CS 405/6001: Game Theory and Algorithmic Mechanism Design

Problem Set 3
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Week 7: Intro to Mechanism Design

1. Prove the following lemma about decisive coalitions, given the conditions of Arrow's theorem (you may assume strict preference profiles): There exists a decisive coalition V containing a single individual.

2. Demonstrate using counterexamples that if any of the conditions for Arrow's theorem are dropped, the Field Expansion Lemma is no longer true. (You may assume strict preference profiles).

3. **[Kendall Distance]** Define the Kendall Distance between two strict preference profiles P_i, P_i' as **the number of unordered pairs $\{a, b\}$ such that P_i and P_i' don't agree on $\{a, b\}$**. For example, with 3 candidates $\{a, b, c\}$ and two preference profiles $a > b > c$ and $c > a > b$, observe that the profiles disagree on $\{a, c\}$ and $\{b, c\}$, so the Kendall distance is 2.

   Show that the Kendall distance forms a valid metric, i.e $d_K(P_1, P_2) + d_K(P_2, P_3) \geq d_K(P_1, P_3)$ for any 3 (strict) preference profiles P_1, P_2, P_3.

4. **[Kemeny Aggregation]** Now, we use the above Kendall distance to define a social welfare function (also called a rank aggregation function) as follows:

   Given strict preference profiles $P_1, P_2, P_3, P_4 \cdots P_k$, the Kemeny Optimal ranking is a ranking R such that $R = \text{argmin}_{\mathcal{P}} \sum_{i=1}^{k} d_K(P_i, R)$, i.e the preference profile that **minimizes the sum of Kendall distances**. If more than one such profile exist, choose any one through some tiebreaking ordering fixed beforehand.

   - Is the above SWF unanimous? Prove or disprove.
   - Is the above SWF dictatorial? Prove or disprove.

5. **[Condorcet Criterition]** Suppose there is a candidate A among the candidates, such that for every pair of candidates A, B, **atleast 50%** of preference profiles prefer A over B. In other words, A can defeat every candidate in head-on voting. Such a candidate is said to be a **Condorcet Winner**. A SWF is **Condorcet Consistent** if whenever there exists a Condorcet winner, it is preferred over all other candidates in the aggregated ranking.

   - Give an example of a preference profile with no Condorcet winner.
   - Show that the Kemeny aggregation function is Condorcet Consistent.

Week 8: Voting and Single Peaked Preferences

1. Consider 3 voters voting for 3 candidates $\{a, b, c\}$ in an election. Every voter gives strict preference over the candidates from the most preferred to the least preferred. Each candidate receives a point if she beats another candidate in a *pairwise election*. A pairwise election considers two candidates at a time, say a and b. In an election between only these two candidates with the current voting profile, if a wins, then it gets a score of 1 and 0 otherwise (ties broken arbitrarily). Considering the pairwise election between all pairs, the candidate who thus amasses the greatest number of points wins the election. If two or more candidates are tied for first place in the number of points, the winner of the election is based on the alphabetical order.

   - Is the above voting rule monotone? If so, prove it. Otherwise, provide a counter-example.
   - Is the above voting rule strategyproof? If so, prove it. Otherwise, provide a counter-example.

2. The following electoral method is used to choose the departmental general secretary of CSE, IIT Bombay: Every student ranks the candidates from the most preferred to the least preferred and places this ranked list in a ballot box. Each candidate receives a number of points equal to the number of students who rank him or her least preferred. The candidate who thus amasses the greatest number of points is then removed from the list of candidates. If two or more candidates are tied for first place in the number of points, the candidate among them whose AADHAR number is greatest is removed from the list of candidates. This candidate is then ignored in the strict preference relations submitted by the students, and the process is repeated as often as is necessary, until only one candidate remains, who is declared the new departmental general secretary of CSE. Assume that there are at least three candidates.

   - Is it possible for the winner of the election to not be the most preferred candidate of any student? Justify your answer.
   - Is it possible for the winner of the election to be ranked least preferred by at least half of the students? Justify your answer.
   - Is it monotonic? Prove why yes, or show by example that it is not monotonic.
   - Is it manipulable? If so, provide an example or prove otherwise that it is not.

3. A committee comprised of 15 members is called upon to choose the prettiest color: red, blue, or yellow. The committee members simultaneously announce their strict preference relations among these three colors. If red is the most-preferred color of at least one committee member, red is declared the prettiest color. Otherwise, if blue is the most-preferred color of at least one committee member, blue is declared the prettiest color. Otherwise, yellow is declared the prettiest color.

   - Is the social choice function described above monotonic? Justify your answer.
   - Is the social choice function described above manipulable? Justify your answer.

4. Consider a two agent model with three alternatives $\{a, b, c\}$. Table 1 shows two preference profiles, $P \equiv (P_1, P_2)$ and $P' \equiv (P'_1, P'_2)$, of the agents. Suppose f is an *onto* and *strategyproof* SCF with $f(P_1, P_2) = a$.

P_1	P_2	P'_1	P'_2
a	c	b	a
b	b	a	b
c	a	c	c

Table 1: Two preference profiles

(A) Suppose the domain of preferences is of unrestricted strict preferences. What is $f(P')$? Explain why. You may use any standard result proved in the class.

(B) Now, suppose that these preferences are generated from a single-peaked preference domain with the intrinsic ordering of the alternatives being $a < b < c$. Are the preference profiles in Table 1 valid single-peaked profiles under this setting?

(C) What will be the value(s) of $f(P')$ in the modified case?

(D) Provide a non-constant (onto and strategyproof) SCF f that has $f(P'_1, P'_2) = a$ in the single-peaked domain.

5. A divisible resource is allocated among a set of agents N. Each agent can be allocated different amounts of the resource. If a_i amount of resource is allocated to the agent i, then he pays a_i^2 for the allocated resource. Moreover, the value that agent i derives for a_i amount of resource is linear in a_i given by $\theta_i a_i$, where θ_i is agent i's private information. In summary, the utility of the agent for the a_i amount of the resource is $u_i(a, \theta_i) = \theta_i a_i - a_i^2$.

   - What type of preference does each agent have over their consumed resource?
   - Is it possible to design a mechanism in this setting such that it reveals the private types of the agents truthfully, and is Pareto efficient and anonymous? Why or why not?
   - If you could find the mechanism, is the allocation returned by that mechanism also envy-free? Why or why not? *[Hint: an envy-free allocation is one in which every agent weakly prefers her own allocated resource than every other agent's allocated resource.]*

Week 9: Mechanisms with Payments

1. In the task sharing domain, we say that an allocation function is **Envy-Free**, if for each pair of agents i and j, the share of i is preferred over the share of j by agent i, i.e $s_i P_i s_j$. If some pair of agents i and j exist such that $s_j P_i s_i$, we say that i *envies* j. Note that when checking if i envies j, we must use i's preference relation.

   - Is the Serial Dictatorship rule envy-free? Prove or Disprove.
   - Is the Sprumont Uniform rule envy-free? Prove or Disprove.

2. **Assignment Game**

   We look at the following problem (called the assignment game) to understand payments and mechanisms in a 2-sided setting:

   Consider a set of n sellers, selling houses at cost $\{c_1, c_2 \cdots c_n\}$ respectively. Let there be n buyers, and let v_{ij} denote the valuation of the buyer i for house j. The mechanism of house-selling proceeds as follows:

   A function $f(v, c)$, where $v = [v_{ij}, i = 1, \ldots, n, j = 1, \ldots, n]$ and $c = (c_i, i = 1, \ldots, n)$, takes all the values v_{ij}, c_i and outputs a perfect matching from sellers to buyers, matching each seller to exactly one buyer. Now, if a buyer b_i is matched to seller s_j, and if $v_{ij} < c_j$, then no trade happens. On the other hand, if $v_{ij} \geq c_j$, the house may be sold at any price between c_j and v_{ij}. If the price agreed is p_j, the buyer's utility value is $v_{ij} - p_j$ and the seller's utility value is $p_j - c_j$. Observe that these utilities are quasi-linear.

   - Model the above problem as a mechanism design with transfers problem as defined in class, i.e specify the set of allocations, specify the valuations for allocations, and finally specify the payments (Note that there may be additional constraints on the payment function here).
   - What does it mean to be allocatively efficient in this setting?
   - Consider 2 buyers and 2 sellers with costs $(3, 5)$ and valuations $\begin{pmatrix} 7 & 6 \\ 10 & 5 \end{pmatrix}$. What are the VCG payments made by each agent, ignoring the additional payment constraints?

3. **Selling an object at the monopoly price**

   Andrew is interested in selling a rare car (whose value in his eyes we will normalize to 0). Assume there are n buyers and that buyer i's private value of the car, V_i, is uniformly distributed over $[0, 1]$. The private values of the buyers are independent. Instead of conducting an auction, Andrew intends on setting a price for the car, publicizing this price, and selling the car only to buyers who are willing to pay this price; if no buyer is willing to pay the price, the car will not be sold, and if more than one buyer is willing to pay the price, the car will be sold to one of them based on a fair lottery that gives each of them equal probability of winning.

   - Find Andrew's expected revenue as a function of the price x that he sets
   - Find the price x^* that maximizes Andrew's expected revenue.

- What is the maximal expected revenue that Andrew can obtain, as a function of n?
- A sealed-bid first price auction is defined as an auction where each buyer sends their private bid to the buyer. The seller selects one among the maximum bidders with equal probability and sells the car to them.

  Compare Andrew's maximal revenue with the revenue he would gain if he sells the car by way of a sealed-bid first-price auction. For which values of n does a sealed-bid first-price auction yield a higher revenue?

4. **Increasing bidders**

   Suppose there are n buyers participating in an auction, where the private values $V_1, V_2, ..., V_n$ are independent and for each $i \in \{1, 2, ..., n\}$, V_i is uniformly distributed over $[0, v_i^*]$. Suppose further that $v_1^* < v_2^* < \; < v_n^*$. Answer the following questions:

   - Determing a selling mechanism that maximizes the seller's expected revenue.
   - What is the seller's expected revenue under this mechanism?
   - What is the probability that buyer n wins the object under this mechanism?

   In the last two items, it suffices to write down the appropriate formula, with no need to solve it explicitly.

Week 10: Clarke mechanism and Combinatorial auctions

1. Use Clark mechanism for the following combinatorial auction problem with three agents and two objects. Compute the efficient allocation and the payments of the agents.

	$\{1\}$	$\{2\}$	$\{1,2\}$
Agent A_1	6	10	10
Agent A_2	8	5	8
Agent A_3	8	0	9

Table 2: Valuations

2. Consider the combinatorial auction problem with $M = \{1, 2, \ldots, m\}$ items and $1, 2, \ldots, n$ agents, where each agent is single-minded. An agent i is said to be single-minded if there exists a $S_i^* \subseteq M$ and a value w_i^* such that the valuation of agent i, $v_i(S) = w_i^*, \forall S \supseteq S_i^*$, and $v_i(S) = 0$ otherwise. Single-mindedness is defined by a pair (S_i^*, w_i^*) for every agent i. Prove that even when the agents are single-minded, the problem of computing the efficient allocation is NP-Hard. *[Hint: Use the independent set problem for reduction.]*

3. Consider an auction of items $1, 2, \ldots, m$ where each bidder is single minded and desires an interval of consecutive items i.e., $S_i = \{j | \, k_i \leq j \leq \ell_i\}$ where $1 \leq k_i \leq \ell_i \leq m$. Prove that in this case the socially efficient allocation can be determined in polynomial time.

4. Consider a seller interested in auctioning two items 1 and 2 to three agents A_1, A_2, and A_3. Let the agents have valuations over combinations of items (bundles) $\{1\}, \{2\}, \{1,2\}$ as shown in Table 3. Note '-' indicates that the agent is not interested in that bundle.

	$\{1\}$	$\{2\}$	$\{1,2\}$
Agent A_1	-	-	10
Agent A_2	2	-	-
Agent A_3	-	2	-

Table 3: Valuations

Consider the pivotal mechanism that guarantees DSIC, AE, and IR for this problem. Can you construct valuations of agents for this problem where two agents can collude such that the pivotal mechanism gives the colluders all the items for free and generates zero revenue?

5. Consider the general combinatorial auction problem where we have $M = \{1, 2, \ldots, m\}$ items that are to be auctioned among $\{A_1, A_2, \ldots, A_n\}$ agents. If the VCG mechanism is used, are the following statements correct? If so, prove them. Otherwise, provide a counterexample.

- *"If the colluders receive all the items at cost 0, then for any positive bid on a bundle of items by a noncolluder, at least two of the colluders receive an item from this bundle."*
- *"Suppose all the items in the auction can be divided among the colluders in such a way that for any positive bid on a bundle of items by a noncolluder, at least two of the colluders receive an item from this bundle. Then the colluders can receive all the items at cost 0."*