CS 228 : Logic in Computer Science

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Quantifier Rank

Let φ be a FO formula over words. Define the quantifier rank of φ denoted $c(\varphi)$

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- $c(\exists x\varphi) = c(\varphi) + 1$
- Quantifier free formulae written in DNF : $C_1 \vee C_2 \vee \cdots \vee C_n$
- Formulae of quantifier rank $k + 1$ written as a disjunction of the conjunction of formulae, each formula of the form $\exists x\varphi, \neg\exists x\varphi$ or φ, with $c(\varphi) \leqslant k$. Eliminate repeated disjuncts/conjunts. (Think Prenex normal form)

Number of FO formulae of rank $\leqslant c$

Let $\mathcal{V}$ be a finite set of first order variables, and let $c \geqslant 0$. There are finitely many FO formulae in normal form (therefore, upto equivalence) with rank $\leqslant c$ over $\mathcal{V}$.

Proof ideas in the following slides.

Number of FO formulae of rank $\leqslant c$

Let $\mathcal{V}$ be a finite set of first order variables. Fix a finite signature τ. Let there be m atomic formulae over τ having variables from $\mathcal{V}$.

- If $\mathcal{V}$ has 2 variables x, y, and τ has $Q_a, S, <$.
- Atomic formulae : $\{Q_a(x), Q_a(y), S(x, y), x < y\}$
- Let $G = \{Q_a(x), \neg Q_a(x), Q_a(y), \neg Q_a(y), S(x, y), \neg S(x, y), x < y, \neg(x < y)\}$ be the closure of all atomic formulae containing all formulae and their negations.
- Each subset of G is a possible conjunct C_i.

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- Each subset of G is a possible conjunct C_i.
- All possible disjuncts using each C_i : formulae in DNF of rank 0

Number of FO formulae of rank $\leqslant c$

Let $\mathcal{V}$ be a finite set of first order variables. Fix a finite signature τ. Let there be m atomic formulae over τ having variables from $\mathcal{V}$.

- $2m$ atomic/negated atomic formulae
- Number of conjunctions C_i possible $\leqslant 2^{2m}$
- Number of formulae in DNF $\leqslant 2^{2^{2m}}$ ($c = 0$)

Rank 1

Let there be p formulae φ of rank 0.

- $2p$ formulae of the form $\exists x \varphi, \neg \exists x \varphi$
- 2^{2p} conjunctions of rank 1
- Conjuncting any one of the p formulae of rank 0 gives all conjuncts of rank 1 : $p2^{2p}$ more
- Possible conjuncts of rank 1 is $q = (p+1)2^{2p}$
- Possible disjuncts of these : 2^q

Some Notation

Given a word $w = a_1 \dots a_n$, and a finite set of variables $\mathcal{V}$, define a $\mathcal{V}$-enriched-word with respect to w as

- $(a_1, U_1)(a_2, U_2) \dots (a_n, U_n)$ where
- $\bigcup_i U_i = \mathcal{V}$
- $U_i \cap U_j = \emptyset$

- A $\mathcal{V}$-enriched-word is over the alphabet $\Sigma \times 2^{\mathcal{V}}$
- $(a, \{x\})(b, \{y, z\})(c, \emptyset)(d, \{u, v\})$ is a $\{x, y, z, u, v\}$-enriched word with respect to the word *abcd*.
- We will refer to $\mathcal{V}$-enriched-word structures as $\mathcal{V}$-structures from here on

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- $w \models Q_a(x)$ iff there exists j such that $a_j = a$ and $x \in S_j$
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  - $(a, \{x\})(b, \{y, z\})(c, \emptyset) \not\models (x = y)$
- $w \models x < y$ iff there exists $i < j$ such that $x \in S_i, y \in S_j$
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  - $(a, \{x\})(b, \{y, z\})(c, \emptyset) \models x < y$
- $w \models \exists x Q_a(x)$ iff there exists i such that $(a_1, S_1) \dots (a_i, S_i \cup \{x\}) \dots (a_n, S_n) \models Q_a(x)$
  - $(b, \{y, z\})(a, \{u\})(c, \emptyset) \models \exists x Q_a(x)$ since $(b, \{y, z\})(a, \{x, u\})(c, \emptyset) \models Q_a(x)$

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Similarly, $(a,\emptyset)(a,\{x\})(b,\{y\}) \models (Q_a(x) \to [(x<y) \wedge Q_b(y)])$ and $(a,\emptyset)(a,\emptyset)(b,\{x,y\}) \models (Q_a(x) \to [(x<y) \wedge Q_b(y)])$

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- $(a, \emptyset)(b, \emptyset) \sim_1 (a, \emptyset)(b, \emptyset)(a, \emptyset)$?
- $\sim_r$ is an equivalence relation
- Finitely many equivalence classes : each class consists of words that behave the same way on formulae of rank $\leqslant r$

Non-Expressibility in FO : The Game Begins

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- Spoiler wants to show that w_1, w_2 are different ($w_1 \nsim_r w_2$)
- Duplicator wants to show that they are same ($w_1 \sim_r w_2$)
- Each player has r pebbles $z_1, \ldots, z_r$

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- A pebble once placed, cannot be removed
- The game ends after r rounds, when both players have used all their pebbles

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  - After round 1, we have two $\{z_1\}$ structures (w_1', w_2')

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- Round 1:
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  - After round 1, we have two $\{z_1\}$ structures (w'_1, w'_2)
- Round 2:

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  - Spoiler continues on the structure w_2'
  - Duplicator gets w_1' to play
  - Spoiler : $(a, \{z_1\})(b, \emptyset)(a, \{z_2\})$
  - Duplicator : $(a, \{z_1, z_2\})(b, \emptyset)$ or $(a, \{z_1\})(b, \{z_2\})$

Winner

- Start with two $\emptyset$ structures (w_1, w_2)

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- Duplicator wins iff for every atomic formula α, $w_1' \models \alpha$ iff $w_2' \models \alpha$
- That is, $w_1' \sim_0 w_2'$
- Spoiler wins otherwise.

Winner

Given two word structures (w_1, w_2), duplicator wins on (w_1, w_2) if for every atomic formula α, $w_1 \models \alpha$ iff $w_2 \models \alpha$

Play continues

- Who won in the earlier play?
- We had
  - $(a, \{z_1\})(b, \emptyset)(a, \{z_2\})$ and $(a, \{z_1, z_2\})(b, \emptyset)$
  - $(a, \{z_1\})(b, \emptyset)(a, \{z_2\}) \models (z_1 < z_2)$
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  - $(a, \{z_1\})(b, \{z_2\}) \not\models Q_a(z_2)$
- Spoiler wins in two rounds
- If the game was played only for one round, who will win?

Unique Winner

Given structures w_1, w_2, and a number of rounds r, exactly one of the players win.

- Think of a tree of all possible sequences of plays.
- The root is (w_1, w_2), and nodes are all possible pairs (w_1', w_2') of obtainable structures.
- Mark a leaf node as S or D depending on whether Spoiler won or Duplicator won.
- An interior node corresponding to a move of Duplicator is labeled D if there any children labeled D; likewise for Spoiler

Logical Equivalence and Winning

Let w_1, w_2 be $\mathcal{V}$-structures and let $r \geqslant 0$. Then $w_1 \sim_r w_2$ iff Duplicator has a winning strategy in the r-round game on (w_1, w_2).

Logical Equivalence and Winning

Assume $w_1 \sim_r w_2$, and induct on r

- Base : $r = 0$ and $w_1 \sim_0 w_2$. Duplicator wins, since by assumption, w_1, w_2 agree on all atomic formulae.

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- Now, let $w_1 \sim_r w_2$, and assume spoiler wins the r-round game on (w_1, w_2).
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  - By assumption, spoiler wins the $r - 1$ round game on (w_1', w_2')

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  - Let ψ be the conjunction of all formulae of rank $r-1$ in normal form that are satisfied by w_1'
  - Then $w_1' \models \psi$, $w_2' \not\models \psi$
  - We thus have

    $$w_1 \models \exists z_1 \psi, w_2 \not\models \exists z_1 \psi$$

    contradicting $w_1 \sim_r w_2$

Logical Equivalence and Winning : Converse

Assume Duplicator wins r-round games on (w_1, w_2) and induct on r

- Base : $r = 0$ and Duplicator wins. Then w_1, w_2 agree on all atomic formulae, and hence $w_1 \sim_0 w_2$

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- Now, let duplicator win in the r round game, but $w_1 \not\sim_r w_2$.
  - $w_1 \not\sim_r w_2 \Rightarrow$ there is some formula ψ, $c(\psi) = r$ such that $w_1 \models \psi$, $w_2 \not\models \psi$

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  - That is, either both w_1', w_2' satisfy φ, or both dont, a contradiction.

FO-definable languages

Assume L is FO-definable, and $L = L(\varphi)$ with rank of φ being k.

- Let $L = \{v_1, v_2, v_3, \ldots\}$ and $\overline{L} = \{w_1, w_2, w_3, \ldots\}$

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- Let $L = \{v_1, v_2, v_3, \dots\}$ and $\overline{L} = \{w_1, w_2, w_3, \dots\}$
- Play a k round game on $v_i \in L$ and $w_j \notin L$. Let ψ_{v_i, w_j} be the formula of rank k that distinguishes the two words.

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- Play a k round game on $v_i \in L$ and $w_j \notin L$. Let ψ_{v_i,w_j} be the formula of rank k that distinguishes the two words.
- Consider the formula

$$
\begin{gathered}
[\psi_{v_1,w_1} \wedge \psi_{v_1,w_2} \wedge \cdots \wedge \psi_{v_1,w_n} \wedge \ldots] \\
\vee \\
[\psi_{v_2,w_1} \wedge \psi_{v_2,w_2} \wedge \cdots \wedge \psi_{v_2,w_n} \wedge \ldots] \\
\vee \\
\vdots
\end{gathered}
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- Each ψ_{vw} has rank atmost k
- Upto equivalence, there are finitely many formulae of rank k
- Hence the disjunction and conjunction are finite
- ψ_L is a proper formula (of finite size)
- ψ_L captures L since each $v \in L$ satisfies $\bigwedge_{w \notin L} \psi_{vw}$ while none of the $w \notin L$ satisfy $\bigwedge_{w \notin L} \psi_{vw}$

FO-definable languages

Given a property $\mathcal{K}$, if for any pair $v \in \mathcal{K}$ and $w \notin \mathcal{K}$, spoiler has a winning strategy in the k-round EF game on v and w, then there is a rank k FO formula $\varphi_{\mathcal{K}}$ that defines the property $\mathcal{K}$.

$$\varphi_{\mathcal{K}} = \bigvee_{v \in \mathcal{K}} \bigwedge_{w \notin \mathcal{K}} \psi_{vw}$$

where ψ_{vw} is as explained in the previous slide.

- Note that k is fixed in the above, and is independent of the choices of the words.

Implications of the Game on FO definability

FO Definability

L is FO definable $\Rightarrow$ there exists an r such that for every (w_1, w_2) pair, such that $w_1 \in L$, $w_2 \notin L$, spoiler wins in r rounds

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Non FO Definability

For all $r \geqslant 0$, there exists a (w_1, w_2) pair with $w_1 \notin L$, $w_2 \in L$, duplicator wins in r rounds $\Rightarrow$ L is not FO definable

$(aa)^*$ is not $FO[<]$ Definable

- Assume that there is a sentence φ that defines words of even length, with $c(\varphi) = r$.
- Then, $a^i \models \varphi$ iff i is even
- Show that for all $r > 0$, $a^{2^r} \sim_r a^{2^r - 1}$

$(aa)^*$ is not $FO[<]$ Definable

- Base case : $(a, \emptyset)(a, \emptyset)$ and $(a, \emptyset)$ for $r = 1$
- In one round, duplicator wins on $(a, \emptyset)(a, \emptyset)$ and $(a, \emptyset)$

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- Base case : $(a, \emptyset)(a, \emptyset)$ and $(a, \emptyset)$ for $r = 1$
- In one round, duplicator wins on $(a, \emptyset)(a, \emptyset)$ and $(a, \emptyset)$
- Consider $(aaaa, aaa)$ for $r = 3$. Who wins?
- Consider $(aaaa, aaa)$ for $r = 2$. Who wins?

$(aa)^*$ is not $FO[<]$ Definable

- Show that for all $k \geqslant 2^r - 1$, duplicator has a winning strategy for the r-round game in (a^k, a^{k+1}), for all $r \geqslant 0$
- Induct on r
- If $r = 1$, then on (a, aa) duplicator wins in one round
- Assume now that the claim is true for $\leqslant r - 1$

$(aa)^*$ is not $FO[<]$ Definable

- Let $k \geqslant 2^r - 1$, and consider the structures

$$(a^k, a^{k+1})$$

- Spoiler puts pebble z_1 in one of the words obtaining

$$(a, \emptyset)^s (a, \{z_1\}) (a, \emptyset)^t$$

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- $s \leqslant \frac{k-1}{2}$ or $t \leqslant \frac{k-1}{2}$

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- Assume $s \leqslant \frac{k-1}{2}$. Duplicator puts her pebble z_1 on the $(s+1)$th letter of the other word obtaining

$$(a, \emptyset)^s (a, \{z_1\})(a, \emptyset)^{t'}$$

where $t' = t + 1$ or $t' = t - 1$.

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  where $t' = t+1$ or $t' = t-1$.
- The structures after round 1 are thus

$$(a,\emptyset)^s(a,\{z_1\})(a,\emptyset)^t, (a,\emptyset)^s(a,\{z_1\})(a,\emptyset)^{t'}$$

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- We have $2^r - 1 \leqslant k = min(t,t') + s + 1 \leqslant min(t,t') + \frac{k-1}{2} + 1$
- Hence $min(t,t') \geqslant \frac{k-1}{2} \geqslant 2^{r-1} - 1$

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- We have $2^r - 1 \leqslant k = min(t,t') + s + 1 \leqslant min(t,t') + \frac{k-1}{2} + 1$
- Hence $min(t,t') \geqslant \frac{k-1}{2} \geqslant 2^{r-1} - 1$
- By inductive hypothesis, duplicator has a winning strategy for the $r-1$ round game on $(a^t, a^{t'})$.

Duplicator's Win

- Use the duplicator's winning strategy for the $r-1$ round game on $(a^t, a^{t'})$, to obtain a winning strategy in $r-1$ rounds on

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- Whenever spoiler plays on a structure on letter $i \leqslant s+1$, duplicator plays on the same position on the other structure
- When spoiler plays at a position $i > s+1$ in either word, duplicator plays in the part of the other word $> s+1$ using her winning strategy in $(a^t, a^{t'})$

Duplicator's Win

- At the end of r rounds, we have structures w'_1, w'_2.
- For $i \leqslant s+1$, pebble z_j appears at position i of w'_1 iff pebble z_j appears at position i of w'_2
- Lets erase the first $s+1$ letters in w'_1, w'_2, obtaining v'_1, v'_2
- v'_1, v'_2 are the words that result after $r' \leqslant r-1$ rounds of play on $(a^t, a^{t'})$. Recall that duplicator won this.
- Show that w'_1, w'_2 satisfy the same atomic formulae

Duplicator's Win

- Atomic Formulae : $Q_a(z_j)$: Both w'_1, w'_2 satisfy this.
- $w'_1 \models z_i < z_j$. If z_i, z_j are in the first $s+1$ letters, then $w'_2 \models z_i < z_j$.
- If z_i, z_j occur in the last $|w'_1| - s - 1$ positions, then $v'_1 \models z_i < z_j$. By duplicator's win in $(a^t, a^{t'})$, $v'_2 \models z_i < z_j$
- If z_i appears among the first $s+1$ letters and z_j after the first $s+1$ letters of w'_1, same is true in w'_2.

Historically Speaking

The games that we saw are due to Ehrenfeucht and Fraïssé

Reference: Finite Automata, Formal Logic and Circuit Complexity, by Howard Straubing.