



भारतीय प्रौद्योगिकी संस्थान मुंबई Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 8

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Slide preparation acknowledgments: C. R. Pradhit and Adit Akarsh

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal



- ▶ The Social Choice Setup
- ▶ The Gibbard-Satterthwaite Theorem
- ▶ Proof of Gibbard-Satterthwaite Theorem
- ▶ Domain Restriction
- ▶ Median Voting Rule
- ▶ Median Voter Theorem: Part 1
- ▶ Median Voter Theorem: Part 2

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- Ways out:
 - ① consider a **social choice** setup
 - ② put restrictions on agent preferences
- **Social choice function** (SCF)

$$f : \mathcal{P}^n \rightarrow A$$

$$A = \{a_1, a_2, \dots, a_m\}$$

Finite set of alternatives

$$N = \{1, 2, \dots, n\}$$

Finite set of players

$$\mathcal{P}$$

Set of all **linear** preference ordering

Examples



- Most representative: **voting**

$$\begin{array}{cccc} & P & & \\ \hline a & a & c & d \\ b & b & b & c \\ c & c & d & b \\ d & d & a & a \end{array} \xrightarrow{f} A = \{a, b, c, d\}$$

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$$\begin{array}{c}
 P \\
 \hline
 a \quad a \quad c \quad d \\
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 - **Borda**: named after French mathematician Jean-Charles de Borda $(m-1, m-2, \dots, 1, 0)$

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 - **k-approval**: $(\underbrace{1, 1, \dots, 1}_k, 0, 0, \dots, 0)$

Examples (contd.)



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P			
a	a	c	d
b	b	b	c
c	c	d	b
d	d	a	a

$$\text{score}(a) = \min\{2(b), 2(c), 2(d)\} = 2$$

$$\text{score}(b) = \min\{2(a), 2(c), 3(d)\} = 2$$

$$\text{score}(c) = \min\{2(a), 2(b), 3(d)\} = 2$$

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- **Copeland**: based on Copeland score = number of wins in pairwise elections

Condorcet consistency



Definition

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30%	30%	40%
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no **scoring rule** is Condorcet consistent

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An SCF f is *Pareto efficient* (PE) if $\forall P$ and $a \in A$, if a is Pareto dominated, then $f(P) \neq a$.



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An SCF f is *unanimous* (UN) if $\forall P$ satisfying $P_1(1) = P_2(1) = \dots = P_n(1) = a$ [$P_i(k)$ is the k -th favorite alternative of i], it holds that $f(P) = a$.



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Which implies which? if the top choice of all voters is the same, say a , all other alternatives are Pareto dominated by a

Desirable properties of SCF (contd.)



Definition (Onto)

An SCF f is *onto* (ONTO) if $\forall a \in A, \exists P^{(a)} \in \mathcal{P}^n$ s.t. $f(P^{(a)}) = a$.

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- Plurality with fixed tie-breaking

$a \succ b \succ c$		
4	4	1
<hr/>		
a	b	c
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 $a \succ b \succ c$, Copeland score = number of wins in pairwise elections

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Strategyproofness and its implications



Definition (Strategyproof)

An SCF is *strategyproof* (SP) if it is not manipulable by any agent at any profile.

Implications: **monotonicity**

- Define **dominated set** of an alternative a at a preference P_i as

$$D(a, P_i) := \{b \in A : aP_ib\}$$



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- The set of alternatives *below* a in P_i

$$P_i = \begin{matrix} b \\ a \\ c \\ d \end{matrix} \Rightarrow D(a, P_i) = \{c, d\}$$



Definition (Monotonicity)

An SCF is *monotone* (MONO) if for every two profiles P and P' that satisfy $f(P) = a$ and $D(a, P_i) \subseteq D(a, P'_i)$, for all $i \in N$, it holds that $f(P') = a$.



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P				P'			
a	a	c	d	c	a	c	d
b	b	b	c	b	c	b	c
c	c	d	b	a	b	d	b
d	d	a	a	d	d	a	a



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d	d	a	a	d	d	a	a

Theorem

An SCF f is **strategyproof** iff it is **monotone**.



- ▶ The Social Choice Setup
- ▶ The Gibbard-Satterthwaite Theorem
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Strategyproofness and Monotonicity



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- Consider the “if” condition of MONO
- P and P' with $f(P) = a$ and $D(a, P_i) \subseteq D(a, P'_i) \forall i \in N$



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- Break the transition from P to P' into n stages:

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$$\begin{array}{ccccc}
 (P_1, P_2, P_3, \dots, P_n) & \rightarrow & (P'_1, P_2, P_3, \dots, P_n) & \rightarrow & (P'_1, P'_2, P_3, \dots, P_n) \\
 P = P^{(0)} & & P^{(1)} & & P^{(2)} \\
 \dots & \rightarrow & (P'_1, \dots, P'_k, P_{k+1}, \dots, P_n) & \rightarrow & (P'_1 \dots P'_n) \\
 & & P^{(k)} & & P^{(n)} = P'
 \end{array}$$

Proof of $SP \Leftrightarrow MONO$



$$\begin{array}{ccccc} (P_1, P_2, P_3, \dots, P_n) & \rightarrow & (P'_1, P_2, P_3, \dots, P_n) & \rightarrow & (P'_1, P'_2, P_3, \dots, P_n) \\ P = P^{(0)} & & P^{(1)} & & P^{(2)} \\ \dots & \rightarrow & (P'_1, \dots, P'_k, P_{k+1}, \dots, P_n) & \rightarrow & (P'_1 \dots P'_n) \\ & & P^{(k)} & & P^{(n)} = P' \end{array}$$

Claim: $f(P^{(k)}) = a, \forall k = 1, \dots, n.$

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Claim: $f(P^{(k)}) = a, \forall k = 1, \dots, n$.

- Suppose not, i.e., $\exists P^{(k-1)}, P^{(k)}, \text{ s.t. } f(P^{(k-1)}) = a, f(P^{(k)}) = b \neq a$



Proof of $SP \Leftrightarrow MONO$

$$\begin{array}{ccccc}
 (P_1, P_2, P_3, \dots, P_n) & \rightarrow & (P'_1, P_2, P_3, \dots, P_n) & \rightarrow & (P'_1, P'_2, P_3, \dots, P_n) \\
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- There can be one of the three cases:
 - 1 $a P_k b$ and $a P'_k b \rightarrow$ voter k misreports $P'_k \rightarrow P_k$

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 - ② $b P_k a$ and $b P'_k a \rightarrow$ voter k misreports $P_k \rightarrow P'_k$



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 - 3 $b P_k a$ and $a P'_k b \rightarrow$ voter k misreports in both



Proof of $SP \Leftrightarrow MONO$

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 - ① $a P_k b$ and $a P'_k b \rightarrow$ voter k misreports $P'_k \rightarrow P_k$
 - ② $b P_k a$ and $b P'_k a \rightarrow$ voter k misreports $P_k \rightarrow P'_k$
 - ③ $b P_k a$ and $a P'_k b \rightarrow$ voter k misreports in both
- Contradiction to f being SP

Proof of $SP \Leftrightarrow MONO$ (contd.)



- For $(SP \Leftarrow MONO)$, we will prove $\neg SP \implies \neg MONO$

Proof of $SP \Leftrightarrow MONO$ (contd.)



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Proof of $SP \Leftrightarrow MONO$ (contd.)

- For $(SP \Leftarrow MONO)$, we will prove $\neg SP \Rightarrow \neg MONO$
- Suppose not, i.e., f is $\neg SP$ but $MONO$
- $\neg SP$ implies that $\exists i, P_i, P'_i, P_{-i}$, s.t. $\underbrace{f(P'_i, P_{-i})}_{b \text{ (say)}} P_i \underbrace{f(P_i, P_{-i})}_{a \text{ (say)}} = b P_i a$



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 - ① $(P_i, P_{-i}) \rightarrow (P''_i, P_{-i})$
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- This concludes the proof

Equivalence of PE, UN, ONTO under SP



Lemma

If an SCF f is MONO and ONTO, then f is PE.

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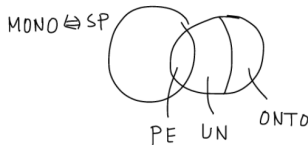


Figure: Relation between SCFs



- Suppose not, i.e. f is MONO and ONTO but not PE then $\exists a, b, P$ s.t., $b \neq P_i a \forall i \in N$ but $f(P) = a$



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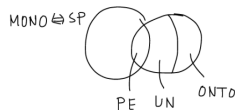


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Corollary: f is SP+PE $\iff f$ is SP+UN $\iff f$ is SP+ONTO



Gibbard-Satterthwaite Theorem



Theorem (Gibbard 1973, Satterthwaite 1975)

Suppose $|A| \geq 3$, f is ONTO and SP iff f is dictatorial.

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So, what did just happen?

- No reasonable voting rule is **truthful**
- Plurality, Borda, Copeland, Maximin, ...
- Crucial: the preferences are **unrestricted**, i.e., all $m!$ preference profiles are in the domain of the SCF f



- ▶ The Social Choice Setup
- ▶ The Gibbard-Satterthwaite Theorem
- ▶ **Proof of Gibbard-Satterthwaite Theorem**
- ▶ Domain Restriction
- ▶ Median Voting Rule
- ▶ Median Voter Theorem: Part 1
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- 3 **Indifference in preferences**: in general, GS theorem does not hold. In the proof, we use some specific constructions. If they are possible, then GS theorem holds.
- 4 **Cardinalization**: GS theorem will hold as long as all possible ordinal ranks are feasible in the cardinal preferences.

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Suppose $|A| \geq 3$, $N = \{1, 2\}$, and f is ONTO and SP, then for every preference profile P , $f(P) \in \{P_1(1), P_2(1)\}$

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- If $P_1(1) = P_2(1)$, then UN implies $f(P) = P_1(1)$ (ONTO \iff UN under SP)
- Say $P_1(1) = a \neq b = P_2(1)$. For contradiction assume $f(P) = c \neq a, b$ (need at least 3 alternatives)

Proof of GS Theorem (contd.)



P_1	P_2	P_1	P'_2	P'_1	P'_2	P'_1	P_2
a	b	a	b	a	b	a	b
\cdot	\cdot	\cdot	a	b	a	b	\cdot
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot

$$f(P_1, P_2) = c(\neq a, b)$$

- Now $f(P_1, P'_2) \in \{a, b\}$ [because all alternatives except b are Pareto dominated by a]

Proof of GS Theorem (contd.)



P_1	P_2	P_1	P'_2	P'_1	P'_2	P'_1	P_2
a	b	a	b	a	b	a	b
\cdot	\cdot	\cdot	a	b	a	b	\cdot
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot

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- Now $f(P_1, P'_2) \in \{a, b\}$ [because all alternatives except b are Pareto dominated by a]
- But if $f(P_1, P'_2) = b$, then player 2 manipulates from P_2 to P'_2 , hence $f(P_1, P'_2) = a$

Proof of GS Theorem (contd.)



P_1	P_2	P_1	P'_2	P'_1	P'_2	P'_1	P_2
a	b	a	b	a	b	a	b
\cdot	\cdot	\cdot	a	b	a	b	\cdot
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Proof of GS Theorem (contd.)



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Proof of GS Theorem (contd.)

P_1	P_2	P_1	P'_2	P'_1	P'_2	P'_1	P_2
a	b	a	b	a	b	a	b
\cdot	\cdot	\cdot	a	b	a	b	\cdot
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot

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a	b	a	b	a	b	a	b
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- Now apply MONO
 - $P'_1, P_2 \rightarrow P'_1, P'_2$ outcome should be b
 - $P_1, P'_2 \rightarrow P'_1, P'_2$ outcome should be a (contradiction)



Proof of GS Theorem (contd.)

Lemma (Two player version of GS theorem)

Suppose $|A| \geq 3$, $N = \{1, 2\}$, and f is ONTO and SP

- Let $P : P_1(1) = a \neq b = P_2(1)$, $P' : P'(1) = c$, $P'_2(1) = d$
- If $f(P) = a$, then $f(P') = c$
- If $f(P) = b$, then $f(P') = d$



Proof of GS Theorem (contd.)

Lemma (Two player version of GS theorem)

Suppose $|A| \geq 3$, $N = \{1, 2\}$, and f is ONTO and SP

- Let $P : P_1(1) = a \neq b = P_2(1)$, $P' : P'_1(1) = c$, $P'_2(1) = d$
- If $f(P) = a$, then $f(P') = c$
- If $f(P) = b$, then $f(P') = d$

Proof: If $c = d$, unanimity proved the lemma. Hence consider $c \neq d$.

cases \downarrow	c	d
1	a	b
2	$\neq a, b$	b
3	$\neq a, b$	$\neq b$
4	a	$\neq a, b$
5	b	$\neq a, b$
6	b	a

- Enough to consider the case: if $f(P) = a \implies f(P') = c$
- The other case is symmetric
- These cases are exhaustive



Proof of GS Theorem (contd.)

Case 1: $c = a, d = b$,

P_1	P_2	P'_1	P'_2	\hat{P}_1	\hat{P}_2
a	b	a	b	a	b
\cdot	\cdot	\cdot	\cdot	b	a
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot

- We know (by previous lemma) $f(P') \in \{a, b\}$

$$\begin{array}{c} P_1 \ P_2 \\ a \end{array} \xrightarrow{\text{MONO}} \begin{array}{c} \hat{P}_1 \ \hat{P}_2 \\ a \end{array}$$

$$\begin{array}{c} P'_1 \ P'_2 \\ b \end{array} \xrightarrow{\text{MONO}} \begin{array}{c} \hat{P}_1 \ \hat{P}_2 \\ b \end{array}$$



Proof of GS Theorem (contd.)

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Proof of GS Theorem (contd.)



Case 2: $c \neq a, b, d = b,$

P_1	P_2	P'_1	P'_2	\hat{P}_1	P_2
a	b	c	b	c	b
\cdot	\cdot	\cdot	\cdot	a	\cdot
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot

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(apply case 1)

agent 1 misreports $\hat{P}_1 \rightarrow P_1$ as $a \hat{P}_1 b$

Proof of GS Theorem (contd.)



Case 2: $c \neq a, b, d = b$,

P_1	P_2	P'_1	P'_2	\hat{P}_1	P_2
a	b	c	b	c	b
\cdot	\cdot	\cdot	\cdot	a	\cdot
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot

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Proof of GS Theorem (contd.)



Case 3: $c \neq a, b$, and $d \neq b$,

P_1	P_2	P'_1	P'_2	\hat{P}_1	\hat{P}_2
a	b	c	d	c	b
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot

- Say $f(P') = d$

$$P' \rightarrow \hat{P}$$

$$f(\hat{P}) = b \text{ (case 2)}$$

$$P \rightarrow \hat{P}$$

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Proof of GS Theorem (contd.)



Case 4: $c = a$, and $d \neq b, a$

P_1	P_2	P'_1	P'_2	\hat{P}_1	\hat{P}_2
a	b	$c = a$	d	a	b
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot

- Say $f(P') = d$

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$$f(\hat{P}) = b \text{ (case 2)}$$

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Proof of GS Theorem (contd.)

Case 5: $c = b$, and $d \neq b, a$

P_1	P_2	P'_1	P'_2	\hat{P}_1	\hat{P}_2
a	b	$c = b$	d	c	d
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot

- Say $f(P') = d$

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Proof of GS Theorem (contd.)

Case 6: $c = b$, and $d = a$

P_1	P_2	P'_1	P'_2	\hat{P}_1	P'_2	\tilde{P}_1	P'_2
a	b	$c = b$	$d = a$	b	a	x	a
\cdot	\cdot	\cdot	\cdot	x	\cdot	\cdot	\cdot
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot	\cdot

$$f(P') = a$$

$$x \neq a, b$$

$$P' \rightarrow (\hat{P}_1 P'_2),$$

$$f(\hat{P}_1 P'_2) = a \text{ (case 1)}$$

$$P' \rightarrow (\tilde{P}_1 P'_2),$$

$$f(\tilde{P}_1 P'_2) = x \text{ (case 3)}$$

- Player 1 manipulates from $\hat{P}_1 P'_1 \rightarrow \tilde{P}_1 P'_2$, since $x \neq a$
- This completes the proof of $n = 2$ agent case
- **$n \geq 3$ agent case:** induction on the number of agents. See Sen (2001): "A direct proof of GS theorem", Economics Letters



- ▶ The Social Choice Setup
- ▶ The Gibbard-Satterthwaite Theorem
- ▶ Proof of Gibbard-Satterthwaite Theorem
- ▶ **Domain Restriction**
- ▶ Median Voting Rule
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GS theorem holds for unrestricted preferences



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- If we reduce the set of feasible preferences from \mathcal{P} to $\mathcal{S} \subset \mathcal{P}$
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 - but there can potentially be more f 's that can be strategyproof on the **restricted domain**

Domain restrictions



- 1 Single peaked preferences
- 2 Divisible goods allocation
- 3 Quasi-linear preferences

Each of these domains have interesting non-dictatorial SCFs that are strategyproof

Single peaked preferences



- Temperature of a room

Single peaked preferences



- Temperature of a room
- For every agent, most comfortable temperature t_i^*

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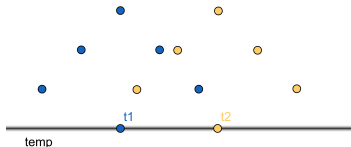


Figure: Single peaked temperature preference

Single peaked preferences



- One **common order** over the alternatives

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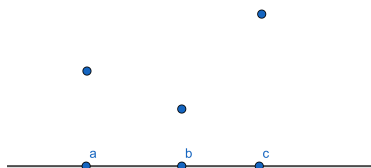


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 - ② consider only one-dimensional single-peakedness



Single peaked preferences

How is it a domain restriction?



Consider $a < b < c$, all possible orderings:

a	b	b	c	a	c
b	a	c	b	c	a
c	c	a	a	b	b

Definition (Single peaked preferences)

A preference ordering P_i (linear over A) of agent i is single-peaked w.r.t. the common order $<$ of the alternatives if

- ① $\forall b, c \in A$ with $b < c \leq P_i(1)$, cP_ib
- ② $\forall b, c \in A$ with $P_i(1) \leq b < c$, bP_ic

Single peaked preferences



- Let \mathcal{S} be the set of single peaked preferences. The SCF: $f : \mathcal{S}^n \rightarrow A$

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How does it circumvent GS theorem?

Answer

Each player's preference has a peak. Suppose, f picks the leftmost peak. For the agent having the leftmost peak, no reason to misreport. For any other agent, the only way she can change the outcome is by reporting her peak to be left of the leftmost – but that is strictly worse than the current outcome.

Repeat this argument for any fixed k^{th} peak from left. Even the rightmost peak choosing SCF is also strategyproof, so is the median ($k = \lceil \frac{n}{2} \rceil$)



- ▶ The Social Choice Setup
- ▶ The Gibbard-Satterthwaite Theorem
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Definition

An SCF $f : S^n \rightarrow A$ is a median voter SCF if there exists $B = \{y_1, y_2, \dots, y_{n-1}\}$ s.t. $f(P) = \text{median}(B, \text{peaks}(P))$ for all preference profiles $P \in S$



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- Phantom voters give a complete spectrum of the median voter SCFs



Theorem (Moulin 1980)

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Proof Sketch:

- if $f(P) = a$ and a player has a peak $P_i(1)$ to the left of a , it has no benefit by misreporting the peak to be on the right of a , which is the only way of changing the outcome of f
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Note: mean does not have this property



Claim

Let p_{\min} and p_{\max} be the leftmost and rightmost peaks of P according to $<$, then f is PE iff $f(P) \in [p_{\min}, p_{\max}]$



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(\impliedby) If $f(P) \in [p_{\min}, p_{\max}]$, then the condition $bP_i f(P), \forall i \in N$ never occurs – there does not exist an alternative b that Pareto dominates $f(P)$. Hence $f(P)$ is PE (vacuously true from definition).

Median voter SCF and Monotonicity



Definition (Monotonicity)

An SCF is *monotone* (MONO) if for every two profiles P and P' that satisfy $f(P) = a$ and $D(a, P_i) \subseteq D(a, P'_i)$, for all $i \in N$, it holds that $f(P') = a$.



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- The relative position of c has improved from P to P' ; if c was the outcome at P , it continues to become the outcome at P'



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P				P'			
a	a	c	d	c	a	c	d
b	b	b	c	b	b	b	c
c	c	d	b	a	c	d	b
d	d	a	a	d	d	a	a

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Theorem

$f \text{ is SP} \implies f \text{ is MONO}$

This proof is similar to the previous one. To prove the reverse implication one needs to argue why the construction is valid in the single peaked domain. **(or provide counterexample)**

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Theorem

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Proof:

- We know $PE \implies UN \implies ONTO$

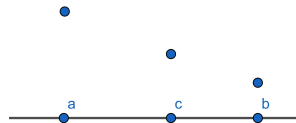


Figure: Arrangement of a, b, c



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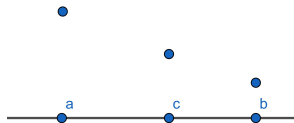


Figure: Arrangement of a, b, c



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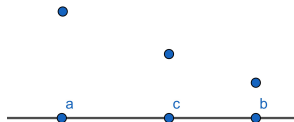


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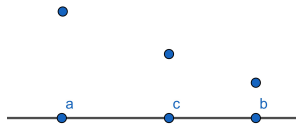


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- Since preferences are single peaked, \exists another alternative $c \in A$, which is a neighbour of b s.t. $c P_i b \forall i \in N$ (c can be a itself)

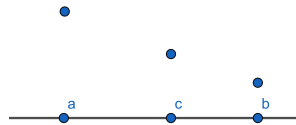


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- ONTO $\implies \exists P'$ s.t. $f(P') = c$



- $\text{ONTO} \implies \exists P' \text{ s.t. } f(P') = c$
- Construct P'' s.t. $P''_i(1) = c, P''(2) = b, \forall i \in N$



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Now, we are interested in non-dictatorial SCFs, hence a necessary property is **anonymity**



- **Anonymity**: outcome insensitive to agent identities



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- **Example**: $N = \{1, 2, 3\}, \sigma : \sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1$

P_1	P_2	P_3			
P_1	P_2	P_3	P_1^σ	P_2^σ	P_3^σ
a	b	b	b	a	b
b	a	c	c	b	a
c	c	a	a	c	c

Anonymity (contd.)



Definition

An SCF $f : \mathcal{S}^n \rightarrow A$ is **anonymous** (ANON) if for every profile P and for every permutation of the agents σ , $f(P^\sigma) = f(P)$

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Dictatorship is not anonymous



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Median Voter Theorem



Seen the equivalence of SP, ONTO, ANON and median voting rule in single peaked domain

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A **strategyproof** SCF f is ONTO and **anonymous** iff it is a median voter SCF.



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Proof: (\Leftarrow)

- Median voter SCF is SP (previous theorem)
- It is **anonymous**: if we permute the agents with peaks unchanged, the outcome does not change
- It is ONTO, pick any arbitrary alternative a , put peaks of all players at a : the outcome will be a irrespective of the positions of the phantom peaks (since there are $(n - 1)$ phantom peaks and n agent peaks)

Proof (contd.)



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\implies Given, $f : \mathcal{S}^n \rightarrow A$ is SP, ANON, and ONTO.

- define, P_i^0 : agent i 's preference with peak at leftmost w.r.t. $<$
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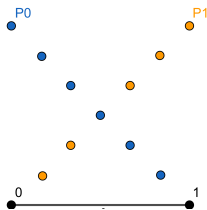


Figure: Two preferences



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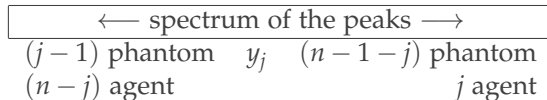
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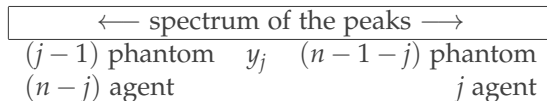
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- Hence, $p_1 \leq \dots \leq p_{n-j} \leq y_j = a \leq p_{n-j+1} \leq \dots \leq p_n$



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- repeat this argument for the first $(n - j)$ agents to get

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- **Claim:** $N = \{1, 2\}$, let P and P' be such that $P_i(1) = P'_i(1), \forall i \in N$. Then $f(P) = f(P')$



Median Voter Theorem: Proof

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- **Claim:** Suppose f satisfies SP, ONTO, ANON, then $f(P) = \text{median}(p_1, \dots, p_n, y_1, \dots, y_{n-1})$
- WLOG, can assume $p_1 \leq p_2 \leq \dots \leq p_n$ due to ANON
- Consider $a = \text{median}(p_1, \dots, p_n, y_1, \dots, y_{n-1})$
- **Case 1:** a is a phantom peak: proved
- **Case 2:** a is an agent peak
- We will prove this for 2 players, the general case repeats this argument
- **Claim:** $N = \{1, 2\}$, let P and P' be such that $P_i(1) = P'_i(1), \forall i \in N$. Then $f(P) = f(P')$
- **Proof:** Let $a = P_1(1) = P'_1(1)$, and $P_2(1) = P'_2(1) = b$. $f(P) = x$ and $f(P'_1, P_2) = y$



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- The only other possibility is that x and y fall on different sides of the peak: **we show that this is not possible.**

Proof (contd.)



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Profile: $(P_1, P_2) = P, P_1(1) = a, P_2(1) = b, y_1$ is the phantom peak, and by assumption, $\text{median}(a, b, y_1)$ is an agent peak

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- We have two cases to consider: $b < a < y_1$ and $y_1 < a < b$



Case 2.1: $b < a < y_1$, by PE $c < a$

- Construct P'_1 s.t. $P'_1(1) = a = P_1(1)$ and $y P'_1 c$ (possible since they are on different sides of a)



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- But $y P'_1 c$ (by construction) and $f(P'_1, P_2) = c$
- Agent 1 manipulates $P'_1 \rightarrow P_1^1$, contradiction to f being SP



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- Construct P'_1 s.t. $P'_1(1) = a = P_1(1)$ and $y P'_1 c$
- $f(P'_1, P_2) = c$ (by claim)



Case 2.2: $y_1 < a < b$, by PE $a < c$

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- $f(P'_1, P_2) = c$ (by claim)
- Consider (P_1^0, P_2) , $P_1^0(1) \leq y_1 < b \implies f(P_1^0, P_2) = y_1$ but $y_1 \ P'_1 \ c$, hence manipulable by agent 1



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- This completes the proof for two agents (case 2)



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- Construct P'_1 s.t. $P'_1(1) = a = P_1(1)$ and $y \succ P'_1 \succ c$
- $f(P'_1, P_2) = c$ (by claim)
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- For the generalization to n players, see Moulin (1980), "On strategyproofness and single-peakedness"



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