CS 228: Logic for Computer Science

Pop Quiz - 13 August 2024

Problem Statement

Let p and q be atomic propositions that take values from the set $\{\texttt{true}, \texttt{false}\}$. Consider the following two formulae:

$$\phi_1 = (p \to \neg\phi_2) \quad \text{and} \quad \phi_2 = (q \to \neg\phi_1)$$

a) Show using natural deduction that $\vdash \phi_1 \lor \phi_2$.
Note: You may use the Law of Excluded Middle (LEM) at most once in the proof.

b) Show that there are exactly two pairs of propositional logic formulae (ϕ_1, ϕ_2) that satisfy the above definitions. Also, give justification for your answer.

Solution.

a) **Proof of $\vdash \phi_1 \lor \phi_2$ using natural deduction:**

1.	$(\phi_1 \lor \neg\phi_1)$	LEM
2.	ϕ_1	Assumption
3.	$\phi_1 \lor \phi_2$	$\lor i$ 2
4.	$\neg\phi_1$	Assumption
5.	q	Assumption
6.	$\neg\phi_1$	Copy 4
7.	$q \to \neg\phi_1$	$\to i$ 5-6
8.	ϕ_2	Definition of ϕ_2
9.	$\phi_1 \lor \phi_2$	$\lor i$ 8
10.	$\phi_1 \lor \phi_2$	$\lor e$ 1, 2-3, 4-9

Note: The solution to this question may not be unique. There are also other ways to prove this sequent using LEM atmost once. Can you try doing this without using LEM at all?

b) For this part of the question, let us try working out the truth table of ϕ_1 and ϕ_2, for different assignments of p and q. Consider the following cases:

- $[\![p]\!] = 0$, $[\![q]\!] = 0$. Since both p and q are `false`, from the definition of ϕ_1 and ϕ_2, $[\![\phi_1]\!] = 1$ and $[\![\phi_2]\!] = 1$.
- $[\![p]\!] = 0$, $[\![q]\!] = 1$. Since p is `false`, ϕ_1 is bound to be `true`. Also, $[\![q]\!] = 1 \implies [\![\phi_2]\!] = [\![\neg\phi_1]\!] = 0$
- $[\![p]\!] = 1$, $[\![q]\!] = 0$. Since q is `false`, ϕ_2 is bound to be `true`. Also, $[\![p]\!] = 1 \implies [\![\phi_1]\!] = [\![\neg\phi_2]\!] = 0$
- $[\![p]\!] = 1$, $[\![q]\!] = 1$. Because of this, $[\![\phi_1]\!] = [\![\neg\phi_2]\!]$ and $[\![\phi_2]\!] = [\![\neg\phi_1]\!]$, i.e ϕ_1 and ϕ_2 are forced to take opposite valuations.

Observe that the various assignments of p and q, constraints to (exactly) two possible truth tables for ϕ_1 and ϕ_2.

p	q	ϕ_1	ϕ_2
0	0	1	1
0	1	1	0
1	0	0	1
1	1	1	0

p	q	ϕ_1	ϕ_2
0	0	1	1
0	1	1	0
1	0	0	1
1	1	0	1

The first truth table corresponds to the pair $(\phi_1, \phi_2) = (p \rightarrow q, \neg q)$, while the second truth table corresponds to the pair $(\phi_1, \phi_2) = (\neg p, q \rightarrow p)$. Thus, there are exactly two semantically distinct pairs (ϕ_1, ϕ_2) satisfying above definitions.