CS 228 : Logic in Computer Science

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Dealing with Equality

Assume φ is in Skolem Normal Form and uses "=". We define a equisatisfiable formula φ_E which does not use "=".

- Let τ be the signature of φ. Let E be a binary relation not in τ.
- Let $\varphi_{\neq}$ be the sentence obtained by replacing all occurrences of $t_1 = t_2$ in φ with $E(t_1, t_2)$.
- Define φ_{ER} to be the sentence

$$\forall x \forall y \forall z (E(x,x) \wedge ((E(x,y) \leftrightarrow E(y,x)) \wedge (E(x,y) \wedge E(y,z) \rightarrow E(x,z)))$$

- For each relation R in τ, define φ_R as

$$\forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_n ((\bigwedge_{i=1}^{n} E(x_i, y_i) \wedge R(x_1, \ldots, x_n)) \rightarrow R(y_1, \ldots, y_n))$$

- Let $\varphi_1 = \bigwedge_{R \in \tau} \varphi_R$

Dealing with Equality

- For each function f in τ, define φ_f as

$$\forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_n ((\bigwedge_{i=1}^{n} E(x_i, y_i) \rightarrow E(f(x_1, \dots, x_n), f(y_1, \dots, y_n)))$$

- Let $\varphi_2 = \bigwedge_{f \in \tau} \varphi_f$
- Let $\psi_E = \varphi_{\neq} \wedge \varphi_{ER} \wedge \varphi_1 \wedge \varphi_2$
- Convert ψ_E to Prenex normal form to obtain φ_E in Skolem normal form

For any formula φ in Skolem normal form, φ is satisfiable iff φ_E is satisfiable

An Example

$\varphi = \forall x[(f(x) \neq x) \wedge (f(f(x)) = x)]$.

- φ is satisfiable : $\mathcal{A} = (\{0,1\}, f^{\mathcal{A}}(0) = 1, f^{\mathcal{A}}(1) = 0)$ and $\mathcal{A} \models \varphi$.
- $\varphi_{\neq} = \forall x[\neg E(f(x), x) \wedge E(f(f(x)), x)]$
- $\varphi_2 = \forall x \forall y[E(x,y) \rightarrow E(f(x), f(y))]$
- Conjunct $\varphi_{\neq}, \varphi_2$ and φ_{ER} and convert to Prenex normal form
- $$\begin{aligned}\varphi_E = \forall x \forall y \forall z[(\neg E(f(x), x) \wedge E(f(f(x)), x)) \wedge \\ (E(x,y) \rightarrow E(f(x), f(y))) \wedge E(x,x) \wedge \\ (E(x,y) \wedge E(y,z) \rightarrow E(x,z))]\end{aligned}$$
- By Herbrand's Theorem, φ_E has a Herbrand model $M = (\{c, f(c), f(f(c)), \ldots, \}, E^M = \{(t,t') \in H(\varphi_E) \mid$ the number of f's in both t, t' have the same parity$\})$
- $M \models \varphi_E$

Herbrand's Method

Given a FO sentence φ, is it satisfiable? Wlg, assume that φ is equality-free and is in Skolem normal form.

- Let $\varphi = \forall x_1 \ldots \forall x_n \psi(x_1, \ldots, x_n)$
- Let $H(\varphi)$ be the Herbrand universe of φ
- Let $E(\varphi) = \{\psi(t_1, \ldots, t_n) \mid t_1, \ldots, t_n \in H(\varphi)\}$ be the set obtained by substituting terms from $H(\varphi)$ for the variables $x_1, \ldots, x_n$ in φ
- φ is satisfiable iff $E(\varphi)$ is satisfiable

Herbrand's Method

- Assume φ is satisfiable. Then $\mathcal{A} \models \forall x_1, \ldots, x_n \psi(x_1, \ldots, x_n)$
- Then $\mathcal{A} \models \psi(t_1, \ldots, t_n)$ where $t_1, \ldots, t_n \in H(\varphi)$
- Then $\mathcal{A} \models \varphi_i$ for all $\varphi_i \in E(\varphi)$
- Hence, $E(\varphi)$ is satisfiable.

Herbrand's Method

- Assume $E(\varphi)$ is satisfiable. $E(\varphi)$ is a set of equality-free sentences.
- By Herbrand's Theorem, there is a Herbrand model M for $E(\varphi)$.
- The Herbrand signature for $E(\varphi)$ is the same as the Herbrand signature of φ.
- The universe of M is $H(\varphi)$. For $t_1, \ldots, t_n \in H(\varphi)$, $M \models \psi(t_1, \ldots, t_n)$
- Then $M \models \forall x_1 \ldots x_n \psi(x_1, \ldots, x_n)$
- Then $M \models \varphi$ and φ is satisfiable.
- φ is unsatisfiable iff $E(\varphi)$ is unsatisfiable.

Checking Unsatisfiability of φ

- $E(\varphi) = \{\varphi_1, \varphi_2, \dots\}$ is a set of quantifier free sentences, so it can be seen as a set of propositional logic formulae

Checking Unsatisfiability of φ

- $E(\varphi) = \{\varphi_1, \varphi_2, \dots\}$ is a set of quantifier free sentences, so it can be seen as a set of propositional logic formulae
- Since φ is in Skolem normal form, each formula $\varphi_i \in E(\varphi)$ is in CNF
- We know that $E(\varphi)$ is unsatisfiable iff $\emptyset \in Res^*(E(\varphi))$
- By Compactness Theorem of propositional logic, there is some finite subset $F = \{\varphi_1, \dots, \varphi_m\} \subseteq E(\varphi)$ such that $\emptyset \in Res^*(F)$
- So if $\emptyset \in Res^*(\{\varphi_1, \dots, \varphi_m\}$ for some finite m, we conclude φ is unsatisfiable

Checking Satisfiability of φ

- If $\emptyset \notin \mathit{Res}^*(\{\varphi_1, \ldots, \varphi_m\}$, then we look at $\mathit{Res}^*(\{\varphi_1, \ldots, \varphi_m, \varphi_{m+1}\}$
- If $\emptyset \notin \mathit{Res}^*(\{\varphi_1, \ldots, \varphi_{m+1}\}$, then we look at $\mathit{Res}^*(\{\varphi_1, \ldots, \varphi_{m+1}, \varphi_{m+2}\}$

  $\vdots$
- If φ is satisfiable, then this procedure will continue.

Wrapping Up

- We have a method to show that a FOL formula φ is unsatisfiable
- First, write φ in equality free Skolem normal form
- Check if $\emptyset \in \mathit{Res}^*(E(\varphi))$, this may take some time
- There is a more systematic resolution for FOL which we do not cover (this also uses Herbrand Theory)
- We also do not cover a direct undecidability proof for the satisfiability of FOL (at least now)