Infinite-dimensional Bayesian inversion for fault slip from surface measurements

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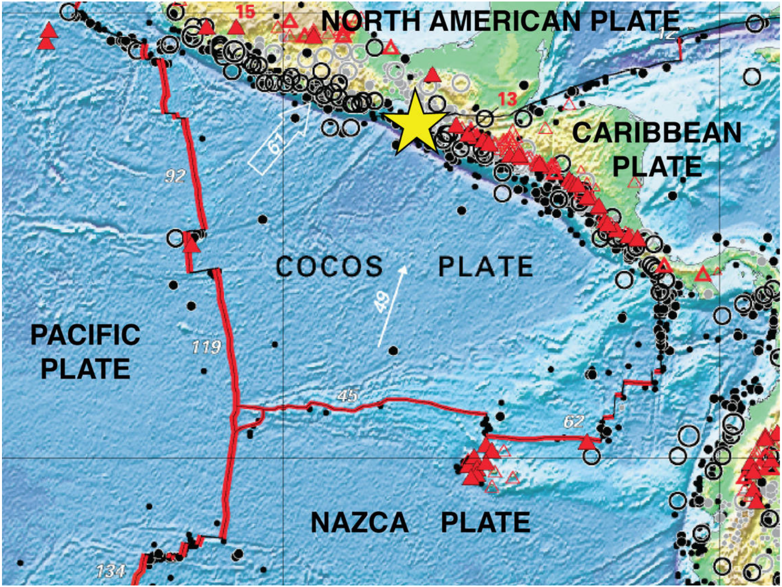
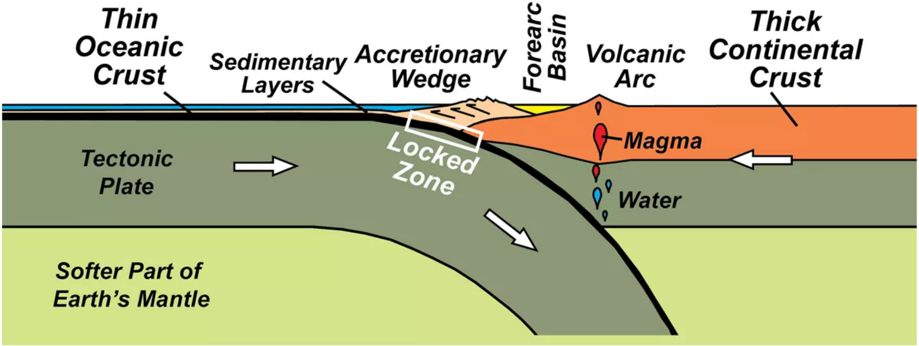


Tom Simkin et al. *This dynamic planet: World map of volcanoes, earthquakes, impact craters and plate tectonics*. Tech. rep. 2006. DOI: 10.3133/i2800



Robert J Lillie. *Oregon's Island In The Sky: Geology Road Guide to Marys Peak*. English. OCLC: 979996650. 2017. ISBN: 9781540611963

Goals

Understand the subduction zone from collected observations.

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Understand the subduction zone from collected observations.

Do so while quantifying measurement uncertainties.

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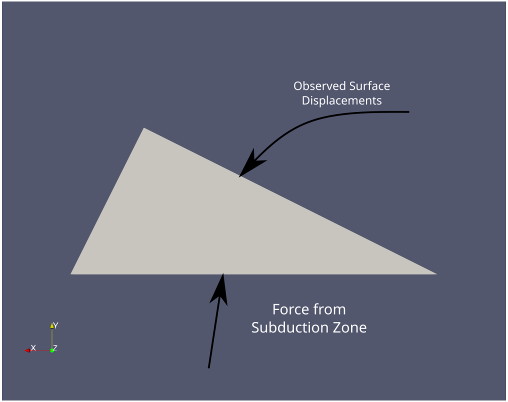
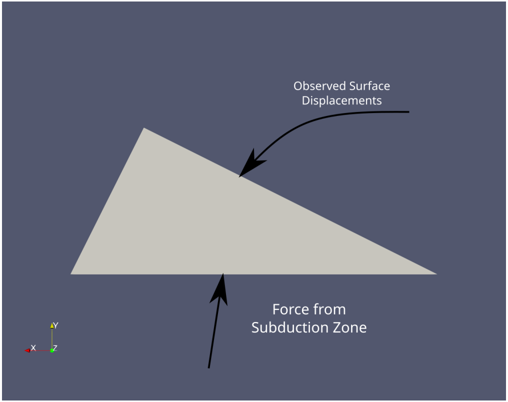
Model Assumptions



- **Governing PDE** (forward model): Linear elasticity

Model Assumptions



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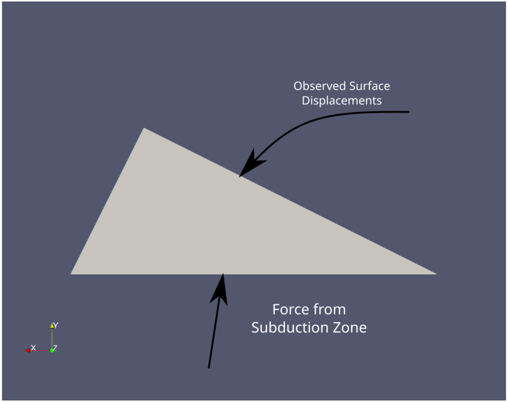

- **Governing PDE** (forward model): Linear elasticity
- **Uncertain parameter**: Displacement on fault plane
- **Inverse Problem**: Given measurements of surface deformation $\mathbf{u}^{\mathrm{obs}}$ reconstruct fault plane displacement.

Forward Model

$$-\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) = \mathbf{0} \text{ in } \Omega,$$

where:

- $\boldsymbol{\sigma}(\boldsymbol{u}) = \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u})$ with
  - $\mathbb{C}[\boldsymbol{\varepsilon}] = 2\mu\boldsymbol{\varepsilon} + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I}$ the fourth-order linear elasticity tensor:
  - $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}\left[\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^{\mathrm{T}}\right]$ the strain tensor.
- μ and λ are known as the Láme constants.

Kimberly Alison McCormack. *Earthquakes, groundwater and surface deformation : Exploring the poroelastic response to megathrust earthquakes*. Aug. 2018. URL: hdl.handle.net/2152/68892

Forward Model

$$
\begin{aligned}
-\boldsymbol{\nabla}\left[\mu(\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^{\mathrm{T}}) + \lambda\boldsymbol{\nabla}\cdot\boldsymbol{u}\mathbf{I}\right] &= \mathbf{0} \quad \text{in } \Omega, && \text{(1a)}\\
\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} &= \mathbf{0} \quad \text{on } \Gamma_t && \text{(1b)}\\
\boldsymbol{u} + \beta\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} &= \boldsymbol{h} \quad \text{on } \Gamma_s && \text{(1c)}\\
\boldsymbol{u}\cdot\boldsymbol{n} &= 0 \quad \text{on } \Gamma_b && \text{(1d)}\\
\delta\mathbf{T}(\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n}) + \mathbf{T}\boldsymbol{u} &= \boldsymbol{m} \quad \text{on } \Gamma_b && \text{(1e)}
\end{aligned}
$$

- $\mathbf{T}$ is the tangential operator $\mathbf{T}\boldsymbol{u} = (\mathbf{I} - \boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{u} = \boldsymbol{u} - (\boldsymbol{n}^{\mathrm{T}}\boldsymbol{u})\boldsymbol{n}$.

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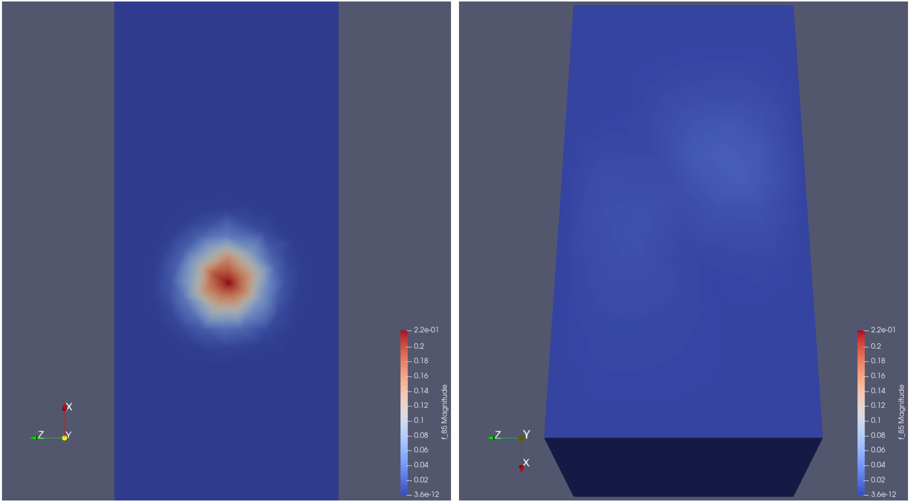
- The right hand side of (1e) is the displacement on the fault plane that is being inverted for.
- (1e) can be understood as a regularized Dirichlet condition.

Mesh

Forward Solution



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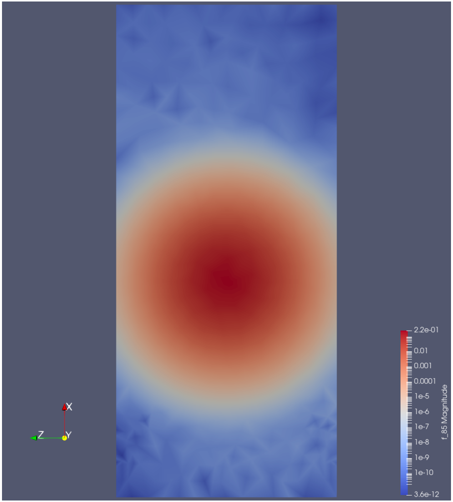

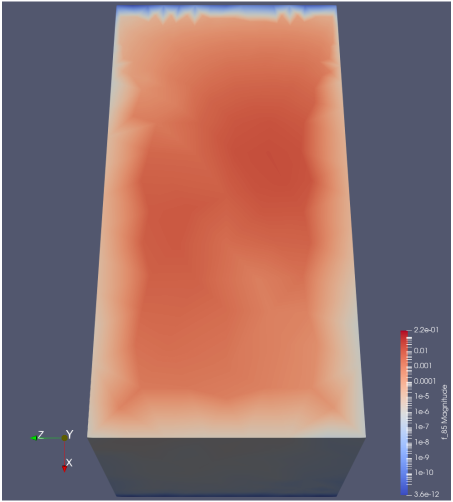

Weak Formulation

Define:

$$\boldsymbol{V} := \{\boldsymbol{u} \in H^1(\Omega)^3 : \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_b\}$$

Then the weak formulation of the forward model is given by:

$$\int_{\Gamma_s} \beta^{-1}(\boldsymbol{u} - \boldsymbol{h}) \cdot \boldsymbol{v} \,\mathrm{d}s + \int_{\Gamma_b} \delta^{-1}(\mathbf{T}\boldsymbol{u} - \boldsymbol{m}) \cdot \boldsymbol{v} \,\mathrm{d}s + \int_{\Omega} \boldsymbol{\varepsilon}(\boldsymbol{u}) : \mathbb{C}[\boldsymbol{\varepsilon}(\boldsymbol{v})] \,\mathrm{d}\boldsymbol{x} = 0, \quad \forall v \in \boldsymbol{V} \quad (2)$$

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The Deterministic Inverse Problem

To reconstruct the fault displacement we construct the PDE-constrained optimization problem:

$$\mathcal{J}(\boldsymbol{m}) = \frac{1}{2}\|\mathcal{B}\boldsymbol{u}(\boldsymbol{m}) - \mathbf{u}^{\mathrm{obs}}\|^2 + \frac{1}{2}\|\mathcal{A}\boldsymbol{m}\|^2$$

where $\boldsymbol{u}$ is given by the solution of the linear elasticity equation.

- $\boldsymbol{u}(\boldsymbol{m})$ is given by the forward model.
- $\mathcal{B} : (L^2(\Omega))^3 \to \mathbb{R}^N$ is an observation operator.
- $\mathbf{u}^{\mathrm{obs}} \in \mathbb{R}^N$ where N is the number of data points.
- $\mathcal{A}$ is some regularization operator.

[1] In literature, called the *parameter-to-observable operator*.

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If we let $\mathcal{S}$ be the forward model operator, i.e. $\boldsymbol{u} = \mathcal{S}\boldsymbol{m}$, and let[1] $\mathcal{F} = \mathcal{B}\mathcal{S}$, then:

$$\mathcal{J}(\boldsymbol{m}) = \frac{1}{2}\|\mathcal{F}\boldsymbol{m} - \mathbf{u}^{\mathrm{obs}}\|^2 + \frac{1}{2}\|\mathcal{A}\boldsymbol{m}\|^2$$

[1] In literature, called the *parameter-to-observable operator*.

Bayesian Inversion in Finite Dimensions

Theorem (Bayes Theorem in Finite Dimensions)

$$\pi_{\text{post}}(\mathbf{m}|\mathbf{u}^{\text{obs}}) \propto \pi_{\text{like}}(\mathbf{u}^{\text{obs}}|\mathbf{m})\pi_{\text{prior}}(\mathbf{m})$$

- Gaussian Prior $\mathbf{m} \sim \mathcal{N}(\mathbf{m}_{\text{pr}}, \mathbf{\Gamma}_{\text{pr}})$.
- Additive Gaussian noise

$$\mathbf{u}^{\text{obs}} = \mathbf{F}\mathbf{m} + \boldsymbol{\eta}, \quad \boldsymbol{\eta} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Gamma}_{\text{noise}})$$

- Gaussian posterior with:

$$\mathbf{m}|\mathbf{u}^{\text{obs}} \sim \mathcal{N}(\mathbf{m}_{\text{post}}, \mathbf{\Gamma}_{\text{post}})$$

where:

$$\mathbf{m}_{\text{post}} = \mathbf{\Gamma}_{\text{post}} \left(\mathbf{F}^{\text{T}} \mathbf{\Gamma}_{\text{noise}}^{-1} \mathbf{u}^{\text{obs}} + \mathbf{\Gamma}_{\text{pr}}^{-1} \mathbf{m}_{\text{pr}} \right)$$

$$\mathbf{\Gamma}_{\text{post}} = \left(\mathbf{F}^{\text{T}} \mathbf{\Gamma}_{\text{noise}}^{-1} \mathbf{F} + \mathbf{\Gamma}_{\text{pr}}^{-1} \right)^{-1}$$

Bayesian Inversion in Infinite Dimensions

Theorem (Bayes Theorem in Infinite Dimensions)

$$\frac{\mathrm{d}\mu_{\mathrm{post}}^{\mathbf{u}^{\mathrm{obs}}}}{\mathrm{d}\mu_{\mathrm{pr}}} \propto \pi_{\mathrm{like}}(\mathbf{u}^{\mathrm{obs}}|\boldsymbol{m})$$

- Gaussian Prior $\boldsymbol{m} \sim \mathcal{N}(\boldsymbol{m}_{\mathrm{pr}}, \mathcal{C}_0)$.
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- Gaussian posterior:

$$\mu_{\mathrm{post}}^{\mathbf{u}^{\mathrm{obs}}} = \mathcal{N}(\boldsymbol{m}_{\mathrm{post}}, \mathcal{C}_{\mathrm{post}})$$

where:

$$\begin{aligned}
\boldsymbol{m}_{\mathrm{post}} &= \mathcal{C}_{\mathrm{post}} \left(\mathcal{F}^* \boldsymbol{\Gamma}_{\mathrm{noise}}^{-1} \mathbf{u}^{\mathrm{obs}} + \mathcal{C}_{\mathrm{pr}}^{-1} \boldsymbol{m}_{\mathrm{pr}} \right) \\
\mathcal{C}_{\mathrm{post}} &= \left(\mathcal{F}^* \boldsymbol{\Gamma}_{\mathrm{noise}}^{-1} \mathcal{F} + \mathcal{C}_{\mathrm{pr}}^{-1} \right)^{-1}
\end{aligned}$$

Returning to Fault Inversion

Slight Wrinkle: Our parameter-to-observable map is affine.

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To find $\boldsymbol{m}_{\text{post}}$, need to minimize:

$$\mathcal{J}(\boldsymbol{m}) = \frac{1}{2}\|\mathcal{B}\boldsymbol{u} - \mathbf{u}^{\text{obs}}\|^2_{\boldsymbol{\Gamma}^{-1}_{\text{noise}}} + \frac{1}{2}\|\mathcal{C}_{\text{pr}}^{-1/2}(\boldsymbol{m} - \boldsymbol{m}_{\text{pr}})\|^2$$

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Define:

$$\mathcal{L}(\boldsymbol{m}, \boldsymbol{u}, \boldsymbol{v}) = \frac{1}{2}\|\mathcal{B}\boldsymbol{u} - \mathbf{u}^{\text{obs}}\|^2_{\boldsymbol{\Gamma}^{-1}_{\text{noise}}} + \frac{1}{2}\|\mathcal{C}_{\text{pr}}^{-1/2}(\boldsymbol{m} - \boldsymbol{m}_{\text{pr}})\|^2 + \int_{\Gamma_s} \beta^{-1}(\boldsymbol{u} - \boldsymbol{h}) \cdot \boldsymbol{v}\,\mathrm{d}s + \int_{\Gamma_b} \delta^{-1}(\mathbf{T}\boldsymbol{u} - \boldsymbol{m}) \cdot \boldsymbol{v}\,\mathrm{d}s + \int_{\Omega} \boldsymbol{\varepsilon}(\boldsymbol{u}) : \mathbb{C}[\boldsymbol{\varepsilon}(\boldsymbol{v})]\,\mathrm{d}\boldsymbol{x}$$

Procedure: Obtain $\boldsymbol{m}_{\text{post}}$ by enforcing $\mathcal{L}_{\boldsymbol{m}} = \mathcal{L}_{\boldsymbol{u}} = \mathcal{L}_{\boldsymbol{v}} = 0$.

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Choice of Prior:

In our case, we choose $\mathcal{C}_{\mathrm{pr}} = \mathcal{A}^{-2} = (-\gamma\Delta + \delta\mathbf{I})^{-2}$.

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$$+ \int_{\Gamma_b} \delta^{-1}(\mathbf{T}\boldsymbol{u} - \boldsymbol{m}) \cdot \boldsymbol{v}\, \mathrm{d}s + \int_{\Omega} \boldsymbol{\varepsilon}(\boldsymbol{u}) : \mathbb{C}[\boldsymbol{\varepsilon}(\boldsymbol{v})]\, \mathrm{d}\boldsymbol{x}$$

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A. M. Stuart. "Inverse problems: A Bayesian perspective". In: *Acta Numerica* 19 (May 2010), pp. 451–559. DOI: 10.1017/s0962492910000061

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FEM Discretization

- Consider finite element space $\mathcal{V}_h = \mathrm{Span}(\phi_1, \ldots, \phi_n)$.
- ϕ_k: Lagrange basis functions.
- For $\mathbf{f} \in \mathcal{V}_h$:

$$\mathbf{f} = \sum_{k=1}^{n} f_k \phi_k$$

In particular, construct $\mathbf{u}, \mathbf{v}, \mathbf{m}, \mathbf{m}_{\mathrm{pr}}, \mathbf{h} \in \mathcal{V}_h$ from $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{m}, \boldsymbol{m}_{\mathrm{pr}}$ and $\boldsymbol{h}$.

- Finite dimensional Hilbert space

$$(\mathcal{V}_h, \langle \cdot, \cdot \rangle_{L^2}) \cong (\mathbb{R}^n, \langle \cdot, \cdot \rangle_M), \qquad \langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^{\mathrm{T}} \mathbf{M} \mathbf{v}$$

- From the Lagrangian we find

$$\begin{bmatrix} \mathbf{C}_{\mathrm{pr}}^{-1} & & -\mathbf{C}^* \\ & \mathbf{B}^* \mathbf{\Gamma}_{\mathrm{noise}}^{-1} \mathbf{B} & \mathbf{A}^* \\ -\mathbf{C} & \mathbf{A} & \end{bmatrix} \begin{bmatrix} \mathbf{m}_{\mathrm{post}} \\ \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{\mathrm{pr}}^{-1} \mathbf{m}_{\mathrm{pr}} \\ \mathbf{B}^* \mathbf{\Gamma}_{\mathrm{noise}}^{-1} \mathbf{u}^{\mathrm{obs}} \\ \mathbf{D} \mathbf{h} \end{bmatrix}$$

where $\mathbf{Au} = \mathbf{Cm} + \mathbf{Dh}$ is the discretized weak form.

Reduced Linear System

Parameter-to-Observable Map

From the discretized weak form we can define:

$$\mathbf{Au} = \mathbf{Cm} + \mathbf{Dh} \implies \mathcal{F}(\mathbf{m}) = \mathbf{BA}^{-1}\left[\mathbf{Cm} + \mathbf{Dh}\right] = \mathcal{G}\mathbf{m} + \mathbf{g}$$

Reduced Linear System

Parameter-to-Observable Map

From the discretized weak form we can define:

$$\mathbf{A}\mathbf{u} = \mathbf{C}\mathbf{m} + \mathbf{D}\mathbf{h} \implies \mathcal{F}(\mathbf{m}) = \mathbf{B}\mathbf{A}^{-1}\left[\mathbf{C}\mathbf{m} + \mathbf{D}\mathbf{h}\right] = \mathcal{G}\mathbf{m} + \mathbf{g}$$

With this:

$$\begin{bmatrix} \mathbf{C}_{\mathrm{pr}}^{-1} & & -\mathbf{C}^* \\ & \mathbf{B}^*\mathbf{\Gamma}_{\mathrm{noise}}^{-1}\mathbf{B} & \mathbf{A}^* \\ -\mathbf{C} & \mathbf{A} & \end{bmatrix} \begin{bmatrix} \mathbf{m}_{\mathrm{post}} \\ \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{\mathrm{pr}}^{-1}\mathbf{m}_{\mathrm{pr}} \\ \mathbf{B}^*\mathbf{\Gamma}_{\mathrm{noise}}^{-1}\mathbf{u}^{\mathrm{obs}} \\ \mathbf{D}\mathbf{h} \end{bmatrix}$$

can be reduced to

$$\left(\mathbf{C}_{\mathrm{pr}}^{-1} + \mathcal{G}^*\mathbf{\Gamma}_{\mathrm{noise}}^{-1}\mathcal{G}\right)\mathbf{m}_{\mathrm{post}} = \mathcal{G}^*\mathbf{\Gamma}_{\mathrm{noise}}^{-1}\left(\mathbf{g} + \mathbf{u}^{\mathrm{obs}}\right) + \mathbf{C}_{\mathrm{pr}}^{-1}\mathbf{m}_{\mathrm{pr}}$$

Tractability of Computation

$$\underbrace{\left[\mathbf{C}_{\mathrm{pr}}^{-1} + \mathcal{G}^* \boldsymbol{\Gamma}_{\mathrm{noise}}^{-1} \mathcal{G}\right]}_{\mathbf{C}_{\mathrm{post}}^{-1}} \mathbf{m}_{\mathrm{post}} = \left[\mathcal{G}^* \boldsymbol{\Gamma}_{\mathrm{noise}}^{-1} \left(\mathbf{g} + \mathbf{u}^{\mathrm{obs}}\right) + \mathbf{C}_{\mathrm{pr}}^{-1} \mathbf{m}_{\mathrm{pr}}\right]$$

How do we make this computation tractable?

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How do we make this computation tractable?

- Exploit the dimensionality reduction in the parameter space
- Use symmetry aware solvers
- Leverage sparsity of
  - matrices induced by differential operators
  - observations

The discretized parameter space

Recall the boundary condition describing the inversion parameter (1e)

$$\delta \mathbf{T}(\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n}) + \mathbf{T}\boldsymbol{u} = \boldsymbol{m}, \quad \text{on } \Gamma_b \subset \mathbb{R}^2$$

By the definition of $\mathbf{T}$:

$$\boldsymbol{m} = \begin{pmatrix} m_x \\ 0 \\ m_z \end{pmatrix}$$

Construct a second finite element space $\mathcal{V}_h'$ for Γ_b and instead choose $\mathbf{m}, \mathbf{m}_{\mathrm{pr}} \in (\mathcal{V}_h', \langle \cdot, \cdot \rangle_{L^2}) \cong (\mathbb{R}^m, \langle \cdot, \cdot \rangle_{\mathbf{M}_p})$.

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- FEniCS data structures not very interoporable over mixed dimensions.

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- FEniCS data structures not very interoporable over mixed dimensions.
- Bijective maps between degrees of freedom and vertices are supported only for 1st order Lagrange elements.

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- FEniCS data structures not very interoporable over mixed dimensions.
- Bijective maps between degrees of freedom and vertices are supported only for 1st order Lagrange elements.
- Need to build custom projector between mixed dimensional topologies and function spaces (Special thanks to Tucker Hartland from UC Merced).

Symmetry: Dealing with the inner product

For $T : (\mathcal{X}, \langle \cdot, \cdot \rangle_{\mathcal{X}}) \to (\mathcal{Y}, \langle \cdot, \cdot \rangle_{\mathcal{Y}})$, the adjoint T^* is defined by the identity:

$$\langle x, T^* y \rangle_{\mathcal{X}} = \langle Tx, y \rangle_{\mathcal{Y}}, \quad \forall (x, y) \in \mathcal{X} \times \mathcal{Y}$$

Hence, in our setting, $(\cdot)^{\mathrm{T}} \neq (\cdot)^*$!

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Hence, in our setting, $(\cdot)^{\mathrm{T}} \neq (\cdot)^*$! Consider the parameter-to-observable map:

$$\mathcal{G} : (\mathbb{R}^m, \langle\cdot,\cdot\rangle_{\mathbf{M}_p}) \to (\mathbb{R}^q, \langle\cdot,\cdot\rangle)$$

Since

$$\langle \mathcal{G}\mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^{\mathrm{T}}\mathcal{G}^{\mathrm{T}}\mathbf{y} = \mathbf{x}^{\mathrm{T}}\mathbf{M}_p\mathbf{M}_p^{-1}\mathcal{G}^{\mathrm{T}}\mathbf{y} = \langle \mathbf{x}, \mathbf{M}_p^{-1}\mathcal{G}^{\mathrm{T}}\mathbf{y}\rangle$$

So

$$\mathcal{G}^* = \mathbf{M}_p^{-1}\mathcal{G}^{\mathrm{T}}$$

Symmetry: Dealing with the inner product

For $T : (\mathcal{X}, \langle \cdot, \cdot \rangle_{\mathcal{X}}) \to (\mathcal{Y}, \langle \cdot, \cdot \rangle_{\mathcal{Y}})$, the adjoint T^* is defined by the identity:

$$\langle x, T^* y \rangle_{\mathcal{X}} = \langle Tx, y \rangle_{\mathcal{Y}}, \quad \forall (x, y) \in \mathcal{X} \times \mathcal{Y}$$

Hence, in our setting, $(\cdot)^{\mathrm{T}} \neq (\cdot)^*$! Consider the parameter-to-observable map:

$$\mathcal{G} : (\mathbb{R}^m, \langle \cdot, \cdot \rangle_{\mathbf{M}_p}) \to (\mathbb{R}^q, \langle \cdot, \cdot \rangle)$$

Since

$$\langle \mathcal{G}\mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathrm{T}} \mathcal{G}^{\mathrm{T}} \mathbf{y} = \mathbf{x}^{\mathrm{T}} \mathbf{M}_p \mathbf{M}_p^{-1} \mathcal{G}^{\mathrm{T}} \mathbf{y} = \langle \mathbf{x}, \mathbf{M}_p^{-1} \mathcal{G}^{\mathrm{T}} \mathbf{y} \rangle$$

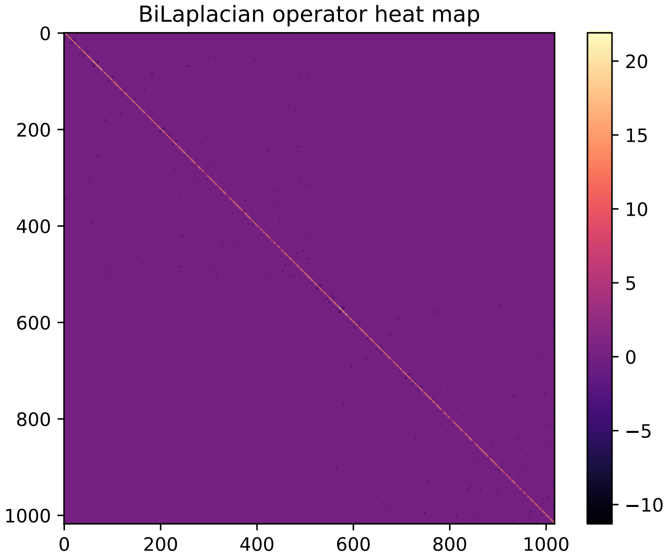
So

$$\mathcal{G}^* = \mathbf{M}_p^{-1} \mathcal{G}^{\mathrm{T}}$$

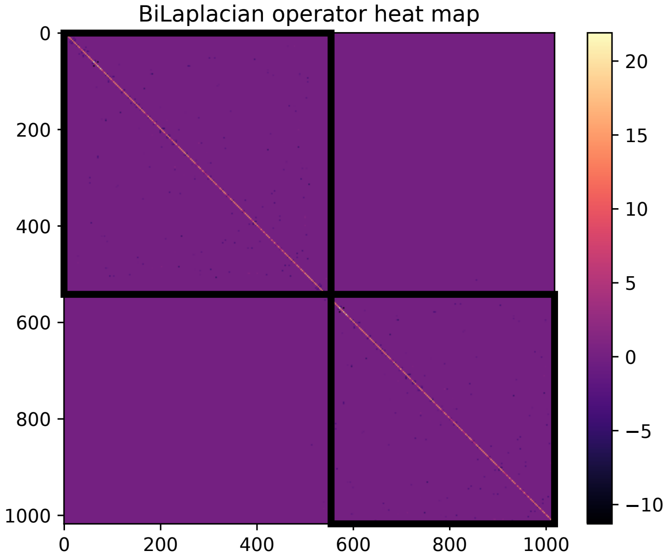
Note:

There are two mass matrices, $\mathbf{M}$ and $\mathbf{M}_p$. Thankfully, only $\mathbf{M}_p$ is necessary.

Sparsity: Matrices induced by differential operators



Sparsity: Matrices induced by differential operators



Sparsity: Observations

$$
\begin{aligned}
\mathbf{C}_{\mathrm{post}} &= \left(\mathcal{G}^* \mathbf{\Gamma}_{\mathrm{noise}}^{-1} \mathcal{G} + \mathbf{C}_{\mathrm{pr}}^{-1}\right) \\
&= \mathbf{C}_{\mathrm{pr}}^{1/2} \left(\mathbf{C}_{\mathrm{pr}}^{1/2} \mathcal{G}^* \mathbf{\Gamma}_{\mathrm{noise}}^{-1} \mathcal{G} \mathbf{C}_{\mathrm{pr}}^{1/2} + \mathbf{I}\right) \mathbf{C}_{\mathrm{pr}}^{1/2} \\
&= \mathbf{C}_{\mathrm{pr}}^{1/2} \left(\tilde{\mathbf{H}}_{\mathrm{misfit}} + \mathbf{I}\right) \mathbf{C}_{\mathrm{pr}}^{1/2}
\end{aligned}
$$

Sparsity: Observations

$$\begin{aligned}
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&= \mathbf{C}_{\mathrm{pr}}^{1/2}\left(\tilde{\mathbf{H}}_{\mathrm{misfit}} + \mathbf{I}\right)\mathbf{C}_{\mathrm{pr}}^{1/2}
\end{aligned}$$

Recall:

$$\mathcal{G} : (\mathbb{R}^m, \langle\cdot,\cdot\rangle_{\mathbf{M}_p}) \to (\mathbb{R}^q, \langle\cdot,\cdot\rangle)$$

Note:

Observations are sparse, so $q \ll m$. Thus $\tilde{\mathbf{H}}_{\mathrm{misfit}}$ is low rank.

Construct:

$$\tilde{\mathbf{H}}_{\mathrm{misfit}} = \mathbf{V}_r \mathbf{\Lambda}_r \mathbf{V}_r^* + \mathcal{O}\left(\sum_{k=r+1}^{m} \lambda_k \right)$$

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Via the Sherman-Morrison-Woodbury formula:

$$\mathbf{D}_r = \mathrm{Diag}\left(\frac{\lambda_1}{1+\lambda_1}, \cdots, \frac{\lambda_r}{1+\lambda_r}\right)$$

$$\begin{aligned}\left(\tilde{\mathbf{H}}_{\mathrm{misfit}} + \mathbf{I}\right)^{-1} &= \mathbf{I} - \mathbf{V}_r \mathbf{D}_r \mathbf{V}_r^* + \mathcal{O}\left(\sum_{k=r+1}^{m} \frac{\lambda_k}{1+\lambda_k}\right) \\ &\approx \mathbf{I} - \mathbf{V}_r \mathbf{D}_r \mathbf{V}_r^*\end{aligned}$$

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Hence:

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Tan Bui-Thanh et al. "A Computational Framework for Infinite-Dimensional Bayesian Inverse Problems Part I: The Linearized Case, with Application to Global Seismic Inversion". In: *SIAM Journal on Scientific Computing* 35.6 (Jan. 2013), A2494–A2523. DOI: 10.1137/12089586x

Sampling from resulting distributions

Note, in order to draw samples from $\mathcal{N}(\mathbf{m}, \mathbf{C})$:

- Find $\mathbf{L}$ where $\mathbf{C} = \mathbf{L}\mathbf{L}^*$
- Sample $n \in \mathcal{N}(\mathbf{0}, \mathbf{I})$.
- Construct sample $\mathbf{s}$ as:

$$\mathbf{s} = \mathbf{m} + \mathbf{L}\mathbf{n}$$

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$$\mathbf{s} = \mathbf{m} + \mathbf{L}\mathbf{n}$$

Since $\mathcal{C}_{\mathrm{pr}} = \mathcal{A}^{-2}$, prior is easy. For posterior:

$$\mathbf{L} = \mathbf{C}_{\mathrm{pr}}^{1/2}(\mathbf{V}_r\mathbf{P}_r\mathbf{V}_r^* + \mathbf{I})\mathbf{M_p}^{-1/2}$$
$$\mathbf{P}_r = \mathrm{Diag}\left(\frac{1}{\sqrt{\lambda_1 + 1} - 1}\right)$$

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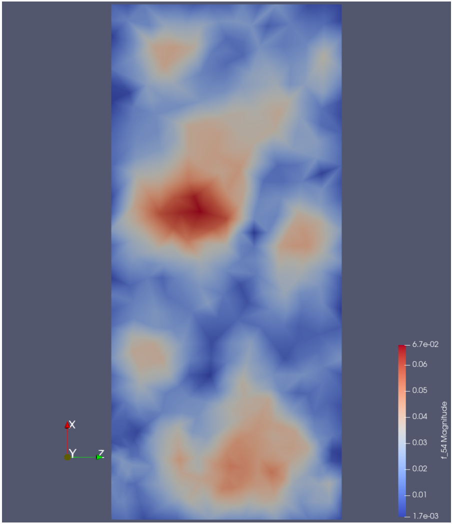
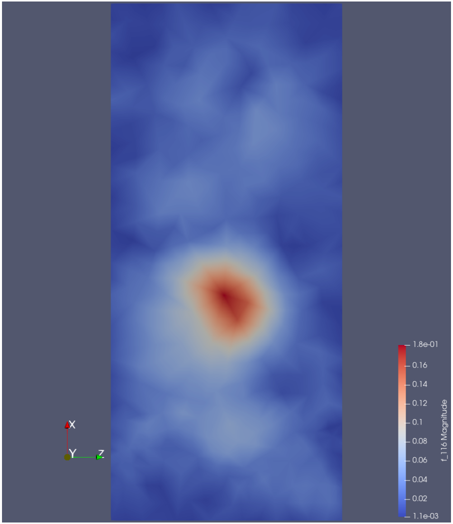
Results

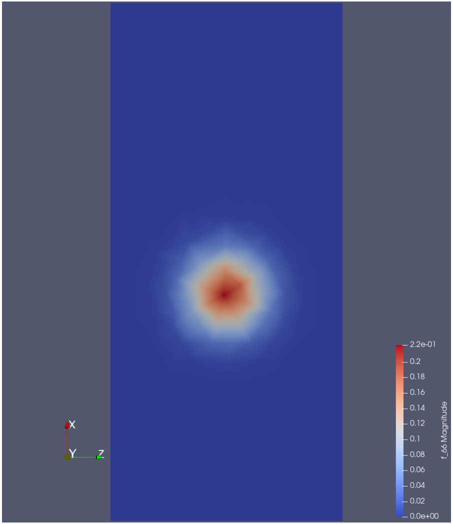


X
Y
Z
6.7e-02
0.06
0.05
0.04
0.03
0.02
0.01
1.7e-03
f_54 Magnitude

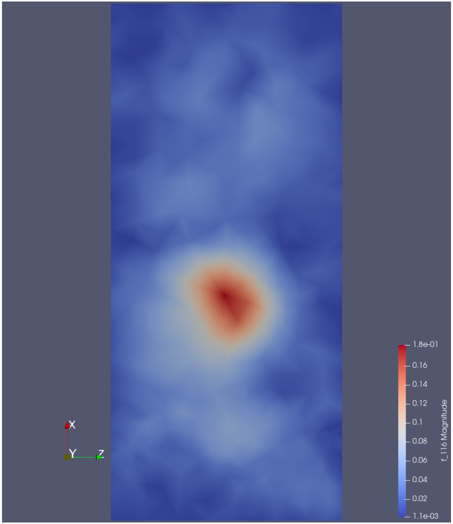


X
Y
Z
1.8e-01
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
1.1e-03
f_116 Magnitude

Results



X
Y
Z
2.2e-01
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0.0e+00
f_66 Magnitude



X
Y
Z
1.8e-01
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
1.1e-03
f_116 Magnitude

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Future Work

This problem is intended as a model problem for future uncertainty quantification endavors. This problem has three interesting qualities:

- It's a reasonable approximation to well-studied physical problem.
- Mixed dimensional formulation in topology and function space.
- It's posterior is provably Gaussian, hence a priori analysis is possible.

The intention is to use this as an additional example for work in

- Different criteria for optimal experimental design
- Hyper-Differential Sensitivity Analysis

Alen Alexanderian et al. "Optimal Design of Large-scale Bayesian Linear Inverse Problems Under Reducible Model Uncertainty: Good to Know What You Don't Know". In: *SIAM/ASA Journal on Uncertainty Quantification* 9.1 (Jan. 2021), pp. 163–184. DOI: 10.1137/20m1347292. URL: https://doi.org/10.1137/20m1347292

Isaac Sunseri et al. "Hyper-differential sensitivity analysis for inverse problems constrained by partial differential equations". In: *Inverse Problems* 36.12 (Dec. 2020), p. 125001. DOI: 10.1088/1361-6420/abaf63

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References I

[1] Alen Alexanderian et al. "Optimal Design of Large-scale Bayesian Linear Inverse Problems Under Reducible Model Uncertainty: Good to Know What You Don't Know". In: *SIAM/ASA Journal on Uncertainty Quantification* 9.1 (Jan. 2021), pp. 163–184. DOI: 10.1137/20m1347292. URL: https://doi.org/10.1137/20m1347292.

[2] Tan Bui-Thanh et al. "A Computational Framework for Infinite-Dimensional Bayesian Inverse Problems Part I: The Linearized Case, with Application to Global Seismic Inversion". In: *SIAM Journal on Scientific Computing* 35.6 (Jan. 2013), A2494–A2523. DOI: 10.1137/12089586x.

[3] Robert J Lillie. *Oregon's Island In The Sky: Geology Road Guide to Marys Peak*. English. OCLC: 979996650. 2017. ISBN: 9781540611963.

[4] Kimberly Alison McCormack. *Earthquakes, groundwater and surface deformation : Exploring the poroelastic response to megathrust earthquakes*. Aug. 2018. URL: hdl.handle.net/2152/68892.

References II

[5] Tom Simkin et al. *This dynamic planet: World map of volcanoes, earthquakes, impact craters and plate tectonics*. Tech. rep. 2006. DOI: 10.3133/i2800.

[6] A. M. Stuart. "Inverse problems: A Bayesian perspective". In: *Acta Numerica* 19 (May 2010), pp. 451–559. DOI: 10.1017/s0962492910000061.

[7] Isaac Sunseri et al. "Hyper-differential sensitivity analysis for inverse problems constrained by partial differential equations". In: *Inverse Problems* 36.12 (Dec. 2020), p. 125001. DOI: 10.1088/1361-6420/abaf63.