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Roll No : 12345
e.g., 190040001

Dept.: CSE
e.g., CSE

IIT Bombay
CS 405/6001: GT&AMD
Midsem, 2024-25-I
Date: September 20, 2024

CS 405/6001: Game Theory and Algorithmic Mechanism Design

Total: $\mathbf{6 + 10 + 9 + 15 = 40}$ **marks,** *Duration:* **2 hours, ATTEMPT ALL QUESTIONS**

Instructions:

1. This question-and-answersheet booklet contains a total of 6 sheets of paper (12 pages, page 2 is blank). Please verify.
2. Write your roll number and department on **every side of every sheet** (except the blank sheet) of this booklet. Use only **black/blue ball-point pen**. The first 5 minutes of additional time is given exclusively for this activity.
3. Write final answers neatly with a pen **only in the given boxes**.
4. Use the rough sheets for scratch works / attempts to solution. **Write only the final solution (which may be a sequence of logical arguments) in a precise and succinct manner in the boxes provided**. Do not provide unnecessarily elaborate steps. The space within the boxes are sufficient for the correct and precise answers.
5. Submit your answerscripts to the teaching staff when you leave the exam hall or the time runs out (whichever is earlier). **Your exam will not be graded if you fail to return the paper**.
6. **This is a closed book, notes, internet exam. No communication device, e.g., cellphones, iPad, etc., is allowed**. Keep it switched off in your bag and keep the bag away from you. If anyone is found in possession of such devices during the exam, that answerscript may be disqualified for evaluation and DADAC may be invoked.
7. One A4 assistance sheet (text **text on both sides**) is allowed for the exam.
8. **After you are done with your exam or the exam duration is over, please DO NOT rush to the desk for submitting your paper**. Please remain seated until we collect all the papers, count them, and give a clearance to leave your seat.

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Problem 1 (6 points). Consider the two-player game given below, in which each player has three pure strategies.

Player 1 \ Player 2	L	C	R
T	0, 0	7, 6	6, 7
M	6, 7	0, 0	7, 6
B	7, 6	6, 7	0, 0

(a) Find all the **pure strategy Nash equilibria (PSNE)** of this game with justification. **2 points.**

This game does not have any PSNE.

(b) Find all the **mixed strategy Nash equilibria (MSNE)** of this game with justification. If you use some results from the class, state them clearly. Also, show how they are being used in concluding that some profiles may/may not be an MSNE. **4 points.**

We will use the MSNE characterization result from the class.

$(\sigma_i^*, \underline{\sigma}_{-i}^*)$ is an MSNE iff

$$u_i(s_i, \underline{\sigma}_{-i}^*) \geqslant u_i(s_i', \underline{\sigma}_{-i}^*) \quad \forall s_i \in \delta(\sigma_i^*), s_i' \in S_i \quad \forall i \in N$$

where $\delta(\sigma_i^*)$ is the support of σ_i^*. --- (1)

We find that this condition is violated for every support that are not the full support of each player. Therefore the only support profile that can have condition (1) satisfied is $\{T, M, B\} \times \{L, C, R\}$.

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Using condition 1 on the support profile

we get $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is the unique

MSNE of this game.

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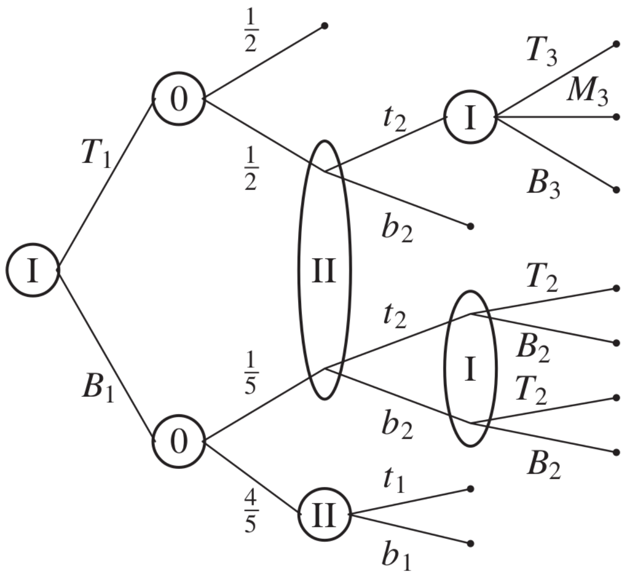
Problem 2 (10 points). Answer the following questions.

(a) Does the following imperfect information extensive form game have **perfect recall**? (Yes/No)

1 point.



Yes.

(b) Explain your answer, i.e., if yes, why it is so, and if not, which property of the perfect recall is violated.

2 points.

Perfect recall requires two conditions to be satisfied.

(1) Every information set of player i intersects every path from the root to a leaf at most once.

(2) Every two path that end in the same information set of player i pass through the same information sets in the same order and in each info set they pick the same action.

Both conditions are satisfied for both players in this game.

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(c) If yes, then find the **behavioral strategy** equivalent to the following mixed strategies of the two players. If not, explain which condition(s) is(are) violated in at most 4 sentences. Given mixed strategy profile (the pure strategy profiles that are not mentioned have probability zero in this mixed strategy):

$$s_I = \left[\frac{3}{7}(B_1B_2M_3), \frac{1}{7}(B_1T_2B_3), \frac{2}{7}(T_1B_2M_3), \frac{1}{7}(T_1T_2T_3)\right]$$
$$s_{II} = \left[\frac{3}{7}(b_1b_2), \frac{1}{7}(b_1t_2), \frac{1}{7}(t_1b_2), \frac{2}{7}(t_1t_2)\right]$$

Please provide only the answer/explanations below. Show only the minimal steps towards your answer and leave out the detailed calculations (which you did in the rough sheets). **4 points.**

In order to find the behavioral strategies, we need the following identities to be satisfied (one at each leaf node). We suppress the information set in the notation to keep it clean (will be clear from the actions)

$$b_1(T_1)\, b_2(t_2)\, b_1(T_3) = \sigma_1(T_1T_3)\,\sigma_2(t_2)$$
$$b_1(T_1)\, b_2(t_2)\, b_1(M_3) = \sigma_1(T_1M_3)\,\sigma_2(t_2)$$
$$b_1(T_1)\, b_2(t_2)\, b_1(B_3) = \sigma_1(\underbrace{T_1B_3}_{=0})\,\sigma_2(t_2)$$
$$b_1(T_1)\, b_2(b_2) = \sigma_1(T_1)\,\sigma_2(b_2)$$
$$b_1(B_1)\, b_2(t_2)\, b_1(T_2) = \sigma_1(B_1T_2)\,\sigma_2(t_2)$$
$$b_1(B_1)\, b_2(t_2)\, b_1(B_2) = \sigma_1(B_1B_2)\,\sigma_2(t_2)$$
$$b_1(B_1)\, b_2(b_2)\, b_1(T_2) = \sigma_1(B_1T_2)\,\sigma_2(b_2)$$
$$b_1(B_1)\, b_2(b_2)\, b_1(B_2) = \sigma_1(B_1B_2)\,\sigma_2(b_2)$$
$$b_1(B_1)\, b_2(t_1) = \sigma_1(B_1)\,\sigma_2(t_1)$$
$$b_1(B_1)\, b_2(b_1) = \sigma_1(B_1)\,\sigma_2(b_1)$$

Solving these carefully: $b_1 = \left[\left(\frac{4}{7}(B_1), \frac{3}{7}(T_1)\right), \left(\frac{3}{4}(B_2), \frac{1}{4}(T_2)\right), \left(0(B_3), \frac{2}{3}(M_3), \frac{1}{3}(T_3)\right)\right]$

$b_2 = \left[\left(\frac{4}{7}(b_1), \frac{3}{7}(t_1)\right), \left(\frac{4}{7}(b_2), \frac{3}{7}(t_2)\right)\right]$

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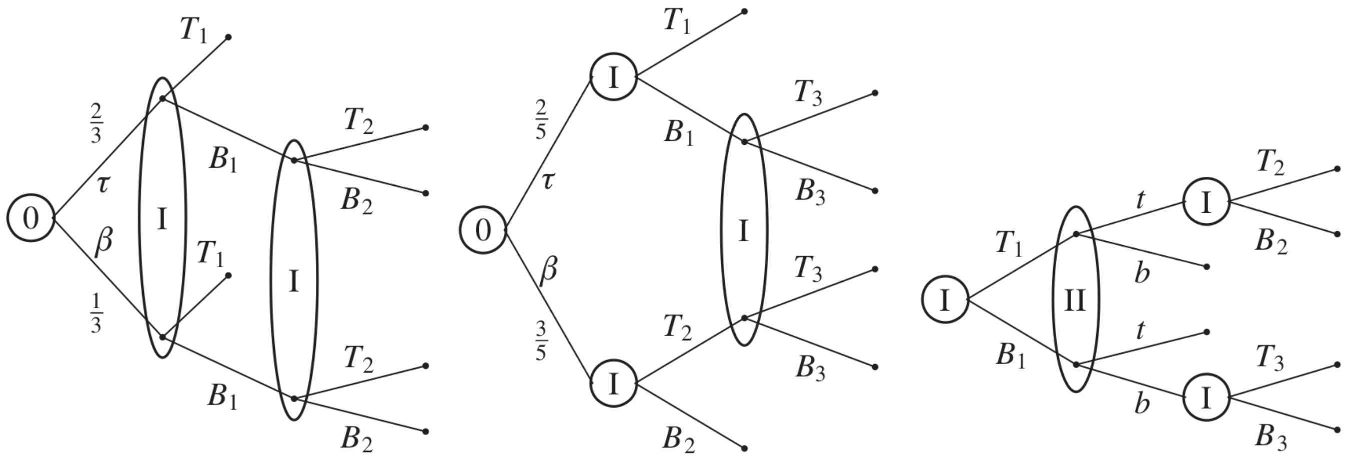
(d) Find whether the following games have perfect/imperfect recall. Explain which players have perfect and which players have imperfect recall, e.g., what type forgetfulness occurs there (if any).



(a) Game A (b) Game B (c) Game C

3 points.

Game A: Perfect recall for both players.

Game B: Player I has imperfect recall, forgets the outcome of the chance move and the move he made earlier.

Game C: Perfect recall for both players

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Problem 3 (9 points). Answer the following questions.

(a) Recall the definition of **weak dominance**. A mixed strategy σ_i of player i is called weakly dominated if there exists a mixed strategy σ_i' of player i satisfying:

- For each strategy $s_{-i} \in S_{-i}$ of the other players,

$$u_i(\sigma_i, s_{-i}) \leqslant u_i(\sigma_i', s_{-i}).$$

- There exists a strategy $t_{-i} \in S_{-i}$ of the other players for which,

$$u_i(\sigma_i, t_{-i}) < u_i(\sigma_i', t_{-i}).$$

Does the set of all weakly dominated strategies of player i form a convex set? (True/False) **1 point.**

True

(b) Explain your previous answer, i.e., provide a counterexample if it is false, or prove it if it is true.

3 points.

Let $\sigma_i^{(1)}$ and $\sigma_i^{(2)}$ are two weakly dominated strategies, that are dominated by $\bar{\sigma}_i^{(1)}$ and $\bar{\sigma}_i^{(2)}$ respectively.

then $u_i(\sigma_i^{(k)}, \sigma_{-i}) \leqslant u_i(\bar{\sigma}_i^{(k)}, \sigma_{-i}) \quad \forall \sigma_{-i}$

and $u_i(\sigma_i^{(k)}, \tilde{\sigma}_{-i}^{(k)}) < u_i(\bar{\sigma}_i^{(k)}, \tilde{\sigma}_{-i}^{(k)})$ for some $\tilde{\sigma}_{-i}^{(k)}$

$k=1,2.$

We know that $u_i(\lambda\sigma_i^{(1)} + (1-\lambda)\sigma_i^{(2)}, \sigma_{-i})$

$= \lambda u_i(\sigma_i^{(1)}, \sigma_{-i}) + (1-\lambda) u_i(\sigma_i^{(2)}, \sigma_{-i}) \quad \forall \sigma_{-i}$

[by linearity of expectation].

$\leqslant \lambda u_i(\bar{\sigma}_i^{(1)}, \sigma_{-i}) + (1-\lambda) u_i(\bar{\sigma}_i^{(2)}, \sigma_{-i}) \quad \forall \sigma_{-i}$

$= u_i(\lambda\bar{\sigma}_i^{(1)} + (1-\lambda)\bar{\sigma}_i^{(2)}, \sigma_{-i}) \quad \forall \sigma_{-i}$

The inequality becomes strict both at $\tilde{\sigma}_{-i}^{(k)}$, $k=1,2$

Hence $\lambda\sigma_i^{(1)} + (1-\lambda)\sigma_i^{(2)}$ is also weakly dominated $\square$

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(c) In the following payoff matrix of a two-person zero-sum game, no pure strategy Nash equilibrium exists.

		Player 2	
		L	R
Player 1	T	a	b
	B	c	d

What inequalities must the numbers a, b, c, d satisfy? **2 points.**

Either $\min\{a,d\} > \max\{b,c\}$

or $\min\{b,c\} > \max\{a,d\}$

(d) Find the MSNE(s) of this game. **3 points.**

let $\left[\left(p(T),(1-p)(B)\right),\left(q(L),(1-q)(R)\right)\right]$ be the MSNE. Using the MSNE characterization theorem, we get

$$p = \frac{d-c}{a-b+d-c}, \quad q = \frac{d-b}{a-c+d-b}$$

[need to show the previous step too for full credit]

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Problem 4 (15 points). An IITB student entering the job market may be **highly skilled (HS)** or **moderately skilled (MS)**. Suppose that one-quarter of the existing students are HS, and the rest are MS. A soon-to graduate student, who knows whether or not s/he is skilled, has the option of crediting the course CS405/6001 before applying for a job. An employer seeking to fill a job opening cannot know whether a student applicant is highly/moderately skilled; all it knows from the transcripts is that the applicant either credited CS405/6001 (C) or did not (NC). The payoff an employer gets from hiring a worker depends solely on the skill of the hired worker (and not on his/her educational level), while the payoff to the student depends on what courses s/he chose to do at IITB, on his/her skills (because skill differences among students lead to different levels of enjoyment in their studies at IITB), and on whether or not s/he gets a job. These payoffs are described in the following tables (where the employer is the row player and a student is the column player, so that a payoff vector of (x, y) represents a payoff of x to the employer and y to the student).

Employer \ Student	C	NC
Hire (H)	$8, 4$	$8, 6$
Not Hire (NH)	$3, 1$	$3, 3$

Payoffs when the student is HS

Employer \ Student	C	NC
Hire (H)	$0, 2$	$0, 6$
Not Hire (NH)	$3, -3$	$3, 3$

Payoffs when the student is MS

(a) Describe the situation above as a Harsanyi incomplete information game. Recall from the classes that an incomplete information game is represented by $\langle N, (\Theta_i)_{i \in N}, P, (\Gamma_\theta)_{\theta \in (\times_{i \in N} \Theta_i)} \rangle$, where the notations have usual meaning. Describe what are the following quantities. **4 × 0.5 points.**

N : $\{1, 2\}$ 1 = Employer , 2 = Student.

Θ_i's, $i \in N$:

$\Theta_1 = \{\theta_1^1\}$ $\Theta_2 = \{HS, MS\}$

Singleton

P :

$P(\theta_1^1, HS) = \frac{1}{4}$, $P(\theta_1^1, MS) = 3/4$

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Γ_θ's, $\theta \in (\times_{i \in N} \Theta_i)$:

For both type profiles $N = \{1, 2\}$, $A_1 = \{H, NH\}$, $A_2 = \{C, NC\}$, utilities are as shown

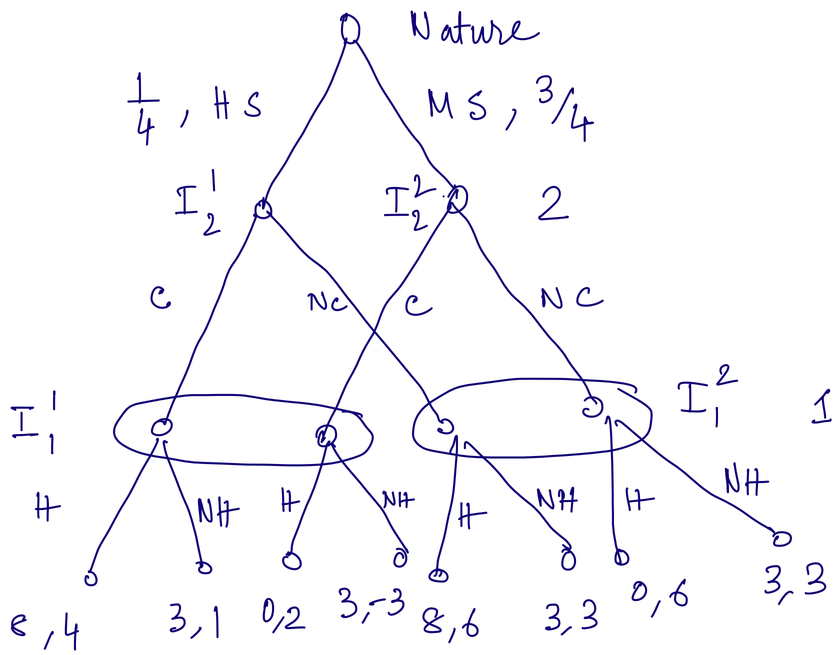
$\Gamma_{\theta_1', HS}$

	C	NC
H	8,4	8,6
NH	3,1	3,3

$\Gamma_{\theta_1', MS}$

	C	NC
H	0,2	0,6
NH	3,-3	3,3

(b) Represent the game using the **extensive form game** representation (whichever EFG representation you find most appropriate). **3 points.**



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(c) What are the pure strategies of each player? **3 points.**

Player 1 : always hire , (H,H) , hire only at C , (H,NH) , hire only at NC (NH,H) , never hire (NH,NH)

Player 2 : (NC,NC) , (NC,C) , (C,NC) , (C,C)

(d) Construct the game matrix with these strategies of the players? **4 points.**

	C, C	C, NC	NC, C	NC, NC
Always hire	2, 2.5	2, 5.5	2, 3	2, 6
Hire only C	2, 2.5	$\boxed{4.25, 3.25}$	0.75, 2.25	3, 3
Hire only NC	3, -2	0.75, 4.75	4.25, -0.75	2, 6
Never hire	3, -2	3, 2.5	3, -1.5	$\boxed{3, 3}$

(e) Find the **Bayesian equilibria in pure strategies** and explain your answer. You may use the result from the class: "*In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium*". **3 points.**

1. Never hire , student not credit in both types
2. Hire only C , student credits if HS
   not credit if MS

END OF QUESTION PAPER. GOOD LUCK!