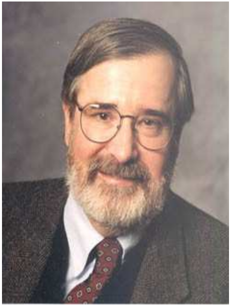
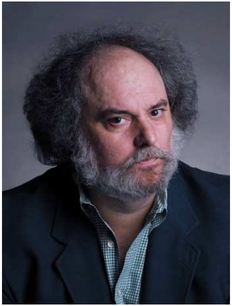
CS 228 : Logic in Computer Science

Krishna. S

Linear Temporal Logic

Model Checking

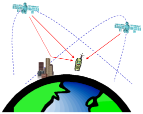
  

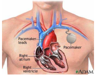
- Year 2007 : ACM confers the Turing Award to the pioneers of Model Checking: Ed Clarke, Allen Emerson, and Joseph Sifakis
- `https://amturing.acm.org/award_winners/clarke_1167964`

Model checking

- Model checking has evolved in last 25 years into a widely used verification and debugging technique for software and hardware.
- Model checking used (and further developed) by companies/institutes such as IBM, Intel, NASA, Cadence, Microsoft, and Siemens, and has culminated in many freely downloadable software tools that allow automated verification.

What is Model Checking?





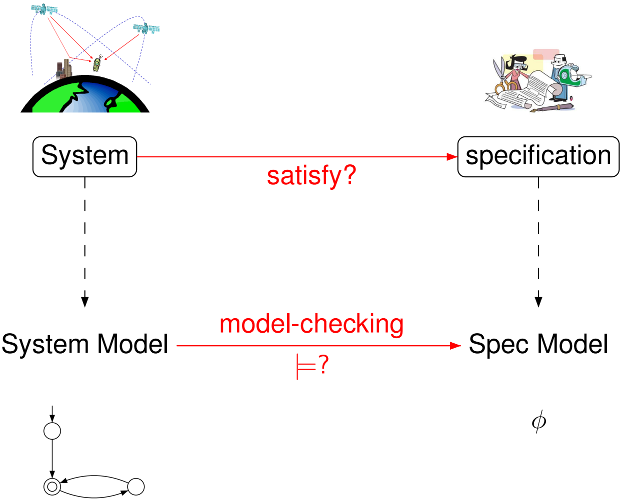
System

satisfy?

specification

good/bad properties

What is Model Checking?



Model Checker as a Black Box

- Inputs to Model checker : A finite state system M, and a property P to be checked.
- Question : Does M satisfy P?
- Possible Outputs
  - Yes, M satisfies P
  - No, here is a counter example!.

What are Models?

Transition Systems

- States labeled with propositions
- Transition relation between states
- Action-labeled transitions to facilitate composition

What are Properties?

Example properties

- Can the system reach a deadlock?
- Can two processes ever be together in a critical section?
- On termination, does a program provide correct output?

Notations for Infinite Words

- Σ is a finite alphabet
- Σ^* set of finite words over Σ
- An infinite word is written as $\alpha = \alpha(0)\alpha(1)\alpha(2)\ldots$, where $\alpha(i) \in \Sigma$
- Such words are called ω-words
- a^ω, $a^7.b^\omega$

Transition Systems

A Transition System is a tuple $(S, Act, \rightarrow, I, AP, L)$ where

- S is a set of states
- Act is a set of actions
- $s \xrightarrow{\alpha} s'$ in $S \times Act \times S$ is the transition relation
- $I \subseteq S$ is the set of initial states
- AP is the set of atomic propositions
- $L : S \rightarrow 2^{AP}$ is the labeling function

Traces of Transition Systems

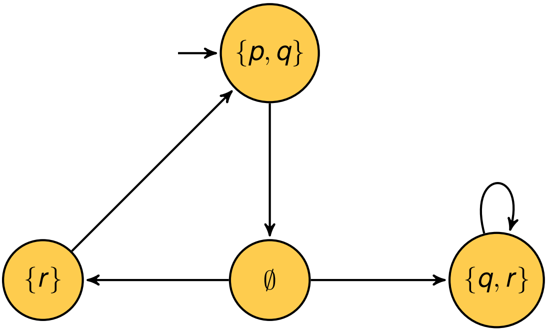
- Labels of the locations represent values of all observable propositions $\in AP$
- Captures system state
- Focus on sequences $L(s_0)L(s_1)\ldots$ of labels of locations
- Such sequences are called traces
- Assuming transition systems have no terminal states,
  - Traces are infinite words over 2^{AP}
  - Traces $\in (2^{AP})^\omega$
  - Go to the example slide and define traces

Traces of Transition Systems

Given a transition system $TS = (S, Act, \rightarrow, I, AP, L)$ without terminal states,

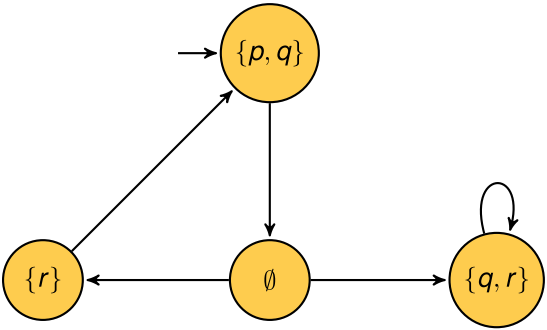
- All maximal executions/paths are infinite
- Path $\pi = s_0 s_1 s_2 \ldots$, $trace(\pi) = L(s_0)L(s_1)\ldots$
- For a set Π of paths, $Trace(\Pi) = \{trace(\pi) \mid \pi \in \Pi\}$
- For a location s, $Traces(s) = Trace(Paths(s))$
- $Traces(TS) = \bigcup_{s \in I} Traces(s)$

Example Traces



$AP = \{p, q, r, t\}$

Example Traces



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- $\{p, q\}\emptyset\,\{q, r\}^\omega$

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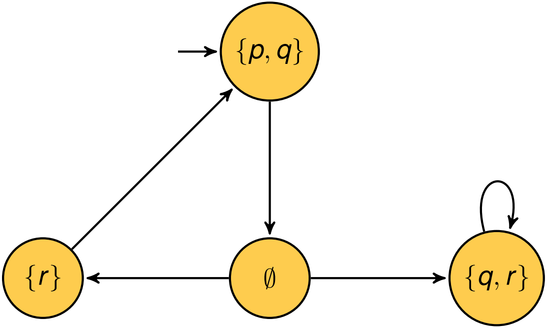

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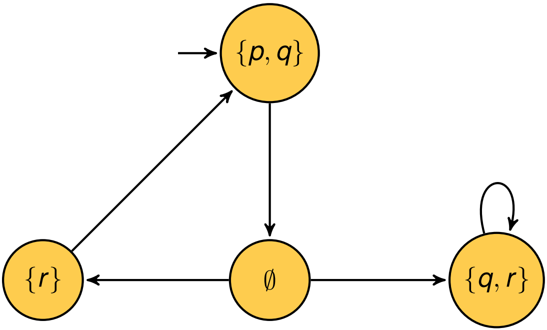

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- $(\{p, q\}\emptyset\{r\})^\omega$
- $(\{p, q\}\emptyset\{r\})^*\ \{p, q\}\emptyset\ \{q, r\}^\omega$

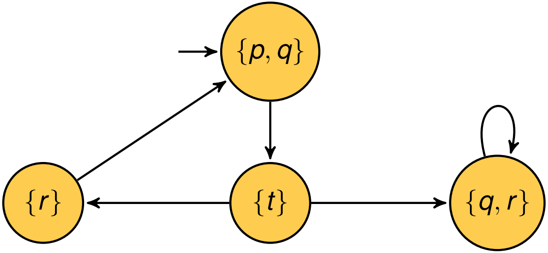
Linear Time Properties

- Linear-time properties specify traces that a TS must have
- A LT property P over AP is a subset of $(2^{AP})^\omega$
- TS over AP satisfies a LT property P over AP

$$TS \models P \text{ iff } Traces(TS) \subseteq P$$

- $s \in S$ satisfies LT property P (denoted $s \models P$) iff $Traces(s) \subseteq P$

Specifying Traces



- Whenever p is true, r will eventually become true
  - $\{A_0A_1A_2\cdots \mid \forall i \geqslant 0, p \in A_i \to \exists j \geqslant i, r \in A_j\}$
- q is true infinitely often
  - $\{A_0A_1A_2\cdots \mid \forall i \geqslant 0, \exists j \geqslant i, q \in A_j\}$
- Whenever r is true, so is q
  - $\{A_0A_1\cdots \mid \forall i \geqslant 0, r \in A_i \to q \in A_i\}$

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- Propositional logic formulae over *AP*
  - $a \in AP$ (atomic propositions)
  - $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$

Syntax of Linear Temporal Logic

Given *AP*, a set of propositions,

- Propositional logic formulae over *AP*
  - $a \in AP$ (atomic propositions)
  - $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$
- Temporal Operators
  - $\bigcirc\varphi$ (Next φ)
  - $\varphi \, \mathsf{U} \psi$ (φ holds until a ψ-state is reached)
- LTL : Logic for describing LT properties

Semantics (On the board)

LTL formulae φ over *AP* interpreted over words $w \in \Sigma^\omega$, $\Sigma = 2^{AP}$,
$w \models \varphi$

Derived Operators

- $\mathit{true} = \varphi \vee \neg\varphi$
- $\mathit{false} = \neg \mathit{true}$
- $\Diamond\varphi = \mathit{true}\, \mathsf{U} \varphi$ (Eventually φ)
- $\Box\varphi = \neg\Diamond\neg\varphi$ (Forever φ)

Precedence

- Unary Operators bind stronger than Binary
- $\bigcirc$ and $\neg$ equally strong
- U takes precedence over $\wedge, \vee, \rightarrow$
  - $a \vee b\, \mathsf{U} c \equiv a \vee (b\, \mathsf{U} c)$
  - $\bigcirc a\, \mathsf{U} \neg b \equiv (\bigcirc a)\, \mathsf{U} (\neg b)$

Examples

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  $\Diamond yellow$
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Semantics over Infinite Words

Given LTL formula φ over AP,

$$L(\varphi) = \{\sigma \in (2^{AP})^\omega \mid \sigma \models \varphi\}$$

Let $\sigma = A_0 A_1 A_2 \ldots$.

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- $\sigma \models \varphi_1 \wedge \varphi_2$ iff $\sigma \models \varphi_1$ and $\sigma \models \varphi_2$

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- $\sigma \models \varphi \mathsf{U} \psi$ iff
  $\exists j \geqslant 0$ such that $A_j A_{j+1} \ldots \models \psi \wedge \forall 0 \leqslant i < j, A_i A_{i+1} \ldots \models \varphi$

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If $\sigma = A_0 A_1 A_2 \ldots$, $\sigma \models \varphi$ is also written as $\sigma, 0 \models \varphi$. This simply means $A_0 A_1 A_2 \ldots \models \varphi$. One can also define $\sigma, i \models \varphi$ to mean $A_i A_{i+1} A_{i+2} \ldots \models \varphi$ to talk about a suffix of the word σ satisfying a property.

Transition System Semantics $TS \models \varphi$

Let $TS = (S, S_0, \rightarrow, AP, L)$ be a transition system, and φ an LTL formula over AP

- For an infinite path fragment π of TS,

$$\pi \models \varphi \text{ iff } \mathit{trace}(\pi) \models \varphi$$

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$$s \models \varphi \text{ iff } \forall \pi \in \mathit{Paths}(s), \pi \models \varphi$$

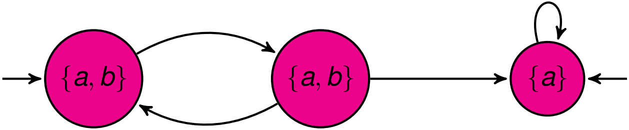
- $TS \models \varphi$ iff $\mathit{Traces}(TS) \subseteq L(\varphi)$

Transition System Semantics $TS \models \varphi$

Assume all states in TS are reachable from S_0.

- $TS \models \varphi$ iff $TS \models L(\varphi)$ iff $Traces(TS) \subseteq L(\varphi)$
- $TS \models L(\varphi)$ iff $\pi \models \varphi \ \forall \pi \in Paths(TS)$
- $\pi \models \varphi \ \forall \pi \in Paths(TS)$ iff $s_0 \models \varphi \ \forall s_0 \in S_0$

Example



- $TS \models \Box a$,

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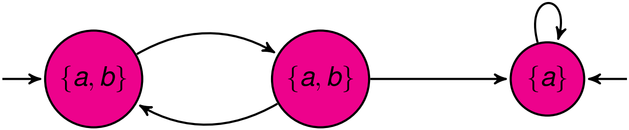

- $TS \models \Box a$,
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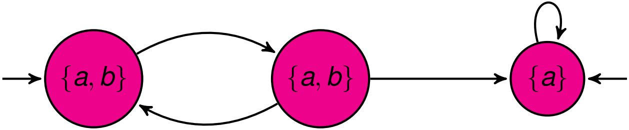

- $TS \models \Box a$,
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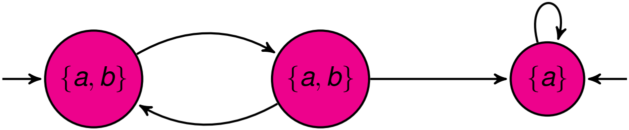
Example



- $TS \models \square a$,
- $TS \not\models \bigcirc(a \wedge b)$
- $TS \not\models (b \, \mathsf{U} (a \wedge \neg b))$
- $TS \models \square(\neg b \rightarrow \square(a \wedge \neg b))$

More Semantics

- For paths π, $\pi \models \varphi$ iff $\pi \not\models \neg\varphi$

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  $\mathit{trace}(\pi) \in L(\varphi)$ iff $\mathit{trace}(\pi) \notin L(\neg\varphi) = \overline{L(\varphi)}$
- $TS \not\models \varphi$ iff $TS \models \neg\varphi$?
  - $TS \models \neg\varphi \rightarrow \forall$ paths π of TS, $\pi \models \neg\varphi$
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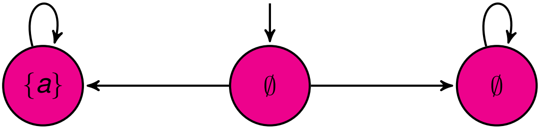
  - Thus, $\forall\pi, \pi \not\models \varphi$. Hence, $TS \not\models \varphi$
  - Now assume $TS \not\models \varphi$
  - Then $\exists$ some path π in TS such that $\pi \models \neg\varphi$
  - However, there could be another path π' such that $\pi' \models \varphi$
  - Then $TS \not\models \neg\varphi$ as well
- Thus, $TS \not\models \varphi \not\equiv TS \models \neg\varphi$.

An Example



$TS \not\models \Diamond a$ and $TS \not\models \Box \neg a$

Equivalence of LTL Formulae

Equivalence

φ and ψ are equivalent ($\varphi \equiv \psi$) iff $L(\varphi) = L(\psi)$.

Expansion Laws

- $\varphi \, \mathsf{U} \psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi \, \mathsf{U} \psi))$
- $\Diamond\varphi \equiv \varphi \vee \bigcirc\Diamond\varphi$
- $\Box\varphi \equiv \varphi \wedge \bigcirc\Box\varphi$

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Distribution

$\bigcirc(\varphi \vee \psi) \equiv \bigcirc\varphi \vee \bigcirc\psi$,
$\bigcirc(\varphi \wedge \psi) \equiv \bigcirc\varphi \wedge \bigcirc\psi$,
$\bigcirc(\varphi \,\mathsf{U} \psi) \equiv (\bigcirc\varphi) \,\mathsf{U} (\bigcirc\psi)$,
$\Diamond(\varphi \vee \psi) \equiv \Diamond\varphi \vee \Diamond\psi$,
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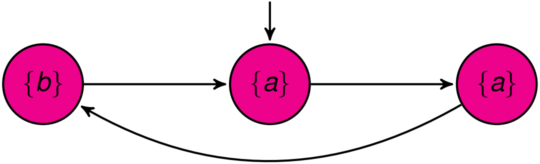
Equivalence of LTL Formulae



$$TS \models \Diamond a \wedge \Diamond b, \; TS \not\models \Diamond(a \wedge b)$$

$$TS \models \Box(a \vee b), \; TS \not\models \Box a \vee \Box b$$

Satisfiability, Model Checking of LTL

Two Questions

Given transition system *TS*, and an LTL formula φ. Does $TS \models \varphi$?
Given an LTL formula φ, is $L(\varphi) = \emptyset$?

How we go about this:

- Translate φ into an automaton A_φ that accepts infinite words such that $L(A_\varphi) = L(\varphi)$.
- Check for emptiness of A_φ to check satisfiability of φ.
- Check if $TS \cap \overline{A_\varphi}$ is empty, to answer the model-checking problem.

ω-automata

An ω-automaton is a tuple $\mathcal{A} = (Q, \Sigma, \delta, q_0, Acc)$ where

- Q is a finite set of states
- Σ is a finite alphabet
- $\delta : Q \times \Sigma \to 2^Q$ is a state transition function (if non-deterministic, otherwise, $\delta : Q \times \Sigma \to Q$)
- $q_0 \in Q$ is an initial state and Acc is an acceptance condition

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Run

A run ρ of $\mathcal{A}$ on an ω-word $\alpha = a_1 a_2 \cdots \in \Sigma^\omega$ is an infinite state sequence $\rho(0)\rho(1)\rho(2)\ldots$ such that

- $\rho(0) = q_0$,
- $\rho(i) = \delta(\rho(i-1), a_i)$ if $\mathcal{A}$ is deterministic,
- $\rho(i) \in \delta(\rho(i-1), a_i)$ if $\mathcal{A}$ is non-deterministic,

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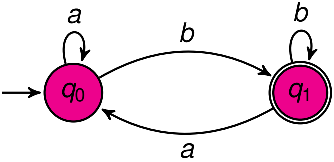
Büchi Acceptance

For Büchi Acceptance, Acc is specified as a set of states, $G \subseteq Q$. The ω-word α is accepted if there is a run ρ of α such that $Inf(\rho) \cap G \neq \emptyset$.

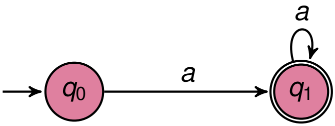
ω-Automata with Büchi Acceptance

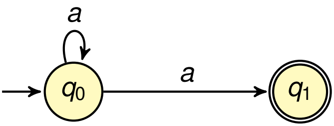


$L(\mathcal{A}) = \{\alpha \in \Sigma^\omega \mid \alpha$ has a run ρ such that $Inf(\rho) \cap G \neq \emptyset$

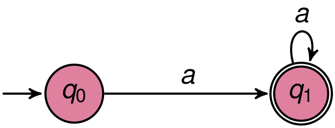
Language accepted=Infinitely many *b*'s.

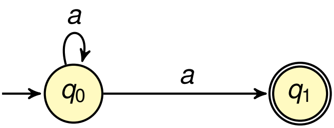
Comparing NFA and NBA

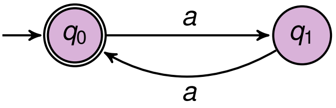


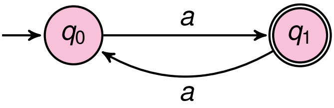


Comparing NFA and NBA

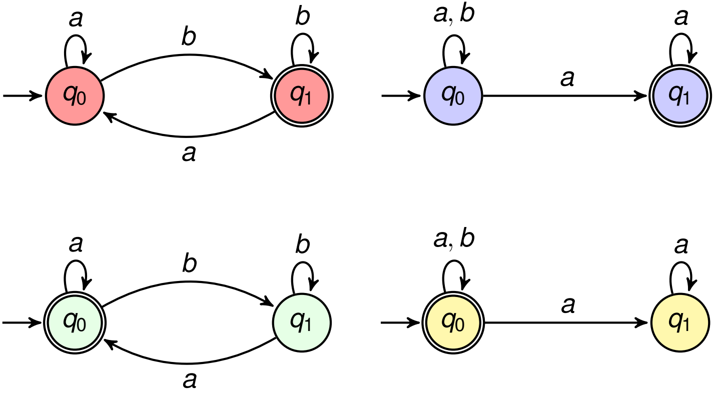








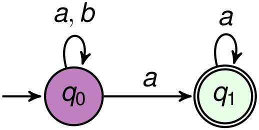
ω-Automata with Büchi Acceptance



- Left (T-B): Inf many b's, Inf many a's
- Right (T-B): Finitely many b's, $(a+b)^\omega$

NBA and DBA

- Is every DBA as expressible as a NBA, like in the case of DFA and NFA?
- Can we do subset construction on NBA and obtain DBA?



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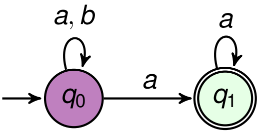

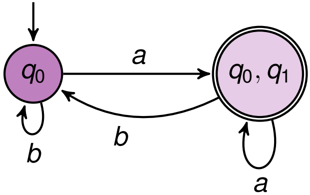

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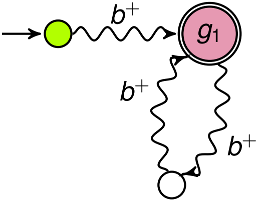

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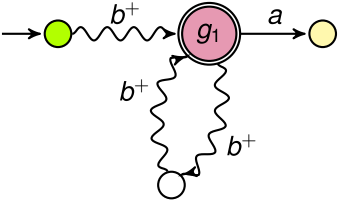

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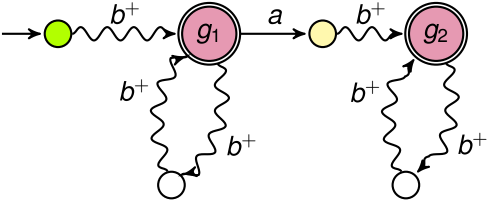

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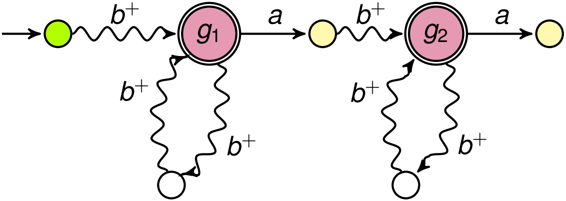

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