Logic in CS Autumn 2024

Problem Sheet 10

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1. Let $AP = \{p, q, r\}$. Formulate the following in LTL:

   (a) Eventually false

   (b) p occurs exactly twice, and q never occurs between two occurrences of p

   (c) If r occurs only finitely often, then p continously occurs from some point

   (d) r is true continously upto somepoint, and at the next point, both p, q hold, and then q and r alternate infinitely often

   (e) Infinitely often there are two consecutive occurrences of p

   (f) Between every consecutive occurrences of p, there is a q, and there is a prefix of r's of even length

Solution

(a) $\Diamond(p \wedge \neg p)$

(b) $\neg p \bigcup (p \wedge \bigcirc((\neg q \wedge \neg p) \bigcup (p \wedge \bigcirc\Box\neg p)))$

(c) $(\Diamond\Box\neg r) \Rightarrow (\Diamond\Box p)$

(d) $r \bigcup (\neg r \wedge p \wedge q \wedge \varphi)$, where $\varphi := \Box((q \Rightarrow \bigcirc r) \wedge (r \Rightarrow \bigcirc q))$

Note: it isn't clear in the question what is meant by q, r alternate infinitely often; We have taken it to mean that if q, r occur, then they are next followed by r, q respectively, and that sets up a chain of an infinite alternation.

(e) $\Box\Diamond(p \wedge \bigcirc p)$

(f) Not expressible in LTL. You can ignore this question.

2. Let TS and TS' be two transition systems without terminal states on the same set of atomic propositions AP. Then show that $Traces(TS) = Traces(TS')$ iff TS and TS' satisfy the same set of LT properties.

Solution

Note that $TS \models P \Leftrightarrow TS \subseteq P$.
If TS and TS' satisfy the same set of LT properties, then for every $P \subseteq (2^{AP})^\omega$, we have $\text{Traces}(TS) \subseteq P \Leftrightarrow \text{Traces}(TS') \subseteq P$. Consequently, taking $P := \text{Traces}(TS) \subseteq (2^{AP})^\omega$ yields $\text{Traces}(TS) \subseteq \text{Traces}(TS')$. Similarly, one may obtain $\text{Traces}(TS') \subseteq \text{Traces}(TS)$, thus proving $\text{Traces}(TS) = \text{Traces}(TS')$.

Conversely, if Traces(TS) = Traces(TS'), then it's obvious that Traces(TS) $\subseteq P \Leftrightarrow$ Traces(TS') $\subseteq P$ for every $P \subseteq (2^{AP})^\omega$, and consequently TS and TS' satisfy the same set of LT properties.

3. Consider a set of atomic propositions AP. Consider the following logic $\mathcal{X}$ defined as follows:

$$\varphi ::== (a \in AP)|\varphi \wedge \varphi|\neg\varphi|\varphi\Delta\varphi$$

with semantics as follows:
Given a word $w = A_0A_1\ldots$ over 2^{AP} and a position $i \in \mathbb{N}$, we define

(a) $w, i \models a$ iff $a \in A_i$ for $a \in AP$

(b) $w, i \models \varphi_1 \wedge \varphi_2$ iff $w, i \models \varphi_1$ and $w, i \models \varphi_2$

(c) $w, i \models \neg\varphi$ iff $w, i \not\models \varphi$

(d) $w, i \models \varphi\Delta\psi$ iff $\exists j > i$, $w, j \models \psi$ and $\forall i < k < j$, $w, k \models \varphi$.

Comment on the equivalence of LTL and $\mathcal{X}$.

Solution

Note that

$$\varphi \, \Delta \, \psi \equiv \bigcirc \left(\varphi \bigcup \psi\right)$$

Consequently, LTL is at least as powerful as $\mathcal{X}$.
Conversely, note that

$$\bigcirc\psi \equiv (p \wedge \neg p) \, \Delta \, \psi$$

since $\perp \bigcup \psi \equiv \psi$. Also,

$$\varphi \bigcup \psi \equiv \psi \vee (\varphi \wedge (\varphi \, \Delta \, \psi))$$

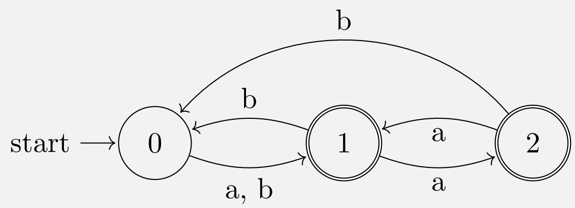
Furthermore, note that every other LTL operator, such as $\Diamond$ and $\Box$, can be written in terms of $\bigcirc$ and $\bigcup$. Consequently, $\mathcal{X}$ is at least as powerful as LTL.
Thus, LTL and $\mathcal{X}$ have equal expressive power, or in other words, any LTL property can be expressed in $\mathcal{X}$, and any $\mathcal{X}$-property can be expressed in LTL.

4. Consider a ω-automaton $(Q, \Sigma, \delta, q_0, Acc)$, and let $\mathcal{G} \subseteq 2^Q$ be a set of good states. An ω-word α is said to be accepted iff there is a run ρ of α such that $Inf(\rho) \in \mathcal{G}$. $\delta : Q \times \Sigma \to 2^Q$ is the transition function.

- Construct a deterministic ω-automata with this acceptance condition that captures the language "Finitely many b's".
- Show that ω-automata with this acceptance condition captures ω-regular languages.
- How do you complement a deterministic ω-automata with this acceptance condition?

Solution

- $\mathcal{G} = \{\{1, 2\}\}$



- Take the usual ω-automata $(Q, \Sigma, \delta, q_0, \text{Acc})$ where $\text{Acc} = \mathcal{G} \subseteq Q$. Then, in the new acceptance condition it corresponds to $(Q, \Sigma, \delta, q_0, \text{Acc}')$, where $\text{Acc}' = \mathcal{G}' \subseteq 2^Q$, with $\mathcal{G}' := \{S \subseteq Q : S \cap \mathcal{G} \neq \emptyset\}$.
  Now take any automaton $\mathcal{A}$ with the new acceptance condition. Consider any $S \in \mathcal{G} \subseteq 2^Q$. Then $\mathcal{A}$ accepts every run ρ such that $\text{Inf}(\rho) = S$. Also, let s be an arbitrary member of S.
  Now, consider the subgraph of $\mathcal{A}$ induced by S. This subgraph can be interpreted as an NFA, whose start state and final state we specify to be s, and which recognizes some regular language $L_{S,s}$. Now note that every run ρ such that $\text{Inf}(\rho) = S$ has an infinite suffix that starts from s, and consists only of states from S. Consequently, every such ρ, after reading some states in $\mathcal{A}$, settles down in the subgraph induced by S. Moreover, since $s \in \text{Inf}(\rho)$, ρ keeps returning to s. Consequently, the word read by ρ is of the form $uL^{\omega}_{S,s}$, where u is some word that is read while traveling from the start state of $\mathcal{A}$ to s.
  We thus obtain a characterization of all words read by ρ such that $\text{Inf}(\rho) = S$: Interpret $\mathcal{A}$ as an NFA with final state s. Let the regular language recognized by this be U_s. Then $U_s L^{\omega}_{S,s}$ is the exact set of words read by runs ρ such that $\text{Inf}(\rho) = S$.
  Consequently, the language accepted by $\mathcal{A}$ is $\bigcup_{S \in G} U_s L^{\omega}_{S,s}$. By the normal form for ω-regular languages, this language is ω-regular.

- We set $\mathcal{G}' := 2^Q \setminus \mathcal{G}$. *Prove that this gives the complement of the deterministic ω-automaton with this acceptance condition as an exercise.*

5. Prove or disprove : A finite set of infinite words is ω-regular.

Solution

False. Consider the ω-word 10100100010000..... Notice that the number of 0's between two consecutive 1's keeps increasing. The singleton language consisting of this word is not regular, since if it is of the normal form for ω-regular languages, it will contain a repeating unit, contradicting the fact that 10100100010000.... is irrational.

6. In class, we discussed the complexity of LTL model checking, where we said it is PSPACE-c. We did not look at the proof details. In this question, we show a weaker lower bound for LTL model checking, namely, it is co-NP hard. Show that the hamiltonian path problem is polynomially reducible to the complement of the LTL modelchecking problem. That is, given

a graph G, we can construct in polytime, a transition system TS and an LTL formula φ such that $TS \not\models \varphi$ iff G has a Hamiltonian path.

Solution

Given graph $G = (V, E)$ synthesize in polynomial time a TS and an LTL formula φ, and show that G has a HP iff $TS \not\models \varphi$. TS is the graph itself, with one new node added, say b such all vertices of G have an edge to b, and b has a self loop. Let the label of a node in the TS be the name of the vertex. Write an LTL formula to capture absence of a HP in G. Assume $V = \{v_1, \ldots, v_n\}$. The formula $\varphi = \neg\psi$ where ψ is

$$(\Diamond v_1 \wedge \Box(v_1 \rightarrow \bigcirc\Box\neg v_1)) \wedge \ldots (\Diamond v_n \wedge \Box(v_n \rightarrow \bigcirc\Box\neg v_n))$$

Show that G has a HP iff $TS \not\models \varphi$.
Assume $TS \not\models \neg\psi$. Then there is a path witnessing ψ. Let π be the path in TS such that $\pi \models \psi$. As $\pi \models \bigwedge_{v \in V}(\Diamond v \wedge \Box(v \rightarrow \bigcirc\Box\neg v))$, π witnesses all vertices of V, and does not repeat any vertex. π has the form $v_{i_1} v_{i_2} \ldots v_{i_n} b^\omega$, $i_1, \ldots, i_n \in \{1, 2, \ldots, n\}$, $i_j \neq i_k$. So G has the HP $v_{i_1} v_{i_2} \ldots v_{i_n}$.
The converse is similar : a HP in G extends to a path $\pi = v_{i_1} v_{i_2} \ldots v_{i_n} b^\omega$ in TS. Clearly, $\pi \models \psi$. So LTL model checking is co-NP hard as HP is NP-complete.

7. Exercises 5.24, 5.23, 5.17, 5.13, 5.7, 5.6, 5.5, 5.2, 5.1, 4.7, 4.15, 4.16, 4.23, 4.24, 4.25 from Baier-Katoen.

Solution

Practice Questions. Left as exercise.