CS 228 : Logic in Computer Science

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Handling Quantifiers : Done on Board

- $\exists x \forall y [x > y \vee \neg Q_a(x)] = \exists x [\neg \exists y [x \leqslant y \wedge Q_a(x)]]$
- Draw the automaton for $[x \leqslant y \wedge Q_a(x)]$
- Project out the y-row
- Determinize it, and complement it
- Fix the x-row : Intersect with $\begin{pmatrix} \Sigma \\ 0 \end{pmatrix}^* \begin{pmatrix} \Sigma \\ 1 \end{pmatrix} \begin{pmatrix} \Sigma \\ 0 \end{pmatrix}^*$
- Project the x-row

Points to Remember

- Given $\varphi(x_1, \ldots, x_n)$, construct automaton for atomic FO formulae over the extended alphabet $\Sigma \times \{0, 1\}^n$
- Intersect with the regular language where every x_i is assigned 1 exactly at one position
- Given a sentence $Q_{x_1} \ldots Q_{x_n} \varphi$, first construct the automaton for the formula $\varphi(x_1, \ldots, x_n)$
- Replace $\forall$ in terms of $\exists$

Points to Remember

- Given the automaton for $\varphi(x_1, \ldots, x_n)$, the automaton for $\exists x_i \varphi(x_1, \ldots, x_n)$ is obtained by projecting out the row of x_i
- This may result in an NFA
- Determinize it and complement it to get a DFA for $\neg\exists x_i \varphi(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$
- Intersect with the regular language where each of $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ are assigned 1 exactly at one position

The Computational Effort

Given NFAs A_1, A_2 each with atmost n states,

- The union has atmost $2n$ states
- Intersection has almost n^2 states
- The complement has atmost 2^n states
- The projection has atmost n states

Cost of determinization : $n+1$ to 2^n

- $\Sigma = \{0, 1\}$, languages where the n^{th} bit from the right is a 1.
- NFA has $n+1$ states.
- Size of corresponding DFA?

The Computational Effort

- $\psi = Q_1 \ldots Q_n \varphi$. If $Q_i = \exists$ for all i, then size of A_ψ is same the size of A_φ.
- When $Q_1 = \exists, Q_2 = \forall, \ldots$: each $\forall$ quantifier can create a 2^n blowup in automaton size
- Size of automaton is

$$2^{2^{2^{2^{2^{2^{n}}}}}}$$

  where the tower height k is the quantifier alternation size.
- This number is indeed a lower bound!

The Automaton-Logic Connection

Given any FO sentence φ, one can construct a DFA A_φ such that $L(\varphi) = L(A_\varphi)$.

Summary

- Given FO formula φ, build an automaton A_φ preserving the language
- Satisfiability of FO reduces to non-emptiness of underlying automaton

Satisfiability to Model Checking

- Satisfiability of FO over words
- Model checking
  - System abstracted as a model DFA/NFA A
  - Specification written in FO as formula φ
  - Does system model $\models \varphi$
  - $L(A) \subseteq L(\varphi)$?
  - $L(A) \cap \overline{L(\varphi)} = \emptyset$?
- FO-definable $\subseteq$ *REG*

Next directions

- Going back to general FO, and discuss the nontermination of the satisfiability checking procedure (Shawn Hedman)
- Inexpressiveness of FO : EF games (Straubing)
- MSO logic that can capture exactly regular languages (Wolfgang Thomas AAT)
- Temporal Logics (only LTL) (Baier-Katoen)
- Immediate next : MSO

Monadic Second Order Logic (MSO)

Symbols in MSO

Formulae of MSO, over signature τ, are sequences of symbols, where each symbol is one of the following:

- The symbol $\perp$ called false
- An element of the infinite set $\mathcal{V}_1 = \{x_1, x_2, \ldots\}$ of first order variables
- An element of the infinite set $\mathcal{V}_2 = \{X_1, X_2, \ldots\}$ of second order variables where each variable has arity 1 (new!)
- Constants and relations from τ
- The connectives $\rightarrow, \wedge, \vee, \neg$
- The quantifiers $\forall, \exists$
- Paranthesis

Well formed Formulae

A well-formed formula (wff) over a signature τ is inductively defined as follows:

- $\perp$ is a wff
- If t_1, t_2 are either variables or constants in τ, then $t_1 = t_2$ is a wff
- If t_i's are terms for $1 \leqslant i \leqslant k$ and R is a k-ary relation symbol in τ, then $R(t_1, \ldots, t_k)$ is a wff
- If t is either a first order variable or a constant, X is a second order variable, then $X(t)$ is a wff
- If φ and ψ are wff, then $\varphi \to \psi, \varphi \wedge \psi, \varphi \vee \psi$ and $\neg\varphi$ are wff
- If φ is a wff and x is a first order variable, then $(\forall x)\varphi$ and $(\exists x)\varphi$ are wff
- If φ is a wff and X is a second order variable, then $(\forall X)\varphi$ and $(\exists X)\varphi$ are wff

Free and Bound Variables

- Free, Bound Variables and Scope same as in FO
- In a wff $\varphi = \forall X \psi$ or $\exists X \psi$ every occurrence of X in ψ is bound
- A sentence is a formula with no free first order and second order variables

Assignments on τ-structures

Assignments

For a τ-structure $\mathcal{A}$, an assignment over $\mathcal{A}$ is a pair of functions (α_1, α_2), where

- $\alpha_1 : \mathcal{V}_1 \to u(\mathcal{A})$ assigns every first order variable $x \in \mathcal{V}_1$ a value $\alpha_1(x) \in u(\mathcal{A})$. If t is a constant symbol c, then $\alpha_1(t)$ is $c^{\mathcal{A}}$.
- $\alpha_2 : \mathcal{V}_2 \to 2^{u(\mathcal{A})}$ assigns to every second order variable $X \in \mathcal{V}_2$, $\alpha_2(X) \subseteq u(\mathcal{A})$.

Binding on a Variable

For an assignment $\alpha = (\alpha_1, \alpha_2)$ over $\mathcal{A}$, and $x \in \mathcal{V}_i$, $i = 1, 2$, $\alpha_i[x \mapsto a]$ is the assignment $\alpha_i[x \mapsto a](y) = \begin{cases} \alpha_i(y), y \neq x, \\ a, y = x \end{cases}$

Satisfaction

We define the relation $\mathcal{A} \models_\alpha \varphi$ (read as φ is true in $\mathcal{A}$ under the assignment α) inductively:

- $\mathcal{A} \not\models_\alpha \bot$
- $\mathcal{A} \models_\alpha t_1 = t_2$ iff $\alpha_1(t_1) = \alpha_1(t_2)$
- $\mathcal{A} \models_\alpha R(t_1, \ldots, t_k)$ iff $(\alpha_1(t_1), \ldots, \alpha_1(t_k)) \in R^{\mathcal{A}}$
- $\mathcal{A} \models_\alpha X(t)$ iff $\alpha_1(t) \in \alpha_2(X)$ (new)
- $\mathcal{A} \models_\alpha (\varphi \to \psi)$ iff $\mathcal{A} \not\models_\alpha \varphi$ or $\mathcal{A} \models_\alpha \psi$
- $\mathcal{A} \models_\alpha (\forall x)\varphi$ iff for every $a \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a]} \varphi$
- $\mathcal{A} \models_\alpha (\forall X)\varphi$ iff for every $S \subseteq u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[X \mapsto S]} \varphi$ (new)

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Recall the signature for the graph structure, $\tau = \{E\}$

- The graph is 3-colorable

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$$\exists X \exists Y \exists Z (\forall x [X(x) \vee Y(x) \vee Z(x)] \wedge$$

$$\forall x \forall y [E(x,y) \rightarrow \{\neg(X(x) \wedge X(y)) \wedge \neg(Y(x) \wedge Y(y)) \wedge \neg(Z(x) \wedge Z(y))\}])$$

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$$\exists I\{\forall x \forall y[(\neg(x = y) \wedge I(x) \wedge I(y)) \rightarrow \neg E(x, y)] \wedge$$

$$\exists x_1 \ldots x_k[\bigwedge_{i \neq j} \neg(x_i = x_j) \wedge \bigwedge_{i} I(x_i)]\}$$

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$$\wedge \forall x[(E(x) \vee O(x)) \wedge \neg(E(x) \wedge O(x))]$$

$$\wedge \forall x \forall y[S(x, y) \wedge O(x) \rightarrow E(y)]$$
$$\wedge \forall x \forall y[S(x, y) \wedge E(x) \rightarrow O(y)]\}$$

MSO on Words : Satisfiability

MSO on Words

- Signature $\tau = (Q_\Sigma, <, S)$, domain or universe = set of positions of a word
- MSO over words: Atomic formulae

$$X(x) | Q_\Sigma(x) | x = y | x < y | S(x, y)$$

- Given a MSO sentence φ, $L(\varphi)$ defined as usual
- A language $L \subseteq \Sigma^*$ is MSO definable iff there is an MSO formula φ such that $L = L(\varphi)$
- Given an MSO sentence φ, is it satisfiable/valid?

MSO Expressiveness

- Clearly, *FO* $\subseteq$ *MSO*
- *MSO*=Regular