Revision on geometry of multivariate Gaussian

$X \sim \mathbb{R}^p$ $\quad \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$ $\quad X \sim N(\underset{p\times 1}{\mu}, \Sigma_{p\times p})$

$$f(x) = \frac{1}{(\sqrt{2\pi})^{d/2} |\Sigma|^{1/2}} e^{-(x-\mu)^T \Sigma^{-1} (x-\mu)}$$

Σ = positive semi-definite matrix.

$\Rightarrow \Sigma^{-1}$ = "

$$\underbrace{(x-\mu)^T \Sigma^{-1} (x-\mu)}_{d_\Sigma(x,\mu)} \geq 0$$

$e^{-d_\Sigma(x,\mu)}$

$x = \mu$ then $d_\Sigma(x,\mu) = 0$

$\Rightarrow f(x)$ is maximized at $x = \mu$.

Analyze the contours of $f(x)$

$\underline{f(x)} = c$

$\Leftrightarrow \quad (x-\mu)^T \Sigma^{-1} (x-\mu) = \Delta^2$

for eg: if $\Delta = 0$ then $x = \mu$.

To understand further the above contour.

exploit the spectral or Eigen decomposition of Σ^{-1}

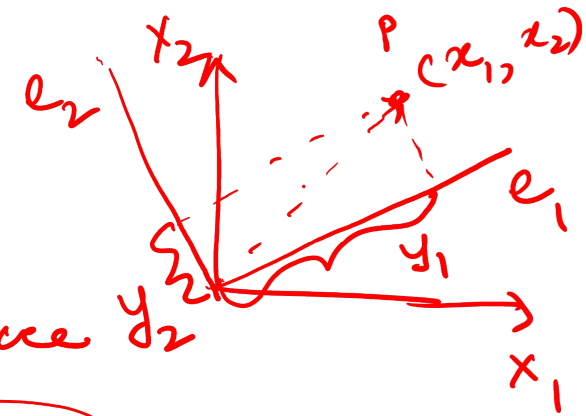
$$\Sigma^{-1} = \frac{1}{\lambda_1} \underbrace{e_1 e_1^T}_{p \times p} + \frac{1}{\lambda_2} e_2 e_2^T + \frac{1}{\lambda_3} e_3 e_3^T + \cdots \frac{1}{\lambda_p} e_p e_p^T$$

$$(x-\mu)^T \left[\frac{1}{\lambda_1} e_1 e_1^T + \cdots \frac{1}{\lambda_p} e_p e_p^T \right] (x-\mu)$$

$$\Leftrightarrow \frac{1}{\lambda_1}\underbrace{(x-\mu)^T e_1}_{\triangleq\, y_1}\,\underbrace{e_1^T(x-\mu)} + \cdots \frac{1}{\lambda_p}\underbrace{(x-\mu)^T e_p}_{\triangleq\, y_p}\, e_p^T(x-\mu) = \Delta^2$$

$$\Leftrightarrow \frac{1}{\lambda_1} y_1^2 + \cdots \frac{1}{\lambda_p} y_p^2 = \Delta^2$$

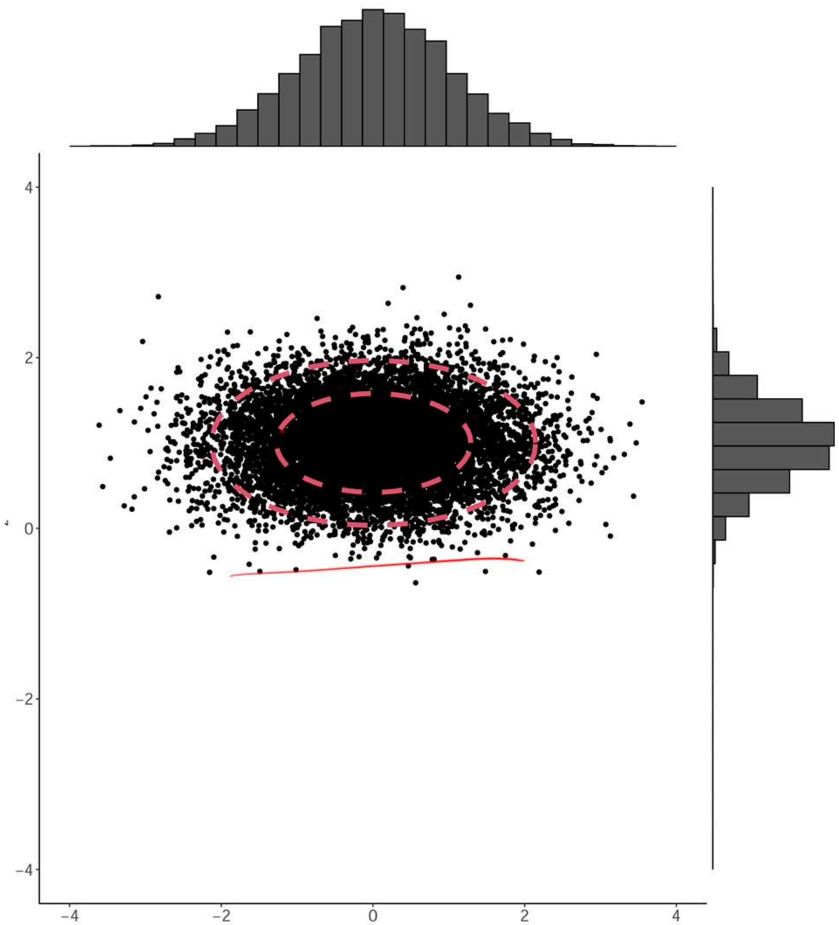
Equation of an ellipse in the eigen vector co-ordinate space



$\Delta\sqrt{\lambda_1}$

$\Delta\sqrt{\lambda_2}$

Example of contours



$$\mu = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 0.2 \end{pmatrix}$$

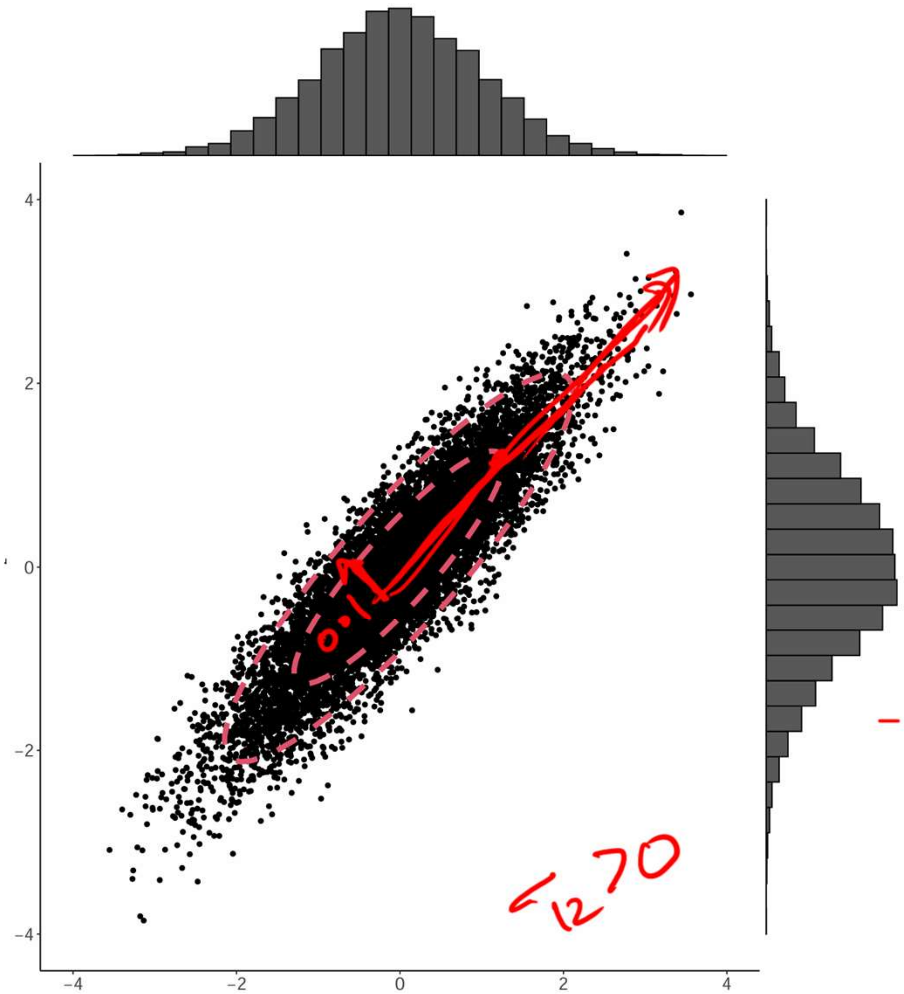
$Ae = \lambda e$

$\Sigma e = \lambda e$

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \lambda_1 = 1$

$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \lambda_2 = 0.2$

Correlated variables



$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \end{pmatrix}$$

Anytime $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{11} \end{pmatrix}$

$\lambda_1 = \sigma_{11} + \sigma_{12}$

$e_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ if $\sigma_{12} > 0$

$e_2 \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

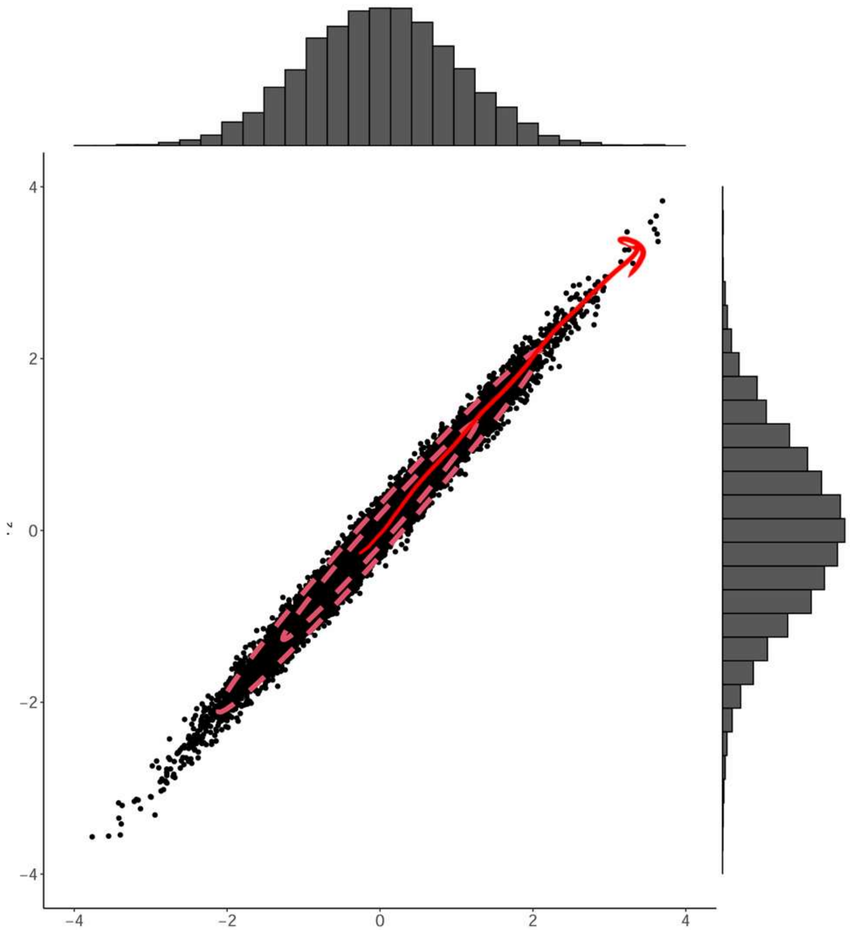
$\lambda_2 = \sigma_{11} - \sigma_{12}$

if $\sigma_{12} < 0$

$e_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$e_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Highly correlated variables



$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

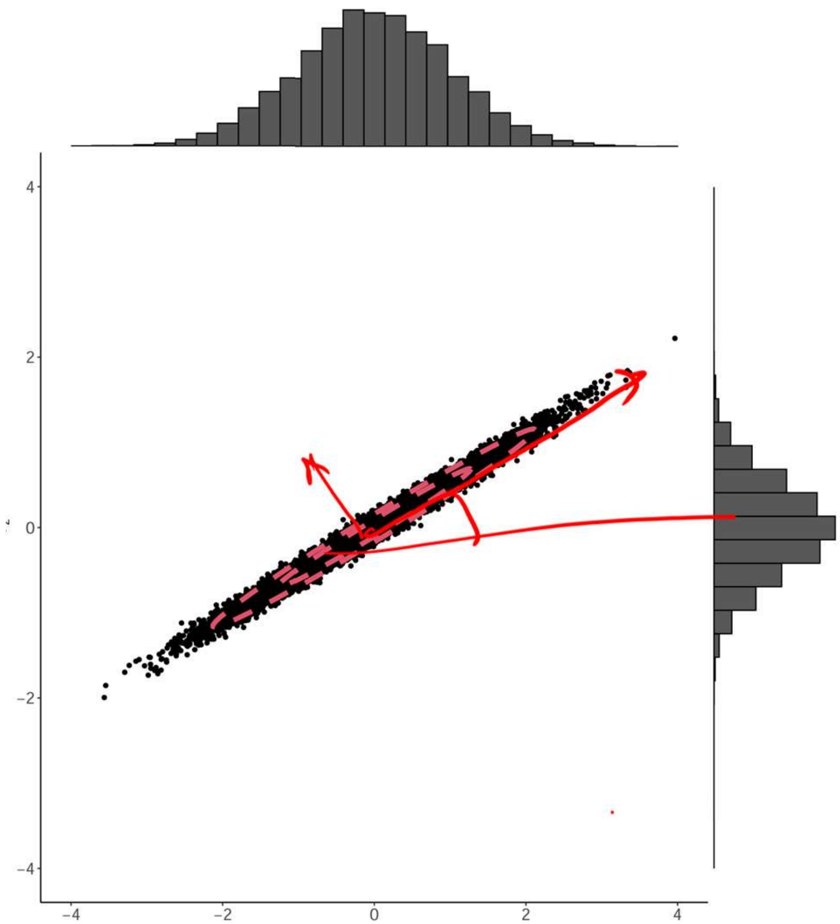
$$\Sigma = \begin{pmatrix} 1 & 0.99 \\ 0.99 & 1 \end{pmatrix}$$

$\lambda_1 = 1.99$

$\lambda_2 = 1 - 0.99$

$= 0.01$

Correlation with different variance. $|\Sigma - \lambda I| = 0$



$$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 1 & 0.54 \\ 0.54 & 0.3 \end{pmatrix}$$

$Cor(Y_1, Y_2) =$
$0.54/\sqrt{(0.3)} =$
0.99

Eigen values and vectors

$\lambda_1 = \underline{1.29} \quad e_1 = \begin{bmatrix} 0.88 \\ 0.48 \end{bmatrix}$

$\lambda_2 = \underline{0.006} \quad e_2 = \begin{bmatrix} -0.48 \\ 0.88 \end{bmatrix}$