Predicates and Quantifiers

Predicate

A predicate is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.

The domain of a predicate variable is the set of all values that may be substituted in the place of the variable.

For ex: Let $P(x)$ be $x^2 > x$ for $x \in R$

$P(2) = 2^2 > 2$ True

$P\left(\frac{1}{2}\right) = \frac{1}{2^2} > \frac{1}{2}$ False

$P\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^2 > -\frac{1}{2}$ True

Quantifiers:

Quantifiers let us talk about how many objects satisfy a property.

Universal Quantifiers ($\forall$)

The statement is true for every element in the domain.

for ex, $\forall x \in N$, $x+0=x$

For all natural numbers x, $x+0$ equals x

For every, For each, For any, Given any, For all

Existential Quantifiers ($\exists$):

There is atleast one element in the domain that makes the statement true.

$\exists x \in Z$, $x^2 = 4$

There exists an integer x such that x squared equals 4.

There exists, There is atleast one, For some, For atleast one

$\neg(\forall x \in D, Q(x)) \equiv \exists x \in D, \neg Q(x)$

$\neg(\exists x \in D, Q(x)) \equiv \forall x \in D, \neg Q(x)$

Statements with Multiple Quantifiers :

When a statement contains more than one kind of quantifier we imagine the actions suggested by the quantifiers as being performed in the order in which the quantifiers occur.

❗ **Conclusion:**

Statement	Loop Analogy	Meaning
$\forall x \in A, \exists y \in B, P(x, y)$	Outer loop on x, inner search y	For every x, at least one good y
$\exists y \in B, \forall x \in A, P(x, y)$	Outer loop on y, inner check all x	There is one y good for all x

Negations of Statements with Two Different Quantifiers

$\sim(\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y)) \equiv \exists x \text{ in } D \text{ such that } \forall y \text{ in } E, \sim P(x, y)$

$\sim(\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, P(x, y)) \equiv \forall x \text{ in } D, \exists y \text{ in } E \text{ such that } \sim P(x, y)$

In statements containing multiple quantifiers, changing the order can significantly change the meaning of the statement.

Logical Notation :

"$\forall x$ in D, $P(x)$" can be written as "$\forall x\ (x \text{ in } D \rightarrow P(x))$," and
"$\exists x$ in D such that $P(x)$" can be written as "$\exists x\ (x \text{ in } D \wedge P(x))$."

Unique Quantifier

$\exists!\, P(x) \equiv \exists x \in D\ [P(x) \wedge \forall y \in D\ (P(y) \rightarrow x = y)]$