Logic in CS Autumn 2024

Problem Sheet 8

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1. **Theorem 1 (Propositional Compactness Theorem)** *Let S be a set of propositional formulae. S is satisfiable iff every finite subset of S is satisfiable.*

   Prove the above theorem using soundness and completeness of natural deduction.

   Solution

   ($\implies$) If S is satisfiable, any subset of S is also satisfiable.
   ($\impliedby$) We will prove the contrapositive. Assume that S is unsatisfiable, then we will show that there is a finite subset of S which also unsatisfiable.

   Since S is unsatisfiable there will be a Natural deduction proof such that $S \vdash \perp$. Now, this proof must be of finite length, hence it can only use some finite number of formulae from S to derive $\perp$. Let $S' \subseteq S$ be the finite subset of formula which we used to derived $\perp$ in the proof above. Then, $S' \vdash \perp$. Since Natural Deduction is sound and complete, $S' \models \perp$, i.e, S' is a finite subset of S which is unsatisfiable

2. **Theorem 2 (Compactness theorem for FOL)** *Let S be a set of formulae in first-order logic. S is satisfiable iff every finite subset of S is satisfiable.*

   Using the fact that FOL is semi-decidable, prove the compactness theorem for FOL.

   Solution

   ($\implies$) If S is satisfiable, any subset of S is also satisfiable.
   ($\impliedby$) We will prove the contrapositive. Assume that S is unsatisfiable, then we will show that there is a finite subset of S which also unsatisfiable.

   Due to the semi-decidability of FOL, $\emptyset \in Res^*(S)$ and we can derive this in finite number of steps. Let $S' \subseteq S$ be the finite subset of formula which we used to derived $\emptyset$, i.e, $\emptyset \in Res^*(S')$. Clearly, $S' \vdash_{resolution} \perp$. Since ground resolution in FOL is also sound and complete, $S' \models \perp$, i.e, S' is a finite subset of S which is unsatisfiable

3. In Problem Sheet 7, we saw how to write the following in MSO: "There is a path from node s to node t in the graph" using the signature $\tau = \{E\}$.

   Show, using compactness theorem, that you cannot capture this using FOL with the same signature.

Solution

In this question we define the length of a path as the number of edges in that path.

Let $\psi(s,t)$ be FOL formula representing "There is a path from node s to t in the graph". Similarly for $n \in \mathbb{N}, n \geq 1$, $\varphi_n(s,t)$ be FOL formula for "There is no path of length n from node s to t in the graph".

Define $S := \{\psi(s,t), \varphi_1(s,t), \varphi_2(s,t), \ldots\}$

Clearly, S is an unsatisfiable set of FOL formulae. Let S' be a finite subset of S. If we S' is empty or singleton sets they are trivially satisfiable.

- **Case 1:** $\psi \notin S'$. Then all the formula in S' corresponds to the statement "There is no path of length m from node s to t in the graph" for some finite number of m's. This is clearly satisfiable by choosing an assignment to s and t such that there is no path between them.

- **Case 2:** $\psi \in S'$. Let N be the largest n such that $\varphi_n(s,t)$ is present in S'. Then S' is satisfiable by choosing an assignment to s and t such that the length of the path between them is $N+1$. (*Think why this is true!*)

Thus, S is a set of unsatisfiable FOL formulae such that any finite subset of S is satisfiable. This contradicts with Compactness Theorem for FOL. Thus the given statement cannot be captured in FOL.