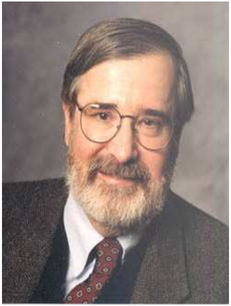
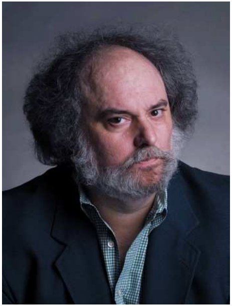
CS 228 : Logic in Computer Science

Krishna. S

Linear Temporal Logic

Model Checking

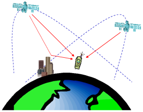
  

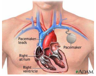
- Year 2007 : ACM confers the Turing Award to the pioneers of Model Checking: Ed Clarke, Allen Emerson, and Joseph Sifakis
- `https://amturing.acm.org/award_winners/clarke_1167964`

Model checking

- Model checking has evolved in last 25 years into a widely used verification and debugging technique for software and hardware.
- Model checking used (and further developed) by companies/institutes such as IBM, Intel, NASA, Cadence, Microsoft, and Siemens, and has culminated in many freely downloadable software tools that allow automated verification.

What is Model Checking?



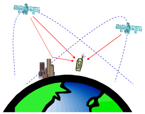


System → specification

satisfy?

good/bad properties

What is Model Checking?



System

satisfy?

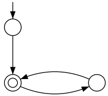
specification

System Model

model-checking

$\models ?$

Spec Model



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Model Checker as a Black Box

- Inputs to Model checker : A finite state system M, and a property P to be checked.
- Question : Does M satisfy P?
- Possible Outputs
  - Yes, M satisfies P
  - No, here is a counter example!.

What are Models?

Transition Systems

- States labeled with propositions
- Transition relation between states
- Action-labeled transitions to facilitate composition

What are Properties?

Example properties

- Can the system reach a deadlock?
- Can two processes ever be together in a critical section?
- On termination, does a program provide correct output?

Notations for Infinite Words

- Σ is a finite alphabet
- Σ^* set of finite words over Σ
- An infinite word is written as $\alpha = \alpha(0)\alpha(1)\alpha(2)\ldots$, where $\alpha(i) \in \Sigma$
- Such words are called ω-words
- a^ω, $a^7.b^\omega$

Transition Systems

A Transition System is a tuple $(S, Act, \rightarrow, I, AP, L)$ where

- S is a set of states
- Act is a set of actions
- $s \xrightarrow{\alpha} s'$ in $S \times Act \times S$ is the transition relation
- $I \subseteq S$ is the set of initial states
- AP is the set of atomic propositions
- $L : S \rightarrow 2^{AP}$ is the labeling function

Traces of Transition Systems

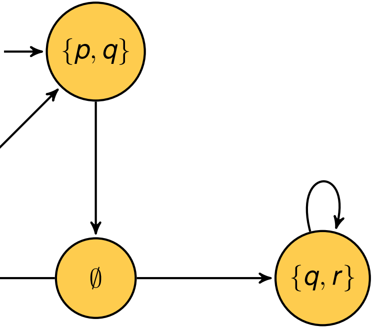
- Labels of the locations represent values of all observable propositions $\in AP$
- Captures system state
- Focus on sequences $L(s_0)L(s_1)\ldots$ of labels of locations
- Such sequences are called traces
- Assuming transition systems have no terminal states,
  - Traces are infinite words over 2^{AP}
  - Traces $\in (2^{AP})^\omega$
  - Go to the example slide and define traces

Traces of Transition Systems

Given a transition system $TS = (S, Act, \rightarrow, I, AP, L)$ without terminal states,

- All maximal executions/paths are infinite
- Path $\pi = s_0 s_1 s_2 \ldots$, $trace(\pi) = L(s_0)L(s_1)\ldots$
- For a set Π of paths, $Trace(\Pi) = \{trace(\pi) \mid \pi \in \Pi\}$
- For a location s, $Traces(s) = Trace(Paths(s))$
- $Traces(TS) = \bigcup_{s \in I} Traces(s)$

Example Traces



$AP = \{p, q, r, t\}$

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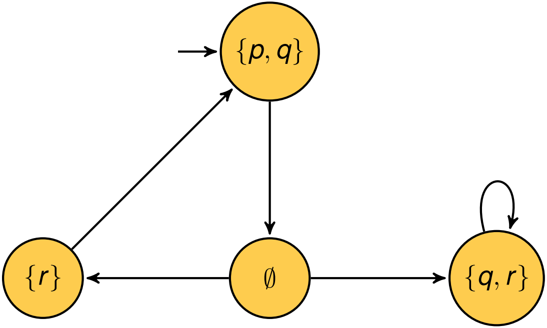
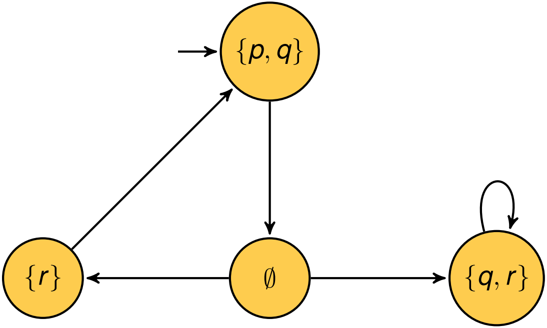


$AP = \{p, q, r, t\}$

- $\{p, q\}\emptyset\{q, r\}^\omega$

Example Traces



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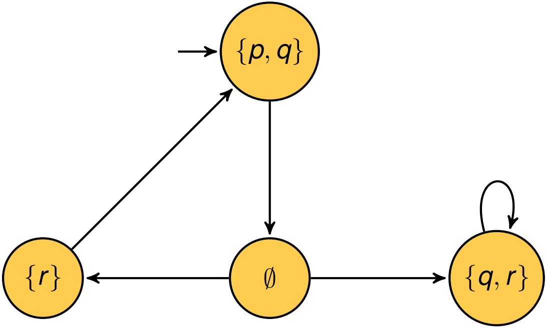

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- $\{p, q\}\emptyset\ \{q, r\}^\omega$
- $(\{p, q\}\emptyset\{r\})^\omega$
- $(\{p, q\}\emptyset\{r\})^*\ \{p, q\}\emptyset\ \{q, r\}^\omega$

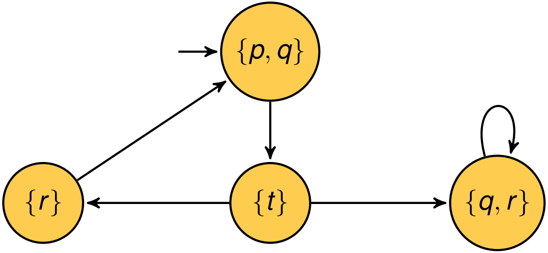
Linear Time Properties

- Linear-time properties specify traces that a TS must have
- A LT property P over AP is a subset of $(2^{AP})^\omega$
- TS over AP satisfies a LT property P over AP

$$TS \models P \text{ iff } Traces(TS) \subseteq P$$

- $s \in S$ satisfies LT property P (denoted $s \models P$) iff $Traces(s) \subseteq P$

Specifying Traces



- Whenever p is true, r will eventually become true
  - $\{A_0 A_1 A_2 \cdots \mid \forall i \geqslant 0, p \in A_i \to \exists j \geqslant i, r \in A_j\}$
- q is true infinitely often
  - $\{A_0 A_1 A_2 \cdots \mid \forall i \geqslant 0, \exists j \geqslant i, q \in A_j\}$
- Whenever r is true, so is q
  - $\{A_0 A_1 \cdots \mid \forall i \geqslant 0, r \in A_i \to q \in A_i\}$

Syntax of Linear Temporal Logic

Given *AP*, a set of propositions,

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- Propositional logic formulae over *AP*
  - $a \in AP$ (atomic propositions)
  - $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$

Syntax of Linear Temporal Logic

Given *AP*, a set of propositions,

- Propositional logic formulae over *AP*
  - $a \in AP$ (atomic propositions)
  - $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$
- Temporal Operators
  - $\bigcirc\varphi$ (Next φ)
  - $\varphi \, \mathsf{U} \psi$ (φ holds until a ψ-state is reached)
- LTL : Logic for describing LT properties

Semantics (On the board)

LTL formulae φ over AP interpreted over words $w \in \Sigma^\omega$, $\Sigma = 2^{AP}$,
$w \models \varphi$

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Derived Operators

- *true* $= \varphi \vee \neg\varphi$
- *false* $= \neg$*true*
- $\Diamond\varphi =$ *true* $\mathsf{U}\varphi$ (Eventually φ)
- $\Box\varphi = \neg\Diamond\neg\varphi$ (Forever φ)

Precedence

- Unary Operators bind stronger than Binary
- $\bigcirc$ and $\neg$ equally strong
- U takes precedence over $\wedge, \vee, \rightarrow$
  - $a \vee b\,\mathsf{U}c \equiv a \vee (b\,\mathsf{U}c)$
  - $\bigcirc a\,\mathsf{U}\neg b \equiv (\bigcirc a)\,\mathsf{U}(\neg b)$

Examples

- Whenever the traffic light is red, it cannot become green immediately:

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- Eventually the traffic light will become yellow
  $\Diamond yellow$
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Examples

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- Eventually the traffic light will become yellow
  $\Diamond \mathit{yellow}$
- Once the traffic light becomes yellow, it will eventually become green
  $\Box(\mathit{yellow} \rightarrow \Diamond \mathit{green})$

Semantics over Infinite Words

Given LTL formula φ over AP,

$$L(\varphi) = \{\sigma \in (2^{AP})^\omega \mid \sigma \models \varphi\}$$

Let $\sigma = A_0 A_1 A_2 \ldots$, with $A_i \subseteq AP$.

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- $\sigma \models a$ iff $a \in A_0$
- $\sigma \models \varphi_1 \wedge \varphi_2$ iff $\sigma \models \varphi_1$ and $\sigma \models \varphi_2$
- $\sigma \models \neg\varphi$ iff $\sigma \not\models \varphi$
- $\sigma \models \bigcirc\varphi$ iff $A_1 A_2 \ldots \models \varphi$
- $\sigma \models \varphi \,\mathsf{U} \psi$ iff
  $\exists j \geqslant 0$ such that $A_j A_{j+1} \ldots \models \psi \wedge \forall 0 \leqslant i < j, A_i A_{i+1} \ldots \models \varphi$

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Let $\sigma = A_0 A_1 A_2 \ldots$, with $A_i \subseteq AP$.

- $\sigma \models \Diamond\varphi$ iff $\exists j \geqslant 0,\ A_j A_{j+1} \ldots \models \varphi$
- $\sigma \models \Box\varphi$ iff $\forall j \geqslant 0,\ A_j A_{j+1} \ldots \models \varphi$
- $\sigma \models \Box\Diamond\varphi$ iff $\forall j \geqslant 0,\ \exists i \geqslant j,\ A_i A_{i+1} \ldots \models \varphi$
- $\sigma \models \Diamond\Box\varphi$ iff $\exists j \geqslant 0,\ \forall i \geqslant j,\ A_i A_{i+1} \ldots \models \varphi$

If $\sigma = A_0 A_1 A_2 \ldots$, $\sigma \models \varphi$ is also written as $\sigma, 0 \models \varphi$. This simply means $A_0 A_1 A_2 \ldots \models \varphi$. One can also define $\sigma, i \models \varphi$ to mean $A_i A_{i+1} A_{i+2} \ldots \models \varphi$ to talk about a suffix of the word σ satisfying a property.

Transition System Semantics $TS \models \varphi$

Let $TS = (S, S_0, \rightarrow, AP, L)$ be a transition system, and φ an LTL formula over AP

- For an infinite path fragment π of TS,

$$\pi \models \varphi \text{ iff } \mathit{trace}(\pi) \models \varphi$$

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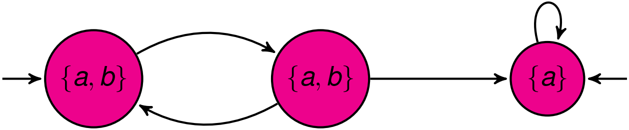
- $TS \models \varphi$ iff $\mathit{Traces}(TS) \subseteq L(\varphi)$

Transition System Semantics $TS \models \varphi$

Assume all states in TS are reachable from S_0.

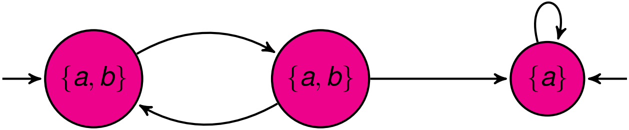
- $TS \models \varphi$ iff $TS \models L(\varphi)$ iff $Traces(TS) \subseteq L(\varphi)$
- $TS \models L(\varphi)$ iff $\pi \models \varphi \; \forall \pi \in Paths(TS)$
- $\pi \models \varphi \; \forall \pi \in Paths(TS)$ iff $s_0 \models \varphi \; \forall s_0 \in S_0$

Example



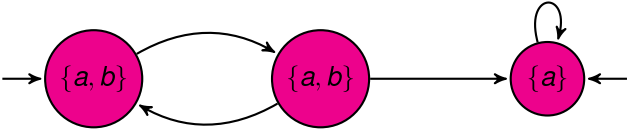
- $TS \models \Box a$,

Example



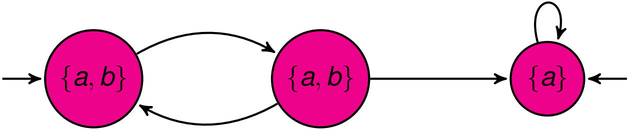
- $TS \models \Box a$,
- $TS \not\models \bigcirc(a \wedge b)$

Example



- $TS \models \Box a$,
- $TS \not\models \bigcirc(a \wedge b)$
- $TS \not\models (b \, \mathsf{U} (a \wedge \neg b))$

Example



- $TS \models \Box a$,
- $TS \not\models \bigcirc(a \wedge b)$
- $TS \not\models (b \,\mathsf{U} (a \wedge \neg b))$
- $TS \models \Box(\neg b \rightarrow \Box(a \wedge \neg b))$

More Semantics

- For paths π, $\pi \models \varphi$ iff $\pi \not\models \neg\varphi$

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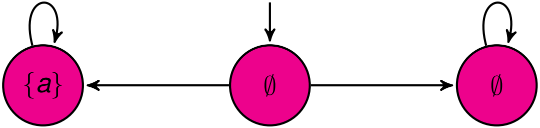
- For paths π, $\pi \models \varphi$ iff $\pi \not\models \neg\varphi$
  $\mathit{trace}(\pi) \in L(\varphi)$ iff $\mathit{trace}(\pi) \notin L(\neg\varphi) = \overline{L(\varphi)}$
- $TS \not\models \varphi$ iff $TS \models \neg\varphi$?

More Semantics

- For paths π, $\pi \models \varphi$ iff $\pi \not\models \neg\varphi$
  trace$(\pi) \in L(\varphi)$ iff *trace*$(\pi) \notin L(\neg\varphi) = \overline{L(\varphi)}$
- $TS \not\models \varphi$ iff $TS \models \neg\varphi$?
  - $TS \models \neg\varphi \rightarrow \forall$ paths π of TS, $\pi \models \neg\varphi$
  - Thus, $\forall\pi$, $\pi \not\models \varphi$. Hence, $TS \not\models \varphi$
  - Now assume $TS \not\models \varphi$
  - Then $\exists$ some path π in TS such that $\pi \models \neg\varphi$
  - However, there could be another path π' such that $\pi' \models \varphi$
  - Then $TS \not\models \neg\varphi$ as well
- Thus, $TS \not\models \varphi \not\equiv TS \models \neg\varphi$.

An Example



$TS \not\models \Diamond a$ and $TS \not\models \Box\neg a$

Equivalence of LTL Formulae

Equivalence

φ and ψ are equivalent ($\varphi \equiv \psi$) iff $L(\varphi) = L(\psi)$.

Expansion Laws

- $\varphi \, \mathsf{U} \psi \equiv \psi \vee (\varphi \wedge \bigcirc(\varphi \, \mathsf{U} \psi))$
- $\Diamond\varphi \equiv \varphi \vee \bigcirc\Diamond\varphi$
- $\Box\varphi \equiv \varphi \wedge \bigcirc\Box\varphi$

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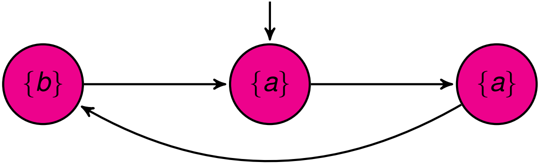
Distribution

$\bigcirc(\varphi \vee \psi) \equiv \bigcirc\varphi \vee \bigcirc\psi$,
$\bigcirc(\varphi \wedge \psi) \equiv \bigcirc\varphi \wedge \bigcirc\psi$,
$\bigcirc(\varphi \,\mathsf{U}\, \psi) \equiv (\bigcirc\varphi) \,\mathsf{U}\, (\bigcirc\psi)$,
$\Diamond(\varphi \vee \psi) \equiv \Diamond\varphi \vee \Diamond\psi$,
$\Box(\varphi \wedge \psi) \equiv \Box\varphi \wedge \Box\psi$

Equivalence of LTL Formulae



$$TS \models \Diamond a \wedge \Diamond b,\ TS \not\models \Diamond(a \wedge b)$$

$$TS \models \Box(a \vee b),\ TS \not\models \Box a \vee \Box b$$

Satisfiability, Model Checking of LTL

Two Questions

Given transition system *TS*, and an LTL formula φ. Does $TS \models \varphi$?
Given an LTL formula φ, is $L(\varphi) = \emptyset$?

ω-automata

An ω-automaton is a tuple $\mathcal{A} = (Q, \Sigma, \delta, q_0, Acc)$ where

- Q is a finite set of states
- Σ is a finite alphabet
- $\delta : Q \times \Sigma \to 2^Q$ is a state transition function (if non-deterministic, otherwise, $\delta : Q \times \Sigma \to Q$)
- $q_0 \in Q$ is an initial state and Acc is an acceptance condition

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Run

A run ρ of $\mathcal{A}$ on an ω-word $\alpha = a_1 a_2 \cdots \in \Sigma^\omega$ is an infinite state sequence $\rho(0)\rho(1)\rho(2)\ldots$ such that

- $\rho(0) = q_0$,
- $\rho(i) = \delta(\rho(i-1), a_i)$ if $\mathcal{A}$ is deterministic,
- $\rho(i) \in \delta(\rho(i-1), a_i)$ if $\mathcal{A}$ is non-deterministic,

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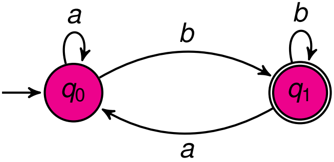
Büchi Acceptance

For Büchi Acceptance, Acc is specified as a set of states, $G \subseteq Q$. The ω-word α is accepted if there is a run ρ of α such that $Inf(\rho) \cap G \neq \emptyset$.

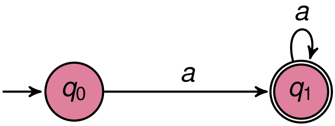
ω-Automata with Büchi Acceptance

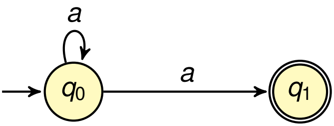


$L(\mathcal{A}) = \{\alpha \in \Sigma^\omega \mid \alpha$ has a run ρ such that $Inf(\rho) \cap G \neq \emptyset$

Language accepted=Infinitely many b's.

Comparing NFA and NBA





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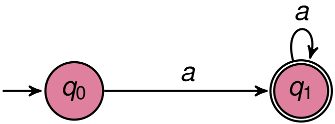
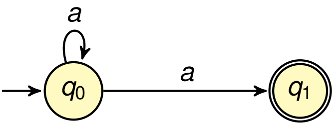
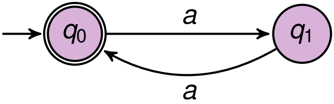
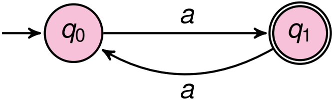
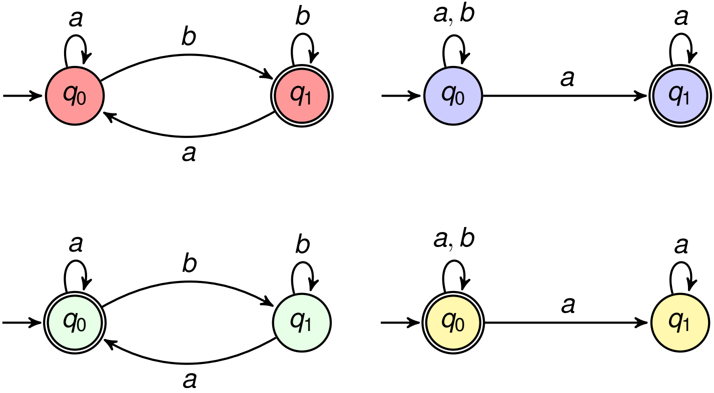








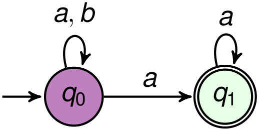
ω-Automata with Büchi Acceptance



- Left (T-B): Inf many *b*'s, Inf many *a*'s
- Right (T-B): Finitely many *b*'s, $(a + b)^\omega$

NBA and DBA

- Is every DBA as expressible as a NBA, like in the case of DFA and NFA?
- Can we do subset construction on NBA and obtain DBA?



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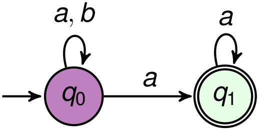
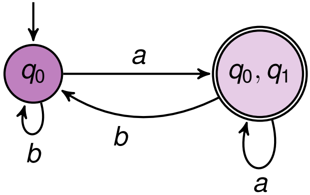
- Can we do subset construction on NBA and obtain DBA?





NBA and DBA

There does not exist a deterministic Büchi automata capturing the language finitely many *a*'s.

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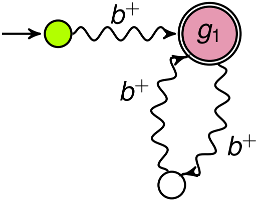

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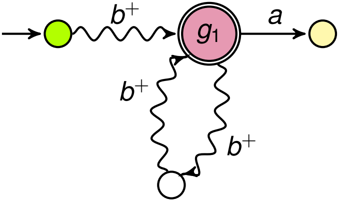

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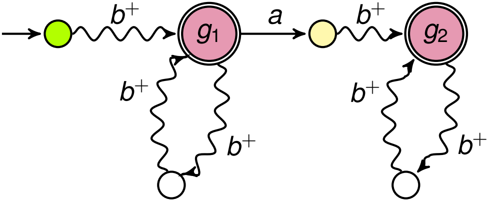

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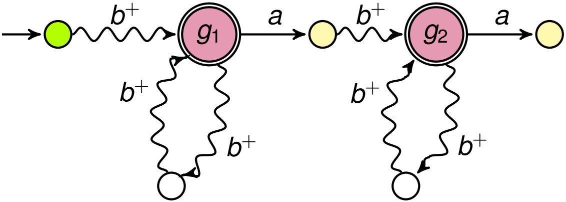

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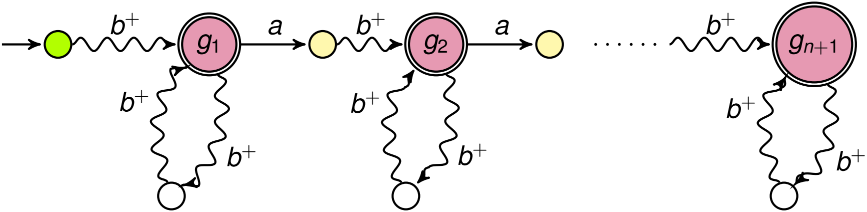

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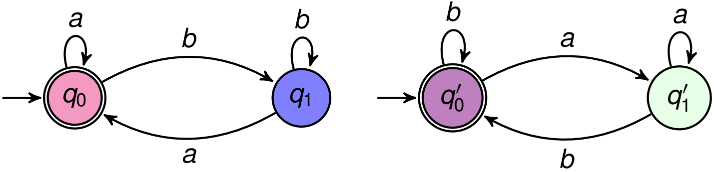
So Far

- ω-automata with Büchi acceptance, also called Büchi automata
- Non-determinism versus determinism

Büchi Acceptance

A language $L \subseteq \Sigma^\omega$ is called ω-regular if there exists a NBA $\mathcal{A}$ such that $L = L(\mathcal{A})$.

Union and Intersection of NBA



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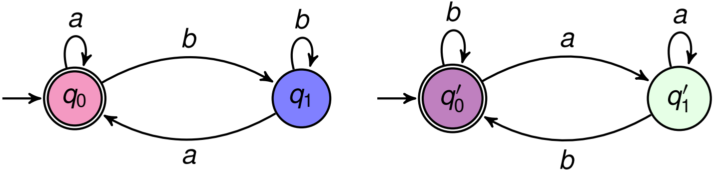
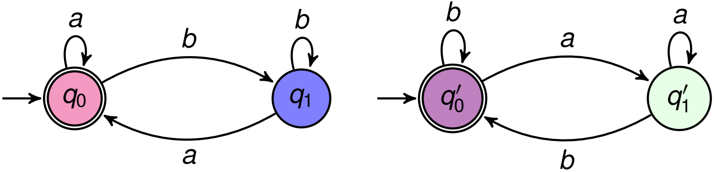


- States as $Q_1 \times Q_2 \times \{1, 2\}$, start state $(q_0, q_0', 1)$

Union and Intersection of NBA



- States as $Q_1 \times Q_2 \times \{1, 2\}$, start state $(q_0, q'_0, 1)$
- $(q_1, q_2, 1) \xrightarrow{a} (q'_1, q'_2, 1)$ if $q_1 \xrightarrow{a} q'_1$ and $q_2 \xrightarrow{a} q'_2$ and $q_1 \notin G_1$
- $(q_1, q_2, 1) \xrightarrow{a} (q'_1, q'_2, 2)$ if $q_1 \xrightarrow{a} q'_1$ and $q_2 \xrightarrow{a} q'_2$ and $q_1 \in G_1$

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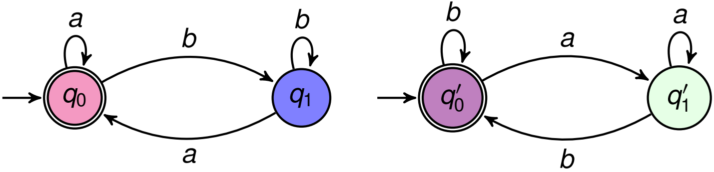

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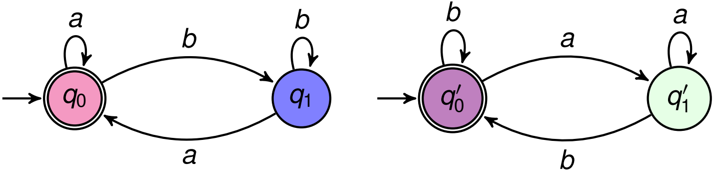

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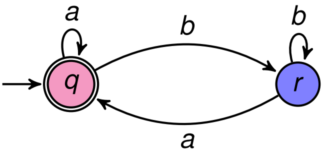
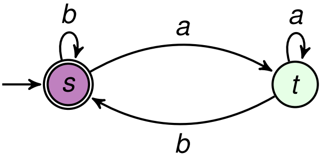
- Good states=$Q_1 \times G_2 \times \{2\}$ or $G_1 \times Q_2 \times \{1\}$

Union and Intersection of NBA





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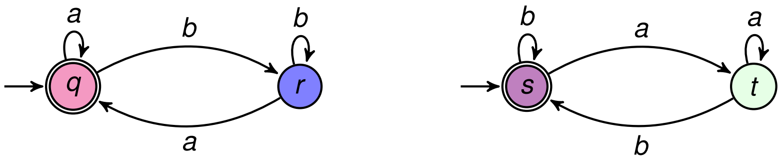

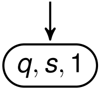

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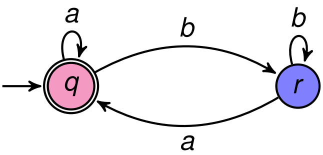

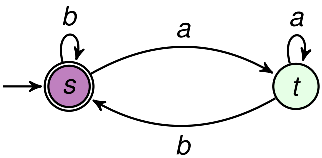

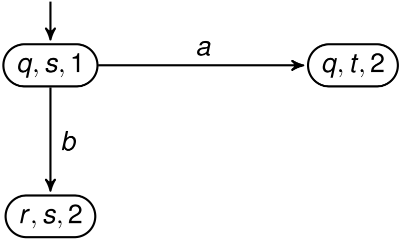

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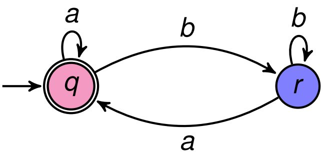

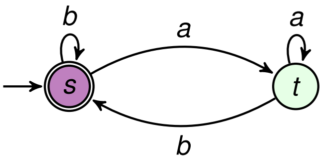

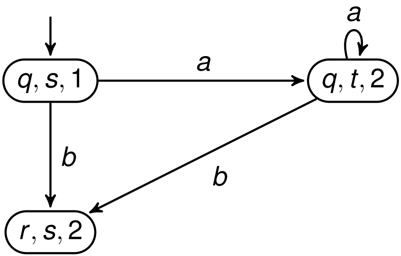

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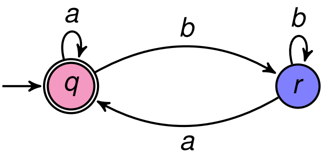

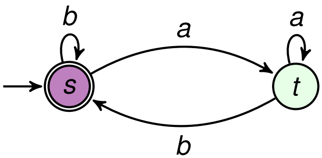

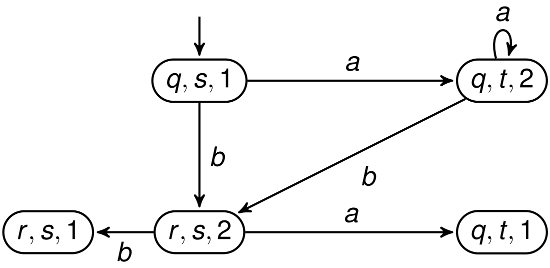

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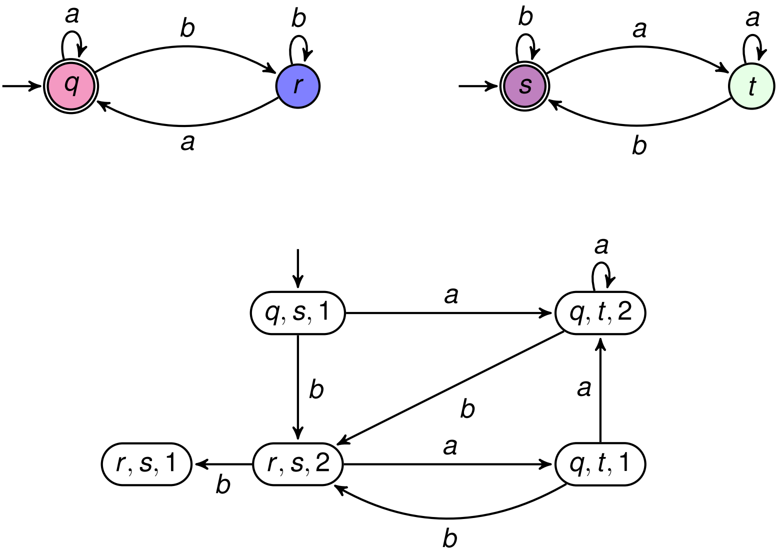

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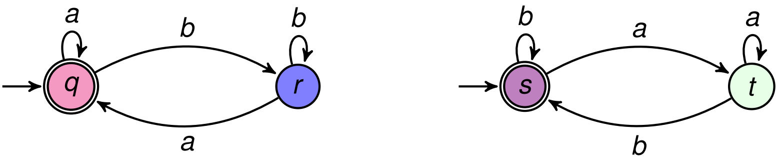

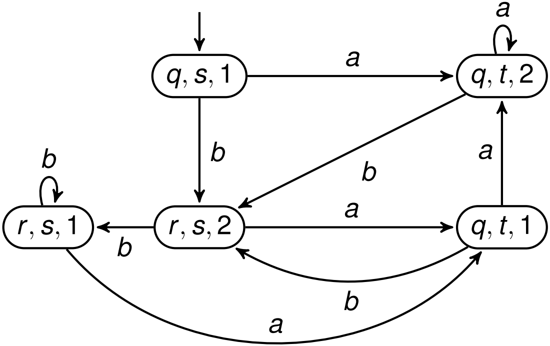

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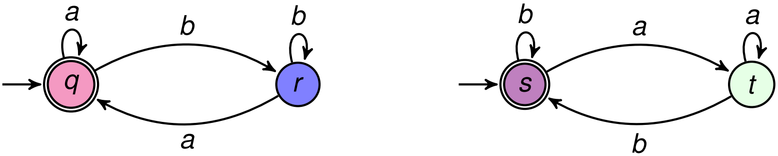
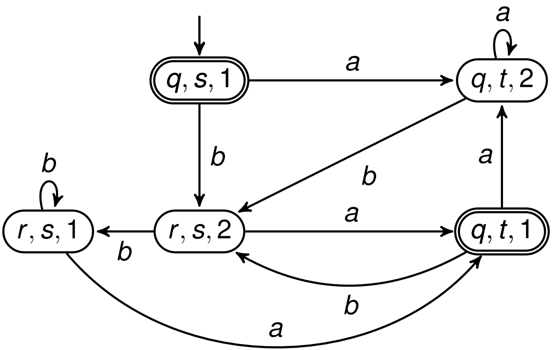


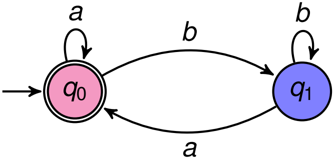


Emptiness

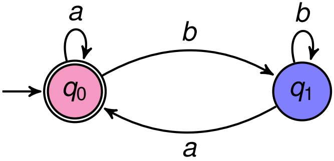
Given an NBA/DBA $\mathcal{A}$, how do you check if $L(\mathcal{A}) = \emptyset$?

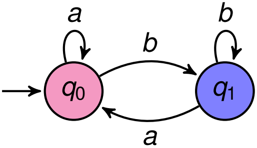
- Enumerate SCCs
- Check if there is an SCC containing a good state

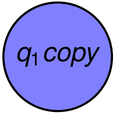
Complementation of DBA



Complementation of DBA







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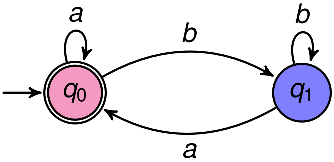

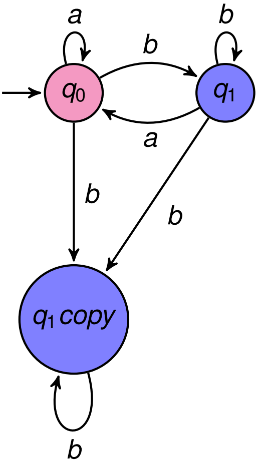

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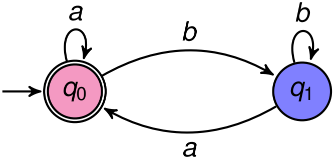

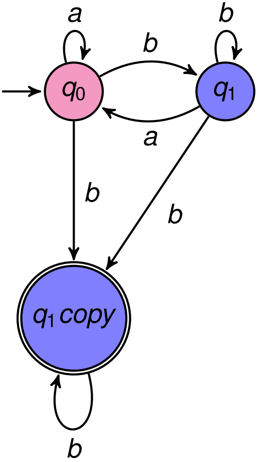

Complementation of DBA

- Given $\mathcal{A}$ is a DBA, and $w \notin L(\mathcal{A})$, then after some finite prefix, the unique run of w settles in bad states.
- Idea for complement: “copy” states of $Q - G$, once you enter this block, you stay there.
- View this as the set of good states, any word w that was rejected by $\mathcal{A}$ has two possible runs in this automaton: the original run, and one another, that will settle in the $Q - G$ copy, and will be accepted.
- What we get now is an NBA for $\overline{L(\mathcal{A})}$, not a DBA.

Complementing NBA non-trivial, can be done.

Normal Form for ω-regular languages

An ω-regular language $L \subseteq \Sigma^\omega$ can be written as $L = \bigcup_{i=1}^{n} U_i V_i^\omega$, where U_i, V_i are regular languages.

One direction : Assume L is accepted by an NBA/DBA.

- Define $U_g = \{w \in \Sigma^* \mid q_0 \xrightarrow{w} g\}$
- Define $V_g = \{w \in \Sigma^* \mid g \xrightarrow{w} g\}$
- Then $L = \bigcup_{g \in G} U_g V_g^\omega$, where U_g, V_g are regular
- Show that U_g, V_g are regular.

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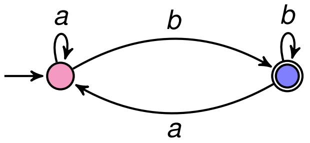

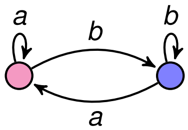

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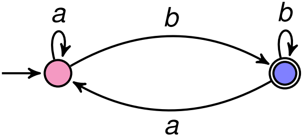

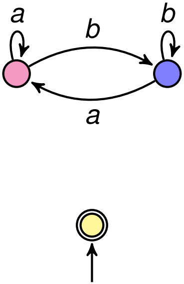

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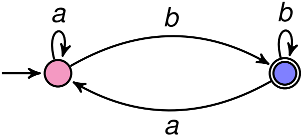

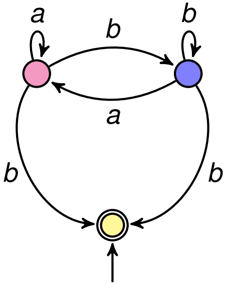

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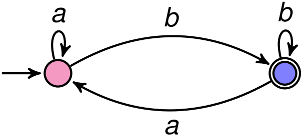

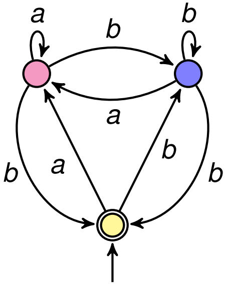

Normal Form for ω-regular languages

1. If V is regular, V^ω is ω-regular
   - Let $D = (Q, \Sigma, q_0, \delta, F)$ be a DFA accepting V
   - Construct NBA $E = (Q \cup \{p_0\}, \Sigma, p_0, \Delta, G)$ such that $G = \{p_0\}$,
   - $\Delta = \delta \cup \{p_0 \in \Delta(q, a) \mid \delta(q, a) \in F\} \cup \{\Delta(p_0, a) = s \mid \delta(q_0, a) = s\}$
2. Show that if U is regular and V^ω is ω-regular, then UV^ω is ω-regular
   - $D = (Q_1, \Sigma, q_0, \delta_1, F)$ be a DFA, $L(D) = U$ and $E = (Q_2, \Sigma, q_0', \delta_2, G)$ be an NBA, $L(E) = V^\omega$.
   - $A = (Q_1 \cup Q_2, \Sigma, q_0, \delta', G)$ NBA such that $\delta' = \delta_1 \cup \delta_2 \cup \{(q, a, q_0') \mid \delta_1(q, a) \in F\}$

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Union of NBA/DBA

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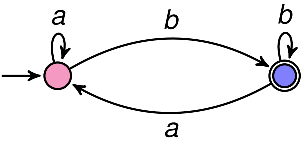

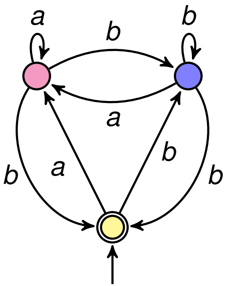

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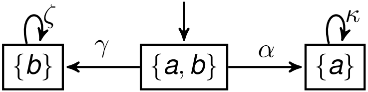
LTL ModelChecking

- Given transition system TS, and LTL formula φ, does $TS \models \varphi$?
- $Tr(TS) \subseteq L(\varphi)$ iff $Tr(TS) \cap \overline{L(\varphi)} = \emptyset$
- First construct NBA $A_{\neg\varphi}$ for $\neg\varphi$.
- Construct product of TS and $A_{\neg\varphi}$, obtaining a new TS, say TS'.
- Check some very simple property on TS', to check $TS \models \varphi$.

An Example $TS \models \varphi$

- Let $\varphi = \Box(a \vee b), \neg\varphi = \Diamond(\neg a \wedge \neg b)$
- Take TS and $A_{\neg\varphi}$, and check the intersection.



An Example $TS \models \varphi$

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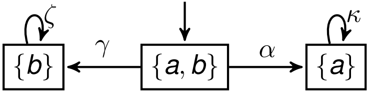

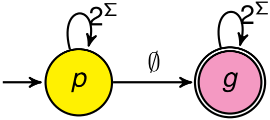

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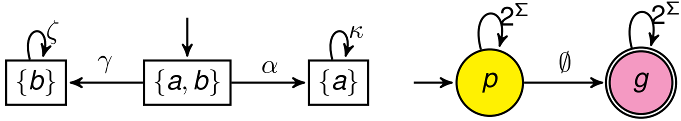

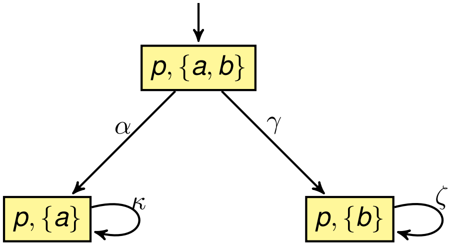

An Example : $TS \not\models \varphi$

- Let $\varphi = \Box(a \vee b), \neg\varphi = \Diamond(\neg a \wedge \neg b)$
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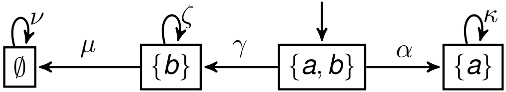

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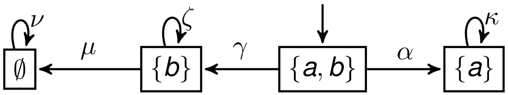

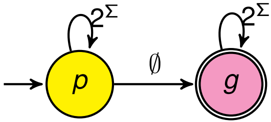

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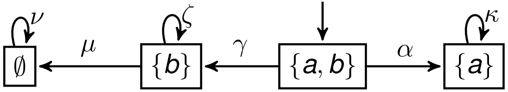
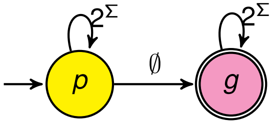
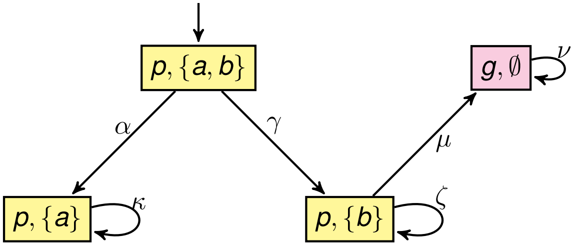






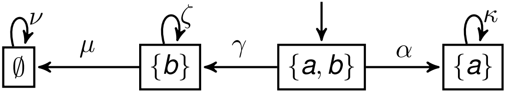
Product of TS and NBA

Given $TS = (S, Act, I, AP, L)$ and $\mathcal{A} = (Q, 2^{AP}, \delta, Q_0, G)$ NBA.
Define $TS \otimes \mathcal{A} = (S \times Q, Act, I', AP', L')$ such that

- $I' = \{(s_0, q) \mid s_0 \in I \text{ and } \exists q_0 \in Q_0, q_0 \stackrel{L(s_0)}{\rightarrow} q\}$
- $AP' = Q$, $L' : S \times Q \rightarrow 2^Q$ such that $L'((s, q)) = \{q\}$
- If $s \stackrel{\alpha}{\rightarrow} t$ and $q \stackrel{L(t)}{\rightarrow} p$, then $(s, q) \stackrel{\alpha}{\rightarrow} (t, p)$

Persistence Properties

Let η be a propositional logic formula over AP. A persistence property P_{pers} has the form $\Diamond\Box\eta$. How will you check a persistence property on a TS?



- For example, $TS \not\models \Diamond\Box(a \vee b)$
- For example, $TS \models \Diamond\Box(a \vee (a \rightarrow b))$

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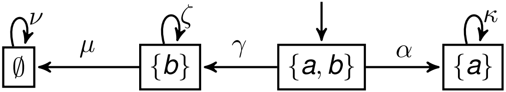

- For example, $TS \not\models \Diamond\Box(a \vee b)$
- For example, $TS \models \Diamond\Box(a \vee (a \rightarrow b))$
- $TS \not\models P_{pers}$ iff there is a reachable cycle in the TS containing a state with a label which satisfies $\neg\eta$.

LTL ModelChecking

- Given TS and LTL formula φ. Does $TS \models \varphi$?
- Construct $A_{\neg\varphi}$, and let $g_1, \ldots, g_n$ be the good states in $A_{\neg\varphi}$.
- Build $TS' = TS \otimes A_{\neg\varphi}$.
- The labels of TS' are the state names of $A_{\neg\varphi}$.
- Check if $TS' \models \Diamond\Box(\neg g_1 \wedge \ldots \neg g_n)$.

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ModelChecking LTL in TS = Check Persistence in TS'

The following are equivalent.

- $TS \models \varphi$
- $Tr(TS) \cap L(A_{\neg\varphi}) = \emptyset$
- $TS' \models \Diamond\Box(\neg g_1 \wedge \ldots \neg g_n)$.

CS 228 : Logic in Computer Science

Krishna. S

GNBA

- Generalized NBA, a variant of NBA
- Only difference is in acceptance condition
- Acceptance condition in GNBA is a set $\mathcal{F} = \{F_1, \ldots, F_k\}$, each $F_i \subseteq Q$
- An infinite run ρ is accepting in a GNBA iff

$$\forall F_i \in \mathcal{F}, \mathit{Inf}(\rho) \cap F_i \neq \emptyset$$

- Note that when $\mathcal{F} = \emptyset$, all infinite runs are accepting
- GNBA and NBA are equivalent in expressive power.

Word View (On the board)

- $w = \{a\}\{a, b\}\{\} \ldots,$
- $\varphi = a \, \mathsf{U}(\neg a \wedge b)$
- Subformulae of $\varphi = \{a, \neg a, b, \neg a \wedge b, \varphi\}$
- Parse trees to compute all subformulae

Closure of φ, $cl(\varphi)$

- $cl(\varphi)$=all subformulae of φ and their negations, identifying $\neg\neg\psi$ to be ψ.
- Example for $\varphi = a\,\mathrm{U}(\neg a \wedge b)$
- $cl(\varphi) = \{a, \neg a, b, \neg b, \neg a \wedge b, \neg(\neg a \wedge b), \varphi, \neg\varphi\}$

Elementary Sets

Let φ be an LTL formula. Then $B \subseteq cl(\varphi)$ is elementary provided:

- B is propositionally and maximally consistent : for all $\varphi_1 \wedge \varphi_2, \psi \in cl(\varphi)$,
  - $\varphi_1 \wedge \varphi_2 \in B \Leftrightarrow \varphi_1 \in B \wedge \varphi_2 \in B$
  - $\psi \in B \Leftrightarrow \neg\psi \notin B$
  - $true \in cl(\varphi) \Rightarrow true \in B$
- B is locally consistent wrt U. That is, for all $\varphi_1 \mathsf{U} \varphi_2 \in cl(\varphi)$,
  - $\varphi_2 \in B \Rightarrow \varphi_1 \mathsf{U} \varphi_2 \in B$
  - $\varphi_1 \mathsf{U} \varphi_2 \in B, \varphi_2 \notin B \Rightarrow \varphi_1 \in B$
- B is elementary : B is propositionally, maximally and locally consistent
- Given a $B \subseteq cl(\varphi)$, how can you check if B is elementary?

Check Elementary

Let $\varphi = a \,\mathsf{U}(\neg a \wedge b)$

- $B_1 = \{a, b, \neg a \wedge b, \varphi\}$

Check Elementary

Let $\varphi = a \, \mathrm{U} (\neg a \wedge b)$

- $B_1 = \{a, b, \neg a \wedge b, \varphi\}$ No, propositionally inconsistent
- $B_2 = \{\neg a, b, \varphi\}$

Check Elementary

Let $\varphi = a \, \mathrm{U} (\neg a \wedge b)$

- $B_1 = \{a, b, \neg a \wedge b, \varphi\}$ No, propositionally inconsistent
- $B_2 = \{\neg a, b, \varphi\}$ No, not maximal as $\neg a \wedge b \notin B_2$, $\neg(\neg a \wedge b) \notin B_2$
- $B_3 = \{\neg a, b, \neg a \wedge b, \neg\varphi\}$

Check Elementary

Let $\varphi = a\,\mathrm{U}(\neg a \wedge b)$

- $B_1 = \{a, b, \neg a \wedge b, \varphi\}$ No, propositionally inconsistent
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- $B_3 = \{\neg a, b, \neg a \wedge b, \neg\varphi\}$ No, not locally consistent for U
- $B_4 = \{\neg a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$

Check Elementary

Let $\varphi = a\,\mathsf{U}(\neg a \wedge b)$

- $B_1 = \{a, b, \neg a \wedge b, \varphi\}$ No, propositionally inconsistent
- $B_2 = \{\neg a, b, \varphi\}$ No, not maximal as $\neg a \wedge b \notin B_2$, $\neg(\neg a \wedge b) \notin B_2$
- $B_3 = \{\neg a, b, \neg a \wedge b, \neg\varphi\}$ No, not locally consistent for U
- $B_4 = \{\neg a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$ Yes, elementary

LTL φ to GNBA G_φ

- States of G_φ are elementary sets B_i
- For a word $w = A_0 A_1 A_2 \ldots$ the sequence of states $\sigma = B_0 B_1 B_2 \ldots$ will be a run for w
- σ will be accepting iff $w \models \varphi$ iff $\varphi \in B_0$
- In general, a run $B_i B_{i+1} \ldots$ for $A_i A_{i+1} \ldots$ is accepting iff $A_i A_{i+1} \ldots \models \psi$ for all $\psi \in B_i$.

LTL to GNBA

- Let $\varphi = \bigcirc a$.
- Subformulae of φ : $\{a, \bigcirc a\}$. Let $A = \{a, \bigcirc a, \neg a, \neg \bigcirc a\}$.
- Possibilities at each state
  - $\{a, \bigcirc a\}$
  - $\{\neg a, \bigcirc a\}$
  - $\{a, \neg \bigcirc a\}$
  - $\{\neg a, \neg \bigcirc a\}$
- Our initial state(s) must guarantee truth of $\bigcirc a$. Thus, initial states: $\{a, \bigcirc a\}$ and $\{\neg a, \bigcirc a\}$

LTL to GNBA

$\{a, \bigcirc a\}$

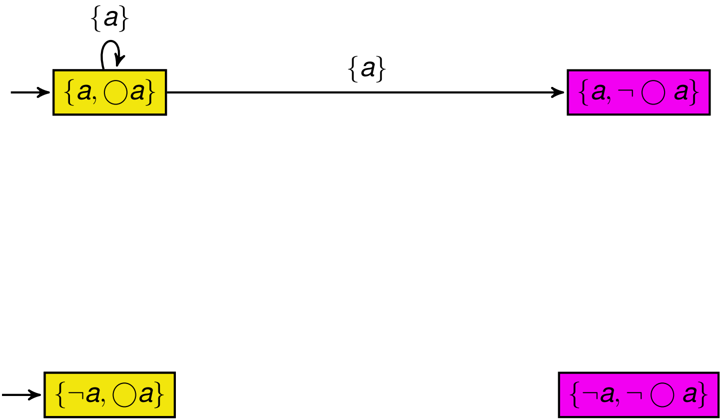
$\{a, \neg \bigcirc a\}$

$\{\neg a, \bigcirc a\}$

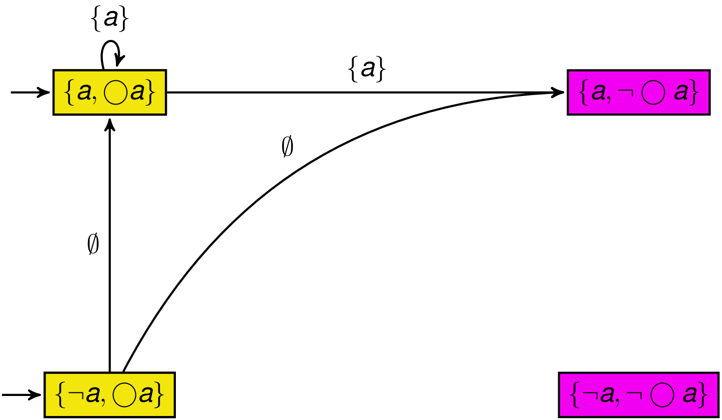
$\{\neg a, \neg \bigcirc a\}$

LTL to GNBA

LTL to GNBA

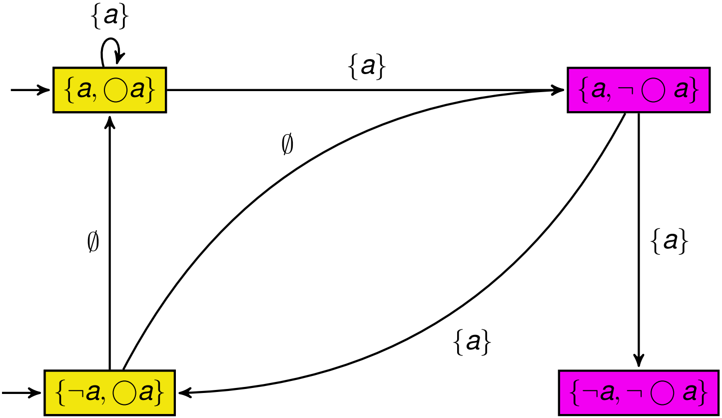

$\{a\}$
$\{a\}$
$\{a, \bigcirc a\}$
$\{a, \neg \bigcirc a\}$
$\{\neg a, \bigcirc a\}$
$\{\neg a, \neg \bigcirc a\}$

LTL to GNBA



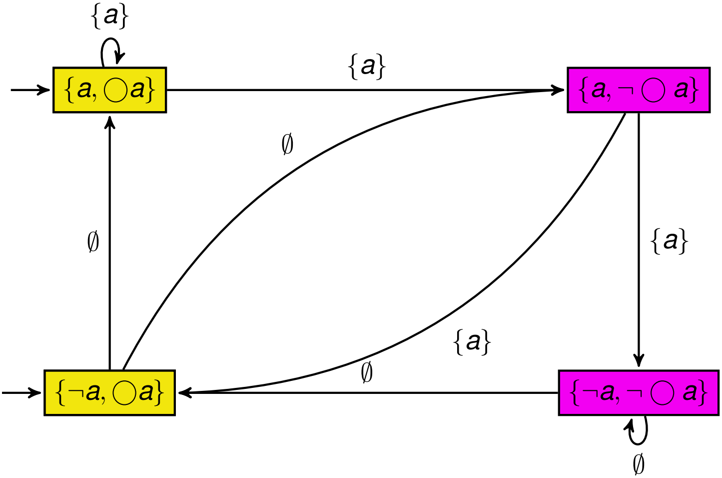
$\{a\}$
$\{a\}$
$\{a, \bigcirc a\}$
$\{a, \neg \bigcirc a\}$
$\emptyset$
$\emptyset$
$\{\neg a, \bigcirc a\}$
$\{\neg a, \neg \bigcirc a\}$

LTL to GNBA



$\{a\}$
$\{a, \bigcirc a\}$
$\{a\}$
$\{a, \neg \bigcirc a\}$
$\emptyset$
$\emptyset$
$\{a\}$
$\{a\}$
$\{\neg a, \bigcirc a\}$
$\{\neg a, \neg \bigcirc a\}$

LTL to GNBA



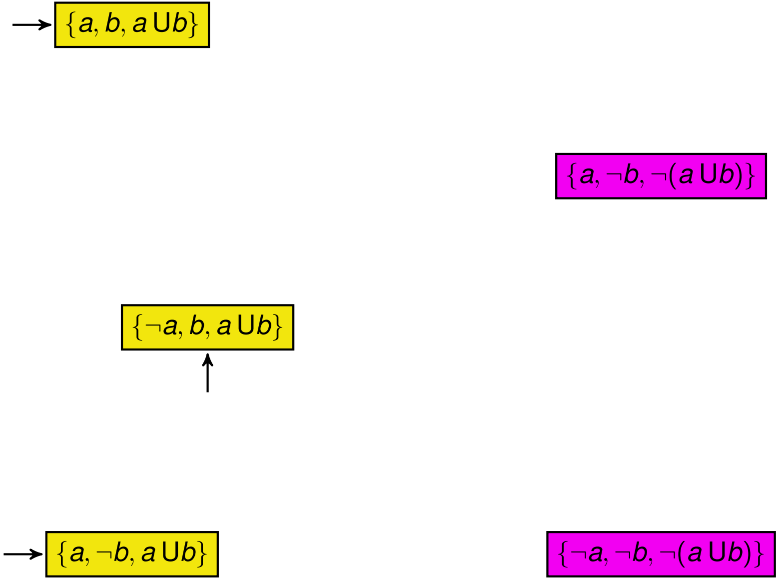
LTL to GNBA

- Claim : Runs from a state labelled set B indeed satisfy B
- No good states. All words having a run from a start state are accepted.
- Automaton for $\neg \bigcirc a$ same, except for the start states.

LTL to GNBA

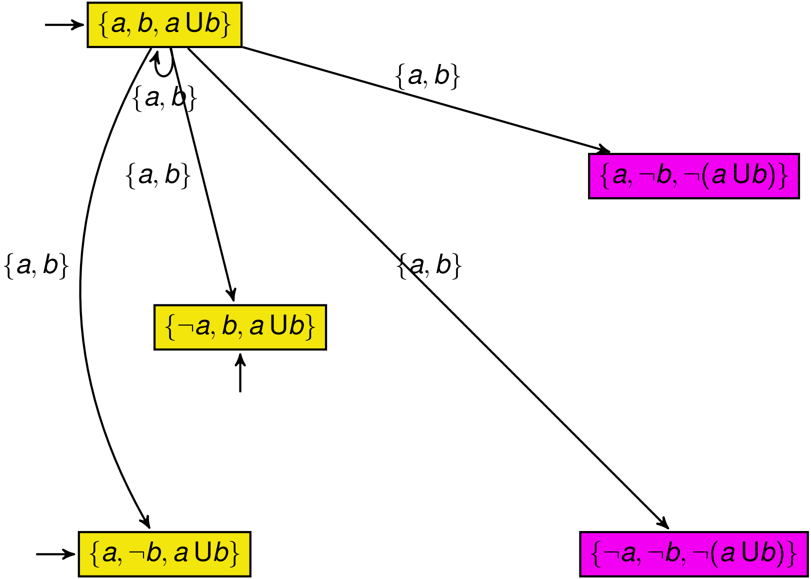
- Let $\varphi = a \,\mathsf{U} b$.
- Subformulae of φ : $\{a, b, a \,\mathsf{U} b\}$. Let $B = \{a, \neg a, b, \neg b, a \,\mathsf{U} b, \neg(a \,\mathsf{U} b)\}$.
- Possibilities at each state
  - $\{a, \neg b, a \,\mathsf{U} b\}$
  - $\{\neg a, b, a \,\mathsf{U} b\}$
  - $\{a, b, a \,\mathsf{U} b\}$
  - $\{a, \neg b, \neg(a \,\mathsf{U} b)\}$
  - $\{\neg a, \neg b, \neg(a \,\mathsf{U} b)\}$
- Our initial state(s) must guarantee truth of $a \,\mathsf{U} b$. Thus, initial states: $\{a, b, a \,\mathsf{U} b\}$ and $\{\neg a, b, a \,\mathsf{U} b\}$ and $\{a, \neg b, a \,\mathsf{U} b\}$.

LTL to GNBA

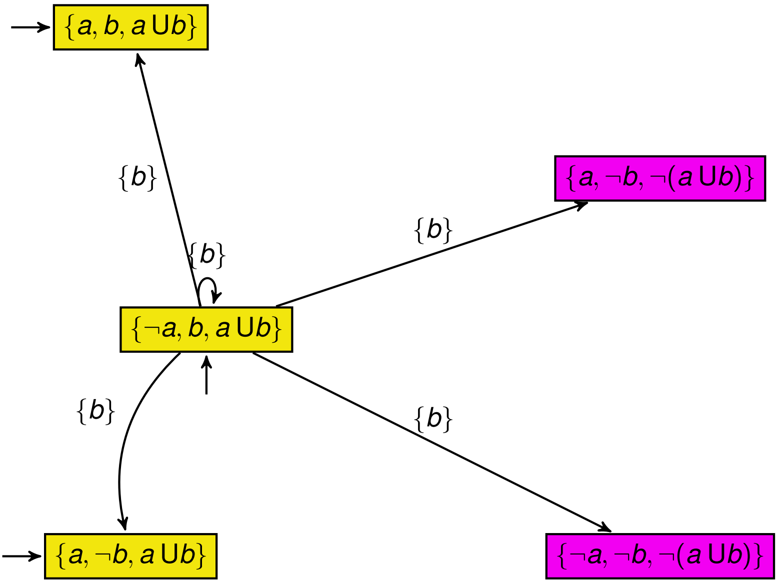


$\{a, b, a \,\mathsf{U}\, b\}$
$\{a, \neg b, \neg(a \,\mathsf{U}\, b)\}$
$\{\neg a, b, a \,\mathsf{U}\, b\}$
$\{a, \neg b, a \,\mathsf{U}\, b\}$
$\{\neg a, \neg b, \neg(a \,\mathsf{U}\, b)\}$

LTL to GNBA

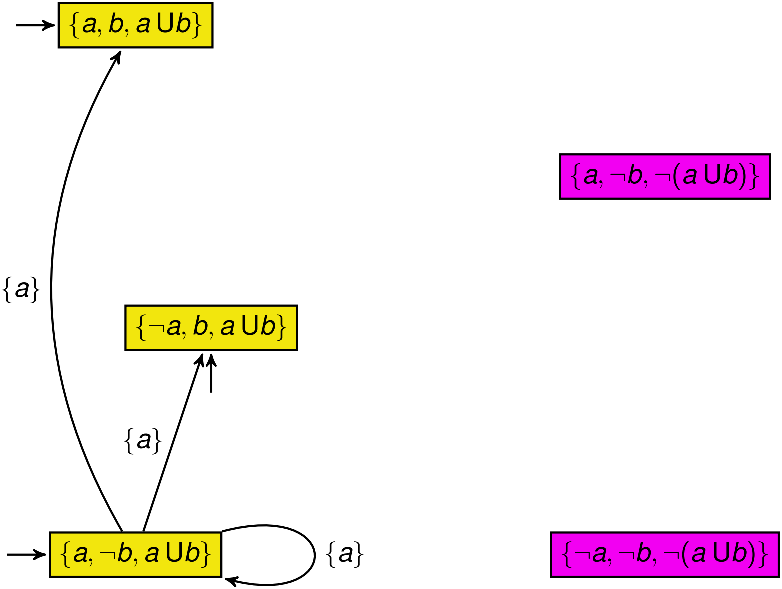


LTL to GNBA



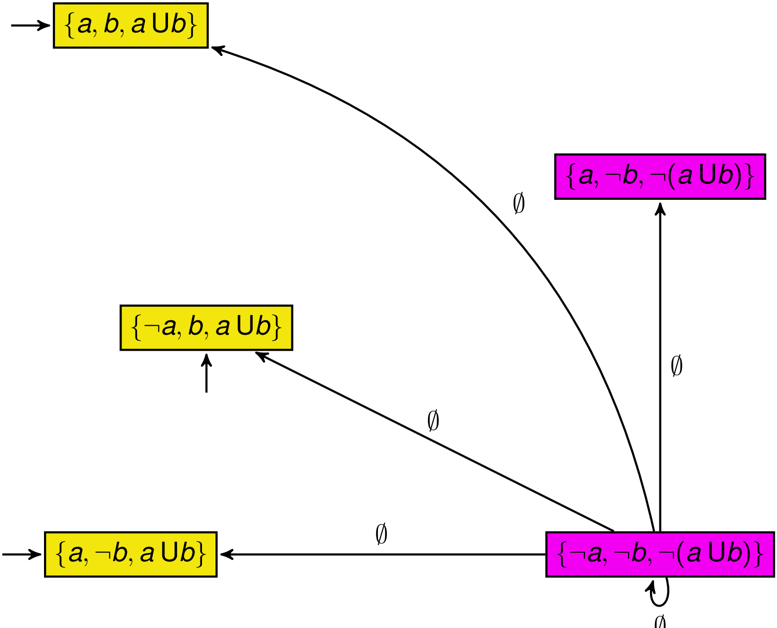
$\{a, b, a\,\mathsf{U}\,b\}$
$\{b\}$
$\{a, \neg b, \neg(a\,\mathsf{U}\,b)\}$
$\{b\}$
$\{b\}$
$\{\neg a, b, a\,\mathsf{U}\,b\}$
$\{b\}$
$\{b\}$
$\{a, \neg b, a\,\mathsf{U}\,b\}$
$\{\neg a, \neg b, \neg(a\,\mathsf{U}\,b)\}$

LTL to GNBA



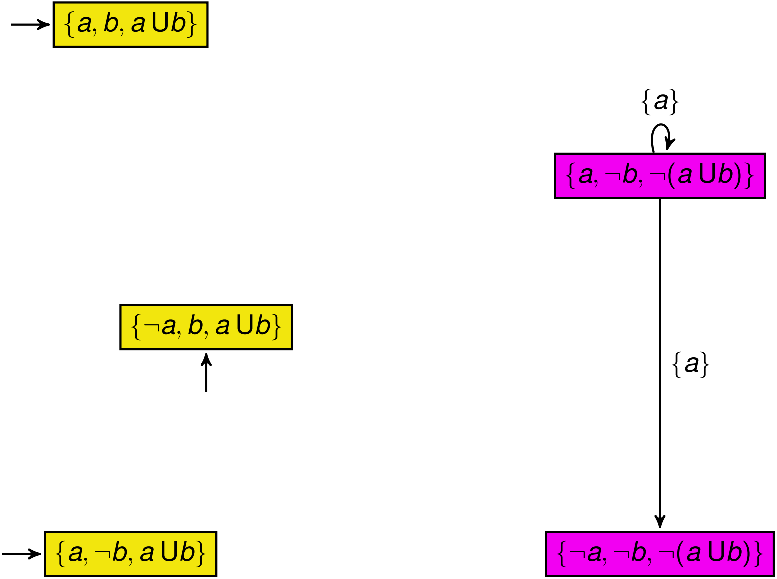
$\{a, b, a\,\mathsf{U}\,b\}$
$\{a, \neg b, \neg(a\,\mathsf{U}\,b)\}$
$\{a\}$
$\{\neg a, b, a\,\mathsf{U}\,b\}$
$\{a\}$
$\{a, \neg b, a\,\mathsf{U}\,b\}$
$\{a\}$
$\{\neg a, \neg b, \neg(a\,\mathsf{U}\,b)\}$

LTL to GNBA



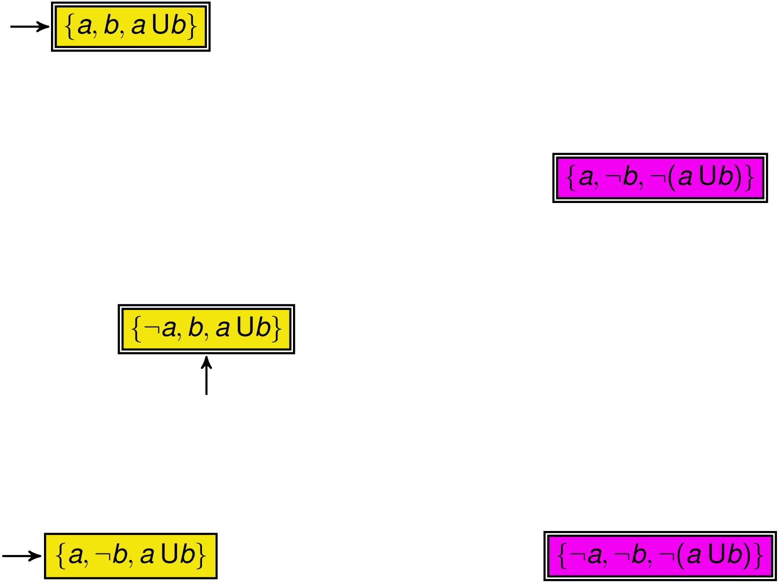
$\{a, b, a\,\mathsf{U}\,b\}$
$\{a, \neg b, \neg(a\,\mathsf{U}\,b)\}$
$\emptyset$
$\emptyset$
$\{\neg a, b, a\,\mathsf{U}\,b\}$
$\emptyset$
$\emptyset$
$\{a, \neg b, a\,\mathsf{U}\,b\}$
$\{\neg a, \neg b, \neg(a\,\mathsf{U}\,b)\}$
$\emptyset$

LTL to GNBA



$\{a, b, a \mathsf{U} b\}$
$\{a\}$
$\{a, \neg b, \neg(a \mathsf{U} b)\}$
$\{\neg a, b, a \mathsf{U} b\}$
$\{a\}$
$\{a, \neg b, a \mathsf{U} b\}$
$\{\neg a, \neg b, \neg(a \mathsf{U} b)\}$

LTL to GNBA : Accepting States



LTL to GNBA

Construct GNBA for $\neg(a \, \mathrm{U} b)$.

LTL to GNBA

- Let $\varphi = a \,\mathsf{U} (\neg a \,\mathsf{U} c)$. Let $\psi = \neg a \,\mathsf{U} c$
- Subformulae of φ : $\{a, \neg a, c, \psi, \varphi\}$. Let $B = \{a, \neg a, c, \neg c, \psi, \neg\psi, \varphi, \neg\varphi\}$.
- Possibilities at each state
  - $\{a, c, \psi, \varphi\}$
  - $\{\neg a, c, \psi, \varphi\}$
  - $\{a, \neg c, \neg\psi, \varphi\}$
  - $\{a, \neg c, \neg\psi, \neg\varphi\}$
  - $\{\neg a, \neg c, \psi, \varphi\}$
  - $\{\neg a, \neg c, \neg\psi, \neg\varphi\}$

LTL to GNBA

$\rightarrow \{a, c, \psi, \varphi\}$

$\{\neg a, \neg c, \psi, \varphi\} \leftarrow$

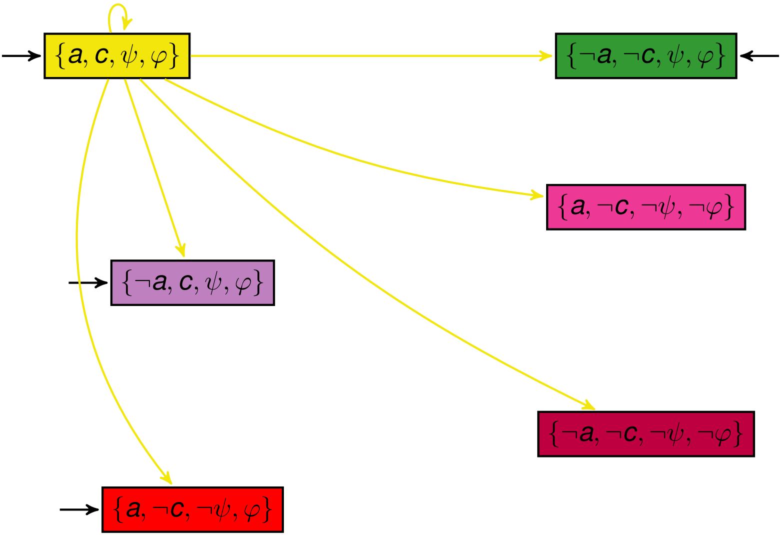
$\{a, \neg c, \neg\psi, \neg\varphi\}$

$\rightarrow \{\neg a, c, \psi, \varphi\}$

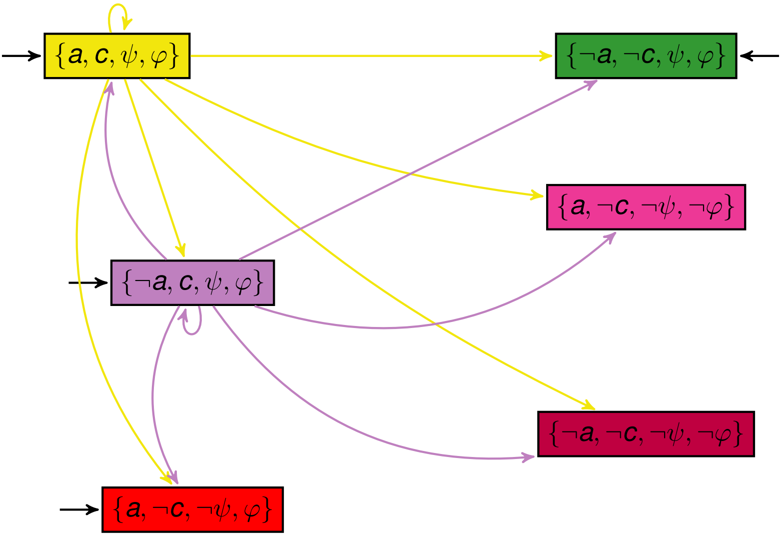
$\{\neg a, \neg c, \neg\psi, \neg\varphi\}$

$\rightarrow \{a, \neg c, \neg\psi, \varphi\}$

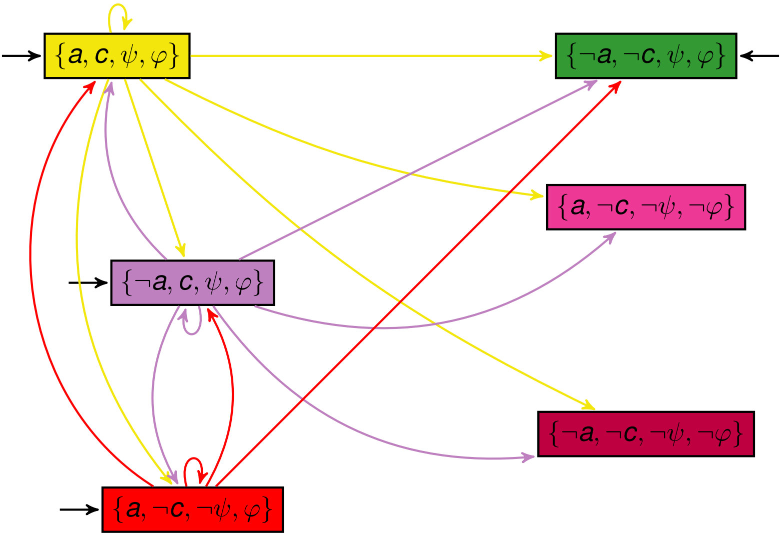
LTL to GNBA


$\{a, c, \psi, \varphi\}$
$\{\neg a, \neg c, \psi, \varphi\}$
$\{a, \neg c, \neg\psi, \neg\varphi\}$
$\{\neg a, c, \psi, \varphi\}$
$\{\neg a, \neg c, \neg\psi, \neg\varphi\}$
$\{a, \neg c, \neg\psi, \varphi\}$

LTL to GNBA


$\{a, c, \psi, \varphi\}$
$\{\neg a, \neg c, \psi, \varphi\}$
$\{a, \neg c, \neg\psi, \neg\varphi\}$
$\{\neg a, c, \psi, \varphi\}$
$\{\neg a, \neg c, \neg\psi, \neg\varphi\}$
$\{a, \neg c, \neg\psi, \varphi\}$

LTL to GNBA



$\{a, c, \psi, \varphi\}$
$\{\neg a, \neg c, \psi, \varphi\}$
$\{a, \neg c, \neg\psi, \neg\varphi\}$
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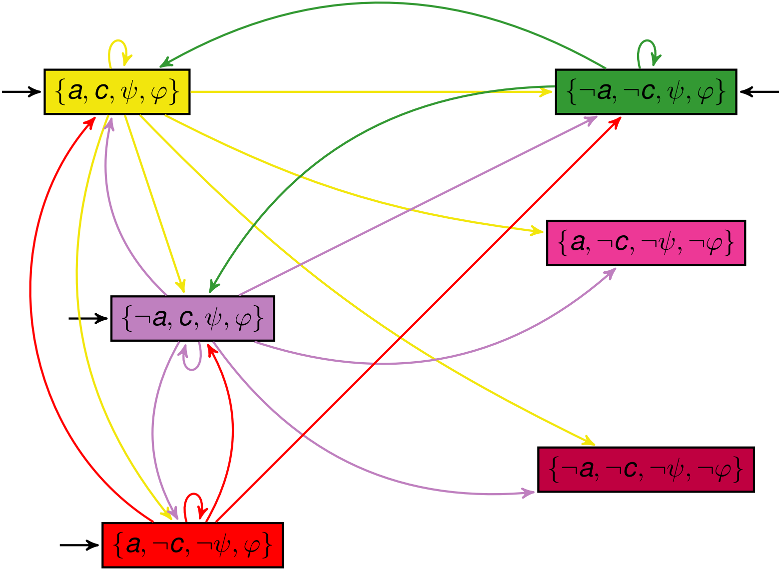
LTL to GNBA



$\{a, c, \psi, \varphi\}$
$\{\neg a, \neg c, \psi, \varphi\}$
$\{a, \neg c, \neg\psi, \neg\varphi\}$
$\{\neg a, c, \psi, \varphi\}$
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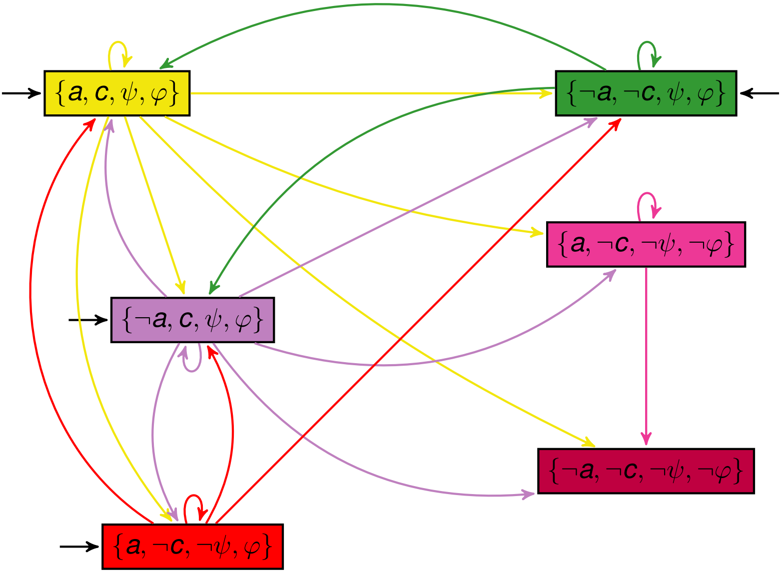
LTL to GNBA


$\{a, c, \psi, \varphi\}$
$\{\neg a, \neg c, \psi, \varphi\}$
$\{a, \neg c, \neg\psi, \neg\varphi\}$
$\{\neg a, c, \psi, \varphi\}$
$\{\neg a, \neg c, \neg\psi, \neg\varphi\}$
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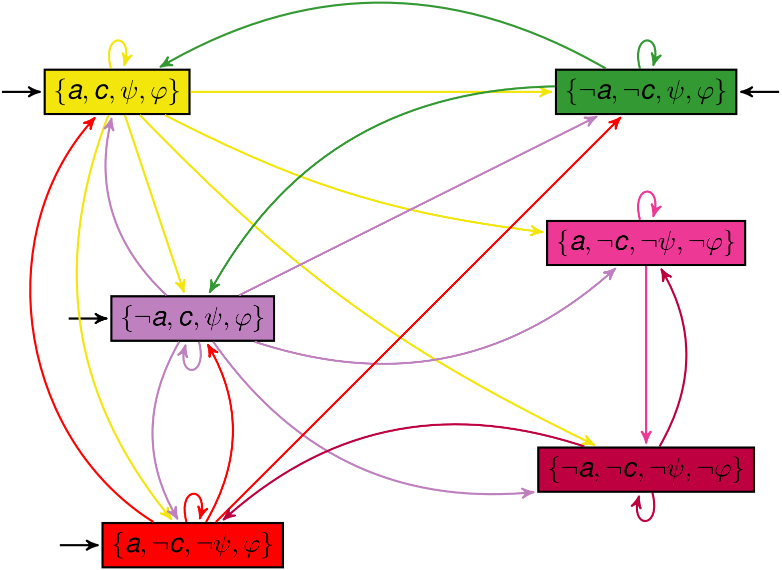
LTL to GNBA



$\{a, c, \psi, \varphi\}$
$\{\neg a, \neg c, \psi, \varphi\}$
$\{a, \neg c, \neg\psi, \neg\varphi\}$
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GNBA Acceptance Condition

- $\psi = \neg a \, \mathsf{U} c$
- $\varphi = a \, \mathsf{U} \psi$
- $F_1 = \{B \mid \psi \in B \rightarrow c \in B\}$
- $F_2 = \{B \mid \varphi \in B \rightarrow \psi \in B\}$
- $\mathcal{F} = \{F_1, F_2\}$

Final States

$\rightarrow$ $\{a, c, \psi, \varphi\} \in F_1, F_2$

$\{\neg a, \neg c, \psi, \varphi\} \in F_2$ $\leftarrow$

$\{a, \neg c, \neg\psi, \neg\varphi\} \in F_1, F_2$

$\rightarrow$ $\{\neg a, c, \psi, \varphi\} \in F_1, F_2$

$\{\neg a, \neg c, \neg\psi, \neg\varphi\} \in F_1, F_2$

$\rightarrow$ $\{a, \neg c, \neg\psi, \varphi\} \in F_1$

Putting Together

- Given φ, build $Cl(\varphi)$, the set of all subformulae of φ and their negations

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  - $\psi \in B \rightarrow \neg\psi \notin B$ and $\psi \notin B \rightarrow \neg\psi \in B$
  - Whenever $\psi_1 \, \mathsf{U} \psi_2 \in Cl(\varphi)$,
    - $\psi_2 \in B \rightarrow \psi_1 \, \mathsf{U} \psi_2 \in B$
    - $\psi_1 \, \mathsf{U} \psi_2 \in B$ and $\psi_2 \notin B \rightarrow \psi_1 \in B$

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Given φ over AP, construct $A_\varphi = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$,

- $Q = \{B \mid B \subseteq Cl(\varphi) \text{ is consistent }\}$
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  - For $C = B \cap AP$, $\delta(B, C)$ is enabled and is defined as :
  - If $\bigcirc\psi \in Cl(\varphi)$, $\bigcirc\psi \in B$ iff $\psi \in \delta(B, C)$
  - If $\varphi_1 \,\mathsf{U} \varphi_2 \in Cl(\varphi)$,
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  $F_{\varphi_1 \mathsf{U} \varphi_2} = \{B \in Q \mid \varphi_1 \mathsf{U} \varphi_2 \in B \to \varphi_2 \in B\}$
- Prove that $L(\varphi) = L(A_\varphi)$

$L(\varphi) \subseteq L(A_\varphi)$

Let $\sigma = A_0 A_1 A_2 \cdots \in L(\varphi)$. Show that there is an accepting run $B_0 A_0 B_1 A_1 B_2 A_2 \ldots$ in A_φ for σ, B_i are the states, such that $B_i = \{\psi \mid A_i A_{i+1} \ldots \models \psi\}$.

Structural induction on φ

- $\varphi = a$. All starting states contain a, and can go to all successor states with all combinations of propositions.
- If $a \in B_i$, every run starting at B_i starts with a. Hence, $A_i A_{i+1} \ldots \models a$

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- $\varphi = \bigcirc a$, then all initial states contain $\bigcirc a$, and all successor states contain a by construction.
- If $\bigcirc a \in B_i$, then by construction, $B_{i+1} \in \delta(B_i, B_i \cap AP)$ iff $a \in B_{i+1}$, for every successor B_{i+1}.
  Then $A_{i+1} \ldots \models a$, and hence $A_i A_{i+1} \ldots \models \bigcirc a$.

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- If $\varphi_1 \mathrm{U} \varphi_2 \in B_i$, then by construction, either $\varphi_2 \in B_i$ or $\varphi_1, \varphi_1 \mathrm{U} \varphi_2 \in B_i$. If $\varphi_2 \in B_i$ then $A_i A_{i+1} \ldots \models \varphi_2$ by induction hypothesis, and hence, $A_i A_{i+1} \ldots \models \varphi_1 \mathrm{U} \varphi_2$.

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- If $\varphi_1, \varphi_1 \mathrm{U} \varphi_2 \in B_i$, then by construction, $B_{i+1} \in \delta(B_i, B_i \cap AP)$ iff $\varphi_2 \in B_{i+1}$ or $\varphi_1, \varphi_1 \mathrm{U} \varphi_2 \in B_{i+1}$. How long can we go like this?

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- If $\varphi_1, \varphi_1 \, \mathsf{U} \varphi_2 \in B_i$, then by construction, $B_{i+1} \in \delta(B_i, B_i \cap AP)$ iff $\varphi_2 \in B_{i+1}$ or $\varphi_1, \varphi_1 \, \mathsf{U} \varphi_2 \in B_{i+1}$. How long can we go like this?
- If $\varphi_2 \in B_k$ for $k > i$, and $\varphi_1, \varphi_1 \, \mathsf{U} \varphi_2 \in B_{i+1}, \ldots, B_{k-1}$ then $A_i A_{i+1} \ldots \models \varphi_1 \, \mathsf{U} \varphi_2$.
- When is $B_i B_{i+1} B_{i+2} \ldots$ an accepting run?
- $B_j \in F_{\varphi_1 \, \mathsf{U} \varphi_2}$ for infinitely many $j \geqslant i$.

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Let $\sigma = A_0A_1A_2\cdots \in L(\varphi)$. Show that there is an accepting run $B_0A_0B_1A_1B_2A_2\ldots$ in A_φ for σ, B_i are the states, such that $B_i = \{\psi \mid A_iA_{i+1}\ldots \models \psi\}$.

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- If $\varphi_1, \varphi_1 \mathsf{U} \varphi_2 \in B_i$, then by construction, $B_{i+1} \in \delta(B_i, B_i \cap AP)$ iff $\varphi_2 \in B_{i+1}$ or $\varphi_1, \varphi_1 \mathsf{U} \varphi_2 \in B_{i+1}$. How long can we go like this?
- If $\varphi_2 \in B_k$ for $k > i$, and $\varphi_1, \varphi_1 \mathsf{U} \varphi_2 \in B_{i+1}, \ldots, B_{k-1}$ then $A_iA_{i+1}\ldots \models \varphi_1 \mathsf{U} \varphi_2$.
- When is $B_iB_{i+1}B_{i+2}\ldots$ an accepting run?
- $B_j \in F_{\varphi_1 \mathsf{U} \varphi_2}$ for infinitely many $j \geqslant i$.
- $\varphi_2 \notin B_j$ or $\varphi_2, \varphi_1 \mathsf{U} \varphi_2 \in B_j$ for infinitely many $j \geqslant i$.
- By construction, there is an accepting run where $\varphi_2 \in B_k$ for some $k \geqslant i$. Hence, $A_iA_{i+1}\ldots \models \varphi_1 \mathsf{U} \varphi_2$.

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Let $\sigma = A_0 A_1 A_2 \cdots \in L(\varphi)$. Show that there is an accepting run $B_0 A_0 B_1 A_1 B_2 A_2 \ldots$ in A_φ for σ, B_i are the states, such that $B_i = \{\psi \mid A_i A_{i+1} \ldots \models \psi\}$.

- If $\neg(\varphi_1 \, \mathsf{U} \varphi_2) \in B_i$, then either $\neg\varphi_1, \neg\varphi_2 \in B_i$ or $\varphi_1, \neg\varphi_2 \in B_i$. If $\neg\varphi_1, \neg\varphi_2 \in B_i$ then $A_i A_{i+1} \ldots \models \neg(\varphi_1 \, \mathsf{U} \varphi_2)$.

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- If $\neg(\varphi_1 \mathsf{U} \varphi_2) \in B_i$, then either $\neg\varphi_1, \neg\varphi_2 \in B_i$ or $\varphi_1, \neg\varphi_2 \in B_i$. If $\neg\varphi_1, \neg\varphi_2 \in B_i$ then $A_i A_{i+1} \ldots \models \neg(\varphi_1 \mathsf{U} \varphi_2)$.
- If $\varphi_1, \neg\varphi_2 \in B_i$, then by construction, $B_{i+1} \in \delta(B_i, B_i \cap AP)$ iff $\varphi_1, \neg\varphi_2 \in B_{i+1}$ or $\neg\varphi_1, \neg\varphi_2 \in B_{i+1}$.

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Let $\sigma = A_0 A_1 A_2 \cdots \in L(\varphi)$. Show that there is an accepting run $B_0 A_0 B_1 A_1 B_2 A_2 \ldots$ in A_φ for σ, B_i are the states, such that $B_i = \{\psi \mid A_i A_{i+1} \ldots \models \psi\}$.

- If $\neg(\varphi_1 \mathsf{U} \varphi_2) \in B_i$, then either $\neg\varphi_1, \neg\varphi_2 \in B_i$ or $\varphi_1, \neg\varphi_2 \in B_i$. If $\neg\varphi_1, \neg\varphi_2 \in B_i$ then $A_i A_{i+1} \ldots \models \neg(\varphi_1 \mathsf{U} \varphi_2)$.
- If $\varphi_1, \neg\varphi_2 \in B_i$, then by construction, $B_{i+1} \in \delta(B_i, B_i \cap AP)$ iff $\varphi_1, \neg\varphi_2 \in B_{i+1}$ or $\neg\varphi_1, \neg\varphi_2 \in B_{i+1}$.
- Either case, $A_i A_{i+1} \ldots \models \neg(\varphi_1 \mathsf{U} \varphi_2)$

$L(A_\varphi) \subseteq L(\varphi)$

For a sequence $B_0 B_1 B_2 \ldots$ of states satisfying

- $B_{i+1} \in \delta(B_i, A_i)$,
- $\forall F \in \mathcal{F}$, $B_j \in F$ for infinitely many j,

we have $\psi \in B_0 \leftrightarrow A_0 A_1 \ldots \models \psi$

- Structural Induction on ψ. Interesting case : $\psi = \varphi_1 \,\mathsf{U} \varphi_2$
- Assume $A_0 A_1 \ldots \models \varphi_1 \,\mathsf{U} \varphi_2$. Then $\exists j \geqslant 0$, $A_j A_{j+1} \ldots \models \varphi_2$ and $A_i A_{i+1} \ldots \models \varphi_1, \varphi_1 \,\mathsf{U} \varphi_2$ for all $i \leqslant j$.
- By induction hypothesis (applied to φ_1, φ_2), we obtain $\varphi_2 \in B_j$ and $\varphi_1 \in B_i$ for all $i \leqslant j$
- By induction on j, $\varphi_1 \,\mathsf{U} \varphi_2 \in B_j, \ldots, B_0$.

$L(A_\varphi) \subseteq L(\varphi)$

For a sequence $B_0 B_1 B_2 \ldots$ of states satisfying

(a) $B_{i+1} \in \delta(B_i, A_i)$,

(b) $\forall F \in \mathcal{F}$, $B_j \in F$ for infinitely many j,

we have $\psi \in B_0 \leftrightarrow A_0 A_1 \ldots \models \psi$

- Conversely, assume $\varphi_1 \,\mathsf{U} \varphi_2 \in B_0$. Then $\varphi_2 \in B_0$ or $\varphi_1, \varphi_1 \,\mathsf{U} \varphi_2 \in B_0$.
- If $\varphi_2 \in B_0$, by induction hypothesis, $A_0 A_1 \ldots \models \varphi_2$, and hence $A_0 A_1 \ldots \models \varphi_1 \,\mathsf{U} \varphi_2$
- If $\varphi_1, \varphi_1 \,\mathsf{U} \varphi_2 \in B_0$. Assume $\varphi_2 \notin B_j$ for all $j \geqslant 0$. Then $\varphi_1, \varphi_1 \,\mathsf{U} \varphi_2 \in B_j$ for all $j \geqslant 0$.
- Since $B_0 B_1 \ldots$ satisfies (b), $B_j \in F_{\varphi_1 \,\mathsf{U} \varphi_2}$ for infinitely many $j \geqslant 0$, we obtain a contradiction.
- Thus, $\exists$ a smallest k s.t. $\varphi_2 \in B_k$. Then by induction hypothesis, $A_i A_{i+1} \ldots \models \varphi_1$ and $A_k A_{k+1} \models \varphi_2$ for all $i < k$
- Hence, $A_0 A_1 \ldots \models \varphi_1 \,\mathsf{U} \varphi_2$.

GNBA Size

- States of A_φ are subsets of $Cl(\varphi)$

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GNBA Size

- States of A_φ are subsets of $Cl(\varphi)$
- Maximum number of states $\leqslant 2^{|\varphi|}$
- Number of sets in $\mathcal{F} = |\varphi|$
- LTL $\varphi \rightsquigarrow$ NBA A_φ : Number of states in $A_\varphi \leqslant |\varphi|.2^{|\varphi|}$
- There is no LTL formula φ for the language

$$L = \{A_0 A_1 A_2 \cdots \mid a \in A_{2i}, i \geqslant 0\}$$

Complexity of LTL Modelchecking

- Given φ, $A_{\neg\varphi}$ has $\leqslant 2^{|\varphi|}$ states
  - $|\varphi|$=size/length of φ, the number of operators in φ
- $TS \otimes A_{\neg\varphi}$ has $\leqslant |TS|.2^{|\varphi|}$ states
- Persistence checking : Checking $\Box\Diamond\eta$ on $TS \otimes A_{\neg\varphi}$ takes time linear in $\eta.|TS \otimes A_{\neg\varphi}|$

∃∀ Automata and the LTL connection

- For finite words