CS 228 : Logic in Computer Science

Krishna. S

Free and Bound Variables in MSO

- Free, Bound Variables and Scope same as in FO
- In a wff $\varphi = \forall X \psi$ or $\exists X \psi$ every occurrence of X in ψ is bound
- A sentence is a formula with no free first order and second order variables

Assignments on τ-structures

Assignments

For a τ-structure $\mathcal{A}$, an assignment over $\mathcal{A}$ is a pair of functions (α_1, α_2), where

- $\alpha_1 : \mathcal{V}_1 \to u(\mathcal{A})$ assigns every first order variable $x \in \mathcal{V}_1$ a value $\alpha_1(x) \in u(\mathcal{A})$. If t is a constant symbol c, then $\alpha_1(t)$ is $c^{\mathcal{A}}$.
- $\alpha_2 : \mathcal{V}_2 \to 2^{u(\mathcal{A})}$ assigns to every second order variable $X \in \mathcal{V}_2$, $\alpha_2(X) \subseteq u(\mathcal{A})$.

Binding on a Variable

For an assignment $\alpha = (\alpha_1, \alpha_2)$ over $\mathcal{A}$, and $x \in \mathcal{V}_i$, $i = 1, 2$, $\alpha_i[x \mapsto a]$ is the assignment $\alpha_i[x \mapsto a](y) = \begin{cases} \alpha_i(y), y \neq x, \\ a, y = x \end{cases}$

Satisfaction

We define the relation $\mathcal{A} \models_\alpha \varphi$ (read as φ is true in $\mathcal{A}$ under the assignment α) inductively:

- $\mathcal{A} \nvDash_\alpha \bot$
- $\mathcal{A} \models_\alpha t_1 = t_2$ iff $\alpha_1(t_1) = \alpha_1(t_2)$
- $\mathcal{A} \models_\alpha R(t_1, \ldots, t_k)$ iff $(\alpha_1(t_1), \ldots, \alpha_1(t_k)) \in R^{\mathcal{A}}$
- $\mathcal{A} \models_\alpha X(t)$ iff $\alpha_1(t) \in \alpha_2(X)$ (new)
- $\mathcal{A} \models_\alpha (\varphi \to \psi)$ iff $\mathcal{A} \nvDash_\alpha \varphi$ or $\mathcal{A} \models_\alpha \psi$
- $\mathcal{A} \models_\alpha (\forall x)\varphi$ iff for every $a \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a]} \varphi$
- $\mathcal{A} \models_\alpha (\forall X)\varphi$ iff for every $S \subseteq u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[X \mapsto S]} \varphi$ (new)

Examples

Recall the signature for the graph structure, $\tau = \{E\}$

- The graph is 3-colorable

Examples

Recall the signature for the graph structure, $\tau = \{E\}$

- The graph is 3-colorable

$$\exists X \exists Y \exists Z (\forall x [X(x) \lor Y(x) \lor Z(x)] \land$$

$$\forall x \forall y [E(x,y) \rightarrow \{\neg(X(x) \land X(y)) \land \neg(Y(x) \land Y(y)) \land \neg(Z(x) \land Z(y))\}])$$

Examples

Recall the signature for the graph structure, $\tau = \{E\}$

- The graph has an independent set of size $\geqslant k$

Examples

Recall the signature for the graph structure, $\tau = \{E\}$

- The graph has an independent set of size $\geqslant k$

$$\exists I\{\forall x \forall y[(\neg(x = y) \wedge I(x) \wedge I(y)) \rightarrow \neg E(x, y)] \wedge$$

$$\exists x_1 \ldots x_k[\bigwedge_{i \neq j} \neg(x_i = x_j) \wedge \bigwedge_i I(x_i)]\}$$

Examples

Recall the signature τ for the word structure, $\tau = \{Q_a, Q_b, <, S\}$ for $\Sigma = \{a, b\}$

- Words of even length

Examples

Recall the signature τ for the word structure, $\tau = \{Q_a, Q_b, <, S\}$ for $\Sigma = \{a, b\}$

▸ Words of even length

$$\exists E \exists O \{\forall x [(\textit{first}(x) \to E(x)) \land (\textit{last}(x) \to O(x))]$$

Examples

Recall the signature τ for the word structure, $\tau = \{Q_a, Q_b, <, S\}$ for $\Sigma = \{a, b\}$

► Words of even length

$$\exists E \exists O \{\forall x[(first(x) \rightarrow E(x)) \wedge (last(x) \rightarrow O(x))]$$

$$\wedge \forall x[(E(x) \vee O(x)) \wedge \neg(E(x) \wedge O(x))]$$

Examples

Recall the signature τ for the word structure, $\tau = \{Q_a, Q_b, <, S\}$ for $\Sigma = \{a, b\}$

- Words of even length

$$\exists E \exists O\{\forall x[(first(x) \rightarrow E(x)) \wedge (last(x) \rightarrow O(x))]$$

$$\wedge \forall x[(E(x) \vee O(x)) \wedge \neg(E(x) \wedge O(x))]$$

$$\wedge \forall x \forall y[S(x,y) \wedge O(x) \rightarrow E(y)]$$
$$\wedge \forall x \forall y[S(x,y) \wedge E(x) \rightarrow O(y)]\}$$

MSO on Words : Satisfiability

MSO on Words

- ▶ Signature $\tau = (Q_\Sigma, <, S)$, domain or universe = set of positions of a word
- ▶ MSO over words: Atomic formulae

$$X(x)|Q_\Sigma(x)|x = y|x < y|S(x,y)$$

- ▶ Given a MSO sentence φ, $L(\varphi)$ defined as usual
- ▶ A language $L \subseteq \Sigma^*$ is MSO definable iff there is an MSO formula φ such that $L = L(\varphi)$
- ▶ Given an MSO sentence φ, is it satisfiable/valid?

MSO Expressiveness

- Clearly, $FO \subseteq MSO$
- MSO=Regular