CS 228 : Logic in Computer Science

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Handling Quantifiers : Done on Board

- $\exists x \forall y [x > y \vee \neg Q_a(x)] = \exists x [\neg \exists y [x \leqslant y \wedge Q_a(x)]]$
- Draw the automaton for $[x \leqslant y \wedge Q_a(x)]$
- Project out the y-row
- Determinize it, and complement it
- Fix the x-row : Intersect with $\begin{pmatrix} \Sigma \\ 0 \end{pmatrix}^* \begin{pmatrix} \Sigma \\ 1 \end{pmatrix} \begin{pmatrix} \Sigma \\ 0 \end{pmatrix}^*$
- Project the x-row

Points to Remember

- Given $\varphi(x_1, \ldots, x_n)$, construct automaton for atomic FO formulae over the extended alphabet $\Sigma \times \{0, 1\}^n$
- Intersect with the regular language where every x_i is assigned 1 exactly at one position
- Given a sentence $Q_{x_1} \ldots Q_{x_n}\varphi$, first construct the automaton for the formula $\varphi(x_1, \ldots, x_n)$
- Replace $\forall$ in terms of $\exists$

Points to Remember

- Given the automaton for $\varphi(x_1, \ldots, x_n)$, the automaton for $\exists x_i \varphi(x_1, \ldots, x_n)$ is obtained by projecting out the row of x_i
- This may result in an NFA
- Determinize it and complement it to get a DFA for $\neg\exists x_i \varphi(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$
- Intersect with the regular language where each of $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ are assigned 1 exactly at one position

The Computational Effort

Given NFAs A_1, A_2 each with atmost n states,

- The union has atmost $2n$ states
- Intersection has almost n^2 states
- The complement has atmost 2^n states
- The projection has atmost n states

Cost of determinization : $n+1$ to 2^n

- $\Sigma = \{0,1\}$, languages where the n^{th} bit from the right is a 1.
- NFA has $n+1$ states.
- Size of corresponding DFA?

The Computational Effort

- $\psi = Q_1 \ldots Q_n \varphi$. If $Q_i = \exists$ for all i, then size of A_ψ is same the size of A_φ.
- When $Q_1 = \exists, Q_2 = \forall, \ldots$: each $\forall$ quantifier can create a 2^n blowup in automaton size
- Size of automaton is

$$2^{2^{2^{2^{2^{2^{n}}}}}}$$

  where the tower height k is the quantifier alternation size.
- This number is indeed a lower bound!

The Automaton-Logic Connection

Given any FO sentence φ, one can construct a DFA A_φ such that $L(\varphi) = L(A_\varphi)$.

Summary

- Given FO formula φ, build an automaton A_φ preserving the language
- Satisfiability of FO reduces to non-emptiness of underlying automaton

Satisfiability to Model Checking

- Satisfiability of FO over words
- Model checking
  - System abstracted as a model DFA/NFA A
  - Specification written in FO as formula φ
  - Does system model $\models \varphi$
  - $L(A) \subseteq L(\varphi)$?
  - $L(A) \cap \overline{L(\varphi)} = \emptyset$?
- FO-definable $\subseteq$ *REG*

Next directions

- Going back to general FO, and discuss the nontermination of the satisfiability checking procedure (Shawn Hedman)
- Inexpressiveness of FO : EF games (Straubing)
- MSO logic that can capture exactly regular languages (Wolfgang Thomas AAT)
- Temporal Logics (only LTL) (Baier-Katoen)
- Immediate next : MSO

Monadic Second Order Logic (MSO)

Symbols in MSO

Formulae of MSO, over signature τ, are sequences of symbols, where each symbol is one of the following:

- The symbol $\perp$ called false
- An element of the infinite set $\mathcal{V}_1 = \{x_1, x_2, \ldots\}$ of first order variables
- An element of the infinite set $\mathcal{V}_2 = \{X_1, X_2, \ldots\}$ of second order variables where each variable has arity 1 (new!)
- Constants and relations from τ
- The connectives $\to, \wedge, \vee, \neg$
- The quantifiers $\forall, \exists$
- Paranthesis

Well formed Formulae

A well-formed formula (wff) over a signature τ is inductively defined as follows:

- $\bot$ is a wff
- If t_1, t_2 are either variables or constants in τ, then $t_1 = t_2$ is a wff
- If t_i's are terms for $1 \leqslant i \leqslant k$ and R is a k-ary relation symbol in τ, then $R(t_1, \ldots, t_k)$ is a wff
- If t is either a first order variable or a constant, X is a second order variable, then $X(t)$ is a wff
- If φ and ψ are wff, then $\varphi \to \psi, \varphi \wedge \psi, \varphi \vee \psi$ and $\neg\varphi$ are wff
- If φ is a wff and x is a first order variable, then $(\forall x)\varphi$ and $(\exists x)\varphi$ are wff
- If φ is a wff and X is a second order variable, then $(\forall X)\varphi$ and $(\exists X)\varphi$ are wff