CS 405/6001: Game Theory and Algorithmic Mechanism Design

Problem Set 4
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1. Consider a 2-player auction setting where an agent's valuation is not independent of the other agent's valuation, assume the agent's valuation is given as follows when the object is allocated to agent i.

$$v_i(\theta_i, \theta_j) = \theta_i + \gamma\theta_j$$

where $\gamma \in (0, 1)$. The utility derived by an agent after reporting their types as θ'_i is given by

$$u_i(\theta') = a_i(\theta')v_i(\theta) - p_i(\theta')$$

a. Construct a payment rule for this mechanism under an efficient allocation such that the mechanism satisfies the following property. (**Hint:** Consider a payment rule that is a linear function of the other agents type $p_i(\theta) = c \cdot \theta_j + k$)

$$u_i(\theta_i, \theta_j) \geqslant u_i(\theta'_i, \theta_j) \quad \forall i = 1, 2$$

b. Comment on whether the mechanism you have designed is DSIC.

Solution sketch:

Consider the following payment scheme $p_i(\theta) = (1 + \gamma)\theta_j$ when the object is allocated to agent i. No payment when the object is not allocated. The mechanism is not DSIC. Consider the players types as $\theta_i > \theta_j$ and player 2's strategy is $\theta'_j > \theta_i$. $u_i(\theta_i, \theta'_j) < u_i(\theta'_i, \theta'_j)$ when $\theta'_i > \theta'_j$.

2. Recall the DaGVa mechanism as discussed in class, assume a uniform prior over $[0, 1]$ and compute the expected utilities of the agents. Check the IIR property for the expected utilities.

Solution sketch: The expected payment is $\frac{\theta_i^2 - \theta_j^2}{2}$. The expected utility is $u_i(\theta) = \frac{3\theta_i + 1}{6}$

3. Consider an auction with m identical items and n agents, agent i has a valuation of v_{ij} on receiving object j. Assume additive valuations ie. when agent i receives objects $j_i, j_2, \ldots, j_k$ the valuation for agent i is given by $\sum_{l=1}^{k} v_{il}$. Evaluate payment schemes for the VCG mechanism under the following conditions.

a. An agent can receive more than one item, and an agent may also receive no item. Note: The players here submits a bid for multiple items at once. For eg. player 1 submits a bid b_i for m_i objects. **Bonus:** Suppose an agent can submit multiple bids, eg, Agent 1 can submit a bid b_{i1} for m_1 objects and b_{i2} for m_2 objects at the same time, a maximum of one bid submitted by an agent is satisfied in the resulting allocation.

b. An agent can receive at most one item or no item.

Proof Sketch:

You are not expected to solve this question completely, for the different allocation constraints the efficient allocation can be written as a Linear Program(LP). The payments will follow using the usual VCG rule based on the efficient allocation, no explicit output is expected here either you will get equations based on the LP solutions from the allocations.

4. Consider an auction where n agents with private values θ_i bid on a single object, the mechanism allocates the object to the highest bidder but requires all agents to pay their bid regardless of the allocation.

$$v_i(\theta_i, a(b_i, b_{-i}) = \begin{cases} \theta_i, & \text{if } a_i = 1 \\ 0, & \text{otherwise.} \end{cases}$$

$$p_i(b, a(b_i, b_{-i})) = b_i$$

a. Is this mechanism DSIC? Does it maximize revenue/welfare?

b.Alternatively consider the situation where the player types are known, is the mechanism revenue/welfare maximizing.

Solution Sketch: The auction described is the All Pay auction, with incomplete and complete information respectively.

a. Assume a uniform prior. It is not DSIC. Assuming a uniform prior you can find the equilibrium bidding strategy similar to first price auction analysis. (If n players seems tricky to resolve try it with 2 bidders.)

b. The complete information setup is also not DSIC.

5. Consider a sealed bid second price auction with an entry fee λ and n buyers, whose private values are independent and uniformly distributed over $[0, 1]$.

a. Find a symmetric equilibrium.

b. Find the seller's expected revenue.

c. Which entry fee maximizes the expected revenue for the seller.

d. How does the entry fee scale as the number of players increases?

Solution Sketch:

There is a very similar solved example in the MSZ book. Example 12.28

6. Fix a bidder i and a profile v_{-i}. Myerson's lemma tells us that incentive compatibility and individual rationality imply two properties:

a. Allocation monotonicity: one's allocation should not decrease as one's value v_i increases.

b. Myerson's payment formula (assuming the normalization $p(0) = 0$):

$$p_i(v_i, v_{-i}) = v_i x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(z, v_{-i})\, dz, \quad \forall i \in N, \forall v_i \in T_i, \forall v_{-i} \in T_{-i}.$$

In a second-price auction, the allocation rule is piecewise constant on any continuous interval. That is, bidder i's allocation function is a Heaviside step function, with discontinuity at $v_i = b^*$, where b^* is the highest bid among all bidders other than i (i.e., $b^* = \max_{j \in N \setminus \{i\}} v_j$):

$$x_i(v_i, \mathbf{v}_{-i}) = \begin{cases} 1, & \text{if } v_i > b^* \\ \frac{1}{2}, & \text{if } v_i = b^* \\ 0, & \text{otherwise.} \end{cases}$$

(In writing $\frac{1}{2}$, we assume all bids other than i's are unique.)

Given this allocation rule, the payment formula tells us what i should pay, if they win:

$$p_i(v_i, \mathbf{v}_{-i}) = v_i x_i(v_i, \mathbf{v}_{-i}) - \int_0^{v_i} x_i(z, \mathbf{v}_{-i})\, dz = v_i(1) - \left(\int_0^{b^*} 0\, dz + \int_{b^*}^{v_i} 1\, dz \right) = b^*.$$

(a) Prove that i's payment can be alternatively expressed as follows:

$$p_i\left(v_i, \mathbf{v}_{-i}\right) = b^* \cdot \left[\textit{jump} \text{ in } x_i\left(\cdot, \mathbf{v}_{-i}\right) \text{ at } b^*\right]$$

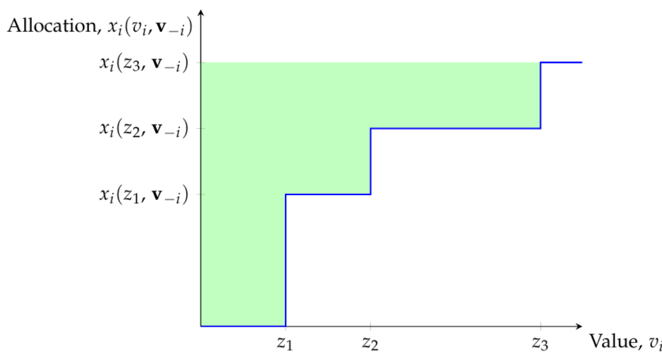

Figure 1: Allocation rule. Shaded area represents payment.

Proof Sketch: Start with the Myerson's payment formula as given in the question and use integration by parts to rewrite the integral term of the payment. In this way, you get a different expression for the payment which has only one integral term within the limits 0 and v_i. Break this limit at the point b^* into 3 parts and proceed further.

(b) Suppose that the allocation rule is piecewise constant on the continuous interval $[0, v_i]$, **and discontinuous at points** $\{z_1, z_2, \ldots, z_\ell\}$ **in this interval.** That is, there are ℓ points at which the allocation jumps from $x(z_j, \mathbf{v}_{-i})$ to $x(z_{j+1}, \mathbf{v}_{-i})$ (refer to the figure 1). Assuming this "jumpy" allocation rule is monotone non-decreasing in value, prove that Myerson's payment rule can be expressed as follows:

$$p_i(v_i, v_{-i}) = \sum_{j=1}^{\ell} z_j \cdot \left[\text{jump in } x_i(\cdot, \mathbf{v}_{-i}) \text{ at } z_j\right].$$

Proof Sketch: Build on the solution of (a) in a similar manner.

7. (a) Take two payment functions p and q that make f DSIC. We define the *Revenue Equivalence result* in single object auction as the following: For every $i \in N$ and every t_{-i} and $s_i, t_i \in T_i$,

$$p_i(s_i, t_{-i}) - q_i(s_i, t_{-i}) = p_i(t_i, t_{-i}) - q_i(t_i, t_{-i})$$

Prove this result using the Myerson's lemma.

Proof Sketch: Use the Myerson payment formula and take the difference between $p_i(s_i, t_{-i})$ and $p_i(t_i, t_{-i})$.

(b) Recall the concept of a *regular virtual valuation*. Suppose the regularity holds for each agent. Consider the following allocation rule f^*. For every type profile $t \in T^n$, $f_i^*(t) = 0$ if $w_i(t_i) < 0$ for all $i \in N$ and else, $f_i^*(t) = 1$ for some $i \in N$ such that $w_i(t_i) \geq 0$, $w_i(t_i) \geq w_j(t_j) \, \forall \, j \in N$.
Prove that there exists payments $(p_1, \ldots, p_n)$ such that $(f^*, p_1, \ldots, p_n)$ is an optimal mechanism.

Proof Sketch: Construct a payment satisfying the revenue equivalence formula (An example payment given in (c)).

(c) Consider the same setting in part (b). For every agent $i \in N$, consider the following payment rule. For every $(t_i, t_{-i}) \in T^n$,

$$p_i^*(t_i, t_{-i}) = \begin{cases} 0 & \text{if } f_i^*(t_i, t_{-i}) = 0 \\ \kappa_i^{f^*}(t_{-i}) & \text{if } f_i^*(t_i, t_{-i}) = 1 \end{cases}$$

where $\kappa_i^{f^*}(t_{-i}) = \inf\{t_i : f_i^*(t_i, t_{-i}) = 1\}$. Prove that the mechanism $(f^*, p_1^*, \ldots, p_n^*)$ is an optimal mechanism.

Proof Sketch: Use the result of (b) and show that the payment in this part satisfies the revenue equivalence formula and hence the mechanism is DSIC and hence BIC.

8. Compute the virtual valuation function of the following distributions:

(a) The uniform distribution on $[0, a]$ with $a > 0$.
Answer: $w(x) = 2x - a$

(b) The exponential distribution with rate $\lambda > 0$ (on $[0, \infty)$).
Answer: $w(x) = x - \frac{1}{\lambda}$

(c) The distribution given by $F(x) = 1 - \frac{1}{(x+1)^c}$ on $[0, \infty)$, where $c > 0$ is some constant.
Answer: $w(x) = \frac{c-1}{c}x - \frac{1}{c}$

9. A valuation distribution meets the *Monotone Hazard Rate (MHR)* condition if its *hazard rate* $\frac{g(x)}{1-G(x)}$ is non-decreasing in x.

   (a) Prove that every distribution meeting the MHR condition is regular.
   Proof Sketch: Use the concept of virtual valuation to prove the same.

   (b) Which of the distributions in Q8 are regular (meaning the virtual valuation function is strictly increasing)? Which of them satisfy the MHR condition?
   Answer: (a) and (b) are both MHR and hence regular but (c) is not.

10. The following figure 2 provides three different allocation functions for auctioning a single indivisible item among two bidders. Each bidder has five *equally likely* valuations and valuations of the agents are *independent*. The number assigned to the some of the cells represents the probability with which agent 1 gets the object. The cells in which no number is written, the probability of agent 1 getting the object at those profiles is zero and it means agent 2 gets the object.

    (a) Answer if the following allocation functions are **Bayesian or dominant strategy implementable** or not. Wherever it is, provide the expected payment π_1 of bidder 1 that Bayesian implements the allocation. Note that

$$\pi_1(t_1) = \mathbb{E}(p_1(t_1, t_2) \mid t_1),$$

    where $p_1(t_1, t_2)$ is the payment of bidder 1 when the bid profile is (t_1, t_2).

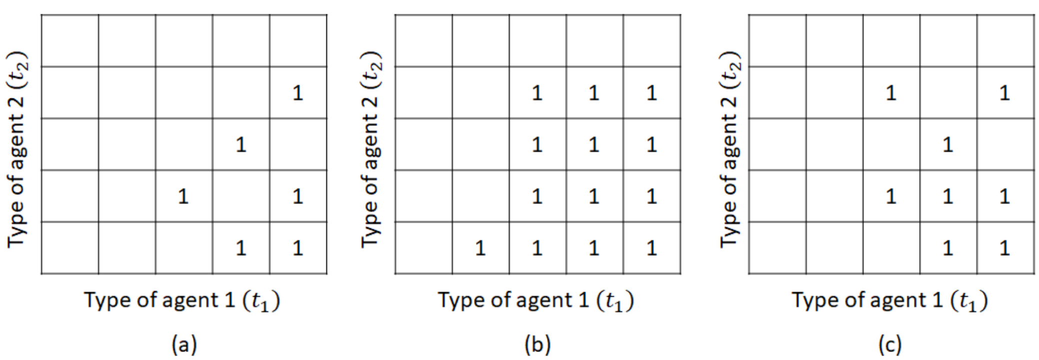

Figure 2: Allocation functions for single item auction.

    Answer: Bayesian implementable: (a), (b)
    Dominant strategy implementable: (b)
    Find the value of the expected payment using Myerson payment formula for $t_1 = 4$ (for example), i.e. compute $\pi_1(4) - \pi_1(0)$.
    (a) 1, (b) $\frac{7}{5}$ (c) NA

    (b) Explain how you arrived at the above set of answers. In particular, which principle(s) did you use to conclude to your answers above.
    Answer: The concepts of NDE and ND are used.