# Problems from TextBook: CS 215, Fall 2024

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This is a sheet containing practice questions. Solve them and ask any doubts regarding them in tutorial classes. Refer to Ross Textbook Edition 3.

## 1 Ross Textbook Chapter 4

Try Questions: 6, 7, 8, 11, 25, 26, 28

## 2 Ross Textbook Chapter 5

Try Questions: 4, 5, 12

## 3 Questions

1. Prove or disprove the following:

   (a) If event B includes the event A, then $P(A) \leq P(B)$.
   **Solution:** Refer Link to StackExchange

   (b) If $P(B \mid A) = P(B \mid \bar{A})$, then A and B are independent.
   **Solution:** Refer Link to StackExchange with A and B swapped

   (c) If P(A) = a and P(B) = b, then $P(A|B) \geq \frac{a+b-1}{b}$
   **Solution:**

   $$P(A \cup B) = P(A) + P(B) - P(AB) \leq 1$$

   which implies

   $$P(AB) \geq a + b - 1$$
   $$\frac{P(AB)}{P(B)} \geq \frac{a + b - 1}{b}$$

2. Find the probability that a randomly written fraction will be irreducible. Assume both numerator and denominator belongs to set of natural numbers.

   **Solution:**

   ### Probability that a Randomly Chosen Fraction is Irreducible

   We want to find the probability that a randomly chosen fraction $\frac{a}{b}$ is irreducible. A fraction is irreducible if the greatest common divisor (GCD) of $a$ and $b$ is 1, meaning $a$ and $b$ have no common divisors other than 1.

**Step 1: Probability that Both $a$ and $b$ are Divisible by a Prime $p$**

Consider any prime number $p$. The probability that a randomly chosen integer $a$ is divisible by $p$ is:

$$\frac{1}{p}$$

This is because every $p$th number is divisible by $p$.
Similarly, the probability that a randomly chosen integer $b$ is divisible by $p$ is also:

$$\frac{1}{p}$$

Since $a$ and $b$ are chosen independently, the probability that both $a$ and $b$ are divisible by $p$ is:

$$\frac{1}{p} \times \frac{1}{p} = \frac{1}{p^2}$$

**Step 2: Probability that GCD of $a$ and $b$ is 1 (i.e., the Fraction is Irreducible)**

For the fraction $\frac{a}{b}$ to be irreducible, there must be no prime number $p$ that divides both $a$ and $b$ simultaneously.
For each prime $p$, the probability that $a$ and $b$ are both *not* divisible by $p$ is:

$$1 - \frac{1}{p^2}$$

To find the probability that $a$ and $b$ have no common prime factors (i.e., the GCD is 1), we multiply these probabilities over all prime numbers:

$$P(\text{irreducible}) = \prod_{\text{prime } p} \left(1 - \frac{1}{p^2}\right)$$

**Step 3: Final Probability (Optional)**

This infinite product converges to a specific value:

$$P(\text{irreducible}) = \prod_{\text{prime } p} \left(1 - \frac{1}{p^2}\right) \approx 0.60793$$

Thus, the probability that a randomly chosen fraction $\frac{a}{b}$ is irreducible is approximately 60.8%.
Step 3 is optional. If interested, refer to Riemann Zeta Function to understand how this series converges to $\frac{6}{\pi^2} \approx 0.60793$.

3. Under what conditions does the following equality hold:

$$P(A) = P(A|B) + P(A|\bar{B})?$$

**Solution:**

$$P(A) = P(AB) + P(A\bar{B}) = P(B)P(A|B) + P(\bar{B})P(A|\bar{B})$$

(Refer Section 3.7.1 from Ross textbook to understand how we write this equation)

(a) A = V
(b) B = U
(c) B = V
(d) B = A
(e) $B = \bar{A}$
where U denotes certain event and V denotes impossible event.

# 4 Solution for text book problems

6) Note first that since $\int f(x)\,dx = 1$, it follows that $\lambda = \frac{1}{100}$; therefore,

$$\int_{50}^{150} f(x)\,dx = e^{-1/2} - e^{-3/2} = 0.3834.$$

Also,

$$\int_{0}^{100} f(x)\,dx = 1 - e^{-1} = 0.6321.$$

7) The probability that a given radio tube will last less than 150 hours is

$$\int_{0}^{150} f(x)\,dx = 1 - \frac{2}{3} = \frac{1}{3}.$$

Therefore, the desired probability is

$$\binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 = 0.3292.$$

8) Since the density must integrate to 1: $c = 2$ and $P(X > 2) = e^{-4} = 0.0183$.

11)

$$P(M \le x) = P(X_1 \le x, X_2 \le x, \ldots, X_n \le x) = \prod_{i=1}^{n} P(X_i \le x) = x^n.$$

Differentiation yields that the probability density function is $nx^{n-1}$, for $0 \le x \le 1$.

25)

(a) $E[X]$, because the randomly chosen student is more likely to have been on a bus carrying a large number of students than on one with a small number of students.

(b)

$$E[X] = 40\left(\frac{40}{148}\right) + 33\left(\frac{33}{148}\right) + 25\left(\frac{25}{148}\right) + 50\left(\frac{50}{148}\right) \approx 39.28$$

$$E[Y] = \frac{40 + 33 + 25 + 50}{4} = 37$$

26) Let $X$ denote the number of games played.

$$E[X] = 2\left[p^2 + (1-p)^2\right] + 3\left[2p(1-p)\right] = 2 + 2p(1-p)$$

Differentiating this and setting the result to 0 gives that the maximizing value of $p$ is such that

$$2 = 4p$$

28)

$$E[X] = a^2 \int_0^\infty x^2 e^{-ax}\,dx = \int_0^\infty y^2 e^{-y}\,\frac{dt}{a} = \frac{2}{a} \quad \text{(upon integrating by parts twice).}$$

**Chapter 5**

4)

$$\binom{4}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right) = \frac{27}{64}$$

5) Need to determine when:

$$6p^2(1-p)^2 + 4p^3(1-p) + p^4 > 2p(1-p) + p^2$$

Algebra shows that this is equivalent to:

$$(p-1)^2(3p-2) > 0$$

This shows that the 4-engine plane is better when $p > \frac{2}{3}$.

12)

$$P(\text{beneficial} \mid 0 \text{ colds}) = \frac{P(0 \text{ colds} \mid \text{beneficial})P(\text{beneficial})}{P(0 \mid \text{ben})P(\text{ben}) + P(0 \mid \text{not ben})P(\text{not ben})}$$

$$= \frac{e^{-2} \cdot \frac{3}{4}}{e^{-2} \cdot \frac{3}{4} + e^{-3} \cdot \frac{1}{4}} = \frac{3e^{-2}}{3e^{-2} + e^{-3}} = 0.8908$$