Exercises for Lecture Course on Modeling and System Identification (MSI)
Albert-Ludwigs-Universität Freiburg – Winter Term 2021-2022

Exercise 3: Optimality Conditions and Linear Least Squares

(to be returned on November 15th, 10:00 a.m.)

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The aim of this sheet is to strengthen your knowledge in least squares estimation, optimality conditions and convexity.

Please hand in the MATLAB tasks through Grader (`http://grader.mathworks.com`). Pen-and-paper exercises can be uploaded on the Ilias course page as a pdf or handed in during the lecture.

Exercise Tasks

1. ON PAPER: Given the function $f(x) : \mathbb{R}^n \to \mathbb{R}$ with $f(x) = x^\top Q\, x + c^\top x$ and fixed $c \in \mathbb{R}^n$.

   (a) Consider the not necessarily symmetric matrix $Q \in \mathbb{R}^{n\times n}$ and compute the gradient $\nabla f(x) \in \mathbb{R}^n$ and the Hessian $\nabla^2 f(x) \in \mathbb{R}^{n\times n}$ of this function for any x.

   (b) If Q is symmetric, what properties does it have to fulfil such that the unique minimizer x^* can be computed?

   (c) Compute the unique minimizer and the minimum function value $f(x^*)$ under the correct assumptions. (3 points)

   Hint: You can re-write a matrix-vector product $b = Qx$ *as* $b_i = \sum_{j=1}^{n} Q_{ij}\, x_j, \quad$ *for* $\quad i = 1, \ldots, n.$

2. ON PAPER: Consider the function $f(x) : \mathbb{R}^2 \to \mathbb{R}$ with

$$f(x) = x^\top \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} x + x^\top \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 7.$$

   (a) Find all points that satisfy the first order necessary conditions (FONC). Which of them is the global minimizer and why?

   (1 points)

3. ON PAPER: In the lecture notes, the sample variance S^2 is defined as

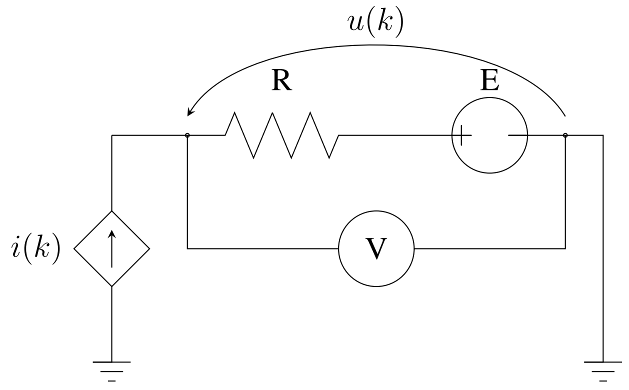
$$S^2 = \frac{1}{N-1} \sum_{n=1}^{N} (Y(n) - M(Y_N))^2,$$

   where $M(Y_N)$ is the sample mean (see lecture notes ch. 2.4 p. 19). Explain, why the division by $N-1$ is preferable over N. (2 points)

   Hint: Calculate the expected value of the sample variance and compare it to the expected value of the mean squared deviations estimator. Also remember that for independent measurements x_j *and* x_k *the following holds:*

$$\mathbb{E}\{x_j x_k\} = \begin{cases} \mathbb{E}\{x_j\}\,\mathbb{E}\{x_k\} = \mu^2, & \text{if } j \neq k \\ \mathbb{E}\left\{x_j^2\right\} = \sigma^2 + \mu^2, & \text{if } j = k \end{cases}$$

4. Consider the following experimental set up to estimate the values of E and R.



You obtain two datasets each containing N measurements of the voltage $u(k)$ for different values of $i(k)$. The first dataset contains $\{u_1(k)\}_{k=1}^{N}$ and $\{i_1(k)\}_{k=1}^{N}$ and the second dataset contains $\{u_2(k)\}_{k=1}^{N}$ and $\{i_2(k)\}_{k=1}^{N}$. For cleaner and simpler notation, we omit the dataset indices, e.g. instead of $u_1(k)$ and $u_2(k)$ we write $u(k)$ but mean both.

We assume that the input measurement $i(k)$ is not affected by noise, but that the measurements $u(k)$ are affected by i.i.d. additive noise $n_{\mathrm{u}}(k)$. Under these assumptions the measurement model is given by:

$$u(k) = m(k) + n_{\mathrm{u}}(k) \text{ where } m(k) = E + R \cdot i(k).$$

Tasks: (4 points)

(a) MATLAB: Load the datasets provided on the website containing the measurements into MATLAB. Plot each dataset in a corresponding plot using the `subplot` command.

(b) ON PAPER: Formulate the problem as a least squares problem where $\theta = \begin{bmatrix} E \\ R \end{bmatrix} \in \mathbb{R}^2$ and define $\Phi \in \mathbb{R}^{N \times 2}$ and $y \in \mathbb{R}^N$ such that the optimizer is given by $\theta^* = \arg\min_\theta \|y - \Phi\theta\|_2^2$.

(c) MATLAB: Use the least squares estimator formulated in the previous subtask to find the experimental values of R and E for each of the two datasets individually.
Plot the linear fits through the respective measurement data.

(d) MATLAB: For each dataset plot a histogram of the residuals defined as $r(k) = m(k) - u(k)$, where $m(k) = E + R \cdot i(k)$ is the voltage determined by the model, and $u(k)$ are the obtained measurements.

(e) ON PAPER: Which dataset is noisier? Give an educated guess of the type of noise.

This sheet gives 10 points in total.