Logic in CS Autumn 2024

Problem Sheet 6

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1. Translate the following formula to rectified form, then to prenex form, and finally to Skolem form:

$$\forall z\, \exists y\, (Q(x, g(y), z) \vee \neg \forall x\, P(x)) \wedge \neg \forall z\, \exists x\, \neg R(f(x, z), z)\,.$$

Solution

Recall that a formula is rectified if no variable appears both bound and free and if each bound variable has exactly one binding occurrence. An equivalent rectified form of the given formula is obtained by renaming some of the bound variables as follows:

$$\forall z\, \exists y\, (Q(x, g(y), z) \vee \neg \forall u\, P(u)) \wedge \neg \forall w\, \exists v\, \neg R(f(v, w), w)\,.$$

To obtain an equivalent prenex form, first push the negations inward:

$$\forall z\, \exists y\, (Q(x, g(y), z) \vee \exists u\, \neg P(u)) \wedge \exists w\, \forall v\, R(f(v, w), w)\,.$$

Then bring the quantifiers to the front (using the equivalences from Lecture 13):

$$\forall z\, \exists y\, \exists u\, \exists w\, \forall v\, ((Q(x, g(y), z) \vee \neg P(u)) \wedge R(f(v, w), w))\,.$$

Finally we obtain an equisatisfiable Skolem form with new unary function symbols f_1, f_2, f_3:

$$\forall z\, \forall v\, ((Q(x, g(f_1(z)), z) \vee \neg P(f_2(z))) \wedge R(f(v, f_3(z)), f_3(z)))$$

2. Are the following claims correct? Justify your answers.

   (a) For any formula F and term t, if F is valid then $F[t/x]$ is valid.

   (b) $\exists x\, (P(x) \to \forall y\, P(y))$ is valid.

   (c) For any formula F and constant symbol c, if $F[c/x]$ is valid and c does not appear in F then $\forall x\, F$ is valid.

Solution

(a) The claim is incorrect. Take the formula F to be $P(x) \to \forall y\, P(x)$. Then F is valid but $F[y/x]$ is the formula $P(y) \to \forall y\, P(y)$, which is not valid.

(b) The claim is correct. Let $\mathcal{A}$ be an arbitrary structure. We consider two cases. The first case is that $\mathcal{A} \models \forall y\, P(y)$. Then $\mathcal{A} \models P(x) \to \forall y\, P(y)$ and hence $\mathcal{A} \models$

$\exists x\,(P(x) \to \forall y\,P(y))$. The second case is that $\mathcal{A} \not\models \forall y\,P(y)$. By definition this means $\mathcal{A}_{[y\mapsto a]} \not\models P(y)$ for some $a \in U_{\mathcal{A}}$. It follows that $a \notin P_{\mathcal{A}}$ and so $\mathcal{A}_{[x\mapsto a]} \not\models P(x)$. This means $\mathcal{A}_{[x\mapsto a]} \models P(x) \to \forall y\,P(y)$ and hence $\mathcal{A} \models \exists x\,(P(x) \to \forall y\,P(y))$.

(c) The claim is correct. We have to show that $\mathcal{A} \models_{\alpha[x\mapsto a]} F$ for any structure $\mathcal{A}$ and all $a \in U^{\mathcal{A}}$.

Assume $F[c/x]$ is valid. Thus, all $\mathcal{A}$ and α are such that $\mathcal{A} \models_\alpha F[c/x]$.

Given such a $\mathcal{A}$ and α, let α' be such that $\alpha'(c) = \alpha(x)$ and let α' agree with α on all others. Then we have

$$\begin{aligned} \mathcal{A} \models_\alpha F \text{ iff } & \mathcal{A} \models_{\alpha'} F && \text{(since } c \text{ is not mentioned in } F\text{)} \\ \text{iff } & \mathcal{A} \models_{\alpha[x\mapsto\alpha'(c)]} F && \text{(identical assignments)} \\ \text{iff } & \mathcal{A} \models_\alpha F[c/x] && \text{(Translation Lemma)} \end{aligned}$$

Since $F[c/x]$ is valid, all assignments assigning any value of $U^{\mathcal{A}}$ to c satisfies $F[c/x]$. Indeed, if we now assign all those possible values to x in F by $\alpha[x \mapsto \alpha'(c)]$ for all possible $\alpha'(c) \in U^{\mathcal{A}}$, then $\mathcal{A} \models_{\alpha[x\mapsto\alpha'(c)]} F$ for all possible values $\alpha'(c) \in U^{\mathcal{A}}$ and any $\mathcal{A}$. Hence, $\mathcal{A} \models_{\alpha[x\mapsto a]} F$ for all $a \in U^{\mathcal{A}}$ and any $\mathcal{A}$. Hence, $\mathcal{A} \models_\alpha \forall x F$ for all $\mathcal{A}, \alpha$. Hence $\forall x F$ is valid.

3. Which of the following languages are regular? Which are FO definable?

(a) The set of words over $\{a, b\}$ which has equal number of occurrences of ab and ba. For example, aba is in the language, while $abab$ is not.

(b) The set of words over $\{a, b, \#\}$ with a single occurrence of #, and every symbol before the # is an a, and all symbols after the # are b's.

(c) The set of strings over $\{a, b\}$ which does not contain any occurrence of ba.

(d) The set of strings over $\{0, 1\}$ such that the second symbol from both ends is 0.

(e) Let $\Sigma = \{\begin{pmatrix} a \\ b \end{pmatrix} \mid a, b \in \{0, 1\}\}$. A string over Σ gives two rows of 0's and 1's. Treat each row as a binary number. The set of words

$$\{w \in \Sigma^* \mid \text{ the top row is larger than the bottom row }\}$$

Solution

(a) Observe that since our alphabet is $\{a, b\}$, this language is equivalent to the set of languages that end and start with the same alphabet. The FO formula for this would be:

$$\forall x\,(\text{first}(x) \implies Q_a(x) \wedge \text{last}(x) \implies Q_a(x)) \vee$$
$$\forall x\,(\text{first}(x) \implies Q_b(x) \wedge \text{last}(x) \implies Q_b(x))$$

where $\text{first}(x) \equiv \neg\exists\, y[y < x]$ and $\text{last}(x) \equiv \neg\exists y\,[x < y]$.

(b) The required FO formula is:

$$\exists x[Q_{\#}(x) \wedge \forall y\,(Q_{\#}(y) \implies y = x) \wedge \forall y\,(y < x \implies Q_a(y)) \wedge$$
$$\forall y\,(x < y \implies Q_b(y))]$$

(c) The required FO formula is:

$$\neg\exists x\exists y\,[S(x,y) \wedge Q_b(x) \wedge Q_a(y)]$$

(d) The required FO formula is:

$$\exists x\exists y\exists z\exists w\;[\text{first}(x) \wedge S(x,y) \wedge Q_0(y) \wedge \text{last}(z) \wedge S(w,z) \wedge Q_0(w)]$$

(e) A necessary and sufficient condition for the top row to be larger than the bottom row is that at the first point of difference (which should exist), the top row should have bit 1 and the bottom row should have bit 0. In FOL, this can be expressed as:

$$\exists x \left[Q_{\binom{1}{0}}(x) \wedge \forall y \left(y < x \implies \left[Q_{\binom{0}{0}}(y) \vee Q_{\binom{1}{1}}(y) \right] \right) \right]$$

You can construct the DFAs for each of these languages, to prove that they are regular, as an exercise.

4. Consider the following FO formulae. In each case, answer the following questions:

- What is $L(\varphi)$?
- What is $\overline{L(\varphi)}$?
- Is $L(\varphi)$ regular?
- Is $\overline{L(\varphi)}$ regular?

(a) $\forall x(x \neq x)$

(b) $\exists x\exists y[x < y \wedge Q_b(x) \wedge Q_a(y) \wedge \forall z[(x < z < y) \rightarrow Q_a(z)]]$

(c) $\exists x[Q_a(x) \wedge \exists y[S(x,y) \wedge \forall z[z \leq y]]]$

(d) $\exists x \forall y[x \leq y \wedge Q_a(x)] \wedge \exists x \forall y[y \leq x \wedge Q_b(x)] \wedge$
$\forall x \forall y[Q_a(x) \wedge S(x,y) \rightarrow Q_b(y)] \wedge \forall x \forall y[Q_b(x) \wedge S(x,y) \rightarrow Q_a(y)]$

Solution

In this question, let $\Sigma = \{a, b\}$. Firstly, note that the for each sub-question, $L(\varphi)$ and $\overline{L(\varphi)}$ will be both regular as all FO definable languages are regular, and FO definable languages are closed under complementation. By inspection, we observe that the answers to (a) and (b) for the languages defined by the FO formulas are:

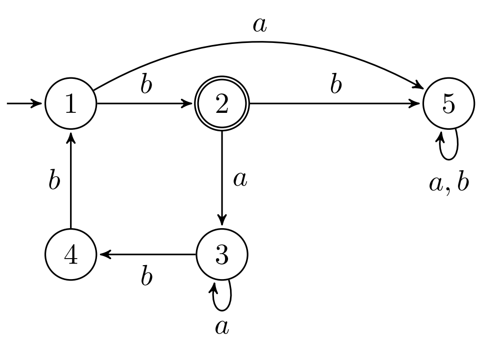
(a) $L(\varphi) = \{\epsilon\}$. Thus, $\overline{L(\varphi)} = \Sigma^* \setminus \{\epsilon\} = \Sigma^+$.

(b) $L(\varphi) = \Sigma^* ba \Sigma^*$. Also, $\overline{L(\varphi)} = a^* b^*$.

(c) $L(\varphi) = \Sigma^* a \Sigma$ and $\overline{L(\varphi)} = \{\epsilon\} \cup \Sigma \cup \Sigma^* b \Sigma$.

(d) $L(\varphi) = a(ba)^* b$ and $\overline{L(\varphi)} = \{\epsilon\} \cup b\Sigma^* \cup \Sigma^* a \cup \Sigma^* aa \Sigma^* \cup \Sigma^* bb \Sigma^*$

5. Consider the following automaton. What is the language L accepted? Can you write an FO formula φ such that $L = L(\varphi)$?



Solution

By observation, $L = b(a^+b^3)^*$ [a] and we also have $L = L(\varphi)$ where φ is

$$\exists x\,[\text{first}(x) \wedge Q_b(x)] \ \wedge\ \forall x[\neg \text{first}(x) \implies ([Q_a(x) \wedge \exists y(x < y \wedge Q_b(y))] \\ \vee\ [C_1(x) \vee C_2(x) \vee C_3(x)])]$$

where $C_i(x)$ expresses that x is the i-th b in the a^+b^3 element, $i = 1, 2, 3$.

$$C_1(x) \equiv \exists p \exists q \exists r\,[Q_a(p) \wedge S(p,x) \wedge Q_b(x) \wedge S(x,q) \wedge Q_b(q) \\ \wedge S(q,r) \wedge Q_b(r) \wedge \forall s(S(r,s) \implies Q_a(s))]$$

$$
\begin{aligned}
C_2(x) \equiv \exists p \exists q \exists r \, [&Q_a(p) \land S(p,q) \land Q_b(q) \land S(q,x) \land Q_b(x) \\
&\land S(x,r) \land Q_b(r) \land \forall s (S(r,s) \implies Q_a(s))]
\end{aligned}
$$

$$
\begin{aligned}
C_3(x) \equiv \exists p \exists q \exists r \, [&Q_a(p) \land S(p,q) \land Q_b(q) \land S(q,r) \land Q_b(r) \\
&\land S(r,x) \land Q_b(x) \land \forall s (S(x,s) \implies Q_a(s))]
\end{aligned}
$$

[a] $a^+ = a^* \setminus \{\epsilon\}$