CS 70: Lecture 3 Notes

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August 30, 2018

Proof By Induction

Goal: Prove $(\forall n \in \mathbb{N})P(n)$

e.g.

$$(\forall n \in \mathbb{N})\sum_{i=0}^{n} i = \frac{n(n+1)}{n}$$

Method:

- Prove $P(o)$ (Base case)

- For arbitrary $k >= 0$, prove $p(k) \implies p(k+1)$, therefore $(\forall n \in \mathbb{N})P(n)holds)$

Induction hypothesis:

Assume P(k), prove P(k+1)

Theorem:

$$(\forall n \in \mathbb{N})\sum_{i=1}^{\infty} i = \frac{n(n+1)}{n}$$

Proof. By induction on n:

Base case: $P(o) : \sum_{i=0}^{n} = 0$

Induction step: For arbitrary k¿= 0, assume P(k): sum i=0 to k i = k(k+1)/2

Prove P(k+1):

$$\sum_{i=0}^{k+1} = \sum_{i=o}^{k} +(k+1)$$

= k(k+1)/2 + (k+1)

= (k+1)(k+2)/2

Hence p(k $\implies p(k+1) forallk >= 0, hence \forall n P(n) by induction$

$\square$

Theorem: For all $n >= 3$, the sum of the interior angles of a polygon with n sides is exactly (n-2)π

Proof. By induction on n.

Base: n= 3 $\implies sumofangles = (3-2)\pi$

Ind. step: assume it holds for any k-gon (k-arbitrary). Prove for (k+1)-gon.

Given any (k+1)-gon, cut off a triangle with a diagonal

Angle sum = (angle sum of a triangle) + (angle sum of a k-gon) = $\pi + (k-2)\pi (usingp(k) = ((k+1)-2)\pi$ $\square$

Note: avoid common mistake of going from k-gon to k+1-gon, you should go backwards (from k+1-gon to k-gon, because that's necessary to prove the arbitrary k+1)

Strengthening the hypothesis

Theorem:

$$\forall ngte1, \sum_{i=1}^{n} \frac{1}{i^2} \leq 2$$

Proof. By induction on n:

Base: n=1:

$$\sum_{i=1}^{1} \frac{1}{i^2} == 1 \leq 2$$

(True)

Ind. Step: for arbitrary k, assume

$$\sum_{i=1}^{k} \frac{1}{i^2} \leq 2$$

Prove same for k+1

$$\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \leq 2 + \frac{1}{(k+1)^2}$$

This doesn't work.

$\square$

Theorem:

$$\forall n gte 1, \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - 1/n$$

Proof. By induction on n:

Base: n=1:

$$\sum_{i=1}^{1} \frac{1}{i^2} == 1 \leq 2 - 1 (True)$$

Ind. Step: for arbitrary k, assume

$$\sum_{i=1}^{k} \frac{1}{i^2} \leq 2 - \frac{1}{k}$$

Prove same for k+1

$$\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

Need:

$$2 - \frac{1}{k} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}$$

Algebraic exercise.

$$-\frac{1}{k} + \frac{1}{k+1} \leq 0$$

Note: You can strengthen an induction proof by trying to prove a stronger claim, because you can use that stronger assumption in your inductive step.

$\square$

Theorem: Any 2^n x 2^n chessboard can be tiled with L-shaped tiles, leaving exactly one hole adjacent to center.

Exercise: $\forall n \geq 1, 3|(2^{2n}-1)$

Proof. Take an arbitrary 2^{k+1} x 2^k board, and chop it in a quarter, and each of those. This leaves 4 holes in each of the quarter centers. We strengthen our claim by changing the phrase "in the center" to "any desired location"

Note: it is important to write "by the induction hypothesis, we can tile all four sub-boards with their desired hole positions" □

Strong Induction

May as well assume all of $P(0), ..., P(k)$ when proving $P(k+1)$ (consider domino visualization)

Theorem: Every n¿1 can be written as a product of primes.

Proof. Induction on n.

Base: n=2: 2 is already a prime, trivial.

Induction step: Assume for all integers $1 < n \leq k$

case i: k+1 is prime, trivially true case ii: k+1 is not prime, then by definition $k+1 = a \cdot b for some 1 < a, b < k+1$

By strong induction hypothesis, both a & b are products of prime. □

Fibonacci Numbers

Definition: F(0)=0
F(1)=1
F(n) = F(n-1)+F(n-2) $\forall n \geq 2$
$0, 1, 1, 2, 3, 5, 8, 13, 21, ...$

Theorem: For all $n \in \mathbb{N}, F(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}$

Where: $\phi = \frac{1+\sqrt{5}}{2} Where : \psi = \frac{1-\sqrt{5}}{2}$

Corollary: For large n, $F(n) \approx \frac{1}{\sqrt{5}}\phi^n = \frac{1}{\sqrt{5}}(1.618...)^n$

$F(n+1)/F(n) \to \phi$ as n $\to \infty$