Infinite-dimensional Bayesian inversion for fault slip from surface measurements

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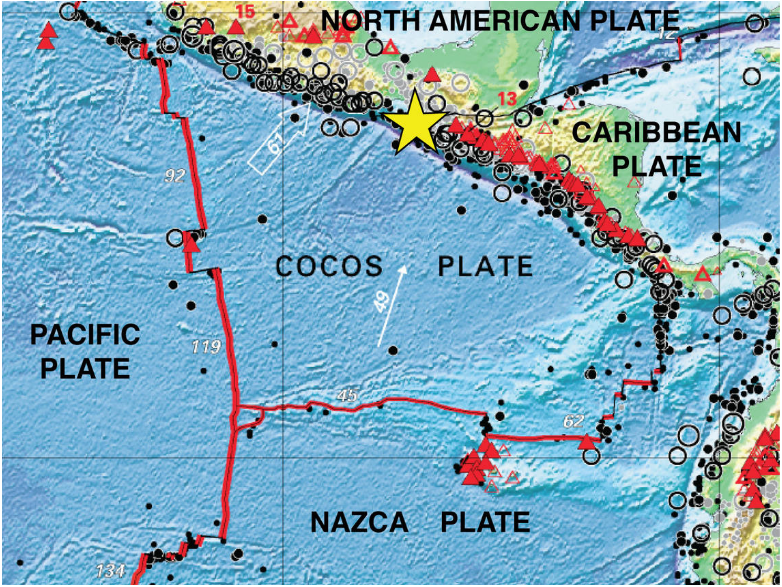
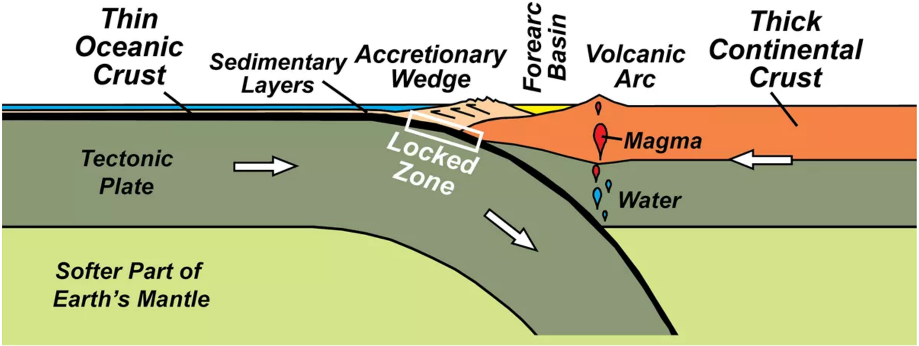


Tom Simkin et al. *This dynamic planet: World map of volcanoes, earthquakes, impact craters and plate tectonics*. Tech. rep. 2006. DOI: 10.3133/i2800



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Goals

Understand the subduction zone from collected observations.

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Do so while quantifying measurement uncertainties.

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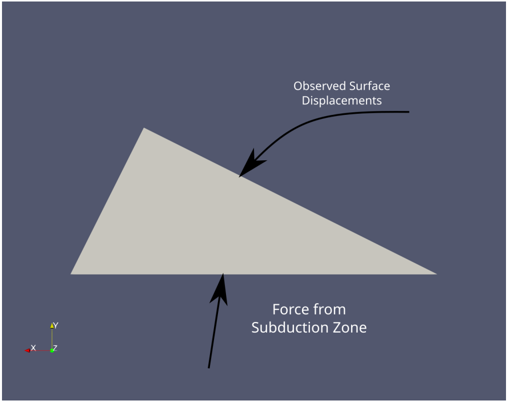
Model Assumptions



For the sake of modeling convienience, assume:

- **Governing PDE** (forward model): Linear elasticity

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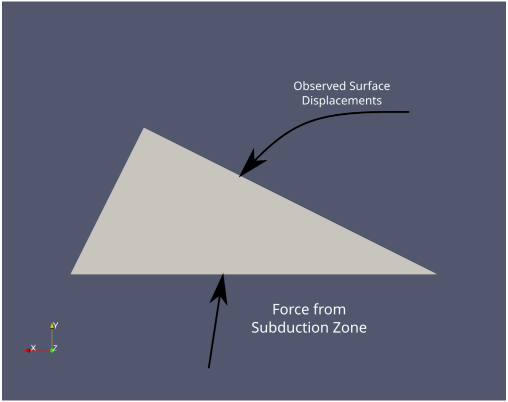

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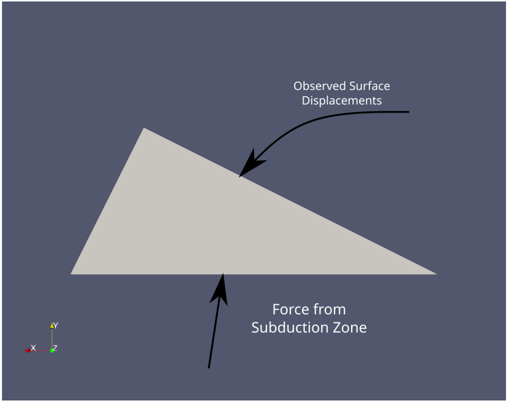

For the sake of modeling convienience, assume:

- **Governing PDE** (forward model): Linear elasticity
- **Uncertain parameter**: Displacement on fault plane
- **Inverse Problem**: Given measurements of surface deformation $\mathbf{u}^{\mathrm{obs}}$ reconstruct fault plane displacement.

Forward Model

$$-\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) = \mathbf{0} \text{ in } \Omega, \tag{1}$$

where:

- $\boldsymbol{\sigma}(\boldsymbol{u}) = \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u})$ with
  - $\mathbb{C}[\boldsymbol{\varepsilon}] = 2\mu\boldsymbol{\varepsilon} + \lambda\, \mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I}$ the fourth-order linear elasticity tensor:
  - $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}\left[\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^{\mathrm{T}}\right]$ the strain tensor.
- μ and λ are known as the Láme constants.

Kimberly Alison McCormack. *Earthquakes, groundwater and surface deformation : Exploring the poroelastic response to megathrust earthquakes*. Aug. 2018. URL: hdl.handle.net/2152/68892

Forward Model

$$-\boldsymbol{\nabla}\left[\mu(\boldsymbol{\nabla}\boldsymbol{u}+(\boldsymbol{\nabla}\boldsymbol{u})^{\mathrm{T}})+\lambda\boldsymbol{\nabla}\cdot\boldsymbol{u}\mathbf{I}\right]=\mathbf{0} \quad \text{in } \Omega, \tag{2a}$$

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n}=\mathbf{0} \quad \text{on } \Gamma_t \tag{2b}$$

$$\boldsymbol{u}+\beta\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n}=\boldsymbol{h} \quad \text{on } \Gamma_s \tag{2c}$$

$$\boldsymbol{u}\cdot\boldsymbol{n}=0 \quad \text{on } \Gamma_b \tag{2d}$$

$$\delta\mathbf{T}(\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n})+\mathbf{T}\boldsymbol{u}=\boldsymbol{m} \quad \text{on } \Gamma_b \tag{2e}$$

- $\mathbf{T}$ is the tangential operator $\mathbf{T}\boldsymbol{u}=(\mathbf{I}-\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{u}=\boldsymbol{u}-(\boldsymbol{n}^{\mathrm{T}}\boldsymbol{u})\boldsymbol{n}$.

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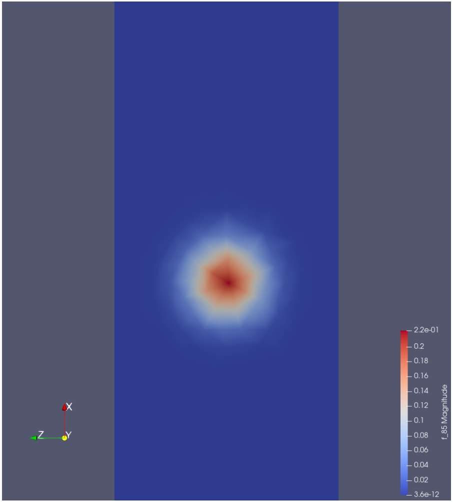
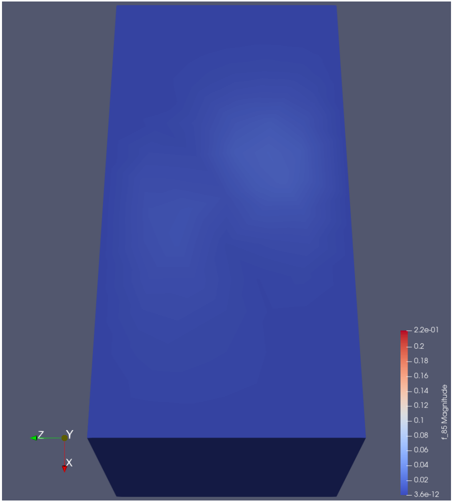
- (2e) can be understood as a regularized Dirichlet condition.

Forward Solution





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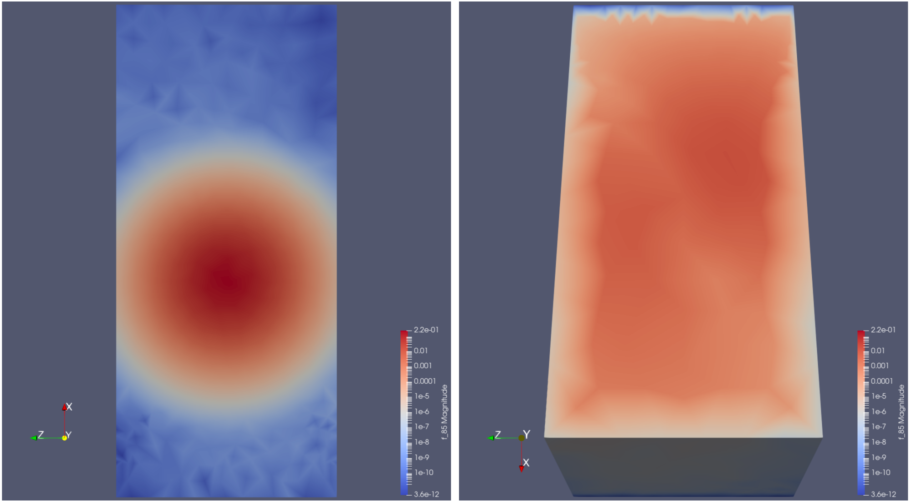

Weak Formulation

Define:

$$\boldsymbol{V} := \{\boldsymbol{u} \in H^1(\Omega)^3 : \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_b\}$$

Then the weak formulation of the forward model is given by:

$$\int_{\Gamma_s} \beta^{-1}(\boldsymbol{u} - \boldsymbol{h}) \cdot \boldsymbol{v} \,\mathrm{d}s + \int_{\Gamma_b} \delta^{-1}(\mathbf{T}\boldsymbol{u} - \boldsymbol{m}) \cdot \boldsymbol{v} \,\mathrm{d}s$$
$$+ \int_{\Omega} \boldsymbol{\varepsilon}(\boldsymbol{u}) : \mathbb{C}[\boldsymbol{\varepsilon}(\boldsymbol{v})] \,\mathrm{d}\boldsymbol{x} = 0, \quad \forall v \in \boldsymbol{V} \quad (3)$$

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The Deterministic Inverse Problem

To reconstruct the fault displacement we construct the PDE-constrained optimization problem:

$$\mathcal{J}(\boldsymbol{m}) = \frac{1}{2}\|\mathcal{B}\boldsymbol{u}(\boldsymbol{m}) - \mathbf{u}^{\mathrm{obs}}\|^2 + \frac{1}{2}\|\mathcal{A}\boldsymbol{m}\|^2$$

where $\boldsymbol{u}$ is given by the solution of the linear elasticity equation.

- $\boldsymbol{u}(\boldsymbol{m})$ is given by the forward model.
- $\mathcal{B} : (L^2(\Omega))^3 \to \mathbb{R}^N$ is an observation operator.
- $\mathbf{u}^{\mathrm{obs}} \in \mathbb{R}^N$ where N is the number of data points.

[1] In literature, called the *parameter-to-observable operator*.

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If we let $\mathcal{S}$ be the forward model operator, i.e. $\boldsymbol{u} = \mathcal{S}\boldsymbol{m}$, and let[1] $\mathcal{F} = \mathcal{B}\mathcal{S}$, then:

$$\mathcal{J}(\boldsymbol{m}) = \frac{1}{2}\|\mathcal{F}\boldsymbol{m} - \mathbf{u}^{\mathrm{obs}}\|^2 + \frac{1}{2}\|\mathcal{A}\boldsymbol{m}\|^2$$

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Bayesian Inversion in Finite Dimensions

Theorem (Bayes Theorem in Finite Dimensions)

$$\pi_{\text{post}}(\mathbf{m}|\mathbf{u}^{\text{obs}}) \propto \pi_{\text{like}}(\mathbf{u}^{\text{obs}}|\mathbf{m})\pi_{\text{prior}}(\mathbf{m})$$

- Gaussian Prior $\mathbf{m} \sim \mathcal{N}(\mathbf{m}_{\text{pr}}, \mathbf{\Gamma}_{\text{pr}})$.
- Additive Gaussian noise

$$\mathbf{u}^{\text{obs}} = \mathbf{F}\mathbf{m} + \boldsymbol{\eta}, \quad \boldsymbol{\eta} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Gamma}_{\text{noise}})$$

- Posterior is therefore Gaussian with:

$$\mathbf{m}|\mathbf{u}^{\text{obs}} \sim \mathcal{N}(\mathbf{m}_{\text{post}}, \mathbf{\Gamma}_{\text{post}})$$

where:

$$\mathbf{m}_{\text{post}} = \mathbf{\Gamma}_{\text{post}} \left(\mathbf{F}^{\text{T}} \mathbf{\Gamma}_{\text{noise}}^{-1} \mathbf{u}^{\text{obs}} + \mathbf{\Gamma}_{\text{pr}}^{-1} \mathbf{m}_{\text{pr}} \right)$$

$$\mathbf{\Gamma}_{\text{post}} = \left(\mathbf{F}^{\text{T}} \mathbf{\Gamma}_{\text{noise}}^{-1} \mathbf{F} + \mathbf{\Gamma}_{\text{pr}}^{-1} \right)^{-1}$$

Corresponds to $\mathcal{J}(\mathbf{m}) = \frac{1}{2}\|\mathcal{F}\mathbf{m} - \mathbf{u}^{\text{obs}}\|^2_{\mathbf{\Gamma}_{\text{noise}}^{-1}} + \frac{1}{2}\|\mathbf{m} - \mathbf{m}_{\text{pr}}\|^2_{\mathbf{\Gamma}_{\text{pr}}^{-1}}$.

Bayesian Inversion in Infinite Dimensions

Theorem (Bayes Theorem in Infinite Dimensions)

$$\frac{\mathrm{d}\mu_{\mathrm{post}}^{\mathbf{u}^{\mathrm{obs}}}}{\mathrm{d}\mu_{\mathrm{pr}}} \propto \pi_{\mathrm{like}}(\mathbf{u}^{\mathrm{obs}}|\boldsymbol{m})$$

- Gaussian Prior $\boldsymbol{m} \sim \mathcal{N}(\boldsymbol{m}_{\mathrm{pr}}, \mathcal{C}_0)$.
- Additive Gaussian noise

$$\mathbf{u}^{\mathrm{obs}} = \mathcal{F}\boldsymbol{m} + \boldsymbol{\eta}, \quad \boldsymbol{\eta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Gamma}_{\mathrm{noise}})$$

- The pair $(\boldsymbol{m}, \mathbf{u}^{\mathrm{obs}})$ is jointly Gaussian with:

$$\mu_{\mathrm{post}}^{\mathbf{u}^{\mathrm{obs}}} = \mathcal{N}(\boldsymbol{m}_{\mathrm{post}}, \mathcal{C}_{\mathrm{post}})$$

where:

$$\boldsymbol{m}_{\mathrm{post}} = \mathcal{C}_{\mathrm{post}}\left(\mathcal{F}^*\boldsymbol{\Gamma}_{\mathrm{noise}}^{-1}\mathbf{u}^{\mathrm{obs}} + \mathcal{C}_{\mathrm{pr}}^{-1}\boldsymbol{m}_{\mathrm{pr}}\right)$$
$$\mathcal{C}_{\mathrm{post}} = \left(\mathcal{F}^*\boldsymbol{\Gamma}_{\mathrm{noise}}^{-1}\mathcal{F} + \mathcal{C}_{\mathrm{pr}}^{-1}\right)^{-1}$$

Corresponds to $\mathcal{J}(\boldsymbol{m}) = \frac{1}{2}\|\mathcal{F}\boldsymbol{m} - \mathbf{u}^{\mathrm{obs}}\|^2_{\boldsymbol{\Gamma}_{\mathrm{noise}}^{-1}} + \frac{1}{2}\|\mathcal{C}^{-1/2}(\boldsymbol{m} - \boldsymbol{m}_{\mathrm{pr}})\|^2$.

Returning to Fault Inversion

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Define:

$$\mathcal{L}(\boldsymbol{m}, \boldsymbol{u}, \boldsymbol{v}) = \frac{1}{2}\|\mathcal{B}\boldsymbol{u} - \mathbf{u}^{\mathrm{obs}}\|^2_{\boldsymbol{\Gamma}^{-1}_{\mathrm{noise}}} + \frac{1}{2}\|\mathcal{C}^{-1/2}(\boldsymbol{m} - \boldsymbol{m}_{\mathrm{pr}})\|^2 + \int_{\Gamma_s} \beta^{-1}(\boldsymbol{u} - \boldsymbol{h}) \cdot \boldsymbol{v}\,\mathrm{d}s$$
$$+ \int_{\Gamma_b} \delta^{-1}(\mathbf{T}\boldsymbol{u} - \boldsymbol{m}) \cdot \boldsymbol{v}\,\mathrm{d}s + \int_{\Omega} \boldsymbol{\varepsilon}(\boldsymbol{u}) : \mathbb{C}[\boldsymbol{\varepsilon}(\boldsymbol{v})]\,\mathrm{d}\boldsymbol{x}$$

Procedure: Obtain $\boldsymbol{m}$ by enforcing $\mathcal{L}_{\boldsymbol{m}} = \mathcal{L}_{\boldsymbol{u}} = \mathcal{L}_{\boldsymbol{v}} = 0$.

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FEM Discretization

- Consider finite element space $\mathcal{V}_h = \mathrm{Span}(\phi_1, \ldots, \phi_n)$.
- ϕ_k: Lagrange basis functions.
- For $\mathbf{f} \in \mathcal{V}_h$:

$$\mathbf{f} = \sum_{k=1}^{n} f_k \phi_k$$

  In particular, construct $\mathbf{u}, \mathbf{v}, \mathbf{m}, \mathbf{m}_{\mathrm{pr}}, \mathbf{h} \in \mathcal{V}_h$ from $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{m}, \boldsymbol{m}_{\mathrm{pr}}$ and $\boldsymbol{h}$.
- Finite dimensional Hilbert space

$$(\mathcal{V}_h, \langle \cdot, \cdot \rangle_{L^2}) \cong (\mathbb{R}^n, \langle \cdot, \cdot \rangle_M), \qquad \langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^{\mathrm{T}} \mathbf{M} \mathbf{v}$$

- From the Lagrangian we find

$$\begin{bmatrix} \mathbf{C}_{\mathrm{pr}}^{-1} & & -\mathbf{C}^* \\ & \mathbf{B}^* \mathbf{\Gamma}_{\mathrm{noise}}^{-1} \mathbf{B} & \mathbf{A}^* \\ -\mathbf{C} & \mathbf{A} & \end{bmatrix} \begin{bmatrix} \mathbf{m} \\ \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{\mathrm{pr}}^{-1} \mathbf{m}_0 \\ \mathbf{B}^* \mathbf{\Gamma}_{\mathrm{noise}}^{-1} \mathbf{u}^{\mathrm{obs}} \\ \mathbf{D} \mathbf{h} \end{bmatrix}$$

  where $\mathbf{A}\mathbf{u} = \mathbf{C}\mathbf{m} + \mathbf{D}\mathbf{h}$ is the discretized weak form.

Reduced Linear System

Parameter-to-Observable Map

From the discretized weak form we can define:

$$\mathbf{Au} = \mathbf{Cm} + \mathbf{Dh} \implies \mathcal{F}(\mathbf{m}) = \mathbf{BA}^{-1}\left[\mathbf{Cm} + \mathbf{Dh}\right] = \mathcal{G}\mathbf{m} + \mathbf{g}$$

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With this:

$$\begin{bmatrix} \mathbf{C}_{\mathrm{pr}}^{-1} & & -\mathbf{C}^* \\ & \mathbf{B}^*\mathbf{\Gamma}_{\mathrm{noise}}^{-1}\mathbf{B} & \mathbf{A}^* \\ -\mathbf{C} & \mathbf{A} & \end{bmatrix} \begin{bmatrix} \mathbf{m} \\ \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{\mathrm{pr}}^{-1}\mathbf{m}_0 \\ \mathbf{B}^*\mathbf{\Gamma}_{\mathrm{noise}}^{-1}\mathbf{u}^{\mathrm{obs}} \\ \mathbf{Dh} \end{bmatrix}$$

can be reduced to

$$\left(\mathbf{C}_{\mathrm{pr}}^{-1} + \mathcal{G}^*\mathbf{\Gamma}_{\mathrm{noise}}^{-1}\mathcal{G}\right)\mathbf{m} = \mathcal{G}^*\mathbf{\Gamma}_{\mathrm{noise}}^{-1}\left(\mathbf{g} + \mathbf{u}^{\mathrm{obs}}\right) + \mathbf{C}_{\mathrm{pr}}^{-1}\mathbf{m}_0$$

Tractability of Computation

$$\underbrace{\left[\mathbf{C}_{\mathrm{pr}}^{-1} + \mathcal{G}^* \mathbf{\Gamma}_{\mathrm{noise}}^{-1} \mathcal{G}\right]}_{\mathbf{C}_{\mathrm{post}}^{-1}} \mathbf{m}_{\mathrm{post}} = \left[\mathcal{G}^* \mathbf{\Gamma}_{\mathrm{noise}}^{-1} \left(\mathbf{g} + \mathbf{u}^{\mathrm{obs}}\right) + \mathbf{C}_{\mathrm{pr}}^{-1} \mathbf{m}_0\right]$$

How do we make this computation tractable?

Tractability of Computation

$$\underbrace{\left[\mathbf{C}_{\mathrm{pr}}^{-1} + \mathcal{G}^* \mathbf{\Gamma}_{\mathrm{noise}}^{-1} \mathcal{G}\right]}_{\mathbf{C}_{\mathrm{post}}^{-1}} \mathbf{m}_{\mathrm{post}} = \left[\mathcal{G}^* \mathbf{\Gamma}_{\mathrm{noise}}^{-1} \left(\mathbf{g} + \mathbf{u}^{\mathrm{obs}}\right) + \mathbf{C}_{\mathrm{pr}}^{-1} \mathbf{m}_0\right]$$

How do we make this computation tractable?

- Use symmetry aware solvers
- Leverage sparsity of
  - matrices induced by differential operators
  - observations
- Exploit the dimensionality reduction in the parameter space

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