CS 228 : Logic in Computer Science

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FO without equality

Let us focus on FO without "=". Recall that "=" is always interpreted as equality.

Herbrand Theorem

Let $\Gamma = \{\varphi_1, \varphi_2, \ldots\}$ be a set of equality-free sentences in Skolem Normal Form. Then Γ is satisfiable iff Γ has a Herbrand model.

If Γ has a Herbrand model, clearly Γ is satisfiable. The converse needs a proof.

The converse

Assume Γ is satisfiable. Let τ_H be the Herbrand signature for Γ.

- Let $\mathcal{A}$ be a τ_H structure such that $\mathcal{A} \models \Gamma$. ($U^{\mathcal{A}}$ need not be the Herbrand universe)
- Let $\mathcal{B}$ be a Herbrand structure over τ_H. ($U^{\mathcal{B}}$ is the Herbrand universe)

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  - How to interpret the relations from τ_H (same as those of τ)?

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  - How to interpret the relations from τ_H (same as those of τ)?
  - Let M interpret relations like $\mathcal{A}$ (not obvious, their universes are not the same.)

Building the Herbrand Model M

- Let R be ant n-ary relation in τ_H (hence in τ).
- For each n-tuple $(t_1, \ldots, t_n)$ with t_i coming from the Herbrand universe $H(\Gamma)$, we must say whether $(t_1, \ldots, t_n) \in R^M$ or not
- Each $t_i \in H(\Gamma)$ is a ground term in τ_H (so variable free).
- Since $\mathcal{A}$ is a structure over τ_H, if $t \in H(\Gamma)$ is a ground term from τ_H, $\mathcal{A}$ interprets t as an element of $U^{\mathcal{A}}$.
- For each n-tuple $(t_1, \ldots, t_n)$, $t_i \in H(\Gamma)$, we know whether $(t_1, \ldots, t_n) \in R^{\mathcal{A}}$ or not
- Define $R^M = R^{\mathcal{A}}$.
- Prove that if $\mathcal{A} \models \varphi$ for any $\varphi \in \Gamma$, then $M \models \varphi$.
- The proof is by induction on the number of quantifiers in φ. Recall that each φ is in Skolem Normal Form.

Base case : φ has 0 quantifiers

$\mathcal{A} \models \varphi$ iff $M \models \varphi$. Do structural induction on φ.

- Assume φ is an atomic formula. Then φ is $R(t_1, \ldots, t_n)$ where R is an n-ary relation from τ_H, and $t_1, \ldots, t_n$ are all terms from $H(\Gamma)$.
- By the construction of M, $R^M = R^{\mathcal{A}}$.
- Hence $M \models \varphi$ iff $\mathcal{A} \models \varphi$.
- Same reasoning holds for $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$ and $\neg\varphi$.
- Hence, $\mathcal{A} \models \varphi$ iff $M \models \varphi$.

Post Inductive Hypothesis

Assume that for any $\psi \in \Gamma$ with $\leqslant k - 1$ quantifiers, if $\mathcal{A} \models \psi$, then $M \models \psi$. Let $\varphi \in \Gamma$ have k quantifiers, $\varphi = \forall x_1 \forall x_2 \ldots \forall x_k \, \zeta(x_1, \ldots, x_k)$ where ζ is quantifier free.

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- Thus, $\mathcal{A} \models \kappa(t)$ for any $t \in H(\Gamma)$.
- By induction hypothesis, $M \models \kappa(t)$ for any $t \in H(\Gamma)$.
- Since $H(\Gamma)$ is the universe of M, $M \models \forall x_1 \kappa(x_1)$. That is, $M \models \varphi$.

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Let φ be in Skolem normal form with equality. Then φ is satisfiable iff there is an equisatisfiable formula ψ in Skolem normal form without equality which has a Herbrand model.