Logic in CS Autumn 2024

Problem Sheet 9

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1. Consider the following set of sentences $\Gamma = \{F_1, F_2, F_3, F_4\}$ such that

$$\begin{aligned} F_1 &= \forall x\, (\forall y\, (C(x,y) \to R(y)) \to H(x)) \\ F_2 &= \forall x\, (G(x) \to R(x)) \\ F_3 &= \forall x\, (\exists y\, (C(y,x) \wedge G(y)) \to G(x)) \\ F_4 &= \neg \forall x\, (G(x) \to H(x))\,. \end{aligned}$$

(a) What is the signature τ of Γ?

(b) Skolemize $F_1, \ldots, F_4$ and obtain $G_1, \ldots, G_4$. What is the signature of $G_1, \ldots, G_4$?

(c) Show that propositional resolution gives $\emptyset$ by resolution applied on ground instances of $G_1, \ldots, G_4$.

Solution

(a) The signature τ contains the binary relation C, unary relations H, R, G.

(b) After Skolemization we have

$$\begin{aligned} G_1 &= \forall x\, ((C(x, f(x)) \vee H(x)) \wedge (\neg R(f(x)) \vee H(x))) \\ G_2 &= \forall y\, (\neg G(y) \vee R(y)) \\ G_3 &= \forall u\, \forall v\, (\neg C(v,u) \vee \neg G(v) \vee G(u)) \\ G_4 &= G(c) \wedge \neg H(c)\,. \end{aligned}$$

Note that we rename the bound variables apart.

(c) The following propositional resolution proof derives $\emptyset$ from ground instances of clauses from G_1, G_2, G_3 and G_4:

1. $\{C(x, f(x)), H(x)\}$ (from G_1)
2. $\{\neg C(v, u), \neg G(v), G(u)\}$ (from G_3)
3. $\{\neg G(x), H(x), G(f(x))\}$ Resolution on 1, 2
4. $\{\neg H(c)\}$ (from G_4)
5. $\{\neg G(c), G(f(c))\}$ Resolution on 3, 4
6. $\{G(c)\}$ (from G_4)
7. $\{G(f(c))\}$ Resolution on 5, 6
8. $\{\neg G(f(y)), R(f(y))\}$ (from G_2)
9. $\{R(f(c))\}$ Resolution on 7, 8
10. $\{\neg R(f(x)), H(x)\}$ (from G_1)
11. $\{H(c)\}$ Resolution on 9, 10
12. $\emptyset$ Resolution on 4, 11

2. Give an example of a finite set of clauses F in first-order logic such that $Res^*(F)$ is infinite.

Solution

Consider $F = \{\{P(0)\}, \{\neg P(x), P(s(x))\}\}$.
Then $Res^*(F) \supseteq \{\{P(0)\}, \{P(s(0))\}, \{P(s(s(0)))\}, \ldots\}$.

3. Give an example of a signature τ that has at least one constant symbol and a τ-formula F (that does not mention equality) such that F is satisfiable but does not have a Herbrand model.

Solution

Consider the signature with constant symbol 0, unary function symbol s, and unary predicate symbol P. Let

$$F = \forall x\, (P(0) \wedge (P(x) \rightarrow P(s(x))) \wedge \exists x\, \neg P(x)\,.$$

Every Herbrand model $\mathcal{H}$ must have $U_{\mathcal{H}} = \{0, s(0), s(s(0)), \ldots\}$ with $P_{\mathcal{H}}$ the total unary relation and hence does not satisfy F.
But F is satisfied by the structure $\mathcal{A}$ with $U_{\mathcal{A}} = \{0, 1\}$, $P_{\mathcal{A}} = \{0\}$, $0_{\mathcal{A}} = 0$ and $s_{\mathcal{A}}$ the identity function.

4. A closed formula is in the class $\exists^*\forall^*$ if it has the form $\exists x_1 \ldots \exists x_m\, \forall y_1 \ldots \forall y_n F$, where F is quantifier-free and $m, n \geq 0$.

(a) Prove that if an $\exists^*\forall^*$-formula over a signature with no function symbols has a model then it has a finite model.

(b) Suggest an algorithm for deciding whether a given $\exists^*\forall^*$-formula over a signature with no function symbols has a model.

(c) Argue that the satisfiability problem for the class of $\forall^*$-formulas that may mention function symbols is undecidable.

Solution

(a) Skolemising, we get a satisfiable formula $G = \forall y_1 \ldots \forall y_n (F^*[c_1/x_1] \ldots [c_m/x_m])$, where $c_1, \ldots, c_m$ are new constant symbols. By Herbrand's Theorem, G has a Herbrand model $\mathcal{H}$. The universe of such a model is the set of closed terms over the signature of G, which has cardinality $m+k$, where k is the number of constant symbols in F. It remains to observe that, by the Translation Lemma,

$$\mathcal{H}_{[x_1 \mapsto c_1] \ldots [x_m \mapsto c_m]}(\forall y_1 \ldots \forall y_n F^*) = \mathcal{H}(\forall y_1 \ldots y_n (F^*[c_1/x_1] \ldots [c_m/x_m])) = \mathcal{H}(G) = 1,$$

and hence $\mathcal{H} \models \exists x_1 \ldots \exists x_m \forall y_1 \ldots \forall y_n F$.

(b) By Part (a) if such a formula has a model then it has a model whose universe has cardinality at most $m+k$. There are finitely such models up to isomorphism, so we can enumerate them and check for each one whether the formula is true in the model.

(c) Given a sentence φ we can compute an equisatisfiable sentence in Skolem form. Since satisfiability is undecidable for general first-order formulas it is also undecidable for Skolem-form formulas. In particular, satisfiability is undecidable for $\forall^*$-formulas.

5. Execute ground resolution to show that the following formula is unsatisfiable:

$$\forall x \forall y ((P(x) \wedge \neg Q(y,y)) \rightarrow Q(x,y)) \wedge \neg \exists x (P(x) \wedge \exists y (Q(y,y) \wedge Q(x,y))) \wedge \exists y (P(y))$$

Solution

Let the clauses of the formula be

$$\begin{aligned}
F_1 &= \forall x \forall y ((P(x) \wedge \neg Q(y,y)) \rightarrow Q(x,y)) \\
F_2 &= \neg \exists x (P(x) \wedge \exists y (Q(y,y) \wedge Q(x,y))) \\
F_3 &= \exists y (P(y))
\end{aligned}$$

After Skolemization, we have

$$\begin{aligned}
G_1 &= \forall x \forall y (\neg P(x) \vee Q(y,y) \vee Q(x,y)) \\
G_2 &= \forall w \forall z (\neg P(w) \vee \neg Q(z,z) \vee \neg Q(w,z)) \\
G_3 &= P(c)
\end{aligned}$$

1. $\{\neg P(x), Q(y,y), Q(x,y)\}$	(from G_1)
2. $\{\neg P(c), Q(y,y), Q(c,y)\}$	(Unifying $x = c$ in 1)
3. $\{P(c)\}$	(from G_3)
4. $\{Q(y,y), Q(c,y)\}$	(Resolution on 2, 3)
5. $\{\neg P(w), \neg Q(z,z), \neg Q(w,z)\}$	(from G_2)
6. $\{\neg P(c), \neg Q(y,y), \neg Q(c,y)\}$	(Unifying $y = z, w = c$ in 5)
7. $\{\neg P(c)\}$	(Resolution on 4, 6)
8. $\emptyset$	(Resolution on 3, 7)