Logic in CS Autumn 2024

Problem Sheet 5

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1. Consider the formula $\varphi = \forall x \exists y R(x,y) \wedge \exists y \forall x \neg R(x,y)$. Show that φ is satisfiable over a structure whose universe is infinite and countable.

Solution

Let $\mathcal{N}$ be the structure having as universe the set $\mathbb{N}$ of natural numbers and which interprets $R(x,y)$ as the successor relation, i.e, $R(x,y) = \{(n, n+1) \mid n \in \mathbb{N}\}$. Observe that $\mathcal{N} \models \varphi$, because every $x \in \mathbb{N}$ has a successor in $\mathbb{N}$, making $\forall x \exists y R(x,y)$ true. But $1 \in \mathbb{N}$ is not the predecessor of any number in $\mathbb{N}$[a], making $\exists y \forall x \neg R(x,y)$ true.

Follow-up Question. What property should R satisfy so that φ is not satisfiable over any structure with infinite and countable universe?

[a] Assuming the set of natural numbers start with 1 and not 0

2. Let τ be a signature consisting of a binary relation P and a unary relation F. Let $\mathcal{F}$ be a structure consisting of a universe of people, $P(x,y)$ is interpreted on $\mathcal{F}$ as "x is a parent of y" and $F(x)$ is interpreted as "x is female". Given the τ-structure $\mathcal{F}$,

   (a) Define a formula $\varphi_B(x,y)$ which says x is a brother of y
   (b) Define a formula $\varphi_A(x,y)$ which says x is an aunt of y
   (c) Define a formula $\varphi_C(x,y)$ which says x and y are cousins
   (d) Define a formula $\varphi_O(x)$ which says x is an only child
   (e) Give an example of a family relationship that cannot be defined by a formula

Solution

(a) Assuming a brother is a distinct non-female who shares a parent in common with you, we have:

$$\varphi_B(x,y) = \neg(x = y) \wedge \neg F(x) \wedge \exists z[P(z,x) \wedge P(z,y)]$$

(b) Assuming an aunt is a female who shares a parent in common with a parent of yours, we have:

$$\varphi_A(x,y) = F(x) \wedge \exists z[P(z,y) \wedge \exists w(P(w,z) \wedge P(w,x))]$$

(c) Assuming cousins are distinct people who have distinct parents who have a parent in common, we can write:

$$\varphi_C(x, y) = \exists a \exists b \exists c[\neg(a = b) \wedge P(a, x) \wedge P(b, y) \wedge P(c, a) \wedge P(c, b)] \wedge \neg(x = y)$$

(d) Assuming an only child is a person whose parents have no other children, we have:

$$\varphi_O(x) = \forall y \forall z[P(z, x) \wedge P(z, y) \implies x = y]$$

(e) Since the only relationships modeled here are parent-child relationships and whether a person is female, a relationship such as marriage, i.e., $\varphi_M(x, y)$, that says x is married to y cannot be defined.

3. Consider the signature τ that has the binary functions $+, \times$. Let $\mathcal{N}$ be the structure over τ having as universe the set $\mathbb{N}$ of natural numbers and which interprets $+, \times$ in the usual way. Construct FO formulae $\mathsf{Zero}(x), \mathsf{One}(x), \mathsf{Even}(x), \mathsf{Odd}(x)$ and $\mathsf{Prime}(x)$ using τ such that

- For any $a \in \mathbb{N}$, $\mathcal{N} \models \mathsf{Zero}(a)$ iff a is zero.
- For any $a \in \mathbb{N}$, $\mathcal{N} \models \mathsf{One}(a)$ iff a is one.
- For any $a \in \mathbb{N}$, $\mathcal{N} \models \mathsf{Even}(a)$ iff a is even.
- For any $a \in \mathbb{N}$, $\mathcal{N} \models \mathsf{Odd}(a)$ iff a is odd.
- For any $a \in \mathbb{N}$, $\mathcal{N} \models \mathsf{Prime}(a)$ iff a is prime.

Goldbach's conjecture says that every even integer greater than 2 is the sum of two primes. Whether or not this is true is an open question in number theory. State Goldbach's conjecture as a FO-sentence over τ.

Solution

- $\mathsf{Zero}(a) = \forall x \ (a + x) = x$
- $\mathsf{One}(a) = \forall x \ (a \times x) = a$
- $\mathsf{Even}(a) = \exists x \ (x + x) = a$
- $\mathsf{Odd}(a) = \neg\mathsf{Even}(a)$
- $\mathsf{Prime}(a) = \neg(\exists x \exists y \ (\neg\mathsf{One}(x) \wedge \neg\mathsf{One}(y) \wedge (x \times y) = a)) \wedge \neg\mathsf{One}(a)$

We can also define a FO formula $\mathsf{Two}(a)$ such that for any $a \in \mathbb{N}$, $\mathcal{N} \models \mathsf{Two}(a)$ iff a is two, as $\mathsf{Two}(a) = \exists x \ (\mathsf{One}(x) \wedge (x + x) = a)$.
Using the above formulae, we can state Goldbach's conjecture as:

$\mathsf{Goldbach} := \forall x \ (\neg\mathsf{Zero}(x) \wedge \neg\mathsf{Two}(x) \wedge \mathsf{Even}(x) \to \exists y \exists z \ \mathsf{Prime}(y) \wedge \mathsf{Prime}(z) \wedge (y+z) = x)$

4. A group is a structure $(G, +, 0)$ where G is a set, $0 \in G$ is a special element called the identity and $+ : G \times G \to G$ is a binary operation such that

   (a) The operation $+$ is associative

   (b) The constant 0 is a right-identity for the operation $+$

   (c) Every element in G has a right inverse: for each $x \in G$, we can find $y \in G$ such that $x + y = 0$

   (d) For any three elements $x, y, z \in G$, if $x + z = y + z$, then $x = y$

   Using a signature $\tau = (c, \mathsf{op})$ where c is a constant and op is a binary function symbol write (a)-(d) in FO.

Solution

We write the following formulae for each one of the above specifications:

(a) $\varphi_a = \forall x \forall y \forall z \; [\mathsf{op}(\mathsf{op}(x, y), z) = \mathsf{op}(x, \mathsf{op}(y, z))]$

(b) $\varphi_b = \forall x \; (\mathsf{op}(x, c) = x)$

(c) $\varphi_c = \forall x \exists y \; (\mathsf{op}(x, y) = c)$

(d) $\varphi_d = \forall x \forall y \forall z \; [(\mathsf{op}(x, z) = \mathsf{op}(y, z)) \to x = y]$

5. Let τ be a signature consisting of the binary function symbol $+$ and a constant 0. We denote by $x + y$ the function $+(x, y)$. Consider the following sentences:

$$\varphi_1 := \forall x \forall y \forall z \; [(x + (y + z)) = ((x + y) + z)]$$

$$\varphi_2 := \forall x \; [(x + 0) = x \wedge (0 + x) = x]$$

$$\varphi_3 := \forall x \; [\exists y \; (x + y = 0) \wedge \exists z (z + x) = 0]$$

   Let ψ be the conjunction of the three sentences.

   (a) Show that ψ is satisfiable by exhibiting a τ-structure.

   (b) Show that ψ is not valid.

   (c) Let α be the sentence $\forall x \forall y \; ((x + y) = (y + x))$. Does α follow as a consequence of ψ? That is, is it the case that $\psi \to \alpha$?

   (d) Show that ψ is not equivalent to any of $\varphi_1 \wedge \varphi_2$, $\varphi_2 \wedge \varphi_3$ and $\varphi_1 \wedge \varphi_3$.

Solution

(a) A structure $\mathcal{A}$ that satisfies ψ can be one with $u(\mathcal{A}) = \mathbb{Z}$, $0_{\mathcal{A}} = 0$ and $+_{\mathcal{A}} = +_{\mathbb{Z}}$.

(b) A structure that does not satisfy ψ can be one where $u(\mathcal{A}) = \mathbb{Z}$, $0_{\mathcal{A}} = 0$ and $+_{\mathcal{A}}(x, y) = 2x + 3y$. You can in fact verify that none of φ_1, φ_2, and φ_3 are satisfied.

(c) An example of a structure $\mathcal{A}$ that does not satisfy $\psi \Rightarrow \alpha$ is one where $u(\mathcal{A}) = \{M \in \mathbb{R}^{n\times n} : |M| \neq 0\}$ (the set of invertible real-valued $n \times n$ matrices), $0_{\mathcal{A}} = I_n$, and $+_{\mathcal{A}}(A, B) = AB$. It can be verified that this structure satisfies ψ but not α (i.e., it is not commutative).

(d) i. A structure $\mathcal{A}$ satisfying $\varphi_1 \wedge \varphi_2$ but not ψ is one where $u(\mathcal{A}) = \mathbb{Z}$, $0_{\mathcal{A}} = 1$, and $+_{\mathcal{A}}(x, y) = xy$.

ii. A structure $\mathcal{A}$ satisfying $\varphi_2 \wedge \varphi_3$ but not ψ is one where $u(\mathcal{A}) = \mathbb{N}$, $0_{\mathcal{A}} = 0$, and $+_{\mathcal{A}}(x, y) = |x - y|$.

iii. A structure $\mathcal{A}$ satisfying $\varphi_1 \wedge \varphi_3$ but not ψ is one where $u(\mathcal{A}) = \mathbb{Z}$, $0_{\mathcal{A}} = 1$, and $+_{\mathcal{A}}(x, y) = x + y$.

6. Explain the difference between the first order prefixes $\exists x \forall y \exists z$ and $\forall x \exists y \forall z$.

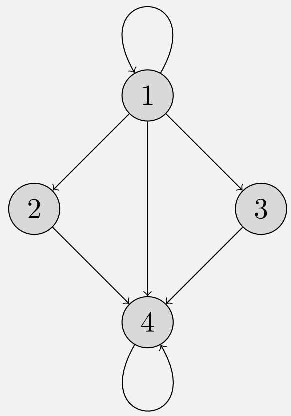
Solution

The first one states that there exists some x in the universe such that for every y in the universe there is some z in the universe (which may depend on y) such that the statement holds.
The second one states that for every x in the universe, there is some y in the universe (which may depend on x) such that for every z in the universe the statement holds.
The difference between the two is illustrated with an example in the following question.

7. Show that the sentences $\forall x \exists y \forall z\ (E(x, y) \wedge E(x, z) \wedge E(y, z))$ and $\exists x \forall y \exists z\ (E(x, y) \wedge E(x, z) \wedge E(y, z))$ are not equivalent by exhibiting a graph which satisfies one but not both of the sentences.

Solution

The first sentence is actually satisfied only by the complete graph K_n, where $E_{\mathcal{A}} = u(\mathcal{A}) \times u(\mathcal{A})$. To see this, assume there is some $(a, b) \notin E_{\mathcal{A}}$. If $\forall x \exists y \forall z\,[E(x, y) \wedge E(x, z) \wedge E(y, z)]$, then we can choose $x = a$, and then, for any y that we choose, choosing $z = b$ will cause $E(x, y) \wedge E(x, z) \wedge E(y, z)$ to not be satisfied, since $E(a, b)$ is not satisfied. This structure will also clearly satisfy the second sentence.
For an example of a structure that satisfies the second sentence but not the first, consider the following graph:



$$u(\mathcal{A}) = \{1, 2, 3, 4\}, \quad E_{\mathcal{A}} = \{(1,1), (1,2), (1,3), (1,4), (2,4), (3,4), (4,4)\}$$

8. For each $n \in \mathbb{N}$, $\exists^{\geq n}$ denotes a counting quantifier. Intuitively, $\exists^{\geq n}$ means that "there exist atleast n such that". FO with counting quantifiers is the logic obtained by adding these quantifiers (for each $n \in \mathbb{N}$) to the fixed symbols of FO. The syntax and semantics are as follows:

   Syntax : For any formula φ of FO with counting quantifiers, $\exists^{\geq n} x\ \varphi$ is also a formula.

   Semantics : $\mathcal{A} \models \exists^{\geq n} x\ \varphi$ iff $\mathcal{A} \models \varphi(a_i)$ for each of n distinct elements $a_1, a_2, \ldots, a_n$ from the universe $u(\mathcal{A})$.

   (a) Using counting quantifiers, define a sentence φ_{45} such that $\mathcal{A} \models \varphi_{45}$ iff $|u(\mathcal{A})| = 45$.

   (b) Define a FO sentence φ (not using counting quantifiers) that is equivalent to the sentence $\exists^{\geq n} x\ (x = x)$.

Solution

(a)

$$\exists^{\geq k} x(x = x) \wedge \neg \exists^{\geq k+1} x(x = x)$$

is an FOL sentence with counting quantifiers that is true iff $|u(\mathcal{A})| = k$.

(b)

$$\exists x_1 \cdots \exists x_n \bigwedge_{1 \leq i < j \leq n} \neg(x_i = x_j)$$

is an FOL sentence equivalent to

$$\exists^{\geq n} x(x = x)$$

9. Write an FO formula that will evaluate to true only over a structure that has atleast n elements and atmost m elements.

Solution

Using the counting quantifiers we discussed earlier, such a sentence would be $\exists^{\geq n} x(x = x) \wedge \neg\exists^{\geq m+1} x(x = x)$. Removing the counting quantifiers, we get the sentence:

$$\left(\exists x_1 \cdots \exists x_n \bigwedge_{1 \leq i < j \leq n} \neg(x_i = x_j)\right) \wedge \neg \left(\exists x_1 \cdots \exists x_{m+1} \bigwedge_{1 \leq i < j \leq m+1} \neg(x_i = x_j)\right)$$

One can show that this sentence is equivalent to:

$$\left(\exists x_1 \cdots \exists x_n \bigwedge_{1 \leq i < j \leq n} \neg(x_i = x_j)\right) \wedge \left(\forall x_1 \cdots \forall x_{m+1} \bigvee_{1 \leq i < j \leq m+1} (x_i = x_j)\right)$$