भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 7

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Slide preparation acknowledgments: C. R. Pradhit and Adit Akarsh

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal

Contents

Mechanism Design (Inverse Game Theory)

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- θ_i : value of i for the object.
- $u_i(x, \theta_i) = a_i \theta_i - p_i$

Social Choice Function

- The designer has an objective and this is captured through a **Social Choice Function (SCF)**.

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Answer: We need mechanisms, but they can be complicated

Mechanisms

Definition

An **indirect** mechanism is a collection of message spaces and a decision rule $\langle M_1, M_2, \ldots, M_n, g \rangle$

- M_i is the message space of agent i
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Answer

Due to a result that will follow.

Weakly Dominant

Definition

In a mechanism $\langle M_1, M_2, \ldots, M_n, g \rangle$, a message m_i is **weakly dominant** for player i at θ_i if

$$u_i(g(m_i, \tilde{m}_{-i}), \theta_i) \geqslant u_i(g(m_i', \tilde{m}_{-i}), \theta_i), \forall \tilde{m}_{-i}, \forall m_i'$$

All subsequent definitions assume cardinal preferences, however they can be replaced with ordinal, e.g., the above one could be defined as

$$g(m_i, \tilde{m}_{-i}) \ \theta_i \ g(m_i', \tilde{m}_{-i}), \forall \tilde{m}_{-i}, \forall m_i'$$

Dominant Strategy Implementable (DSI)

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An SCF $f : \Theta \to X$ is **implemented in dominant strategies** by $\langle M_1, M_2, \ldots, M_n, g \rangle$ if

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- $\exists$ message mappings $s_i : \Theta_i \to M_i$, s.t, $s_i(\theta_i)$ is a dominant strategy for agent i at θ_i, $\forall \theta_i \in \Theta_i$, $\forall i \in N$.

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We call this an indirect implementation, i.e., SCF f is **dominant strategy implementable (DSI)** by $\langle M_1, M_2, \ldots, M_n, g \rangle$.

Dominant Strategy Incentive Compatible DSIC)

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A direct mechanism $\langle \Theta_1, \Theta_2, \ldots, \Theta_n, f \rangle$ is **dominant strategy incentive compatible (DSIC)** if

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- To find if an SCF f is dominant strategy implementable, we need to search over all possible indirect mechanisms $\langle M_1, M_2, \ldots, M_n, g \rangle$.
- But luckily, there is a result that reduces the search space.

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Eq. (1) holds for all $m_i', \tilde{m}_{-i}$, in particular, $m_i' = s_i(\theta_i'), \tilde{m}_{-i} = s_{-i}(\tilde{\theta}_{-i})$ where θ_i' and $\tilde{\theta}_{-i}$ are arbitrary.

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$$u_i(\underbrace{g(s_i(\theta_i), s_{-i}(\tilde{\theta}_{-i}))}_{=f(\theta_i, \tilde{\theta}_{-i})}, \theta_i) \geqslant u_i(\underbrace{g(s_i(\theta_i'), s_{-i}(\tilde{\theta}_{-i}))}_{=f(\theta_i', \tilde{\theta}_{-i})}, \theta_i)$$

$\Rightarrow f$ is DSIC. □

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- Recall : Bayesian games $\langle N, (M_i)_{i \in N}, (\Theta_i)_{i \in N}, P, (\Gamma_\theta)_{\theta \in \Theta} \rangle$

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Observation: If an SCF f dominant strategy implementable, then it is Bayesian implementable.

Bayesian Incentive Compatible

Definition

A direct mechanism $\langle \Theta_1, \Theta_2, \dots, \Theta_n, f \rangle$ is **Bayesian Incentive Compatible (BIC)** if

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Revelation Principle for Bayesian Implementable SCFs

Revelation Principle (for Bayesian implementable SCFs)

If an SCF f is implementable in Bayesian equilibrium, then f is BIC.

- Proof idea is similar to the DSI, with expected utilities at appropriate places.
- For truthfulness of these two kinds, we will only consider incentive compatibility.
- These results hold even for ordinal preferences and mechanisms.
- Detailed proof: homework

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Arrow's Social Welfare Function Setup

Question

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Objective: create **social preferences** from **individual preferences**

- Finite set of alternatives $A = \{a_1, a_2, \ldots, a_m\}$
- Finite set of players $N = \{1, 2, \ldots, n\}$
- Each player i has a **preference relation** R_i over A (A binary relation over A, aR_ib means alternative a is at least as good as b to i
- Properties of R_i
  1. **Completeness:** for every pair of alternatives $a, b \in A$, either aR_ib or bR_ia or both

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  3. **Transitivity:** if aR_ib and bR_ic, then aR_ic, $\forall a, b, c \in A$ and $i \in N$

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  2. **symmetric** part I_i
- Example:

$$R_i = \begin{bmatrix} a \\ b, c \\ d \end{bmatrix} = \{(a, b), (a, c), (a, d), (b, c), (c, b), (b, d), (c, d)\}$$

$$\Rightarrow P_i = \begin{bmatrix} a & a \\ b & c \\ d & d \end{bmatrix} = \{(a, b), (a, c), (a, d), (b, d), (c, d)\}, \qquad I_i = \{(b, c), (c, b)\}$$

Arrovian Social Welfare Function (ASWF)

$$F : \mathcal{R}^n \to \mathcal{R} \qquad \text{domain and co-domain are both rankings}$$

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Pareto or Unanimity

Definition (Weak Pareto)

An ASWF F satisfies **weak Pareto** if the following holds for all $a, b \in A$ and for every strict preference profile R:

$$[aP_ib, \ \forall i \in N] \implies [a\hat{F}(R)b].$$

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Which property implies the other?

Independence of Irrelevant Alternatives

- We say $R_i, R_i' \in \mathcal{R}$ **agree** on $\{a, b\}$ for agent i if

$$aP_ib \Leftrightarrow aP_i'b, \; bP_ia \Leftrightarrow bP_i'a, \; aI_ib \Leftrightarrow aI_i'b$$

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An ASWF F satisfies **independence of irrelevant alternatives** (IIA) if for all $a, b \in A$, and for every pair of preference profiles R and R', if $R|_{a,b} = R'|_{a,b}$, then $F(R)|_{a,b} = F(R')|_{a,b}$.

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If the relative positions of two alternatives are the same in two different preference profiles, then the aggregate should also match the relative positions of those two alternatives

Example

If the relative positions of two alternatives are the same in two different preference profiles, then the aggregate should also match the relative positions of those two alternatives

R			
a	a	c	d
b	c	b	c
c	b	a	b
d	d	d	a

R'			
d	c	b	b
a	a	c	a
b	b	a	d
c	d	d	c

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R					R'			
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c	b	a	b		b	b	a	d
d	d	d	a		c	d	d	c

- IIA says $F(R)|_{a,b} = F(R')|_{a,b}$
- Simple aggregation rules, e.g., **scoring rules**: each position of each agent gets a score $(s_1, s_2, \dots, s_m), s_i \geqslant s_{i+1}, i = 1, 2, \dots, m-1$, the final ordering is in the decreasing order of the scores

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- One special scoring rule: **plurality**, $s_1 = 1, s_i = 0, i = 2, \ldots, m$.

Satisfaction of IIA

Question

Does plurality satisfy IIA?

R

a	a	c	d
b	c	b	c
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$$
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Check: $aF_{\texttt{plu}}(R)b$, but $bF_{\texttt{plu}}(R')a$, even though $R|_{a,b} = R'|_{a,b}$

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Question

Does **dictatorship** satisfy IIA?

A **dictatorship** ASWF is where there exists a pre-determined agent d and $F^d(R) = R_d$

Arrow's impossibility result

Theorem (Arrow 1951)

For $|A| \geqslant 3$, if an ASWF F satisfies WP and IIA, then it must be dictatorial.

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We cannot aggregate reasonably even when there is no truthfulness constraint

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Observation: $D_G(a, b) \implies \overline{D}_G(a, b)$

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Note: these two lemmas immediately proves the theorem

Field expansion lemma

Lemma

Let F satisfy WP and IIA, then $\forall a, b, x, y, G \subseteq N, G \neq \emptyset, a \neq b, x \neq y$

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7. $\overline{D}_G(a, b) \Rightarrow D_G(b, a)$

Proof of FEL

- **Case 1:** $\overline{D}_G(a, b) \Rightarrow D_G(a, y), y \neq a, b$

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- Construct R' s.t.

G		$N \setminus G$	
a	a	b	b
$\vdots$	$\vdots$	$\vdots$	$\vdots$
b	b	a	y
$\vdots$	$\vdots$	$\vdots$	$\vdots$
y	y	y	a

positions of a and y in $N \setminus G$ s.t. $R'|_{a,y} = R|_{a,y}$

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- Construct R' s.t.

$$\begin{array}{cc||cc}
\multicolumn{2}{c}{G} & \multicolumn{2}{c}{N \setminus G} \\
\hline\hline
a & a & b & b \\
\vdots & \vdots & \vdots & \vdots \\
b & b & a & y \\
\vdots & \vdots & \vdots & \vdots \\
y & y & y & a
\end{array}$$

positions of a and y in $N \setminus G$ s.t. $R'|_{a,y} = R|_{a,y}$

- $\overline{D}_G(a,b) \Rightarrow a\hat{F}(R')b$

Proof of FEL

- **Case 1:** $\overline{D}_G(a,b) \Rightarrow D_G(a,y), y \neq a,b$
- Pick an arbitrary $R \in \mathcal{R}^n$, s.t., aP_iy, $\forall i \in G$
- Need to show: $a\hat{F}(R)y$
- Construct R' s.t.

$$
\begin{array}{cc||cc}
\multicolumn{2}{c}{G} & \multicolumn{2}{c}{N \setminus G} \\
\hline\hline
a & a & b & b \\
\vdots & \vdots & \vdots & \vdots \\
b & b & a & y \\
\vdots & \vdots & \vdots & \vdots \\
y & y & y & a
\end{array}
$$

positions of a and y in $N \setminus G$ s.t. $R'|_{a,y} = R|_{a,y}$

- $\overline{D}_G(a,b) \Rightarrow a\hat{F}(R')b$
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$$\begin{array}{cc||cc}
\multicolumn{2}{c}{G} & \multicolumn{2}{c}{N \setminus G} \\
\hline\hline
a & a & b & b \\
\vdots & \vdots & \vdots & \vdots \\
b & b & a & y \\
\vdots & \vdots & \vdots & \vdots \\
y & y & y & a
\end{array}$$

positions of a and y in $N \setminus G$ s.t. $R'|_{a,y} = R|_{a,y}$

- $\overline{D}_G(a,b) \Rightarrow a\hat{F}(R')b$
- WP over b, y, $\Rightarrow b\hat{F}(R')y$, transitivity $\Rightarrow a\hat{F}(R')y$
- IIA $\Rightarrow a\hat{F}(R)y$. Hence, $D_G(a,y)$

Proof of FEL (contd.)

- **Case 2:** $\overline{D}_G(a,b) \Rightarrow D_G(x,b), x \neq a, b$

Proof of FEL (contd.)

- **Case 2:** $\overline{D}_G(a,b) \Rightarrow D_G(x,b), x \neq a, b$
- Pick an arbitrary $R \in \mathcal{R}^n$, s.t., xP_ib, $\forall i \in G$

Proof of FEL (contd.)

- **Case 2:** $\overline{D}_G(a,b) \Rightarrow D_G(x,b), x \neq a, b$
- Pick an arbitrary $R \in \mathcal{R}^n$, s.t., xP_ib, $\forall i \in G$
- Need to show: $x\hat{F}(R)b$

Proof of FEL (contd.)

- **Case 2:** $\overline{D}_G(a,b) \Rightarrow D_G(x,b), x \neq a, b$
- Pick an arbitrary $R \in \mathcal{R}^n$, s.t., xP_ib, $\forall i \in G$
- Need to show: $x\hat{F}(R)b$
- Construct R' s.t.

$$
\begin{array}{cc||cc}
\multicolumn{2}{c}{G} & \multicolumn{2}{c}{N \setminus G} \\
\hline\hline
x & x & x & b \\
\vdots & \vdots & \vdots & \vdots \\
a & a & b & x \\
\vdots & \vdots & \vdots & \vdots \\
b & b & a & a
\end{array}
$$

positions of x and b in $N \setminus G$ s.t. $R'|_{x,b} = R|_{x,b}$

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- Need to show: $x\hat{F}(R)b$
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$$\begin{array}{cc||cc} \multicolumn{2}{c}{G} & \multicolumn{2}{c}{N \setminus G} \\ \hline\hline x & x & x & b \\ \vdots & \vdots & \vdots & \vdots \\ a & a & b & x \\ \vdots & \vdots & \vdots & \vdots \\ b & b & a & a \end{array}$$

positions of x and b in $N \setminus G$ s.t. $R'|_{x,b} = R|_{x,b}$

- $\overline{D}_G(a,b) \Rightarrow a\hat{F}(R')b$

Proof of FEL (contd.)

- **Case 2:** $\overline{D}_G(a,b) \Rightarrow D_G(x,b), x \neq a,b$
- Pick an arbitrary $R \in \mathcal{R}^n$, s.t., xP_ib, $\forall i \in G$
- Need to show: $x\hat{F}(R)b$
- Construct R' s.t.

$$\begin{array}{cc||cc} \multicolumn{2}{c}{G} & \multicolumn{2}{c}{N \setminus G} \\ \hline\hline x & x & x & b \\ \vdots & \vdots & \vdots & \vdots \\ a & a & b & x \\ \vdots & \vdots & \vdots & \vdots \\ b & b & a & a \end{array}$$

positions of x and b in $N \setminus G$ s.t. $R'|_{x,b} = R|_{x,b}$

- $\overline{D}_G(a,b) \Rightarrow a\hat{F}(R')b$
- WP over x,a, $\Rightarrow x\hat{F}(R')a$, transitivity $\Rightarrow x\hat{F}(R')b$

Proof of FEL (contd.)

- **Case 2:** $\overline{D}_G(a,b) \Rightarrow D_G(x,b), x \neq a,b$
- Pick an arbitrary $R \in \mathcal{R}^n$, s.t., xP_ib, $\forall i \in G$
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$$
\begin{array}{cc||cc}
\multicolumn{2}{c}{G} & \multicolumn{2}{c}{N \setminus G} \\
\hline\hline
x & x & x & b \\
\vdots & \vdots & \vdots & \vdots \\
a & a & b & x \\
\vdots & \vdots & \vdots & \vdots \\
b & b & a & a
\end{array}
$$

positions of x and b in $N \setminus G$ s.t. $R'|_{x,b} = R|_{x,b}$

- $\overline{D}_G(a,b) \Rightarrow a\hat{F}(R')b$
- WP over x, a, $\Rightarrow x\hat{F}(R')a$, transitivity $\Rightarrow x\hat{F}(R')b$
- IIA $\Rightarrow x\hat{F}(R)b$. Hence, $D_G(x,b)$

Proof of FEL (other cases)

- **Case 3:** $\overline{D}_G(a,b) \overset{\text{(case 1)}}{\Longrightarrow} D_G(a,y)\ (y \neq a,b) \overset{\text{(definition)}}{\Longrightarrow} \overline{D}_G(a,y) \overset{\text{(case 2)}}{\Longrightarrow} D_G(x,y)\ (x \neq a,y)$
- **Case 4:** $\overline{D}_G(a,b) \overset{\text{(case 2)}}{\Longrightarrow} D_G(x,b)\ (x \neq a,b) \overset{\text{(definition)}}{\Longrightarrow} \overline{D}_G(x,b) \overset{\text{(case 1)}}{\Longrightarrow} D_G(x,a)\ (x \neq a,b)$
- **Case 5:** $\overline{D}_G(a,b) \overset{\text{(case 1)}}{\Longrightarrow} D_G(a,y)\ (y \neq a,b) \overset{\text{(definition)}}{\Longrightarrow} \overline{D}_G(a,y) \overset{\text{(case 2)}}{\Longrightarrow} D_G(b,y)\ (y \neq a,b)$
- **Case 6:** $\overline{D}_G(a,b) \overset{\text{(case 2)}}{\Longrightarrow} D_G(x,b)\ (x \neq a,b) \overset{\text{(definition)}}{\Longrightarrow} \overline{D}_G(x,b) \overset{\text{(case 2)}}{\Longrightarrow} D_G(a,b)$
- **Case 7:** $\overline{D}_G(a,b) \overset{\text{(case 5)}}{\Longrightarrow} D_G(b,y)\ (y \neq a,b) \overset{\text{(definition)}}{\Longrightarrow} \overline{D}_G(b,y) \overset{\text{(case 1)}}{\Longrightarrow} D_G(b,a)$

Group contraction lemma

Lemma

Let F satisfy WP and IIA, and let $G \subseteq N, G \neq \emptyset, |G| \geqslant 2$ be decisive. Then $\exists G' \subset G, G' \neq \emptyset$ which is also decisive.

Proof:

- $G, |G| \geqslant 2$ is given. Let $G_1 \subset G, G_2 = G \setminus G_1$, $G_1, G_2 \neq \emptyset$, arbitrary.

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Proof:

- $G, |G| \geqslant 2$ is given. Let $G_1 \subset G, G_2 = G \setminus G_1, G_1, G_2 \neq \emptyset$, arbitrary.
- Construct R

G_1	G_2	$N \setminus G$
a	c	b
b	a	c
c	b	a

$aP_ib, \forall i \in G$ and G decisive $\Rightarrow a\hat{F}(R)b$

Group contraction lemma

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a	c	b
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$$aP_ib, \forall i \in G \text{ and } G \text{ decisive } \Rightarrow a\hat{F}(R)b$$

- Where can c stand in $F(R)$ w.r.t. a? We will show in every possible case, either G_1 or G_2 will be decisive

Proof of GCL

Case 1: $a\hat{F}(R)c$

G_1	G_2	$N \setminus G$
a	c	b
b	a	c
c	b	a

have seen $\Rightarrow a\hat{F}(R)b$

- Consider G_1

Proof of GCL

Case 1: $a\hat{F}(R)c$

G_1	G_2	$N \setminus G$
a	c	b
b	a	c
c	b	a

have seen $\Rightarrow a\hat{F}(R)b$

- Consider G_1
- $aP_ic, \forall i \in G_1, cP_ia, \forall i \in N \setminus G_1$

Proof of GCL

Case 1: $a\hat{F}(R)c$

G_1	G_2	$N \setminus G$
a	c	b
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have seen $\Rightarrow a\hat{F}(R)b$

- Consider G_1
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- Consider each R' where the above relation holds

Proof of GCL

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- Consider G_1
- $aP_ic, \forall i \in G_1, cP_ia, \forall i \in N \setminus G_1$
- Consider each R' where the above relation holds
- by IIA $a\hat{F}(R')c$

Proof of GCL

Case 1: $a\hat{F}(R)c$

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- Consider G_1
- $aP_ic, \forall i \in G_1, cP_ia, \forall i \in N \setminus G_1$
- Consider each R' where the above relation holds
- by IIA $a\hat{F}(R')c$
- Hence $\overline{D}_{G_1}(a,c) \overset{\text{(FEL)}}{\Longrightarrow} D_{G_1}$

Proof of GCL (contd.)

Case 2: $\neg(a\hat{F}(R)c) \implies cF(R)a$

- $a\hat{F}(R)b$ and $cF(R)a$ give $c\hat{F}(R)b$

G_1	G_2	$N \setminus G$
a	c	b
b	a	c
c	b	a

have seen $\Rightarrow a\hat{F}(R)b$

Proof of GCL (contd.)

Case 2: $\neg(a\hat{F}(R)c) \implies cF(R)a$

- $a\hat{F}(R)b$ and $cF(R)a$ give $c\hat{F}(R)b$
- Consider G_2

G_1	G_2	$N \setminus G$
a	c	b
b	a	c
c	b	a

have seen $\Rightarrow a\hat{F}(R)b$

Proof of GCL (contd.)

Case 2: $\neg(a\hat{F}(R)c) \implies cF(R)a$

G_1	G_2	$N \setminus G$
a	c	b
b	a	c
c	b	a

have seen $\Rightarrow a\hat{F}(R)b$

- $a\hat{F}(R)b$ and $cF(R)a$ give $c\hat{F}(R)b$
- Consider G_2
- $cP_ib, \forall i \in G_2, bP_ic, \forall i \in N \setminus G_2$

Proof of GCL (contd.)

Case 2: $\neg(a\hat{F}(R)c) \implies cF(R)a$

G_1	G_2	$N \setminus G$
a	c	b
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G_1	G_2	$N \setminus G$
a	c	b
b	a	c
c	b	a

have seen $\Rightarrow a\hat{F}(R)b$

- $a\hat{F}(R)b$ and $cF(R)a$ give $c\hat{F}(R)b$
- Consider G_2
- $cP_ib, \forall i \in G_2, bP_ic, \forall i \in N \setminus G_2$
- Consider each R' where the above relation holds
- by IIA $c\hat{F}(R')b$
- Hence $\overline{D}_{G_2}(c,b) \overset{\text{(FEL)}}{\implies} D_{G_2}$

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