A Well-Balanced Stochastic Galerkin Method for PDEs with Random Forcing

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[**Jin2015**]

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As we described in the introduction, the scheme we will be working with combines parts of a well-balanced scheme and stochastic galerkin method. As we are working a system of the form

$$u_t + f(u)_x = -b'(x)u \tag{1}$$

At the steady state we have that $u_t = 0$, and so (**??**) becomes the steady state condition

$$f(u)_x + b'(x)u = 0 \tag{2}$$

We will assume that our flux function f is such that $\frac{f'(\xi)}{\xi}$ is a well-defined number for all ξ. We can now define an operator $D : \mathbb{R} \to \mathbb{R}$ by

$$D(u) = \int_0^u \frac{f'(\xi)}{\xi} d\xi$$

Now if $u(x)$ is a strong solution to **??** and $u > 0$, then by Liebniz rule

$$D(u(x))_x + b'(x) = \frac{f'(u(x))u_x}{u(x)} + b'(x) = \frac{f(u(x))_x}{u(x)} + b'(x) = 0$$

This shows that if u is a strong solution to (**??**) and $u > 0$, then $D(u(x)) + q(x)$ is equal to some constant. If we assume that D is monotone so that $D'(x) \neq 0$, then we can see that

$$u(x) = \frac{f(u)_x}{D'(x)}$$

If we adopt the finite volume framework, then we shall have N_x cells with uniform mesh size of Δx. For $j = 1, \dots, N_x$, we let x_j represent the point at the center of the cell and let $x_{j+1/2}$ represent the point at the cell interface. Since we will have a uniform mesh, then $\Delta x = x_{j+1/2} - x_{j-1/2}$. We also have a uniform discretization in time and let t^n represent the discretization at then n^{th} level and $\Delta t = t^n - t^{n-1}$. The cell average at t^n over $[x_{j-1/2}, x_{j+1/2}]$ is given by

$$u_j^n = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t^n) dx$$

At the cell interfaces we will have

$$D_{j+1/2} = D(u_{j+1/2})$$

$$f_{j+1/2} = f(u_{j+1/2})$$
$$b_{j+1/2} = b(x_{j+1/2})$$

Then discrete analog to (**??**) will be

$$\frac{f_{j+1/2} - f_{j-1/2}}{\Delta x} = -\frac{(b_{j+1/2} - b_{j-1/2})}{\Delta x}\Big(\frac{f_{j+1/2} - f_{j-1/2}}{D_{j+1/2} - D_{j-1/2}}\Big)$$
$$D_{j+1/2} - D_{j-1/2} = -(b_{j+1/2} - b_{j-1/2})$$
$$D_{j+1/2} + b_{j+1/2} = D_{j-1/2} + b_{j-1/2} = \text{constant}$$

In particular for the Burgers' equation we have that $f(u) = \frac{u^2}{2}$, so $D(u) = u$ and so

$$\frac{f_{j+1/2} - f_{j-1/2}}{D_{j+1/2} - D_{j-1/2}} = \frac{u^2_{j+1/2} - u^2_{j-1/2}}{2(u_{j+1/2} - u_{j-1/2})} = \frac{u_{j+1/2} + u_{j-1/2}}{2}$$

It follows that the following semi-discrete method for the Burgers' equation is well-balanced.

$$\partial_t u_j + \Big(\frac{u^2_{j+1/2} - u^2_{j-1/2}}{\Delta x}\Big) = -\Big(\frac{b_{j+1/2} - b_{j-1/2}}{\Delta x}\Big)\Big(\frac{u_{j+1/2} + u_{j-1/2}}{2}\Big) \tag{3}$$

It is described in [**Jin2001**] that for a general f that the scheme is well balanced.

$$\partial_t u_j + \Big(\frac{f_{j+1/2} - f_{j-1/2}}{\Delta x}\Big) = -\Big(\frac{b_{j+1/2} - b_{j-1/2}}{\Delta x}\Big)\Big(\frac{u_{j+1/2} + u_{j-1/2}}{2}\Big) \tag{4}$$

In particular, (**??**) is a special case of this general formulation. This gives us the formulation for the deterministic well-balancing.

For the stochastic case, we incorporate some uncertainty into our bottom topography via a random variable $Z : \Omega \to \mathbb{R}$ with law P. For every instance of our random variable Z, let $u(x, t; Z)$ be the function $(x, t) \to u(x, t; Z)$ which solves

$$u_t(x, t; Z) + f(u(x, t; Z))_x = -b(x; Z)u(x, t; Z)$$

If we suppose that we have appropriate integrability conditions so that $u(x, t; Z), Z \in L^2(P)$ holds for all (x, t), then we can use that $u(x, t; Z)$ will be $Z-$measurable. Consider

$$\mathcal{P}_Z = \{\phi(Z) : \phi \text{ is a polynomial of any order}\}$$

Then $\mathcal{P}_Z$ is a dense vector subspace in the space of all $Z-$measurable functions. Since $\mathcal{P}_Z \subset L^2(P)$ by our integrability assumption on Z and that $L^2(P)$ is a Hilbert space, then we can find an orthonormal basis $\{\Phi_m(Z)\}_{m\in\mathbb{N}}$ of $\mathcal{P}_Z$. Thus $\{\Phi_m(Z)\}_{m\in\mathbb{N}}$ is dense in the space $Z-$measurable functions. Then since $u(x, t; Z)$ is $Z-$measurable, then we can find coefficients $\hat{u}_m(x, t)$ such that $P-$almost everywhere we have that

$$u(x, t; Z) = \sum_{m\in\mathbb{N}} \hat{u}_m(x, t)\Phi_m(Z) \qquad \hat{u}_m(x, t) = \mathbb{E}[u(x, t; Z)\Phi_m(Z)]$$

$$b(x;Z) = \sum_{m\in\mathbb{N}} \hat{b}_m(x)\Phi_m(Z) \qquad\qquad \hat{b}_m(x) = \mathbb{E}[b(x;Z)\Phi_m(Z)]$$

We want our scheme to respect our stochastic PDE almost everywhere, this is equivalent to enforcing the condition

$$\mathbb{E}[(\partial_t u(x,t;Z) + f(u(x,t;Z))_x)\Phi_m(Z)] = \mathbb{E}[-b'(x,Z)u(x,t;Z)\Phi_m(Z)] \tag{5}$$

holds for all m. In the discrete setting, we can only take a finite number of basis elements, so let N represent the number of nodes that we are considering. Then the galerkin approximation of u and b in terms of Z is given by

$$u_N(x,t;Z) = \sum_{m=1}^{N+1} \hat{u}_m(x,t)\Phi_m(Z)$$
$$b_N(x;Z) = \sum_{m=1}^{N+1} \hat{b}(x)\Phi_m(Z)$$

If we fix t, then let $u_{N,j} = u_N(x_j,t;Z)$ where the right hand side is the cell average over $[x_{j-1/2}, x_{j+1/2}]$. We take $u_{j+1/2} \approx u_j$ and similarly for the other components to get that with (**??**) we have that scheme

$$\partial_t u_{N,j} + \Big(\frac{f(u_{N,j}) - f(u_{N,j-1})}{\Delta x}\Big) = -\Big(\frac{b_{N,j} - b_{N,j-1}}{\Delta x}\Big)\Big(\frac{u_{N,j} + u_{N,j-1}}{2}\Big)$$

If we then enforce the condition (**??**), we derive the following condition on the scheme we construct

$$\mathbb{E}\Big[\Big(\partial_t u_{N,j} + \Big(\frac{f(u_{N,j}) - f(u_{N,j-1})}{\Delta x}\Big)\Big)\Phi_m(Z)\Big] = -\mathbb{E}\Big[\Big(\frac{b_{N,j} - b_{N,j-1}}{\Delta x}\Big)\Big(\frac{u_{N,j} + u_{N,j-1}}{2}\Big)\Phi_m(Z)\Big] \tag{6}$$

If we set $\mathbf{u}_j = (\hat{u}_{1,j}, \dots, \hat{u}_{N+1,j})^T$, define E to be the tensor with components $e_{klm} = \mathbb{E}[\Phi_k\Phi_l\Phi_m]$ and D to be the tensor with components $d_{klmnp} = \mathbb{E}[\Phi_k\Phi_l\Phi_m\Phi_n\Phi_p]$ and where $1 \le k,l,m,n,p \le N+1$. We then define matrices A_j, S, B to satisfy

$$A_{kl,j} = \sum_{m=1}^{N+1} e_{klm}\hat{u}_{m,j}$$
$$B_{kl,j} = \sum_{m=1}^{N+1} e_{klm}\hat{b}_{m,j}$$
$$S_{kl,j} = \sum_{m,n,p=1}^{N+1} d_{klmnp}\hat{u}_{m,j}\hat{u}_{n,j}\hat{u}_{p,j}$$

If $f(u) = \frac{u^2}{2}$, then a scheme which satisfies (**??**) and (**??**) is

$$\partial_t \mathbf{u}_j + \frac{A_j\mathbf{u}_j - A_{j-1}\mathbf{u}_{j-1}}{2\Delta x} = -\frac{(B_j - B_{j-1})(\mathbf{u}_j + \mathbf{u}_{j-1})}{2\Delta x}$$

If $f(u) = \frac{u^4}{4}$, then a scheme which satisfies the stochastic condition and the well-balancing condition is

$$\partial_t \mathbf{u}_j + \frac{s_j \mathbf{u}_j - S_{j-1}\mathbf{u}_{j-1}}{4\Delta x} = -\frac{(B_j - B_{j-1})(\mathbf{u}_j + \mathbf{u}_{j-1})}{2\Delta x}$$

4 Results

5 Conclusion