Direct Proof: Quotient Remainder Theorem

The Quotient Remainder Theorem:

Given any integer n and a positive integer d, there exist unique integer q and r such that

$$n = dq + r \text{ and } 0 \leq r < d$$

n div d = Integer quotient (q)
n mod d = Non-negative integer remainder (r)
$0 \leq r < d$

Parity:

The fact that any integer is either odd or even is called the parity property.

Theorem: Any two consecutive integers have opposite parity.

The Triangle Inequality: (Also an example of proof by cases)

$$|x| + |y| \geq |x + y|$$

similar In a triangle sum of two sides is greater than 3rd side.

Case I: $x > 0$, $y > 0$

$|x| + |y| = x + y$ $\qquad |x| + |y| \geq |x + y|$

$|x + y| = x + y$

Case II: $x > 0$, $y < 0$ $\qquad |x| + |y| \geq |x + y|$

$|x| + |-y| = x + y$

$|x - y|$, Here $x - y$ or $y - x$

Case III: $x < 0$, $y > 0$ $\qquad$ Similar to 3 $\qquad |x| + |y| \geq |x + y|$

Case IV: $x < 0$, $y < 0$ $\qquad |-x| + |-y| = x + y$

$|-x-y| = |-(x+y)| = x+y$

$\therefore$ Combining all 4 cases, $|x|+|y| \geq |x+y|$

Unique Factorisation Theorem: (Fundamental Theorem of Arithmetic)

Every integer greater than 1 can be uniquely factored into prime numbers, upto the order of factors.

For ex:- $60 = 2^2 \times 3 \times 5$

No other set of primes will multiply to 60.

Good Problems:

1) Find integers q and r such that $n = dq + r$ and $0 \leq r < d$.

$n = -45, \; d = 11$

$-45 = 11 \times -5 + 10$

Properties of Divisibility:

1) Divisibility by 1 and itself:

For any real no. a, $1 | a$ and $a | a$

2) Zero Property:

For any integer a, $a | 0$

3) Transitivity:

If $a | b$ and $b | c$ then $a | c$

4) Additivity / Subtractivity:

If $a | b$ and $a | c$ then,

$a | (b+c)$ and $a | (b-c)$

5) Multiplicativity:

If $a \mid b$ then $a \mid bc$ for $c \in \mathbb{Z}$

6) Preservation under Multiplication:

If $a \mid b$ then $ac \mid bc$ for $c \neq 0$

7) Linear Combinations:

If $a \mid b$ and $a \mid c$ then $a \mid (mb + nc)$
for $m, n \in \mathbb{Z}$