi_1 \mathsf{U} \varphi_2 \in B_i$, then by construction, $B_{i+1} \in \delta(B_i, B_i \cap AP)$ iff $\varphi_2 \in B_{i+1}$ or $\varphi_1, \varphi_1 \mathsf{U} \varphi_2 \in B_{i+1}$. How long can we go like this?

$L(\varphi) \subseteq L(A_\varphi)$

Let $\sigma = A_0 A_1 A_2 \cdots \in L(\varphi)$. Show that there is an accepting run $B_0 A_0 B_1 A_1 B_2 A_2 \ldots$ in A_φ for σ, B_i are the states, such that $B_i = \{\psi \mid A_i A_{i+1} \ldots \models \psi\}$.

- If $\varphi_1 \mathsf{U} \varphi_2 \in B_i$, then by construction, either $\varphi_2 \in B_i$ or $\varphi_1, \varphi_1 \mathsf{U} \varphi_2 \in B_i$. If $\varphi_2 \in B_i$ then $A_i A_{i+1} \ldots \models \varphi_2$ by induction hypothesis, and hence, $A_i A_{i+1} \ldots \models \varphi_1 \mathsf{U} \varphi_2$.
- If $\varphi_1, \varphi_1 \mathsf{U} \varphi_2 \in B_i$, then by construction, $B_{i+1} \in \delta(B_i, B_i \cap AP)$ iff $\varphi_2 \in B_{i+1}$ or $\varphi_1, \varphi_1 \mathsf{U} \varphi_2 \in B_{i+1}$. How long can we go like this?
- If $\varphi_2 \in B_k$ for $k > i$, and $\varphi_1, \varphi_1 \mathsf{U} \varphi_2 \in B_{i+1}, \ldots, B_{k-1}$ then $A_i A_{i+1} \ldots \models \varphi_1 \mathsf{U} \varphi_2$.
- When is $B_i B_{i+1} B_{i+2} \ldots$ an accepting run?
- $B_j \in F_{\varphi_1 \mathsf{U} \varphi_2}$ for infinitely many $j \geqslant i$.

$L(\varphi) \subseteq L(A_\varphi)$

Let $\sigma = A_0 A_1 A_2 \cdots \in L(\varphi)$. Show that there is an accepting run $B_0 A_0 B_1 A_1 B_2 A_2 \ldots$ in A_φ for σ, B_i are the states, such that $B_i = \{\psi \mid A_i A_{i+1} \ldots \models \psi\}$.

- If $\varphi_1 \, \mathsf{U} \varphi_2 \in B_i$, then by construction, either $\varphi_2 \in B_i$ or $\varphi_1, \varphi_1 \, \mathsf{U} \varphi_2 \in B_i$. If $\varphi_2 \in B_i$ then $A_i A_{i+1} \ldots \models \varphi_2$ by induction hypothesis, and hence, $A_i A_{i+1} \ldots \models \varphi_1 \, \mathsf{U} \varphi_2$.
- If $\varphi_1, \varphi_1 \, \mathsf{U} \varphi_2 \in B_i$, then by construction, $B_{i+1} \in \delta(B_i, B_i \cap AP)$ iff $\varphi_2 \in B_{i+1}$ or $\varphi_1, \varphi_1 \, \mathsf{U} \varphi_2 \in B_{i+1}$. How long can we go like this?
- If $\varphi_2 \in B_k$ for $k > i$, and $\varphi_1, \varphi_1 \, \mathsf{U} \varphi_2 \in B_{i+1}, \ldots, B_{k-1}$ then $A_i A_{i+1} \ldots \models \varphi_1 \, \mathsf{U} \varphi_2$.
- When is $B_i B_{i+1} B_{i+2} \ldots$ an accepting run?
- $B_j \in F_{\varphi_1 \, \mathsf{U} \varphi_2}$ for infinitely many $j \geqslant i$.
- $\varphi_2 \notin B_j$ or $\varphi_2, \varphi_1 \, \mathsf{U} \varphi_2 \in B_j$ for infinitely many $j \geqslant i$.
- By construction, there is an accepting run where $\varphi_2 \in B_k$ for some $k \geqslant i$. Hence, $A_i A_{i+1} \ldots \models \varphi_1 \, \mathsf{U} \varphi_2$.

$L(\varphi) \subseteq L(A_\varphi)$

Let $\sigma = A_0 A_1 A_2 \cdots \in L(\varphi)$. Show that there is an accepting run $B_0 A_0 B_1 A_1 B_2 A_2 \ldots$ in A_φ for σ, B_i are the states, such that $B_i = \{\psi \mid A_i A_{i+1} \ldots \models \psi\}$.

- If $\neg(\varphi_1 \,\mathsf{U} \varphi_2) \in B_i$, then either $\neg\varphi_1, \neg\varphi_2 \in B_i$ or $\varphi_1, \neg\varphi_2 \in B_i$. If $\neg\varphi_1, \neg\varphi_2 \in B_i$ then $A_i A_{i+1} \ldots \models \neg(\varphi_1 \,\mathsf{U} \varphi_2)$.

$L(\varphi) \subseteq L(A_\varphi)$

Let $\sigma = A_0 A_1 A_2 \cdots \in L(\varphi)$. Show that there is an accepting run $B_0 A_0 B_1 A_1 B_2 A_2 \ldots$ in A_φ for σ, B_i are the states, such that $B_i = \{\psi \mid A_i A_{i+1} \ldots \models \psi\}$.

- If $\neg(\varphi_1 \mathrm{U} \varphi_2) \in B_i$, then either $\neg\varphi_1, \neg\varphi_2 \in B_i$ or $\varphi_1, \neg\varphi_2 \in B_i$. If $\neg\varphi_1, \neg\varphi_2 \in B_i$ then $A_i A_{i+1} \ldots \models \neg(\varphi_1 \mathrm{U} \varphi_2)$.
- If $\varphi_1, \neg\varphi_2 \in B_i$, then by construction, $B_{i+1} \in \delta(B_i, B_i \cap AP)$ iff $\varphi_1, \neg\varphi_2 \in B_{i+1}$ or $\neg\varphi_1, \neg\varphi_2 \in B_{i+1}$.

$L(\varphi) \subseteq L(A_\varphi)$

Let $\sigma = A_0 A_1 A_2 \cdots \in L(\varphi)$. Show that there is an accepting run $B_0 A_0 B_1 A_1 B_2 A_2 \ldots$ in A_φ for σ, B_i are the states, such that $B_i = \{\psi \mid A_i A_{i+1} \ldots \models \psi\}$.

- If $\neg(\varphi_1 \mathsf{U} \varphi_2) \in B_i$, then either $\neg\varphi_1, \neg\varphi_2 \in B_i$ or $\varphi_1, \neg\varphi_2 \in B_i$. If $\neg\varphi_1, \neg\varphi_2 \in B_i$ then $A_i A_{i+1} \ldots \models \neg(\varphi_1 \mathsf{U} \varphi_2)$.
- If $\varphi_1, \neg\varphi_2 \in B_i$, then by construction, $B_{i+1} \in \delta(B_i, B_i \cap AP)$ iff $\varphi_1, \neg\varphi_2 \in B_{i+1}$ or $\neg\varphi_1, \neg\varphi_2 \in B_{i+1}$.
- Either case, $A_i A_{i+1} \ldots \models \neg(\varphi_1 \mathsf{U} \varphi_2)$

$L(A_\varphi) \subseteq L(\varphi)$

For a sequence $B_0 B_1 B_2 \ldots$ of states satisfying

- $B_{i+1} \in \delta(B_i, A_i)$,
- $\forall F \in \mathcal{F}$, $B_j \in F$ for infinitely many j,

we have $\psi \in B_0 \leftrightarrow A_0 A_1 \ldots \models \psi$

- Structural Induction on ψ. Interesting case : $\psi = \varphi_1 \,\mathsf{U} \varphi_2$
- Assume $A_0 A_1 \ldots \models \varphi_1 \,\mathsf{U} \varphi_2$. Then $\exists j \geqslant 0$, $A_j A_{j+1} \ldots \models \varphi_2$ and $A_i A_{i+1} \ldots \models \varphi_1, \varphi_1 \,\mathsf{U} \varphi_2$ for all $i \leqslant j$.
- By induction hypothesis (applied to φ_1, φ_2), we obtain $\varphi_2 \in B_j$ and $\varphi_1 \in B_i$ for all $i \leqslant j$
- By induction on j, $\varphi_1 \,\mathsf{U} \varphi_2 \in B_j, \ldots, B_0$.

$L(A_\varphi) \subseteq L(\varphi)$

For a sequence $B_0 B_1 B_2 \ldots$ of states satisfying

(a) $B_{i+1} \in \delta(B_i, A_i)$,

(b) $\forall F \in \mathcal{F}$, $B_j \in F$ for infinitely many j,

we have $\psi \in B_0 \leftrightarrow A_0 A_1 \ldots \models \psi$

- Conversely, assume $\varphi_1 \mathsf{U} \varphi_2 \in B_0$. Then $\varphi_2 \in B_0$ or $\varphi_1, \varphi_1 \mathsf{U} \varphi_2 \in B_0$.
- If $\varphi_2 \in B_0$, by induction hypothesis, $A_0 A_1 \ldots \models \varphi_2$, and hence $A_0 A_1 \ldots \models \varphi_1 \mathsf{U} \varphi_2$
- If $\varphi_1, \varphi_1 \mathsf{U} \varphi_2 \in B_0$. Assume $\varphi_2 \notin B_j$ for all $j \geqslant 0$. Then $\varphi_1, \varphi_1 \mathsf{U} \varphi_2 \in B_j$ for all $j \geqslant 0$.
- Since $B_0 B_1 \ldots$ satisfies (b), $B_j \in F_{\varphi_1 \mathsf{U} \varphi_2}$ for infinitely many $j \geqslant 0$, we obtain a contradiction.
- Thus, $\exists$ a smallest k s.t. $\varphi_2 \in B_k$. Then by induction hypothesis, $A_i A_{i+1} \ldots \models \varphi_1$ and $A_k A_{k+1} \models \varphi_2$ for all $i < k$
- Hence, $A_0 A_1 \ldots \models \varphi_1 \mathsf{U} \varphi_2$.

GNBA Size

- States of A_φ are subsets of $Cl(\varphi)$

GNBA Size

- States of A_φ are subsets of $Cl(\varphi)$
- Maximum number of states $\leqslant 2^{|\varphi|}$

GNBA Size

- States of A_φ are subsets of $Cl(\varphi)$
- Maximum number of states $\leqslant 2^{|\varphi|}$
- Number of sets in $\mathcal{F} = |\varphi|$

GNBA Size

- States of A_φ are subsets of $Cl(\varphi)$
- Maximum number of states $\leqslant 2^{|\varphi|}$
- Number of sets in $\mathcal{F} = |\varphi|$
- LTL $\varphi \rightsquigarrow$ NBA A_φ : Number of states in $A_\varphi \leqslant |\varphi|.2^{|\varphi|}$
- There is no LTL formula φ for the language

$$L = \{A_0 A_1 A_2 \cdots \mid a \in A_{2i}, i \geqslant 0\}$$

Complexity of LTL Modelchecking

- Given φ, $A_{\neg\varphi}$ has $\leqslant 2^{|\varphi|}$ states
  - $|\varphi|$=size/length of φ, the number of operators in φ
- $TS \otimes A_{\neg\varphi}$ has $\leqslant |TS|.2^{|\varphi|}$ states
- Persistence checking : Checking $\Box\Diamond\eta$ on $TS \otimes A_{\neg\varphi}$ takes time linear in $\eta.|TS \otimes A_{\neg\varphi}|$

∃∀ Automata and the LTL connection

- For finite words