CS 228 : Logic in Computer Science

Krishna, S

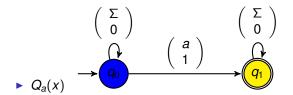
Formulae to DFA

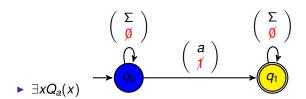
▶ Given $\varphi(x_1, ..., x_n)$, a FO formula over Σ , consider the extended alphabet

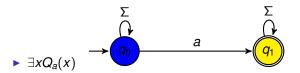
$$\Sigma' = \Sigma \times \{0, 1\}^n$$

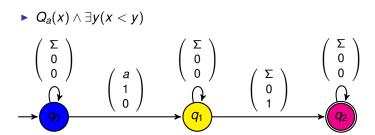
- Assign values to x_i at every position as seen in the cases of atomic formulae
- ▶ Keep in mind that every x_i can be assigned 1 at a unique position

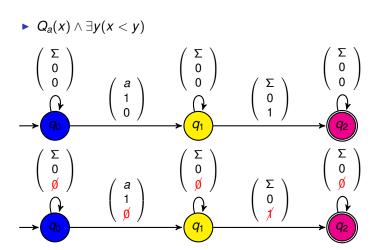
Quantifiers





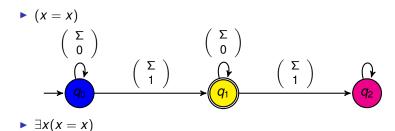




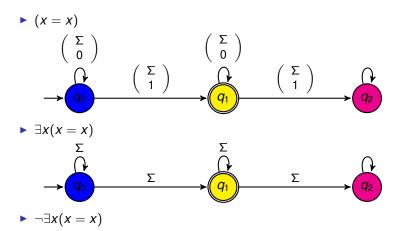


Handling Quantifiers: $\forall x (x \neq x)$

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Handling Quantifiers: Summary

- ▶ Let $L \subseteq (\Sigma \times \{0,1\}^n)^*$ be defined by $\varphi(x_1,\ldots,x_n)$.
- ▶ Let $f: (\Sigma \times \{0,1\}^n)^* \to (\Sigma \times \{0,1\}^{n-1})^*$ be the projection $f(w, c_1, ..., c_n) = (w, c_1, ..., c_{n-1}).$
- ▶ Then $\exists x_n \varphi(x_1, \dots, x_{n-1})$ defines f(L).

Handling Quantifiers : Done on Board

- $\exists x \forall y [x > y \lor \neg Q_a(x)] = \exists x [\neg \exists y [x \leqslant y \land Q_a(x)]]$
- ▶ Draw the automaton for $[x \le y \land Q_a(x)]$
- Project out the y-row
- Determinize it, and complement it
- ► Fix the *x*-row : Intersect with $\begin{pmatrix} \Sigma \\ 0 \end{pmatrix}^* \begin{pmatrix} \Sigma \\ 1 \end{pmatrix} \begin{pmatrix} \Sigma \\ 0 \end{pmatrix}^*$
- ▶ Project the *x*-row

Points to Remember

- ▶ Given $\varphi(x_1, ..., x_n)$, construct automaton for atomic FO formulae over the extended alphabet $\Sigma \times \{0, 1\}^n$
- ► Intersect with the regular language where every x_i is assigned 1 exactly at one position
- ▶ Given a sentence $Q_{x_1} \dots Q_{x_n} \varphi$, first construct the automaton for the formula $\varphi(x_1, \dots, x_n)$
- ► Replace ∀ in terms of ∃

Points to Remember

- ► Given the automaton for $\varphi(x_1, ..., x_n)$, the automaton for $\exists x_i \varphi(x_1, ..., x_n)$ is obtained by projecting out the row of x_i
- This may result in an NFA
- ▶ Determinize it and complement it to get a DFA for $\neg \exists x_i \varphi(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$
- ▶ Intersect with the regular language where each of $x_1, ..., x_{i-1}, x_{i+1}, ..., x_n$ are assigned 1 exactly at one position

The Computational Effort

Given NFAs A_1 , A_2 each with atmost n states,

- ▶ The union has atmost 2*n* + 1 states
- ▶ Intersection has atmost n² states
- ▶ The complement has atmost 2ⁿ states
- ► The projection has atmost *n* states

The Computational Effort

- ▶ $\psi = Q_1 \dots Q_n \varphi$. If $Q_i = \exists$ for all i, then size of A_{ψ} is same the size of A_{φ} .
- ▶ When $Q_1 = \exists, Q_2 = \forall, \dots$: each \forall quantifier can create a 2^n blowup in automaton size
- Size of automaton is

where the tower height k is the quantifier alternation size.

▶ This number is indeed a lower bound!

The Automaton-Logic Connection

Given any FO sentence φ , one can construct a DFA A_{φ} such that $L(\varphi) = L(A_{\varphi})$.

Summary

- ▶ Given FO formula φ , build an automaton A_{φ} preserving the language
- Satisfiability of FO reduces to non-emptiness of underlying automaton