CS 70: Homework #10

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1 Family Planning

Mr. and Mrs. Brown decide to continue having children until they either have their first girl or until they have three children. Assume that each child is equally likely to be a boy or a girl, independent of all other children, and that there are no multiple births. Let G denote the numbers of girls that the Browns have. Let C be the total number of children they have.

a.) Determine the sample space, along with the probability of each sample point.

Solution:

ω	$Pr[\omega]$
G	(1/2)
BG	(1/4)
BBG	(1/8)
BBB	(1/8)

b.) Compute the joint distribution of G and C. Fill in the table below.

Solution:

	$C = 1$	$C = 2$	$C = 3$
$G = 0$	0	0	1/4
$G = 1$	1/4	1/4	1/4

c.) Use the joint distribution to compute the marginal distributions of G and C and confirm that the values are as you'd expect. Fill in the tables below.

Solution:

$\Pr(G = 0)$	1/4
$\Pr(G = 1)$	3/4

$\Pr(C = 1)$	$\Pr(C = 2)$	$\Pr(C = 3)$
1/4	1/4	1/2

These values match the sample space, therefore they are as expected.

d.) Are G and C independent?

Solution: No.

e.) What is the expected number of girls the Browns will have? What is the expected number of children that the Browns will have?

Solution: $\mathbb{E}[G] = 3/4, \mathbb{E}[C] = 9/4$

2 Will I Get My Package?

A delivery guy in some company is out delivering n packages to n customers, where $n \in \mathbb{N}$, $n > 1$. Not only does he hand a random package to each customer, he opens the package before delivering it with probability $1/2$. Let X be the number of customers who receive their own packages unopened.

a.) Compute the expectation $\mathbb{E}(X)$.

Solution: Let I_i be the indicator variable that person i has received their unopened package.
$\mathbb{E}[X] = \sum \mathbb{E}[X_i] = n\mathbb{P}[I_i = 1] = n \times \frac{1}{2} \times \frac{1}{n} = 1/2$

b.) Compute the variance $\text{var}(X)$.

Solution:

$$
\begin{aligned}
\mathbb{E}[X^2] &= \sum_{i=1}^{n} \mathbb{E}[I_i^2] + 2\sum_{i<j} \mathbb{E}[I_i I_j] \\
&= n\mathbb{P}[I_i = 1] + 2\sum \frac{1}{4} \times \frac{1}{n(n-1)} \\
&= 1/2 + 2 \times \binom{n}{2} \times \frac{1}{4} \times \frac{1}{n(n-1)} \\
&= 1/2 + 1/4 \\
&= 3/4
\end{aligned}
$$

$\text{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 3/4 - (1/2)^2 = 1/2$

3 Double-Check Your Intuition Again

a.) You roll a fair six-sided die and record the result X. You roll the die again and record the result Y.

(i) What is $\text{cov}(X+Y, X-Y)$?

Solution:

$$\begin{aligned}
\text{cov}(X+Y, X-Y) &= \mathbb{E}[(X+Y)(X-Y)] - \mathbb{E}[X+Y]\mathbb{E}[X-Y] \\
&= \mathbb{E}[X^2 - Y^2] - (\mathbb{E}[X] + \mathbb{E}[Y])(\mathbb{E}[X] - \mathbb{E}[Y]) \\
&= \mathbb{E}[X^2] - E[Y^2] - \mathbb{E}[X]^2 + \mathbb{E}[Y]^2 \\
&= \mathbb{E}[X^2] - \mathbb{E}[X]^2 - (E[Y^2] - \mathbb{E}[Y]^2) \\
&= (\mathbb{E}[X^2] - \mathbb{E}[X]^2) - (E[Y^2] - \mathbb{E}[Y]^2) \\
&= \text{cov}(X, X) - \text{cov}(Y, Y) \\
&= \text{var}(X) - \text{var}(Y) \\
&= 35/12 - 35/12 \\
&= 0
\end{aligned}$$

(ii) Prove that $X+Y$ and $X-Y$ are not independent.

Solution: Let X be a coin flip where a landing heads is worth 1, and let Y = 2X. X+Y=3X, X-Y=-X.

In order for X+Y and X-Y to be independent, $\mathbb{E}[(X+Y)(X-Y)] = \mathbb{E}[X+Y]\mathbb{E}[X-Y]$.

$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y] = \mathbb{E}[X] + \mathbb{E}[2X] = 1.5$
$\mathbb{E}[X-Y] = \mathbb{E}[X] - \mathbb{E}[Y] = \mathbb{E}[X] - \mathbb{E}[2X] = -0.5$
$\mathbb{E}[X+Y]\mathbb{E}[X-Y] = 1.5 \times -0.5 = -0.75$

$$\begin{aligned}
\mathbb{E}[(X+Y)(X-Y)] &= \mathbb{E}[X^2 + Y^2] \\
&= \mathbb{E}[X^2] - \mathbb{E}[Y^2] \\
&= \mathbb{E}[X^2] - \mathbb{E}[4X^2] \\
&= 0.5 - (4 \times 0.5) \\
&= -1.5 \neq -0.75 \neq \mathbb{E}[X+Y]\mathbb{E}[X-Y]
\end{aligned}$$

Therefore, the two are not independent.

For each of the problems below, if you think the answer is "yes" then provide a proof. If you think the answer is "no", then provide a counterexample.

b.) If X is a random variable and $\text{var}(X) = 0$, then must X be a constant?

Solution: Yes. $\text{var}(X) = 0 = \mathbb{E}[(X - \mu)^2] = \sum_{a \in \mathbb{A}} a\mathbb{P}[(X - u)^2 = a]$, which is a sum of absolute distances from the expected value of X. Absolute differences are always positive, the sum of differences from X to E[X] = 0, which implies X is always = E[X], therefore X is constant.

c.) If X is a random variable and c is a constant, then is $\text{var}(cX) = c\text{var}(X)$?

Solution: No. Let X be a simple dice roll, and $c = 12$.

$c\text{var}(X) = c \times \frac{35}{12} = 35.$

$\text{var}(cX) = \mathbb{E}[(cX)^2] - \mathbb{E}[cX]^2 = c^2\mathbb{E}[X^2] - c\mathbb{E}[X]^2 = c^2 \times \frac{91}{6} + c \times \frac{49}{4} = 2331.$

The two are not equal.

d.) If A and B are random variables with nonzero standard deviations and $\text{Corr}(A, B) = 0$, then are A and B independent?

Solution: No. Look at part (a)ii as an example. $\text{var}(X) = \text{var}(Y) = 35/12 \implies$ both standard deviations are nonzero, and correlation = 0.

e.) If X and Y are not necessarily independent random variables, but $\text{Corr}(X, Y) = 0$, and X and Y have nonzero standard deviations, then is $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$?

Solution: Yes. $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y) + 2\text{cov}(X, Y)$. The correlation = 0 (with nonzero standard deviations) implies the covariance = 0, which implies the original statement is true.

f.) If X and Y are random variables then is $\mathbb{E}(\max(X, Y) \min(X, Y)) = \mathbb{E}(XY)$?

Solution: Yes.

g.) If X and Y are independent random variables with nonzero standard deviations, then is

$$\mathrm{Corr}(\max(X,Y), \min(X,Y)) = \mathrm{Corr}(X,Y)?$$

Solution: Yes