SC 639 (Autumn 2020) - Mathematical Structures for Control

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June 28, 2021

Outline

1. Isomorphism
2. Rank and Nullity
3. Inner Product and Norm
4. Approximation

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3. Inner Product and Norm
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Isomorphism

Isomorphism

A bijective linear transformation between two finite dimensional subspaces is called an **isomorphism**.

- Let $A : V \to W$ be a linear mapping between two subspaces.
- Subspaces V and W are called isomorphic to each other if the A is surjective and injective.
- If A is an isomorphism, then $dim(V) = dim(W)$.

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Range space and Null space

Range Space

Let $A : V \to W$ be a linear transformation. The **range space (R(A))** is given by following.

$$R(A) = \{w \in W | w = Av, \forall v \in V\} \tag{1}$$

Null space

Let $A : V \to W$ be a linear transformation. The **null space (N(A))** is given by following.

$$N(A) = \{v \in V | Av = 0, \forall v \in V\} \tag{2}$$

- R(A) is a subspace of W.
- N(A) is a subspace of V.
- A is one-one if $N(A) = \{0\}$

Rank and Nullity

- The dimension of Range space of A is called **Rank of A ($\rho(A)$)**.
- The dimension of Null space of A is called **Nullity of A ($\nu(A)$)**.

Rank-Nullity Theorem

Let $A : V \rightarrow W$ be linear with $dim(V) = k$. Then,

$$\rho(A) + \nu(A) = k \tag{3}$$

Proof

- Let $\nu(A) = m$ and basis for null space of A be $\{u_1, u_2, \dots, u_m\}$.
- Extend the set as $\{u_1, u_2, \dots, u_n, w_1, w_2, \dots, w_n\}$ to be a basis of the vector space V. Thus $dim(V) = m + n$
- We need to prove that $R(A)$ is finite and $dim(R(A)) = n$
- Let $v \in V$

$$v = a_1 u_1 + a_2 u_2 + \ldots + a_n u_n + b_1 w_1 + b_2 w_2 + \ldots + b_n w_n$$

applying transformation A on both sides,

$$Av = b_1 Aw_1 + b_2 Aw_2 + \ldots + b_n Aw_n$$

Here, Au_i disappears because $u_i \in N(A)$

- This implies $\{Aw_1, Aw_2, \dots, Aw_n\}$ spans the entire range of A.

Proof

- Now, we show $\{Aw_1, Aw_2, \ldots, Aw_n\}$ is linearly independent.

$$c_1Aw_1 + c_2Aw_2 + \ldots + c_nAw_n = 0$$

Then,

$$A(c_1w_1 + c_2w_2 + \ldots + c_nw_n) = 0$$

hence, $(c_1w_1 + c_2w_2 + \ldots + c_nw_n) \in N(A)$

$$(c_1w_1 + c_2w_2 + \ldots + c_nw_n) = (d_1u_1 + d_2u_2 + \ldots + d_nu_n)$$

- This implies all c and d are zero because $\{u_1, u_2, \ldots, u_n, w_1, w_2, \ldots, w_n\}$ is linearly independent.
- Thus,$\{Aw_1, Aw_2, \ldots, Aw_n\}$ is linearly independent and spans the range of A.

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Norm

Norm

Given a vector space V over a field $\mathcal{F}$, a norm on V is a non-negative valued function $||\cdot|| : V \rightarrow$ with following properties.

Let, $v, w \in V$ and if a is a scalar then,

1. $||av|| = |a| \cdot ||v||$,
2. $||v|| \geq 0$, with equality if and only if $v = 0_V$,
3. $||v + w|| \leq ||v|| + ||w||$,

Examples of Norm

1. Let $v \in \mathbb{R}^n$. The p-norm is given by,

$$||v||_p = \sqrt[p]{\sum_{i=1}^{n} |v_i|^p}$$

2. Let $A \in \mathbb{R}^{m \times n}$. The Frobenius norm is given by, $||A|| = \sqrt{Tr(A^T A)}$
3. Let V be the vector space consisting of all continuous functions on $[a, b]$ and $f \in V$. Then,$||f||_1 = \int_a^b |f(x)|dx$
4. The 2-norm of a signal $f(t)$ is $||f||_2 = \left(\int_{-\infty}^{\infty} f(t)^2 dt \right)^{1/2}$

Inner Product

Inner Product

An inner product in a vector space is a numerically valued function of ordered pair of vectors v and w, such that

1. $\langle \alpha_1 v_1 + \alpha_2 v_2, w \rangle = \alpha_1 \langle v_1, w \rangle + \alpha_2 \langle v_2, w \rangle$
2. $\langle v, w \rangle = \overline{\langle w, v \rangle}$;
3. $\langle v, v \rangle \geq 0; \qquad \langle v, v \rangle = 0$ if and only if $v = 0$

Examples:

1. Let $u, v \in V = \mathbb{R}^n$. Then, $\langle u, v \rangle = u^T v$ is an inner product.
2. Let V be the vector space consisting of all continuous functions and $f, g \in V$. Then, $\langle f, g \rangle = \int_0^1 f(t)g(t)\, dt$ is an inner product.
3. Let $u, v \in^2$. Then $\langle u, v \rangle = u_1 v_1 + 10 u_2 v_2$ is an inner product.

Norm induced by inner product

Norm (induced by inner product)

Let V be an inner product space. The norm of a vector $v \in V$ is defined to be a scalar $||v|| = \sqrt{\langle v, v\rangle}$.

Let V be an inner product space. If $v, w \in V$ and if a is a scalar then, it satisfies following additional properties.

1. $||v+w||^2 + ||v-w||^2 = 2(||v||^2 + ||w||^2)$
2. $||v-w|| \geq |||v|| - ||w|||$

Bessel's Inequality

Theorem

Let V be an inner product space and $u = \{u_1, u_2, \ldots, u_n\}$ be an orthonormal set in V. Let $x \in V$ be arbitary. Define $u \in V$ by $u = \sum_{k=1}^{n} \langle x, u_k \rangle u_k$. Then,

$$||u||^2 = \sum_{k=1}^{n} |\langle x, u_k \rangle|^2 \leqslant ||x||^2 \tag{4}$$

Proof.

Let $g = \sum_{k=1}^{n} \langle x, u_k \rangle u_k$

$$||g||^2 = \langle g, g \rangle = \sum_{k=1}^{n} \langle x, u_k \rangle^2 ||u_k||^2 = \sum_{k=1}^{n} \langle x, u_k \rangle^2 \tag{5}$$

For each $x \in V$ we have that,

$$\begin{aligned} 0 \leq ||x - g||^2 &= \langle x - g, x - g \rangle \\ &= ||x||^2 - 2\langle x, g \rangle = ||g||^2 \\ &= ||x||^2 - 2\langle x, \sum_{k=1}^{n} \langle x, u_k \rangle u_k \rangle + ||g||^2 \\ &= ||x||^2 - 2\sum_{k=1}^{n} |\langle x, u_k \rangle|^2 + ||g||^2 \\ &= ||x||^2 - ||g||^2 \end{aligned} \tag{6}$$

Therefore, $||g||^2 \leq ||x||^2$, which implies that, $||g|| \leq ||x||$.

$$\sum_{k=1}^{n} |\langle x, u_k \rangle|^2 \leqslant ||x||^2 \tag{7}$$

Cauchy-Schwarz Inequality

Recall: We may write a vector u as a scalar multiple of a nonzero vector v, plus a vector orthogonal to the vector v.

$$u = \frac{\langle u, v\rangle}{||v||^2} v + \left(u - \frac{\langle u, v\rangle}{||v||^2} v\right) \tag{8}$$

Theorem

Let V be an inner product space and $u, v \in V$. Then,

$$|\langle u, v\rangle| \leqslant ||u||\,||v|| \tag{9}$$

Proof.

Let $u, v \in V$. If $v = 0$, then both sides of (9) equal 0 and the desired inequality holds. Thus we assume that $v \neq 0$. Consider orthogonal decomposition,

$$u = \frac{\langle u, v\rangle}{||v||^2} v + w$$

By Pythagorean theorem,

$$\begin{aligned} ||u||^2 &= ||\frac{\langle u, v\rangle}{||v||^2} v||^2 + ||w||^2 \\ &= \frac{|\langle u, v\rangle|^2}{||v||^2} + ||w||^2 \\ &\geq \frac{|\langle u, v\rangle|^2}{||v||^2} \end{aligned}$$

Multiplying both sides of this inequality by $||v||^2$ and taking square root gives the Cauchy-Schwarz inequality. □

Adjoint Transformation

- Let V be any vector space and V' be its dual space. A is a linear transformation V.
- For each fixed $y \in V'$, the function y' defined by $y'(x) = [Ax, y]$ is a linear functional on V.
- The adjoint $A' : V' \to V$ is defined by the following property.

$$[Ax, y] = [x, A'y] \tag{10}$$

Adjoint Transformation (based on inner product)

Let V and W be inner product spaces and $A : V \to W$ be a linear transformation. A linear transformation $A' : W \to V$ satisfying the condition $\langle A(v), w \rangle = \langle v, A'(w) \rangle$ for all $V \in V$ and $W \in W$ is called an adjoint transformation of A.

Let V, W and Y be inner product spaces. Let $A : v \to W$, $B : v \to W$ and $C : W \to Y$ be linear transformations having adjoints and k be a scalar. Then,

1. $(A + B)' = A' + B'$
2. $(kA)' = \bar{k}A'$
3. $(CA)' = A'C'$
4. $(A')' = A$

Riesz Representation Theorem

Riesz Representation Theorem

Let V be an inner product space. If $\delta \in V'$ then there exists a unique vector $y \in V$ satisfying $\delta(v) = \langle v, y \rangle$ for all $v \in V$.

Example: Let $n > 1$ be an integer and let V be the subspace of $\mathcal{P}_n$ consisting of all polynomial functions of degree at most n, on which we have an inner product defined by $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\, dt$. Let $\delta \in V'$ be a linear functional defined by $\delta : f \mapsto f(0)$. Then there exists a polynomial function $p \in V$ satisfying the condition $f(0) = \int_{-1}^{1} f(t)p(t)\, dt$ for all $f \in V$.

Projection

Projection

If V is the direct sum of $\mathcal{M}$ and $\mathcal{N}$, so that every $z \in V$ maybe written, uniquely, in form of $z = x + y$, with $x \in \mathcal{M}$ and $y \in \mathcal{N}$, the projection on $\mathcal{M}$ along $\mathcal{N}$ is the transformation E defined by, $Ez = x$.

- A linear transformation E is a projection on some subspace if and only if it is idempotent, that is, $E^2 = E$.

Proof: If E is the projection on $\mathcal{M}$ along $\mathcal{N}$, and if $z = x + y$, then

$$E^2 z = EEz = Ex = x = Ez \tag{11}$$

Conversely, suppose $E^2 = E$. Let $\mathcal{N}$ be a set of all vectors $z \in V$ such that $Ez = 0$. Let $\mathcal{M}$ be a set of all vectors $z \in V$ such that $Ez = z$.
For an arbitary z we have,

$$z = Ez + (1 - E)z$$

If we write $Ez = x$ and $(1 - E)z = y$, then $Ex = E^2 z = Ez = x$ and $Ey = E(1 - E)z = Ez - E^2 z = 0$, so that $x \in \mathcal{M}$ and $y \in \mathcal{N}$. This proves $V = \mathcal{M} \bigoplus \mathcal{N}$ and that the projection on $\mathcal{M}$ along $\mathcal{N}$ is E.

Properties of Projections

- If E is a projection on $\mathcal{M}$ along $\mathcal{N}$, then $\mathcal{M}$ and $\mathcal{N}$ are, respectively, the sets of all solutions of equations $Ez = z$ and $Ez = 0$.
- A linear transformation E is a projection if and only if $1 - E$ is a projection. If E is a projection on $\mathcal{M}$ along $\mathcal{N}$ then $1 - E$ is the projection on $\mathcal{N}$ along $\mathcal{M}$.
- $E_1 + E_2$ is a projection $\iff$ $E_1E_2 = E_2E_1 = 0$; then $E = E_1 + E_2$ is a projection on $\mathcal{M}$ along $\mathcal{N}$, where $\mathcal{M} = \mathcal{M}_1 \bigoplus \mathcal{M}_2$ and $\mathcal{N} = \mathcal{N}_1 \cap \mathcal{N}_2$
- $E_1 - E_2$ is a projection $\iff$ $E_1E_2 = E_2E_1 = E_2$; then $E = E_1 - E_2$ is a projection on $\mathcal{M}$ along $\mathcal{N}$, where $\mathcal{M} = \mathcal{M}_1 \cap \mathcal{N}_2$ and $\mathcal{N} = \mathcal{N}_1 \bigoplus \mathcal{M}_2$
- If $E_1E_2 = E_2E_1 = E$, then E is a projection on $\mathcal{M}$ along $\mathcal{N}$, where $\mathcal{M} = \mathcal{M}_1 \cap \mathcal{M}_2$ and $\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2$
- If $\mathcal{M}$ is a subspace invariant under linear transformation A, then $EAE = AE$ for every projection E on $\mathcal{M}$. Conversely, if $EAE = AE$ for some projection E on $\mathcal{M}$, then $\mathcal{M}$ is invariant under linear transformation A.

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Least Squares Method

- Let a linear system be given by $Ax = b$, where $A \in^{m\times n}, x \in^m$ and $b \in^n$ then, the aim is to find best approximation of b which is in the range space of A.
- The problem is converted to minimizing the error $e = ||Ax - b||^2$.
- The solution $\hat{x}$ is which minimizes e is the same as locating the point $p = A\hat{x}$ that is closer to b than any other point in the column space of A.
- The error vector e must be perpendicular to the column space.
- All vectors perpendicular to the column space lie in the left nullspace. Thus the error vector e must be in the nullspace of A^T:

$$\begin{aligned} A^T(b - A\hat{x}) &= 0 \\ \hat{x} &= (A^T A)^{-1} A^T b \\ p = A\hat{x} &= A\left((A^T A)^{-1} A^T b\right) \end{aligned} \tag{12}$$

Example

Let a linear system be $Ax = b$ where,

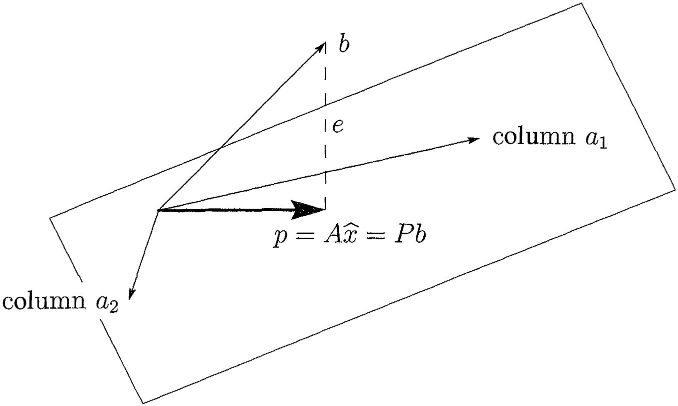
$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} b = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$$

Here, b does not lie in the column space of A. So we have to find a best approximation which lies in column span of A.
By (12) we get,

$$p = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$$

Geometrically, p is an orthogonal projection of b on column space of A.

Geometric representation of least squares approximation



Example

Problem: Sheldon wants to buy an apartment. But he wants to make sure that he pays the right market value for the apartment. So he decides to do a survey and tabulates them as under.

Area (in hundred Sq. ft)	12	10	15	6
No. of Bedrooms	2	3	4	1
Price(in Lacs)	70	75	85	50

Now, Sheldon has finalized a 750 Sq. ft. and 3 bedroom apartment. What do you think he should pay? (For 0 Sq. ft. and 0 bedroom apartment the price should be zero.)

Example

$$
\begin{aligned}
12x + 2y &= 70 \\
10x + 3y &= 75 \\
15x + 4y &= 85 \\
60x + 1y &= 50
\end{aligned}
$$

We write it as

$$AX = b$$

where,

$$A = \begin{pmatrix} 12 & 2 \\ 10 & 3 \\ 15 & 4 \\ 6 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \end{pmatrix} \quad b = \begin{pmatrix} 70 \\ 75 \\ 85 \\ 50 \end{pmatrix}$$

$$
\begin{aligned}
X &= (A^T A)^{-1} A^T b \\
&= \begin{pmatrix} 5.800 \\ 1.967 \end{pmatrix}
\end{aligned}
$$

Example

Now we have to find the value of a 750 sq. ft. and 3 bedroom apartment.

$$
\begin{aligned}
Price &= 7.5x + 3y \\
&= 49.4
\end{aligned}
$$

Hence, Sheldon should pay 49.4 Lacs for the apartment.

References

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Thank You