An Investigation into Parareal

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May 19, 2019

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With regards to the stability analysis, we aim to try and prove some results on the ”stability function” $R(z)$ as seen in Levecue's [3] chapter five through eight and in Staff [5]. We heavily follow the arguments in these sources to derive such results.

Suppose we have the following ordinary differential equation:

$$\begin{cases} u' = \mu u, \quad \mu < 0, t > 0 \\ u(0) = u_0 \end{cases} \tag{1}$$

We would like to analyze the stability of parareal on this system with a course operator $\mathcal{G}$ and a fine operator $\mathcal{F}$.

5.1 A stability function | Inspired by Euler Methods

First we restrict ourselves to the infamous explicit and implicit Euler, with the hope of deriving some stability criteria for them. Recall, that the explicit and implicit euler scheme is:

$$u_{n+1} = u_n + \mu \Delta t u_n = (1 + \mu \Delta t) u_n \tag{Explicit Euler}$$

$$u_{n+1} = u_n + \mu \Delta t u_{n+1} = (1 - \mu \Delta t)^{-1} u_n \tag{Implicit Euler}$$

It's easy to see, after unrolling the recurrence on u_n, that these translate to $u_{n+1} = (1+\mu\Delta t)^n u_0$ or $(1-\mu\Delta t)^{-n}$ for explicit and implicit Euler respectively. Calling $R_e(z) = 1+z$ and $R_i(z) = (1-z)^{-1}$, we can analyze the stability of the method through these, specifically we desire that $\left|R(z)\right| \leq 1$, so that $R(z)^n$ doesn't blow up in n.

With respect to Parareal, let's try and write our iteration in the form $\lambda_n^k = H(n,k)\lambda_0$. Applying the explicit Euler iteration to **??**, we see that it goes to:

$$\begin{aligned}\lambda_{n+1}^{k+1} &= \mathcal{G}(t^{n+1}, t^n, \lambda_n^{k+1}) + \mathcal{F}(t^{n+1}, t^n, \lambda_n^k) - \mathcal{G}(t^{n+1}, t^n, \lambda_n^k) \\ &= R_e(\mu\Delta t)\lambda_n^{k+1} + R_e(\mu\delta t)^s\lambda_n^k - R_e(\mu\Delta t)\lambda_n^k\end{aligned}$$

where we say $s = \Delta t/\delta t$, i.e. how many fine steps are needed to make one course step. Combining like terms results in:

$$\lambda_{n+1}^{k+1} = R_e(\mu\Delta t)\lambda_n^{k+1} + \left(R_e(\mu\delta t)^s - R_e(\mu\Delta t)\right)\lambda_n^k$$

Now, if we were to note the terms on λ, we note that we have something very similar to the recurrence relation on combinations $\binom{n}{k} = \binom{n}{k-1} + \binom{n-1}{k-1}$. Exploiting that relationship, we can unroll our recursion into:

$$\lambda_{n+1}^{k+1} = \left(\sum_{i=0}^{k}\binom{n}{i}\left[R_e(\mu\delta t)^s - R_e(\mu\Delta t)\right]^i R_e(\mu\Delta t)^{n-i}\right)\lambda_0 = H_e(\mu, n, k, \delta t, \Delta t)\lambda_0$$

In doing this, we notice that any method for which we could write as $u_{n+1} = R(z)u_n$ will have a very similar stability region, with $R_e \to R$. See figure 1 for the regions of stability under the Forward Euler method.

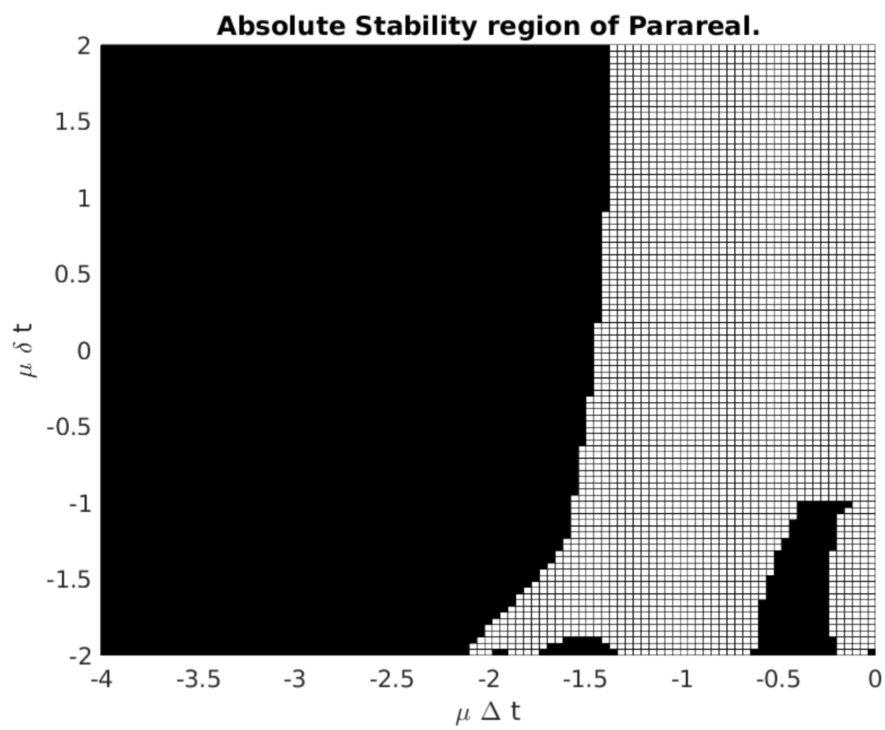

Figure 1

6 Convergence Analysis

Now down to the main point, we would like to analyze and confirm the convergence of the Parareal method. First we derive a theoretical result, and then confirm it through numerical expirements.

6.1 Theoretical Convergence

Suppose we have the following ordinary differential equation:

$$\begin{cases} u' = f(t,u), \quad t > 0 \\ u(0) = u_0 \end{cases} \tag{2}$$

In addition suppose we have the course operator $\mathcal{G}(t^{n+1}, t^n, u^n)$ and the fine operator $\mathcal{F}(t^{n+1}, t^n, u^n)$ with the following properties:

1. On $\mathcal{G}(t^{n+1}, t^n, u^n)$:
   - Suppose this operator has order m.
   - Suppose it's Lipschitz in the initial condition:

   $$\left\|\mathcal{G}(t^{n+1}, t^n, u) - \mathcal{G}(t^{n+1}, t^n, v)\right\| \leq C\|u - v\|$$

   In particular we write $C = (1 + L\Delta t)$.
2. With respect to $\mathcal{F}(t^{n+1}, t^n, u^n)$, we suppose it's accurate enough to be assumed to be the true solution u^*. This means that if $\mathcal{G}$ is accurate with order m to the true solution, then it too will be so to $\mathcal{F}$.

Then we can prove the following theorem:

Theorem. *The parareal method with course operator $\mathcal{G}$ and fine operator $\mathcal{F}$ has order of accuracy mk, where $k-1$ is the number of parareal iterations made. [1] [2]*

Proof. We proceed via induction on k and n. Suppose $k = 1$, then it is trivial, this is the course operator, and for $n = 0$, this is the initial condition which we know to any accuracy.

Now suppose for $k, n > 1$, that we know:

$$\left\|u(t^n) - u_k^n\right\| \leq \|u_0\| \, C(\Delta t)^{mk},$$

We want to show that:

$$\left\|u(t^n) - u_{k+1}^n\right\| \leq \|u_0\| \, C(\Delta t)^{m(k+1)}$$

To proceed, recall that $\mathcal{F}$ is assumed to be a good approximation for $u(t^n)$, so we may write:

$$\begin{aligned}
\left\|u(t^n) - u_{k+1}^n\right\| &= \left\|\mathcal{F}(u(t^{n-1})) - \mathcal{G}(u_{k+1}^{n-1}) - \mathcal{F}(u_k^{n-1}) + \mathcal{G}(u_k^{n-1})\right\| \\
&= \left\|\mathcal{G}(u(t^{n-1})) + \delta\mathcal{G}(u(t^{n-1})) - \mathcal{G}(u_{k+1}^{n-1}) - \delta\mathcal{G}(u_k^{n-1})\right\| \\
&\leq \left\|\mathcal{G}(u(t^{n-1})) - \mathcal{G}(u_{k+1}^{n-1})\right\| + \left\|\delta\mathcal{G}(u(t^{n-1})) - \delta\mathcal{G}(u_k^{n-1})\right\| \\
&\leq (1 + L\Delta t)\left\|u(t^{n-1}) - u_{k+1}^{n-1}\right\| + C(\Delta t)^{m+1}\left\|u(t^{n-1}) - u_k^{n-1}\right\| \\
&\leq (1 + L\Delta t)\left\|u(t^{n-1}) - u_{k+1}^{n-1}\right\| + C(\Delta t)^{m+1}(\Delta t)^{mk}\|u_0\| \\
&\leq (1 + L\Delta t)\left\|u(t^{n-1}) - u_{k+1}^{n-1}\right\| + C(\Delta t)^{m(k+1)+1}\|u_0\|
\end{aligned}$$

At this point, note that the left hand term is the approximation of u at the previous time step, which we can assume to also be of the order $m(k+1)$. Therefore, we can say that $\left\|u(t^n) - u_{k+1}^n\right\| = \mathcal{O}(\Delta t^{m(k+1)})$, completing our inductive step. □

6.2 Numerical Results and Validation

Here we seek to confirm the theoretical order derived in the last section (from [1] [2]). To see a clear example, consier the Euler methods introduced back in section **??**. Recall that the explicit Euler method is of order 1, so theoretically if we have a fine method of high enough order, we should see a method whose order is k, for an order k Parareal. And indeed, as we expect, we do indeed see such such results, see figure 2 for the numerical plots.

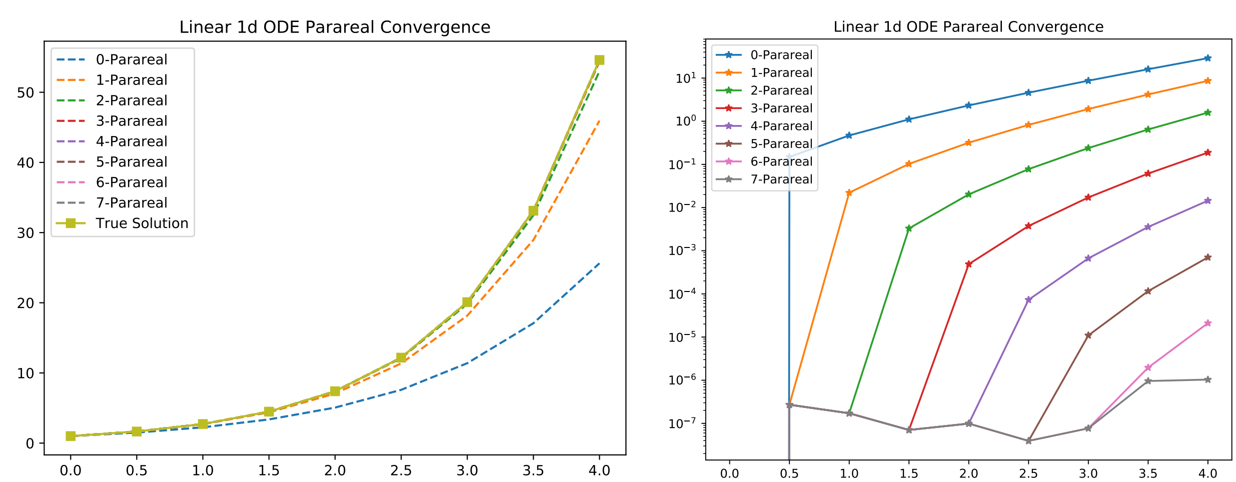

Figure 2: The figure on the left is showing the resulting solutions to the ODE $u' = u, u(0) = 1$ for $t \in [0, 4]$, and the right is showing the errors under the 1 norm. Critically, this plot reveals that our implementation of Parareal conforms to the theoreical convergence rate, which validates our method. The last iterate has strange behavior, but this is because I could not decrease the accuracy of the fine integrator (forward Euler) anymore without running out of memory.

Using this information, we might begin to consider how can we leverage this mk convergence rate. It's immediately clear, that if we want k to be as small as possible (as noted in the efficiency section), that we want m to be as large as possible. However, since m corresponds to the course integrator, we have to understand this optimization problem between efficiency and accuracy to achieve the desired order of accuracy.

7 Conclusion

References

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