Logic in CS Autumn 2024

Problem Sheet 7

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1. (a) Let A be a DFA accepting the language L. Is the reverse of all the strings accepted by $L(A)$ regular?

   (b) Let L be a regular language over $\{a, b, c\}$. Define the projection of L with respect to $\{b, c\}$ denoted $L \downarrow \{b, c\}$ as the language

   $$\{w' \mid w' \text{ is obtained from } w \in L \text{ after deleting all occurrences of symbol } a\}$$

   Is $L \downarrow \{b, c\}$ regular?

   (c) Show that every NFA can be converted into an equivalent one with a single accepting state.

   (d) Let $N = (Q, \Sigma, \delta, q_0, F)$ be a DFA. Construct an automaton $N_1 = (Q, \Sigma, \delta_1, q_0, F_1)$ as follows:

   - $F_1 = F \cup \{q_0\}$
   - Define δ_1 such that for any $q \in Q$ and $a \in \Sigma \cup \{\epsilon\}$,

   $$\delta_1(q, a) = \delta(q, a) \text{ for } q \notin F \text{ or } a \neq \epsilon$$

   $$\delta_1(q, a) = \delta(q, a) \cup \{q_0\} \text{ for } q \in F \text{ and } a = \epsilon$$

   Is $L(N_1) = (L(N))^*$?

   (e) Let L be a regular language. Is the language $L_{\frac{1}{2}}$, the set of first halves of strings in L regular? Formally,

   $$L_{\frac{1}{2}} = \{x \mid \exists y, |x| = |y|, xy \in L\}$$

   (f) Let L be a regular language. Is the language $Cuberoot(L)$ defined as $\{w \mid w^3 \in L\}$ regular?

   (g) Let L be a regular language. Consider the language L' defined as

   $$\{w \in L \mid \text{no proper prefix of } w \text{ is in } L\}$$

   Show that L' is regular.

   (h) For any language L, define $cycle(L) = \{uv \mid vu \in L\}$ as the set of all cyclic shifts of words accepted by L. As an example, if $abcd \in L$, then all its cyclic shifts $abcd, dabc, cdab, bcda$ are also in L. Show that if L is regular, so is $cycle(L)$.

2. Let $L_n = \{x \mid x \text{ is a binary number that is a multiple of } n\}$. Show that for all $n \geq 1$, L_n is regular.

3. Recall that we defined an angelic acceptance condition for NFAs in class : a word w is accepted whenever it has atleast one accepting run. Under this, we showed that the languages accepted by NFAs are regular. Consider the following *devilish* acceptance condition, which says that an NFA M accepts a word x iff every possible computation of M on x ends in an accept state. Show that NFAs with the devilish acceptance condition recognize the class of regular languages.

4. Write second order logic formulae to capture the following:
   (a) There is a path from node s to node t in the graph. The signature is $\tau = \{E\}$.
   (b) Every bounded non empty set has a least upper bound. The signature is $\tau = \{\leq\}$

5. Let Σ be a finite alphabet. The atomic formulae in MSO defined over Σ^* are $x = y, x < y, S(x, y), X(x)$ and $Q_a(x)$, $a \in \Sigma$. Consider the following logic called MSO_0 having atomic formulae of the following forms:

$$Sing(X), X \subseteq Y, X < Y, S(X, Y), Q_a(X)$$

   where

   - $Sing(X)$ means that X is a SO variable of cardinality 1;
   - $X \subseteq Y$ means that every element of the SO variable X is contained in the SO variable Y;
   - $X < Y$ means that SO variables X, Y have cardinality 1, and that the element in Y is greater than the element in X;
   - $S(X, Y)$ means that SO variables X, Y have cardinality 1, and Y contains the successor of the element in X; and,
   - $Q_a(X)$ means that all positions in X are decorated by $a \in \Sigma$.

   If φ is an atomic formula in MSO, then $\varphi \wedge \varphi, \neg\varphi, \varphi \vee \varphi$, $\forall x\ \varphi$ and $\forall X\ \varphi$ are formulae in MSO. Similarly, if φ is an atomic formula in MSO_0, then, $\varphi \wedge \varphi, \neg\varphi, \varphi \vee \varphi$ and $\forall X\ \varphi$ are formulae in MSO_0.
   Compare the expressiveness of MSO and MSO_0.

6. For the formula $\exists x \forall y (x < y \rightarrow Q_a(y))$ give an equivalent MSO_0 formula. Also draw the equivalent DFA following the steps done in class.

7. Consider the following NFA $N = (\{0, 1, 2, 3\}, \{a, b\}, \Delta, \{0\}, \{1\})$ with $\Delta(0, b) = \{1\}$, $\Delta(1, a) = \{2\}$, $\Delta(2, a) = \{2\}$, $\Delta(2, b) = \{3\}$ and $\Delta(3, b) = \{0\}$. Write an MSO formula with two SO variables that characterizes $L(N)$.