CS 405/6001: Game Theory and Algorithmic Mechanism Design

Autumn 2024
Check course webpage for registration policy

Swaprava Nath

IIT Bombay

Let us play a game: Neighboring King(Queen)dom's Dilemma

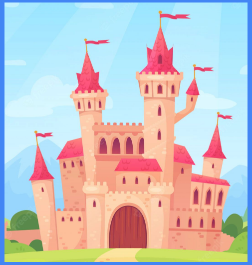
Each kingdom can invest either in Agriculture or War – but not both

- If both choose Agri – happiness is 5 for each
- If both choose War – happiness is 1 for each
- If one chooses Agri, but the other War – the Agri kingdom stand to lose everything and War kingdom gets happiness more than 5

What should the King / Queen do?

What is collectively best?

(Agriculture, Agriculture)



	Agriculture	War
Agriculture		
War		

Decisions are simultaneous

War is a Dominant Strategy for Queen as well as King

(War, War) is a Dominant Strategy Equilibrium

Professor’s Dilemma

- More effort in setting difficult exam papers
- Attentive in class leads to better learning
- Gives a positive feedback to the class
- Attentive = Positive feedback
- Not attentive = Negative feedback

Equilibrium

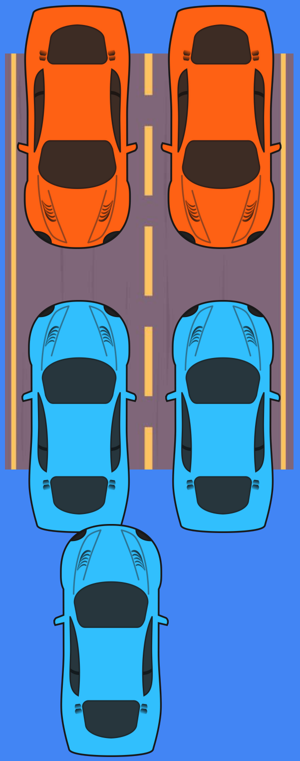
	Negative feedback	Positive feedback
Exams: hard		
Exams: easy		

Decisions are simultaneous

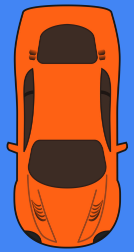
Easy exam is a Dominant Strategy for professor

Being attentive is a Dominant Strategy for student

Another game: Traffic Movement



Does this game have a dominant strategy equilibrium?

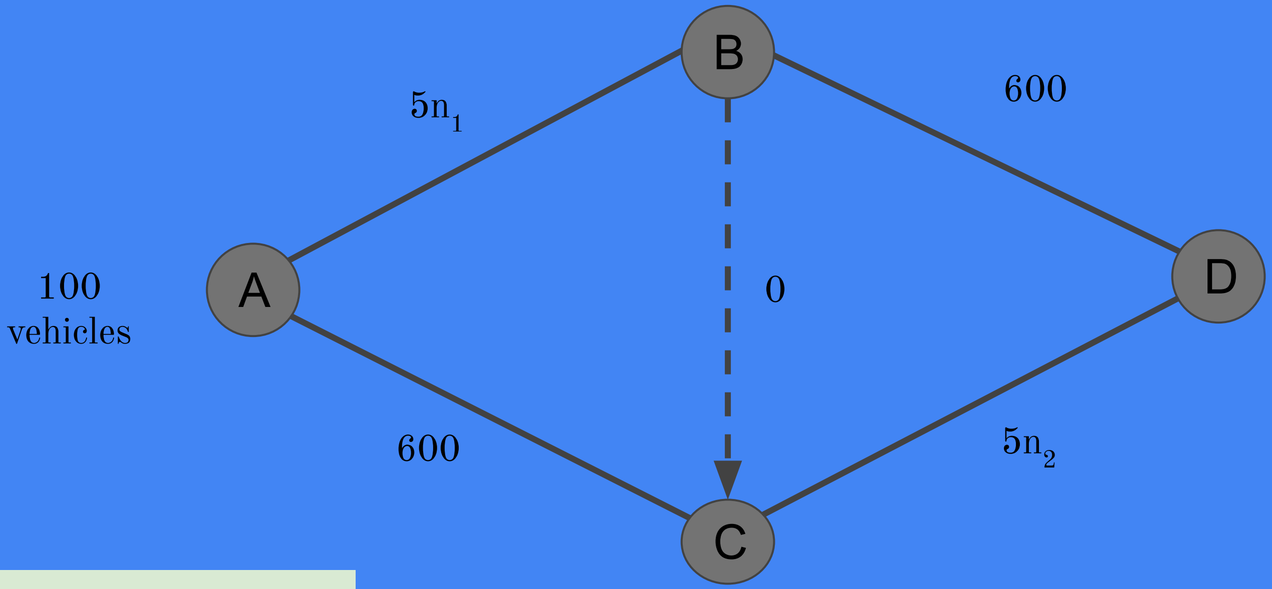


The Nash Equilibrium (John Nash, 1951)

	Left	Right
Left		
Right		

Equilibrium here is a strategy profile from where no player wants to unilaterally deviate

Adding resources (blindly) does not improve the society



n_k : number of vehicles on road k, $k = 1, 2$

Equilibrium (original)

50 vehicles each on paths ABD and ACD

Time for each vehicle = 850

Equilibrium (after)

100 vehicles on path ABCD

Time for each vehicle = 1000

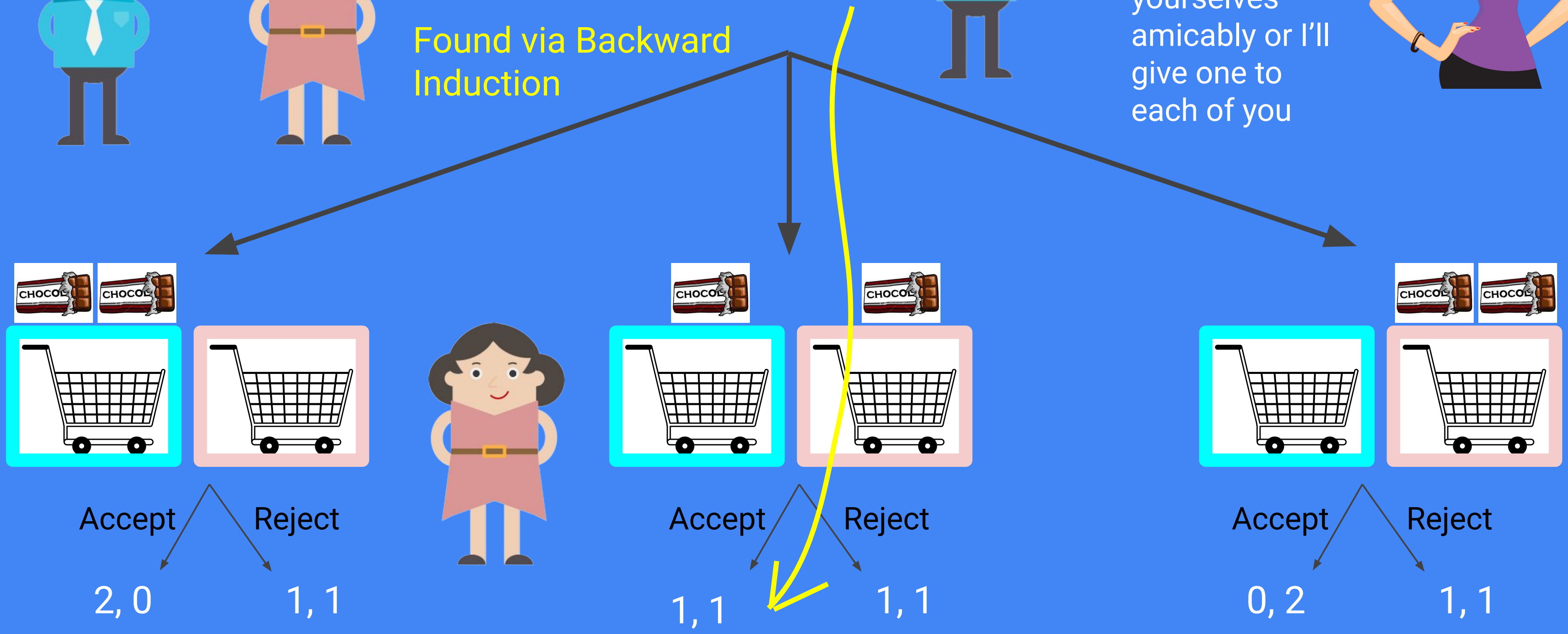
Design is important

All games do not end in the same round

Subgame Perfect
Nash Equilibrium

Found via Backward
Induction

Settle yourselves amicably or I'll give one to each of you



Example: Auction

Question: how to allocate the object to the individual who (truly) values it the most?

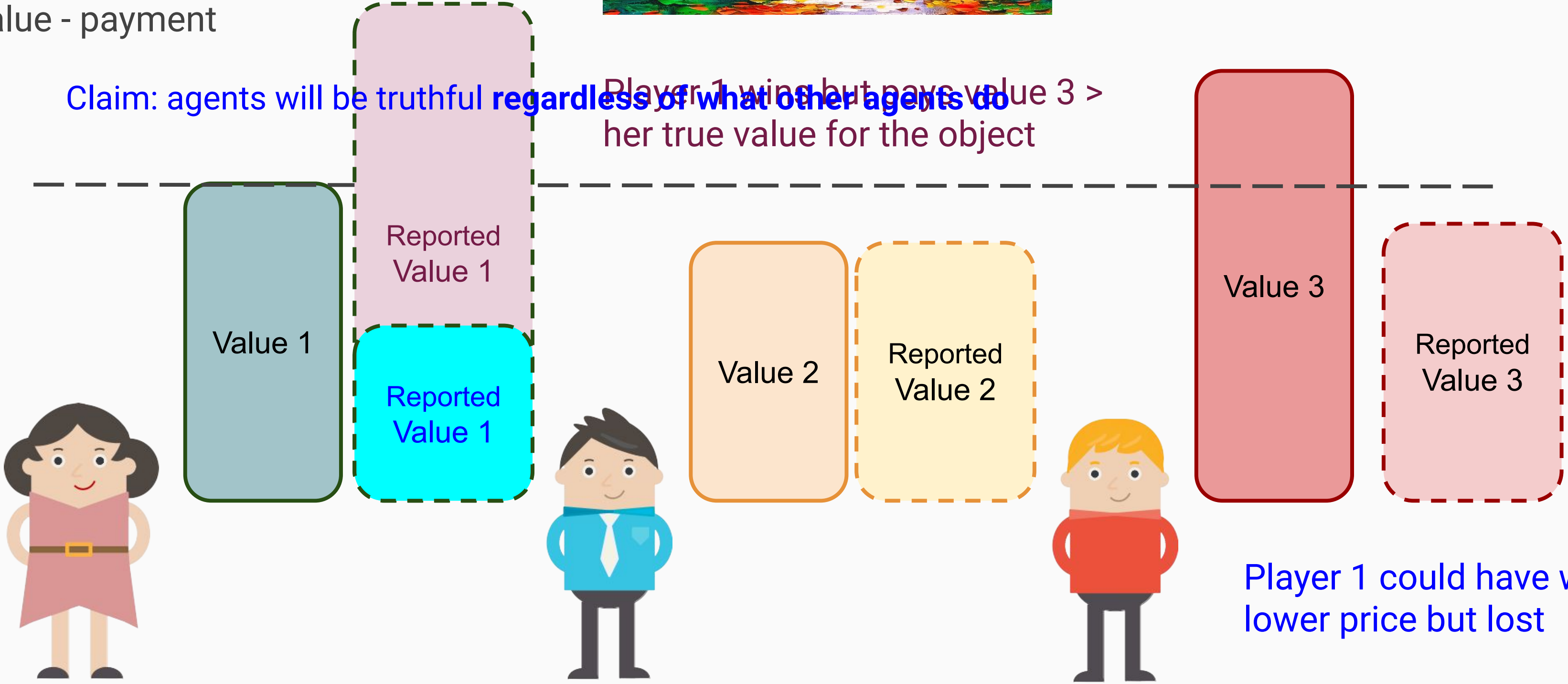
Agents want to maximize:
value - payment

- Ask everyone to bid
- Highest bidder wins the object
- Pays **second** highest bid

Second price auction

Claim: agents will be truthful **regardless of what other agents do**

Player 1 wins but pays value 3 > her true value for the object



Player 1 could have won at a lower price but lost

Can we design algorithms for a better society?

Fair division

Equal amount may have different values for an agent

Heterogeneous

Any fractional allocation is feasible

Divisible

Differing preferences

Different agents may have different preferences for the same piece

Proportional division

For each agent i

$v_i(A_i) \geq 1/n$

Each agent gets at least the average share

Envy free division

For each pair of agents i, j

$v_i(A_i) \geq v_i(A_j)$

Each agent likes her own share than others

Normalization: for each i, $v_i([0,1]) = 1$

"I cut, you choose" algorithm

Proportional?

Yes, agent 1 cuts ½, and agent 2 picks the larger

Envy-free?

Yes, agent 1 gets ½, which is the same as the other piece in his view
agent 2 picks first, can't envy the other piece

Fair division of indivisible objects

All are indivisible objects

Faculty retires and wants to give away his/her belongings to the department staff / existing faculty

Items:
1. Books
2. Shelves
3. Furnitures
4. Wall decor
5. Table decor
6. Electronic gadgets
7. Many more …

Notice that Envy-free allocation is no longer possible

Consider a single item and two agents

Envy free upto one good allocation

For each pair of agents i, j

$v_i(A_i) \geq v_i(A_j \setminus x_j)$, for some $x_j \in A_j$

Each agent likes her own share upto all but one item of every other agent

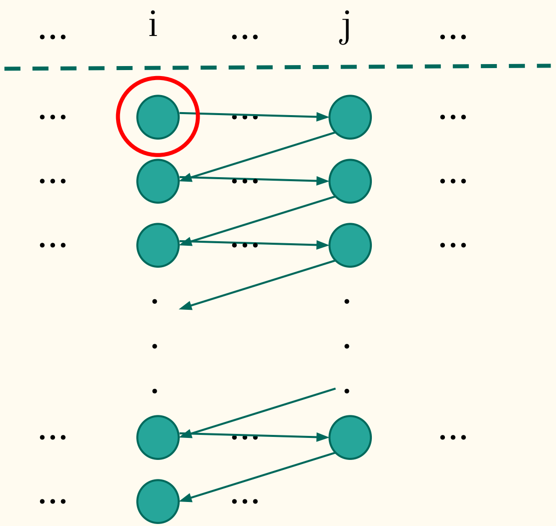
Envy free upto one good (EF-1)

All are indivisible objects

Always exists and computable in polynomial time!

Example for additive valuations: Round-Robin Algorithm

Place the agents in any arbitrary order, and ask them to pick their favorite remaining item



i does not envy j, since it picks before j

j may envy i, but not if the first item i picked is dropped

Round-Robin achieves EF-1 for additive valuations

Envy free upto one good allocation

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Each agent likes her own share upto all but one item of every other agent

- A game is an interaction between agents who want to maximize their utilities
- Game theory predicts the outcome of a game
- This is a predictive approach
- Mechanism design tries to design the game with desirable outcomes
- This is a prescriptive approach

Applications

- Online advertising – google, facebook, etc.
- Stable matching
- Kidney exchanges
- Automated priority scheduling
- Peer-grading
- Airlines scheduling
- Many more …

What you will learn in this course

- You will be equipped with a general purpose tool to analyze strategic behavior in multi-agent interaction
- Mathematically capture the situations of strategic agent modeling and interaction
- Design protocols / mechanisms that satisfy desirable economic and computational properties
- Applications in multi-agent environment like sponsored advertisements, crowdsourcing, social networks, internet-based trade

What you will need to follow this course

- Familiarity with formal mathematical reasoning
- Probability theory (detailed)
- Calculus
- Basics of computational complexity
- Moderate familiarity with computer programming (in any language)

Evaluation

- CS 405: Two quizzes -- 20% weightage for each, no project
- CS 6001: One course project (groups of size ≤ 3) -- 30% weightage, no quizzes
- One midsem and one endsem exam -- 30% weightage for each (for CS 405), 35% weightage for each (for CS 6001)
- Offline exams – as scheduled by the institute

Exam Schedule

- August 28: Quiz 1 – during class times
- Midsem: Institute scheduled (between 14 - 22 September)
- October 25: Quiz 2 – during class times
- Endsem: Institute scheduled (between 11 - 23 November)

Problem sets

- Will be given about a week before every exam (2 quizzes and 2 exams)
- You are supposed to solve all the problems
- The problem sets are NOT exhaustive – should look for other problems of similar kind from books, lecture notes, internet resources, past question papers, etc.

Tutorials

- On some Saturday/Sunday before each exam
- You may ask to revisit some topics or problem solving from the problems sets / otherwise
- If the problem is outside problem sets, post them on Piazza at least two days before bringing them to tutorials (failing this, such problems will not be discussed)
- Before asking the TA to solve a problem, you should show till what distance you have got

Team – so far ...

Swaprava Nath
Lead instructor

Ramsundar
Anandanarayanan, TA

Drashthi Doshi
TA

Sayantika Mandal
TA

Isha Arora
TA

Karan Godara
TA

Ameya Vikrama Singh
TA

Course content delivery

- The course will be taught in a **regular** classroom mode
- Lecture videos will be posted on the course website as additional resources
- Any additional reading materials will be posted on the course webpage
- Online discussion forum to clarify the doubts on the topics discussed

https://www.cse.iitb.ac.in/~swaprava/cs6001_07_2024.html

Piazza access code: `cs6001_2024`

Content sources

- Game Theory — Michael Maschler, Eilon Solan, Shmuel Zamir
- Multiagent Systems — Y. Shoham and K. Leyton Brown, Cambridge University Press
- Game Theory and Mechanism Design — Y. Narahari, World Scientific and IISc Press
- Lecture notes on Theory of Mechanism Design, by Debasis Mishra, Indian Statistical Institute, New Delhi
- Lecture notes on Individual and Collective Choice, by Arunava Sen, Indian Statistical Institute, New Delhi
- Preprints of Introduction to Economics and Computation, David C Parkes and Sven Seuken
- Several research papers
- Lecture notes (non-reviewed) on the course webpage

भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 1

Swaprava Nath

Slide preparation acknowledgments: Ramsundar Anandanarayanan and Harshvardhan Agarwal

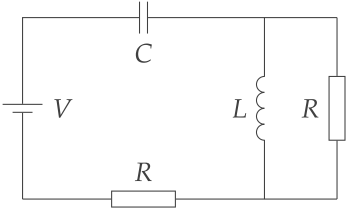
ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal

Contents

Typical Engineering Courses

- Circuit **analysis**



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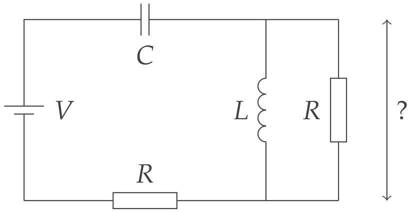

analysis

Typical Engineering Courses

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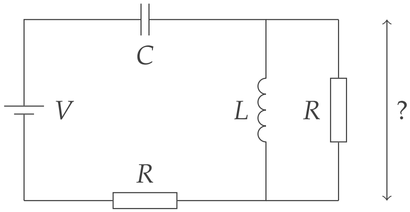

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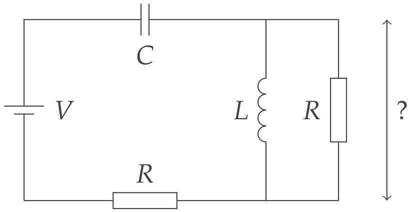

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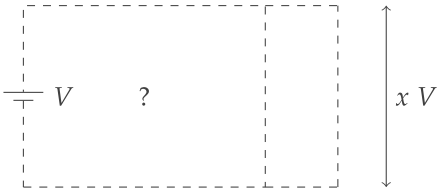

synthesis

Similarly ...

$\boxed{\text{Given Game}} \longrightarrow$

Similarly ...

$$\boxed{\text{Given Game}} \longrightarrow \text{Outcomes ?}$$

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Given Game $\longrightarrow$ Outcomes ?

Game Theory

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Given Game ⟶ Outcomes ? ⟶ Outcomes

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Game Theory

Game ? $\longrightarrow$ Outcomes

Mechanism Design

Similarly ...

Given Game $\longrightarrow$ Outcomes ?

Game Theory

Game ? $\longrightarrow$ Outcomes

Mechanism Design

- Social **analysis** and **synthesis**

Contents

Game: Neighboring Kingdom's Dilemma

		Rashtrakuta: Agri	Rashtrakuta: War
Pala	Agri	5, 5	0, 6
Pala	War	6, 0	1, 1

Game: Neighboring Kingdom's Dilemma

		Rashtrakuta	
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Question

What is a reasonable outcome of this game?

Game

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- Games can be of many *kinds* and *representations*:
  Normal form, **Extensive form**, **Static**, **Dynamic**, **Repeated**, **Stochastic**, ...

Game Theory

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- A player is **intelligent** if she knows the rules of the game perfectly and can pick (i.e., has that computational ability) the *best* action considering that there are other rational and intelligent players in the game
- **Goal of game theory:** **predict** the outcomes of a game (refer to the dilemma game)

Assumptions of Game Theory

This course is an axiomatic analysis of multi-agent behavior – and the axioms are as follows

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  - ... ad infinitum

Implication of CK: Blue-eyed islander problem

- **Location**: an isolated island (does not have any reflecting device)

[1]This person is correct beyond any question. Whatever he says must be true.

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Question

How does common knowledge percolate?

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Percolation of Common Knowledge

Let us think in steps

- If there was **one** blue-eyed man

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Assumption in Game Theory

The fact that all players are rational and intelligent is a common knowledge

Contents

Game of Chess

Description

- Two-player game: White (**W**) and Black (**B**) – 16 pieces each

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  1. **Win** for **W**: if W captures B king
  2. **Win** for **B**: if B captures W king
  3. **Draw**: everything else, e.g., if nobody has legal moves but kings are not in check, both players agree to a draw, board position is such that nobody can win, ...

Natural Questions from Game Theorist's perspective

Question

Does **W** have a winning strategy?
i.e., a plan of moves such that it wins **irrespective** of the moves of **B**?

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- Neither may be possible – not synonymous with the end of the game

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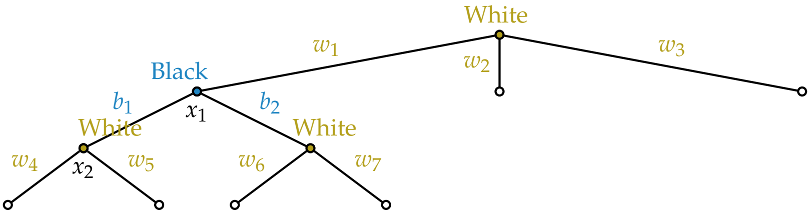
  - $x_k \to x_{k+1}$
    k even – created by a single action of W
    k odd – created by a single action of B

What is a strategy? (contd.)

Board positions may repeat in this tree, but a vertex is unique – **game situation**



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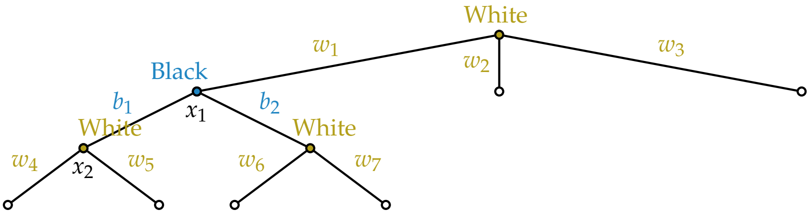

Strategy: mapping from **game situation** to action, i.e., what action to take at every vertex of this game tree

a complete plan to play the game at every game situation

What is a strategy? (contd.)

Definition (Strategy)

A **strategy** for **W** is a function s_W that associates every game situation $(x_0, x_1, x_2, \ldots, x_k) \in H$ (set of all game situations), k even, with a board position x_{k+1} such that the move $x_k \to x_{k+1}$ is a single valid move for W.

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Can a player guarantee an outcome?

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- Not obvious if such strategies exist.

Contents

An Early Result (von Neumann, 1928)

Theorem

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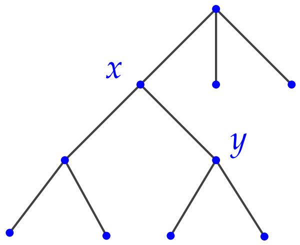
Chess would have been a boring game if any of these answers were known

Setup of the Proof

- Each vertex in the tree represents a game situation, the edges represent actions



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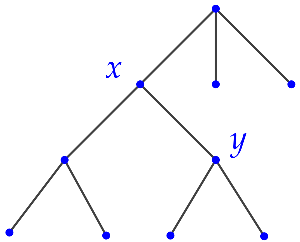

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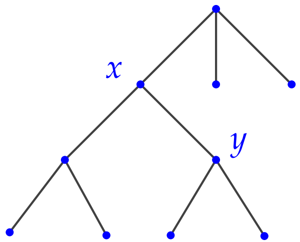

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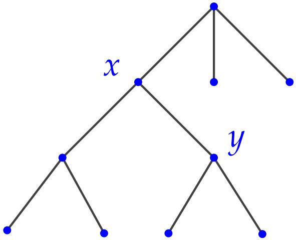

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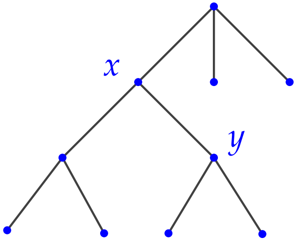

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Does the Theorem hold for $n_x = 1$?

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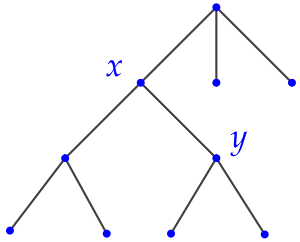

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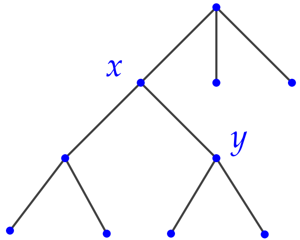

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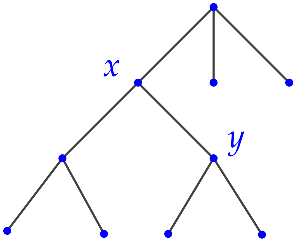
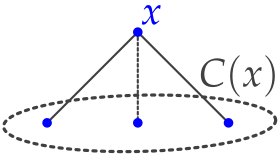


Extend to $n_x > 1$



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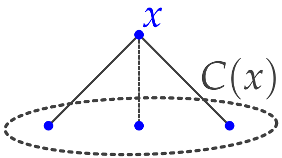

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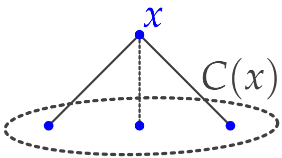

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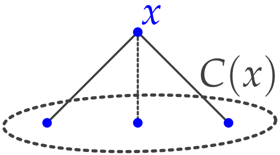

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- Let $C(x)$ denote vertices reachable from x in one move

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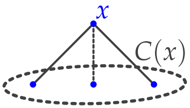

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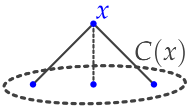

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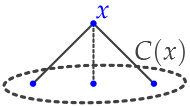

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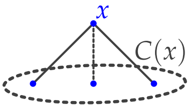

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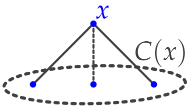

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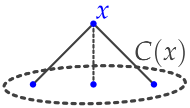

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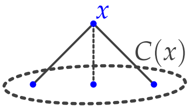

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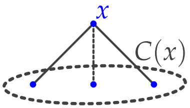

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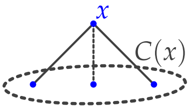

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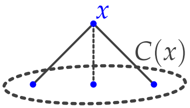
  - Hence **W** picks action to go to y', where **B** can only guarantee a draw (induction hypothesis)



भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

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Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 2

Swaprava Nath

Slide preparation acknowledgments: Ramsundar Anandanarayanan and Harshvardhan Agarwal

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal

Contents

Normal Form Games

- It is a representation technique for games – particularly suitable for **static games**
- In a *static game*, the players interact only once with each other

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- **Normal form** representation is a tuple $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$
- If S_i is finite $\forall i \in N$, this is called a finite game.

Example: Penalty Shoot Game

Shooter \ Goalkeeper	L	C	R
L	$-1, 1$	$1, -1$	$1, -1$
C	$1, -1$	$-1, 1$	$1, -1$
R	$1, -1$	$1, -1$	$-1, 1$

Example: Penalty Shoot Game

		Goalkeeper L	Goalkeeper C	Goalkeeper R
Shooter	L	$-1, 1$	$1, -1$	$1, -1$
	C	$1, -1$	$-1, 1$	$1, -1$
	R	$1, -1$	$1, -1$	$-1, 1$

- $N = \{1, 2\}$, $1 = \text{Shooter}, 2 = \text{Goalkeeper}$

Example: Penalty Shoot Game

Goalkeeper

Shooter	L	C	R
L	$-1, 1$	$1, -1$	$1, -1$
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- $N = \{1, 2\}, 1 = \text{Shooter}, 2 = \text{Goalkeeper}$
- $S_1 = S_2 = \{L, C, R\}$

Example: Penalty Shoot Game

Goalkeeper (columns), Shooter (rows)

	L	C	R
L	$-1, 1$	$1, -1$	$1, -1$
C	$1, -1$	$-1, 1$	$1, -1$
R	$1, -1$	$1, -1$	$-1, 1$

- $N = \{1, 2\}, 1 = \text{Shooter}, 2 = \text{Goalkeeper}$
- $S_1 = S_2 = \{L, C, R\}$
- $u_1(L, L) = -1, u_1(L, C) = u_1(L, R) = 1$

Example: Penalty Shoot Game

Goalkeeper

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- $u_2(L, L) = 1, u_2(L, C) = u_2(L, R) = -1$

Example: Penalty Shoot Game

Goalkeeper

Shooter	L	C	R
L	$-1, 1$	$1, -1$	$1, -1$
C	$1, -1$	$-1, 1$	$1, -1$
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- $N = \{1, 2\}, 1 = \text{Shooter}, 2 = \text{Goalkeeper}$
- $S_1 = S_2 = \{L, C, R\}$
- $u_1(L, L) = -1, u_1(L, C) = u_1(L, R) = 1$
- $u_2(L, L) = 1, u_2(L, C) = u_2(L, R) = -1$
- (loosely) $u_1(X, X) = -1 = -u_2(X, X), u_1(X, Y) = -u_2(X, Y) = 1$

Contents

Domination in NFGs

		Player 2 L	Player 2 C	Player 2 R
Player 1	U	$1, 0$	$1, 3$	$3, 2$
Player 1	D	$-1, 6$	$0, 5$	$3, 3$

Domination in NFGs

		Player 2 L	Player 2 C	Player 2 R
Player 1	U	$1, 0$	$1, 3$	$3, 2$
Player 1	D	$-1, 6$	$0, 5$	$3, 3$

Question

Will a **rational** Player 2 ever play R?

Dominated Strategy

Definition (Strictly Dominated Strategy)

A strategy $s_i' \in S_i$ of player i is **strictly dominated** if there exists another strategy $s_i \in S_i$ such that **for every strategy profile** $s_{-i} \in S_{-i}$ of the other players, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$.

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Definition (Weakly Dominated Strategy)

A strategy $s_i' \in S_i$ of player i is **weakly dominated** if there exists another strategy $s_i \in S_i$ such that **for every strategy profile** $s_{-i} \in S_{-i}$ of the other players $u_i(s_i, s_{-i}) \geqslant u_i(s_i', s_{-i})$ and **there exists some** $\tilde{s}_{-i} \in S_{-i}$ such that $u_i(s_i, \tilde{s}_{-i}) > u_i(s_i', \tilde{s}_{-i})$.

Dominated Strategy

		Player 2		
		L	C	R
Player 1	U	$1, 0$	$1, 3$	$3, 2$
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Dominated Strategy

		Player 2		
		L	C	R
Player 1	U	1, 0	1, 3	3, 2
	D	−1, 6	0, 5	3, 3

Strictly / Weakly dominated strategy?

Dominated Strategy

		Player 2 L	C	R
Player 1	U	1, 0	1, 3	3, 2
	D	−1, 6	0, 5	3, 3

Strictly / Weakly dominated strategy?

R is strictly dominated (by C) while D is weakly dominated (by U)

Dominant Strategy

A strategy s_i' can be dominated by s_i, and a different strategy s_i'' can be dominated by $\tilde{s}_i$

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Examples of **dominant strategy**

- Neighboring kingdom's dilemma
- Indivisible item for sale

Neighboring Kingdom's Dilemma

		Rashtrakuta	
		Agri	War
Pala	Agri	5, 5	0, 6
	War	6, 0	1, 1

Neighboring Kingdom's Dilemma

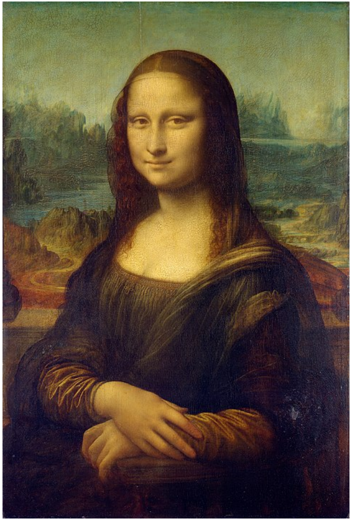
		Rashtrakuta: Agri	Rashtrakuta: War
Pala	Agri	5, 5	0, 6
Pala	War	6, 0	1, 1

Question

Is there a dominant strategy in this game? Which kind?

Indivisible Item for Sale

- Two players value an indivisible item as v_1 and v_2 respectively



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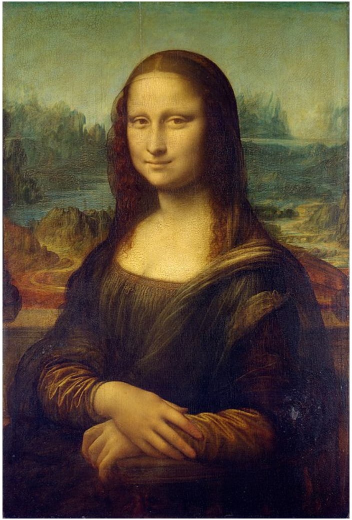

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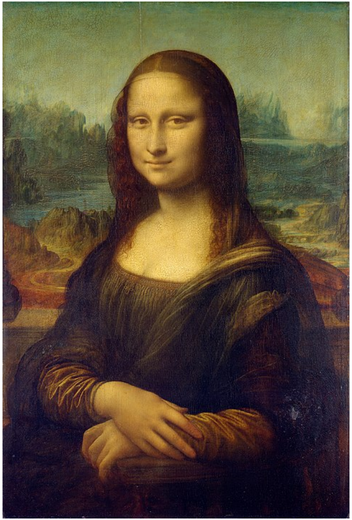

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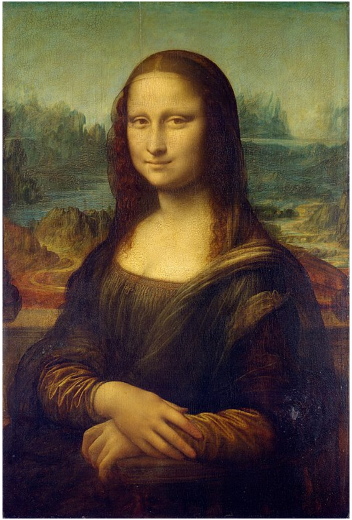

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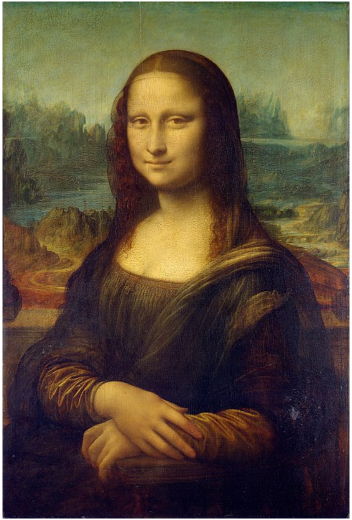
- utility of losing player = 0



Indivisible Item for Sale

Normal form representation of the game

- $N = \{1, 2\}$, $S_1 = S_2 = [0, M]$
- Agents pick s_i, while their **real** value for the item is v_i, and s_i may **not** be the same as v_i

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$$u_1(s_1, s_2) = \begin{cases} v_1 - s_2 & \text{if } s_1 \geqslant s_2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$u_2(s_1, s_2) = \begin{cases} v_2 - s_1 & \text{if } s_1 < s_2 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

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Definition (Weak Domination)

A strategy $s_i \in S_i$ of player i weakly dominates $s_i' \in S_i$ if **for every strategy profile** $s_{-i} \in S_{-i}$ of the other players $u_i(s_i, s_{-i}) \geqslant u_i(s_i', s_{-i})$ and **there exists some** $\tilde{s}_{-i} \in S_{-i}$ such that $u_i(s_i, \tilde{s}_{-i}) > u_i(s_i', \tilde{s}_{-i})$. $[\tilde{s}_{-i} = \tilde{s}_{-i}(s_i, s_i')]$

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Dominant Strategy Equilibrium

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A strategy profile $(s_1^*, s_2^*, \ldots, s_n^*)$ is a strictly (weakly) dominant strategy equilibrium (SDSE/WDSE) if s_i^* is strictly (weakly) dominant strategy $\forall i \in N$.

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Example of **dominant strategy equilibrium**

Player 2 (columns), Player 1 (rows)

	D	E
A	5, 5	0, 5
B	5, 0	1, 1
C	4, 0	1, 1

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Question

What kind of equilibrium in this game?

Existence of Dominant Strategies

Not guaranteed!

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Player 2

Player 1	L	R
L	1, 1	0, 0
R	0, 0	1, 1

Co-ordination game

Existence of Dominant Strategies

Not guaranteed!

		Player 2	
		L	R
Player 1	L	1, 1	0, 0
	R	0, 0	1, 1

Co-ordination game

		Friend 2	
		F	C
Friend 1	F	2, 1	0, 0
	C	0, 0	1, 2

Football or Cricket Game

Existence of Dominant Strategies

Not guaranteed!

Player 1 \ Player 2	L	R
L	1, 1	0, 0
R	0, 0	1, 1

Co-ordination game

Friend 1 \ Friend 2	F	C
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Football or Cricket Game

If **dominance** cannot explain a reasonable outcome – refine equilibrium concept

Contents

Nash Equilibrium (Nash 1951)

No player gains by a unilateral deviation

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No player gains by a unilateral deviation

Definition (Nash Equilibrium)

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Football or Cricket Game

Best Response View

- A best response of a player i against the strategy profile s_{-i} of other players is a strategy that gives the maximum utility i.e.,

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Question

Relationship between SDSE, WDSE, PSNE?

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Question

Relationship between SDSE, WDSE, PSNE?

Answer

SDSE $\implies$ WDSE $\implies$ PSNE

How to find equilibrium?

- Rational players do not play **dominated strategies**

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Player 2

Player 1		L	C	R
	T	1, 2	2, 3	0, 3
	M	2, 2	2, 1	3, 2
	B	2, 1	0, 0	1, 0

- Order T, R, B, C $\rightarrow (M, L) : (2, 2)$

How to find equilibrium?

- Rational players do not play **dominated strategies**
- To obtain rational outcomes eliminate dominated strategies
- For strictly dominated strategies the order of elimination does not matter **(homework)**
- It matters for weakly dominated strategies – some reasonable outcomes are also eliminated

Player 2 (columns), Player 1 (rows)

	L	C	R
T	1,2	2,3	0,3
M	2,2	2,1	3,2
B	2,1	0,0	1,0

- Order T, R, B, C $\rightarrow (M, L) : (2, 2)$
- Order B, L, C, T $\rightarrow (M, R) : (3, 2)$

Contents

Risk Aversion of Players

		Player 2: L	Player 2: R
Player 1	T	$2, 1$	$1, -20$
	M	$3, 0$	$-10, 1$
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What if the other player does not pick an equilibrium action (Nash)?

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Question

What if the other player does not pick an equilibrium action (Nash)?

Picking T is less risky for player 1

Max-min Strategy

Definition

The worst case optimal choice is **max-min strategy**

$$u_i(s_i, s_{-i})$$

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Max-min value (utility at the max-min strategy) of player i is given by

$$\underline{v}_i = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

$$u_i(s_i^{\max\min}, t_{-i}) \geqslant \underline{v}_i, \qquad \forall t_{-i} \in S_{-i}$$

Max-min and Dominant Strategies

Theorem

*If s_i^* is **dominant strategy** for player i, then it is a **max-min strategy** for player i as well.*

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Let s_i^* be dominant strategy for player i

$$u_i(s_i^*, s_{-i}) \geqslant u_i(s_i', s_{-i}),\ \forall s_{-i} \in S_{-i}, \forall s_i' \in S_i \tag{5}$$

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$\square$

Max-min and PSNE

Theorem

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□

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- Dominance cannot explain all outcomes; games may not have dominant strategies
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Question

What happens to stability and security when some strategies are eliminated?

Iterated elimination of dominated strategies (contd.)

Player 1 \ Player 2	L	C	R
T	1, 2	2, 3	0, 3
M	2, 2	2, 1	3, 2
B	2, 1	0, 0	1, 0

Iterated elimination of dominated strategies (contd.)

		Player 2		
		L	C	R
Player 1	T	1, 2	2, 3	0, 3
	M	2, 2	2, 1	3, 2
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- Order T, R, B, C $\rightarrow (M, L) : (2, 2)$

Iterated elimination of dominated strategies (contd.)

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Iterated elimination of dominated strategies (contd.)

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Player 1	T	1, 2	2, 3	0, 3
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Iterated elimination of dominated strategies (contd.)

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Question

Does it change the maxmin value?

Iterated elimination of dominated strategies (contd.)

Player 2

Player 1		L	C	R
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Consider in the above example: elimination of dominated strategy B for player 1

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Maxmin values	Player 1	Player 2
Before	2	0
After	2	2

Maxmin value is not affected for the player whose **dominated strategy** is removed

A Result for Iterated Elimination

Theorem

Consider an NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, and let $s'_j \in S_j$ be a dominated strategy. Let G' be the residual game after removing s'_j. Then, the maxmin value of j in G' is equal to her maxmin value in G.

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- Maxmin is the 'max' of all 'min's

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- Maxmin is the ‘max’ of all ‘min’s
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A Result for Iterated Elimination

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Consider an NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, and let $s_j' \in S_j$ be a dominated strategy. Let G' be the residual game after removing s_j'. Then, the maxmin value of j in G' is equal to her maxmin value in G.

Intuition

- Maxmin is the 'max' of all 'min's
- Elimination affects one 'min'
- But that does not affect the 'max' since the strategy was dominated

Proof

Maxmin value of player j in G $$\underline{v}_j = \max_{s_j \in S_j} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j})$$

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Suppose t_j dominates s'_j in $G, t_j \in S_j \setminus \{s'_j\},$ then, $\qquad u_j(t_j, s_{-j}) \geqslant u_j(s'_j, s_{-j}), \forall s_{-j} \in S_{-j}$

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$\implies$ $\max_{s_j \in S_j \setminus \{s'_j\}} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}) \geqslant \min_{s_{-j} \in S_{-j}} u_j(t_j, s_{-j}) \geqslant \min_{s_{-j} \in S_{-j}} u_j(s'_j, s_{-j})$

Proof (contd.)

$$\underline{v}_j \quad \text{[maxmin value of } j \text{ in } G]$$

$$= \max_{s_j \in S_j} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j})$$

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$$
\begin{aligned}
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\end{aligned}
$$

Proof (contd.)

$$
\begin{aligned}
&\underline{v}_j \quad [\text{maxmin value of } j \text{ in } G] \\
&= \max_{s_j \in S_j} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}) \\
&= \max \left\{ \max_{s_j \in S_j \setminus \{s_j'\}} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}), \min_{s_{-j} \in S_{-j}} u_j(s_j', s_{-j}) \right\} \\
&= \max_{s_j \in S_j \setminus \{s_j'\}} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}), \text{ because of the previous inequality} \\
&= \underline{v}_j' \quad [\text{maxmin value of } j \text{ in } G']
\end{aligned}
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Preservation of PSNE

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Consider G and $\hat{G}$ are games before and after elimination of a strategy (not necessarily dominated). If s^* is a PSNE in G and survives in $\hat{G}$, then s^* is a PSNE in $\hat{G}$ too.

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Consider G and $\hat{G}$ are games before and after elimination of a strategy (not necessarily dominated). If s^* is a PSNE in G and survives in $\hat{G}$, then s^* is a PSNE in $\hat{G}$ too.

Intuition

PSNE was a maxima of utility of i among the strategies of i. Removing other strategies does not affect maximality.

Proof: exercise.

Can new equilibrium be generated?

Theorem

Consider NFG G. Let $\hat{s}_j$ be a weakly dominated strategy of j. If $\hat{G}$ is obtained from G eliminating $\hat{s}_j$, then every PSNE of $\hat{G}$ is a PSNE of G.

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No new PSNE if the eliminated strategy is dominated

But old PSNEs could be killed: saw in the previous example

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In the game $\hat{G}$, modified strategy sets are $\hat{S}_j = S_j \setminus \{\hat{s}_j\}$, $\hat{S}_i = S_i, \forall i \neq j$

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Since s^* is a PSNE in $\hat{G}$ and $t_j \in \hat{S}_j$

$$u_j(s_j^*, s_{-j}^*) \geqslant u_j(t_j, s_{-j}^*) \geqslant u_j(\hat{s}_j, s_{-j}^*)$$

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Contents

Matrix games: *two player zero-sum* games

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A NFG $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ with $N = \{1, 2\}$ and $u_1 + u_2 \equiv 0$

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Why called **matrix** game?

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Possible to represent the game with only one matrix considering the utilities of player 1; player 2's utilities are negative of this matrix

Example: Penalty shoot game

		Player 2: L	Player 2: R
Player 1	L	$-1, 1$	$1, -1$
	R	$1, -1$	$-1, 1$

Example: Penalty shoot game

Player 1 \ Player 2	L	R
L	$-1, 1$	$1, -1$
R	$1, -1$	$-1, 1$

$$\Longrightarrow \qquad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} =: u$$

Player 1	L	R	$maxmin$
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CS 6001: Game Theory and Algorithmic Mechanism Design

Week 3

Swaprava Nath

Slide preparation acknowledgments: Onkar Borade and Rounak Dalmia

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal

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Another example

		Player 2		
		L	C	R
Player 1	T	$3, -3$	$-5, 5$	$-2, 2$
	M	$1, -1$	$4, -4$	$1, -1$
	B	$6, -6$	$-3, 3$	$-5, 5$

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Player 1	L	C	R
T	$3, -3$	$-5, 5$	$-2, 2$
M	$1, -1$	$4, -4$	$1, -1$
B	$6, -6$	$-3, 3$	$-5, 5$

Player 1	L	C	R	maxmin
T	3	-5	-2	-5
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Two examples together

Player 1

	L	R	maxmin
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What are the PSNEs for the above games?

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Answer

Left: no PSNE; **Right:** (M,R)

Saddle point

Saddle point of a matrix

The value is simultaneously the maximum in its column and minimum in its row i.e., maximum for player 1 and minimum for player 2

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What are the saddle points for the previous two games?

Saddle point

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L	-1	1
R	1	-1

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		L	R
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For the first example: no saddle point, for the second: (M,R)

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Answer

For the first example: no saddle point, for the second: (M,R)

Theorem

In a matrix game with utility matrix u, (s_1^*, s_2^*) is a saddle point iff it is a PSNE.

Saddle point and PSNE

Proof.

Consider (s_1^*, s_2^*) to be a saddle point. By definition of saddle point, this happens iff $u(s_1^*, s_2^*) \geqslant u(s_1, s_2^*), \forall s_1 \in S_1$ and $u(s_1^*, s_2^*) \leqslant u(s_1^*, s_2), \forall s_2 \in S_2$. Since, $u \equiv u_1 \equiv -u_2$, the above is equivalent to $u_1(s_1^*, s_2^*) \geqslant u_1(s_1, s_2^*), \forall s_1 \in S_1$ and $u_2(s_1^*, s_2^*) \geqslant u_2(s_1^*, s_2), \forall s_2 \in S_2 \Leftrightarrow (s_1^*, s_2^*)$ is a PSNE. $\square$

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$$\underline{v} = \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2) \qquad \textbf{maxmin}$$

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Question

How are the maxmin and minmax values related?

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Lemma

For matrix games $\overline{v} \geqslant \underline{v}$.

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$$u(s_1, s_2) \geqslant \min_{t_2 \in S_2} u(s_1, t_2), \ \forall s_1, s_2, \qquad \text{definition of min}$$

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$$\Rightarrow \max_{t_1 \in S_1} u(t_1, s_2) \geqslant \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2), \ \forall s_2 \in S_2 \qquad \text{RHS was dominated for each } s_1$$

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\Rightarrow \min_{t_2 \in S_2} \max_{t_1 \in S_1} u(t_1, t_2) &\geqslant \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2) && \text{RHS was a constant}
\end{aligned}
$$

□

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Earlier examples and security values (contd.)

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PSNE Theorem

Define the following strategies

$$s_1^* \in \arg \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2), \qquad \textbf{maxmin strategy of player 1}$$

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Theorem

A game has a PSNE (equivalently, a saddle point) if and only if $\overline{v} = \underline{v} = u(s_1^*, s_2^*)$, where s_1^* and s_2^* are maxmin and minmax strategies for players 1 and 2 respectively.

Corollary: (s_1^*, s_2^*) is a PSNE

Proof of the PSNE Theorem

Proof

($\implies$) let (s_1^*, s_2^*) is a PSNE $\implies \overline{v} = \underline{v} = u(s_1^*, s_2^*)$ and s_1^* and s_2^* are maxmin and minmax

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$$\begin{aligned} \implies u(s_1^*, s_2^*) &\geqslant \max_{t_1 \in S_1} u(t_1, s_2^*) \\ &\geqslant \min_{t_2 \in S_2} \max_{t_1 \in S_1} u(t_1, t_2), \text{ since } s_2^* \text{ is a specific strategy} \\ &= \overline{v} \end{aligned}$$

Proof of the PSNE Theorem

Proof

($\implies$) let (s_1^*, s_2^*) is a PSNE $\implies \overline{v} = \underline{v} = u(s_1^*, s_2^*)$ and s_1^* and s_2^* are maxmin and minmax
Since (s_1^*, s_2^*) is a PSNE, $u(s_1^*, s_2^*) \geqslant u(s_1, s_2^*), \forall s_1 \in S_1$.

$$
\begin{aligned}
\implies u(s_1^*, s_2^*) &\geqslant \max_{t_1 \in S_1} u(t_1, s_2^*) \\
&\geqslant \min_{t_2 \in S_2} \max_{t_1 \in S_1} u(t_1, t_2), \text{ since } s_2^* \text{ is a specific strategy} \\
&= \overline{v}
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Similarly, using the same argument for player 2, we get $\underline{v} \geqslant u(s_1^*, s_2^*)$

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But $\overline{v} \geqslant \underline{v}$ (from the previous lemma), hence

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$$\begin{aligned} & u(s_1^*, s_2^*) \geqslant \overline{v} \geqslant \underline{v} \geqslant u(s_1^*, s_2^*) \\ \implies & u(s_1^*, s_2^*) = \overline{v} = \underline{v} \end{aligned}$$

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Also implies that the maxmin for 1 and minmax for 2 are s_1^* and s_2^* respectively.

Proof of the PSNE Theorem (contd.)

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($\Longleftarrow$) i.e. $\overline{v} = \underline{v} = u(s_1^*, s_2^*)$ and s_1^* and s_2^* are maxmin and minmax $\implies$ (s_1^*, s_2^*) is a PSNE

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But $v = u(s_1^*, s_2^*)$. Substitute and get that (s_1^*, s_2^*) is a PSNE.

Contents

Mixed Strategies

Mixed strategy: *probability distribution* over the set of strategies of that player

Player 2

Player 1	L	R
L	$-1, 1$	$1, -1$
R	$1, -1$	$-1, 1$

Mixed Strategies

Mixed strategy: *probability distribution* over the set of strategies of that player

		Player 2 L	Player 2 R
Player 1	$\frac{2}{3}$ L	$-1, 1$	$1, -1$
	$\frac{1}{3}$ R	$1, -1$	$-1, 1$

Mixed Strategies

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		Player 2	
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$$\Delta A = \{p \in [0,1]^{|A|} : \sum_{a \in A} p(a) = 1\}$$

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- Mixed strategy set of player 1: $\Delta S_1 = \Delta\{L, R\}$, $(\frac{2}{3}, \frac{1}{3}) \in \Delta S_1$

Mixed Strategies (contd.)

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$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \cdots \sum_{s_n \in S_n} \sigma_1(s_1) \cdot \sigma_2(s_2) \cdots \sigma_n(s_n) \; u_i(s_1, s_2, \ldots, s_n)$$

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- We are *overloading* u_i to denote the utility at *pure* and *mixed* strategies
- Utility at a mixed strategy is the **expectation** of the utilities at pure strategies; all the rules of expectation hold, e.g., linearity, conditional expectation, etc.

Example

Player 2

Player 1		$\frac{4}{5}$ L	$\frac{1}{5}$ R
	$\frac{2}{3}$ L	$-1, 1$	$1, -1$
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$u_1(\sigma_1, \sigma_2)$

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$$u_1(\sigma_1, \sigma_2) = \frac{2}{3} \cdot \frac{4}{5} \cdot (-1)$$

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Contents

Mixed Strategies Nash Equilibrium

Definition (Mixed Strategy Nash Equilibrium)

A **mixed strategy Nash equilibrium (MSNE)** is a mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$, s.t.

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geqslant u_i(\sigma_i, \sigma_{-i}^*), \ \forall \sigma_i \in \Delta S_i \text{ and } \forall i \in N.$$

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Question

Relation between **PSNE** and **MSNE**?

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Answer

PSNE $\implies$ MSNE

An Alternative Definition

Theorem

A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$, *is an* ***MSNE*** *if and only if* $\forall s_i \in S_i$ *and* $\forall i \in N$

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Proof.

$(\Rightarrow)$: The pure strategy s_i is a special case of the mixed strategy, the mixed strategy with s_i having probability 1. Inequality holds by definition of MSNE

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($\Leftarrow$) Pick an arbitrary mixed strategy σ_i of player i

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Examples of MSNE

Question

Is the mixed strategy profile an **MSNE**?

		Player 2	
		$\frac{4}{5}$ L	$\frac{1}{5}$ R
Player 1	$\frac{2}{3}$ L	$-1, 1$	$1, -1$
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- To answer this, we need to show that there does not exist any better mixed strategy for the player

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- Expected utility of player 2 from $L = 2/3 \cdot 1 + 1/3 \cdot (-1) = 1/3$
- Expected utility of player 2 from $R = 2/3 \cdot (-1) + 1/3 \cdot 1 = -1/3$

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Player 2 (columns), Player 1 (rows)

	$\frac{4}{5}$ L	$\frac{1}{5}$ R
$\frac{2}{3}$ L	$-1, 1$	$1, -1$
$\frac{1}{3}$ R	$1, -1$	$-1, 1$

- Expected utility will increase if some probability is transferred from R to L
- $\Rightarrow$ the current profile is **not** an MSNE
- Some balance in the utilities is needed

Examples of MSNE

Question

Is the mixed strategy profile an **MSNE**?

Player 2

Player 1		$\frac{1}{2}$ L	$\frac{1}{2}$ R
	$\frac{1}{2}$ L	$-1, 1$	$1, -1$
	$\frac{1}{2}$ R	$1, -1$	$-1, 1$

- Expected utility will increase if some probability is transferred from R to L
- $\Rightarrow$ the current profile is **not** an MSNE
- Some balance in the utilities is needed
- **Does there exist any improving mixed strategy?**

Contents

How to find an MSNE

Definition (Support of mixed strategy/probability distribution)

For mixed strategy σ_i, the subset of strategy set of i on which σ_i has a positive mass is called the **support** of σ_i and is denoted by $\delta(\sigma_i)$. Formally, $\delta(\sigma_i) = \{s_i \in S_i : \sigma_i(s_i) > 0\}$.

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Using the definition of support, here is a characterization of MSNE

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A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ *is an MSNE iff*[a] $\forall i \in N$

[a]This is a shorthand for 'if and only if'.

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2. $u_i(s_i, \sigma_{-i}^*) \geqslant u_i(s_i', \sigma_{-i}^*), \forall s_i \subseteq \delta(\sigma_i^*), s_i' \notin \delta(\sigma_i^*)$.

[a] This is a shorthand for 'if and only if'.

Implication

Consider Penalty Shoot Game

		Goalkeeper L	Goalkeeper R
Shooter	L	$-1, 1$	$1, -1$
Shooter	R	$1, -1$	$-1, 1$

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Case 1: support profile $(\{L\}, \{L\})$: for player 1 , $s'_1 = R$ – violates condition 2

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Case 1: support profile $(\{L\}, \{L\})$: for player 1 , $s_1' = R$ – violates condition 2

Case 2: support profile $(\{L, R\}, \{L\})$ – symmetric for the other case

For Player 1, the expected utility has to be the same for L and R - not possible – violates condition 1

Implication

Case 3: support profile $(\{L, R\}, \{L, R\})$: condition 2 is vacuously satisfied

Implication

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For condition 1, let player 1 chooses L w.p. p and player 2 choose L w.p. q

Implication

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For player 1:

$$u_1(L, (q, 1-q)) = u_1(R, (q, 1-q)) \Rightarrow (-1)q + 1 \cdot (1-q) = 1 \cdot q + (-1)(1-q) \Rightarrow q = \frac{1}{2}$$

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$$u_2((p, 1-p), L) = u_2((p, 1-p), R) \Rightarrow p = \frac{1}{2}$$

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MSNE =

$$\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$$

Exercises

Player 2

Player 1	F	C
F	2, 1	0, 0
C	0, 0	1, 2

Player 2

Player 1	F	C	D
F	2, 1	0, 0	1, 1
C	0, 0	1, 2	2, 0

Contents

MSNE Characterization Theorem

Theorem

A mixed strategy profile is an MSNE iff $\forall i \in N$

1. $u_i(s_i, \sigma^*_{-i})$ *is identical* $\forall s_i \in \delta(\sigma^*_i)$,
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Observations:

- $\max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma_{-i}) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i})$
  maximizing w.r.t. a distribution $\Leftrightarrow$ whole probability mass at max

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- $\max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma_{-i}) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i})$
  maximizing w.r.t. a distribution $\Leftrightarrow$ whole probability mass at max
- If $(\sigma^*_i, \sigma^*_{-i})$ is an MSNE, then

$$\max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma^*_{-i}) = \max_{s_i \in S_i} u_i(s_i, \sigma^*_{-i}) = \max_{s_i \in \delta(\sigma^*_i)} u_i(s_i, \sigma^*_{-i})$$

  the maximizer must lie in $\delta(\sigma^*_i)$ – if not, then put all probability mass on that $s'_i \notin \delta(\sigma^*_i)$ that has the maximum value of the utility – $(\sigma^*_i, \sigma^*_{-i})$ is not a MSNE

Proof of MSNE Characterization Theorem

($\Rightarrow$) Given $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) = \max_{s_i \in \delta(\sigma_i^*)} u_i(s_i, \sigma_{-i}^*) \tag{1}$$

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By definition of expected utility

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{s_i \in S_i} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) = \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) \tag{2}$$

Proof of MSNE Characterization Theorem

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Equations (1) and (2) are equal, i.e., max is equal to positive weighted average – can happen only when all values are same: **proves condition 1**

Proof (contd.)

For **condition 2**: Suppose for contradiction, there exists $s_i \in \delta(\sigma_i^*)$ and $s_i' \notin \delta(\sigma_i^*)$ s.t. $u_i(s_i, \sigma_{-i}^*) < u_i(s_i', \sigma_{-i}^*)$

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We can shift the probability mass $\sigma^*(s_i)$ to s_i', this new mixed strategy gives a strict higher utility to player i: contradicts MSNE

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This completes the proof of the necessary direction.

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$(\Leftarrow)$ Given the 2 conditions of the theorem, need to show that $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE

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For **condition 2**: Suppose for contradiction, there exists $s_i \in \delta(\sigma_i^*)$ and $s_i' \notin \delta(\sigma_i^*)$ s.t. $u_i(s_i, \sigma_{-i}^*) < u_i(s_i', \sigma_{-i}^*)$

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This completes the proof of the necessary direction.

$(\Leftarrow)$ Given the 2 conditions of the theorem, need to show that $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE

$$\begin{array}{lll} \text{Let} & u_i(s_i, \sigma_{-i}^*) = m_i(\sigma_{-i}^*), \forall s_i \in \delta(\sigma_i^*) & \textbf{condition 1} \\ \text{Note} & m_i(\sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) & \textbf{condition 2} \end{array}$$

Proof (contd.)

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*), \qquad \textbf{by definition of } \delta(\sigma_i^*)$$

Proof (contd.)

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\begin{aligned}
u_i(\sigma_i^*, \sigma_{-i}^*) &= \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*), && \textbf{by definition of } \delta(\sigma_i^*) \\
&= m_i(\sigma_{-i}^*) && \textbf{previous conclusion}
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This proves the sufficient direction. The result yields an algorithmic way to find MSNE

Contents

MSNE characterization theorem to algorithm

Consider a NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$

MSNE characterization theorem to algorithm

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The total number of supports of $S_1 \times S_2 \times S_3 \cdots \times S_n$ is
$K = (2^{|S_1|} - 1) \times (2^{|S_2|} - 1) \times \cdots \times (2^{|S_n|} - 1)$

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For every support profile $X_1 \times X_2 \times \cdots X_n$, where $X_i \subseteq S_i$, solve the following feasibility program

Program

$$w_i = \sum_{s_{-i}\in S_{-i}} \Big(\prod_{j\neq i} \sigma_j(s_j)\Big) \cdot u_i(s_i, s_{-i}), \forall s_i \in X_i, \forall i \in N$$

$$w_i \geqslant \sum_{s_{-i}\in S_{-i}} \Big(\prod_{j\neq i} \sigma_j(s_j)\Big) \cdot u_i(s_i, s_{-i}), \forall s_i \in S_i \setminus X_i, \forall i \in N$$

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Remarks on the algorithm

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- This is not a linear program unless $n = 2$
- For general game, there is no poly-time algorithm
- Problem of finding an MSNE is PPAD-complete [Polynomial Parity Argument on Directed graphs] [1]

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MSNE and Dominance

The previous algorithm can be applied to a smaller set of strategies by removing the dominated strategies

Is there a dominated strategy in this game? Domination can be via mixed strategies too

Player 1 \ Player 2	L	R
T	4, 1	2, 5
M	1, 3	6, 2
B	2, 2	3, 3

MSNE and Dominance

Theorem

If a pure strategy s_i is strictly dominated by a mixed strategy $\sigma_i \in \Delta S_i$, then in every MSNE of the game, s_i is chosen with probability zero.

So, We can remove such strategies without loss of equilibrium

Contents

Existence of MSNE

Definition (Finite Games)

A game is said to be **finite** when the number of players is finite, and each player has a finite set of strategies.

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Proof requires a few tools and a result from real analysis. Proof is separately given in the course webpage.

Existence of MSNE

Some background for understanding the proof.

- A set $S \subseteq \mathbb{R}^n$ is **convex** if $\forall x, y \in S$ and $\forall \lambda \in [0,1]$, $\lambda x + (1-\lambda)y \in S$.

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- A set $S \subseteq \mathbb{R}^n$ is **bounded** if $\exists x_0 \in \mathbb{R}^n$ and $R \in (0, \infty)$ s.t. $\forall x \in S, ||x - x_0||_2 < R$.

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A result from real analysis (proof omitted):

Brouwer's fixed point theorem

If $S \subseteq \mathbb{R}^n$ is **convex** and **compact** and $T : S \to S$, is **continuous** then T has a fixed point, i.e., $\exists\, x^* \in S$ s.t. $T(x^*) = x^*$.

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CS 6001: Game Theory and Algorithmic Mechanism Design

Week 4

Swaprava Nath

Slide preparation acknowledgments: Onkar Borade and Rounak Dalmia

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal

Contents

Recap

- MSNE $\rightarrow$ weakest notion of equilibrium so far

Recap

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- Existence is guaranteed for finite games

Recap

- MSNE $\rightarrow$ weakest notion of equilibrium so far
- Existence is guaranteed for finite games
- Finding MSNE is computationally expensive

Contents

Correlated Strategy and Equilibrium

Alternative approach - entry of a **mediating** agent/device

Why do we need such an agent?

- Alternative explanation of player rationality

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Alternative approach - entry of a **mediating** agent/device

Why do we need such an agent?

- Alternative explanation of player rationality
- Utility for all players may get better
- Computational tractability

Correlated Strategy and Equilibrium

Player 1 \ Player 2	Wait	Go
Wait	$0, 0$	$1, 2$
Go	$2, 1$	$-10, -10$

Busy cross road game

Correlated Strategy and Equilibrium

		Player 2: Wait	Player 2: Go
Player 1	Wait	$0, 0$	$1, 2$
Player 1	Go	$2, 1$	$-10, -10$

Busy cross road game

Nash solutions for the above are

- One waits and the other goes, or
- Large probability of waiting

Correlated Strategy and Equilibrium

- In practice, something else happens

Correlated Strategy and Equilibrium

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- A traffic light guides the players, and the players agree to this plan (**Why?**)

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- The **trusted third party** is called the **mediator**

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- If the strategies are **enforceable** then it is an equilibrium (**correlated**)

Correlated Strategy and Equilibrium (contd.)

Definition (Correlated Strategy)

A **correlated strategy** is a mapping $\pi : S_1 \times S_2 \times \cdots \times S_n \rightarrow [0,1]$ s.t. $\sum_{s \in S} \pi(s) = 1$.

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Correlated Strategy and Equilibrium (contd.)

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Discussions:

- The mediator suggests the actions after running its randomization device π
- Every agent's best response is to follow it if others are also following it

Examples

		Friend 2	
		F	C
Friend 1	F	2, 1	0, 0
	C	0, 0	1, 2

Football or Cricket Game

Examples

Friend 1 \ Friend 2	F	C
F	2, 1	0, 0
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Football or Cricket Game

MSNE: $\left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$

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Car 1 \ Car 2	Wait	Go
Wait	0, 0	1, 2
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Question

Are there other CEs of this game?

Contents

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CE finding is to solve a set of linear equations

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Two set of constraints

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m^n inequalities

Computing Correlated Equilibrium (contd.)

- The inequalities together represent a **feasibility linear program** that is easier to compute than MSNE

[1] take log of both quantities to understand this point

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- **MSNE** : total number of support profiles = $O(2^{mn})$
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- Moreover, this can also be used to optimize some objective function, e.g., maximize the sum of utilities of the players

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Comparison with the previous equilibrium notions

Theorem

For every ***MSNE*** σ^**, there exists a* ***CE*** π^*

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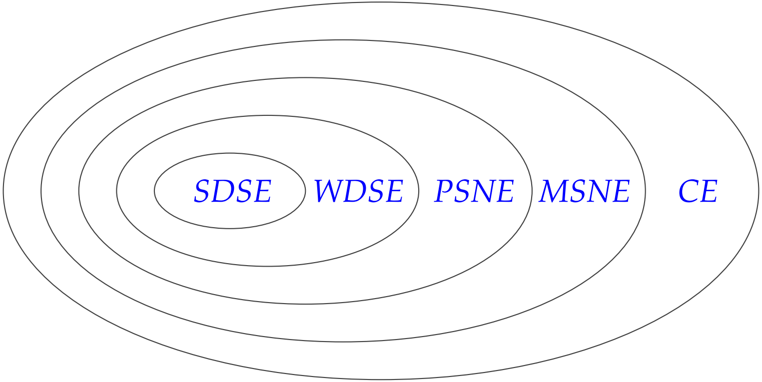
Venn diagram of games having equilibrium



Familiar Game: Neighboring Kingdom's Dilemma

Player 1 \ Player 2	Agri	War
Agri	5, 5	0, 6
War	6, 0	1, 1

Question

What is the CE of this game?

Summary so far

- Normal form games

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Richer representation of games

- More appropriate for multi-stage games, e.g. **chess**

Richer representation of games

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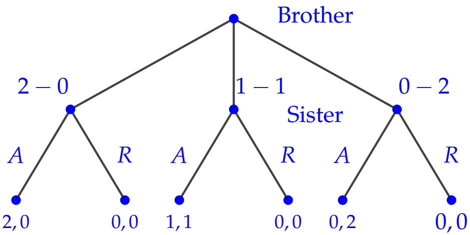
- Players interact in a sequence - the sequence of actions is the history of the game

Contents

Perfect Information Extensive Form Games (PIEFG)

- Brother-sister Chocolate Division
- **Disagreement** $\rightarrow$ both chocolates taken away

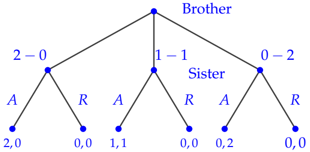


Perfect Information Extensive Form Games (PIEFG)

Formal capture

$$\text{PIEFG } \langle N, A, H, X, P, (u_i)_{i \in N} \rangle$$

- N: a set of players



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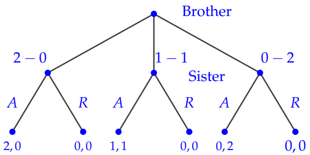

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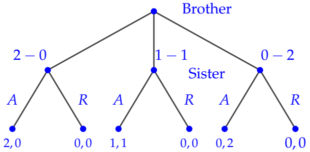

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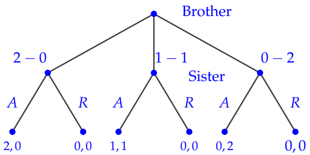

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- N: a set of players
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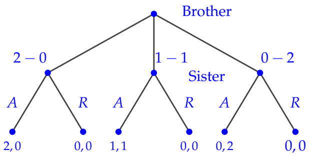

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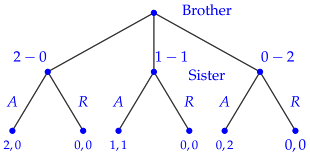

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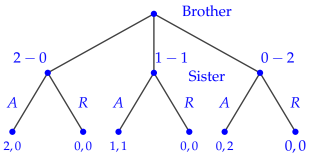

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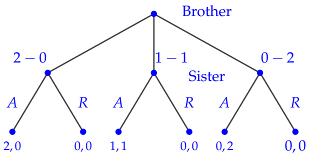

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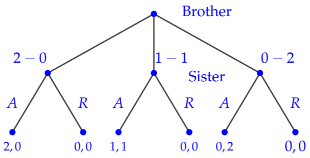

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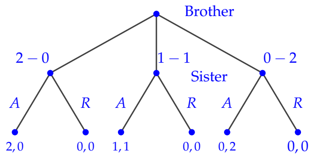

Perfect Information Extensive Form Games (PIEFG)

The **strategy** of a player in an EFG is a tuple of actions at every history where the player plays, i.e.,

$$S_i = \times_{\{h \in H : P(h) = i\}} X(h)$$

Remember:

- Strategy is a **complete contingency plan** of the player

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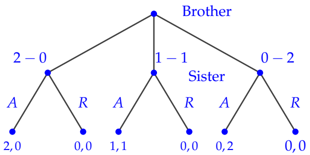
Remember:

- Strategy is a **complete contingency plan** of the player
- It enumerates potential actions a player can take at every node where she can play, even though some combination of actions may never be executed together

Perfect Information Extensive Form Games (PIEFG)

- $N = \{1, 2\}$ – Brother and Sister respectively



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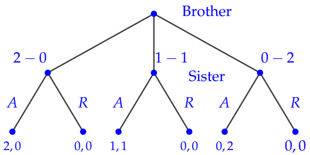

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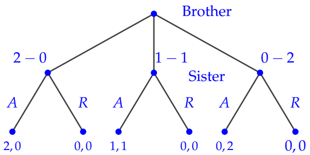

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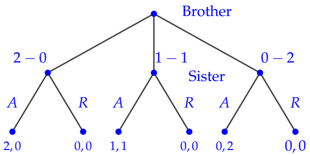

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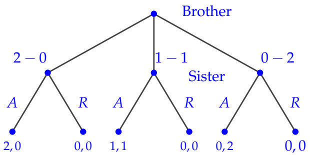

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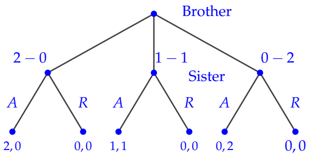

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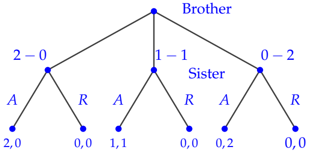

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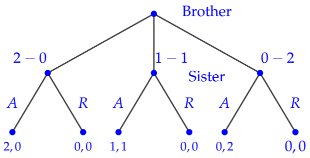

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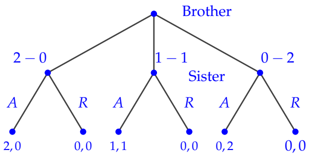

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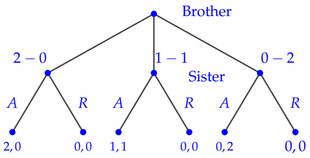

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- $S_1 = \{2-0, 1-1, 0-2\}$
- $S_2 = \{A, R\} \times \{A, R\} \times \{A, R\} = \{AAA, AAR, ARA, ARR, RAA, RAR, RRA, RRR\}$

Transforming PIEFG into NFG

Once we have the S_1 and S_2, the game can be represented as an NFG

Brother \ Sister	AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR
2-0	2, 0	2, 0	2, 0	2, 0	0, 0	0, 0	0, 0	0, 0
1-1	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0
0-2	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0

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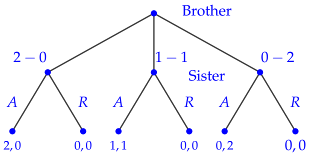
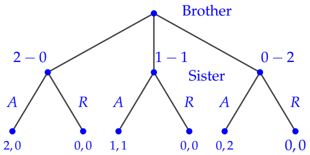
Brother \ Sister	AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR
2-0	2, 0	2, 0	2, 0	2, 0	0, 0	0, 0	0, 0	0, 0
1-1	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0
0-2	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0



- Nash equilibrium like $(2-0, RRA)$ not quite reasonable, e.g., why R at $1-1$?

Transforming PIEFG into NFG

		Sister							
		AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR
Brother	2-0	2, 0	2, 0	2, 0	2, 0	0, 0	0, 0	0, 0	0, 0
	1-1	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0
	0-2	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0



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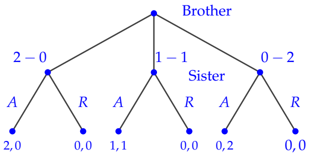
2-0	$2, 0$	$2, 0$	$2, 0$	$2, 0$	$0, 0$	$0, 0$	$0, 0$	$0, 0$
1-1	$1, 1$	$1, 1$	$0, 0$	$0, 0$	$1, 1$	$1, 1$	$0, 0$	$0, 0$
0-2	$0, 2$	$0, 0$	$0, 2$	$0, 0$	$0, 2$	$0, 0$	$0, 2$	$0, 0$



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		AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR
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	1-1	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0
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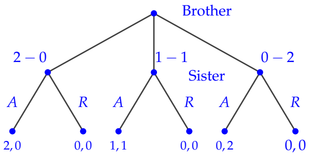

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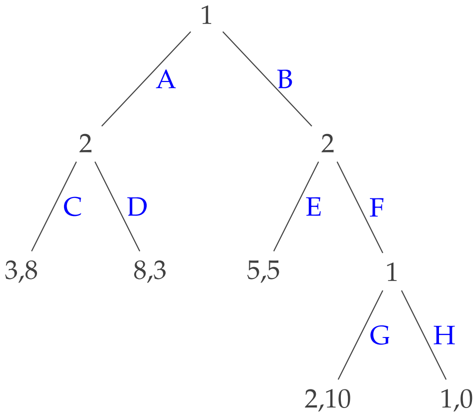
- Hence this equilibrium concept (PSNE) is not good enough for predicting outcomes in PIEFGs
- Also the representation of a sequential game as NFG has huge redundancy – EFG is succinct

Contents

PIEFG to NFG

Equilibrium guarantees are weak for PIEFG in an NFG representation



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PIEFG to NFG

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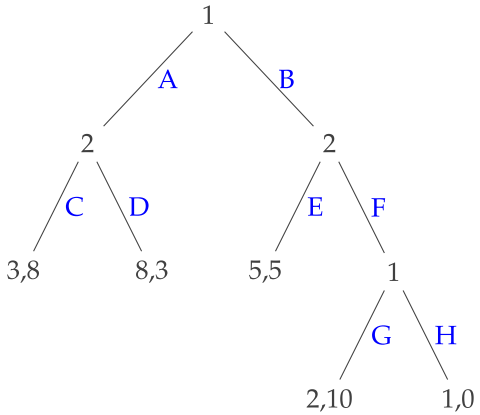

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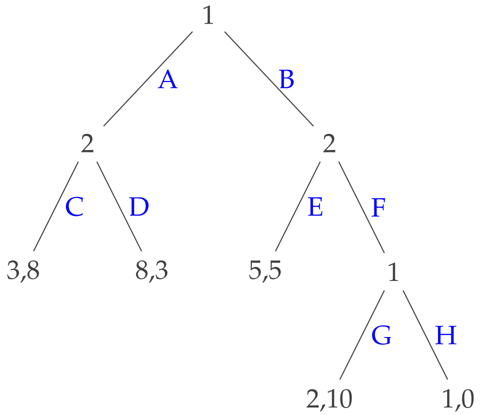

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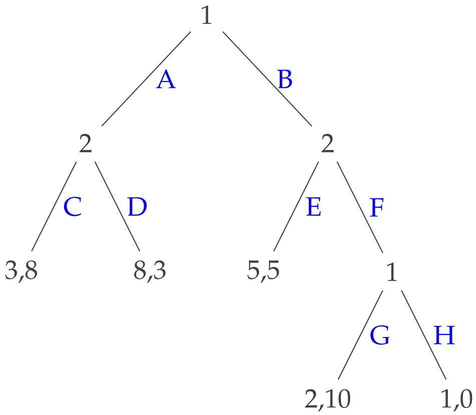

- Strategies of Player 1 : AG, AH, BG, BH
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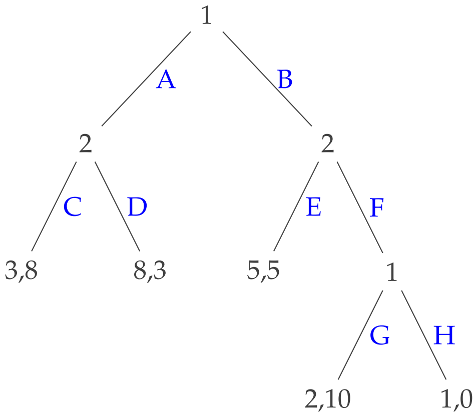

- Strategies of Player 1 : AG, AH, BG, BH
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- PSNEs?
- (AG, CF), (AH, CF), (BH, CE) – is there any non-credible threat
- Better notion of rational outcome will be that which considers a history and ensures utility maximization

Subgame and subgame perfection

Subgame: Game rooted at an intermediate vertex

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Definition (Subgame)

The subgame of a PIEFG G rooted at a history h is the *restriction* of G to the descendants of h.

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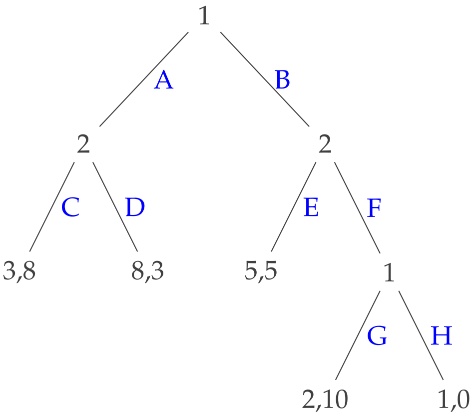
Subgame perfection: Best response at every subgame

Definition (Subgame Perfect Nash Equilibrium (SPNE))

A subgame perfect Nash Equilibrium (SPNE) of a PIEFG G is a strategy profile $s \in S$ s.t. for every subgame G' of G, the restriction of s to G' is a PSNE of G'

Example



- PSNEs : (AH, CF), (BH, CE), (AG, CF)

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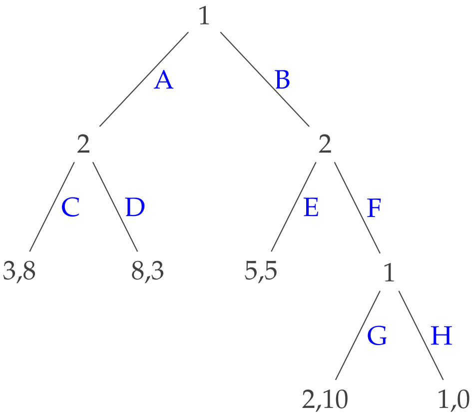

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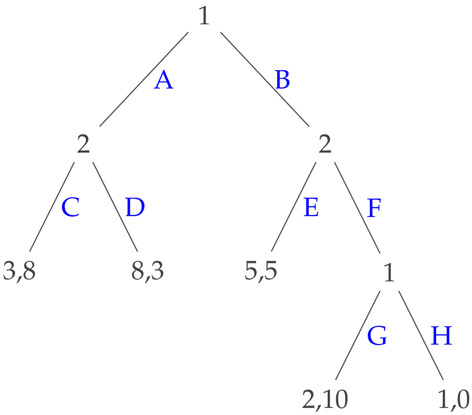

- PSNEs : (AH, CF), (BH, CE), (AG, CF)
- Are they all SPNEs?
- How to compute them?

Subgame Perfection

Algorithm 1: Backward Induction

Function *BACK_IND*(*history h*):

    if $h \in Z$ **then**

        return $u(h), \emptyset$

    $best\_util_{P(h)} \longleftarrow -\infty$

    foreach $a \in X(h)$ **do**

        $util\_at\_child_{P(h)} \longleftarrow BACK\_IND((h, a))$

        if $util\_at\_child_{P(h)} > best\_util_{P(h)}$ **then**

            $best\_util_{P(h)} \longleftarrow util\_at\_child_{P(h)}, best\_action_{P(h)} \longleftarrow a$

    return $best\_util_{P(h)}, best\_action_{P(h)}$

Contents

Limitations of SPNE

The idea of subgame perfection inherently is based on backward induction

Advantages:

- SPNE is guaranteed to exist in finite PIEFGs (requires proof)

Disdvantages and criticisms:

Limitations of SPNE

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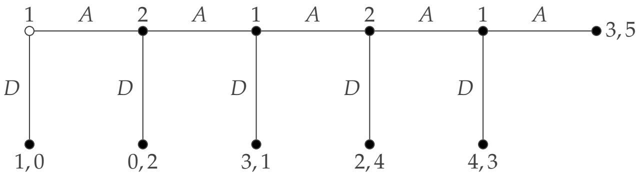
- Cognitive limit of real players may prohibit playing an SPNE

Centipede game



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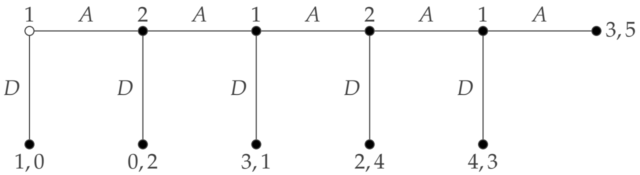

Question

What is/are the SPNE(s) of this game?

Question

What is the problem with that prediction ?

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- Works in explaining outcomes in certain games, but there is another way to extend this idea
- Using the idea of **belief** of the players

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Indian Institute of Technology Bombay

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CS 6001: Game Theory and Algorithmic Mechanism Design

Week 5

Swaprava Nath

Slide preparation acknowledgments: Ramsundar Anandanarayanan and Harshvardhan Agarwal

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal

Contents

Games with Imperfect Information

The story so far

- Games discussed so far (EFGs) are of perfect information

[a] https://rbc.jhuapl.edu/

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  - several games have states that are unknown to certain agents, e.g., card games like poker, reconnaissance blind chess[a]
  - not possible to represent simultaneous move games using EFGs

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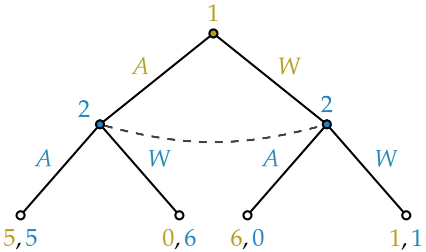
Games with Imperfect Information

		Kingdom 2: Agri	Kingdom 2: War
Kingdom 1	Agri	5, 5	0, 6
	War	6, 0	1, 1

Neighboring Kingdom's Dilemma



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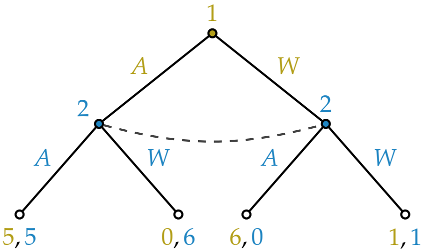

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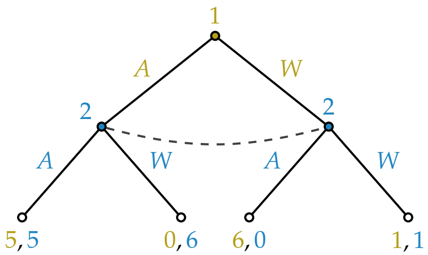

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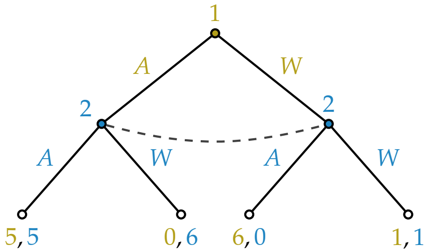

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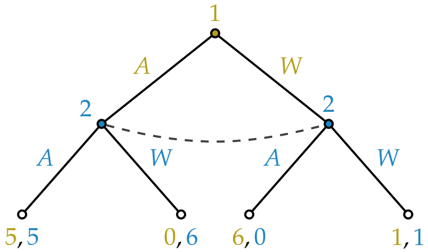

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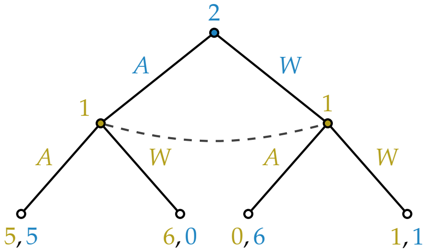
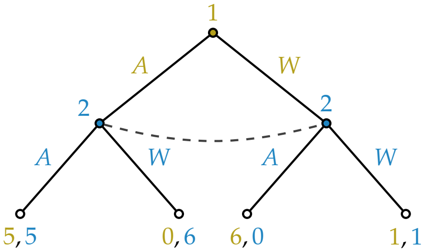
- Kingdom 2 does not know which node/history the game is in
- These indistinguishable histories form an **information set** for player 2.
- More general representation than PIEFG since information sets can be singleton

Games with Imperfect Information





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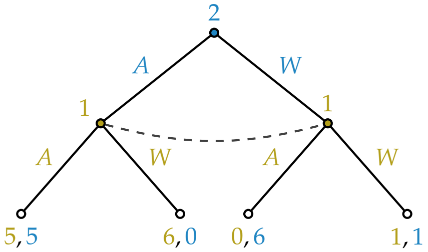

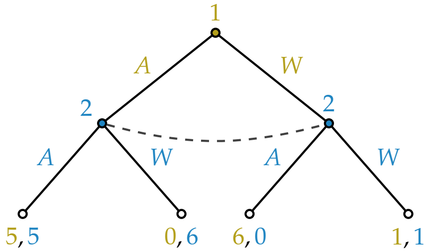

- The Neighboring Kingdom's dilemma can also be represented with the information set of player 1 being non-singleton.

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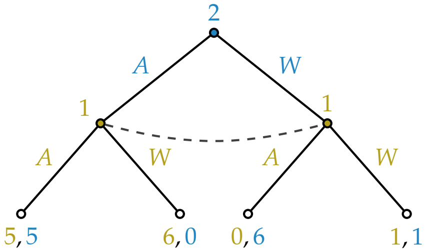

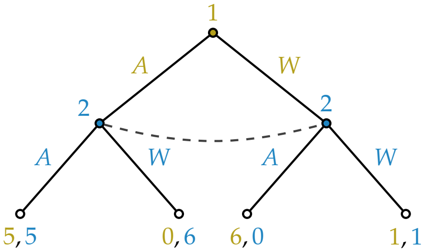

- The Neighboring Kingdom's dilemma can also be represented with the information set of player 1 being non-singleton.
- IIEFGs are not unique for a given simultaneous move game

Games with Imperfect Information

Definition (IIEFG)

An IIEFG is tuple $\langle N, A, H, X, P, (u_i)_{i \in N}, (I_i)_{i \in N} \rangle$

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- I_i^js are called an **information set** of player i and I_i is the collection of information sets of i.
- At an information set, the player and her available actions are the same.
- The player is uncertain about which history in the information set is reached.

Games with Imperfect Information (contd.)

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- Some differences with PIEFG
  - Since actions at an information set are identical, X (action set function) can be defined over I_i^js i.e., $X(h) = X(h') = X(I_i^j), \forall h, h' \in I_i^j$

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- Some differences with PIEFG
  - Since actions at an information set are identical, X (action set function) can be defined over $I_i^j s$ i.e., $X(h) = X(h') = X(I_i^j), \forall h, h' \in I_i^j$
  - Strategies can also be defined over information sets, i.e., strategy set of a player $i \in N$ is defined as the Cartesian product of actions available to i at her information sets $S_i = \times_{I' \in I_i} X(I') = \times_{j=1}^{j=k(i)} X(I_i^j)$

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- With IIEFG, NFGs can be represented using EFGs, although not very succinct.
- IIEFG is a richer representation than both NFG and PIEFG.

Example of Information Addition

- Consider the two-player zero-sum game comprised of the following two stages

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- Each of the following matrices are chosen w.p. $\frac{1}{2}$, but no player sees the realization of this randomization process

Player II

Player I	L	R
T	0	$\frac{1}{2}$
B	0	1

Matrix G_1

Player II

Player I	L	R
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B	$\frac{1}{2}$	0

Matrix G_2

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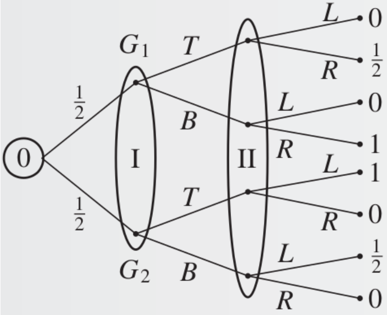
- **What is the extensive form representation?**

Example (Contd.)

- EFG:



Example (Contd.)

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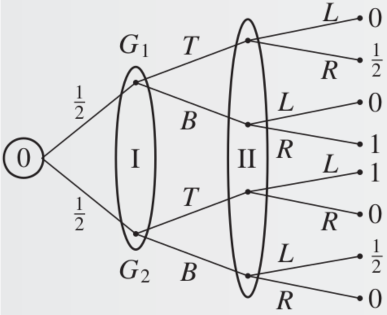
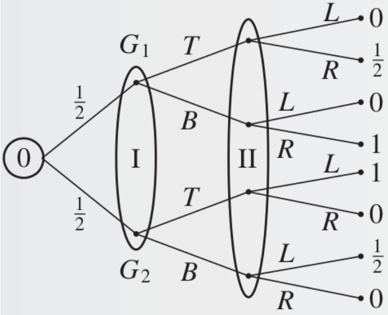


- **What is the normal form representation?**

Example (Contd.)

- EFG $\Rightarrow$ NFG:



		Player II	
		L	R
Player I	T	$\frac{1}{2}$	$\frac{1}{4}$
	B	$\frac{1}{4}$	$\frac{1}{2}$

Example (Contd.)

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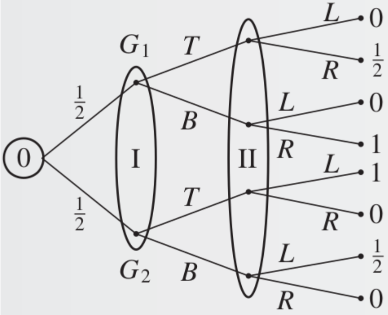

Player I \ Player II	L	R
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- **What is an MSNE of this game?**

Example (Contd.)

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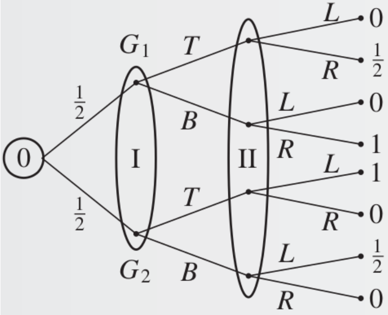
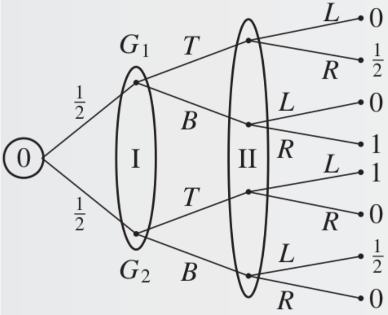


		Player II	
		L	R
Player I	T	$\frac{1}{2}$	$\frac{1}{4}$
	B	$\frac{1}{4}$	$\frac{1}{2}$

- **What is an MSNE of this game?**
- **What is the value of this game?**

Example (Contd.)

- EFG $\Rightarrow$ NFG:



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		L	R
Player I	T	$\frac{1}{2}$	$\frac{1}{4}$
	B	$\frac{1}{4}$	$\frac{1}{2}$

- **What is an MSNE of this game?**
- **What is the value of this game?**
- **MSNE:** $\left(\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right)\right)$, **value** $= \frac{3}{8}$

Same Example: More Information to Player I

- What happens if Player I is informed (but Player II is not) which matrix was chosen,

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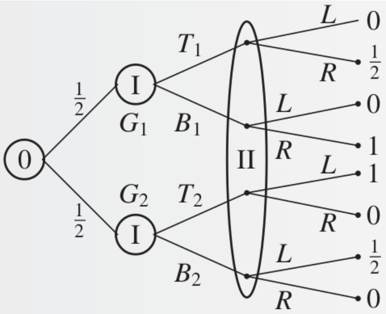

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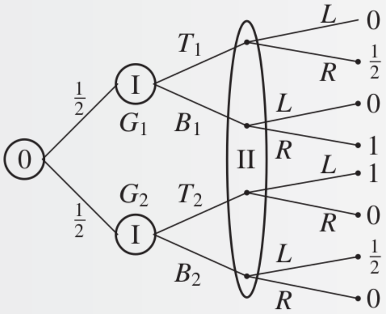
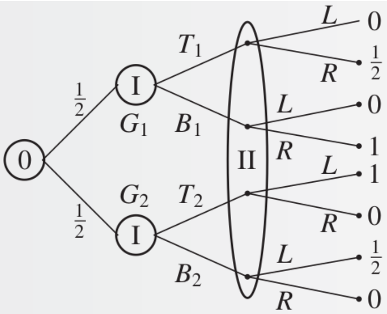


- **What are the strategies now? What is the NFG representation?**

Example (Contd.)

- EFG $\Rightarrow$ NFG:



Player II

Player I	L	R
T_1T_2	$\frac{1}{2}$	$\frac{1}{4}$
T_1B_2	$\frac{1}{4}$	$\frac{1}{4}$
B_1T_2	$\frac{1}{2}$	$\frac{1}{2}$
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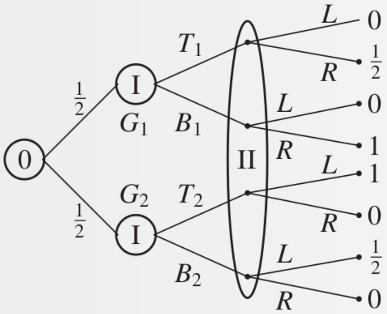

		Player II	
		L	R
Player I	T_1T_2	$\frac{1}{2}$	$\frac{1}{4}$
	T_1B_2	$\frac{1}{4}$	$\frac{1}{4}$
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- **What is an MSNE and value of this game?**

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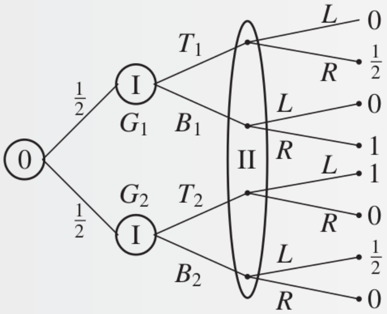

Player I \ Player II	L	R
T_1T_2	$\frac{1}{2}$	$\frac{1}{4}$
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- **What is an MSNE and value of this game?**
- **MSNE:** $((1(B_1T_2)),(p,1-p))\,,p \in [0,1]$, **value =** $\frac{1}{2}$

Result on Information Addition in Matrix Games

Theorem

Let Γ be a two-player zero-sum game in extensive form and let Γ' be the game derived from Γ by splitting several information sets of Player I. Then the value of the game Γ' in mixed strategies is greater than or equal to the value of Γ in mixed strategies.

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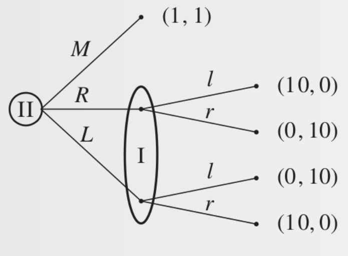
Proof: exercise

How about General-sum Games?



Game A

- Find the MSNE of this game!

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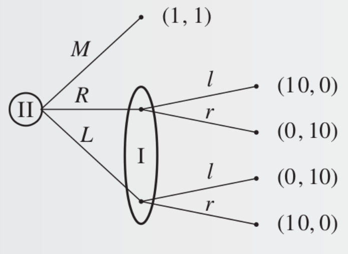
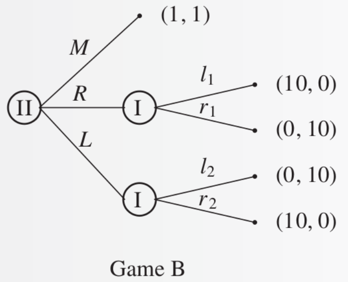


Game A

- Find the MSNE of this game!
- $\left((\frac{1}{2}(l), \frac{1}{2}(r)), (\frac{1}{2}(L), \frac{1}{2}(R), 0(M)) \right) \implies$ expected payoff = $(5, 5)$

Player I gets more information



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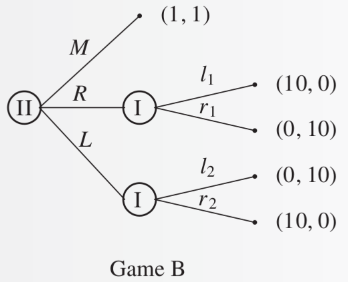

- Find the MSNE of this game!
- $((1(l_1 r_2)), (0(L), 0(R), 1(M))) \implies$ expected payoff = $(1, 1)$

Contents

Randomized Strategies in IIEFGs

- Strategy set of i : $S_i = \times_{j=1}^{j=k(i)} X(I_i^j)$

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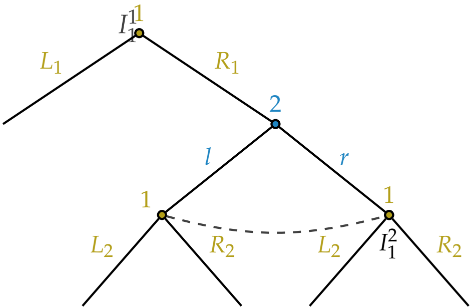
Randomized Strategies in IIEFGs



- Strategies?

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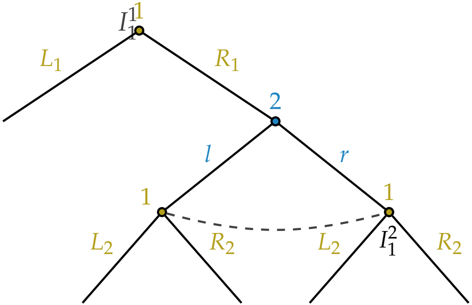

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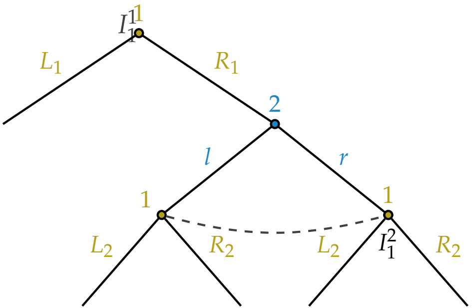

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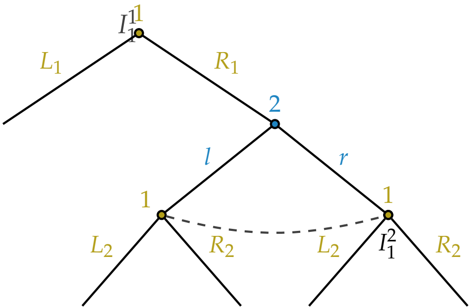

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- Mixed Strategy σ_1, $\sigma_1(L_1L_2)$, $\sigma(L_1R_2)$, $\sigma(R_1L_2)$, $\sigma(R_1R_2)$.
- Behavioral Strategy b_1, $b_1(I_1^1) \in \Delta(L_1, R_1), b_1(I_1^2) \in \Delta(L_2, R_2), b_2(I_2^1) \in \Delta(l, r)$

Behavioral Strategy

Definition

A behavioral strategy of a player in an IIEFG is a function that maps each of her information sets to a probability distribution over the set of possible actions **at that information set**.

Mixed and Behavioral strategy

Question

What is the relation between mixed and behavioral strategies?

Mixed and Behavioral strategy

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What is the relation between mixed and behavioral strategies?

- In this example, MSs live in $\mathbb{R}^4$, BSs live in two $\mathbb{R}^2$ spaces
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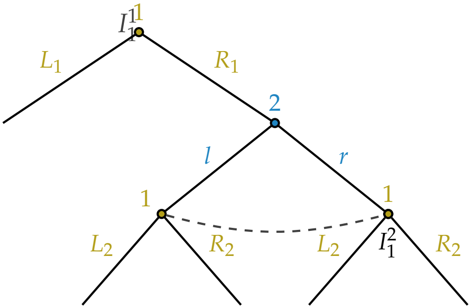
Equivalence in terms of the probability of reaching a vertex/history x

- Say $\rho(x;\sigma)$ is the probability of reaching a node x under mixed strategy profile σ
- Similarly, $\rho(x;b)$ is the same for behavioral strategy profile b

Example



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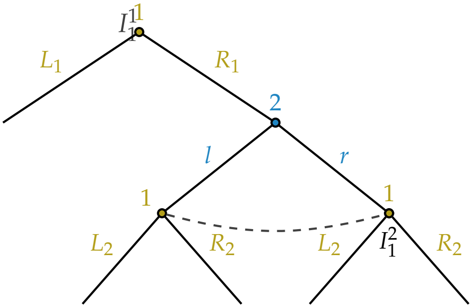

$$\rho(x;\sigma) = \sigma_1(R_1)\sigma_2(r)$$

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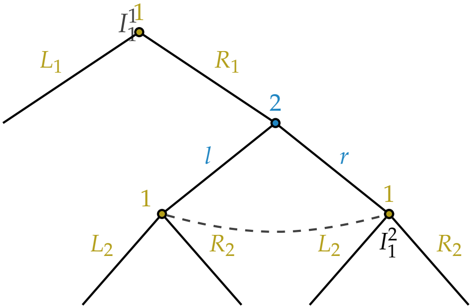

$$\begin{aligned}\rho(x;\sigma) &= \sigma_1(R_1)\sigma_2(r) \\ &= (\sigma_1(R_1L_2) + \sigma_1(R_1R_2)) \cdot \sigma_2(r)\end{aligned}$$

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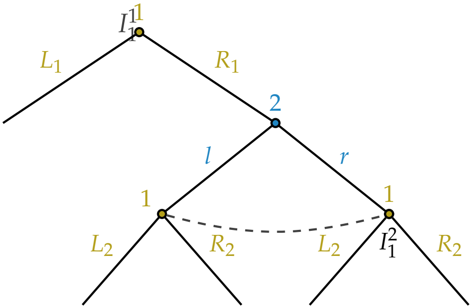

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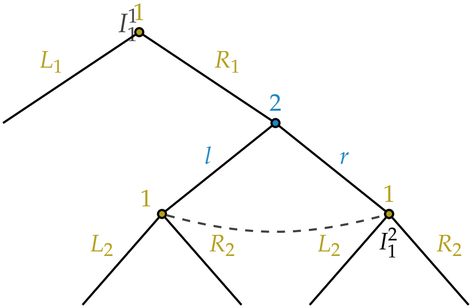

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Players can choose different kind of strategies

$$\rho(x;\sigma_1, b_2) = (\sigma_1(R_1L_2) + \sigma_1(R_1R_2)) \cdot b_2(I_2^1)(r)$$

Equivalence Definition

Definition

A mixed strategy σ_i and a behavioural strategy b_i of a player i in an IIEFG are **equivalent** if for every mixed/behavioral strategy ξ_{-i} of the other players and every vertex x in the game tree.

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Example (in the game above)

Equivalent strategies induce same probability of reaching a vertex.

$$\begin{aligned}
b_1(I_1^1)(L_1) &= \sigma_1(L_1 L_2) + \sigma_1(L_1 R_2) \\
b_1(I_1^1)(R_1) &= \sigma_1(R_1 L_2) + \sigma_1(R_1 R_2) \\
b_1(I_1^2)(L_2) &= \sigma_1(L_2 | R_1) \\
b_1(I_1^2)(R_2) &= \sigma_1(R_2 | R_1)
\end{aligned}$$

We call b_1 and σ_1 are equivalent.

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It is enough to check the equivalence only at the leaf nodes.

Reason: Pick an arbitrary non-leaf node, the probability of reaching that node is equal to the sum of the probabilites of reaching the leaf nodes in its subtree.

More on Equivalent Strategies

This argument can be extended further

Theorem (Utility Equivalence)

If σ_i and b_i are equivalent, then for every mixed/behavioural strategy vector of the other players ξ_{-i}, the following holds,

$$u_j(\sigma_i, \xi_{-i}) = u_j(b_i, \xi_{-i}), \ \forall j \in N.$$

Repeat the argument for any equivalent mixed and behavioral strategy profiles.

Corollary

Let σ and b are equivalent, i.e., σ_i and b_i are equivalent $\forall i \in N$, then $u_i(\sigma) = u_i(b)$.

Contents

Equivalence of strategies in IIEFGs

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Why behavioral strategies are desirable?

Answer

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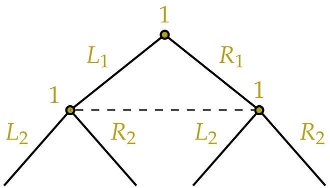
Can we construct one from another?
OR
Does equivalence always hold?

Equivalence of strategies in IIEFGs (Example 1)

Player remembers that it made a move but forgets which move



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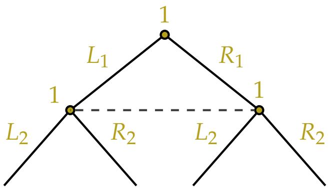

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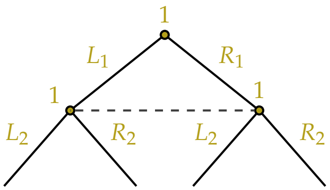

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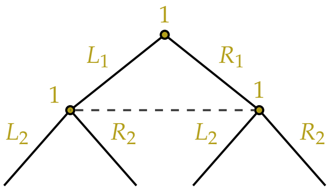

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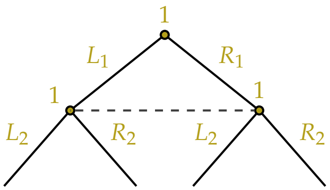

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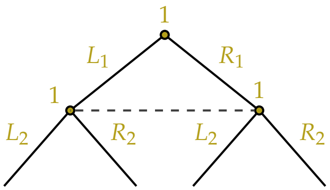

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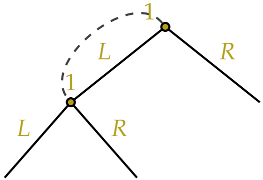
Player forgets whether it made a move or not



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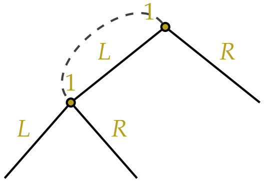

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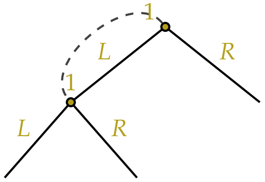

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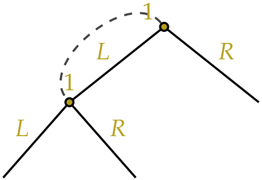

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Answer

The equivalence does not hold if the players are forgetful

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Observation from a graph viewpoint

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2. action at x_1 leading to x: the unique edge emanating from x_1 that is on the path from root to x

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Observation from a graph viewpoint

1. Let x be a non-root node
2. action at x_1 leading to x: the unique edge emanating from x_1 that is on the path from root to x
3. In example 2, there is a node that has a path from the root to itself that crosses the same information set twice

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Observation from a graph viewpoint

1. Let x be a non-root node
2. action at x_1 leading to x: the unique edge emanating from x_1 that is on the path from root to x
3. In example 2, there is a node that has a path from the root to itself that crosses the same information set twice
4. If the path from the root to x passes through vertices x_1 and x_1' that are in the same information set of player i, and the action leading to x at x_1 and x_1' is different, then no **pure strategy** can ever lead to x

Equivalence of strategies in IIEFGs

Question

When does behavioral strategy have no equivalent mixed strategy?

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5. Since mixed strategy is a randomization over pure strategies, every mixed strategy will put zero probability mass on x but behavioral strategy randomizes on every vertex **independently**, hence x may be reached in behavioral strategies with a positive probability

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The last observation can be stated as a lemma

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Lemma

If there exists a path from the root to some vertex x that passes through the same information set at least twice, and if the action leading to x is **not** *the same at each of those vertices, then the player at the information set has a behavioral strategy that has no equivalent mixed strategy.*

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Theorem (6.11 of MSZ)

Consider an IIEFG such that every vertex has at least two actions. Every behavioral strategy has an equivalent mixed strategy for a player iff each information set of that player intersects every path emanating from the root at most once.

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Proof.

Homework. Reading exercise from MSZ. □

Contents

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To formalize (i.e., to set the conditions when the equivalence holds), we need to formalize the **forgetfulness** of a player.

- saw few examples of players' forgetfulness.
- our conditions need to ensure that none of the previous types of forgetfulness happens.

Behavioral Strategy equivalent to Mixed Strategy

Definition (Choice of same action at an information set)

Let $X = (x^0, x^1, \ldots, x^K)$ and $\hat{X} = (x_0, \hat{x}^1, \ldots, \hat{x}^L)$ be two paths in the game tree.

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'Leading to' may not be a relation between parent and child nodes, it can be any descendant of the former since the path is unique in a tree.

Games with Perfect Recall

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Player i has perfect recall if the following conditions are satisfied

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Rephrasing

For every I_i^j of player i and every pair of vertices $x, y \in I_i^j$, if the decision vertices of i are $x_i^1, x_i^2, \ldots, x_i^L = x$ and $y_i^1, y_i^2, \ldots, y_i^{L'} = y$ respectively for the two paths from the root to x and y, then

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1. $L = L'$,
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3. $a_i(x_i^l \to x_i^{l+1}) = a_i(y_i^l \to y_i^{l+1}), \forall l = 1, 2, \ldots, L-1.$

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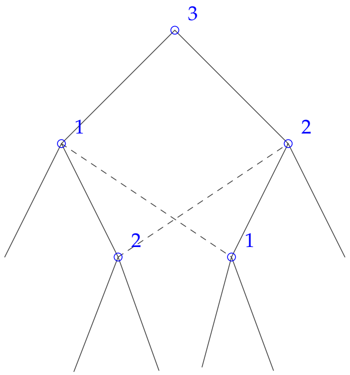
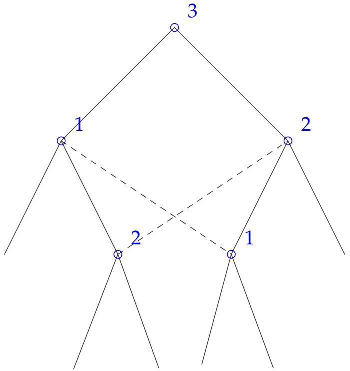
A game has **perfect recall** if every player has a perfect recall.

Note: Definition of perfect recall subsumes the condition where every behavioral strategy has equivalent mixed strategy

Examples

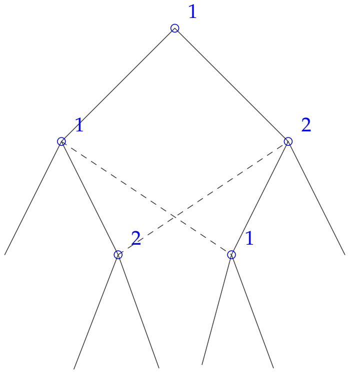


Examples



Game with Perfect Recall: This example satisfies the conditions of the definitions.

Examples



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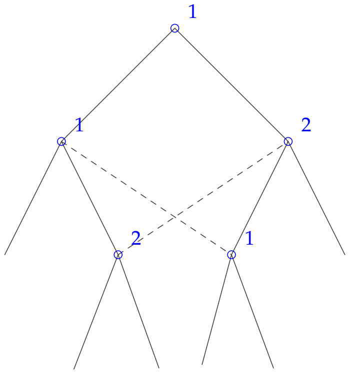

Game with Imperfect Recall: Player 1 takes two different actions at the first information set to reach two different vertices of the second information set.

Implications of Perfect Recall

Let $S_i^*(x)$ be the set of pure strategies of player i at which he chooses actions leading to x, i.e., intersections of members of S_i with the path from root to x.

Theorem

If i is a player with perfect recall and x and x' are the two vertices in the same information set of i, then $S_i^*(x) = S_i^*(x')$.

The above conclusion comes from the same sequence of information sets and same actions. The next implication of mixed and behavioral strategies.

Implications of Perfect Recall

Theorem (Kuhn 1957)

In every IIEFG, if i is a player with perfect recall, then for every mixed strategy of i, there exists a behavioral strategy.

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- Proof left as reading exercise (MSZ Theorem 6.15)

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- The converse is already true since the sufficient condition for that is already subsumed in the definition of perfect recall.
- Proof left as reading exercise (MSZ Theorem 6.15)
- The proof is constructive. It starts with the mixed strategy and constructs the behavioral strategies such that the probabilities of reaching a leaf are same. The arguments show that such a construction is always possible because of perfect recall.

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CS 6001: Game Theory and Algorithmic Mechanism Design

Week 6

Swaprava Nath

Slide preparation acknowledgments: Ramsundar Anandanarayanan and Harshvardhan Agarwal

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal

Contents

Equilibrium notions in IIEFG

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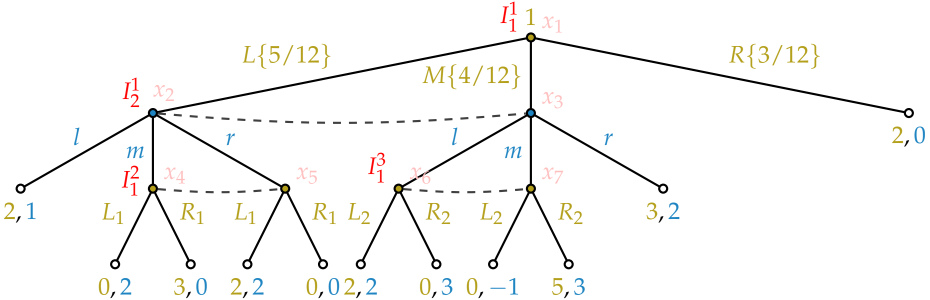
Belief

It is the conditional probability distribution over the histories in an information set - conditioned on reaching the information set.

Example: An IIEFG with perfect recall

EX 7.38 MSZ: An IIEFG with perfect recall, i.e., mixed and behavioral strategies are equivalent.



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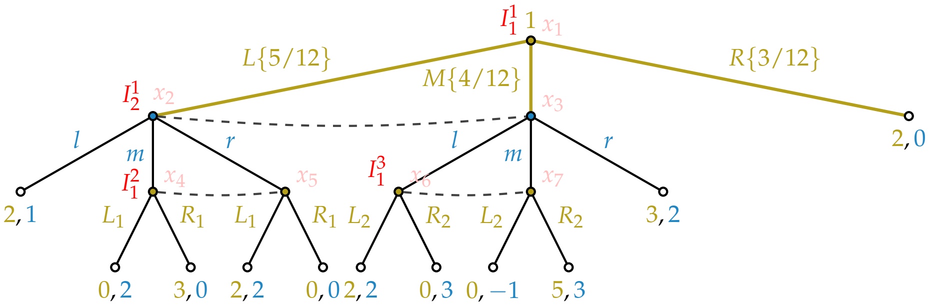

Consider the behavioral strategy profile: σ_1, at $I_1^1(L\{5/12\}, M\{4/12\}, R\{3/12\})$

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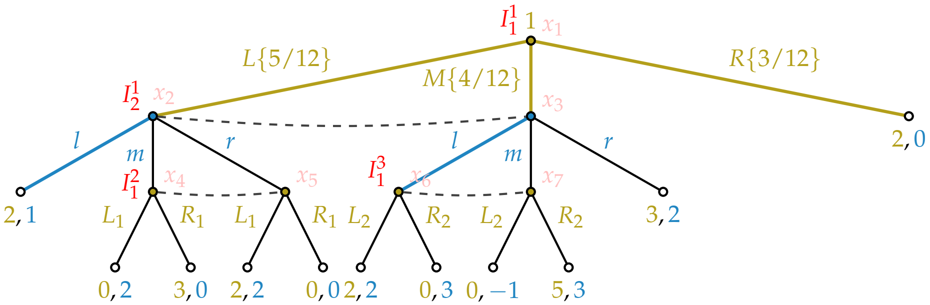

Consider the behavioral strategy profile: σ_2, at $I_2^1(l\{1\}, m\{0\}, r\{0\})$ choose l

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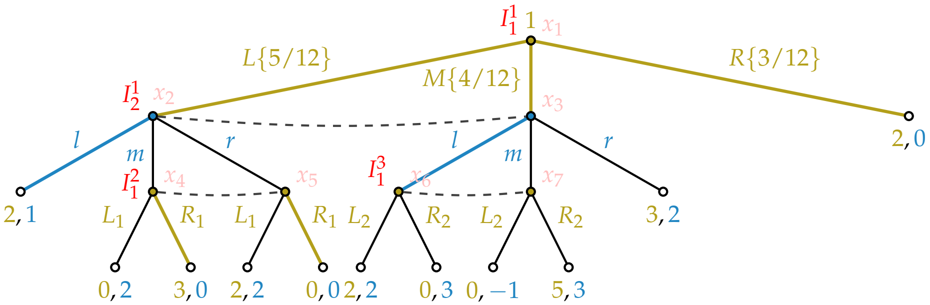

Consider the behavioral strategy profile: σ_1, at $I_1^2(L_1\{0\}, R_1\{1\})$ choose R_1

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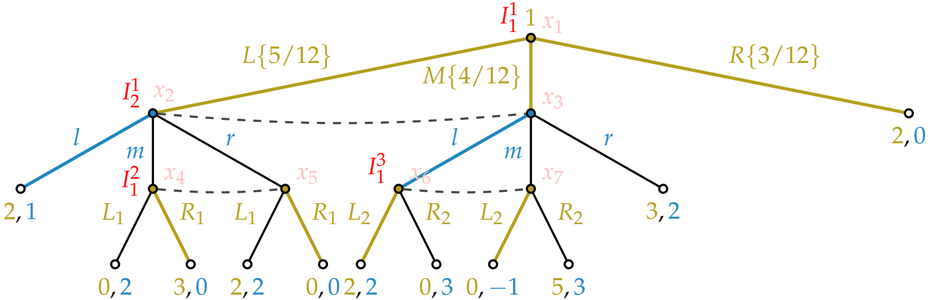

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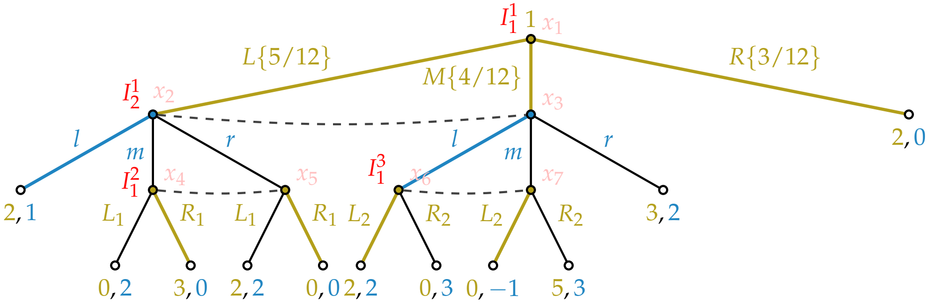
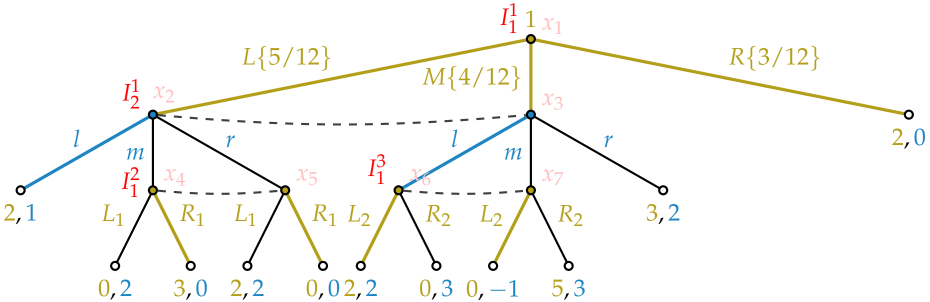


Question

Is this an equilibrium?
which implies

- Are the Bayesian beliefs consistent with P_σ - that visits vertex x with probability $P_\sigma(x)$?
- The actions and beliefs are consistent for every player, i.e., maximizes their expected utility?

Example: An IIEFG with perfect recall



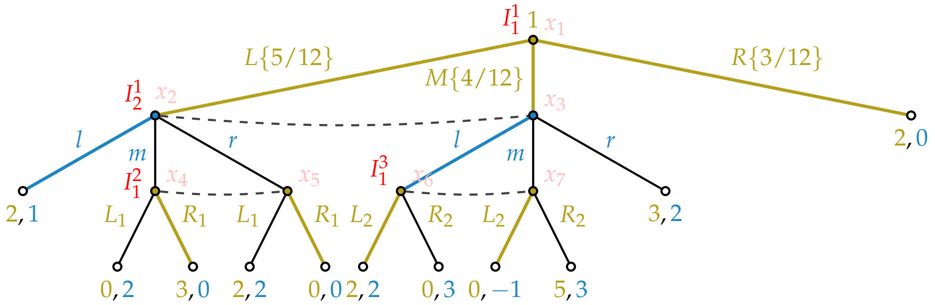
Sequential rationality

Choose an action maximizing expected utility at each information set.

The strategy vector σ induces the following probabilities to the vertices.

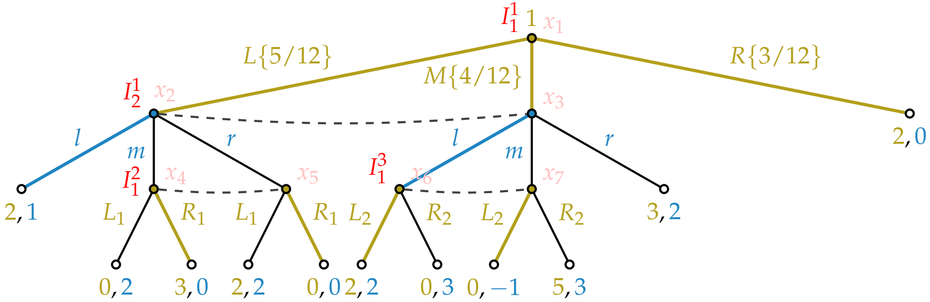
$$P_\sigma(x_2) = 5/12, P_\sigma(x_3) = 4/12, P_\sigma(x_4) = 0, P_\sigma(x_5) = 0, P_\sigma(x_6) = 4/12, P_\sigma(x_7) = 0$$

Example: An IIEFG with perfect recall



- Player 1 at information set I_1^3, believes that x_6 is reached with probability 1.
- If the belief was $> 2/7$ in favor of x_7, player 1 should have chosen R_2

Example: An IIEFG with perfect recall



- Player 2 at I_2^1 believes the x_3 is reached w.p. $P_\sigma(x_3|I_2^1) = P_\sigma(x_3)/(P_\sigma(x_2) + P_\sigma(x_3)) = 4/9$
- Similarly $P_\sigma(x_2|I_2^1) = 5/9$

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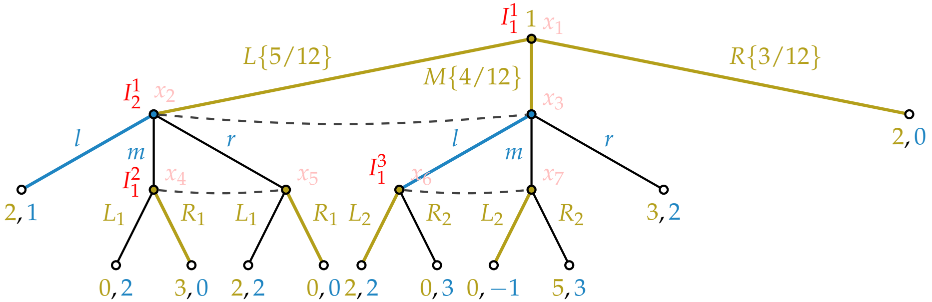


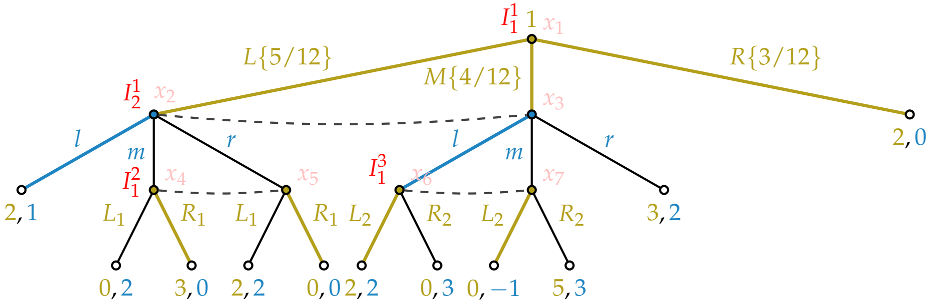
Question

Is the action of player 2 sequentially rational w.r.t.her belief?

Answer

By picking l, expected utility=$5/9 \times 1 + 4/9 \times 2 = 13/9$, larger than any other choice of action.

Example: An IIEFG with perfect recall



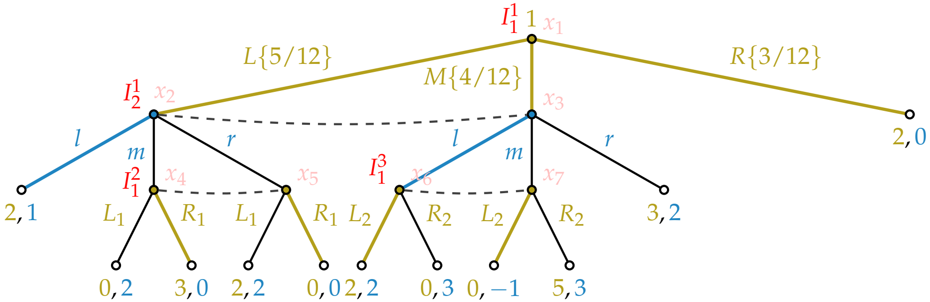
Question

Given all information, what is the sequentially rational strategy for player 1 at I_1^1

Answer

L, M, R all give the same expected utility for player 1 (utility $= 2$).

Example: An IIEFG with perfect recall



Thus, mixed/behavioral strategy profile σ is sequentially rational for all players.

Formal definitions

Belief

Let the information sets of player i be $I_i = \{I_i^1, I_i^2, I_i^3, ..., I_i^{k(i)}\}$.
The belief of player i is a mapping $\mu_i^j : I_i^j \to [0,1]$ s.t., $\sum_{x \in I_i^j} \mu_i^j(x) = 1$

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Bayesian belief

A belief $\mu_i = \{\mu_i^1, \mu_i^2, ..., \mu_i^{k(i)}\}$ of player i is Bayesian w.r.t.to the behavioral strategy σ, if it is derived from σ using Bayes rule, i.e.,

$$\mu_i^j(x) = P_\sigma(x) / \sum_{y \in I_i^j} P_\sigma(y), \forall x \in I_i^j, \forall j = 1, 2, 3, ..., k(i)$$

Formal definitions

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A strategy σ_i of player i at an information set I_i^j is sequentially rational given σ_{-i} and partial belief μ_i^j if

$$\sum_{x \in I_i^j} \mu_i^j(x) u_i(\sigma_i, \sigma_{-i} | x) \geqslant \sum_{x \in I_i^j} \mu_i^j(x) u_i(\sigma_i', \sigma_{-i} | x)$$

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- Sequential rationality is a refinement of Nash Equilibrium.
- The notion coincides with SPNE when applied to PIEFGs

Formal definitions

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In a PIEFG, a behavioral strategy profile σ is an SPNE iff the tuple $(\sigma, \hat{\mu})$ is sequentially rational.

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Equilibrium with Sequential rationality

Perfect Bayesian Equilibrium: An assessment (σ, μ) is PBE if $\forall i \in N$

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Formal definitions

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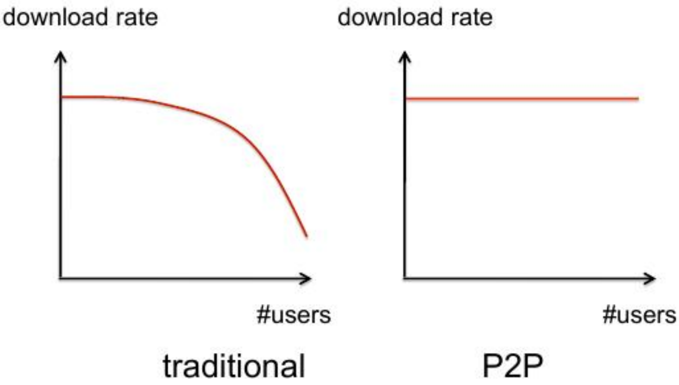
- Often represented only with σ, since μ is obtained from σ
- Self-enforcing (like the SPNE) in a Bayesian way.

Contents

Peer to Peer[1]



[1] Slides of this section are adapted from CS186, Harvard

Desired Properties and Terminology

- Scalability

Terminology:

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- Failure resilience

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Early P2P Technologies

Napster (1999 - 2001)

- Centralized database
- Users download music from each other

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Gnutella (2000 -)

- Get list of IP addresses of peers from set of known peers (no server)
- To get a file: Query message broadcast by peer A to known peers
- Query response: sent by B if B has the desired file (routed back to requestor)
- A can then download directly from B

The File Sharing Game

Player 2

	Share	Free-ride	
Player 1	$2, 2$	$-1, 3, -1$	$0, 0$

(Gnutella) File Sharing Game

The File Sharing Game (Contd.)

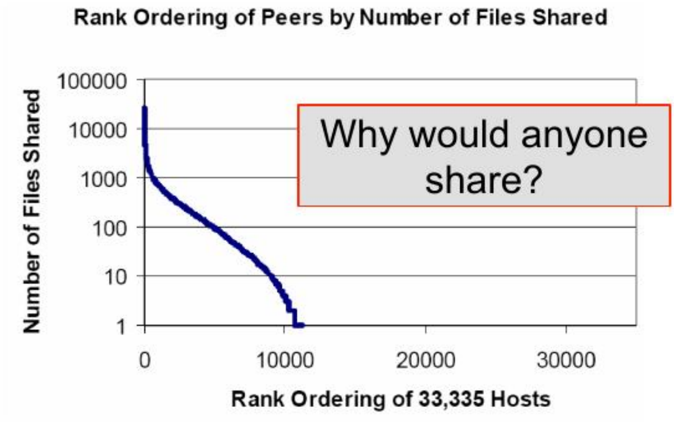

Image courtesy: Adar and Huberman (2000)

Incentives for Client Developers

- Client developers can ensure file sharing
- But competition among the developers

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- But competition among the developers

- 85% peers free-riding by 2005; Gnutella less than 1% of ww P2P traffic by 2013
- Few other P2P systems met the same fate

New Protocol

BitTorrent (2001 -)

- Approx 85% of P2P traffic in US
- File sharing
- Also used for S/W distribution (e.g., Linux)

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- Approx 85% of P2P traffic in US
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Key innovations

- Break file into pieces: A repeated game!
- "If you let me download, I'll reciprocate."

BitTorrent Schematic

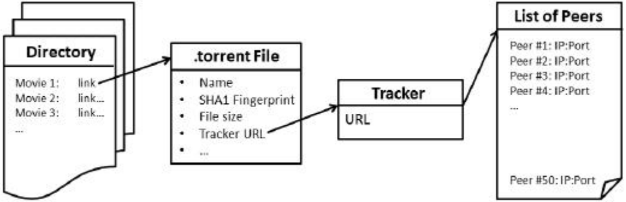

Figure 5.4.: Starting a download process in the BitTorrent protocol: 1) A user goes to a searchable directory to find a link to a .torrent file corresponding to the desired content; 2) the .torrent file contains metadata about the content, in particular the URL of a tracker; 3) the tracker provides a list of peers participating in the swarm for the content (i.e., their IP address and port); 4) the user's BitTorrent client can now contact all these peers and download content.

Image courtesy: Parkes and Seuken (2017)

BitTorrent Optimistic Unchoking Algorithm

Tracker is a centralized entity that controls the traffic, tracks the connection between peers and their speed of upload, download etc.

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- Every three time periods, optimistically unchoke a random peer from the neighborhood who is currently choked, and leave that peer unchoked for three time periods.

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Forcing a repeated game by fragmenting the files

The leecher-seeder game is a repeated Prisoners' Dilemma

Strategy of the seeder is tit-for-tat

Illustration

Illustration

Strategic Behaviors

- How often to contact tracker?
- Which pieces to reveal?
- How many upload slots, which peers to unchoke, at what speed?
- What data to allow others to download?
- Possible goals: min upload, max download speed, some balance

Attacks on BitTorrent

- BitThief
- Strategic piece revealer
- BitTyrant

BitThief

- Goal: download files without uploading
- Keep asking for peers from tracker, grow neighborhood quickly
- Exploit the optimistic unchoking part
- Never upload!

BitThief

- Goal: download files without uploading
- Keep asking for peers from tracker, grow neighborhood quickly
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- Never upload!
- Fix: modify the tracker (block same IP address within 30 minutes).

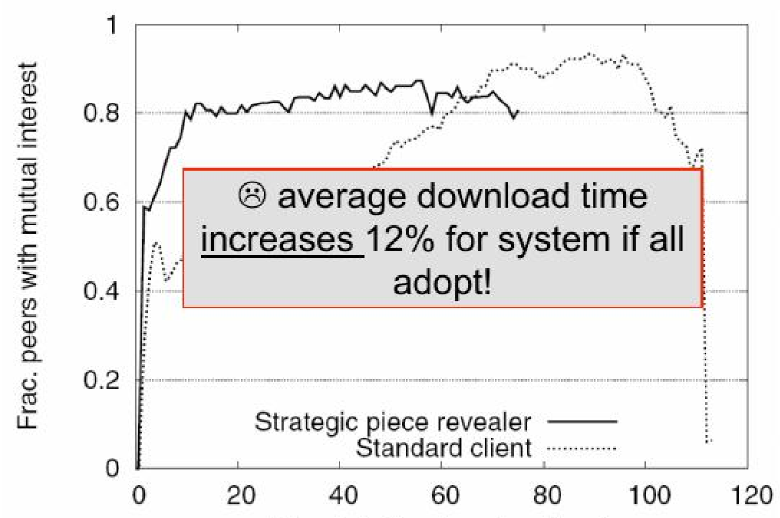
Ref: Locher et al., “Free Riding in BitTorrent is Cheap”, HotNets 2006

Strategic Piece Revealer

- Reference client: tell neighbors about new pieces, use "rarest-first" to request
- Manipulator strategy: reveal most common piece that reciprocating peer does not have!
- Try to protect a monopoly, keep others interested

Ref: Levin et al., "BitTorrent is an Auction: Analyzing and Improving BitTorrent's Incentives", SIGCOMM 2008

Strategic Piece Revealer



☹ average download time increases 12% for system if all adopt!

Summary

- P2P demonstrates importance of game-theory in computer systems
- Early systems were easily manipulated
- BitTorrent's innovation was to break files into pieces, enabling TitForTat.
- Still some vulnerabilities, but generally very successful example of incentive-based protocol design.

Contents

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- **Non-cooperative games**

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- Cooperative games - Players form coalitions and utilities are defined over coalitions
- Other types of games - repeated, stochastic etc.

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- There can be some chance moves but probabilities are known

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- Also called **Bayesian games**

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Football game (two competing teams)

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WW profile (rows: ARG, columns: FRA)

ARG \ FRA	ATT	DEF
ATT	1, 1	2, 0
DEF	0, 2	0, 0

WD profile (rows: ARG, columns: FRA)

ARG \ FRA	ATT	DEF
ATT	2, 0	2, 1
DEF	0, 1	1, 0

DD profile (rows: ARG, columns: FRA)

ARG \ FRA	ATT	DEF
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DEF	0, 1	$-1, -1$

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  i.e., marginals for every type is positive (otherwise we can prune the type set)
- Γ_θ: NFG for the type profile $\theta \in \Theta$ i.e., $\Gamma_\theta = \langle N, (A_i(\theta))_{i \in N}, (u_i(\theta))_{i \in N} \rangle$
  $u_i : A \times \Theta \to \mathbb{R}$, $A = \times_{i \in N} A_i$ [We assume $A_i(\theta) = A_i, \forall \theta$]

Bayesian games

Stages of a Bayesian game

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- Player i realizes a payoff of $u_i(a_i, a_{-i}; \theta_i, \theta_{-i})$

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Strategy and Utilities

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Strategy is a plan to map type to action.

$$s_i : \Theta_i \to A_i \qquad \textbf{Pure}$$

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- Ex-interim utility
- Ex-post utility (for complete information game)

Ex-ante Utility

Definition (Ex-ante utility)

Expected utility before observing own type.

$$\begin{aligned} u_i(\sigma) &= \sum_{\theta \in \Theta} P(\theta) u_i(\sigma(\theta); \theta) \\ &= \sum_{\theta \in \Theta} P(\theta) \sum_{(a_1, a_2, \ldots, a_n) \in A} \prod_{j \in N} \sigma_j(\theta_j)[a_j] u_i(a_1, \ldots a_n; \theta_1, \ldots \theta_n) \end{aligned}$$

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P(\theta_{-i} | \theta_i) = \frac{P(\theta_i, \theta_{-i})}{\sum_{\tilde{\theta}_{-i} \in \Theta_{-i}} P(\theta_i, \tilde{\theta}_{-i})}
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This is why we needed every marginal to be positive – otherwise that type can be removed from its type set

Ex-interim utility

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Expected utility after observing one's own type.

$$u_i(\sigma|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i}|\theta_i) u_i(\sigma(\theta); \theta)$$

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Relation between the two utilities is given by

$$u_i(\sigma) = \sum_{\theta_i \in \Theta_i} P(\theta_i) u_i(\sigma|\theta_i)$$

Example 1: Two Player Bargaining Game

- Player 1 : seller, type : price at which he is willing to sell

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- If the bid of the seller is smaller or equal to that of the buyer, trade happens at a price average of the two bids. Else, trade does not happen.

Example 1: Two Player Bargaining Game

Suppose type generation is independent and uniform over Θ_1, Θ_2 respectively,

$$P(\theta_2|\theta_1) = P(\theta_2) = \frac{1}{100}, \forall \theta_1, \theta_2$$

$$P(\theta_1|\theta_2) = P(\theta_1) = \frac{1}{100}, \forall \theta_1, \theta_2$$

$$u_1(a_1, a_2; \theta_1, \theta_2) = \begin{cases} \frac{a_1+a_2}{2} - \theta_1 & \text{if } a_2 \geqslant a_1 \\ 0 & \text{otherwise} \end{cases}$$

$$u_2(a_1, a_2; \theta_1, \theta_2) = \begin{cases} \theta_2 - \frac{a_1+a_2}{2} & \text{if } a_2 \geqslant a_1 \\ 0 & \text{otherwise} \end{cases}$$

Common Prior : $P(\theta_1, \theta_2) = \frac{1}{1000}, \forall \theta_1, \theta_2$

Example 2: Sealed Bid Auction

Two players, both willing to buy an object. Their values and bids lie in [0,1].

Allocation Function:

$$O_1(b_1, b_2) = \begin{cases} 1 & \text{if } b_1 \geqslant b_2 \\ 0 & \text{ow} \end{cases} \qquad O_2(b_1, b_2) = \begin{cases} 1 & \text{if } b_2 > b_1 \\ 0 & \text{ow} \end{cases}$$

Beliefs

$$f(\theta_2 | \theta_1) = 1, \forall \theta_1, \theta_2$$
$$f(\theta_1 | \theta_2) = 1, \forall \theta_1, \theta_2$$
$$f(\theta_1, \theta_2) = 1, \forall \theta_1, \theta_2$$

$$u_i(b_1, b_2; \theta_1, \theta_2) = O_i(b_1, b_2)(\theta_i - b_i)$$

Winner pays for his bid.

Contents

Equilibrium concepts in Bayesian games

Ex-ante: before observing her own type

Nash Equilibrium (σ^*, P): $u_i(\sigma_i^*, \sigma_{-i}^*) \geqslant u_i(\sigma_i', \sigma_{-i}^*), \forall \sigma_i', \forall i \in N$

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- The RHS of the definition can be replaced by a pure strategy $a_i, \forall a_i \in A_i$. The reason is exactly the same as that of MSNE (these definitions are equivalent)

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- NE takes expectation over $P(\theta)$
- BE takes expectation over $P(\theta_{-i} | \theta_i)$

Equivalence of equilibrium concepts

Theorem

In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium

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Proof.

For the forward direction, suppose (σ^*, P) is a Bayesian equilibrium, consider

$$\begin{aligned} u_i(\sigma_i', \sigma_{-i}^*) &= \sum_{\theta_i \in \Theta_i} P(\theta_i) u_i(\sigma_i'(\theta_i), \sigma_{-i}^* | \theta_i) \\ &\leqslant \sum_{\theta_i \in \Theta_i} P(\theta_i) u_i(\sigma_i^*(\theta_i), \sigma_{-i}^* | \theta_i), \text{ since } (\sigma^*, P) \text{ is a BE} \\ &= u_i(\sigma_i^*, \sigma_{-i}^*) \end{aligned}$$

□

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For the reverse direction, proof by contradiction. Suppose (σ^*, P) is not a Bayesian equilibrium i.e., there exists some $i \in N$, some $\theta_i \in \Theta_i$, some $a_i \in A_i$, s.t.

$$u_i(a_i, \sigma^*_{-i}|\theta_i) > u_i(\sigma^*_i(\theta_i), \sigma^*_{-i}|\theta_i)$$

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Construct the strategy $\hat{\sigma}_i$ s.t.,

$$\hat{\sigma}_i(\theta'_i) = \sigma^*_i(\theta'_i), \forall \theta'_i \in \Theta_i \setminus \{\theta_i\}$$

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$$\hat{\sigma}_i(\theta_i)[a_i] = 1, \hat{\sigma}_i(\theta_i)[b_i] = 0, \forall b_i \in A_i \setminus \{a_i\}$$

$\square$

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Reverse direction proof continued ...

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$$u_i(\hat{\sigma}_i, \sigma^*_{-i}) = \sum_{\tilde{\theta}_i \in \Theta_i} P(\tilde{\theta}_i) u_i(\hat{\sigma}_i(\tilde{\theta}_i), \sigma^*_{-i} | \tilde{\theta}_i)$$

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u_i(\hat{\sigma}_i, \sigma^*_{-i}) &= \sum_{\tilde{\theta}_i \in \Theta_i} P(\tilde{\theta}_i) u_i(\hat{\sigma}_i(\tilde{\theta}_i), \sigma^*_{-i} | \tilde{\theta}_i) \\
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Hence, $(\sigma^*_i, \sigma^*_{-i})$ is not a Nash equilibrium □

Existence of Bayesian Equilibrium

Theorem

Every finite Bayesian game has a Bayesian equilibrium.

[Finite Bayesian game: set of players, action set and type set are finite]

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Proof.

Proof idea: Transform the Bayesian game into a complete information game treating each type as a player, and invoke Nash Theorem for the existence of equilibrium - which is a BE in the original game. [See addendum for details] □

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$$f(\theta_2|\theta_1) = 1, \forall \theta_1, \theta_2$$
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First Price Auction

- If $b_1 \geqslant b_2$ payer 1 wins and pays her bid otherwise, player 2 wins and pays her bid.

$$u_1(b_1, b_2, \theta_1, \theta_2) = (\theta_1 - b_1)T\{b_1 \geqslant b_2\}$$
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- $b_1 = s_1(\theta_1), b_2 = s_2(\theta_2)$
  Assume $s_i(\theta_i) = \alpha_i\theta_i, \alpha_i > 0, i = 1, 2$

First Price Auction

To find the BE, we need to find the s_i^* (or α_i^*) that maximizes the ex-interim utility of player i. i.e.

$$max_{\sigma_i} u_i(\sigma_i, \sigma_{-i}^* | \theta_i)$$

For player 1, this reduces to

$$\begin{aligned} max_{\sigma_i} u_i(\sigma_i, \sigma_{-i}^* | \theta_i) &= max_{b_1 \in [0, \alpha_2]} \int_0^1 f(\theta_2 | \theta_1)(\theta_1 - b_1) I\{b_1 \geqslant \alpha_2 \theta_2\} d\theta_2 \\ &= max_{b_1 \in [0, \alpha_2]} (\theta_1 - b_1) \frac{b_1}{\alpha_2} \\ \implies b_1 &= \begin{cases} \frac{\theta_1}{2} \text{if } \alpha_2 > \frac{\theta_1}{2} \\ \alpha_2 \text{otherwise} \end{cases} \end{aligned}$$

First Price Auction

From this we get,

$$s_1^*(\theta_1) = min\{\frac{\theta_1}{2}, \alpha_2\}$$
$$s_2^*(\theta_2) = min\{\frac{\theta_2}{2}, \alpha_1\}$$

If $\alpha_1 = \alpha_2 = \frac{1}{2}$, then $(\frac{\theta_1}{2}, \frac{\theta_2}{2})$ is a BE.
In the Bayesian Game induced by uniform prior on first price auction, bidding half the true value is a Bayesian equilibrium.

Second Price Auction

Highest bidder wins but pays the second highest bid.

$$u_1(b_1, b_2, \theta_1, \theta_2) = (\theta_1 - b_2)T\{b_1 \geqslant b_2\}$$
$$u_2(b_1, b_2, \theta_1, \theta_2) = (\theta_2 - b_1)T\{b_1 < b_2\}$$

Player 1 has to maximize

$$\begin{aligned}
&= \int_0^1 f(\theta_2|\theta_1)(\theta_1 - s_2(\theta_2))I\{b_1 \geqslant s_2(\theta_2)\}d\theta_2 \\
&= \int_0^1 1 \cdot (\theta_1 - \alpha_2\theta_2)I\{\theta_2 \leq \frac{b_1}{\alpha_2}\}d\theta_2 \\
&= \frac{1}{\alpha_2}(b_1\theta_1 - \frac{\theta_1^2}{2})
\end{aligned}$$

This is maximized when $b_1 = \theta_1$. Similarly for $b_2 = \theta_2$.

Second Price Auction

If the distribution of θ_1 and θ_2 were arbitrary but independent, the maximization problem would have been

$$\int_0^{\frac{b_1}{\alpha_2}} f(\theta_2)(\theta_1 - \alpha_2\theta_2)d\theta_2 = \theta_1 F\left(\frac{b_1}{\alpha_2}\right) - \alpha_2 \int_0^{\frac{b_1}{\alpha_2}} \theta_2 f(\theta_2)d\theta_2$$

Differentiating wrt b_1, we get

$$\theta_1 \frac{1}{\alpha_2} f\left(\frac{b_1}{\alpha_2}\right) - \alpha_2 \cdot \frac{b_1}{\alpha_2} f\left(\frac{b_1}{\alpha_2}\right) \frac{1}{\alpha_2} = 0 \implies \frac{1}{\alpha_2} f\left(\frac{b_1}{\alpha_2}(b_1 - \theta_1)\right) = 0 \tag{1}$$

$$\implies b_1 = \theta_1 \textbf{if} f\left(\frac{b_1}{\alpha_2}\right) > 0 \tag{2}$$

Similarly for 2.
For any independent positive prior, bidding true type is a BE of the induced Bayesian game in Second Price Auction.

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