CS 228: Logic in CS Autumn 2024

Quiz 1

Total Marks: 25 *21 August 2024*

1. Answer each of the following questions with a short justification. Below, φ, ψ represent wffs. **Answers without justification will fetch NO marks, even if your answer is correct (1 mark each)**

   (a) Let φ be a tautology. Then the formula $\psi \rightarrow \varphi$ is also a tautology. [True/False]

   (b) If $\varphi \rightarrow \psi$ is equivalent to $\perp$, then so is its contrapositive. [True/False]

   (c) If φ and ψ are equivalent, then so are $\neg\varphi$ and $\neg\psi$. [True/False]

   (d) The formula $\perp \rightarrow \varphi$ is a valid formula. [True/False]

   (e) There is an assignment α under which $p \rightarrow q$ and $q \rightarrow p$ can both evaluate to false. [True/False]

   (f) It is possible for an implication $\varphi \rightarrow \psi$ and its contrapositive $\neg\psi \rightarrow \neg\varphi$ to have different truth values. [True/False]

   (g) For propositional variables p and q, is $(p \rightarrow q) \rightarrow [(p \rightarrow q) \rightarrow q]$ a tautology?

   (h) If $\varphi \models \psi$ and φ and ψ have no variables in common, then comment on the satisfiability/validity of φ and ψ.

Solution

(a) True. If φ is a tautology, it is true under every interpretation. Therefore, from the semantics of $\rightarrow$ it follows that $\psi \rightarrow \varphi$ will also be true under every interpretation since φ is always true regardless of ψ.

(b) True. If $\varphi \rightarrow \psi$ is equivalent to $\perp$ (false), then $\varphi \rightarrow \psi$ is always false. Its contrapositive, $\neg\psi \rightarrow \neg\varphi$, will also be false in the same circumstances. This is because the contrapositive of an implication has the same truth value as the original implication (this can be verified from truth table).

(c) True. If φ and ψ are equivalent, they have the same truth value in all cases. Therefore, $\neg\varphi$ and $\neg\psi$ will also have the same truth value in all cases, making them equivalent as well.

(d) True. The formula $\perp \rightarrow \varphi$ is always true because an implication with a false antecedent is always true regardless of the consequent.

(e) False. Clear from the truth table

(f) False. Evident from the truth table

(g) No. For $[\![p]\!] = [\![q]\!] = 0$, the formula evaluates to False

(h) Either φ is UNSAT, otherwise, ψ is Valid

Rubrics

For each of the above question:

- 0 marks if the answer is wrong / No justification is provided / Justification is incomplete or incorrect
- 1 mark for correct answer with proper justification.

Note. In questions where it is written"Clear from Truth table", you are expected to draw the truth table and draw a conclusion.

2. **(2 marks)** We have discussed in class that the validity of a CNF can be determined in polynomial time. Suppose you are provided with an algorithm, CNF-VAL, which takes a CNF formula as input and returns whether the formula is valid or not, in polynomial time. We claim to use CNF-VAL to check the satisfiability of any CNF formula as follows: First, we run CNF-VAL on the given CNF formula. If the output is `VALID`, we conclude that the formula is `SAT`. If not, we negate the formula, convert it into CNF, and run CNF-VAL again. If the output is `VALID` on the negated formula, we conclude the original formula is `UNSAT`; otherwise, it is `SAT`. Given that CNF-VAL operates in polynomial time, it follows that the satisfiability of any CNF formula can also be determined in polynomial time.

   Is this reasoning correct? (Note: For this question, we do not consider the empty CNF.)

Solution

The reasoning is **incorrect**. This is because,if CNF-VAL returns not `VALID`, and we negate it, we get a DNF, and converting a DNF into CNF *can possibly* take exponential time. So there is no guarantee that the whole procedure terminates in polynomial time

Rubrics

- 0 marks. If the answer is that "reasoning is correct" / answer is that "reasoning is incorrect" with incorrect explanation.

  One common argument is that if the reason is correct, then the satisfiability of any CNF can be done in poly time, but since SAT problem is NP complete, this is not possible. Though the overall idea is correct, this does not get any marks as this doesn't comment anything on the procedure mentioned in the question.
- 2 mark. If the answer is that "reasoning is incorrect", with correct explanation.
- Partial marks also awarded as is necessary.

3. **(5 marks) Prove or disprove** : From any CNF formula φ, one can compute in polynomial time an equisatisfiable formula $\psi_1 \wedge \psi_2$ where ψ_1 is a Horn formula and ψ_2 is a 2-CNF formula.

   If you think the statement is true, explain how to compute ψ_1 and ψ_2, and show that $\psi_1 \wedge \psi_2$ is equisatisfiable with φ. Otherwise, give a counterexample and prove that in this case no such ψ_1 and ψ_2 can exist.

Solution

The statement is True. Suppose φ has the propositional variables $p_1, \ldots, p_n$. Introduce new propositional variables $q_1, \ldots, q_n$ and consider the formula

$$\psi = \varphi^* \wedge (p_1 \vee q_1) \wedge (\neg p_1 \vee \neg q_1) \wedge \cdots \wedge (p_n \vee q_n) \wedge (\neg p_n \vee \neg q_n)$$

where φ^* is obtained from φ by replacing each positive literal p_i with $\neg q_i$. Note that φ^* has only negative literals and hence is a Horn formula.

Rubrics

- 0 marks: If the statement is taken to be False / statement is taken to be True and incorrect explanation is provided.

  One common mistake is that many of you claimed that since HornSAT and 2-CNFSat are both poly time, if there exist some ψ_1 and ψ_2 as in the question, then the satisfiability of any CNF can be done in poly time. This argument is wrong. Because, if we run HornSAT on ψ_1 and 2-CNFSat on ψ_2 separately, and both return `SAT`, we cannot be sure that $\psi_1 \wedge \psi_2$ is `SAT`, because ψ_1 and ψ_2 may not be true for the same assignment. (consider the case where $\psi_1 = p$ and $\psi_2 = \neg p$, but $\psi_1 \wedge \psi_2$ is `UNSAT`)

- 5 marks: If ψ_1 and ψ_2 are explicitly stated or, a polynomial time method for computing them is given. (3 marks for this), and if it is proven that $\psi_1 \wedge \psi_2$ is equisatisfiable with φ (2 marks for this).

- Partial marks are also awarded, as and when it is necessary

4. **(5 marks)** A perfect matching in an undirected graph $G = (V, E)$ is a subset of edges $M \subseteq E$ such that every vertex $v \in V$ is an endpoint of exactly one edge in M. Given a finite graph G, describe how to obtain a propositional logic formula φ_G such that φ_G is satisfiable iff G has a perfect matching. φ_G must be computable in time polynomial in $|V|$. **Answers without proper explanation are bound to lose marks. Make sure you properly indicate what each propositional variable mean.**

Solution

Introduce a propositional variable p_e for each edge $e \in E$. For each vertex $v \in V$, let $E(v)$ be the set of edges with v as an endpoint. Then the formula is

$$\bigwedge_{v \in V} \Big(\bigvee_{e \in E(v)} p_e \quad \wedge \bigwedge_{e, e' \in E(v), e \neq e'} (\neg p_e \vee \neg p_{e'}) \Big)$$

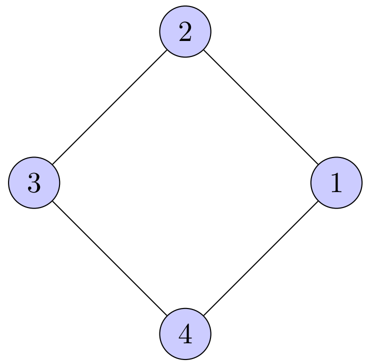

Figure 1: Cyclic Graph with 4 Nodes

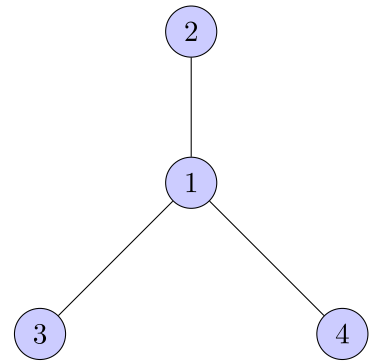

Figure 2: 3-Star Graph

Rubrics

- 0 marks:
  - If formula is blatantly wrong.
  - If variables and what they represent are ambiguous/not very clearly stated (no explanation of thought process at cribs counts).
  - Most incorrect formulas seen will fail on either Figure 1 (which has a perfect matching) or Figure 2 (which does not have a perfect matching). Do not crib if it does not pass both tests above.
- 5 marks: If a correct formula is shown, the set of propositional variables is clearly indicated, and the equivalence of `SAT` on this formula and the perfect matching problem is shown in a reasonable manner (for example, clearly explained meanings of the propositional variables).
- Partial marks are also awarded, as and when it is necessary.

5. **(5 marks)** Using natural deduction, show that $\neg y \vdash [x \leftrightarrow ((\neg x \wedge y) \vee (x \wedge \neg y))]$

   For the purpose of this question you may use $\leftrightarrow$ introduction rule as follows:

$$\frac{\varphi \to \psi \quad \psi \to \varphi}{\varphi \leftrightarrow \psi} \leftrightarrow\text{-introduction}$$

   where φ and ψ are wffs.

   Make sure that you use only basic proof rules (like $\wedge-i$, $\vee-i$, MP etc.), or derived rules like (MT, LEM, etc.), which are discussed in class. or $\leftrightarrow$ introduction rule described above. Using other semantic equivalences are bound to lose marks.

Solution

1.	$\neg y$	premise
2.	x	assumption
3.	$x \wedge \neg y$	$\wedge - i$ 1, 2
4.	$(\neg x \wedge y) \vee (x \wedge \neg y)$	$\vee - i_2$ 3
5.	$x \to ((\neg x \wedge y) \vee (x \wedge \neg y))$	$\to i$ 2-4
6.	$(\neg x \wedge y) \vee (x \wedge \neg y)$	assumption
7.	$(\neg x \wedge y)$	assumption
8.	y	$\wedge - e_2$ 7
9.	$\perp$	$\perp - i$ 1, 8
10.	x	$\perp - e$ 9
11.	$(x \wedge \neg y)$	assumption
12.	x	$\wedge - e_1$ 11
13.	x	$\vee - e$ 6, 7-9, 11-12
14.	$((\neg x \wedge y) \vee (x \wedge \neg y)) \to x$	$\to i$ 6-13
15.	$x \leftrightarrow ((\neg x \wedge y) \vee (x \wedge \neg y))$	$\leftrightarrow i$ 5,14

Rubrics

- 5 marks. For completely correct proof, ignoring small careless errors.
- 1 mark deducted everywhere where semantics has been used. If 5 or more such places exist, total of 0 marks awarded.
- 1 mark deducted for each incorrect usage of proof rules.
- Partial marks awarded as necessary