An Investigation into Parareal

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May 17, 2019

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Now down to the main point, we would like to analyze and confirm the convergence of the Parareal method. First we derive a theoretical result, and then confirm it through numerical expirements.

6.1 Theoretical Convergence

Suppose we have the following ordinary differential equation:

$$\begin{cases} u' = f(t,u), \quad t > 0 \\ u(0) = u_0 \end{cases}$$

In addition suppose we have the course operator $\mathcal{G}(t^{n+1}, t^n, u^n)$ and the fine operator $\mathcal{F}(t^{n+1}, t^n, u^n)$ with the following properties:

1. On $\mathcal{G}(t^{n+1}, t^n, u^n)$:
   - Suppose this operator has order m.
   - Suppose it's Lipschitz in the initial condition:

     $$\left\|\mathcal{G}(t^{n+1}, t^n, u) - \mathcal{G}(t^{n+1}, t^n, v)\right\| \leq C\|u - v\|$$

     In particular we write $C = (1 + L\Delta t)$.
2. With respect to $\mathcal{F}(t^{n+1}, t^n, u^n)$, we suppose it's accurate enough to be assumed to be the true solution u^*. This means that if $\mathcal{G}$ is accurate with order m to the true solution, then it too will be so to $\mathcal{F}$.

Then we can prove the following theorem: [1] [2]

Theorem. *The parareal method with course operator $\mathcal{G}$ and fine operator $\mathcal{F}$ has order of accuracy mk, where k is the number of parareal iterations made.*

Proof. We proceed via induction on k. Suppose $k = 1$, then it is trivial, this is the course operator.

Now suppose for $k > n$, that we know:

$$\left\|u(t^n) - u_k^n\right\| \leq \|u_0\| \, C(\Delta t)^{mk}$$

We want to show that:

$$\left\|u(t^n) - u_{k+1}^n\right\| \leq \|u_0\| \, C(\Delta t)^{m(k+1)}$$

To proceed, recall that $\mathcal{F}$ is assumed to be a good approximation for $u(t^n)$, so we may write:

$$\begin{aligned}
\left\|u(t^n) - u_{k+1}^n\right\| &= \left\|\mathcal{F}(u(t^{n-1})) - \mathcal{G}(u_{k+1}^{n-1}) - \mathcal{F}(u_k^{n-1}) + \mathcal{G}(u_k^{n-1})\right\| \\
&= \left\|\mathcal{G}(u(t^{n-1})) + \delta\mathcal{G}(u(t^{n-1})) - \mathcal{G}(u_{k+1}^{n-1}) - \delta\mathcal{G}(u_k^{n-1})\right\| \\
&\leq \left\|\mathcal{G}(u(t^{n-1})) - \mathcal{G}(u_{k+1}^{n-1})\right\| + \left\|\delta\mathcal{G}(u(t^{n-1})) - \delta\mathcal{G}(u_k^{n-1})\right\| \\
&\leq (1 + L\Delta t)\left\|u(t^{n-1}) - u_{k+1}^{n-1}\right\| + C(\Delta t)^{m+1}\left\|u(t^{n-1}) - u_k^{n-1}\right\| \\
&\leq (1 + L\Delta t)\left\|u(t^{n-1}) - u_{k+1}^{n-1}\right\| + C(\Delta t)^{m+1}(\Delta t)^{mk}\|u_0\| \\
&\leq (1 + L\Delta t)\left\|u(t^{n-1}) - u_{k+1}^{n-1}\right\| + C(\Delta t)^{m(k+1)+1}\|u_0\|
\end{aligned}$$

Therefore, we can say that $\left\|u(t^n) - u_{k+1}^n\right\| \leq \mathcal{O}(\Delta t^{m(k+1)})$. □

6.2 Numerical Results and Validation

7 Conclusion

References

[1] Guillaume Bal. *On the Convergence and the Stability of the Parareal Algorithm to solve Partial Differential Equations*. Columbia University, APAM

[2] Scott Fields. *Parareal Methods*. `http://www.cfm.brown.edu/people/jansh/page5/page10/page40/assets/Field_Talk.pdf`