SC639-MSC : Sequence and Series in Normed Linear Spaces

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Prerequisites : Metric Spaces and Normed spaces

A metric defines a notion of distance between points in a set (or between points in a set with another set or even between sets). Let X be a set, a metric is a mapping

$$d : X \times X \to \mathbb{R}_{\geq 0}$$

$$X \times X \ni (f, g) \mapsto d(f, g) \in \mathbb{R}_{\geq 0}$$

such that it satisfies

- (M1) $d(f, g) \geq 0$, forall $f, g \in X$ and $d(f, g) = 0 \iff f = g$.
- (M2) $d(f, g) = d(g, f)$ forall $f, g \in X$.
- (M3) $d(f, g) \leq d(f, h) + d(h, g)$ forall $f, g, h \in X$ (Triangle Inequality)

And the pair (X, d) is called a metric space.

- Next we will recall the definition of a "Norm" and a "Normed space" and see how a metric can be naturally "induced" from a norm.

Normed Spaces

Norm provides a notion of length of vector in vector space. Let $(X, \mathbb{K}, +, \cdot)$ be a vector space over the field $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$, a norm is a mapping defined as

$$||\cdot|| : X \to \mathbb{R}_{\geq 0}$$

$$X \ni x \mapsto ||x|| \in \mathbb{R}_{\geq 0}$$

which satisfies

- (N1) $||x|| \geq 0$ forall $x \in X$ and $||x|| = 0 \iff x = 0$
- (N2) $||\alpha x|| = |\alpha|\,||x||$ forall $x \in X, \alpha \in \mathbb{K}$
- (N3) $||x + y|| \leq ||x|| + ||y||$ forall $x, y \in X$ (Triangle Inequality)

The pair $(X, ||\cdot||)$ is called a normed vector space.

Theorem 1

Let $(X, ||\cdot||)$ is a normed vector space. Then the mapping

$$(X \times X) \ni (x, y) \mapsto d(x, y) := ||x - y|| \in \mathbb{R}_{\geq 0} \tag{1}$$

defines a metric on X, and (X, d) is a metric space.

Normed spaces (Contd)

Proof of Theorem 1 :

We just need to check properties (M1) to (M3).

- (M1) $||x-y|| \geq 0$ by property of the norm, and $||x-y|| = 0 \implies x = y$
- (M2) $||x-y|| = |(-1)|\,||y-x|| = ||y-x||$
- (M3) let $x, y, z \in X$ then
  $d(x,z) = ||x-z|| = ||x-y+y-z|| \leq ||x-y|| + ||y-z|| = d(x,y) + d(y,z)$

A Remark!

Note the metric defined in (1) satisfies the following properties :

$$(P1)\ \text{(Translation invariance)}\ d(x+w, y+w) = d(x,y)$$

$$(P2)\ \text{(Homogeneity)}\ d(tx, ty) = |t|d(x,y), \forall t \in \mathbb{K}$$

Note : So every normed space is a metric space if we define the metric as (1) but the converse is not true, it's true only when the metric satisfies the above two properties (**Exercise 1 :** Can you think of an example?)

Examples

Ex 1 : Euclidean space $(\mathbb{R}^n, ||\cdot||)$ of cector $x = (x_i)_{i=1}^n$ with the metric

$$d(x,y) = ||x-y|| := \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

Exercise 2 : Show this is indeed a norm.
Ex 2 : Define the space

$$\mathbb{L}^p(\Omega) := \left\{ f : \Omega \to \mathbb{K} : \int_\Omega |f|^p < +\infty \right\}$$

with the norm

$$\mathbb{L}^p \ni f \mapsto ||f||_p := \left(\int_\Omega |f|^p \mathrm{d}x \right)^{1/p} \in \mathbb{R}_{\geq 0} \tag{2}$$

and the induced metric

$$\mathbb{L}^p \times \mathbb{L}^p \ni (f,g) \mapsto d(f,g) := ||f-g||_p \in \mathbb{R}_{\geq 0}$$

Examples

Ex 3 : Define the sequence space with index $I = \mathbb{N}$

$$\ell^p(I) := \left\{ (x_i)_{i\in I} : x_i \in \mathbb{K},\ \text{forall } i \in I \text{ and } \sum_{i\in I} |x_i|^p < +\infty \right\}$$

with the norm

$$\ell^p \ni x \mapsto ||x||_{\ell_p} := \left(\sum_{i\in I} |x_i|^p \right)^{1/p} \in \mathbb{R}_{\geq 0} \tag{3}$$

and the induced metric

$$\ell^p \times \ell^p \ni (x, y) \mapsto d(x, y) := ||x - y||_{\ell_p} \in \mathbb{R}_{\geq 0}$$

Exercise 3 : At this stage proving (2) and (3) indeed are valid norms is non-trivial, so take $p = 1, 2$ and ∞ and proceed.
Exercise 4 : Prove that $\mathbb{L}^2[a, b]$ and $\ell^2(\mathbb{N})$ are linear vector spaces, where $-\infty < a < b < +\infty$.

Examples

Ex 4 : Consider the space of continuous functions

$$\mathcal{C}([a,b],\mathbb{K}) := \{f : [a,b] \to \mathbb{K} : \text{f is continuous}\} \tag{4}$$

with the norm

$$\mathcal{C}[a,b] \ni f \mapsto ||f||_\infty := \sup_{a \le x \le b} |f(x)| \in \mathbb{R}_{\ge 0} \tag{5}$$

$$(f,g) \mapsto d(f,g) := \sup_{a \le x \le b} |f-g|(x) =: ||f-g||_\infty \in \mathbb{R}_{\ge 0}$$

Ex. 5 : Consider the space $\mathcal{C}^1[a,b]$ i.e space of all functions $f(x)$ defined on $[a,b]$ which are continuous and have continuous first derivatives. We define norm on $\mathcal{C}^1[a,b]$ by :

$$\mathcal{C}^1[a,b] \ni f \mapsto ||f||_{\mathcal{C}^1} := \sup_{x \in [a,b]} |f(x)| + \sup_{x \in [a,b]} |f'(x)| \tag{6}$$

Frame Title

A Remark on Ex. 5

What (6) really says is: two functions in $\mathcal{C}^1[a,b]$ are regarded as close together if both the functions themselves and their first derivatives are close together since

$$||f-g||_{\mathcal{C}^1} < \epsilon$$

implies that

$$|f(x)-g(x)| < \epsilon \text{ and } |f'(x)-g'(x)| < \epsilon$$

for all $a \leq x \leq b$

Ex. 6 : The space $\mathcal{C}^k[a,b]$

The space $\mathcal{C}^k[a,b]$, as you can already guess is the space of continuous functions which have continuous derivatives upto order $k \in \mathbb{Z}$ inclusive, and the norm :

$$\mathcal{C}^k[a,b] \ni f \mapsto ||f||_{\mathcal{C}^k} := \sum_{i=0}^{k} \sup_{x\in[a,b]} |f^i(x)| \in \mathbb{R} \tag{7}$$

Examples

Exercise 5 : (British railroad metric) consider the mertric space (X, d) s.t

$$X \times X \ni (x, y) \mapsto d(x, y) := \begin{cases} 0 & x = y \\ |x| + |y| & x \neq y \end{cases}$$

Do you think that this is an induced metric (i.e genrated form some norm)?
Hint: Try to see if property $(P1)$ and $(P2)$ checks out.
Before jumping to the main theme, we revise some definitions of **supremum and infimum** of subsets of $\mathbb{R}$ and present some examples for clarity.

Supremum and Infimum

Definition : Supremum

Consider a subset E of $\mathbb{R}$.

- E is bounded above if there exists a number $\beta \in \mathbb{R}$ such that
$$x \leq \beta \ , \ \forall x \in E \tag{8}$$
.
- If E is bounded above, the smallest number β satisfying (8) is called the supremum of E, and is written using one of the following three equivalent notations:
$$\sup E = \sup_{x \in E} x = \sup \{x | x \in E\} \, . \tag{9}$$
- For a set E that is not bounded above, we put $\sup E = \infty$.

Definition : Infimum

Consider a subset E of $\mathbb{R}$.

- E is bounded below if there exists a number $\alpha \in \mathbb{R}$ such that

$$\alpha \leq x \ , \ \forall x \in E \tag{10}$$

.

- If E is bounded below, the largest number α satisfying (10) is called the infimum of E, and is written using one of the following three equivalent notations:

$$\inf E = \inf_{x \in E} x = \inf \{x | x \in E\} \, . \tag{11}$$

- For a set E that is not bounded above, we put $\inf E = -\infty$.

In particular case, when $f : A \to \mathbb{R}$, we consider the set

$$E := \{f(x) : x \in A\}$$

Where A is called "Domain" and E is called "Range" of the function f.
Now

$$\sup E = \sup_{x \in A} f(x) = \sup \{f(x) : x \in A\} \tag{12}$$

A Warning

A warning is in order. In general, the number $\sup_{x \in A} f(x)$ does not need to be a function value for the function f; that is, there might not exist an $x_0 \in A$ such that

$$f(x_0) = \sup_{x \in A} f(x) \tag{13}$$

In case an $x_0 \in A$ satisfying (13) exists, we write

$$\max_{x \in A} f(x) = \sup_{x \in A} f(x) \tag{14}$$

Supremum and Infimum : Two Examples

Ex. 7 : Consider the function

$$[0,2] \ni x \mapsto f(x) := x^2 \in \mathbb{R}$$

It's clear that

$$\sup_{x\in[0,2]} f(x) = f(2) = 4$$

The supremum value is attained and we may write

$$\sup_{x\in[0,2]} f(x) = \max_{x\in[0,2]} f(x)$$

Ex. 8 : Now consider

$$[0,2[\ni x \mapsto f(x) := x^2 \in \mathbb{R}$$

Then

$$\sup_{x\in[0,2[} f(x) = 4$$

But the supremum is not attained

A Few Remarks

- Note that the supremum and the infimum are *unique* if they exists. For example if M_1 and M_2 are two least upper bounds for set E, then the definition implies that $M_1 \leq M_2$ and $M_2 \leq M_1$ so $M_1 = M_2$.
- The existence of *sup* of every set bounded above, and the existence of *inf* of every set bounded below, is a consequence of *completeness* of $\mathbb{R}$, in fact it is equivalent to it.

Some definitions

Open and closed Balls

Let (X, d) be a metric space. For $x \in X$ and $r > 0$, we let

$$\mathcal{B}(x,r) := \{y \in X : d(x,y) := \|x-y\| < r\} \tag{15}$$

Denotes the **open ball** of radius r centered at x. Similarly, the closed ball in the metric space (X, d) is defined as

$$\overline{\mathcal{B}}(x,r) := \{y \in X : d(x,y) := \|x-y\| \leq r\} \tag{16}$$

Open and closed sets

A set $A \subset (X, d)$ is said to be **open** if around each point $x \in A$ one can find $r > 0$ such that the open ball $\mathcal{B}(x, r)$ is completely contained in A i.e $\mathcal{B}(x, r) \subseteq A$.
Sets whose complement are open are called **closed** sets. We urge the reader to check the fact that : (a) ***open balls are open sets (b) closed balls are closed sets***

Alternative characterization of open and closed sets

- Let (X, d) be a metric space, and $A \subseteq X$. A point $x \in A$ is an **interior point** of A if $\mathcal{B}(x, r) \subseteq A$ for some $r > 0$. The **interior** of A which will be denoted by $\operatorname{int} A$, is the set of all interior points of A. Then, we say that the set A is ***open*** if $\operatorname{int} A = A$.
- $x \in X$ (possibly, $x \notin A$) is a **limit/accumulation/closure point** of A if
$$(\mathcal{B}(x, r) \setminus \{x\}) \cap A \neq \emptyset \quad \forall r > 0$$
Another way to say the same thing is : x is limit point of A if *every* neighbourhood of x contains a point $y \neq x$ such that $y \in A$. Then, we say the set A is **closed** if A *contains all of it's limit points.*

An Example

Let's see how the open balls looks like for example is $\mathbb{R}^2$, with the metric $d_p((x,y),(x',y')) := \{|x-x'|^p + |y-y'|^p\}^{1/p} =: \|(x,y)-(x',y')\|_p$. Specifically take $(x',y') = (0,0) \in \mathbb{R}^2$, then the open ball centered at origin with radius say 1, for the metric d_p is the set

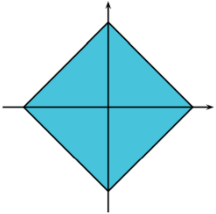
$$\begin{aligned} \mathcal{B}_{d_p}(\mathbf{0},1) &:= \left\{(x,y) \in \mathbb{R}^2 : d_p\left((x,y),(0,0)\right) < 1\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 : \sqrt[p]{|x|^p + |y|^p} < 1\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 : |x|^p + |y|^p < 1\right\} \end{aligned}$$

See Fig.(18), with different values of p's.

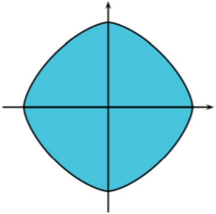
Exercise 5.1 How does open balls looks like in the metric space $(\mathcal{C}[a,b], d_\infty(f,g))$. Where $d_\infty(f,g) := \sup_{x\in[a,b]} |f(x)-g(x)|$ with the associated norm as you already know $\|f\|_\infty := \sup_{x\in[a,b]} |f(x)|$.

Open balls for various values of p

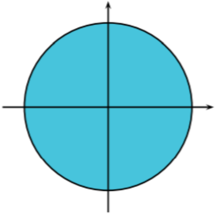
The diagram below shows this region is for various values of p.



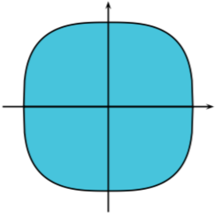
$\{\,(x,y)\mid |x|+|y|<1\,\}$
(the case $p=1$)



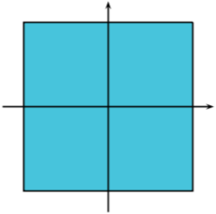
$\{\,(x,y)\mid |x|^{3/2}+|y|^{3/2}<1\,\}$
(the case $p=3/2$)



$\{\,(x,y)\mid |x|^{2}+|y|^{2}<1\,\}$
(the case $p=2$)



$\{\,(x,y)\mid |x|^{3}+|y|^{3}<1\,\}$
(the case $p=3$)



$\{\,(x,y)\mid \max(|x|,|y|)<1\,\}$
(the case $p=\infty$)

Sequences in normed vector spaces

Definition : Sequence

Consider a normed vector space $(X, ||\cdot||)$. A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is a mapping $f : \mathbb{N} \to X$ defined as $\mathbb{N} \ni n \mapsto f(n) := x_n \in X$. If the "range" of the sequence in real numbers i.e $X := \mathbb{R}$ then the explicit mapping is $f : \mathbb{N} \to \mathbb{R}$, we call it a sequence of real numbers.

Convergence

A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is said to converge to a limit point x if

$$\forall \epsilon > 0 \ : \exists N(\epsilon) \in \mathbb{N} \ : \ \forall n \geq N(\epsilon) \ : \ ||x_n - x|| < \epsilon \tag{17}$$

Note: the integer N depends on ϵ, since smaller ϵ generally requires larger N. One also writes (17) as $x_n \to x$ as $n \to \infty$ or $\lim\limits_{n\to\infty} x_n = x$

Exercise 6 : Consider (1) $\lim\limits_{n\to\infty} n \sin\left(\frac{1}{n}\right) = 1$ (2) $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.
Prove convergence using definition (17)

A remark on previous definition of convergence

Let us make an remark on converging sequences in the context of metric spaces, and normed spaces. We start with a metric space (X, d). We take a sequence $(x_n) \subset X$, and convergence in the sense of metric is :

$$\forall\, \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : x_n \in \mathcal{B}(x, \epsilon)$$

And this is equivalent to (as per definition of open balls)

$$\forall\, \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : d(x_n, x) < \epsilon$$

And in case of normed space, using equivalence of norm and metric we may write

$$\forall \epsilon > 0 \; : \exists N \in \mathbb{N} \; : \; \forall n \geq N(\epsilon) \; : \; ||x_n - x|| < \epsilon$$

These three establishes an equivalence of various definitions, and we may use any of these freely.

An illustrative example

Example

Consider the Sequence

$$\mathbb{N} \ni n \mapsto x_n := \frac{n+1}{n} \in \mathbb{R}$$

Does the sequence converge? To which value? (Note : Here we're talking about the metric space $(\mathbb{R}, d(x,y) := |x-y|)$ or alternatively the normed space $(\mathbb{R}, |\cdot|)$.

Scratch work : Based on your intuition, you already have a candidate right? It's 1. But why? And how to show this using the definition? We do some scratch work first before writing the proof. Our **claim** : $x_n \to 1$. Note that from Definition (17), we would want to bound the expression

$$|x_n - 1| = \left|\frac{n+1}{n} - 1\right| = \left|\frac{1}{n}\right|$$

The definition says, you're given an ϵ, and your task is to find an $N \in \mathbb{N}$, s.t $|\frac{1}{n}| < \epsilon$.

Frame Title

We're still doing the scratch work

- From the above expression you can take the absolute value off (as it's always positive) and then we have $n > \frac{1}{\epsilon}$.
- Now, can you guess if there's an index N, beyond which this condition holds? what's N?
- How about N be anything bigger than $\frac{1}{\epsilon}$, then things works out. So, here's the proof :

Proof

Let $\epsilon > 0$, choose $N := \left(\left\lceil \frac{1}{\epsilon} \right\rceil + 1\right)$ [1] (Note that, we're adding this one because we want strict inequalities). If $n \geq N$ then also $n > \frac{1}{\epsilon}$, hence $\frac{1}{n} < \epsilon$. Then $|x_n - 1| = \left|\frac{1}{n}\right| < \epsilon$, and we're done. ■

[1] $\lceil \cdot \rceil$ is the ceiling function, for example $\lceil \pi \rceil = 4$, $\lceil 1.006 \rceil = 2$, $\lceil -10.26 \rceil = -10$

Some results

Proposition 1

If a sequence converges it's limit is unique.

Proof : Let $(X, ||\cdot||)$ is a normed vector space, assume $X \ni x_n \to x$ and $X \ni x_n \to y$. Then

$$||x - y|| = ||x - x_n + x_n - y|| \leq ||x - x_n|| + ||x_n - y|| \to 0$$

Thus $x = y$, as desired. ■

A Warning !!

Note that the concept of convergence is always relative to a given metric space/Normed vector space. The assertion "The sequence $(x_n)_{n\in\mathbb{N}}$ is convergent" to be meaningful requires a metric/Norm and a space X where this metric is defined. When the reference space is in doubt, we may say for clarification that $(x_n)_{n\in\mathbb{N}}$ is convergent in X. This remark is not as trivial as it sounds, see the following example.

Boundedness of a sequence

Ex. 9 : Consider the space $(X, ||\cdot||_X)$, take a sequence $x_n := \frac{1}{n}, n \in \mathbb{N}$, does it converge? Well it depends :

- Yes, if $X := \mathbb{R}$ with the norm $||x||_{\mathbb{R}} := |x|$, one has $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$
- No, if $X :=]0,1]$ with $||x||_{]0,1]} := |x|$
- Yes, if $X :=]0,1]$ with $]0,1]\times]0,1] \ni (x,y) \mapsto d(x,y) := \left|e^{2\pi i x} - e^{2\pi i y}\right|$. The sequence converges to one i.e $\lim\limits_{n\to\infty} x_n = 1$

Boundedness

A subset $A \subset X$ is **bounded** if $\exists M > 0$ s.t $||x|| \leq M$ forall $x \in A$. Similarly a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is **bounded** iff $\exists K \in \mathbb{R}$ such that $\forall n \in \mathbb{N} : ||x_n|| \leq K$ or equivalently $\sup_{n\in\mathbb{N}} ||x_n|| < +\infty$.

Boundedness (Continued)

Proposition 2

Let $(X, ||\cdot||)$ is a normed vector space, let x_n be a sequaence in X such that $x_n \to x$ as $n \to \infty$, then x_n is bounded.

Proof : In order to prove boundedness we want to find $M \in \mathbb{R}$ s.t $\forall n \in \mathbb{N} : ||x_n - x|| \leq M$. Since x_n is convergent it is true that : $\forall \epsilon > 0 : \exists N \in \mathbb{N} : n \geq N \implies ||x_n - x|| < \epsilon$. In particular this is true for $\epsilon = 1$, that is $\exists N_1 \in \mathbb{N} : \forall n \geq N_1 : ||x_n - x|| < 1$. Now if we set

$$K := \max \{||x_1 - x||, ||x_2 - x||, \ldots, ||x_{N_1} - x||, 1\}$$

the claim follows. ■

(New) Another Proof for Proposition 2

In case some reader finds the proof of the last proposition difficult to grasp, we present a easier version to make the logic clear.
So we have a sequence $(x_n)_{n\in\mathbb{N}} \subset (\mathbb{R}, |\cdot|)$ which converges to x. Let $\varepsilon = 1$, then there exists $N \in \mathbb{N}$ such that whenever $n \geq N$ we have $|x_n - x| < 1$.
Now using triangle inequality we get $|x_n| \leq |x_n - x| + |x| < 1 + |x|$ for all $n \geq N$. Now define $K := \max\{|x_1|, |x_2|, \ldots, |x_N|, 1 + |x|\}$ and clearly, $|x_n| \leq K$ for all $n \in \mathbb{N}$, and so $(x_n)_{n\in\mathbb{N}}$ is bounded. ■

Continuity

Next, we introduce the crucial notion of continuity of function $f : \mathbb{R}^n \to \mathbb{R}^m$

Continuity

A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **continuous** at $x_0 \in \mathbb{R}^n$, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $y \in \mathcal{B}(x_0, \delta) \implies f(y) \in \mathcal{B}(f(x_0), \epsilon)$ Note δ may depend on both x_0 and ϵ, and the map f is said to be continuous if it is continuous for every $x_0 \in \mathbb{R}^n$ (as our choice of x_0 was arbitrary)

Note that these definitions can be easily extended to any normed spaces without any effort. In the next section we present the NVS-version of continuity, which is basically the same as the previous definition

Continuity

Continuity in NVS : Definition 1

Let X and Y be normed spaces over the field $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$, the operator

$$A : M \subseteq X \to Y$$

is called sequentially continuous iff for each sequence $(u_n) \in M$

$$\lim_{n\to\infty} u_n = u \text{ with } u \in M \implies \lim_{n\to\infty} Au_n = Au \tag{18}$$

Continuity in NVS : Definition 2

The operator A is called continuous iff for each $u \in \mathbb{M}$ and each ϵ there exist $\delta(\epsilon, u) > 0$ such that

$$||u - v|| < \delta(\epsilon, u) \text{ and } v \in M \implies ||Av - Au|| < \epsilon \tag{19}$$

Some Continuous operators

A Remark

- In these definitions if $\delta(\epsilon, u) = \delta(\epsilon)$ i.e independent of u then A is called **uniformly continuous**.
- Definition (18) and (19) are equivalent i.e

$$A \text{ is continuous} \iff A \text{ is sequentially continuous}$$

Proposition 3 : Continuity of Operations

Let $(X, ||\cdot||)$ be a NVS, and $\alpha_n \in \mathbb{K}$ is a sequence s.t $\lim_{n\to\infty} \alpha_n = \alpha$, and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$. Then

- prove that $\forall\ x, y \in X\ ||x - y|| \geq \big|\, ||x|| - ||y||\, \big|$ (Reverse Triangle Ineq.)
- Prove $||\cdot|| : X \to \mathbb{R}_{\geq 0}$ is a continuous map.
- $\lim\limits_{n\to\infty} x_n + y_n = x + y$ (+ operation is continuous)
- $\lim\limits_{n\to\infty} \alpha_n x_n = \alpha x$

Continuous operators

Proof

- (a)Using triangular inequality (M3) and (M2)

$$(T1) \qquad ||x|| = ||x - y + y|| \leq ||x - y|| + ||y||$$

  and

$$(T2) \quad ||y|| \leq ||y - x|| + ||x|| = ||x - y|| + ||x||$$

  (T1) and (T2) together proves the claim.
- (b) Define

$$f(\cdot) := || \cdot || : X \to \mathbb{R}_{\geq 0}$$

  Let $\epsilon > 0$ then forall $x, y \in X$ and $||x - y|| < \delta$ we need to show $|f(x) - f(y)| < \epsilon$, to this end choose $\delta := \epsilon$ then using result from (a)

$$\big| ||x|| - ||y|| \big| \leq ||x - y|| < \epsilon$$

  as desired.

Continuous operators

Proof continued

- (c) We have $X \ni x_n \to x$ and $X \ni y_n \to y$, then $||x_n - x|| \to 0$ as $n \to \infty$ and $||y_n - y|| \to 0$ as $n \to \infty$, so we have

$$\|(x_n + y_n) - (x + y)\| \leq \|x_n - x\| + \|y_n - y\| \to 0 \text{ as } n \to \infty \tag{20}$$

$$(x_n + y_n) \to (x + y) \ \text{ as } n \to \infty$$

So $+ : X \times X \to X$ is continuous, according to definition (18)

- (d) Notice : $||x_n - x|| \to 0$ as $n \to \infty$ and $||\alpha_n - \alpha|| \to 0$ as $n \to \infty$

$$\begin{aligned}
\|(\alpha_n x_n - \alpha\, x)\| &\leq \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha\, x\| \\
&\leq \|\alpha_n x_n - \alpha_n x\| + \|\alpha_n x - \alpha\, x\| \\
&= |\alpha_n| \|x_n - x\| + \|x\| |\alpha_n - \alpha| \\
&\leq K \|x_n - x\| + \|x\| \|\alpha_n - \alpha\| \\
&\to 0 \text{ as } n \to \infty \quad \blacksquare
\end{aligned} \tag{21}$$

A Remark

A Remark

As a result of (b), for a sequence $x_n : \mathbb{N} \to X$ we have:

$$\text{If } x_n \to x \text{ as } n \to \infty \implies ||x_n|| \to ||x||$$

which can be written as

$$\lim_{n\to\infty} x_n = x \implies \lim_{n\to\infty} ||x_n|| = ||x|| \underbrace{=}_{\star} ||\lim_{n\to\infty} x_n||$$

$\star$: Because $||\cdot||$ is continuous limit operation commutes with norm.

Theorem 2

Let X be a NLS, then the following results holds :

- (a) If $x_n \to x$ and $k \in \mathbb{R}$ then $(kx_n) \to kx$ as $n \to \infty$.
- (b) If $0 \neq x_n \to x \neq 0$, $\forall n \in \mathbb{N}$, then $\left(\frac{1}{x_n}\right) \to \left(\frac{1}{x}\right)$

Exercise 7 : Prove Theorem 2

Sub-sequences

Defintion (Sub-sequence)

Let (x_n) be a sequence, where $x_n = f(n)$ and $f : \mathbb{N} \to X$. A sequence (y_k) where $y_k = g(k)$ and $g : \mathbb{N} \to X$ is a sub-sequence of (x_n) if there is a strictly increasing function $\phi : \mathbb{N} \to \mathbb{N}$ such that $g = f \circ \phi$. In this case we write $\phi(k) = n_k$ and $y_k = x_{n_k}$

Ex : 10 A sub-sequence of $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ is the sequence $\left(\frac{1}{k^2}\right)_{k\in\mathbb{N}}$

Proposition 4

A sequence converges iff every subsequence converges.

The forward direction needs a bit of work. Let $\mathbb{N} \ni n \mapsto (x_n) \to x$, then for all $\varepsilon > 0$, there exists a $N := N(\varepsilon) \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ whenever $n \geq N$ or equivalently $\mathcal{B}(x, \varepsilon) \ni x_n : n \geq N$. Now by definition of subsequence $n_k \geq k$ for all $k \in \mathbb{N}$, so $n_k \geq N$ whenever $k \geq N$ which implies $x_{n_k} \in \mathcal{B}(x, \varepsilon) \implies d(x_{n_k}, x) < \varepsilon$ for all $n_k \geq N$. This proves the forward direction. The reverse direction is trivial because a sequence is a subsequence of itself $(n_k = k)$. ■

Monotonic sequences

Definition (Monotonic sequences) A sequence (x_n) of real-numbers is said to be (a) *monotonically non decreasing* if $x_n \leq x_{n+1} : n \in \mathbb{N}$ (b) *monotonically non increasing* if $x_n \geq x_{n+1} : n \in \mathbb{N}$.

Monotonicity and Convergence

If $(x_n)_{n\in\ltimes}$ is a bounded monotonic sequence of real numbers then x_n converges.

Proof : Without loss of generality let's assume that (x_n) is a bounded non-decreasing sequence, i.e $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$. Since (x_n) is bounded then there exists a $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. This boundedness implies that $\{x_n : n \in \mathbb{N}\}$ has a least upper bound (completeness of $\mathbb{R}$). Let $x := \sup_{n\in\mathbb{N}} x_n$. We claim that $x_n \to x$. To show this let $\varepsilon > 0$ be given and x is an upper bound for $\{x_n : n \in \mathbb{N}\}$, then $x - \varepsilon$ is not an upper bound. Which implies that, $\exists\, N(\varepsilon) \in \mathbb{N}$ such that $x_N > (x - \varepsilon)$. Now the sequence is monotonic so for $n \geq N$ we have $x_n > x - \varepsilon$. Again x is an upper bound, so $x_n \leq x$ for all $n \in \mathbb{N}$. Now we have

$$x - \varepsilon < x_n \leq x \;\;:\;\; \forall n \in \mathbb{N}$$

Which shows that $x_n \to x$, as desired. ■

Limit supremum and infimum

Remark : *monotone* non decreasing sequence converges to it's supremum (which could be ∞), and a *monotone* non increasing sequence converges to it's infimum (which could be $-\infty$). Thus, provided we allow for convergence to $\pm\infty$, all monotone sequences converge.

Intuition

- Consider (x_n), an arbitrary sequence of real numbers. We construct a new sequence (y_n) by taking the supremum of successively truncated *tails* of the original sequence, $y_n := \sup\{x_k : k \geq n\}$.
- Note, y_n is monotinically non increasing because the supremum is taken over smaller sets of larger n's. Therefor the seq. (y_n) has a limit, which we call the *lim sup* of the sequence (x_n), and denote by $\limsup x_n$.
- Similarly we can define a limit infimum too by taking infimum of successively truncated tails of (x_n).

limsup and liminf

Definition

Let (x_n) be a sequence of real numbers, then

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left[\sup\{x_k : k \geq n\}\right];$$
$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \left[\inf\{x_k : k \geq n\}\right] \tag{22}$$

Provided that we allow values $\pm\infty$, in contrast to limits, the $\limsup$ and $\liminf$ always exists. $\limsup$ of a sequence whose terms are bounded from above and $\liminf$ of a sequence whose terms are bounded from below are finite.

An example and an exercise

Example : If $x_n = (-1)^n$, then $\liminf_{n\to\infty} x_n = -1$ and $\limsup_{n\to\infty} x_n = 1$. The sequence doesn't have a limit as $\limsup$ and $\liminf$ are not equal. A good exercise for the reader is to prove the following : sequence (x_n) converges if and only if

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = \lim_{n\to\infty} x_n$$

Cauchy sequences

A little preparation

So far we have seen convergence with respect to some limits, the whole idea of "Cauchy sequence" is to talk about convergence of sequences without explicitly identifying their limits, which which gives us freedom to "control" the tail of such sequences.

Definition (Cauchy sequences)

A sequence $(x_n)_{n\in\mathbb{N}} \subset (X, ||\cdot||)$ is called a Cauchy sequence if

$$\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n, m \geq N : ||x_n - x_m|| < \epsilon \tag{23}$$

Proposition 5 : In a NLS every convergent sequence is a Cauchy sequence.
Exercise 9 : Prove Proposition 5.
Exercise 10 : Prove that a subset $\mathcal{M} \subset (X, ||\cdot||)$ is bounded $\iff$ $||\lambda_n x_n|| \to 0$ for any $x_n \in \mathcal{M}$ and any scalar $\lambda_n \to 0$

A visual proof

Consider the Exercise 9, plus another one : ***(a)*** *In a normed space every convergent sequence is a Cauchy sequence* ***(b)*** *If* (x_n) *is the Cauchy sequence in* (X, d) *and if there exists a subsequence* (x_{n_k}) *such that* $x_{n_k} \to x$ *for* $k \to +\infty$ *in* (X, d)*, then* $x_n \to x$ *for* $n \to \infty$ *in* (X, d)

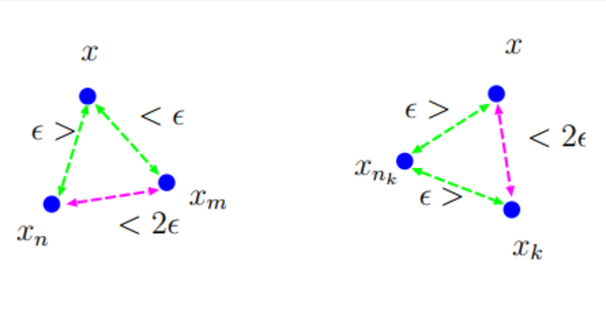


Figure: **(a)** As distances $d(x_n, x) < \epsilon$ and $d(x, x_m) < \epsilon$, then by the triangle inequality $d(x_n, x_m) < 2\epsilon$ (the left-hand schema). **(b)** As $d(x_{n_k}, x_k) < \epsilon$ and $d(x_{n_k}, x) < \epsilon$, then by the triangle inequality $d(x_k, x) < 2\epsilon$ (the right-hand schema).

(Can be skipped) The proof of 9-(b)

Suppose that (X, d_X) is a metric space. Show that a Cauchy sequence in X converges if and only if it has a convergent subsequence.

Here's the *mathy* proof for both direction : ($\Longrightarrow$): If (x_n) is a Cauchy sequence in X and $x_n \to x$, then every subsequence of (x_n) also converges to x. ($\Longleftarrow$): Suppose that the subsequence (x_{n_k}) of (x_n) converges to $x \in X$. Then for a given $\varepsilon > 0$, we can find integer $N := N(\varepsilon) \in \mathbb{N}$ such that

$$d_X(x_{n_k}, x) < \varepsilon \text{ and } d_X(x_n, x_m) < \varepsilon : \forall\, n_k, n, m \geq N.$$

Then

$$d_X(x_n, x) \leq d_X(x_n, x_{n_k}) + d_X(x_{n_k}, x) < \varepsilon : \forall\, n \geq N,$$

so $x_n \to x \in (X, d_X)$.

Remark:

Note the part "if it has ***a*** convergent subsequence"; this is not true in general for other sequences. Can you give an example of such sequence?

Examples

Ex : 11 $\mathbb{N} \ni n \mapsto x_n := \frac{1}{n}$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$
Proof : We want to show that $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\forall m, n \geq N$, then $|x_n - x_m| < \epsilon$, our first step is to choose N such that $N > \frac{2}{\epsilon}$. If $m, n \geq N$ then

$$n \geq N > \frac{2}{\epsilon} \implies \frac{1}{n} \leq \frac{1}{N} < \frac{\epsilon}{2}$$

and similarly

$$m \geq N > \frac{2}{\epsilon} \implies \frac{1}{m} \leq \frac{1}{N} < \frac{\epsilon}{2}$$

Therefore applying the triangle inequality we have that

$$|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \blacksquare$$

Exercise 11 : Check if the following sequences are Cauchy or not (a) $a_n := \frac{1}{n^2} : n \in \mathbb{N}$ $\quad (b)$ $b_n := (-1)^n : n \in \mathbb{N}$

Results

Proposition 6

If (x_n) is a Cauchy sequence in a NVS, then the *sequence of norms* $(||x_n||)_{n\in\mathbb{N}}$ converges.

Proof : Since $\big|||x|| - ||y||\big| \leq ||x - y||$ we have $\big|||x_m|| - ||x_n||\big| \leq ||x_m - x_n|| \to 0$, as $m, n \to \infty$. This shows that the seq of norms is Cauchy sequence of real numbers, hence it is convergent. ∎

Corollary 1

Every Cauchy sequence is bounded

Proof : The proof follows directly form Proposition 2, where we have proved that every convergent sequence is bounded. do you think the converse is true ? consider the case Exercise 11 (b). ∎

A Warning !!

We have seen that every convergent sequence in a general metric space/Normed linear space (NLS) is Cauchy, but the converse is NOT true, i.e every Cauchy sequence in a metric space/NLS X need not converge (in X with respect to the norm $||\cdot||_X$ or the metric d_X).

Counter Examples

Sequence is Cauchy but not convergent

Ex 12: Consider the sequence of rational numbers

$$S_n := \{s_1 = 1, s_2 = 1.4, s_3 = 1.414, \ldots\}$$

clearly this sequence is Cauchy and converges to $\sqrt{2}$ but $\sqrt{2} \notin \mathbb{Q}$.
Ex 13: Consider $a_n := \left(1 + \frac{1}{n}\right)^n$ which is again a sequence of rational numbers, a_n is Cauchy but $\lim_{n\to\infty} a_n = e \notin \mathbb{Q}$
Ex 14: Define $\mathcal{P}[0,1] := \{\text{Space of polynomials on } [0,1]\}$ Define norm as

$$\mathcal{P}[0,1] \ni x \mapsto ||x||_{\mathcal{P}} := \max_{x\in[0,1]}|\mathcal{P}(x)|$$

Now define a sequence

$$\mathbb{N} \ni n \mapsto p_n(x) := 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \in \mathcal{P}[0,1]$$

It's easy to check (p_n) is Cauchy but it converges to the exponential function e^x which is NOT a polynomial.

A bit of "Completeness"

Remedy

So what can we do with such spaces ? How to fill up the "holes" like $\sqrt{2} \notin \mathbb{Q}$? There exists surgical procedures to fill up "holes" in Real Analysis, which is called "Completion-procedure". Using completion we can make spaces such as $\mathbb{Q}$, complete (a space where every Cauchy sequence converges), this construction is same as "Dedekind cut" which is a procedure of constructing real numbers, and the completion of $\mathbb{Q}$ is $\mathbb{R}$. These Complete metric spaces/Normed spaces are called **"Banach Spaces"** (named after Stephan Banach) in the theory of Functional Analysis.

But, this is not of our concern right now !

The Bolzano-Weierstrass Theorem

Every bounded sequence of real numbers has a convergent subsequence.

Proof : We prove this for $(\mathbb{R}, d(x,y) := |x-y|)$. Let (x_n) be a bounded sequence of real numbers. Let

$$M := \sup_{n\in\mathbb{N}} x_n, \quad m := \inf_{n\in\mathbb{N}} x_n$$

and define the closed interval $I_0 := [m, M]$. Now, divide I_0 in half of two closed intervals where $L_0 := [m, (m+M)/2]$ and $R_0 := [(m+M)/2, M]$, at least one of the intervals L_0, R_0 contains infinitely many terms of the sequence; meaning that $x_n \in L_0$ or $x_n \in R_0$ for infinitely many $n \in \mathbb{N}$ (even if the terms themselves are repeated).

Choose I_1 to be one of the intervals of L_0, R_0 that contains infinitely many terms and choose $n_1 \in \mathbb{N}$ such that $x_{n_1} \in I_1$. Divide $I_1 := L_1 \cup R_1$ into half of two closed intervals. One or both of L_1, R_1 contains infinitely many terms of the sequence. Choose I_2 to be one of these intervals and choose $n_2 > n_1$ such that $x_{n_2} \in I_2$. This is always possible as I_2 contains infinitely many terms of the sequence. Continuing this way we get a nested sequence of intervals $I_1 \supset I_2 \supset \cdots$ of length $|I_k| := 2^{-k}(M-m)$, together with a subsequence x_{n_k} such that $x_{n_k} \in I_k$.

The Bolzano-Weierstrass Theorem (Continued)

Let $\varepsilon > 0$ be given. Since $|I_k| \to 0$ as $k \to \infty$, there exists $K \in \mathbb{N}$ such that $|I_k| < \varepsilon$ for all $k > K$. Furthermore, since $x_{n_k} \in I_K$ for all $k > K$, we have $|x_{n_j} - x_{n_k}| < \varepsilon$ for all $j, k > K$. This proves that (x_n) is a Cauchy sequence, and therefor it converges by part-(b) of Exercise 9. ■

Cauchy sequences on $\mathbb{R}$

A special case

If $(x_n)_{n\in\mathbb{N}}$ is a sequence of real numbers, then $(x_n)_{n\in\mathbb{N}}$ is convergent $\iff$ $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Proof : ($\implies$) this direction is already proved, or rather was left as an exercise, which is not difficult to prove. We look at the converse direction. ($\impliedby$) Suppose that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. We want to show that $(x_n)_{n\in\mathbb{N}}$ is thus convergent to some real number in $\mathbb{R}$. Now since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence it follows that $(x_n)_{n\in\mathbb{N}}$ is bounded. Since $(x_n)_{n\in\mathbb{N}}$ is bounded, it follows from The Bolzano-Weierstrass Theorem that there exists a sub-sequence of $(x_n)_{n\in\mathbb{N}}$, call it $(x_{n_k})_{k\in\mathbb{N}}$ that converges to some real number L. So we're basically done here, as Bolzano-Weierstrass guarantees existence of one such converging subsequence, and that's all we need. Because, we've proved that if there exists a converging subsequence of a Cauchy sequence then the whole sequence converges. so $\left(x_n\right)_{n\in\mathbb{N}}$ converges to the same limit L. Thus $\mathbb{R}$ is a complete metric space/Banach space. $\blacksquare$

Series and their convergence in NVS

Definition (Series)

Let $(X, ||\cdot||)$ be a NLS, let (x_n) be a sequence in X, then the expression $\sum_{n=1}^{\infty} x_n$ is called the series associated with the sequence or simply a series.

Definition (Convergence of a series)

Suppose $(X, ||\cdot||)$ is a normed space and (x_n) is a sequence in X: Then we say $\displaystyle\sum_{n=1}^{\infty} x_n$ converges in X iff $\displaystyle s := \lim_{N\to\infty} \sum_{n=1}^{N} x_n$ exists in X otherwise we say $\displaystyle\sum_{n=1}^{\infty} x_n$ diverges. We often let $\displaystyle S_N := \sum_{n=1}^{N} x_n$ and refer to $\{S_N\}_{N=1}^{\infty} \subset X$ as the sequence of partial sums.

Convergence of a series

Now we consider the case of series of non negative real numbers.

Theorem 4

Let (x_n) be a sequence of numbers s.t $x_n \geq 0 : \forall n \in \mathbb{N}$. The series $\displaystyle\sum_{n=1}^{\infty} x_n$ converges if and only if the partial sums are bounded.

Proof : The partial sums $S_n := \sum_{n=1}^{N} x_n$ of the above series forms a monotonically non decreasing sequence, from where the claim follows. ■

Theorem 5

if $\sum_{n=1}^{\infty} x_n$ converges then $\lim_{n\to\infty} x_n = 0$

Proof : We have $\sum_{n=1}^{\infty} x_n < +\infty$ without loss of generality we can say $\sum_{n=1}^{\infty} x_n \to L$. Then

$$\lim_{n\to\infty} \sum_{k=1}^{n} x_k = \lim_{n\to\infty} \left(x_n + \sum_{k=1}^{n-1} x_k \right) \tag{24}$$

$$= \lim_{n\to\infty} x_n + \lim_{n\to\infty} \sum_{k=1}^{n-1} x_k \tag{25}$$

However as $n \to \infty$, $\sum_{k=1}^{n-1} x_k \to L$ hence

Comparison Test

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series of real numbers with $a_n, b_n \geq 0 : \forall n \in \mathbb{N}$. Assume there exists a positive constant C (independent of n) such that $0 \leq a_n \leq C b_n : \forall n \geq N$, then

- If $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.
- If $\sum_{n=1}^{\infty} a_n$ diverges, then so does $\sum_{n=1}^{\infty} b_n$.

Proof : (a) Replacing finite number of terms in a_n by zero (think! why?), we may assume that $a_n \leq C b_n : n$. Then

$$a_1 + \cdots + a_n \leq C(b_1 + \cdots + b_n) \leq C \sum_{k=1}^{\infty} b_k$$

This is true forall n. Hence partial sums of the series $\sum a_n$ are bounded, and this series converges by Theorem 4.
(b) The next statement is the contrapositive of (a). ■

Theorem 6 (The Integral test)

Let $f :]0, \infty[\to \mathbb{R}$, assume that (a)$f(x) \geq 0 : \forall x$ (b) f is non increasing and (c)

$$\int_1^\infty f(x)\mathrm{d}x = \lim_{B\to\infty} \int_1^B f(x)\mathrm{d}x$$

exists. Then the series $\sum_{n=1}^{\infty} f(n)$ converges, and if the integral diverges the series also diverges.

Proof : For all $n \geq 2$ we have $f(n) \leq \int_{n-1}^{n} f(x)\mathrm{d}x$. Hence if the integral converges

$$f(2) + \cdots + f(n) \leq \int_1^n f(x)\mathrm{d}x \leq \int_1^\infty f(x)\mathrm{d}x$$

Hence the partial sums of the series are bounded and the series converges. Suppose conversely the series diverges.

Then $\forall n \geq 1$ we have $f(n) \geq \int_n^{n+1} f(x)\mathrm{d}x$ and consequently

$$f(1) + \cdots + f(n) \geq \int_1^{n+1} f(x)\mathrm{d}x$$

the right hand side becomes large as $n \to \infty$, so the series diverges which is a contradiction. So we've proved our claim. ■

Ex. 15 : An example (convergence of the p-series)

We define

$$\zeta(p) := \sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} +\infty & p \leq 1 \\ < +\infty & p > 1 \end{cases}$$

This result can be easily proved using Integral test.

Historical remark (million dollar problem)

$\zeta(s)$ is known as the "Riemann's zeta function" which has deep connection to prime number theorem, moreover Riemann stated the "Riemann's Hypothesis" saying that the non-real zeroes of $\zeta(s)$ must have $\mathfrak{Re}\, s = \frac{1}{2}$. Proving this will get you one million dollars.

Cauchy criteria of series

We say a series $\sum_n x_n$ satisfies the Cauchy criterion if its sequence (s_n) of partial sums is a Cauchy sequence i.e

$$\forall \epsilon > 0 : N : m, n > N \implies |s_n - s_m| < \epsilon \tag{27}$$

Nothing is lost in this definition if we impose the restriction $n > m$. Moreover, it is only a notational matter to work with $m - 1$ where $m \geq n$ instead of m where $m < n$. Therefore (27) is equivalent to

$$\forall \epsilon > 0 : N : n \geq m > N \implies |s_n - s_{m-1}| < \epsilon \tag{28}$$

Since $s_n - s_{m-1} = \sum_{k=m}^{n} x_k$, condition (28) can be rewritten

$$\forall \epsilon > 0 : N : n \geq m > N \implies \left| \sum_{k=m}^{n} x_k \right| < \epsilon \tag{29}$$

We will use definition version-(29) mostly.

Absolute convergence

Theorem 7 (Absolute convergence)

Let $(X, ||\cdot||)$ be a complete NLS (recall: Completeness means every Cauchy sequences in X converges to a limit in X with respect to the norm $||\cdot||$ on X), then

$$\text{(Absolute convergence)} \sum_{n=1}^{\infty} ||x_n|| < +\infty \implies \text{(Convergence)} \sum_{n=1}^{\infty} x_n < +\infty \tag{30}$$

This is called absolute convergence of $\sum_{n=1}^{\infty} x_n$

Proof : The sequence $s_n := \sum_{k=1}^{n} x_k$ of partial sums is clearly Cauchy, and completeness of X implies it must converge to some limit $s \in X$. This is clear: $\forall \epsilon > 0 : \exists N \in \mathbb{N} : n, m \geq N$ and assume $n \geq m$ we have

$$|s_n - s_m| = \left\| \sum_{k=m+1}^{n} x_k \right\| \leq \sum_{k=m+1}^{n} ||x_k|| < \epsilon$$

As desired. ■

Non-absolute convergence

Theorem 8 (Non-absolute convergence)

Let a_n be a sequence of numbers s.t (a) $a_n \geq 0 : \forall n$ (b) a_n decreases monotonically and (c) $\lim_{n\to\infty} a_n = 0$. Then the series $\sum_n (-1)^n a_n$ converges, and

$$\left| \sum_{n=1}^{\infty} (-1)^n a_n \right| \leq a_1$$

Proof : Let us assume $a_1 > 0$, so that we can write the series in the form

$$b_1 - c_1 + b_2 - c_2 + b_3 - c_3 + \cdots$$

with $b_n, c_n \geq 0$ and $b_1 = a_1$. Let

$$s_n := b_1 - c_1 + b_2 - c_2 + \cdots + b_n$$

$\mathrm{t}_n := b_1 - c_1 + b_2 - c_2 + \cdots + b_b - c_n$Then $s_{n+1} = s_n - c_n + b_{n+1}$. Since $0 \leq b_{n+1} \leq c_n$ it follows that $s_{n+1} \leq s_n$ and thus

$$s_1 \geq s_2 \geq s_3 \geq \cdots$$

and similarly

$$t_1 \leq t_2 \leq t_3 \leq \cdots$$

Proof of theorem 8 continued

i.e s_n and t_n forms a decreasing and increasing sequence respectively. Since $t_n = s_n - c_n$ and $c_n \geq 0$ it follows that $t_n \leq s_n$ so that we have the following inequalities

$$s_1 \geq s_2 \geq \cdots \geq s_n \geq \cdots \geq t_n \geq \cdots \geq t_2 \geq t_1$$

Given ϵ there exists $N \in \mathbb{N}$ s.t if $n \geq N$ then $0 \leq s_m - t_n < \epsilon$. And if $m \geq n$ then $|s_n - s_m| \leq s_n - t_n \leq \epsilon$.
Hence the series converges, the limit being viewed as either the greatest lower bound of (s_n) or the least upper bound of (t_n). Finally observe this limit lies between s_1 and

$$t_1 = s_1 - c_1 = b_1 - c_1 \geq 0 \tag{31}$$

Which proves our claim. ■

Exercise 12 :

Prove $\zeta(1) = H_n := \sum_{n=1}^{\infty} \frac{1}{n}$ i.e the "Harmonic series" in Ex. 15 is divergent.

Ex. 16 :

Show that the series $\sum_{n=1}^{\infty} \frac{1}{n}(-1)^n$ converges non-absolutely

Proof : Compare with the series $\sum_n (-1)^n a_n$ in theorem 8. We see that

- (a) $a_n = 1/n$ and $a_n \geq 0 : \forall n \in \mathbb{N}$
- (b) $a_{n+1} = 1/(n+1)$, clearly a_n is monotonically decreasing
- (c) $\lim_{n\to\infty}(1/n) = 0$.

So, by theorem 8, it converges. But

$$\sum_n \left| \frac{1}{n}(-1)^n \right| = \sum_n \frac{1}{n} = H_n \tag{32}$$

Which is divergent (Exercise 12), implying that the series converges non-absolutely. ■

Two important tests

Theorem 9 : Root test

Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{C}$ and let $\alpha := \limsup_{n\to\infty} |a_n|^{1/n}$. Then

- If $\alpha < 1$ then $\sum_{n=1}^{\infty} |a_n| < \infty$ and $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
- If $\alpha > 1$; then $\limsup_{n\to\infty} |a_n| + \infty$ and $\sum_{n=1}^{\infty} a_n$ diverges.
- If $\alpha = 1$, the test fails.

Theorem 10 : Ratio test

Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{C}$ and let $\alpha := \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. Then

- if $\alpha < 1$ then $\sum_{n=1}^{\infty} |a_n| < \infty$ and $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
- If $\left|\frac{a_{n+1}}{a_n}\right| \geq 1 : \forall n \geq N \in \mathbb{N}$, $\sum_{n=1}^{\infty} a_n$ diverges.
- If $\alpha = 1$, the test fails.

Proofs of theorem 9 and 10 are left as an exercise to the reader.

Introduction to the power series

Definition (power series)

Let $(Z := \mathbb{C}, ||\cdot||)$ be a complete normed linear space. Given $z_0 \in \mathbb{C}$ and a sequence $(a_n)_{n=0}^{\infty} \subset Z$, the series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{33}$$

is called a power series. If we look at the series from the origin i.e $z_0 = 0$ then we call it a "Maclaurin series" which is of the form

$$\sum_{n=0}^{\infty} a_n z^n \tag{34}$$

Radius of convergence

The radius of convergence (ROC) of either of the series is defined to be $R := 1/\alpha$, where

$$\alpha := \limsup_{n\to\infty} ||a_n||^{1/n} \in [0, +\infty]$$

Power series

Theorem 11

If R is the ROC of the power series in (33) then,

- If $|z - z_0| < R$, the series converges.
- If $|z - z_0| > R$, the series diverges.
- If $|z - z_0| = R$, we can't say anything for sure.

Proof : Let

$$\begin{aligned}\rho(z) &:= \limsup_{n\to\infty} ||a_n(z - z_0)^n||^{1/n} = |z - z_0| \limsup_{n\to\infty} ||a_n||^{1/n} \\ &= \frac{z - z_0}{R}\end{aligned}$$

By the root test, Theorem 9, we know that the power series in (33) converges if $\rho(z) <$ and diverges if $\rho(z) > 1$ and the test fails if $\rho(z) = 1$. ■

corollary 2 : An important consequence

If $\mu := \lim_{n\to\infty} \frac{||a_{n+1}||}{||a_n||}$ exists, the ROC of the power series (17) is

$$R = \frac{1}{\mu} = \lim_{n\to\infty} \frac{||a_n||}{||a_{n+1}||}$$

Proof : Now define

$$\begin{aligned} \rho(z) &= \lim_{n\to\infty} \frac{||a_{n+1}(z-z_0)^{n+1}||}{||a_n(z-z_0)^n||} = |z-z_0| \lim_{n\to\infty} \frac{||a_{n+1}||}{||a_n||} \\ &= \mu \cdot |z-z_0| \end{aligned} \tag{35}$$

Now from the ratio-test (Theorem 10)

- $\rho(z) < 1 \implies |z-z_0| < \frac{1}{\mu} \ : \sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges
- $\rho(z) > 1 \implies |z-z_0| > \frac{1}{\mu} \ : \sum_{n=0}^{\infty} a_n(z-z_0)^n$ diverges

So $R = 1/\mu$ ■

Exercise 13 : Find ROC of the following series (trivial cases)

(a) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ (b) $\sum_{n=0}^{\infty} x^n$ (c) $\sum_{n=0}^{\infty} \frac{x^n}{n}$ (d)$\sum_{n=0}^{\infty} \frac{x^n}{n^2}$ (e) $\sum_{n=0}^{\infty} n!x^n$

Ex 16 : Find ROC, two non-trivial example

(a) $\displaystyle\sum_n \frac{x^n}{n(\log n)^2}$ (b) $\displaystyle\sum_n \frac{x^n}{n^{\log n}}$

Solution (using ratio test) : Since

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)\log^2(n+1)}}{\frac{1}{n\log^2 n}} = \lim_{n\to\infty} \frac{n}{n+1} \times \left(\frac{\log n}{\log(n+1)}\right)^2$$

and since $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{\log n}{\log(n+1)} = 1$, the radius of convergence of the series is equal to 1. Note that $\log(n+1) = \log(n) + \log\left(\frac{n+1}{n}\right)$ and that $\lim_{n\to\infty} \log\left(\frac{n+1}{n}\right) = 0$; it's easy to deduce from this that $\lim_{n\to\infty} \frac{\log n}{\log(n+1)} = 1$ indeed. Solution to (b) is similar, try it yourself.

An alternate solution using root test

(a) $\sum_n \frac{x^n}{n(\log n)^2}$ (b) $\sum_n \frac{x^n}{n^{\log n}}$

Solution (using root test) :
First series:

$$\sqrt[n]{a_n} = \frac{1}{\sqrt[n]{n(\ln n)^2}} = \exp\left(-\frac{\ln n + 2\ln(\ln n)}{n}\right) \to \exp(0) = 1.$$

So, the radius of convergence is equal to one.
Second series:

$$\sqrt[n]{a_n} = \frac{1}{n^{\ln n/n}} = \exp\left(-\frac{(\ln n)^2}{n}\right) \to \exp(0) = 1.$$

Again, radius of convergence is equal to one.

Next stop : Sequence and series of functions

In what follows we will consider the normed vector space $(X, ||\cdot||) = (\mathbb{R}, |\cdot|)$

Definition (pointwise convergence)

Let $D \subseteq \mathbb{R}$, consider $(f_n)_{n\in\mathbb{N}}$ a sequence of functions defined on D, i.e $\mathbb{N} \ni n \mapsto f_n(x) \in \mathbb{R}$. We say (f_n) is pointwise convergent (PWC) to a limiting function $f : D \subseteq \mathbb{R} \to R$ if :

$$\forall \epsilon > 0, \forall x \in D, \exists N \in \mathbb{N} : n \geq N : |f_n(x) - f(x)| < \epsilon \tag{36}$$

A few remarks are in order

- Don't get confused with the mappings, note $f_n : \mathbb{N} \to \mathbb{R}$ which is the whole sequence, and when you fix a $n \in \mathbb{N}$ we have the mapping $D \subseteq \mathbb{R} \ni x \mapsto (f_n)(x) := f_n(x) \in \mathbb{R}$: for a fixed n
- In (20) Careful with the order of the quantifiers i.e $(\forall), (\exists), (:)$ etc, misplacing them will change the meaning of assertion. Note in (36) the integer $N = N(\epsilon, x)$, i.e it depends on both x and ϵ which is clear from the order of quantifiers.

PWC : It's not as good as you think

We present some interesting example to show what can go wrong

Example 17 : Each f_n is bounded but PW limit is not

Suppose $]0,1[\ni x \mapsto f_n(x) := \frac{n}{nx+1}$

Since $x \neq 0$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{x+1/n} = \frac{1}{x}$$

So $f_n \to f$ pointwise where $f :]0,1[\to \mathbb{R}$ is given by

$$f(x) = \frac{1}{x}$$

We have $|f_n(x)| < n : \ \forall x \in]0,1[$, so each f_n is bounded in $]0,1[$, but the pointwise limit f is not. Thus, the **PWC does not**, in general, **preserve boundedness**

PWC : It's not as good as you think

Example 18 : PWC does not preserve continuity

$[0,1] \ni x \mapsto f_n(x) := x^n$

If $0 \leq x < 1$, then $x^n \to 0$ as $n \to \infty$, while if $x = 1$, then $x^n \to 1$ as $n \to \infty$. So $f_n \to f$ pointwise where

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

Although each f_n is continuous on $[0,1]$, the pointwise limit f is not (discontinuous at 1). Thus, in general pointwise convergence doesn't preserve continuity.

PWC : It's not as good as you think

Example 19 : Not well behaved when differentiated

$\mathbb{R} \ni x \mapsto f_n(x) := \frac{\sin nx}{x} \in \mathbb{R}$

Clearly $f_n \to 0$ pointwise in $\mathbb{R}$. The sequence f_n' of derivatives $f_n'(x) := \cos nx$ does not converge pointwise on $\mathbb{R}$; for example

$$f_n'(\pi) = (-1)^n$$

Does not converge as $n \to \infty$. Thus in general, one cannot differentiate a pointwise convergence sequence. (Note : this is not only limited to PWC sequences)

What's next

In the next section we introduce a stronger notion of convergence: uniform convergence (UC). The difference between PWC and UC is analogous to the difference between continuity and uniform continuity.

Uniform convergence : A stronger notion

Definition : Uniform convergence

Let $D \subseteq \mathbb{R}$, consider $(f_n)_{n\in\mathbb{N}}$ a sequence of functions defined on D, i.e $\mathbb{N} \ni n \mapsto f_n(x) \in \mathbb{R}$. We say (f_n) is uniformly convergent (UC) to a limiting function $f : D \subseteq \mathbb{R} \to R$ if :

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N}\,, \forall x \in D : n \geq N : |f_n(x) - f(x)| < \epsilon \tag{37}$$

Clearly, here $N = N(\epsilon)$, i.e there is no dependence on x

Obviously, uniform convergence implies pointwise convergence.

Theorem 12 : Cauchy Criteria

The sequence of functions defined on D, converges uniformly on D if and only if for every $\epsilon > 0, \exists N \in \mathbb{N}$ such that $m, n \geq N$, $x \in D$ implies $|f_n(x) - f_m(x)| \leq \epsilon$

Cauchy criteria for uniform convergence

Proof : Suppose $f_n \xrightarrow{unif} f \in D$, where f is the limit function. Then

$$\exists N \in \mathbb{N} : n \geq N, x \in D \implies |f_n(x) - f_(x)| < \frac{\epsilon}{2} \tag{38}$$

So that, if $n, m \geq N, x \in D$

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \epsilon \tag{39}$$

Conversely, if Cauchy criterion folds $\{f_n(x)\}$ converges for every x, to a limit we may call $f(x)$, so the convergence is uniform. Now for $\epsilon > 0$ and choose N s.t (38) holds, fix n and let $m \to \infty$ in (38). Since $f_m(x) \to f(x)$ as $m \to \infty$ we have

$$|f_n(x) - f(x)| < \epsilon \tag{40}$$

For every $n \geq N$ and $\forall x \in D$, this completes the proof. ■

Some useful testing techniques for UC

Theorem 13 : A very useful one

Suppose $\lim_{n\to\infty} f_n(x) = f(x) \quad x \in D$, put

$$M_n := ||f_n - f||_\infty = \sup_{x\in D} |f_n(x) - f(x)| \tag{41}$$

Then $f_n \xrightarrow{unif} f \in D$ iff $M_n \to 0$ as $n \to \infty$

Exercise 14: Prove Theorem 13, Hint: immediate consequence of definition (37).

Theorem 14 : Another very useful one

$D \subseteq \mathbb{R}$ and let $f_n(x) : D \to \mathbb{R}$ is continuous forall $n \in \mathbb{N}$. If $f_n \xrightarrow{unif} f \in D$, then f is continuous.

Note : For proof of above theorem refer [ROSS]. Theorem 14 tells us, if we find the pointwise limit function of some seq $f_n(x)$ is NOT continuous, then we automatically know that the convergence can't be uniform.

Uniform convergence and boundedness

Theorem 15

Suppose that $f_n(x) : X \to \mathbb{R}$ is bounded on X for every $n \in \mathbb{N}$ and $f_n \xrightarrow{unif} f$ on X. Then $f : X \to \mathbb{R}$ is bounded on X

Proof : Taking $\epsilon = 1$ in the definition of UC, we find that $\exists N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < 1 \text{ for all } x \in X \text{ if } n > N$$

Choose some $n > N$, then since f_n is bounded, there exists a constant $M \geq 0$ such that

$$|f_n(x)| \leq M \text{ for all } x \in X$$

it follows that

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| < 1 + M \text{ for all } x \in X$$

meaning that f is bounded on X.

Uniform convergence and integration

Theorem 16

Let f_n be a seq of continuous functions on $[a, b]$ and suppose $f_n \xrightarrow{unif} f$ on $[a, b]$, then

$$\lim_{n\to\infty} \int_a^b f_n(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x \tag{42}$$

Proof : By Theorem 14 f is continuous, so the functions $(f_n - f)$ are all integrable on $[a, b]$. Let $\epsilon > 0$, since $f_n \xrightarrow{unif} f$ on $[a, b]$, $\exists N$ s.t $|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$, $\forall x \in [a, b]$ and $n \geq N$. Then

$$\begin{aligned} \left|\int_a^b f_n(x)\mathrm{d}x - \int_a^b f(x)\mathrm{d}x\right| &= \left|\int_a^b f_n(x) - f(x)\mathrm{d}x\right| \\ &\leq \int_a^b |f_n(x) - f(x)|\mathrm{d}x \\ &\leq \int_a^b \frac{\epsilon}{b-a}\mathrm{d}x = \epsilon \quad \blacksquare \end{aligned} \tag{43}$$

Uniform convergence and differentiation

Theorem 17

Let (f_n) be a sequence of differentiable functions on an interval $[a,b]$, $a<b$. Assume that each (f_n') is continuous, and that the sequence (f_n') converges uniformly to a function g. Assume also that there exists one point $x_0 \in [a,b]$ such that the sequence of $\{f_n(x_0)\}$ converges. Then the sequence (f_n) converges uniformly to a function f which is differentiable and $f' = g$

Proof : For each n, there exist a c_n such that

$$f_n(x) = \int_a^x f_n' + c_n \quad \forall x \in [a,b] \tag{44}$$

Let $x = x_0$. Taking the limit as $n \to \infty$ shows that the sequence of numbers (c_n) converges, say to a number c. For an arbitrary x, we take the limit as $n \to \infty$ and apply Theorem 16. We see that the sequence (f_n) converges pointwise to a function f such that

$$f(x) = \int_a^x g + c \tag{45}$$

Proof of Theorem 17. contd.

On the other hand, this convergence is uniform, because

$$\left|\int_a^x f_n' - \int_a^x g\right| = \left|\int_a^x (f_n' - g)\right| \leq (b-a)\left\|f_n' - g\right\| \tag{46}$$

So $||f_n - f|| \leq (b-a)||f_n' - g|| + |c_n - c|$, this proves the theorem. ∎

Ex. 20

we define a sequence $[1,\infty[\ni x \mapsto f_n(x) := \frac{nx}{1+nx}$. Comment of PWC and UC.

Solution : Here $D := [1,\infty[\subset \mathbb{R}$. Consider fixed $x \in [1,\infty[$, then

$$f_n(x) = \frac{nx}{1+nx} = \frac{x}{x+(1/n)} \to 1, \text{ as } n \to \infty$$

so $f_n \xrightarrow{pointwise} f(x) = 1$. To check UC we make use of Thrm. 13, we have

$$\begin{aligned} \lim_{n\to\infty} ||f_n - f||_\infty &= \lim_{n\to\infty} \sup_{x\in[1,\infty[} \left| \frac{nx}{1+nx} - 1 \right| \\ &= \lim_{n\to\infty} \sup_{x\in[1,\infty[} \left| \frac{1}{1+nx} \right| \\ &= \lim_{n\to\infty} \frac{1}{1+n} = 0 \end{aligned} \tag{47}$$

So $f_n \xrightarrow{unif} f(x) = 1$

Examples : Functional sequences

Ex. 21

we define a sequence $[0,\infty[\ni x \mapsto f_n(x) := \frac{nx}{1+nx}$. Comment of PWC and UC.

Solution : Observe at $x=0$ $f_n(x)=0 : \forall n \in \mathbb{N}$, otherwise $f_n(x) \to f(x) = 1$. So we have a pointwise limit function

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & \text{otherwise} \end{cases}$$

To check UC, observe

$$|f_n(x) - f(x)| = \begin{cases} 0 & x = 0 \\ \left|\frac{nx}{1+nx} - 1\right| = \frac{1}{1+nx} & \text{otherwise} \end{cases}$$

So

$$\sup_{x\in[0,\infty[} |f_n(x) - f(x)| = \frac{1}{1+n\cdot 0} = 1 \nrightarrow 0$$

Implying $f_n(x) \nrightarrow f(x)$ uniformly. Also, from Thrm. 14 it's evident convergence can't be uniform as the functional limit is discontinuous.

Ex. 22

Define $]0,\pi[\ni x \mapsto f_n(x) := \sin^n x$, Comment on PWC, UC

Solution : Note $\sin x = 1, x = \pi/2$ and $\sin x \in]0,1[, x \in]0,\pi/2[\cup]\pi/2,\pi[$, so

$$f_n(x) := \sin^n x \xrightarrow{n\to\infty} f(x) = \begin{cases} 1 & x = \frac{\pi}{2} \\ 0 & x \in]0,1[, x \in]0,\pi/2[\cup]\pi/2,\pi[\end{cases}$$

Which is dis-continuous at $x = \pi/2$, implying no uniform convergence according to Thrm. 14.

Examples : Functional sequences

Ex. 23

$]0,\infty[\ni x \mapsto f_n(x) := nxe^{-nx^2}$, check for PWC, UC

Solution : Fix $x \in]0,\infty[$

$$\lim_{n\to\infty} \frac{nx}{e^{-nx^2}} = \lim_{n\to\infty} \frac{x}{x^2 e^{nx^2}} \quad \left(\frac{\infty}{\infty}\right) \to 0$$

$f_n(x) \xrightarrow{PW} f(x) = 0$. UC in this case is a little tricky. Consider

$$||f_n - f||_\infty = \sup_{x\in]0,\infty[} |nxe^{-nx^2}| \tag{48}$$

Let $y(x) := nxe^{-nx^2}$ find the extremizer by solving

$$y'(x) = 0 \implies x = \pm\frac{1}{\sqrt{2n}}$$

Finally

$$\lim_{n\to\infty} ||f_n - f||_\infty \Big|_{x=1/\sqrt{2n}} = \lim_{n\to\infty} \sqrt{\frac{n}{2}} e^{-1/2} \to \infty$$

So we don't have UC in this case

Weierstrass M-test

Theorem 18: Weierstrass M-test

Recall $\mathcal{C}[a,b] := \{f : [a,b] \to \mathbb{R} :\ \text{f is continuous}\}$, let $f_n \in \mathcal{C}[a,b]$, suppose $||f_n||_\infty \leq M_n : n = 1, 2, \ldots$ and suppose that $\sum_{n=1}^{\infty} M_n$ converges in $\mathbb{R}$. Then $\sum_{n=1}^{\infty} f_n$ converges in the $||\cdot||_\infty$ norm i.e it converges uniformly to a continuous function $f \in \mathcal{C}[a,b]$

Ex. 24

(a) $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$

Solution : Use M-test with $M_n := \frac{1}{n^2}$ clearly the series $\sum_{n=1}^{\infty} M_n$ converges, in-fact to $\frac{\pi^2}{6}$ (A fun exercise). So the series converges uniformly and absolutely on any interval $[a,b] \subset \mathbb{R}$

Further Exercises for practice

Check for uniform convergence

$$(1) \sum_{n=1}^{\infty} \frac{x^n}{1+x^n}\ ,\ x \in [0,1] \quad (2) \sum_{n=1}^{\infty} \frac{\cos^n x}{n}\ ,\ x \in]-\pi,\pi[$$

$$(3) \sum_{n=0}^{\infty} x^2(1-x^2)^n\ ,\ x \in [-1,1] \quad (4) \sum_{n=1}^{\infty} (\cos x)^2 (\sin x)^{2n}\ ,\ x \in]0,\pi[$$

$$(5) \sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3}\ ,\ x \in [0,1] \quad (6) \sum_{n=1}^{\infty} \left(\frac{1}{2}+x\right)^n \left(\frac{1}{2}-x\right)^n\ ,\ x \in \left]-\frac{1}{2},\frac{1}{2}\right[$$

$$(7) \sum_{n=1}^{\infty} (n+1)x^n\ ,\ x \in \left]-\frac{1}{2},\frac{1}{2}\right[\quad (8) \sum_{n=1}^{\infty} \left(\frac{x}{1+x^2}\right)^n\ ,\ x \in [0,\infty[$$

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Thank You