A Well-Balanced Stochastic Galerkin Method for Scalar Hyperbolic Laws with Random Forcing.

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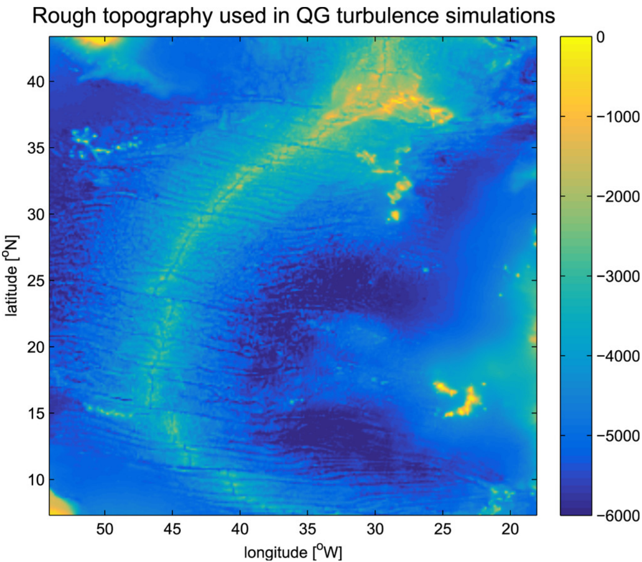


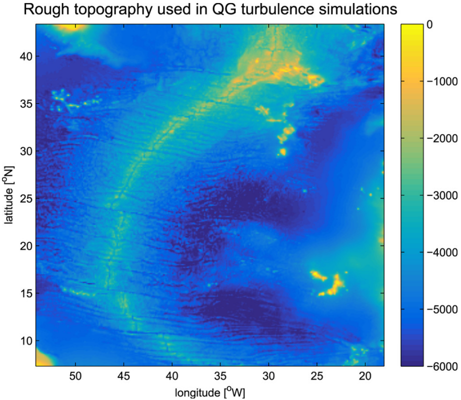
David S. Trossman et al. "The Role of Rough Topography in Mediating Impacts of Bottom Drag in Eddying Ocean Circulation Models". In: *Journal of Physical Oceanography* 47.8 (Aug. 2017), pp. 1941–1959. DOI: 10.1175/jpo-d-16-0229.1

Difficulties in real-world simulation

If the given bottom topography given was our source term, we would have:

- Uncertainty in measurement
- Low regularity

The former can have a non-negligible effect on simulation and the latter can pose difficulties in capturing the true steady state solution.



Idea: Combine previous ideas for stochastic Galerkin via generalized polynomial chaos and well-balanced interface methods.

Shi Jin, Dongbin Xiu, and Xueyu Zhu. "A Well-Balanced Stochastic Galerkin Method for Scalar Hyperbolic Balance Laws with Random Inputs". In: *Journal of Scientific Computing* 67.3 (Nov. 2015), pp. 1198–1218. DOI: 10.1007/s10915-015-0124-2

Table of Contents

Introduction to stochastic Galerkin via gPC

Consider a general SDE with random inputs:

$$\partial_t u = \mathcal{L}(t, x, u, z; b(x, z)) \tag{1}$$

where, for convenience, let $z \in I_z \subset \mathbb{R}$ parameterize the random input.

Introduction to stochastic Galerkin via gPC

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where, for convenience, let $z \in I_z \subset \mathbb{R}$ parameterize the random input. We seek to approximate u with the gPC expansion:

$$u(x, t, z) = u_M(x, t, z) = \sum_{m=1}^{M} \hat{u}_m(t, x)\Phi_m(z)$$

$$b(x, z) = u_M(x, z) = \sum_{m=1}^{M} \hat{b}_m(t, x)\Phi_m(z)$$

where $\{\Phi_m(z)\} \subset \mathbb{P}_M$ are the orthonormal polynomials satisfying

$$\int \Phi_i(z)\Phi_j(z)\rho(z)\,\mathrm{d}z = \delta_{ij}, \quad 1 \leq i, j \leq M$$

Naive Stochastic Galerkin for Burgers

Let's focus on Burgers' equation with a random source term

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = -b'(x, z) u \tag{2}$$

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Consider the uniform discretization:

- $(x_{j+1/2})_{j=1}^{N_x}$ with $\Delta x = x_{j+1/2} - x_{j-1/2}$;
- $(t_n)_{n=1}^{N_t}$ with $\Delta t = t^n - t^{n-1}$.

with:

$$u_j^n = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t^n) \, \mathrm{d}x$$

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$$u_j^n = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t^n) \, \mathrm{d}x$$

A natural way to discretize Burgers is:

$$\partial_t u_j + \frac{u_j^2 - u_{j-1}^2}{2\Delta x} = -\frac{b_j - b_{j-1}}{\Delta x} u_j \tag{3}$$

where we've applied an upwind flux assuming $u > 0$ for brevity.

Naive Stochastic Galerkin for Burgers

Multiplying both sides of the discretization by Φ_j and taking an expectation:

$$\mathbb{E}\left[\left(\frac{\partial}{\partial t}u_{M,j} + \frac{u_{M,j}^2 - u_{N,j-1}^2}{2\Delta x}\right)\Phi_m(z)\right] = -\mathbb{E}\left[\frac{b_{M,j} - b_{M,j-1}}{\Delta x}u_{M,j}\Phi_m(z)\right]$$

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Finally, substitute in their gPC expansions and and use orthonormality:

Naive Stochastic Galerkin for Burgers

$$\partial_t \hat{\mathbf{u}}_j + \frac{\mathbf{A}_j \hat{\mathbf{u}}_j - \mathbf{A}_{j-1} \hat{\mathbf{u}}_{j-1}}{2\Delta x} = -\frac{(\mathbf{B}_j - \mathbf{B}_{j-1})}{2\Delta x} \hat{\mathbf{u}}_j \tag{4}$$

where:

$$\hat{\mathbf{u}} = (\hat{u}_1, \ldots, \hat{u}_M)^{\mathrm{T}} \qquad \hat{\mathbf{b}} = (\hat{b}_1, \ldots, \hat{b}_M)^{\mathrm{T}}$$

$$[\mathbf{A}_j]_{mn} = \mathbb{E}[u_{N,j} \Phi_m \Phi_n] = \sum_{k=1}^{M} \hat{u}_{k,j} e_{kmn}$$

$$[\mathbf{B}_j]_{mn} = \mathbb{E}[b_{N,j} \Phi_m \Phi_n] = \sum_{k=1}^{M} \hat{b}_{k,j} e_{kmn}$$

with $e_{kmn} = \mathbb{E}[\Phi_k \Phi_m \Phi_n]$.

Test Problems

Impose the following initial/boundary conditions:

$$\begin{cases} u(x,0) = 0, & \forall x > 0 \\ u(0,t) = 2, & \forall t > 0 \end{cases}$$

With the following bottom functions:

$$b_1(x,z) = \begin{cases} (2+z)\cos(\pi x), & 4.5 \le x \le 5.5 \\ 0, & \text{otherwise} \end{cases}$$
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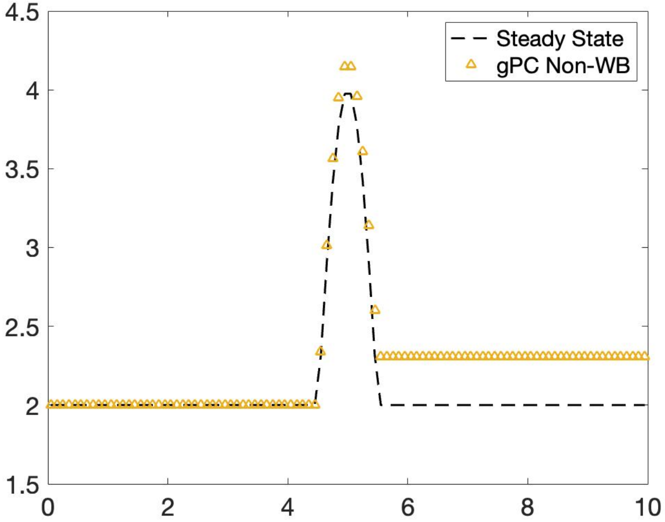
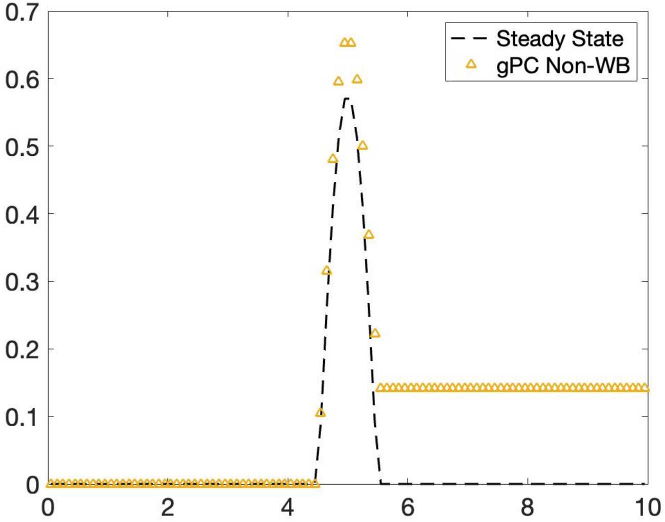
Note:, b_1 is continuous and b_2 is discontinuous.

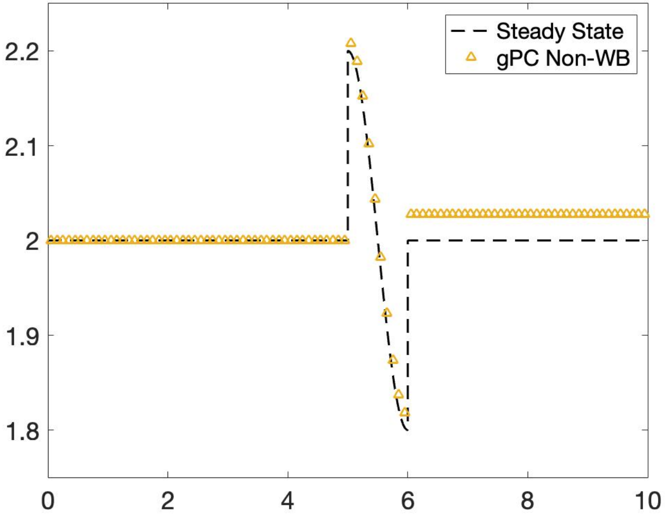
Mean: Why well-balanced matters



Standard Deviation: Why well-balanced matters



Mean: Discontinuous bottom topography



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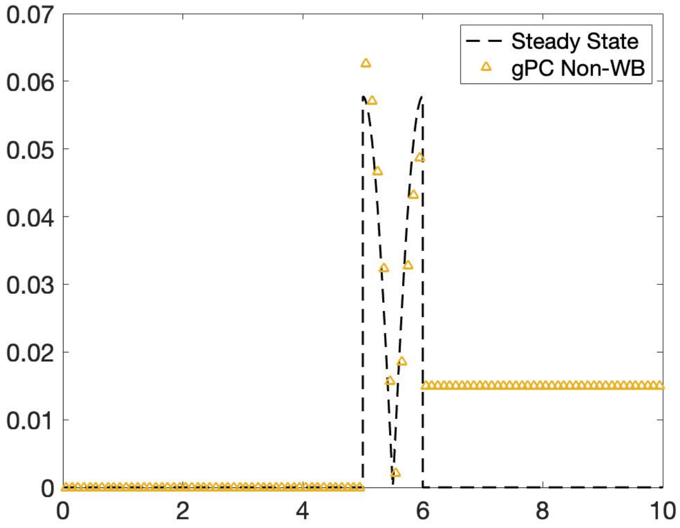

Table of Contents

Determininistic Model Problem

Given the scalar conservation law with source term:

$$\partial_t u + \partial_x f(u) = -b'(x)q(u). \tag{5}$$

we have the steady-state equation

$$\partial_x f(u) + -b'(x)q(u) = 0. \tag{6}$$

which, supposing a smooth solution, can be written into the form:

$$\begin{cases} D(x) + b(x) = \text{constant.} \\ D(x) = \int_0^{u(x)} \frac{f'(s)}{q(s)} \, \mathrm{d}s \end{cases} \tag{7}$$

which we call the steady-state condition.

A Well-Balanced Numerical Scheme

Definition

A numerical scheme is called well-balanced (WB) if it can preserve the steady-state condition (7) either exactly, or formally with at least second order accuracy.

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Construct the semi-discrete interface method proposed in [Jin01]:

$$\partial_t u_j + \frac{f_{j+1/2} - f_{j-1/2}}{\Delta x} = -\frac{b_{j+1/2} - b_{j-1/2}}{\Delta x} \underbrace{\frac{q_{j+1/2} + q_{j-1/2}}{2}}_{\text{Critical Difference}} \tag{8}$$

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or if $D(x)$ is monotone: (Recall:$D = \int_0^{u(x)} f'(s)/q(s)\,\mathrm{d}s$)

$$\partial_t u_j + \frac{f_{j+1/2} - f_{j-1/2}}{\Delta x} = -\frac{b_{j+1/2} - b_{j-1/2}}{\Delta x} \frac{f_{j+1/2} + f_{j-1/2}}{D_{j+1/2} - D_{j-1/2}} \tag{9}$$

Hence, considering the steady state solution:

$$\frac{f_{j+1/2} - f_{j-1/2}}{\Delta x} + \frac{b_{j+1/2} - b_{j-1/2}}{\Delta x} \frac{f_{j+1/2} + f_{j-1/2}}{D_{j+1/2} - D_{j-1/2}} = 0$$

Hence, considering the steady state solution:

$$
\begin{aligned}
&\frac{f_{j+1/2} - f_{j-1/2}}{\Delta x} + \frac{b_{j+1/2} - b_{j-1/2}}{\Delta x} \frac{f_{j+1/2} + f_{j-1/2}}{D_{j+1/2} - D_{j-1/2}} = 0 \\
\implies &D_{j+1/2} - D_{j-1/2} + b_{j+1/2} - b_{j-1/2} = 0
\end{aligned}
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&\frac{f_{j+1/2} - f_{j-1/2}}{\Delta x} + \frac{b_{j+1/2} - b_{j-1/2}}{\Delta x} \frac{f_{j+1/2} + f_{j-1/2}}{D_{j+1/2} - D_{j-1/2}} = 0 \\
\implies\ & D_{j+1/2} - D_{j-1/2} + b_{j+1/2} - b_{j-1/2} = 0 \\
\implies\ & D_{j+1/2} + b_{j+1/2} = \text{constant}
\end{aligned}
$$

The steady state condition (7) is preserved *exactly* at the cell interface!

Stochastic Well-Balanced Schemes

Stochastic WB

Let $\mathcal{S}$ be a numerical scheme for (10), which results in a solution $v(z) \in V_z$, where V_z is a finite dimensional linear function space.

- A numerical scheme $\mathcal{S}$ is called *strongly well-balanced* if it preserves the steady state condition either exactly or formally with at least second order accuracy for almost every z.
- It is *weakly well-balanced* if it satisfies the weak form of the steady state (in the sense of Galerkin) with at least second order accuracy.

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Claim: The previous interface method will be well-balanced for Burgers.

Table of Contents

Burgers' equation

Consider Burgers' equation with a random source term

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We apply the interface method to find

$$\partial_t u_j + \frac{u_j^2 - u_{j-1}^2}{2\Delta x} = -\frac{b_j - b_{j-1}}{\Delta x} \frac{u_j + u_{j-1}}{2} \tag{11}$$

where we've applied an upwind flux assuming $u > 0$ for brevity.

sWB Burgers

Multiplying both sides of the discretization by Φ_j and taking an expectation:

$$\mathbb{E}\left[\left(\frac{\partial}{\partial t}u_{N,j} + \frac{u_{N,j}^2 - u_{N,j-1}^2}{2\Delta x}\right)\Phi_m(z)\right]$$

$$= -\mathbb{E}\left[\left(\frac{b_{N,j} - b_{N,j-1}}{\Delta x}\right)\left(\frac{u_{N,j} - u_{N,j-1}}{2}\right)\Phi_m(z)\right]$$

Finally, substitute in their gPC expansions and and use orthonormality:

sWB Burgers Full Scheme

$$\partial_t \hat{\mathbf{u}}_j + \frac{\mathbf{A}_j \hat{\mathbf{u}}_j - \mathbf{A}_{j-1} \hat{\mathbf{u}}_{j-1}}{2\Delta x} = -\frac{(\mathbf{B}_j - \mathbf{B}_{j-1})(\hat{\mathbf{u}}_j + \hat{\mathbf{u}}_{j-1})}{2\Delta x} \tag{12}$$

where:

$$\hat{\mathbf{u}} = (\hat{u}_1, \dots, \hat{u}_M)^{\mathrm{T}} \qquad \hat{\mathbf{b}} = (\hat{b}_1, \dots, \hat{b}_M)^{\mathrm{T}}$$

$$[\mathbf{A}_j]_{mn} = [\mathbf{A}(\hat{\mathbf{u}}_j)] = \mathbb{E}[u_{M,j} \Phi_m \Phi_n] = \sum_{k=1}^{M} \hat{u}_{k,j} e_{kmn}$$

$$[\mathbf{B}_j]_{mn} = [\mathbf{A}(\hat{\mathbf{b}}_j)] = \mathbb{E}[b_{M,j} \Phi_m \Phi_n] = \sum_{k=1}^{M} \hat{b}_{k,j} e_{kmn}$$

with $e_{kmn} = \mathbb{E}[\Phi_k \Phi_m \Phi_n]$.

The steady state governing equation is:

$$\partial_x(u^2/2) + b'(x,z)u = 0$$

which, under it's gPC approximation goes to:

$$\mathbb{E}\left[(u_M^2/2)_x \Phi_m\right] = -\mathbb{E}\left[(b_M)_x u_M \Phi_m\right] \implies \frac{\partial}{\partial x}(\mathbf{Au}) + \mathbf{B}'(x)\mathbf{u} = 0$$

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Our scheme:

$$\partial_t \hat{\mathbf{u}}_j + \frac{\mathbf{A}_j\hat{\mathbf{u}}_j - \mathbf{A}_{j-1}\hat{\mathbf{u}}_{j-1}}{2\Delta x} = -\frac{(\mathbf{B}_j - \mathbf{B}_{j-1})(\hat{\mathbf{u}}_j + \hat{\mathbf{u}}_{j-1})}{2\Delta x}$$

reduces to

$$\frac{\mathbf{A}_j\hat{\mathbf{u}}_j - \mathbf{A}_{j-1}\hat{\mathbf{u}}_{j-1}}{2\Delta x} + \frac{(\mathbf{B}_j - \mathbf{B}_{j-1})(\hat{\mathbf{u}}_j + \hat{\mathbf{u}}_{j-1})}{2\Delta x} = 0$$

This is sWB via the same procedure as the deterministic system.

Table of Contents

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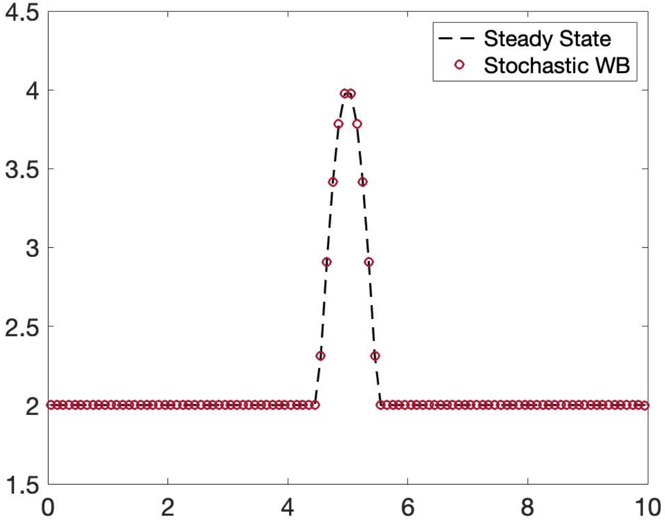

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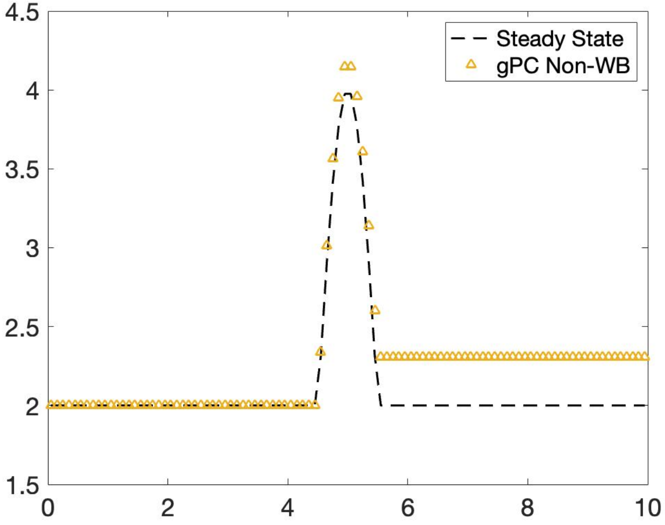
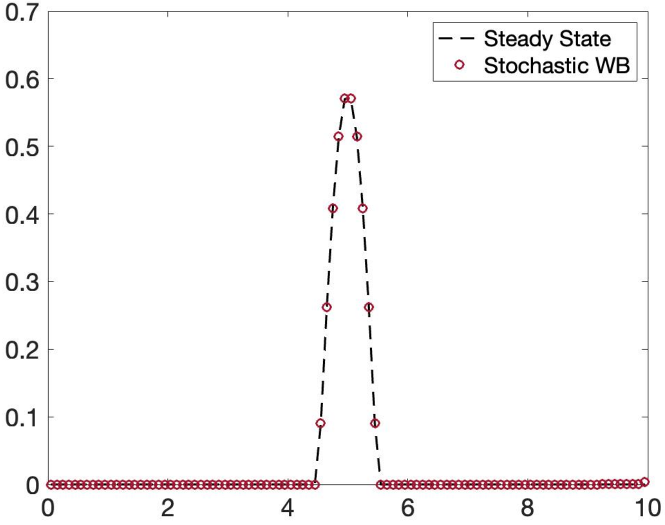


Standard Deviation: Why well-balanced matters



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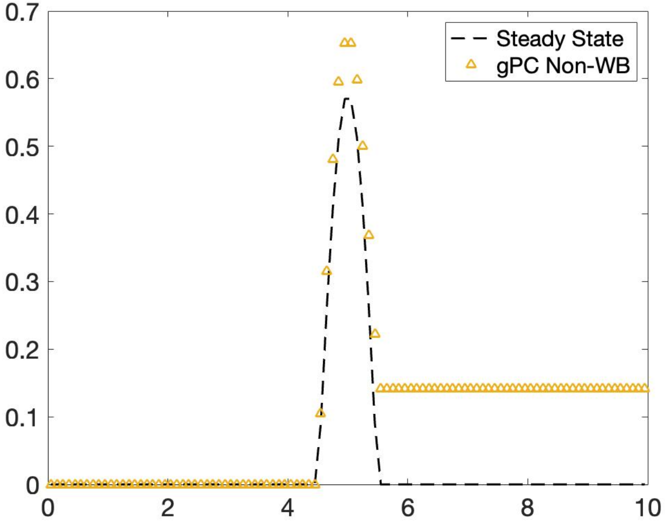
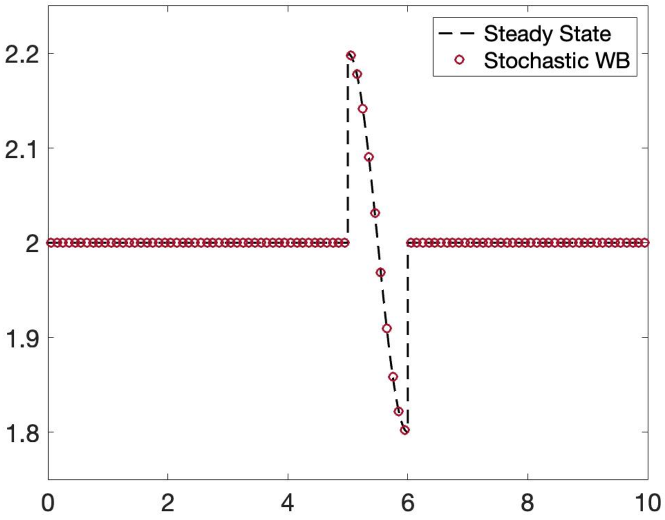


Mean: Discontinuous bottom topography



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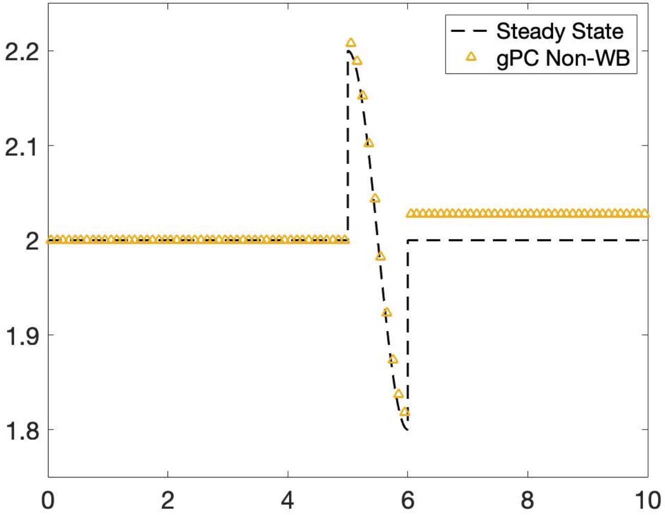

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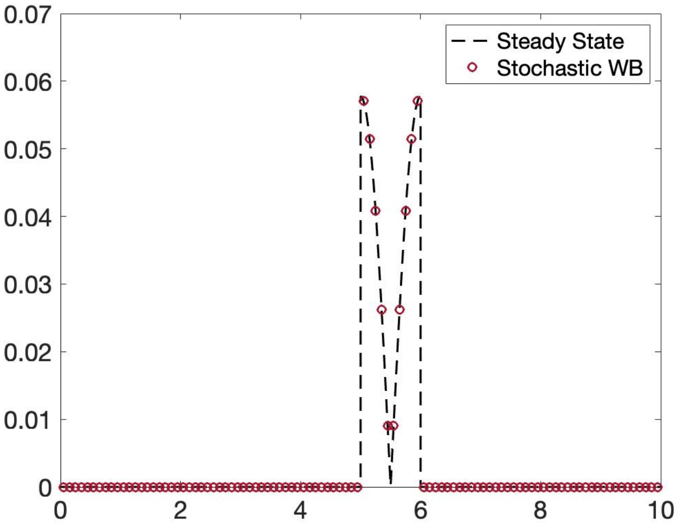

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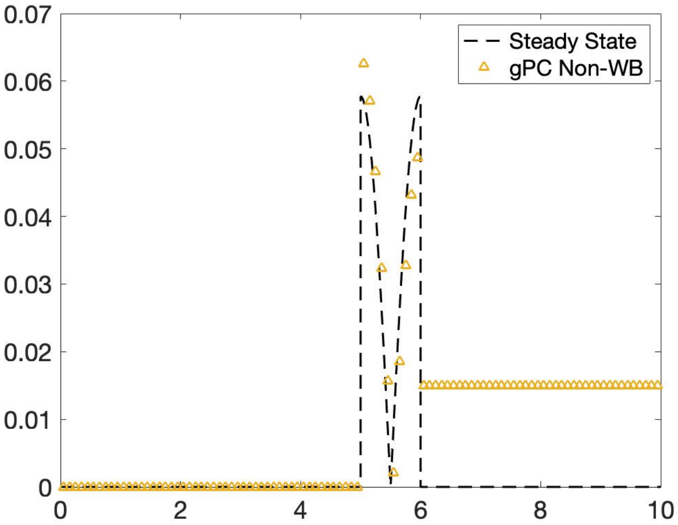


An alternative flux

Likewise, with flux $f(u) = u^4/4$ we can derive a similar method:

$$\partial_t \hat{\mathbf{u}}_j + \frac{\mathbf{S}_j \hat{\mathbf{u}}_j - \mathbf{S}_{j-1}\hat{\mathbf{u}}_{j-1}}{4\Delta x} = -\frac{(\mathbf{B}_j - \mathbf{B}_{j-1})(\hat{\mathbf{u}}_j + \hat{\mathbf{u}}_{j-1})}{2\Delta x} \tag{13}$$

where:

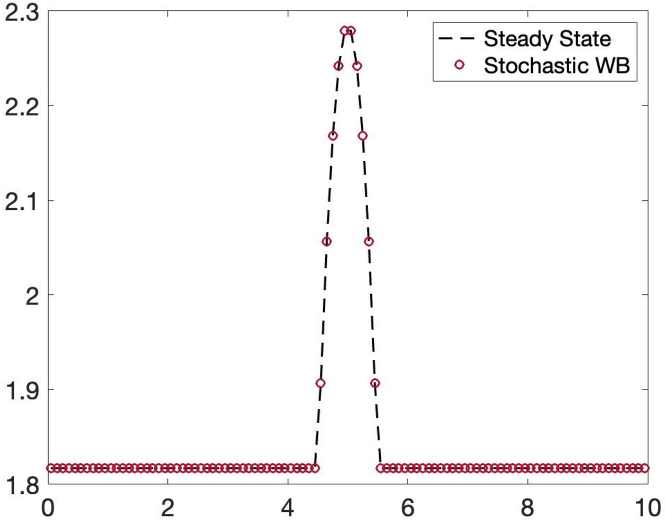
$$\hat{\mathbf{u}} = (\hat{u}_1, \ldots, \hat{u}_M)^{\mathrm{T}} \qquad \hat{\mathbf{b}} = (\hat{b}_1, \ldots, \hat{b}_M)^{\mathrm{T}}$$

$$[\mathbf{S}_j]_{mn} = \mathbb{E}[u_{N,j}^3 \Phi_m \Phi_n] = \sum_{p,q,r}^{M} \hat{u}_{p,j}\hat{u}_{q,j}\hat{u}_{r,j} d_{pqrmn}$$

$$[\mathbf{B}_j]_{mn} = \mathbb{E}[b_{N,j}\Phi_m\Phi_n] = \sum_{k=1}^{M} \hat{b}_{k,j} e_{kmn}$$

with $e_{kmn} = \mathbb{E}[\Phi_k \Phi_m \Phi_n]$ and $d_{pqrmn} = \mathbb{E}[\Phi_p \Phi_q \Phi_r \Phi_m \Phi_n]$. (Agony)

Mean: $f(u) = u^4/4$



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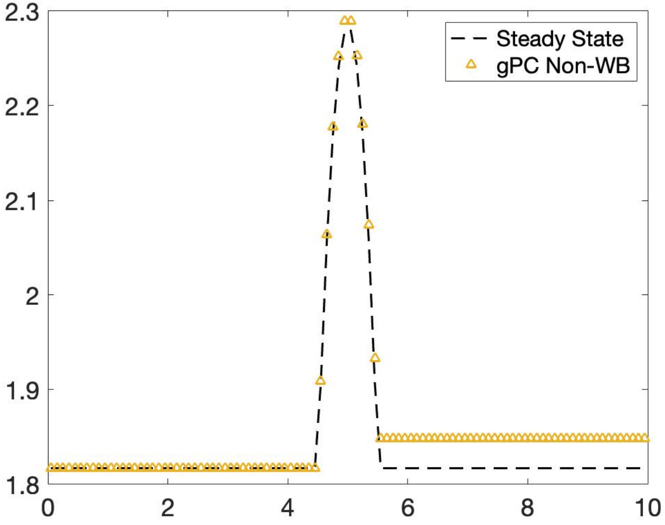

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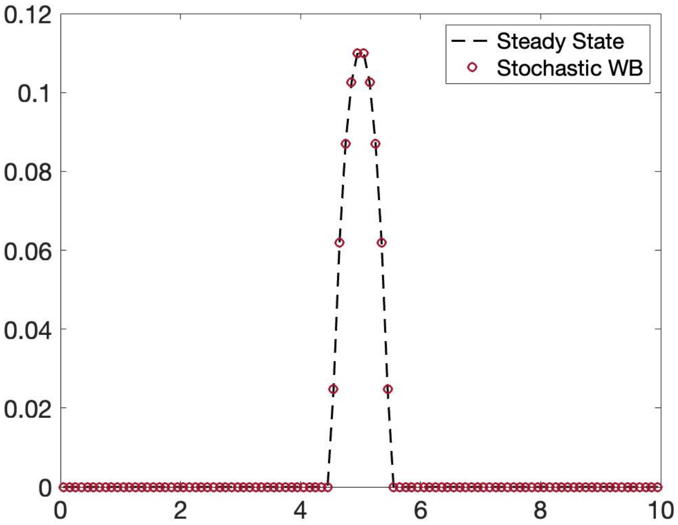

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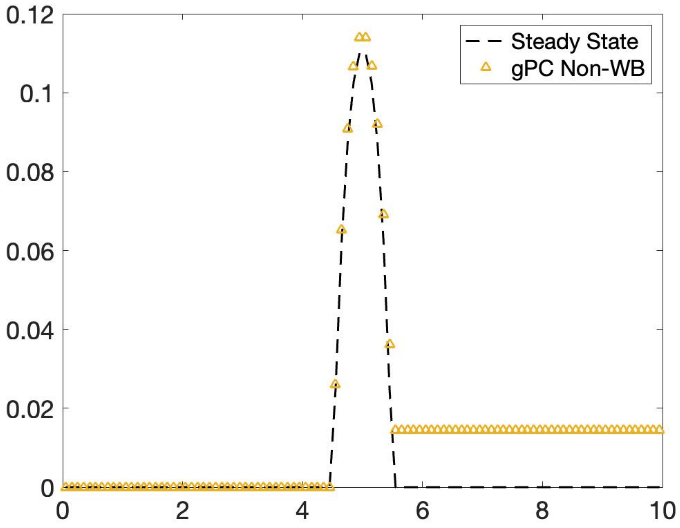

Table of Contents

Numerical Considerations

$$\partial_t \hat{\mathbf{u}}_j + \frac{\mathbf{A}_j \hat{\mathbf{u}}_j - \mathbf{A}_{j-1} \hat{\mathbf{u}}_{j-1}}{2\Delta x} = -\frac{(\mathbf{B}_j - \mathbf{B}_{j-1})(\hat{\mathbf{u}}_j + \hat{\mathbf{u}}_{j-1})}{2\Delta x}$$

There are a couple of points to note regarding this scheme:

- $\hat{u}$ has $N_x \times M$ entries;
- $\mathbf{B}$ is constant in time $\implies$ can be computed before time evolution
- $\mathbf{A}$ depends on time $\implies$ must be computed per time step.
- $\mathbf{A}_j$ and $\mathbf{B}_j$ are symmetric.

Based on profiling, the computation of $\mathbf{A}$ dominates runtime. In the $f(u) = u^4/4$ scenario, $\mathbf{S}$ is extremely costly in the same way.

Table of Contents

Discussion

In summary, we've implemented a method which:

- Handles a random input in a physically derived forcing term
- Can be generalized to different forms of fluxes other than Burgers
- Is well-balanced in a stochastic sense
- Handles discontinuous solutions resulting from discontinuous forcing.

References

[Jin01] Shi Jin. "A steady-state capturing method for hyperbolic systems with geometrical source terms". In: *ESAIM: Mathematical Modelling and Numerical Analysis* 35.4 (July 2001), pp. 631–645. DOI: 10.1051/m2an:2001130.

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[Tro+17] David S. Trossman et al. "The Role of Rough Topography in Mediating Impacts of Bottom Drag in Eddying Ocean Circulation Models". In: *Journal of Physical Oceanography* 47.8 (Aug. 2017), pp. 1941–1959. DOI: 10.1175/jpo-d-16-0229.1.