Write your roll number so legibly that a machine can read it

Roll No : 12345
e.g., 190040001

Dept.:
e.g., CSE

IIT Bombay
CS 405/6001: GT&AMD
Quiz 2, 2024-25-I
Date: October 25, 2024

CS 405/6001: Game Theory and Algorithmic Mechanism Design

Total: $\mathbf{11 + 18 + 6 = 35}$ **marks**, *Duration:* **1 hour, ATTEMPT ALL QUESTIONS**

Instructions:

1. This question-and-answersheet booklet contains a total of 5 sheets of paper (10 pages, pages 2 and 10 are blank). Please verify.
2. Write your roll number and department on **every side of every sheet** (except the blank sheet) of this booklet. Use only **black/blue ball-point pen**. The first 5 minutes of additional time is given exclusively for this activity.
3. Write final answers neatly with a pen **only in the given boxes**.
4. Use the rough sheets for scratch works / attempts to solution. **Write only the final solution (which may be a sequence of logical arguments) in a precise and succinct manner in the boxes provided**. Do not provide unnecessarily elaborate steps. The space within the boxes are sufficient for the correct and precise answers.
5. Submit your answerscripts to the teaching staff when you leave the exam hall or the time runs out (whichever is earlier). **Your exam will not be graded if you fail to return the paper**.
6. **This is a closed book, notes, internet exam. No communication device, e.g., cellphones, iPad, etc., is allowed**. Keep it switched off in your bag and keep the bag away from you. If anyone is found in possession of such devices during the exam, that answerscript may be disqualified for evaluation and DADAC may be invoked.
7. One A4 assistance sheet (**text only on one side**) is allowed for the exam.
8. **After you are done with your exam or the exam duration is over, please DO NOT rush to the desk for submitting your paper**. Please remain seated until we collect all the papers, count them, and give a clearance to leave your seat.

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Problem 1 (11 points). For each of the following games, where Player I is the row player and Player II is the column player, find all the equilibria in mixed strategies, and all the equilibrium payoffs. You do not need to show the calculations here.

	L	R
T	5, 16	15, 8
B	16, 7	8, 15

Game 1

	L	R
T	8, 3	10, 1
B	6, −6	3, 5

Game 2

(a) Game 1 equilibria. **4 points.**

no equilibrium with pure strategy. The only mixed Nash equilibrium is

$$\left(\left(\frac{1}{2}(T), \frac{1}{2}(B)\right), \left(\frac{7}{18}(L), \frac{11}{18}(R)\right)\right)$$

(b) Game 1 equilibria payoffs. **2 points.**

Payoffs are $(11 \cdot 11, 11 \cdot 5)$

(c) Game 2 equilibria. **3 points.**

~~(T, L) is the SDSE, hence is the unique MSNE~~

T strictly dominates B for player 1, so B can be eliminated. After the removal, L dominates R for player 2. Hence, the elimination of strictly dominated strategies leads to a pure strategy profile (T,L) which is the unique MSNE of this game.

(d) Game 2 equilibria payoffs. **2 points.**

$(8, 3)$

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Problem 2 (18 points). A city council is trying to decide on one for the **four** possible public alternatives. For simplicity, let us denote the alternatives as a, b, c, d. Suppose there are only **two** voters representing the two major localities of the city and their votes will detemine the final decision. Consider the following possible voting schemes and answer the questions below.

Mechanism 1 (M1). If both voters select the same alternative as being the most preferred, that alternative is chosen. Otherwise, if there is only one alternative that both voters rank within their top two most-preferred alternatives, that alternative is chosen. Otherwise, a is chosen.

Mechanism 2 (M2). If voter 1 selects a as being the most preferred, that alternative is chosen. Otherwise, the most preferred alternative of voter 2 is chosen.

Mechanism 3 (M3). If both voters' top choices agree, then that alternative is chosen. Otherwise, if both voters' *second* top choices agree, then that alternative is chosen. Otherwise, if both voters' *third* top choices agree, then that alternative is chosen. Otherwise, if both voters' *fourth* top choices agree, then that alternative is chosen. Otherwise, a is chosen.

(a) Consider the properties *Pareto efficiency*, *Unanimity*, *Monotonicity*, and *Strategyproofness*, and fill the following table with "yes/no" against those properties for the given mechanisms.

0.5 × 12 points.

Mechanisms	*Pareto efficient?*	*Unanimous?*	*Monotone?*	*Strategyproof?*
M1	no	yes	no	no
M2	yes	yes	no	no
M3	no	yes	no	no

(b) Explain each of your answers above, i.e., if yes, explain in not more than three sentences why it is so, and if not, provide an example, in this and the following parts of this question. No credit will be given for explanation if the corresponding conclusion is incorrect. In this part, start with **M1 and Pareto efficiency**. **1 point.**

P_1	P_2
b	c
a	d
c	b
d	a

according to the rule of M1, a is the winner, but $b \, P_i \, a \;\; \forall i = 1, 2$

not PE

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(c) **M1 and Unanimity**. **1 point.**

Yes, by definition of the mechanism.

(d) **M1 and Monotonicity**. **1 point.**

P_1	P_2
b	c
a	d
c	b
d	a

outcome $\rightarrow$ a

P_1	P_2'
b	c
a	b
c	a
d	d

outcome $\rightarrow$ b

but a's relative position has improved in P_1, P_2'

not monotone

(e) **M1 and Strategyproofness**: **1 point.**

The same example above works. Player 2 manipulates from $P_2 \rightarrow P_2'$

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(f) **M2 and Pareto efficiency**. **1 point.**

If $\exists$ pair of alternatives $\tilde{a}, \tilde{b}$ s.t. $\tilde{a} P_i \tilde{b}$ $\forall i = 1, 2$, then $\tilde{b}$ can't be the most preferred alternative for any voter. But the mechanism chooses either player 1 or 2's top choice. Hence $\tilde{b}$ will never be chosen. Hence PE.

(g) **M2 and Unanimity**. **1 point.**

PE $\Rightarrow$ UN. Hence this is also unanimous.

(h) **M2 and Monotonicity**. **1 point.**

P_1	P_2
c	b
a	c
b	d
d	a

$\xrightarrow{\text{out}}$ b

P_1'	P_2
a	b
b	c
c	d
d	a

$\xrightarrow{\text{out}}$ a

b's relative position improved but the outcome changed. Not monotone

(i) **M2 and Strategyproofness:** **1 point.**

The same example as above

$P_1 \rightarrow P_1'$ player 1 gains. not strategyproof.

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(j) **M3 and Pareto efficiency.** **1 point.**

P_1	P_2
b	c
a	b
d	a
c	d

$\xrightarrow{\text{out}}$ a

but $b P_i a \; \forall i = 1, 2$

so this is not PE

(k) **M3 and Unanimity.** **1 point.**

It is unanimous by definition.

(l) **M3 and Monotonicity.** **1 point.**

P_1	P_2
b	c
a	b
d	a
c	d

$\longrightarrow$ a

P_1	P_2'
b	b
a	c
d	a
c	d

$\longrightarrow$ b

a's relative position weakly improved (actually stayed same), yes outcome changed

not monotone

(m) **M3 and Strategyproofness:** **1 point.**

The same example as above

$P_2 \rightarrow P_2'$ player 2 gains. Manipulable.

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Problem 3 (6 points). First Price Auction with Private Values. Consider a first-price sealed-bid auction of an object between two bidders who are targeting to maximize their expected utility. Each bidder i (for $i = 1, 2$) simultaneously submits a bid $b_i \geqslant 0$. The bidder who submits the highest bid **receives the object and pays her bid**; both bidders win with equal probability in case they submit the same bid. Before the auction takes place, each bidder i privately observes the realization of her random type t_i that is drawn independently from a uniform distribution over the interval $[0, 1]$. The fact that the types of both agents are distributed as $U[0, 1]$ is a common knowledge. The actual **valuation** of the object to bidder i is equal to $t_i + 0.5$ if she wins the object, and zero otherwise. Therefore, the payoff of bidder i is given by

$$u_i = \begin{cases} t_i + 0.5 - b_i & \text{if } b_i > b_j, j \neq i, \\ \frac{1}{2}(t_i + 0.5 - b_i) & \text{if } b_i = b_j, j \neq i, \\ 0 & \text{if } b_i < b_j, j \neq i. \end{cases}$$

(a) Derive the **linear, symmetric Bayesian Nash equilibrium** for this game. Show the derivation in the most precise manner and enclose the final answer in a box. **Linear** bid implies that each bidder i uses an strategy of the form $b_i = \alpha_i t_i + \beta_i$. **Symmetric** Bayesian NE implies that at the equilibrium, $\alpha_i = \alpha$ and $\beta_i = \beta$, for all $i = 1, 2$. **3 + 1 points.**

Consider the payoff of player 1 (the analysis for player 2 is similar)

$u_1 = P(b_1 > b_2)(t_1 + \frac{1}{2} - b_1)$, since $P(b_1 = b_2) = 0$ for any continuous distribution.

$= P(b_1 > \alpha_2 t_2 + \beta_2)(t_1 + \frac{1}{2} - b_1)$ --- (1)

Consider $b_1 > \alpha_2 t_2 + \beta_2 \Rightarrow t_2 < \frac{1}{\alpha_2}(b_1 - \beta_2)$ for $\alpha_2 > 0$

$P\left(t_2 < \frac{1}{\alpha_2}(b_1 - \beta_2)\right) = \frac{1}{\alpha_2}(b_1 - \beta_2)$, if $\frac{b_1 - \beta_2}{\alpha_2} \in [0, 1]$

We will solve the bidding strategies in an unconstrained manner and then check whether these two conditions get satisfied. Hence from eq (1),

$u_1 = \frac{1}{\alpha_2}(b_1 - \beta_2)(t_1 + \frac{1}{2} - b_1) = \frac{1}{\alpha_2}\left[(t_1 + \frac{1}{2} + \beta_2) b_1 - b_1^2 - \beta_2(t_1 + \frac{1}{2})\right]$

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maximizing wrt b_1, we get from the first order conditions,

$$b_1^* = \frac{1}{2}\left(t_1 + \frac{1}{2} + \beta_2\right) = \alpha_1 t_1 + \beta_1$$

Similarly, $$b_2^* = \frac{1}{2}\left(t_2 + \frac{1}{2} + \beta_1\right) = \alpha_2 t_2 + \beta_2$$ (2)

Since the equilibrium is symmetric, $\alpha_1 = \alpha_2 = \alpha$

$\beta_1 = \beta_2 = \beta$. Plugging them in (2), we get

$$\frac{1}{2} t_1 + \frac{1}{4} + \beta/2 = \alpha t_1 + \beta$$

$$\frac{1}{2} t_2 + \frac{1}{4} + \beta/2 = \alpha t_2 + \beta$$

easy to see $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$ satisfies all of them.

$$\boxed{b^*(t) = \frac{1}{2} t + \frac{1}{2}}$$

(b) What is the expected payoff of bidder i conditioned on her type t_i in this equilibrium? **2 points.**

using the above expression for the utility

$$E[u_i \mid t_i] = P(t_i > t_j)\left(t_i + \frac{1}{2} - b_i\right) = \left[t_i \cdot \frac{t_i}{2}\right] = \frac{1}{2} t_i^2$$

END OF QUESTION PAPER. GOOD LUCK!