Logic in CS Autumn 2024

Problem Sheet 1

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1. There are three suspects for a murder: A, B, and C.

   - A says "I did not do it. The victim was an old acquaintance of B's. But C hated him."
   - B states "I did not do it. I did not even know the guy. Besides I was out of town all that week."
   - C says "I did not do it. I saw both A and B downtown with the victim that day; one of them must have done it."

   Assume that the two innocent men are telling the truth, but that the guilty man might not be. Who did it? Deduce the answer by encoding in propositional logic and finding a solution.

Solution

All three of them say, "I did not do it". So, this does not give any useful information to us. (*Why*?) Let us define the following propositions:

$$p_1 : \text{victim is acquaintance of B}$$

$$p_2 : \text{B was out of town}$$

Then, we can encode the statements of A, B, and C as $a = p_1$, $b = \neg p_1 \wedge p_2$, and $c = \neg p_2$ respectively. Since we know at least two people tell the truth, the formula, $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$ has to be `true`.

$$a \wedge b = (p_1 \wedge (\neg p_1 \wedge p_2)) \equiv \bot$$

$$b \wedge c = ((\neg p_1 \wedge p_2) \wedge \neg p_2)) \equiv \bot$$

Hence, $c \wedge a$ is `true` $\Rightarrow$ c is `true` and a is `true` $\Rightarrow$ B is guilty

2. In an island, there are three tribes : the Knights, Knaves and Normals. The Knights always speak truth, the Knaves always lie, while the Normals lie sometimes and speak truth sometimes. On a visit to this island, I met two inhabitants A and B. A told me that B is a knight and B told me that A is a knave. Prove, using natural deduction, that one of them told the truth but is not a knight, or that one of them told a lie but is not a knave.

Solution

Note that if B is a knight, then A must be a knave, which makes her statement that B is a knight false, which contradicts the fact that B is a knight. Therefore B is not a knight. This means A's statement is false, which means A is also not a knight. So both are either normals or knaves, and A's statement is false. Let us take some cases now:

(a) If A is a knave, then B's statement is true, which means B must be a normal. This means one of them (B) told the truth but is not a knight.

(b) If B is a knave, then A must be a normal. This means one of them (A) told a lie but is not a knave.

(c) If both A and B are normals, then both their statements are false, which means one of them (either) told a lie but is not a knave.

Therefore, by cases, we can say that one of them told the truth but is not a knight, or that one of them told a lie but is not a knave.
This natural language proof can also be directly translated into a formal proof.

3. Let F, G and H be formulas and let $\mathbf{S}$ be a set of formulas. Which of the following statements are true? Justify your answer.

(a) If F is unsatisfiable, then $\neg F$ is valid.
(b) If $F \to G$ is satisfiable and F is satisfiable, then G is satisfiable.
(c) $P_1 \to (P_2 \to (P_3 \to \ldots (P_n \to P_1) \ldots))$ is valid.
(d) $\mathbf{S} \models F$ and $\mathbf{S} \models \neg F$ cannot both hold.
(e) If $\mathbf{S} \models F \vee G$, $\mathbf{S} \cup \{F\} \models H$ and $\mathbf{S} \cup \{G\} \models H$, then $\mathbf{S} \models H$.

Solution

(a) True. Let $\mathcal{A}$ be an arbitrary assignment. Since F is unsatisfiable we have $\mathcal{A}[\![F]\!] = 0$ and thus $\mathcal{A}[\![\neg F]\!] = 1$.

(b) False. A counterexample is $P \to \texttt{false}$ for an atomic proposition P.

(c) True. Consider an assignment $\mathcal{A}$. If $\mathcal{A}[\![P_1]\!] = 0$ then the outermost implication is true. If $\mathcal{A}[\![P_1]\!] = 1$ then, arguing from the inside outwards, all the implication subformulas are true.

(d) False. If $\mathbf{S}$ is unsatisfiable then $\mathbf{S} \models F$ and $\mathbf{S} \models \neg F$ for any F.

(e) True. Let $\mathcal{A}$ be a model of $\mathbf{S}$. Since $\mathbf{S} \models F \vee G$, $\mathcal{A}$ is a model of F or a model of G. In the first case, since $\mathbf{S} \cup \{F\} \models H$, $\mathcal{A}$ is a model of H. Likewise in the second case, since $\mathbf{S} \cup \{G\} \models H$, $\mathcal{A}$ is a model of H. Since the two cases are exhaustive, $\mathcal{A}$ is a model of H. Thus every model of $\mathbf{S}$ is a model of H.

4. The Pigeon Hole Principle states that if there are $n+1$ pigeons sitting amongst n holes then there is atleast one hole with more than one pigeon sitting in it. For $i \in \{1, 2, .., n+1\}$ and $j \in \{1, 2, ..., n\}$, let the atomic proposition $P(i, j)$ indicate that the i-th pigeon is sitting in the j-th hole.

   Write out a propositional logic formula that states the Pigeon Hole Principle.

Solution

Let $P(i, j)$ represent the proposition that the i^{th} pigeon is sitting in the j^{th} hole, where $i \in \{1 \dots n+1\}$ and $j \in \{1 \dots n\}$.
The Pigeonhole principle states that, if there are $n+1$ pigeons and n holes, and every pigeon sits in exactly one hole, then there is a hole occupied by more than one pigeon.
To convert this into a PL formula, let us convert each side of the implication into PL first.
Every pigeon sits in at least one hole can be expressed in PL as:

$$\bigwedge_{i=1}^{n+1} \bigvee_{j=1}^{n} P(i, j)$$

Here the inner disjunction refers to the i^{th} pigeon sitting in some hole, and the outer conjuction makes it so that every pigeon must sit in some hole. Call this condition F.
We also need no pigeon to sit in multiple holes. Say pigeon i sits in holes j and k with $j < k$. The formula $P(i, j) \wedge P(i, k)$ represents this scenario. There exists a pigeon sitting in multiple holes therefore becomes:

$$\bigvee_{i=1}^{n+1} \bigvee_{\substack{j,k=1 \\ j<k}}^{n} (P(i, j) \wedge P(i, k))$$

Here, the inner disjunction refers to the i^{th} pigeon sitting in multiple holes and the outer disjunction refers to there existing a pigeon sitting in multiple holes.
Negating this, we get the condition for no pigeon to sit in multiple holes:

$$\bigwedge_{i=1}^{n+1} \bigwedge_{\substack{j,k=1 \\ j<k}}^{n} (\neg P(i, j) \vee \neg P(i, k))$$

Call this condition G.
Now, say hole k is occupied by pigeons i and j with $i < j$. We then have $P(i, k) \wedge P(j, k)$.
There exists a hole occupied by more than one pigeon therefore becomes:

$$\bigvee_{k=1}^{n} \bigvee_{\substack{i,j=1 \\ i<j}}^{n+1} (P(i, k) \wedge P(j, k))$$

Here, the inner disjunction refers to the k^{th} hole being occupied by more than one pigeon and the outer disjunction refers to there existing a hole occupied by multiple pigeons. Call this condition H.
The Pigeonhole Principle therefore becomes:

$$F \wedge G \implies H$$

5. Prove formally $\vdash [(p \to q) \to q] \to [(q \to p) \to p]$

Solution

1.	$(p \to q) \to q$	assumption
2.	$q \to p$	assumption
3.	$\neg p$	assumption
4.	$\neg q$	MT 2, 3
5.	$\neg(p \to q)$	MT 1, 4
6.	p	assumption
7.	$\bot$	$\bot i$ 3, 6
8.	q	$\bot e$ 7
9.	$p \to q$	$\to i$ 6-8
10.	$\bot$	$\bot i$ 5, 9
11.	$\neg\neg p$	$\neg i$ 3, 10
12.	p	$\neg\neg e$ 11
13.	$(q \to p) \to p$	$\to i$ 1-12
14.	$((p \to q) \to q) \to ((q \to p) \to p)$	$\to i$ 1-13

6. Let $\mathcal{H}$ be a given set of premises. If $\mathcal{H} \vdash (A \to B)$ and $\mathcal{H} \vdash (C \vee A)$, then show that $\mathcal{H} \vdash (B \vee C)$ where A, B, C are wffs.

Solution

1.	$\mathcal{H}$	premises
2.	$A \to B$	as $\mathcal{H} \vdash (A \to B)$
3.	$C \vee A$	as $\mathcal{H} \vdash (C \vee A)$
4.	C	assumption
5.	$B \vee C$	$\vee i$ 3
6.	A	assumption
7.	B	MP 1, 5
8.	$B \vee C$	$\vee i$ 6
9.	$B \vee C$	$\vee e$ 2, 3-4, 5-7

7. Let $\mathcal{H}$ be a given set of premises. If $\mathcal{H} \vdash (A \to C)$ and $\mathcal{H} \vdash (B \to C)$, then show that $\mathcal{H} \vdash ((A \vee B) \to C)$. Here, A, B and C are wffs.

Solution

1.	$\mathcal{H}$	premises
2.	$A \to C$	as $\mathcal{H} \vdash (A \to C)$
3.	$B \to C$	as $\mathcal{H} \vdash (B \to C)$
4.	$A \vee B$	assumption
5.	A	assumption
6.	C	MP 1, 4
7.	B	assumption
8.	C	MP 2, 6
9.	C	$\vee e$ 3, 4-5, 6-7
10.	$(A \vee B) \to C$	$\to i$ 3-8

8. Show that a truth assignment α satisfies the wff

$$(\ldots(x_1 \leftrightarrow x_2) \leftrightarrow \cdots \leftrightarrow x_n)$$

iff $\alpha(x_i) = \texttt{false}$ for an even number of i's, $1 \leq i \leq n$.

Solution

Base Case. For $n = 1$, the wff will be (x_1), which is `true` iff $\alpha(x_1) = \texttt{true}$.
Induction Hypothesis. Assume α satisfies the wff $(\ldots(x_1 \leftrightarrow x_2) \leftrightarrow \cdots \leftrightarrow x_{n-1})$ iff $\alpha(x_i) = \texttt{false}$ for an even number of i's, $1 \leq i \leq n-1$.
Now consider $(\ldots(x_1 \leftrightarrow x_2) \leftrightarrow \cdots \leftrightarrow x_n)$. Here, $\alpha(x_n)$ can be `true` or `false`.

- If $\alpha(x_n) = \texttt{true}$, and the wff $(\ldots(x_1 \leftrightarrow x_2) \leftrightarrow \cdots \leftrightarrow x_n)$ is `true` $\Leftrightarrow$ $(\ldots(x_1 \leftrightarrow x_2) \leftrightarrow \cdots \leftrightarrow x_{n-1})$ is `true` $\Leftrightarrow \alpha(x_i) = \texttt{false}$ for an even number of i's, $1 \leq i \leq n-1 \Leftrightarrow \alpha(x_i) = \texttt{false}$ for an even number of i's, $1 \leq i \leq n$

- If $\alpha(x_n) = \texttt{false}$, and the wff $(\ldots(x_1 \leftrightarrow x_2) \leftrightarrow \cdots \leftrightarrow x_n)$ is `true` $\Leftrightarrow$ $(\ldots(x_1 \leftrightarrow x_2) \leftrightarrow \cdots \leftrightarrow x_{n-1})$ is `false` $\Leftrightarrow \alpha(x_i) = \texttt{false}$ for an odd number of i's, $1 \leq i \leq n-1 \Leftrightarrow \alpha(x_i) = \texttt{false}$ for an even number of i's, $1 \leq i \leq n$

9. Of the following three formulae, which tautologically imply which?

(a) $x \leftrightarrow y$
(b) $(\neg((x \leftrightarrow y) \to (\neg y \to x)))$
(c) $((\neg x \vee y) \wedge (x \vee \neg y))$

Solution

A wff p is said to *tautologically imply* a wff q if there is no truth assignment α which makes p `true` and q `false`. In this question, this can be seen by looking at the truth tables of given formulae:

x	y	$x \leftrightarrow y$	$(\neg((x \leftrightarrow y) \to (\neg y \to x)))$	$((\neg x \vee y) \wedge (x \vee \neg y))$
1	1	1	0	1
1	0	0	0	0
0	1	0	0	0
0	0	1	1	1

Clearly, from the truth tables we can conclude that:

- $x \leftrightarrow y$ tautologically implies $((\neg x \vee y) \wedge (x \vee \neg y))$
- $((\neg x \vee y) \wedge (x \vee \neg y))$ tautologically implies $x \leftrightarrow y$
- $(\neg((x \leftrightarrow y) \to (\neg y \to x))))$ tautologically implies $x \leftrightarrow y$
- $(\neg((x \leftrightarrow y) \to (\neg y \to x))))$ tautologically implies $((\neg x \vee y) \wedge (x \vee \neg y))$

10. Let $\mathcal{L}$ be a formulation of propositional logic in which the sole connectives are negation and disjunction. The rules of natural deduction corresponding to disjunction and negation (also includes double negation) are available. For any wffs A, B and C, let $\neg(A \vee B) \vee (B \vee C)$ be an axiom of $\mathcal{L}$. Show that any wff of $\mathcal{L}$ is a theorem of $\mathcal{L}$.

Solution

An axiom of a proof system is a formula that can always be taken as `true`. A theorem is a logical consequence of axioms, i.e a wff in a proof system is a theorem if it can be derived from axioms using the proof rules of the system. Let P be any wff of $\mathcal{L}$. We shall apply the axiom by choosing $A = \neg P$, $B = \perp$ and $C = P$.

1.	$\neg(\neg P \vee \bot) \vee (\bot \vee P)$	premise (axiom)
2.	$\neg(\neg P \vee \bot)$	assumption
3.	$\neg P$	assumption
4.	$\neg P \vee \bot$	$\vee i$ 3
5.	$\bot$	$\bot i$ 2, 4
6.	$\neg\neg P$	$\bot e$ 3-5
7.	P	$\neg\neg e$ 6
8.	$(\bot \vee P)$	assumption
9.	$\bot$	assumption
10.	P	$\bot e$
11.	P	assumption
12.	P	$\vee e$ 8, 9-10, 11
13.	P	$\vee e$ 1, 2-7, 8-12

Notice that we can also derive any wff P, if we can derive $\bot$.

11. Let $\mathcal{P}$ denote propositional logic. Suppose we add to $\mathcal{P}$ the axiom schema $(A \to B)$ for wffs A, B of $\mathcal{P}$. Comment on the consistency of the resulting logical system obtained. A logic system $\mathcal{P}$ is inconsistent if it is capable of producing $\bot$ using the rules of natural deduction.

Solution

The resulting logical system is inconsistent, since we can produce $\bot$ as follows. Let φ be any wff of $\mathcal{P}$.

1.	$(\varphi \vee \neg\varphi) \to \bot$	premise (axiom)
2.	$\varphi \vee \neg\varphi$	LEM
3.	$\bot$	MP 1, 2