Temporal Vaccination Games Under Resource Constraints

Abhijin Adiga[†] and **Anil Vullikanti**[*,†]
[†]Network Dynamics and Simulation Science Laboratory,
Biocomplexity Institute of Virginia Tech
[*]Department of Computer Science, Virginia Tech
Email: {abhijin, akumar}@vbi.vt.edu

Abstract

The decision to take vaccinations and other protective interventions for avoiding an infection is a natural game-theoretic setting. Most of the work on vaccination games has focused on decisions at the start of an epidemic. However, a lot of people defer their vaccination decisions, in practice. For example, in the case of the seasonal flu, vaccination rates gradually increase, as the epidemic rate increases. This motivates the study of temporal vaccination games, in which vaccination decisions can be made more than once. An important issue in the context of temporal decisions is that of resource limitations, which may arise due to production and distribution constraints. While there has been some work on temporal vaccination games, resource constraints have not been considered.

In this paper, we study temporal vaccination games for epidemics in the SI (susceptible-infectious) model, with resource constraints in the form of a repeated game in complex social networks, with budgets on the number of vaccines that can be taken at any time. We find that the resource constraints and the vaccination and infection costs have a significant impact on the structure of Nash equilibria (NE). In general, the budget constraints can cause NE to become very inefficient, and finding efficient NE as well as the social optimum are NP-hard problems. We develop algorithms for finding NE and approximating the social optimum. We evaluate our results using simulations on different kinds of networks.

Introduction

Despite a lot of progress in medical diagnostics and pharmaceutical tools, infectious diseases remain a major challenge for governments all over the world. Even the annual influenza epidemic in the US has a significant social and economic burden, which is estimated to exceed $87.1 billion (e.g., (Molinari et al. 2007)). For many diseases, especially the annual influenza, there exist vaccines, though their efficacy might be quite variable. However, taking a vaccine has a cost (economic cost, inconvenience and health effects). Further, an individual can get protected without any intervention if enough other people he/she comes in contact with in the population are protected— this is referred to as *herd immunity* in mathematical epidemiology, and is a natural setting for a game-theoretical analysis. This has been a very active area of research both in epidemiology and network security, e.g. (Bauch and Earn 2004; Grossklags, Christin, and Chuang 2008; Khouzani, Sarkar, and Altman 2011; Reluga and Galvani 2011; Saha, Adiga, and Vullikanti 2014); see the related works section for a more detailed discussion.

In practice, most people do not take a preventive vaccine before the start of the epidemic, and instead wait for some time. Often, the vaccination rate grows with the epidemic outbreak rate. There are many different and complex reasons for vaccination decisions being made at different times, and understanding this remains a big open problem, as well as an important public health issue. Almost all the work on vaccination games only considers vaccination decision made only at the start of the epidemic, in a simultaneous game setting. The only studies on temporal vaccination are by (Reluga and Galvani 2011; Adiga, Venkataramanan, and Vullikanti 2016). The work by (Reluga and Galvani 2011) uses a differential equation approach, which assumes simplified and homogeneous connectivity among individuals.

We build on the approach of (Adiga, Venkataramanan, and Vullikanti 2016), which studies this as a repeated game on a network, in the SI (Susceptible-Infectious) model of epidemics. They characterize Nash equilibria in such games, and show that these exhibit interesting temporal structure, such as a large fraction of nodes defer their vaccination decisions. An important open question from their work is the effect of resource constraints— they assume there are no bounds on the number of individuals who can get vaccinated at any time, and this is reflected in the solutions they find. In practice, there are important resource constraints of various kinds, e.g., capacity of hospitals and pharmacies in administering vaccines, or the production capacity–see e.g., (CDC ; Orenstein and Schaffner 2008). In this paper, we study the temporal vaccination problem with resource constraints at different times. Our main contributions are:

1. We formalize the temporal vaccination game with resource constraints, BUDGETTEMPVACC, as a multi-stage game on a network, and study the structure of Nash equilibria (NE) in such games. There can be multiple NE even when the network of interactions is a tree, and finding one with the minimum cost is NP-complete.
2. We show that many nodes defer their vaccination deci-

sions, and the budget constraints lead to very significant differences from the solution of (Adiga, Venkataramanan, and Vullikanti 2016). Specifically, there exist families of instances, where small changes in the budgets (while keeping all other components of the problem fixed) lead to very high inefficiencies. Further, unlike their formulation (in which there is no need to consider more than two time steps), we find that there can be nodes that choose to vaccinate at each possible time step.

3. Computing the social optimum turns out to be a challenging optimization problem. We show that it is NP-hard to approximate within a factor of $O(n^{\alpha})$ for any $\alpha \in (0,1)$, without violating any budget constraints. For the special case of BUDGETTEMPVACC, with only two times at which vaccination decisions are made, we design an algorithm that approximately satisfies the budget constraints.
4. We study the characteristics of NE in different kinds of networks experimentally. We use best response strategies, and find that they usually converge to NE quickly. Corroborating our theoretical results, we find very high sensitivity of the solution cost, as well as the number of nodes that defer vaccination decisions, to the budget constraints and vaccination delays.

Some of the details, including proofs and experimental results are presented in Appendix.

Preliminaries and Model

We extend the approach of (Adiga, Venkataramanan, and Vullikanti 2016) and formally define BUDGETTEMPVACC as a repeated game on an instance $(G, \mathbf{p}, \mathcal{T}, \mathcal{B}, \mathbf{C}, \mathbf{L})$ in the following manner:

1. V is a set of $n = |V|$ players, who form the nodes of a graph $G = (V, E)$, with an edge $(u, v) \in E$ if the epidemic can spread from u to v;
2. $p_i = \Pr[\text{source of infection is node } i]$, with $\sum_i p_i = 1$.
3. $\mathcal{T} = \{t_0 = 0, \cdots, t_k\}$ is a set of time instants, at which the vaccination decisions will be made;
4. $\mathbf{B} = \{B^t \mid t \in \mathcal{T}\}$ specifies the vaccination budget, where B^t is the number of vaccines allocated for time t;
5. $\mathbf{C} = \{C_v^t \mid v \in V, t \in \mathcal{T}\}$ is the set of vaccination costs,
6. $\mathbf{L} = \{L_v \mid v \in V\}$ is the set of infection costs.

Epidemic model and strategies. We assume nodes are in Susceptible (S) and Infectious (I) states, and the epidemic spreads according to a simple discrete time **SI** (Easley and Kleinberg 2010). In this model, if node v gets infected at time t, each uninfected neighbor u will be infected at time $t+1$, unless it gets vaccinated. The vaccine is assumed to have 100% efficacy, and starts protecting the node immediately after taking it. A node's strategy is to decide if and when to get vaccinated. The strategy of node v at time 0 (before the source of the infection is known) is denoted by $Y(v, \cdot, 0) \in \{0,1\}$, with $Y(v, \cdot, 0) = 1$ if v gets vaccinated at time 0. The source of the infection is known after time $t = 0$, and $Y(v, s, t)$ denotes the strategy of node v at time t given that the source of infection is s. If node v gets vaccinated at time

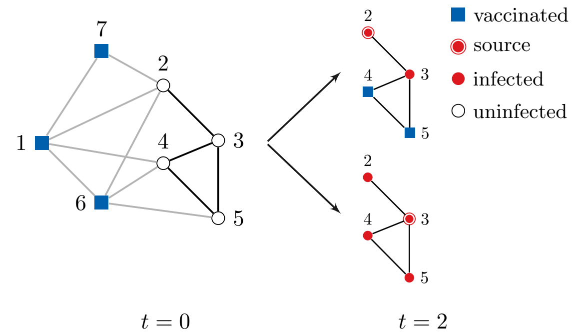

Figure 1: Example NE: Uniform vaccination cost $C = 0.5$ and infection cost $L = 1$ for all nodes, $\mathcal{T} = \{0, 2\}$, $p_2 = 0.5$, $p_3 = 0.5$, $B^0 = 3$, $B^2 = 2$.

t, it incurs a cost C_v^t, whereas if it gets infected at any time, it incurs a cost L_v. Throughout, we assume that $L_v > C_v^t$ for all t, so that it is always cheaper to get vaccinated, instead of getting infected. The complete strategy vector is denoted by $Y(\cdot)$, and the expected cost associated with it is denoted by $\text{cost}(Y)$.

Example. Fig. 1 shows an example where $Y(1, \cdot, 0) = Y(6, \cdot, 0) = Y(7, \cdot, 0) = 1$ and $Y(v, \cdot, 0) = 0$ for $v = 2, 3, 4, 5$. For $t = 2$, $Y(4, 2, 2) = Y(5, 2, 2) = 1$ and $Y(v, s, 2) = 0$ for all other v and s combinations. Note that if node 1 decides to become insecure, then, the cost incurred is $p_2 L + p_3 C > 0.5 = C$. The same holds for 6 and 7. At $t = 2$, if 2 is the source, then only 4 and 5 can be saved, while if 3 is the source, none can be saved. By definition no other node can be the source.

Resource constraints. We assume at most B^t people can be vaccinated at time t–this can capture resource constraints, e.g., due to production or distribution limits. Therefore, a strategy vector $Y(\cdot)$ is *feasible* if for any time t: (1) $\sum_v Y(v, \cdot, 0) \leqslant B^0$, if $t = 0$, and (2) $\sum_v Y(v, s, t) \leqslant B^t$, for any source s. Let $\mathcal{F}$ denote the set of all feasible strategy vectors.

Stages of BUDGETTEMPVACC. This involves the following rounds:

1. At time $t = 0$, all the nodes play a simultaneous vaccination game to decide whether to get vaccinated or not. As mentioned earlier, the vaccination takes effect immediately. If node v gets vaccinated at this time, we denote this by $Y(v, \cdot, 0) = 1$. As mentioned above, at most B^0 nodes can get vaccinated at this time, so that $\sum_v Y(v, \cdot, 0) \leqslant B^0$.
2. A randomly chosen node $s \in V$ is selected to be the source of the epidemic. We assume that if $Y(s, \cdot, 0) = 1$, it remains immune, and the infection does not spread. If $Y(s, \cdot, 0) = 0$, then s gets infected and the infection spreads to each uninfected neighbor in subsequent times. We also assume perfect information, so all nodes know about the source s.
3. For each $t = 1, 2, \ldots$, we have the following two steps:

   (a) If $t \in \mathcal{T}$, a simultaneous vaccination game is played at time t, and each node v decides whether to get vaccinated at this time or not— this is denoted by $Y(v, s, t) \in \{0, 1\}$, with 1 denoting vaccination and s the source. Further, for feasibility, we have $\sum_v Y(v,s,t) \leqslant B^t$.

(b) Let I_{t-1} denote the set of nodes which are infected at time $t-1$. For each node $u \in I_{t-1}$, each uninfected neighbor $v \in N(u)$ will get infected at time t, (i.e., $v \notin I_{t-1}$), unless v gets vaccinated at or before time t. Recall that vaccination takes precedence over infection in our model. Define set $I_t = I_{t-1} \cup \{v :\ v \text{ gets infected at time } t\}$ to be the set of all infected nodes till this time.

4. The game stops at time t if there are no more uninfected nodes that can be infected from their neighbors, and $t' \notin \mathcal{T}$ for all $t' \geqslant t$ (i.e., there are no more vaccination games to be played). Each node v incurs cost L_v if it ever got infected, i.e., $v \in I_t$. It incurs cost $C_v^{t'}$ if it got vaccinated at time $t' \leqslant t$. The overall cost for node v is the expectation over all possible choices of the source.

Cost of a feasible strategy. For every $Y \in \mathcal{F}$, we define the cost incurred by $v \in V(G)$ given strategy Y as: $\text{cost}(v,Y) = C_v^0 Y(v,\cdot,0) + \sum_{s\in V} p_s\big(\sum_{T\in\mathcal{T}} C_v^T Y(v,s,T) + L_v I(v,s,Y)\big)$, where $I(s,v,Y) = 1$ if v gets infected due to s in the strategy Y. We define $\text{cost}(Y) = \sum_v \text{cost}(v,Y)$.

Nash equilibria and social optimum. For a strategy profile $Y(\cdot)$, let $Y_{-v}(\cdot)$ be the strategy profile for all the remaining players. We say that a strategy $Y(\cdot)$ is a Nash equilibrium (NE) (Leyton-Brown and Shoham 2008) if for each $v \in V$: $\text{cost}(v,Y') \geqslant \text{cost}(v,Y)$ where Y' is any strategy profile such that $Y'_{-v}(\cdot) = Y_{-v}(\cdot)$, i.e., $Y'(\cdot)$ has the same strategies as $Y(\cdot)$ for all other players $v' \neq v$. In other words, no player v can reduce its expected cost by unilaterally changing its strategy, given that the other players' strategies are fixed. We define the *social optimum* as a strategy $Y(\cdot)$ that has the minimum cost, over the space of all possible strategies— this is not necessarily (and is not usually) a pure NE. Therefore, the cost of a pure NE relative to the social cost is an important measure, and the maximum such ratio over all possible pure NE is known as the *price of anarchy* (Koutsoupias and Papadimitriou 1999).

Source probability. For simplifying our discussion, henceforth, we will assume that the source is chosen uniformly at random, i.e., $p_s = \frac{1}{n}$ for all $s \in V$. Most of our results extend to general source distributions.

Characterization of Nash Equilibria

First we will define valid strategy.

Definition 1. *A strategy Y is* valid *if it satisfies the budget constraints, i.e., $\sum_v Y(v,\cdot,0) \leqslant B^0$ and $\forall t \in \mathcal{T} \setminus \{0\}$, $\sum_v Y(v,s,t) \leqslant B^t$.*

From the definition of BUDGETTEMPVACC, $Y(v,s,T) = 1$ implies that a vaccine is "reserved" for node v if s is the source. However, even if s is the source, it is possible that it is secure (i.e., $Y(s,\cdot,0) = 1$) or the infection never reaches v (because of other nodes that chose to be vaccinated at time T). In both cases, v will not incur the cost of C_v/n. This notion has important implications for the structure of NE, in the sense that it does not hurt a node v to choose $Y(v,s,T) = 1$ if the budget constraints allow for this.

We now discuss a characterization of the pure NE for BUDGETTEMPVACC for the special case where $\mathcal{T} = \{0,T\}$. We start with some definitions. Let $G[V']$ be the subgraph of G induced by the set V' of nodes. Let $V_0(Y) = \{v : Y(v,\cdot,0) = 1\}$ and $V_s(Y) = \{v : Y(v,s,T) = 1\}$ be the sets of nodes that are vaccinated at time 0 and time T when the source is s, respectively. Let $C(v,G')$ be the connected component containing node v in the graph G'. For $v,s \in V$, let $I(v,s,G[V - V_0(Y) - V_s(Y)]) = 1$ if $s \in C(v,G')$, $Y(v,\cdot,0) = 0$ and $Y(s,\cdot,0) = 0$, i.e., $I(\cdot)$ is the indicator of the event "v will be infected if s is the source." Let $d(x,y,G)$ denote the distance of x from y in G. Let $A(v,Y)$ be the set of potential sources s for which it is still possible for node v to get vaccinated at time T, i.e. $A(v,Y) = \{s \mid Y(v,s,T) = 0, |V_s(Y)| < B^T, d(v,s,G[V - V_0(Y) - V_s(Y)) \geqslant T, I(v,s,G[V - V_0(Y) - V_s(Y)]) = 1\}$.

Lemma 2. *Consider an instance $(G,\mathbf{p},\mathcal{T} = \{0,T\},\mathcal{B},\mathbf{C},\mathbf{L})$ of* BUDGETTEMPVACC. *A valid strategy $Y(\cdot)$ is a pure NE iff*

1. *If $Y(v,\cdot,0) = 0$ then either (a) $\sum_s I(v,s,G[V - V_0(Y) - V_s(Y)])\frac{L_v}{n} \leqslant C_v$ or (b) $|V_0| = B^0$.*
2. *If $Y(v,\cdot,0) = 1$ then the following conditions hold: (a) $\sum_s I(v,s,G[V - V_0(Y) - V_s(Y) \cup \{v\}])\frac{L_v}{n} > C_v$, and (b) $\sum_{s\in V-A(v,Y)} I(v,s,G[V - V_0(Y) - V_s(Y))\frac{L_v}{n} > C_v$.*
3. *If $Y(v,s,T) = 0$ and $Y(v,\cdot,0) = 0$ then either (a) $I(v,s,G[V - V_0(Y) - V_s(Y)]) = 0$ or (b) $|V_s(Y)| = B^T$.*

The proof is in Appendix.

Existence and Complexity of finding NE

In this section, we first propose a best-response algorithm to compute NE (if it exists) for the special case $\mathcal{T} = \{0,T\}$. This will be followed by results on minimum cost pure NE and the price of anarchy.

Best response strategy: Consider an instance $(G,\mathbf{p},\mathcal{T} = \{0,T\},\mathcal{B},\mathbf{C},\mathbf{L})$ of BUDGETTEMPVACC.

1. We start with a feasible strategy $Y(\cdot)$ (one possibility is to start with all nodes insecure).
2. If there exists node v such that $Y(v,\cdot,0) = 0$ and $\sum_s I(v,s,G[V - V_0(Y) - V_s(Y)])\frac{L_v}{n} > C_v$, then:
   (a) If $|V_0| < B^0$, then switch the strategy of node v: set $Y(v,\cdot,0) = 1$. Set $Y(v,s,T) = 0$ for all s.
   (b) If $|V_0| = B^0$: For each $s \in A(v,Y)$, we set $Y(v,s,T) = 1$.
3. If there exists node v such that $Y(v,\cdot,0) = 1$ and $\sum_{s\in V-A(v,Y)} I(v,s,G[V - V_0(Y) - V_s(Y) \cup \{v\}])\frac{L_v}{n} \leqslant C_v$, we switch the strategy of node v and set $Y(v,\cdot,0) = 0$. Then, for $s \in A(v,y)$, we set $Y(v,s,T) = 1$.

When there are no budget constraints a pure NE need not exist (Adiga, Venkataramanan, and Vullikanti 2016). However, for $B^T = 0$ and uniform vaccination and infection costs, it can be shown that the best response strategy always converges to a pure NE. The proof is based on a potential function argument by (Aspnes, Chang, and Yampolskiy 2006), where they prove the same result for the special case where B^0 is unbounded. The same proof holds for $B^0 < n$.

Lemma 3. *For an instance of* BUDGETTEMPVACC*, where (1) $\mathcal{T} = \{0, T\}$, and (2) vaccination and infection costs are uniform for all the nodes (i.e., there exist c, L such that $C_v = C$ and $L_v = L$ for all $v \in V$), and (3) $B^T = 0$ for all $t \in \mathcal{T}$, the above algorithm converges to a pure NE.*

In Experimental Results section, we discuss the performance of the best response strategy on several networks.

Minimum cost NE. By a reduction from the Firefighter problem (Finbow et al. 2007), we showed that it is hard to compute the minimum cost NE.

Lemma 4. *Finding a minimum cost pure NE is NP-complete even on instances of* BUDGETTEMPVACC *where G is a tree.*

The proof is in Appendix.

Price of anarchy (PoA). When there are no budget constraints, it was shown in (Adiga, Venkataramanan, and Vullikanti 2016) that the price of anarchy can be $\Omega(n)$. Clearly this holds for BUDGETTEMPVACC as well. However, with the budget constraints, the PoA is $\Omega(n)$ even for trees.

Lemma 5. *There exist instances $\mathcal{I} = (G, \mathbf{p}, \mathcal{T}, \mathcal{B}, \mathbf{C}, \mathbf{L})$ of* BUDGETTEMPVACC*, where G is a tree with root s, $p_s = 1$ and $B^0 = 0$ for which the price of anarchy is $\Omega(n)$.*

Effect of budget constraints

We now show that the budget constraints have a very significant impact on the cost of a NE, as well as on the social optimum.

Lemma 6. *There exist instances $\mathcal{I} = (G, \mathbf{p}, \mathcal{T}, \mathcal{B}, \mathbf{C}, \mathbf{L})$ of* BUDGETTEMPVACC*, for which there exist strategies $Y(\cdot)$ and $Y'(\cdot)$ such that:*

1. *$Y(\cdot)$ is a NE for $\mathcal{I}$ that satisfies the budget constraints,*
2. *$Y'(\cdot)$ is a NE for the instance $\mathcal{I}'$ obtained by removing all the budget constraints for $T \neq 0$ in $\mathcal{T}$ in instance $\mathcal{I}$ (i.e., setting $B^T = \infty$ for all $T \in \mathcal{T}, T \neq 0$), and*
3. *$cost(Y)/cost(Y') = \Theta(n)$.*

The proof is in Appendix.

Extending the instance constructed in the above proof, we can see that in the absence of budget constraints, for each random source s, *all* nodes at distance T from s choose to get vaccinated, where $T = \min\{t \in \mathcal{T} : t > 0\}$ is the first time in $\mathcal{T}$ when the vaccination decisions can be made. As a result, it suffices to consider only one time step in $\mathcal{T}$ at which the vaccination game needs to be played, as was observed in (Adiga, Venkataramanan, and Vullikanti 2016). In contrast, because of the budget constraints in an instance of BUDGETTEMPVACC, vaccination decisions might be made in each round. Further, nodes far away from the source might choose to get vaccinated, which might cause a large outbreak. Further, it does not help to increase the budget at time $t = 0$ alone, as discussed in the next lemma.

Lemma 7. *There exist instances $\mathcal{I} = (G, \mathbf{p}, \mathcal{T}, \mathcal{B}, \mathbf{C}, \mathbf{L})$ of* BUDGETTEMPVACC*, with $B^0 = \infty$, for which removing the budget constraint B^T (i.e., setting $B^T = \infty$) can reduce the cost of NE by a factor of $\Theta(n)$.*

Finding the social optimum

Since BUDGETTEMPVACC generalizes the formulation studied in (Adiga, Venkataramanan, and Vullikanti 2016), it is easy to verify that finding the social optimum is NP-complete. We show that even approximating the social optimum of BUDGETTEMPVACC is very hard. This motivates bicriteria approximation algorithms, in which the budget constraints can be violated.

Theorem 8. *The social optimum of an instance $(G, \mathbf{p}, \mathcal{T}, \mathcal{B}, \mathbf{C}, \mathbf{L})$, of* BUDGETTEMPVACC *cannot be approximated within a factor of n^α in polynomial time for $\alpha \in (0, 1)$, unless $P = NP$.*

The proof is by a reduction from a game-theoretic version of the FireFighter problem studied in (Anshelevich et al. 2009) (see Appendix).

Bicriteria Approximation Algorithm

Approximating the social optimum. We now discuss an approximation algorithm for computing the social optimum for the special case of $\mathcal{T} = \{0, T\}$. Our algorithm builds on (Hayrapetyan et al. 2005), and involves a linear-programming (LP) rounding approach. We use the following notation below: (1) $x(j, s)$ is an indicator variable, which is 1 if node j gets infected due to source s; (2) $y(j)$ is an indicator which is 1 if node j is vaccinated at time 0; (3) $y(j, s)$ is an indicator which is 1 if node j is vaccinated at time T, when the source is s; (4) $P^d(s, j)$ is the set of paths from s to j of length d; (5) $P^{<d}(s, j)$ is the set of paths from s to j of length less than d.

We now describe an integer programming formulation $\mathcal{P}$ for the social optimum below. Let, $f(\mathbf{x}, \mathbf{y}) = \sum_v C_v^0 y(v) + \frac{1}{n} \sum_{s,v} C_v^T y(v, s) + \frac{1}{n} \sum_{s,j} L_j x(j, s)$.

$\min_{\mathbf{y}} f(\mathbf{x}, \mathbf{y})$ such that

$$\sum_{v \in p} y(v) \geqslant 1 - x(j, s), \forall s, j, \forall p \in P^{<T}(s, j) \quad (1)$$

$$\sum_{v \in p} y(v) + y(j, s) \geqslant 1 - x(j, s), \forall s, j, \forall p \in P^T(s, j) \quad (2)$$

$$y(j) + y(j, s) \geqslant x(u, s) - x(j, s), \forall u \in N(j), \forall s, j \quad (3)$$

$$x(s, s) = 1 - y(s)$$

$$\sum_{v} y(v) \leqslant B^0 \quad (4)$$

$$\sum_{v} y(v, s) \leqslant B^t, \ \forall s, t \quad (5)$$

$$y(v), y(v, s), x(i, j) \in \{0, 1\}, \ \ \forall s, v, i, j \in V \quad (6)$$

Constraint (1) ensures that if a node j is within T hops of a source s, if node j does not get infected due to source s, then at least some node on every path from s to j of path length less than T has a vaccinated node. In constraint (2) for paths of length T, node j could be vaccinated at time T to ensure the same effect. Constraint (3) ensures that if j is not vaccinated at either 0 or T, then, it gets infected if any of its neighbors does. Budget constraints are captured by (4) and (5).

Lemma 9. *The optimum solution* $(\mathbf{x}, \mathbf{y})$ *to the program* $\mathcal{P}$ *is the social optimum of the* BUDGETTEMPVACC *instance.*

We describe an approximation algorithm based on relaxing and rounding the solution of $\mathcal{P}$ for the special case where $B^0 = 0$. For $A \subseteq V$, let $N(A) = \{v \notin A \mid \exists u \in A, u \text{ is adjacent to } v\}$. The algorithm involves the following steps:

1. Solve a linear relaxation $\mathcal{P}_R$ with the constraints $y(i), y(i,j), x(i,j) \in [0,1]$, for all i, j. Let $\mathbf{x}^f, \mathbf{y}^f$ denote the fractional solution to this relaxed program.
2. We round $\mathbf{x}^f, \mathbf{y}^f$ to an integral solution in the following manner:
   (a) For each j, s, define $x'(j,s) = \min\{2x^f(j,s), 1\}$, $y'(j) = \min\{2y^f(j), 1\}$, $y'(j,s) = \min\{2y^f(j,s), 1\}$.
   (b) For each j, if $y'(j) \geqslant \frac{1}{T-1}$, set $Y(j, \cdot, 0) = 1$.
   (c) For each j, s, if $y'(j,s) \geqslant \frac{1}{T}$, set $Y(j,s,T) = 1$.
   (d) For each $s \in V$, pick "suitable" $r_s \in [0,1]$ (as discussed in the proof of Theorem 10)
      i. Let $A(s, r_s) = \{j : x'(j,s) \geqslant r_s\}$ and $C(s, r_s) = N(A(s, r_s)) - A(s, r_s)$.
      ii. For each $j \in C(s, r_s)$, define $Y(j,s,T) = 1$.

We show below that this randomized strategy $Y(\cdot)$ satisfies the budget constraints approximately, and cost(Y) is within a constant factor of the fractional LP value. Ensuring that all the budget constraints are satisfied is a challenging open problem.

Theorem 10. *Suppose* $B^0 = B^T = B$. *The solution* $Y(\cdot)$ *constructed by the above algorithm satisfies the following properties: (1)* $\sum_v Y(v, \cdot, 0) \leqslant (2T) \cdot B$, *(2) for each* s, $\sum_v Y(v,s,T) \leqslant (8T) \cdot B$, *(3) cost*$(Y) \leqslant (8T) \cdot f(\mathbf{x}^f, \mathbf{y}^f)$.

Experimental Results

We now explore the structure of the NE obtained through the best response strategy. In particular, we study how it behaves with respect to the budget at time 0, B_0, time T, the budget at time T, B_T, and the vaccination cost C. The algorithm was applied on three synthetic graphs: Erdős-Rényi with 100 nodes and average degree 7 (ER); random power-law graph generated using the Chung-Lu model with power-law index $\gamma = 2.5$, 93 nodes and average degree 4 (CL), and a random regular graph with 100 nodes and average degree 4 (RR). We chose these networks because they are commonly used in social network analysis (see, e.g., (Newman 2003)), and together model both homogeneous and heterogeneous degree networks. Recall that the best response algorithm requires computing $I(v,s,Y)$ each time a node changes its strategy. For this we need to use the all-pairs shortest path algorithm. Therefore, we study relatively small networks here.

To keep the framework simple, we assumed uniform vaccination cost $C < 1$ and infection cost $L = 1$ for all nodes. We ran the best response algorithm for various values of B_0, B_T, T, and C. The results presented are averaged across 20 iterations, each producing a NE. Where the results are consistent across networks, plots are shown for only CL network. The remaining plots are in Appendix.

Structure of NE with respect to B_0 and B_T. The results are shown in Fig. 2. We used two criteria to study the effects of B_0 and B_T, viz. average cost of the NE (cost(Y)) and the average number of nodes which vaccinate at time 0 ($\mathbb{E}[|V_0|]$). Comparing column 1 to column 3, we note that across all networks, the effect of B_0 on the cost(Y) is more pronounced than that of B_T. The plots in column 2 are quite interesting in the sense that they exhibit a threshold phenomenon. We note that $\mathbb{E}[|V_0|]$ increases steadily (possibly linearly) with B_0 and then tapers off. The point at which it hits a plateau is a function of B_T. Qualitatively, these plots indicate that greater the guarantee of vaccine availability at a later stage, the higher the inclination to postpone vaccination. Possible directions from here are to determine when these regime changes occur, exploring the role of network structure in this behavior.

Effect of vaccination cost C. We observe that the vaccination cost has a significant influence on the average cost of the NE—see Fig. 3 first plot, which shows the average cost of solution vs. B_0 for different values of T and B_T. We have only shown results for CL due to lack of space.

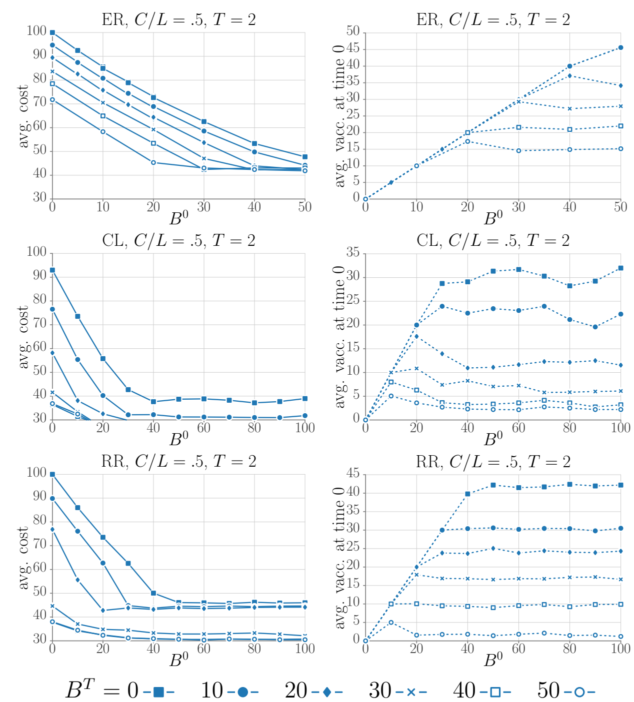

Figure 2: The effect of budget constraints on the structure of NE from the best response algorithm: Each row corresponds to a particular network. The first column is a plot of average cost of the NE vs. B^0, for various values of B^T. The second column corresponds to similar plots for average number of nodes vaccinating at time 0.

Convergence time. We observed that the best response algorithm generally converges quickly and shows little variation with budget constraints (Fig. 3 second plot). However, it would be interesting to see how it performs as the network size increases.

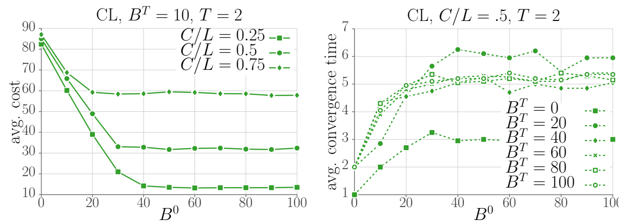

Figure 3: Best response algorithm on Chung-Lu graph CL: Plot (1) corresponds to effect of C on the average cost. Plot (2) shows how average convergence time varies with respect to budget constraints.

Effect of time T. From Fig. 4, we observe that the effect of T depends on B^T; higher the B^T, the more sensitive is the cost to T. Note that when B^T is small, even if the second stage of vaccination happens early, it is not enough to secure enough nodes from the source. A possible future direction is to quantify this effect based on network properties such as density, expansion, etc.

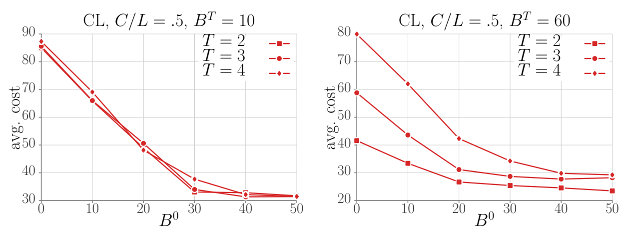

Figure 4: The effect of T on the cost of NE. Plots (1) and (2) show cost vs. B^0 for $B^T = 10$ and 60 respectively.

Related Work

There is a large literature on vaccination games, which can be broadly partitioned into differential equation based or network based models. We summarize them below.

One of the earliest works on vaccination games is by (Bauch and Earn 2004), who use a differential equation based model to study vaccination decisions made at the start of the epidemic. They show that the vaccinated fraction in NE can be expressed in terms of the reproductive number, R_0, which is the expected number of secondary infections caused by an infected individual. Vaccination is a special kind of behavioral change in the context of epidemics. There has been a lot of research on different kinds of game formulations arising out of epidemic behaviors, e.g., (Bauch and Earn 2004; Khouzani, Sarkar, and Altman 2012; Khouzani, Altman, and Sarkar 2012; Khouzani, Sarkar, and Altman 2011; Galvani, Reluga, and Chapman 2007; Reluga 2010; Reluga and Galvani 2011; Trajanovski et al. 2015; Manfredi and D'Onofrio 2013). Reluga et al. (Reluga and Galvani 2011) develop a general approach that combines population games with Markov decision process, and allows decisions at different times. There has also been work on repeated game formulations that take information and past experience into account. (Conforth et al. 2011) study the effects of vaccination decisions based on past epidemics, and find that individuals with number of contacts above a threshold get vaccinated, whereas individuals with fewer than these many contacts do not, leading to erratic flu behavior. (Bauch and Bhattacharyya 2012) use an evolutionary game theory based model to study the feedback between disease prevalence and vaccinating behavior, especially in the context of vaccine scares.

A different area of research involves the use of network based models. One of the earliest works in this direction is (Aspnes, Rustagi, and Saia 2007). They formalize the network version of the vaccination game problem of (Bauch and Earn 2004), and design algorithms for computing NE and the social optimum. This turns out to be a challenging problem, and they develop an approximation algorithm for the social optimum, that is based on linear programming rounding. This approximation bound was improved by (Chen, David, and Kempe 2010; Kumar et al. 2010). All these approaches have assumed a simplistic model of epidemic spread, which models highly contagious diseases. There has been work on more realistic SIS (Susceptible-Infectious-Susceptible) models, e.g., (Omic, Orda, and Mieghem 2009; Trajanovski et al. 2015). An alternative approach was studied by (Saha, Adiga, and Vullikanti 2014), who develop a formulation based on the spectral radius (the first eigenvalue of the network), in which the utility is based on whether or not the spectral radius is above a threshold or not. One limitation of all of these works is that they only consider vaccination decisions at the start of the epidemic, in a one-shot simultaneous game formulation. The work most closely related to our paper is by (Adiga, Venkataramanan, and Vullikanti 2016), who formalize a temporal vaccination game, in which vaccination decisions can be done at multiple times. However, there are no budget constraints in their formulation. As mentioned earlier, this has a very significant impact on both the structure and complexity of the problem.

Conclusions

In this paper, we formalize the temporal vaccination problem with resource constraints on complex networks, as a repeated game BudgetTempVacc. This captures many realistic aspects of epidemic spread in real networks. The budget constraints have a very significant impact on the structure of the game solutions, as well as on the complexity of finding equilibria. A significant fraction of nodes that do get vaccinated, choose to defer their vaccination decision— the specific effects depend on the network structure, the budgets, and the vaccination and infection costs. Computing properties of such repeated games becomes more challenging, compared to those of standard vaccination games. Some of the interesting open problems that arise out of our work include: (1) understanding networks in which pure NE exist and can be computed efficiently, and (2) extending our results to other epidemic models, and improving the algorithmic bounds. This work has been partially supported by the following grants: DTRA Grant HDTRA1-11-1-0016, DTRA CNIMS Contract HDTRA1-11-D-0016-0010, NSF

ICES CCF-1216000, NSF NETSE Grant CNS-1011769, NIH MIDAS Grant 5U01GM070694, NSF DIBBS Grant ACI-1443054.

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Appendix

$G(V,E)$	simple undirected graph with node set V and edge set E
n	number of vertices in G
$G[V']$	subgraph induced by $V' \subseteq V$
$N(v,G)$	neighborhood of v in G
$\mathcal{T}$	set of time instants at which vaccine decisions can be made
Y	strategy vector
$Y(v,s,t)$	strategy for node v at time t; $Y(v,\cdot,0)$ for time 0
C_v^t	Cost for node v to vaccinate at time t
$\mathbf{C}$	$\{C_v^t \mid v \in V(G), t \in \mathcal{T}\}$
L_v	Cost of infection for node v
$\mathbf{B}$	$\{B^t \mid t \in \mathcal{T}\}$, budget for each t
$\mathbf{L}$	$\{L_v \mid v \in V(G)\}$
cost(v,Y)	expected cost for node v associated with strategy Y
cost(Y)	expected cost associated with strategy Y

Table 1: A summary of the notation used in the paper.

Proof. (of Lemma 2) If $Y(v,\cdot,0)=0$, for each source s, node v gets infected if $I(v,s,G[V-V_0(Y)-V_s(Y)])=1$. In this case, the expected infection cost for node v is $\sum_s I(v,s,G[V-V_0(Y)-V_s(Y)])\frac{L_v}{n}$. Since strategy $Y(\cdot)$ is a NE, it must be the case that either this expected infection cost is less than C_v (so that v has no incentive to switch the strategy $Y(v,\cdot,0)$), or $|V_0(Y)|=B^0$, in which case v cannot switch. Therefore, statement (1) follows.

If $Y(v,\cdot,0)=1$, node v can switch its strategy by becoming unvaccinated at time 0, and/or choosing to get vaccinated at time T, relative to some sources. Condition 2(a) implies that the infection cost for node v is larger than C_v if it becomes unvaccinated at time 0, and does not get vaccinated at time T. The set $A(v,Y)$ is the set of sources s relative to which node v could have any benefit in getting vaccinated at time T. Condition 2(b) implies that even if v gets vaccinated at time T relative to all the sources in $A(v,Y)$, the expected infection cost would still be larger than C_v, and so it has no benefit in switching.

If $Y(v,s,T)=0$ and $Y(v,\cdot,0)=0$ then it must be the case that v does not get infected due to source s, i.e., $I(v,s,G[V-V_0(Y)-V_s(Y)])=0$ (statement 3(a)) or, node v cannot get vaccinated at time T because of budget constraints (Statement 3(b)). □

Proof. (of Lemma 4) The proof is by a reduction from the Firefighter problem (Finbow et al. 2007). An instance of this problem consists of an undirected graph $G=(V,E)$, a root $r \in V$, a parameter k and a budget B. The objective is to select a sequence of sets V_i, to secure at each time $i=1,2,\ldots$, such that $|V_i| \leqslant B$, and at most k nodes are burned because of a direct path from the root r. (Finbow et al. 2007) show that determining if there exists a strategy for securing at most B nodes at each time step, so that at most k nodes are burned is NP-complete. □

Proof. (of Lemma 6) We show that the lemma holds even for simple instances where G is a tree, $p_s=1$ for a node $s \in V$, and $\mathcal{T}=\{0,1\}$, with uniform vaccination and infection costs C and L, respectively. Let $B^0=0$ and $B^1=o(n)$. Let $Y(\cdot)$ be a strategy in which B^1 nodes that are farthest away from s are vaccinated at time $T=1$. This implies all nodes $v \in V-\{s\}-V_1(Y)$ get infected, and $Y(\cdot)$ is a NE since no other node can switch its strategy because of the budget constraint. Therefore, cost$(Y)=|B^1|C+\Theta(nL)$.

Next, consider the instance $\mathcal{I}'$ obtained from $\mathcal{I}$ by setting $B^1=\infty$. Let $Y'(\cdot)$ be the strategy in which $V_1(Y')=N(s)$, i.e., all neighbors of s are vaccinated at time 1. It can be verified that $Y'(\cdot)$ is a NE for instance $\mathcal{I}'$ and cost$(Y')=d_s(G)C$. Since $C<L$, the lemma follows. □

Proof. (of Theorem 8) We show the hardness for an instance of BudgetTempVacc with $B^0=0$ and $B^t=1$ for each t— this is a game-theoretic version of the FireFighter problem studied in (Anshelevich et al. 2009). We consider the Save-y problem on instance $(G=(V,E),x,y)$, which involves deciding whether there is a sequence $v_1,\ldots,v_k$ of vaccination at time $1,\ldots,k$ such that y does not get infected from x. This problem is NP-complete, as mentioned in (Anshelevich et al. 2009). We construct a graph $G'=(V',E')$ with $V'=V\cup T$, where T consists of $n^\beta-n$ nodes distinct from V, for constant β specified later. We define $E'=E\cup\{e=(u,t): u\in N(y)\}$, where $N(y)$ is the set of neighbors of y in graph G. We set $p_x=1$ and $p_i=0$ for all $i\in V'-\{x\}$, so that the source is always x. Finally, we set $C=L$ for all nodes.

We observe that if there exists a feasible solution for Save-y, then there is a strategy $Y(\cdot)$ for the above instance of BudgetTempVacc of cost at most $2nC$. This is because the strategy $Y(\cdot)$ that corresponds to the optimal solution for the Save-y instance ensures that no nodes in T get infected. Therefore, at most n nodes are vaccinated, and at most n nodes in $V'-T$ are infected, leading to a total cost of at most $2nC$.

Next, suppose there exists no feasible solution for the Save-y instance. Then, for any strategy $Y(\cdot)$ for the BudgetTempVacc instance, all nodes in T get infected, so that the cost of the strategy is at least $(n^\beta-n)C$. As in (Anshelevich et al. 2009), we select $\beta>\max\{\frac{\ln 2n^2}{(1-\alpha)\ln n},3\}$, which ensures that $n^\beta-n>n^{\alpha\beta}2n$.

Let $Y^{opt}(\cdot)$ denote the optimum solution to the above BudgetTempVacc instance, and suppose there were an approximation algorithm $\mathcal{A}$ that gave an n^α approximation to the social optimum. Let ϕ denote the cost of the solution returned by algorithm $\mathcal{A}$ on the above instance. We have two cases: (1) Suppose cost$(Y^{opt})\leqslant 2nC$: then $\phi\leqslant|V'|^\alpha 2n=n^{\alpha\beta}2nC<(n^\beta-n)C$. In this case, we infer that the Save-y instance is feasible; (2) Suppose cost$(Y^{opt})>2nC$. From the above discussion, it follows that cost$(Y^{opt})\geqslant(n^\beta-n)C$ in this case, and we infer that the Save-y instance is infeasible. □

Proof. (of Lemma 9) First, let $Y(\cdot)$ be any feasible strategy vector. Define $X(\cdot)$ in the following manner. Let $A(s)$ denote the set of nodes that are infected from source s. We have $X(s,s)=1-Y(s,\cdot,0)$, and $A(s)=\phi$ if s is vaccinated at time 0. For each $j\in A(s)$, we have $X(j,s)=1$, and for

$j \notin A(s)$, we have $X(j,s) = 0$. For each node $j \notin A(s)$, it must be the case that for each path $p \in P^{<T}(s,j)$, some node $v \in p$ must be vaccinated at time 0 (this could be the source s as well). Therefore, we must have $\sum_{v \in p} Y(v,\cdot,0) \geqslant 1 = 1 - X(j,s)$, since $X(j,s) = 0$ in this case. Note that in this case, vaccinations at time T do not take effect. Next, for any path $p \in P^T(s,j)$, node j could get protected by getting vaccinated at time T as well. Therefore, if $j \notin A(s)$, we have $\sum_{v \in p} Y(v,\cdot,0) + Y(j,s,T) \geqslant 1 = 1 - X(j,s)$. For each node $j \in N(A(s))$, there must exist a node $u \in N(j) \cap A(s)$ which is infected, and so it must be the case that j is vaccinated either at time 0, i.e., $Y(j,0) = 1$, or at time T, i.e., $Y(j,s,T) = 1$ (in this case, additionally on all paths p of length $< T$ from s to j, some node must be vaccinated); therefore, condition $Y(j,\cdot,0) + Y(j,s,T) \geqslant 1 = X(u,s)$ in this case, for some $u \in A(s) \cap N(j)$, which is an infected neighbor of j. Next, for all $j \in V - A(s) - N(A(s))$, such nodes do not get vaccinated at any time, so $Y(j,\cdot,0) + Y(j,s,T) = 0$ for such j. Finally, for all nodes $j \in A(s)$, since $X(j,s) = 1$, it follows that $Y(j,\cdot,0) + Y(j,s,T) \geqslant 0 \geqslant X(u,s) - X(j,s)$ for any $u \in N(j)$. Therefore, the condition $Y(j,\cdot,0) + Y(j,s,T) \geqslant X(u,s) - X(j,s)$ holds for all nodes, which corresponds to constraint (3) of program $\mathcal{P}$. Further, since $Y(\cdot)$ is a feasible strategy, the budget constraints (4) and (5) of program $\mathcal{P}$ are satisfied. Therefore, $X(\cdot)$ and $Y(\cdot)$ defined this way satisfy the constraints of program $\mathcal{P}$.

Next, let x, y be a feasible integral solution to program $\mathcal{P}$. Since all the variables are non-negative, and the objective is minimization, all constraints must be satisfied with equality. For node s, $x(s,s) = 0$ if $y(s) = 1$. Define $A(s) = \{u : x(u,s) = 1\}$. For any $j \in N(A(s)) - A(s)$, we have $x(j,s) = 0$ and $x(u,s) = 1$ for some $u \in N(j) \cap A(s)$. Constraint (3) implies that $y(j) + y(j,s) \geqslant x(u,s) - x(j,s) = 1$, so that node j must be vaccinated at time 0 or at time T. Next, if a node j has $x(j,s) = 0$, then for every path $p \in P^{<T}(s,j)$, we have $\sum_{v \in p} y(v) \geqslant 1 - x(j,s) = 1$, so that at least one node from such path must be vaccinated by constraint (1). Similarly, for every path $p \in P^T(s,j)$, we have $\sum_{v \in p} y(v) + y(j,s) \geqslant 1 - x(j,s) = 1$, so that either some node from such path is vaccinated at time 0, or j is vaccinated at time T, by constraint (2). Finally, (4) and (5) imply budget constraints. Therefore, any feasible solution to $\mathcal{P}$ corresponds to a feasible strategy vector and an outcome of BUDGETTEMPVACC. $\square$

Proof. (of Theorem 10) Observe that constraints (1), (2) and (3) are satisfied by equality, since the variables are non-negative and the objective is minimization.

First, consider j, s such that there exists a path $p \in P^{<T}(s,j)$. If $x^f(j,s) \leqslant 1/2$, for some node $v \in p$ we have $Y(v,\cdot,0) = 1$, since $\sum_{v \in p} y^f(v) \geqslant 1/2$, so that there exists $v \in p$ with $y^f(v) \geqslant \frac{1}{2(T-1)}$. Similarly, if there exists a path $p \in P^T(s,j)$, we have either $Y(v,\cdot,0) = 1$ for some $v \in p$ or $Y(j,s,T) = 1$, since $\sum_{v \in p} y^f(v) + y^f(j,s) \geqslant 1/2$, so that either $y^f(v) \geqslant \frac{1}{2T}$ for some $v \in p$, or $y^f(j,s) \geqslant \frac{1}{2T}$.

Next, let $A_0(s)$ denote the set of nodes infected in the first T time steps after s becomes the source, after the nodes with $Y(v,\cdot,0) = 1$ are vaccinated at time 0, and nodes j with $Y(j,s,T) = 1$, connected to s by a path of length T are vaccinated at time T. For all $v \in A_0(s)$, it must be the case that $x^f(v,s) \geqslant 1/2$. If not, at least one node on any path p from s to v of length $\leqslant T$ is vaccinated by our above rounding, so that v would not get infected within the first T time steps.

For each s, consider the fractional solution $x'(v,s) = \min\{2x^f(v,s), 1\}$, and $y'(\cdot)$. Pick a random $r_s \in [0,1]$, and let $A(s, r_s)$ and $C(s, r_s)$ be as defined in our rounding step. We will prove below that $\Pr[Y(j,s,T) = 1] = y'(j) + y'(j,s)$, and $\Pr[X(j,s) = 1] = x'(j,s)$. This implies, $E[\sum_j C_j^T Y(j,s,T) + \sum_j L_j X(j,s)] \leqslant 2\sum_j C_j^T y^f(j,s) + 2\sum_j L_j x^f(j,s)$ and $E[\sum_j Y(j,s,T)] \leqslant 2B$. By Markov inequality, $\sum_j C_j^T Y(j,s,T) + \sum_j L_j X(j,s) \leqslant 8(\sum_j C_j^T y^f(j,s) + \sum_j L_j x^f(j,s))$ and $\sum_j Y(j,s,T) \leqslant 8B$ hold simultaneously with probability at least $1/2$. This implies, there exists $r_s \in [0,1]$ for which both these properties hold. We choose that r_s in the approximation step.

Finally, we prove that $\Pr[Y(j,s,T) = 1] = y'(j) + y'(j,s)$, and $\Pr[X(j,s) = 1] = x'(j,s)$. Since r_s is chosen uniformly at random, $j \in A(s, r_s)$ if r_s is chosen from $[0, x'(j,s)]$, which occurs with probability $x'(j,s)$. Next, observe that for any $j \notin A_0(s)$, $y'(j) + y'(j,s) = \min_{u \in N(j)} x'(u,s) - x'(j,s)$. Order the neighbors $N(j) = \{u_1, \ldots, u_k\}$ with $x'(u_1,s) \leqslant \ldots x'(u_k,s)$. Let i be the smallest index such that $x'(u_i,s) \geqslant x'(j,s)$. Then, $j \in C(s, r_s)$ if $r_s \in [x'(j,s), x'(u_i,s)]$, which occurs with probability $x'(u_i,s) - x'(j,s) = \min\{x^f(u,s) - x^f(j,s) : u \in N(j)\}$. We have $y^f(j) + y^f(j,s) = \min\{x^f(u,s) - x^f(j,s) : u \in N(j)\}$, as mentioned earlier. Finally, since $\sum_j y'(j) + y'(j,s)$, the statement follows. $\square$

Results

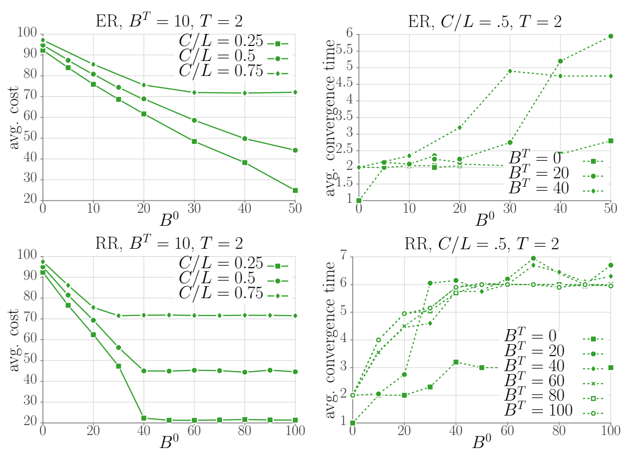

Figure 5: Continuation of Fig. 3: Plot (1) corresponds to effect of C on average cost. Plot (2) shows how average convergence time varies with respect to budget constraints.

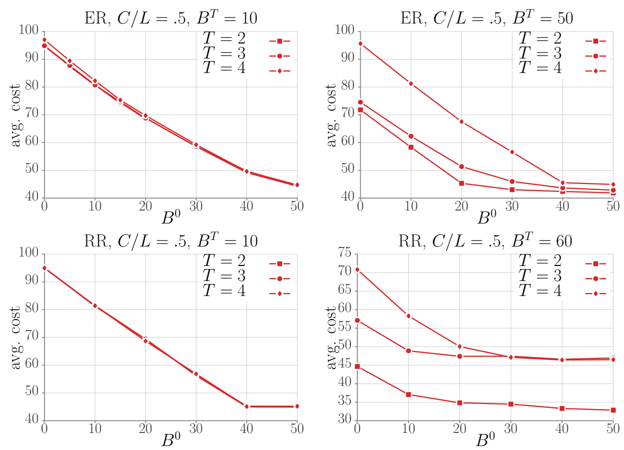

Figure 6: Continuation of Fig. 4. The effect of T on the cost of NE. Plots (1) and (2) are show cost vs. B^0 for a low and high value of B^T respectively.