Physics 5a: Homework 1

Abhijay Bhatnagar

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Continued from last week

New Material

Linear transformations

For

$$y = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, v_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} -2 \\ 2 \\ 6 \end{pmatrix}$$

Is y a linear combination of v_1 and v_2? We can use RREF to find whether there are scalars c_1, c_2 s.t. $cv_1 + cv_2 = y$.

Definition: Given $v_1, ...v_n \in \mathbb{R}^m$, the span of $v_1, ..., v_n$ is denoted $span\{v_1, ..., v_n\} \in \mathbb{R}^m$, which is the set of all linear combinations of $v_1, ..., v_n$.

e.g.: $span\{\underline{0}\} = \{\underline{0}\}$
e.g. 2: $span\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\} = cv : c \in \mathbb{R}^n$
e.g. 3:

$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$span\{v_1, v_2\} = \{c_1v_1 + c_2v_2 : c_1, c_2 \in \mathbb{R}\} = \mathbb{R}^2$

Proof.

Goal: $span\{v_1, v_2\} = \mathbb{R}^2 \iff$
For every b in R2, b is a linear comb of $v_1, v_2 \iff$
for ever b in R2, RREF (1,1,b1),(0,1,b2) is consistent.

b=(b1),(b2) $\square$

e.g. 4

$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$span\{v_1, v_2\} = \{c_1 v_1 + c_2 v_2 : c_1, c_2 \in \mathbb{R}\} = \mathbb{R}^2$
Intuitively, this is all points reachable by moving along v1, v2 (2d plane containing the vectors)

Question: Why is the span $span\{v_1, v_2\} \notin R^3$?

A1.) Cross product not in span

A2.) No Vector perpendicular to plane of span is in span

A3.) etc...

Linear Algebra Answer: $b \in span\{v_1, v_2\} \iff$

$$\begin{matrix} 1 & 0 & b_1 \\ 1 & 0 & b_2 \\ 0 & 0 & b_3 \end{matrix}$$

is consistent.

Using RREF, there are 3 cases:

1. no pivots
2. 1 pivot, top right
3. 2 pivots

Since b3 is an undefined variable, we can define it as nonzero in all three cases so that the RREF is inconsistent, which means no matter what b_1, b_2 are, we can find a b_3 s.t. $\underline{b}$ is a vector that is not in the span.

Theorem: Equivalence Theorem

$v_1, ..., v_n$ spans $\mathbb{R}^m \iff x_1 v_1 + ... + x_n v_n = b$ has solution for every $b \in R^m$.

$\iff$

$$\begin{matrix} 1 & ... & 1 & \\ v_1 & ... & v_n & b \\ 1 & ... & 1 & \end{matrix}$$

is consistent for every b in Rm

$\iff$

{A, B} has a pivot in every row of A =

$$\begin{array}{ccc} 1 & ... & 1 \\ v_1 & ... & v_n \\ 1 & ... & 1 \end{array}$$

, aka need to block last augmented column with a pivot in every row