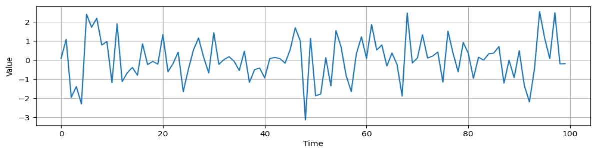
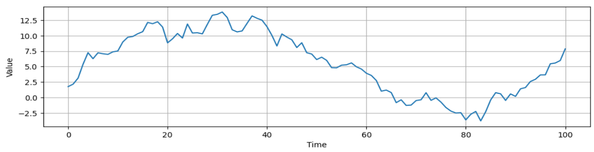
Practice Problems

November 13, 2024

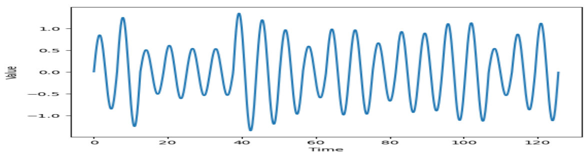
1. Identify if the following time series are stationary or not.



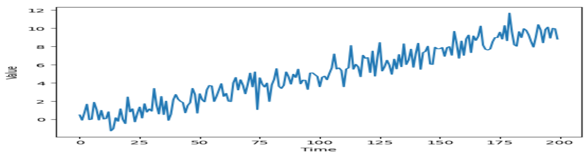
(a) **Ans.** Yes, for weak stationarity, we need to check if the mean, variance, and lag covariance remain the same with shifts in time.



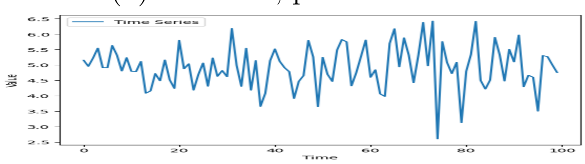
(b) **Ans:** No, presence of a trend.



(c) **Ans:** No. As the peaks are of different heights the mean does not remain constant.



(d) **Ans:** No, presence of a trend.



(e) **Ans:** No, variance is not constant as it increases towards the end.

Figure 1: Which of these are stationary?

2. For the following time series, calculate the coefficients for an AR(1) model.

t	1	2	3	4	5	6	7	8	9	10
x_t	10	9	8	7	16	14	7	13	20	18

Ans. *We want to minimize the sum of squared errors.*
For an AR(1) process

$$x_t = \phi_0 + \phi_1 x_{t-1} + w_t$$

$$SSE = \sum_{t=2}^{n} (x_t - \phi_0 - \phi_1 x_{t-1})^2$$

Differentiate with respect to ϕ_0 and set to 0 to get;

$$\sum_{t=2}^{n}(x_t - \phi_0 - \phi_1 x_{t-1}) = 0$$

Solving for ϕ_0;

$$\phi_0 = \frac{1}{n-1}\sum_{t=2}^{n}(x_t - \phi_1 x_{t-1})$$

The equation for ϕ_1 is:

$$\sum_{t=2}^{n} x_{t-1}(x_t - \phi_0 - \phi_1 x_{t-1}) = 0$$

This simplifies to:

$$\phi_1 = \frac{\sum_{t=2}^{n} x_{t-1}x_t - (n-1)\phi_0\bar{x}}{\sum_{t=2}^{n} x_{t-1}^2}$$

Where $\bar{x}$ is the mean of the time series.

For the given time series $\{10, 9, 8, 7, 16, 14, 7, 13, 20, 18\}$, we calculate the ϕ_0 and ϕ_1 to be :

$$\phi_0 = 11.40 \quad \textit{and} \quad \phi_1 = 0.069$$

3. When $x_t = \eta + \phi_1 x_{t-2} + w_t$ work out the values of mean, variance, auto-correlation, and partial auto-correlation.

   Ans. *We are given the process:*

$$x_t = \eta + \phi_1 x_{t-2} + w_t$$

Taking the expectation on both sides of the equation, we get:

$$\mathbb{E}[x_t] = \mathbb{E}[\eta + \phi_1 x_{t-2} + w_t]$$

Since $\mathbb{E}[w_t] = 0$ (the mean of white noise is zero), we have:

$$\mathbb{E}[x_t] = \eta + \phi_1 \mathbb{E}[x_{t-2}]$$

Using weak stationarity assumptions we get;

$$\mathbb{E}[x_t] = \frac{\eta}{1-\phi_1}$$

For the variance of x_t, take variance on both side and use stationarity assumptions to get;

$$Var(x_t) = \frac{\sigma_w^2}{1-\phi_1^2}$$

For ACF Calculation check appendix here.
For PACF Calculation check slides 1-13 here.

4. Express an AR(1) model as an MA(q) model for an adequate value of q.

Ans. *We can express x_{t-1} as:*

$$x_{t-1} = \phi_1 x_{t-2} + w_{t-1}$$

Substitute this into the original equation:

$$x_t = \phi_1(\phi_1 x_{t-2} + w_{t-1}) + w_t$$

Expand the equation,

$$x_t = \phi_1^2 x_{t-2} + \phi_1 w_{t-1} + w_t$$

Continuing this way;

$$x_t = \sum_{i=0}^{\infty} \theta_i w_{t-i}$$

5. Say Y_t is a weakly stationary time series. Is ΔY_t a weakly stationary time-series? (here Δ is the first difference operator)

Ans. *Yes. Differencing is a safe operator i.e. it does not introduce any new non stationarity.*

6. Say ΔY_t is a weakly stationary time series. Is Y_t a weakly stationary time-series?

Ans. *No. Take a series with trend, differencing might remove the non-stationary behavior but the original series isn't stationary.*

7. Which of the following models will have the same ACF to the MA(1) model $x_t = w_t + \theta w_{t-1}$?

- $x_t = w_t - \theta w_{t-1}$
- $x_t = w_t + \frac{1}{\theta} w_{t-1}$
- $x_t = w_t + |\theta| w_{t-1}$
- None of them

Ans. *We are given the MA(1) model:*

$$x_t = w_t + \theta w_{t-1}$$

The autocorrelation function (ACF) of an MA(1) model is defined as:

$$\rho(k) = \frac{Cov(x_t, x_{t-k})}{\sqrt{Var(x_t) \cdot Var(x_{t-k})}}$$

For the MA(1) model $x_t = w_t + \theta w_{t-1}$, the ACF is:

$$\rho(0) = 1$$

$$\rho(1) = \frac{\theta}{1+\theta^2}$$

$$\rho(k) = 0 \quad for \quad (k > 1)$$

For proof check appendix of link. Using this we can check each of the above options. Only

$$x_t = w_t + \frac{1}{\theta} w_{t-1}$$

has same ACF as the original model.