



CS 228 : Logic in Computer Science

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GNBA

- ▶ Generalized NBA, a variant of NBA
- ▶ Only difference is in acceptance condition
- ▶ Acceptance condition in GNBA is a set $\mathcal{F} = \{F_1, \dots, F_k\}$, each $F_i \subseteq Q$
- ▶ An infinite run ρ is accepting in a GNBA iff

$$\forall F_i \in \mathcal{F}, \text{Inf}(\rho) \cap F_i \neq \emptyset$$

- ▶ Note that when $\mathcal{F} = \emptyset$, all infinite runs are accepting
- ▶ GNBA and NBA are equivalent in expressive power.

Word View (On the board)

- ▶ $w = \{a\}\{a, b\}\{\}\dots,$
- ▶ $\varphi = a \text{ U } (\neg a \wedge b)$
- ▶ Subformulae of $\varphi = \{a, \neg a, b, \neg a \wedge b, \varphi\}$
- ▶ Parse trees to compute all subformulae

Closure of φ , $cl(\varphi)$

- ▶ $cl(\varphi)$ =all subformulae of φ and their negations, identifying $\neg\neg\psi$ to be ψ .
- ▶ Example for $\varphi = a \cup (\neg a \wedge b)$
- ▶ $cl(\varphi) = \{a, \neg a, b, \neg b, \neg a \wedge b, \neg(\neg a \wedge b), \varphi, \neg\varphi\}$

Elementary Sets

Let φ be an LTL formula. Then $B \subseteq cl(\varphi)$ is elementary provided:

- ▶ B is maximally consistent : for all $\varphi_1 \wedge \varphi_2, \psi \in cl(\varphi)$,
 - ▶ $\varphi_1 \wedge \varphi_2 \in B \Leftrightarrow \varphi_1 \in B \wedge \varphi_2 \in B$
 - ▶ $\psi \in B \Leftrightarrow \neg\psi \notin B$
 - ▶ $true \in cl(\varphi) \Rightarrow true \in B$
- ▶ B is locally consistent wrt U . That is, for all $\varphi_1 U \varphi_2 \in cl(\varphi)$,
 - ▶ $\varphi_2 \in B \Rightarrow \varphi_1 U \varphi_2 \in B$
 - ▶ $\varphi_1 U \varphi_2 \in B, \varphi_2 \notin B \Rightarrow \varphi_1 \in B$
- ▶ B is elementary : B is maximally and locally consistent
- ▶ Given a $B \subseteq cl(\varphi)$, how can you check if B is elementary?

Check Elementary

Let $\varphi = a \text{ U } (\neg a \wedge b)$

► $B_1 = \{a, b, \neg a \wedge b, \varphi\}$

Check Elementary

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- ▶ $B_2 = \{\neg a, b, \varphi\}$

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- ▶ $B_2 = \{\neg a, b, \varphi\}$ No, not maximal as $\neg a \wedge b \notin B_2$, $\neg(\neg a \wedge b) \notin B_2$
- ▶ $B_3 = \{\neg a, b, \neg a \wedge b, \neg \varphi\}$

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- ▶ $B_3 = \{\neg a, b, \neg a \wedge b, \neg \varphi\}$ No, not locally consistent for \cup
- ▶ $B_4 = \{\neg a, \neg b, \neg(\neg a \wedge b), \neg \varphi\}$

Check Elementary

Let $\varphi = a \cup (\neg a \wedge b)$

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- ▶ $B_3 = \{\neg a, b, \neg a \wedge b, \neg \varphi\}$ No, not locally consistent for \cup
- ▶ $B_4 = \{\neg a, \neg b, \neg(\neg a \wedge b), \neg \varphi\}$ Yes, elementary

LTL φ to GNBA G_φ

- ▶ States of G_φ are elementary sets B_i
- ▶ For a word $w = A_0A_1A_2\dots$ the sequence of states $\sigma = B_0B_1B_2\dots$ will be a run for w
- ▶ σ will be accepting iff $w \models \varphi$ iff $\varphi \in B_0$
- ▶ In general, a run $B_iB_{i+1}\dots$ for $A_iA_{i+1}\dots$ is accepting iff $A_iA_{i+1}\dots \models \psi$ for all $\psi \in B_i$.

LTL to GNBA

- ▶ Let $\varphi = \bigcirc a$.
- ▶ Subformulae of φ : $\{a, \bigcirc a\}$. Let $A = \{a, \bigcirc a, \neg a, \neg \bigcirc a\}$.
- ▶ Possibilities at each state
 - ▶ $\{a, \bigcirc a\}$
 - ▶ $\{\neg a, \bigcirc a\}$
 - ▶ $\{a, \neg \bigcirc a\}$
 - ▶ $\{\neg a, \neg \bigcirc a\}$
- ▶ Our initial state(s) must guarantee truth of $\bigcirc a$. Thus, initial states: $\{a, \bigcirc a\}$ and $\{\neg a, \bigcirc a\}$

LTL to GNBA

$\{a, \bigcirc a\}$

$\{a, \neg \bigcirc a\}$

$\{\neg a, \bigcirc a\}$

$\{\neg a, \neg \bigcirc a\}$

LTL to GNBA

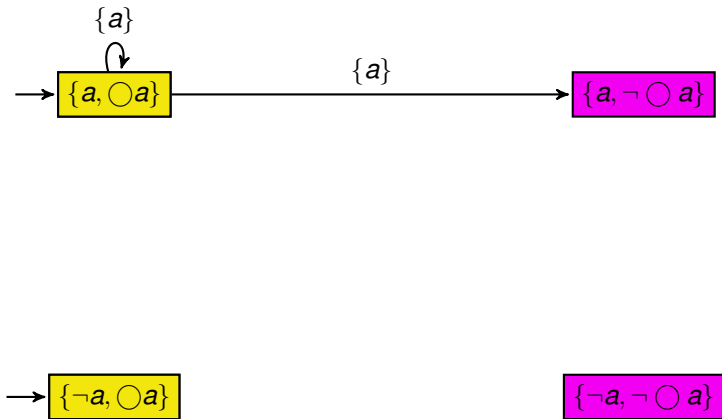
→ $\{a, \bigcirc a\}$

$\{a, \neg \bigcirc a\}$

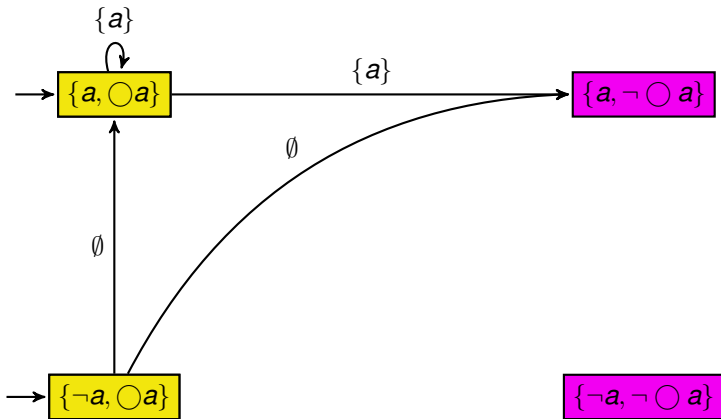
→ $\{\neg a, \bigcirc a\}$

$\{\neg a, \neg \bigcirc a\}$

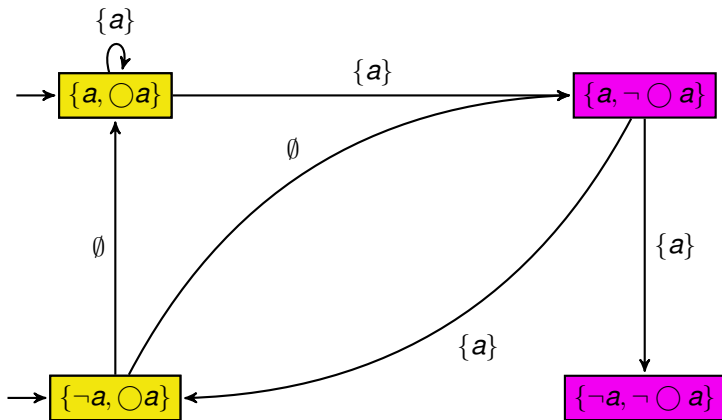
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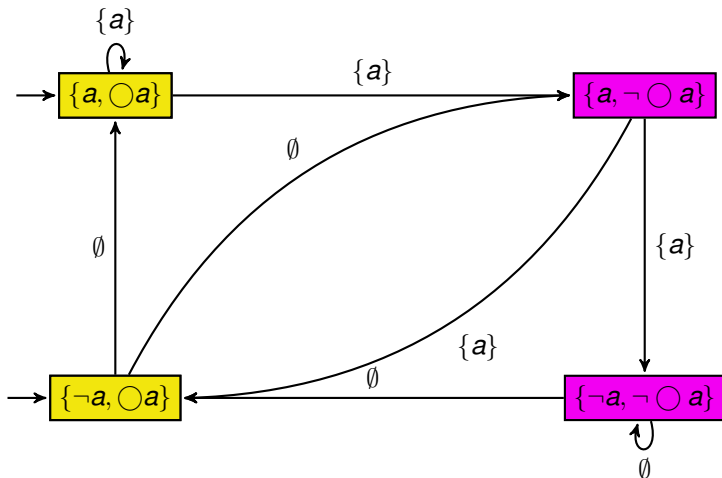
LTL to GNBA



LTL to GNBA



LTL to GNBA



LTL to GNBA

- ▶ Claim : Runs from a state labelled set B indeed satisfy B
- ▶ No good states. All words having a run from a start state are accepted.
- ▶ Automaton for $\neg \bigcirc a$ same, except for the start states.

LTL to GNBA

- ▶ Let $\varphi = a \text{ Ub}$.
- ▶ Subformulae of $\varphi : \{a, b, a \text{ Ub}\}$. Let $B = \{a, \neg a, b, \neg b, a \text{ Ub}, \neg(a \text{ Ub})\}$.
- ▶ Possibilities at each state
 - ▶ $\{a, \neg b, a \text{ Ub}\}$
 - ▶ $\{\neg a, b, a \text{ Ub}\}$
 - ▶ $\{a, b, a \text{ Ub}\}$
 - ▶ $\{a, \neg b, \neg(a \text{ Ub})\}$
 - ▶ $\{\neg a, \neg b, \neg(a \text{ Ub})\}$
- ▶ Our initial state(s) must guarantee truth of $a \text{ Ub}$. Thus, initial states: $\{a, b, a \text{ Ub}\}$ and $\{\neg a, b, a \text{ Ub}\}$ and $\{a, \neg b, a \text{ Ub}\}$.

LTL to GNBA

→ {a, b, a U b}

{a, ¬b, ¬(a U b)}

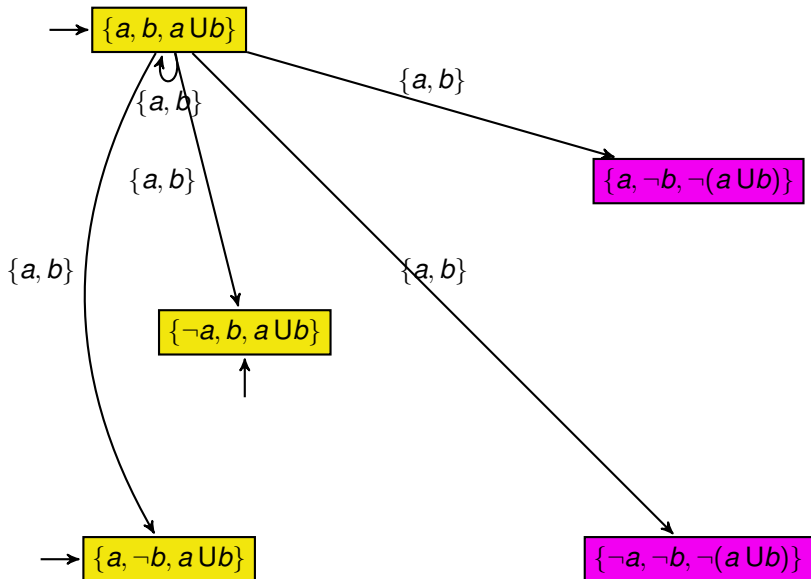
{¬a, b, a U b}



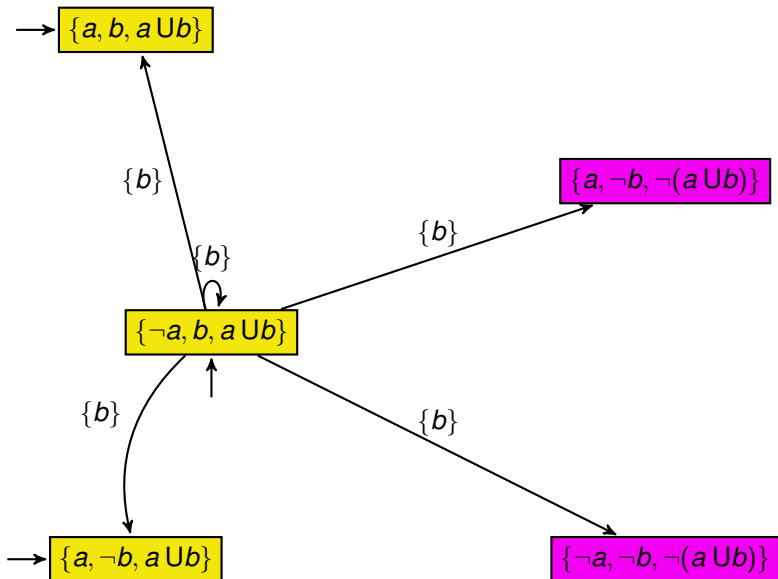
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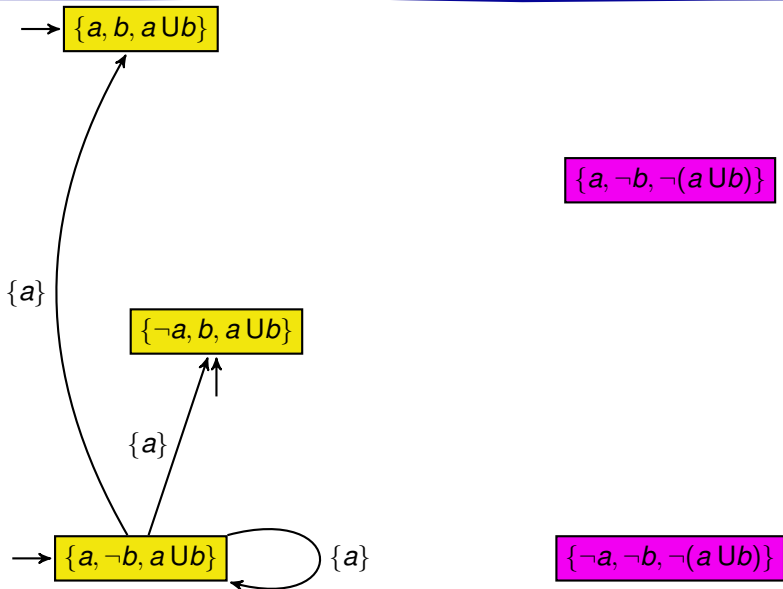
LTL to GNBA



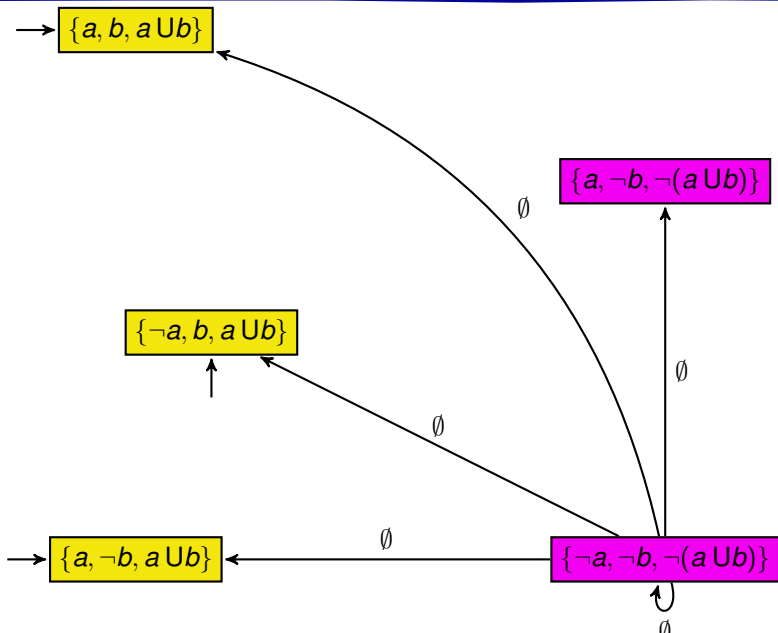
LTL to GNBA



LTL to GNBA



LTL to GNBA



LTL to GNBA

→ $\{a, b, a \cup b\}$

$\{\neg a, b, a \cup b\}$



→ $\{a, \neg b, a \cup b\}$

$\{a\}$



$\{a, \neg b, \neg(a \cup b)\}$

$\{a\}$



$\{\neg a, \neg b, \neg(a \cup b)\}$

LTL to GNBA : Accepting States

→ $\{a, b, a \cup b\}$

$\{a, \neg b, \neg(a \cup b)\}$

$\{\neg a, b, a \cup b\}$



→ $\{a, \neg b, a \cup b\}$

$\{\neg a, \neg b, \neg(a \cup b)\}$

LTL to GNBA

Construct GNBA for $\neg(a \text{ U } b)$.

LTL to GNBA

- ▶ Let $\varphi = a \text{ U } (\neg a \text{ U } c)$. Let $\psi = \neg a \text{ U } c$
- ▶ Subformulae of $\varphi : \{a, \neg a, c, \psi, \varphi\}$. Let $B = \{a, \neg a, c, \neg c, \psi, \neg\psi, \varphi, \neg\varphi\}$.
- ▶ Possibilities at each state
 - ▶ $\{a, c, \psi, \varphi\}$
 - ▶ $\{\neg a, c, \psi, \varphi\}$
 - ▶ $\{a, \neg c, \neg\psi, \varphi\}$
 - ▶ $\{a, \neg c, \neg\psi, \neg\varphi\}$
 - ▶ $\{\neg a, \neg c, \psi, \varphi\}$
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LTL to GNBA

→ $\{a, c, \psi, \varphi\}$

$\{\neg a, \neg c, \psi, \varphi\}$ ←

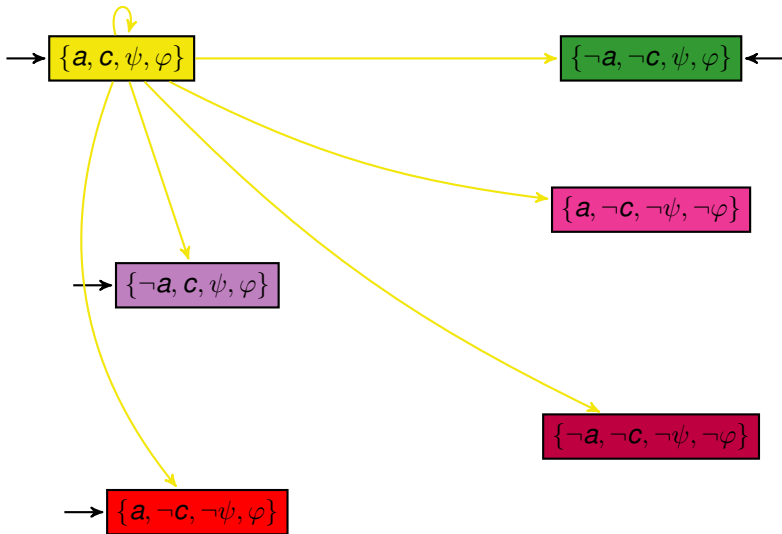
→ $\{\neg a, c, \psi, \varphi\}$

$\{a, \neg c, \neg \psi, \neg \varphi\}$

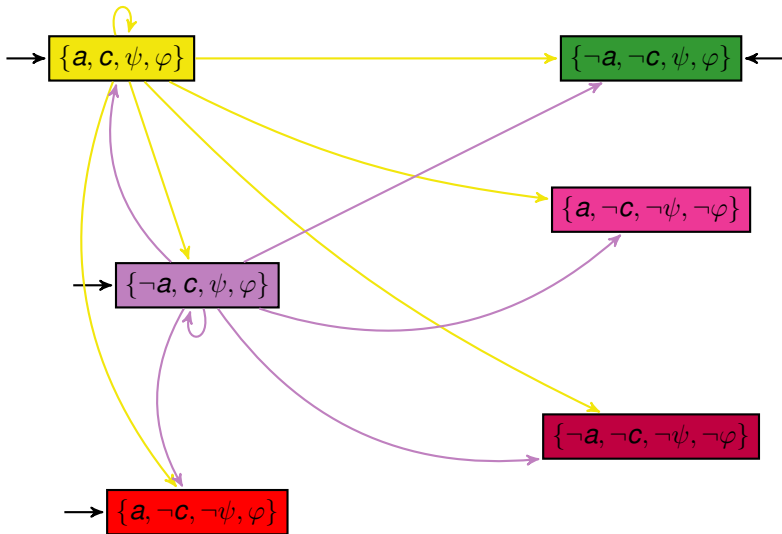
→ $\{a, \neg c, \neg \psi, \varphi\}$

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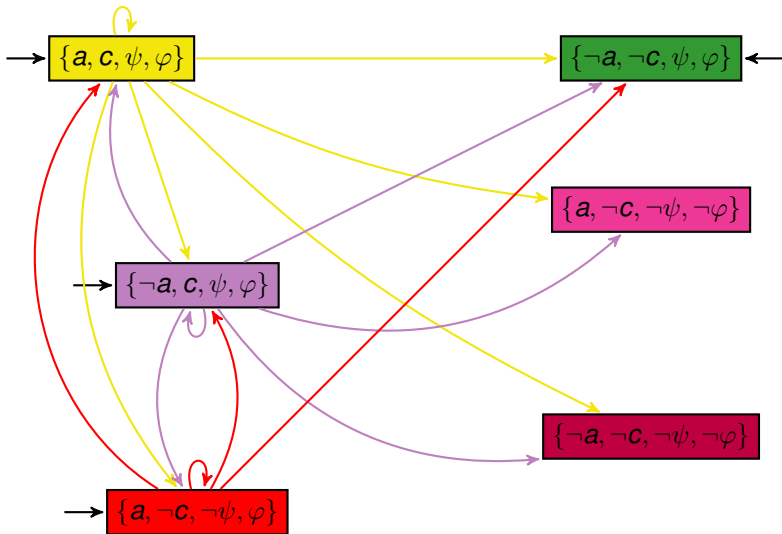
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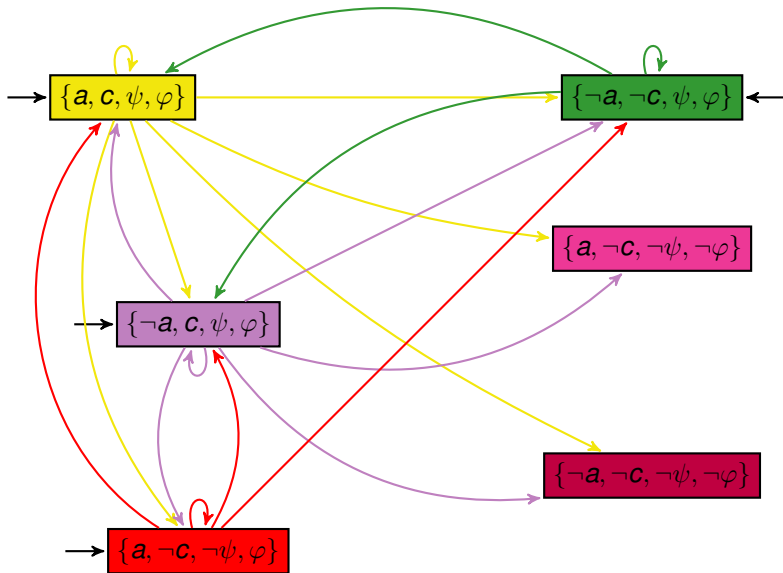
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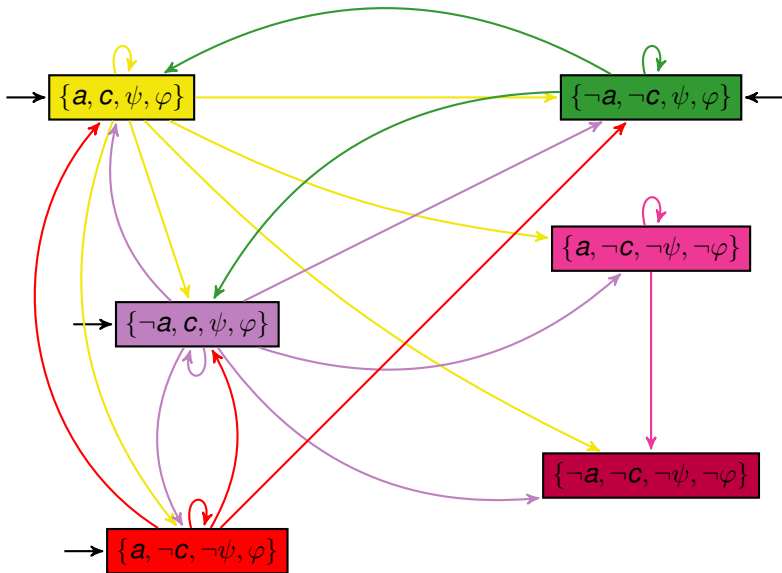
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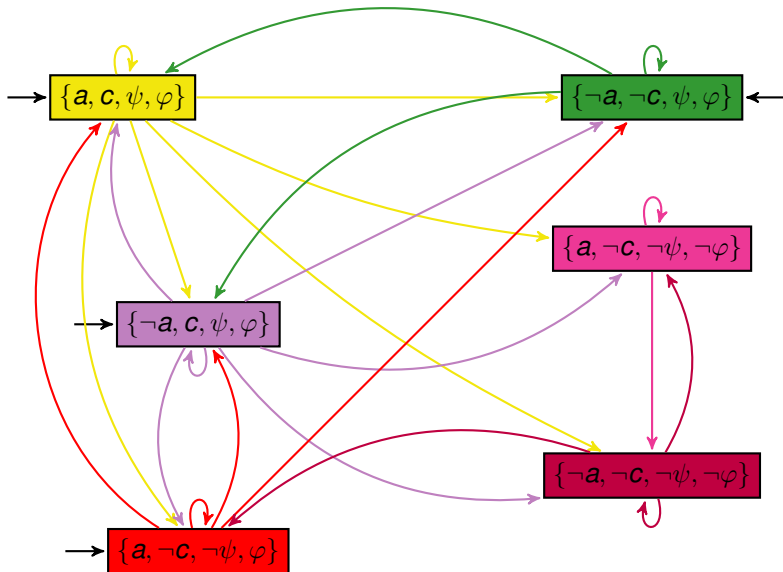
LTL to GNBA



LTL to GNBA



LTL to GNBA



GNBA Acceptance Condition

- ▶ $\psi = \neg a \mathbf{U} c$
- ▶ $\varphi = a \mathbf{U} \psi$
- ▶ $F_1 = \{B \mid \psi \in B \rightarrow c \in B\}$
- ▶ $F_2 = \{B \mid \varphi \in B \rightarrow \psi \in B\}$
- ▶ $\mathcal{F} = \{F_1, F_2\}$

Final States

$$\rightarrow \{a, c, \psi, \varphi\} \in F_1, F_2$$

$$\{\neg a, \neg c, \psi, \varphi\} \in F_2 \leftarrow$$

$$\{a, \neg c, \neg \psi, \neg \varphi\} \in F_1, F_2$$

$$\rightarrow \{\neg a, c, \psi, \varphi\} \in F_1, F_2$$

$$\{\neg a, \neg c, \neg \psi, \neg \varphi\} \in F_1, F_2$$

$$\rightarrow \{a, \neg c, \neg \psi, \varphi\} \in F_1$$

Putting Together

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- ▶ Consider those $B \subseteq CI(\varphi)$ which are **consistent**
 - ▶ $\varphi_1 \wedge \varphi_2 \in B \leftrightarrow \varphi_1 \in B \text{ and } \varphi_2 \in B$
 - ▶ $\psi \in B \rightarrow \neg\psi \notin B$ and $\psi \notin B \rightarrow \neg\psi \in B$
 - ▶ Whenever $\psi_1 \cup \psi_2 \in CI(\varphi)$,
 - ▶ $\psi_2 \in B \rightarrow \psi_1 \cup \psi_2 \in B$
 - ▶ $\psi_1 \cup \psi_2 \in B$ and $\psi_2 \notin B \rightarrow \psi_1 \in B$

Putting Together

Given φ over AP , construct $A_\varphi = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$,

- ▶ $Q = \{B \mid B \subseteq Cl(\varphi) \text{ is consistent} \}$
- ▶ $Q_0 = \{B \mid \varphi \in B\}$
- ▶ $\delta : Q \times 2^{AP} \rightarrow 2^Q$ is such that

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 - ▶ For $C = B \cap AP$, $\delta(B, C)$ is enabled and is defined as :
 - ▶ If $\bigcirc\psi \in Cl(\varphi)$, $\bigcirc\psi \in B$ iff $\psi \in \delta(B, C)$
 - ▶ If $\varphi_1 \cup \varphi_2 \in Cl(\varphi)$,
 $\varphi_1 \cup \varphi_2 \in B$ iff $(\varphi_2 \in B \vee (\varphi_1 \in B \wedge \varphi_1 \cup \varphi_2 \in \delta(B, C)))$

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 $\varphi_1 \mathbf{U} \varphi_2 \in B$ iff $(\varphi_2 \in B \vee (\varphi_1 \in B \wedge \varphi_1 \mathbf{U} \varphi_2 \in \delta(B, C)))$
- ▶ $\mathcal{F} = \{F_{\varphi_1 \mathbf{U} \varphi_2} \mid \varphi_1 \mathbf{U} \varphi_2 \in Cl(\varphi)\}$, with
 $F_{\varphi_1 \mathbf{U} \varphi_2} = \{B \in Q \mid \varphi_1 \mathbf{U} \varphi_2 \in B \rightarrow \varphi_2 \in B\}$

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- ▶ $\mathcal{F} = \{F_{\varphi_1 \mathbf{U}\varphi_2} \mid \varphi_1 \mathbf{U}\varphi_2 \in Cl(\varphi)\}$, with
 $F_{\varphi_1 \mathbf{U}\varphi_2} = \{B \in Q \mid \varphi_1 \mathbf{U}\varphi_2 \in B \rightarrow \varphi_2 \in B\}$
- ▶ Prove that $L(\varphi) = L(A_\varphi)$

GNBA Size

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- ▶ States of A_φ are subsets of $C/(\varphi)$
- ▶ Maximum number of states $\leq 2^{|\varphi|}$
- ▶ Number of sets in $\mathcal{F} = |\varphi|$
- ▶ LTL $\varphi \rightsquigarrow$ NBA A_φ : Number of states in $A_\varphi \leq |\varphi|.2^{|\varphi|}$
- ▶ There is no LTL formula φ for the language

$$L = \{A_0A_1A_2 \cdots \mid a \in A_{2i}, i \geq 0\}$$

Complexity of LTL Modelchecking

- ▶ Given φ , $A_{\neg\varphi}$ has $\leq 2^{|\varphi|}$ states (to be proved)
- ▶ $TS \otimes A_{\neg\varphi}$ has $\leq |TS| \cdot 2^{|\varphi|}$ states
- ▶ Persistence checking : Checking $\Box\Diamond\eta$ on $TS \otimes A_{\neg\varphi}$ takes time linear in $\eta \cdot |TS \otimes A_{\neg\varphi}|$

A Weak Lower Bound

The hamiltonian path problem is polynomially reducible to the complement of the LTL modelchecking problem.

- ▶ Given graph $G = (V, E)$ synthesize in polynomial time a TS and an LTL formula φ
- ▶ Show that G has a HP iff $TS \not\models \varphi$

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- ▶ Given graph $G = (V, E)$ synthesize in polynomial time a TS and an LTL formula φ
- ▶ Show that G has a HP iff $TS \not\models \varphi$
- ▶ TS is the graph itself, with one new node added, say b such all vertices of G have an edge to b , and b has a self loop. Let the label of a node in the TS be the name of the vertex.
- ▶ Write an LTL formula to capture absence of a HP in G . Assume $V = \{v_1, \dots, v_n\}$.
- ▶ The formula $\varphi = \neg\psi$ where ψ is

$$(\Diamond v_1 \wedge \Box(v_1 \rightarrow \bigcirc \Box \neg v_1)) \wedge \dots (\Diamond v_n \wedge \Box(v_n \rightarrow \bigcirc \Box \neg v_n))$$

- ▶ Show that G has a HP iff $TS \not\models \varphi$.

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Assume $TS \not\models \neg\psi$. Then there is a path witnessing ψ .

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- ▶ π has the form $v_{i_1} v_{i_2} \dots v_{i_n} b^\omega$, $i_1, \dots, i_n \in \{1, 2, \dots, n\}$, $i_j \neq i_k$.

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- ▶ So G has the HP $v_{i_1} v_{i_2} \dots v_{i_n}$.

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- ▶ So G has the HP $v_{i_1} v_{i_2} \dots v_{i_n}$.
- ▶ The converse is similar : a HP in G extends to a path $\pi = v_{i_1} v_{i_2} \dots v_{i_n} b^\omega$ in TS . Clearly, $\pi \models \psi$.

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- ▶ The converse is similar : a HP in G extends to a path $\pi = v_{i_1} v_{i_2} \dots v_{i_n} b^\omega$ in TS . Clearly, $\pi \models \psi$.
- ▶ So LTL model checking is co-NP hard as HP is NP-complete.
- ▶ Actual complexity of LTL model checking : PSPACE-complete.
For this, show that given a LBTM M and a word w , construct in poly time a TS and an LTL formula φ such that M accepts w iff $TS \models \varphi$.