Seismic Inversion in the Bayesian Framework for Infinite-Dimensional Inverse Problems

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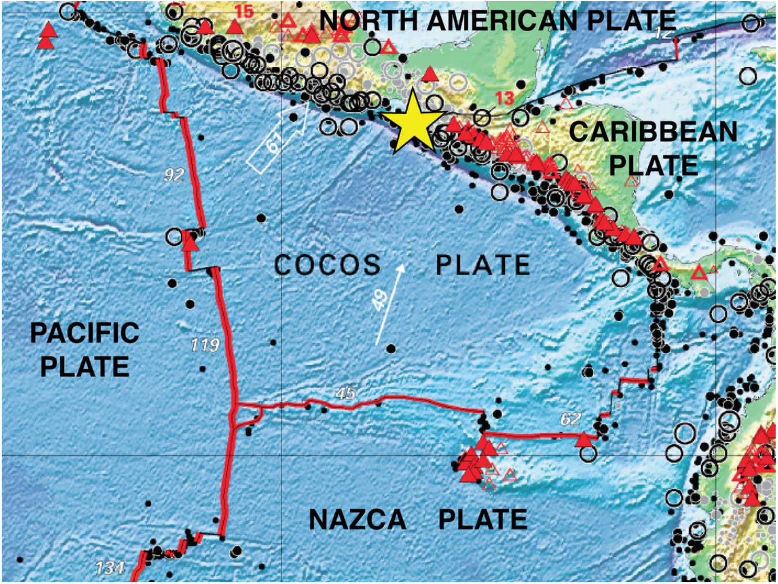

Figure: [Coc06]

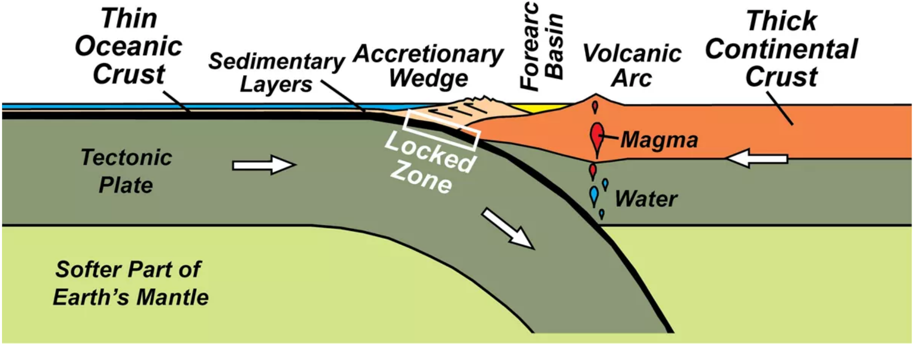

Figure: [Lil17]

Goals

Understand the subduction zone from collected observations.

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Do so while quantifying measurement uncertainties.

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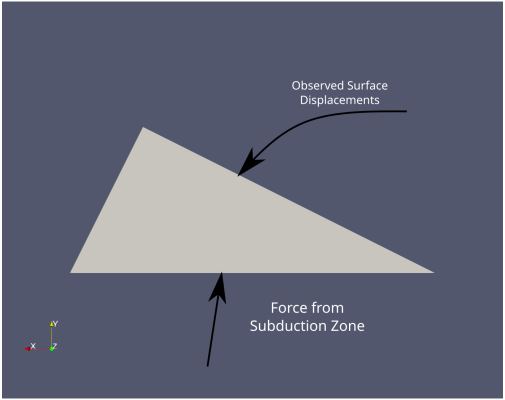
Model Assumptions



For the sake of modeling convienience, assume:

- **Governing PDE** (forward model): Linear elasticity

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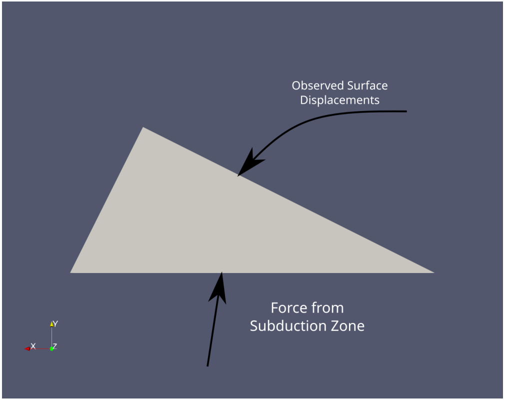

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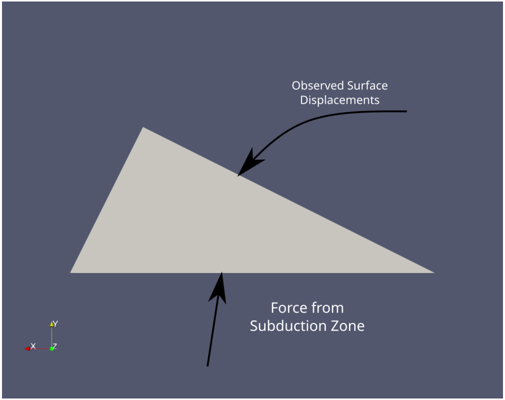

For the sake of modeling convienience, assume:

- **Governing PDE** (forward model): Linear elasticity
- **Uncertain parameter**: Displacement on fault plane
- **Inverse Problem**: Given measurements of surface deformation $\mathbf{u}^{\mathrm{obs}}$ reconstruct fault plane displacement.

Forward Model

$$-\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) = \mathbf{0} \text{ in } \Omega, \tag{1}$$

where:

- $\boldsymbol{\sigma}(\boldsymbol{u}) = \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u})$ with
  - $\mathbb{C}[\boldsymbol{\varepsilon}] = 2\mu\boldsymbol{\varepsilon} + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I}$ the fourth-order linear elasticity tensor:
  - $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}\left[\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^{\mathrm{T}}\right]$ the strain tensor.
- μ and λ are known as the Láme constants.

Forward Model

$$-\boldsymbol{\nabla}\left[\mu(\boldsymbol{\nabla}\boldsymbol{u}+(\boldsymbol{\nabla}\boldsymbol{u})^{\mathrm{T}})+\lambda\boldsymbol{\nabla}\cdot\boldsymbol{u}\mathbf{I}\right]=\mathbf{0}\quad\text{in }\Omega, \tag{2a}$$

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n}=\mathbf{0}\quad\text{on }\Gamma_t \tag{2b}$$

$$\boldsymbol{u}+\beta\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n}=\boldsymbol{h}\quad\text{on }\Gamma_s \tag{2c}$$

$$\boldsymbol{u}\cdot\boldsymbol{n}=0\quad\text{on }\Gamma_b \tag{2d}$$

$$\delta\mathbf{T}(\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n})+\mathbf{T}\boldsymbol{u}=\boldsymbol{m}\quad\text{on }\Gamma_b \tag{2e}$$

- $\mathbf{T}$ is the tangential operator $\mathbf{T}\boldsymbol{u}=(\mathbf{I}-\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{u}=\boldsymbol{u}-(\boldsymbol{n}^{\mathrm{T}}\boldsymbol{u})\boldsymbol{n}$.

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Forward Model

$$
\begin{aligned}
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\boldsymbol{u}+\beta\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} &= \boldsymbol{h} \quad \text{on } \Gamma_s && \text{(2c)}\\
\boldsymbol{u}\cdot\boldsymbol{n} &= 0 \quad \text{on } \Gamma_b && \text{(2d)}\\
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\end{aligned}
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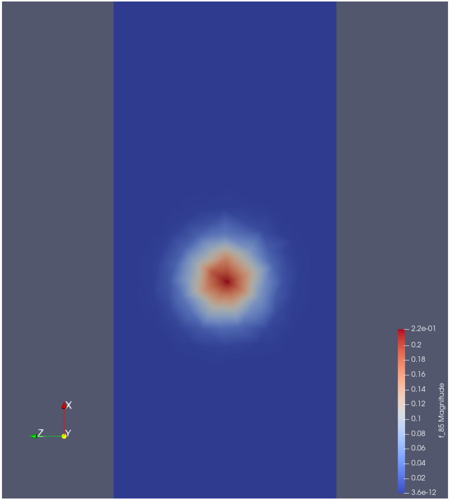
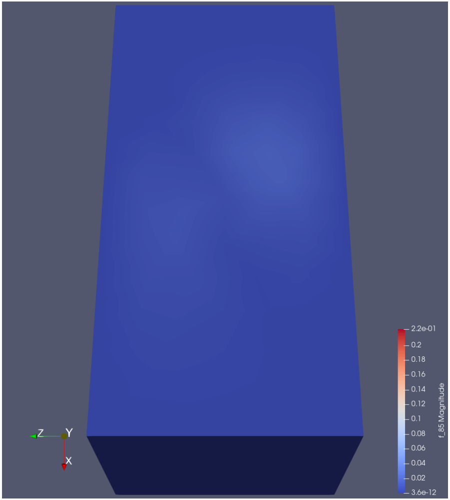
- The right hand side of (2e) is the displacement on the fault plane that is being inverted for.
- (2e) can be understood as a regularized Dirichlet condition.

Forward Solution





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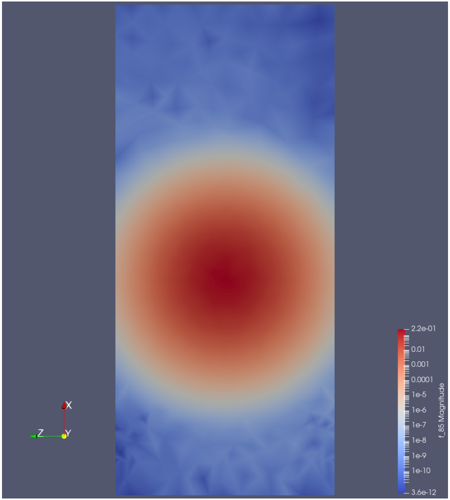

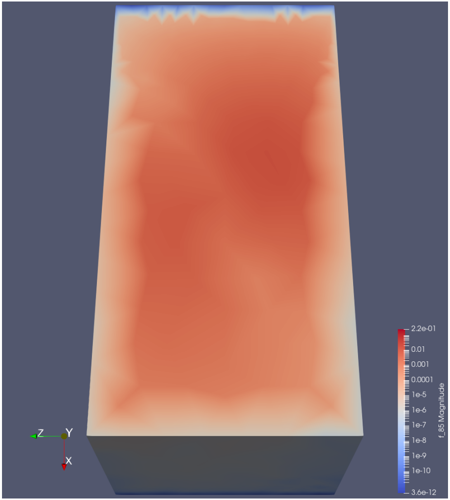

Weak Formulation

Define:

$$\boldsymbol{V} \coloneqq \{\boldsymbol{u} \in H^1(\Omega)^3 : \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_b\}$$

Then the weak formulation of the forward model is given by:

$$\int_{\Gamma_s} \beta^{-1}(\boldsymbol{u} - \boldsymbol{h}) \cdot \boldsymbol{v} \, \mathrm{d}s + \int_{\Gamma_b} \delta^{-1}(\mathbf{T}\boldsymbol{u} - \boldsymbol{m}) \cdot \boldsymbol{v} \, \mathrm{d}s + \int_{\Omega} \boldsymbol{\varepsilon}(\boldsymbol{u}) : \mathbb{C}[\boldsymbol{\varepsilon}(\boldsymbol{v})] \, \mathrm{d}\boldsymbol{x} = 0, \quad \forall v \in \boldsymbol{V} \tag{3}$$

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The Deterministic Inverse Problem

To reconstruct the fault displacement we construct the PDE-constrained optimization problem:

$$\mathcal{J}(\boldsymbol{m}) = \min_{\boldsymbol{m}} \frac{1}{2}\|\mathcal{B}\boldsymbol{u}(\boldsymbol{m}) - \mathbf{u}^{\mathrm{obs}}\|^2 + \frac{1}{2}\|\mathcal{A}\boldsymbol{m}\|^2$$

where $\boldsymbol{u}$ is given by the solution of the linear elasticity equation.

- $\boldsymbol{u}(\boldsymbol{m})$ is given by the forward model.
- $\mathcal{B} : (L^2(\Omega))^3 \to \mathbb{R}^N$ is an observation operator.
- $\mathbf{u}^{\mathrm{obs}} \in \mathbb{R}^N$ where N is the number of data points.

[1] In literature, called the *parameter-to-observable operator*.

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If we let $\mathcal{S}$ be the forward model operator, i.e. $\boldsymbol{u} = \mathcal{S}\boldsymbol{m}$, and let[1] $\mathcal{F} = \mathcal{B}\mathcal{S}$, then:

$$\mathcal{J}(\boldsymbol{m}) = \min_{\boldsymbol{m}} \frac{1}{2}\|\mathcal{F}\boldsymbol{m} - \mathbf{u}^{\mathrm{obs}}\|^2 + \frac{1}{2}\|\mathcal{A}\boldsymbol{m}\|^2$$

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