

CS 70: Lecture 3 Notes

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Proof By Induction

Goal: Prove $(\forall n \in \mathbb{N})P(n)$

e.g.

$$(\forall n \in \mathbb{N}) \sum_{i=0}^n i = \frac{n(n+1)}{2}$$

Method:

- Prove $P(0)$ (Base case)
- For arbitrary $k \geq 0$, prove $p(k) \implies p(k+1)$, therefore $(\forall n \in \mathbb{N})P(n)$ holds

Induction hypothesis:

Assume $P(k)$, prove $P(k+1)$

Theorem:

$$(\forall n \in \mathbb{N}) \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

Proof. By induction on n :

Base case: $P(0) : \sum_{i=0}^0 i = 0$

Induction step: For arbitrary $k \geq 0$, assume $P(k)$: $\sum_{i=0}^k i = k(k+1)/2$

Prove $P(k+1)$:

$$\sum_{i=0}^{k+1} i = \sum_{i=0}^k i + (k+1)$$

$$= k(k+1)/2 + (k+1)$$

$$= (k+1)(k+2)/2$$

Hence $p(k) \implies p(k+1)$ for all $k \geq 0$, hence $\forall n P(n)$ by induction

□

Theorem: For all $n \geq 3$, the sum of the interior angles of a polygon with n sides is exactly $(n-2)\pi$

Proof. By induction on n .

Base: $n=3 \implies \text{sum of angles} = (3-2)\pi$

Ind. step: assume it holds for any k -gon (k -arbitrary). Prove for $(k+1)$ -gon.

Given any $(k+1)$ -gon, cut off a triangle with a diagonal

Angle sum = (angle sum of a triangle) + (angle sum of a k -gon) = $\pi + (k-2)\pi$ (using $p(k) = ((k+1)-2)\pi$) □

Note: avoid common mistake of going from k -gon to $k+1$ -gon, you should go backwards (from $k+1$ -gon to k -gon, because that's necessary to prove the arbitrary $k+1$)

Strengthening the hypothesis

Theorem:

$$\forall n \geq 1, \sum_{i=1}^n \frac{1}{i^2} \leq 2$$

Proof. By induction on n :

Base: $n=1$:

$$\sum_{i=1}^1 \frac{1}{i^2} = 1 \leq 2$$

(True)

Ind. Step: for arbitrary k , assume

$$\sum_{i=1}^k \frac{1}{i^2} \leq 2$$

Prove same for $k+1$

$$\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^k \frac{1}{i^2} + \frac{1}{(k+1)^2} \leq 2 + \frac{1}{(k+1)^2}$$

This doesn't work.

□

Theorem:

$$\forall n \geq 1, \sum_{i=1}^n \frac{1}{i^2} \leq 2 - 1/n$$

Proof. By induction on n:

Base: n=1:

$$\sum_{i=1}^1 \frac{1}{i^2} = 1 \leq 2 - 1/1 \text{ (True)}$$

Ind. Step: for arbitrary k, assume

$$\sum_{i=1}^k \frac{1}{i^2} \leq 2 - \frac{1}{k}$$

Prove same for k+1

$$\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^k \frac{1}{i^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

Need:

$$2 - \frac{1}{k} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}$$

Algebraic exercise.

$$-\frac{1}{k} + \frac{1}{k+1} \leq 0$$

Note: You can strengthen an induction proof by trying to prove a stronger claim, because you can use that stronger assumption in your inductive step.

□

Theorem: Any $2^n \times 2^n$ chessboard can be tiled with L-shaped tiles, leaving exactly one hole adjacent to center.

Exercise: $\forall n \geq 1, 3 \mid (2^{2n} - 1)$

Proof. Take an arbitrary $2^{k+1} \times 2^k$ board, and chop it in a quarter, and each of those. This leaves 4 holes in each of the quarter centers. We strengthen our claim by changing the phrase "in the center" to "any desired location"

Note: it is important to write "by the induction hypothesis, we can tile all four sub-boards with their desired hole positions" \square

Strong Induction

May as well assume all of $P(0), \dots, P(k)$ when proving $P(k+1)$ (consider domino visualization)

Theorem: Every $n \geq 1$ can be written as a product of primes.

Proof. Induction on n .

Base: $n=2$: 2 is already a prime, trivial.

Induction step: Assume for all integers $1 < n \leq k$

case i: $k+1$ is prime, trivially true case ii: $k+1$ is not prime, then by definition $k+1 = a \cdot b$ for some $1 < a, b < k+1$

By strong induction hypothesis, both a & b are products of prime. \square

Fibonacci Numbers

Definition: $F(0)=0$

$F(1)=1$

$F(n) = F(n-1) + F(n-2) \quad \forall n \geq 2$

0, 1, 1, 2, 3, 5, 8, 13, 21, ...

Theorem: For all $n \in \mathbb{N}$, $F(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}$

Where: $\phi = \frac{1+\sqrt{5}}{2}$ Where: $\psi = \frac{1-\sqrt{5}}{2}$

Corollary: For large n , $F(n) \approx \frac{1}{\sqrt{5}} \phi^n = \frac{1}{\sqrt{5}} (1.618\dots)^n$

$F(n+1)/F(n) \rightarrow \phi$ as $n \rightarrow \infty$

Exercise: Complete proof of theorem by induction on n