# Computing Eigenvalue Sensitivities for Sensitivity Analysis of the Information Gain in Bayesian Linear Inverse Problems

Abhijit Chowdhary and Alen Alexanderian

Department of Mathematics
North Carolina State University

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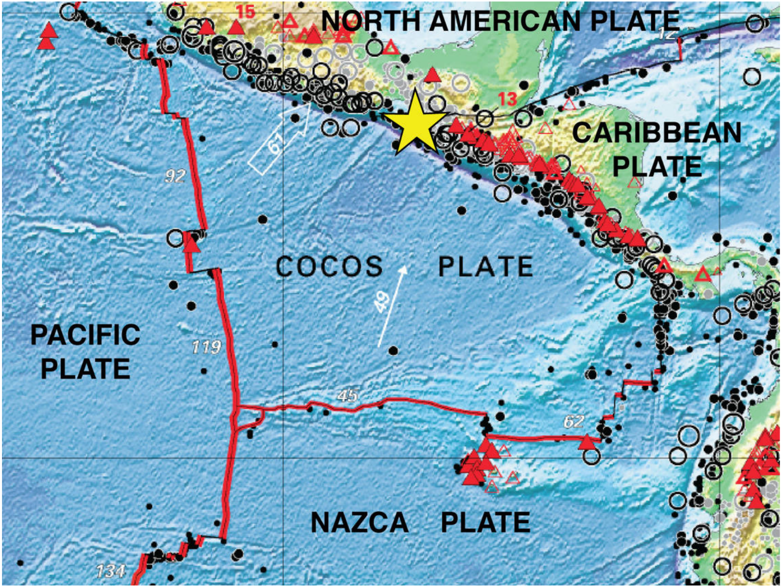
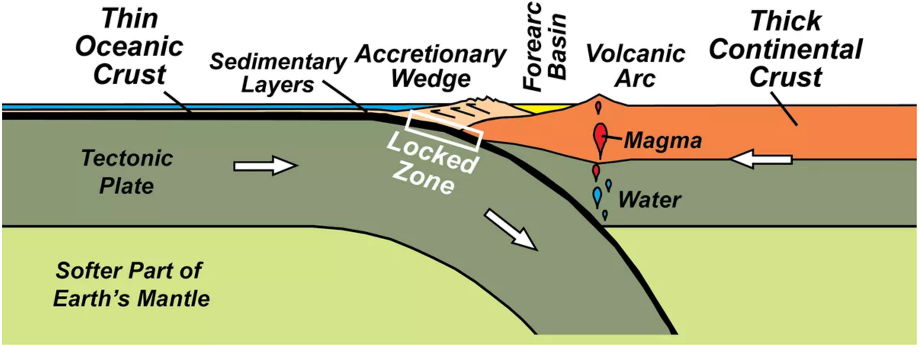



Tom Simkin et al. *This dynamic planet: World map of volcanoes, earthquakes, impact craters and plate tectonics*. Tech. rep. 2006. DOI: 10.3133/i2800




Robert J Lillie. *Oregon's Island In The Sky: Geology Road Guide to Marys Peak*. English. OCLC: 979996650. 2017. ISBN: 9781540611963

# Goals

Understand the subduction zone from collected observations.

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Do so while quantifying measurement uncertainties.

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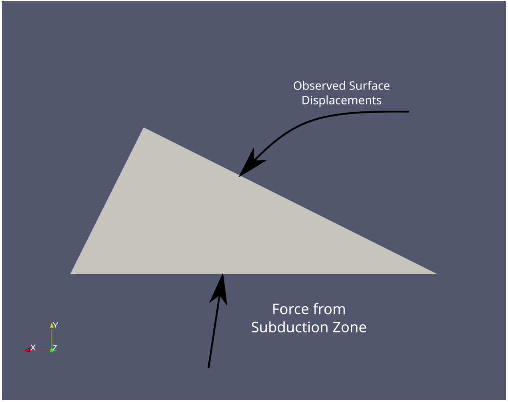
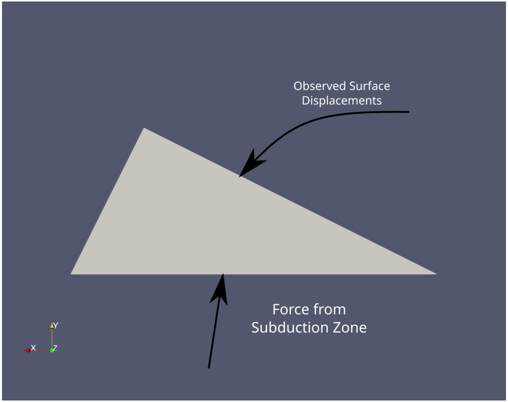
# Model Assumptions




- **Governing PDE** (forward model): Linear elasticity

# Model Assumptions

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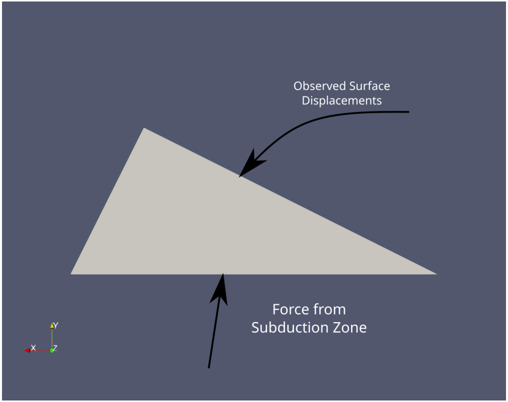


- **Governing PDE** (forward model): Linear elasticity
- **Uncertain parameter**: Displacement on fault plane
- **Inverse Problem**: Given measurements of surface deformation $\mathbf{u}^{\text{obs}}$ reconstruct fault plane displacement.

# Forward Model

$$-\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) = \mathbf{0} \text{ in } \Omega,$$

where:

- $\boldsymbol{\sigma}(\boldsymbol{u}) = \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u})$ with
  - $\mathbb{C}[\boldsymbol{\varepsilon}] = 2\mu\boldsymbol{\varepsilon} + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I}$ the fourth-order linear elasticity tensor:
  - $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}\left[\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^{\mathrm{T}}\right]$ the strain tensor.
- $\mu$ and $\lambda$ are the Láme constants.

Kimberly Alison McCormack. *Earthquakes, groundwater and surface deformation : Exploring the poroelastic response to megathrust earthquakes*. Aug. 2018. URL: hdl.handle.net/2152/68892

# Forward Model

$$
\begin{aligned}
-\boldsymbol{\nabla}\left[\mu(\boldsymbol{\nabla}\boldsymbol{u}+(\boldsymbol{\nabla}\boldsymbol{u})^{\mathrm{T}})+\lambda\boldsymbol{\nabla}\cdot\boldsymbol{u}\mathbf{I}\right] &= \mathbf{0} \quad \text{in } \Omega,\\
\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} &= \mathbf{0} \quad \text{on } \Gamma_t\\
\boldsymbol{u}+\beta\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} &= \boldsymbol{h} \quad \text{on } \Gamma_s\\
\boldsymbol{u}\cdot\boldsymbol{n} &= 0 \quad \text{on } \Gamma_b\\
\delta\mathbf{T}(\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n})+\mathbf{T}\boldsymbol{u} &= \boldsymbol{m} \quad \text{on } \Gamma_b
\end{aligned}
$$

- $\mathbf{T}$ is the tangential operator $\mathbf{T}\boldsymbol{u} = (\mathbf{I}-\boldsymbol{n}\otimes\boldsymbol{n})\boldsymbol{u} = \boldsymbol{u}-(\boldsymbol{n}^{\mathrm{T}}\boldsymbol{u})\boldsymbol{n}$.

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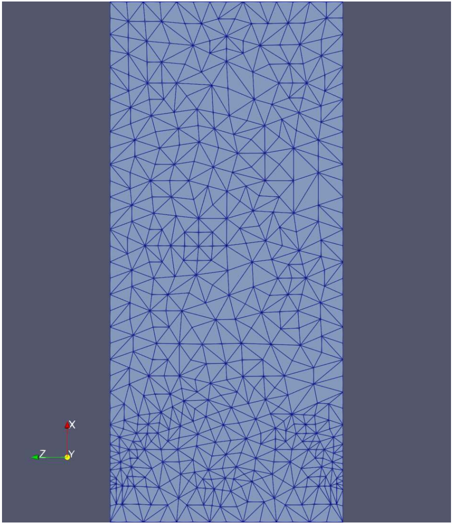
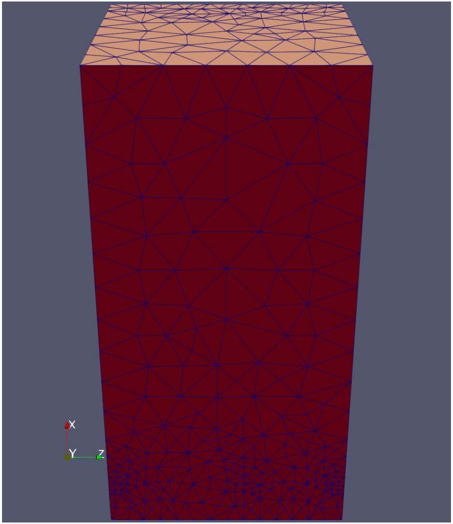
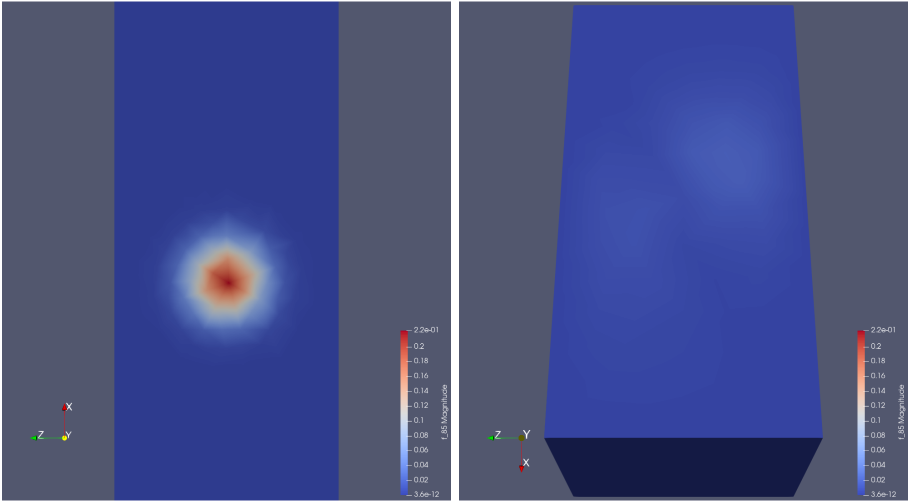
- $\boldsymbol{m}$ is the displacement on the fault plane that is being inverted for.

# Mesh



# Forward Solution



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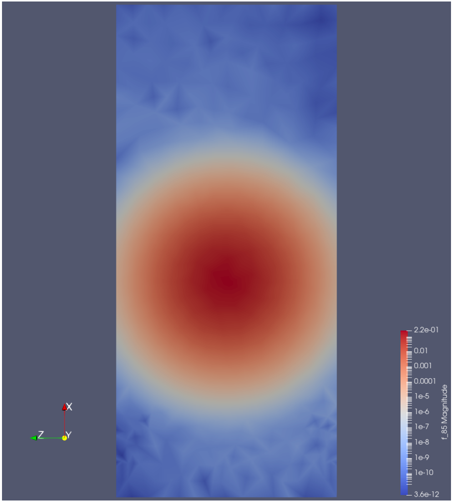


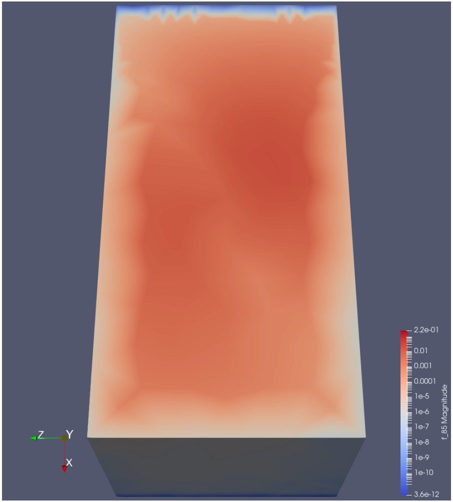

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# The Deterministic Inverse Problem

To reconstruct the fault displacement we construct the PDE-constrained optimization problem:

$$\mathcal{J}(\boldsymbol{m}) = \frac{1}{2}\|\boldsymbol{\mathcal{B}}\boldsymbol{u}(\boldsymbol{m}) - \mathbf{u}^{\mathrm{obs}}\|^2 + \frac{1}{2}\|\mathcal{R}\boldsymbol{m}\|^2$$

where $\boldsymbol{u}$ is given by the solution of the linear elasticity equation.

- $\boldsymbol{u}(\boldsymbol{m})$ is given by the forward model.
- $\boldsymbol{\mathcal{B}} : (L^2(\Omega))^3 \to \mathbb{R}^N$ is an observation operator.
- $\mathbf{u}^{\mathrm{obs}} \in \mathbb{R}^N$ where $N$ is the number of data points.
- $\mathcal{R}$ is some regularization operator.

[1] In literature, called the *parameter-to-observable operator*.

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- $\mathcal{R}$ is some regularization operator.

If we let $\boldsymbol{\mathcal{S}}$ be the forward model operator, i.e. $\boldsymbol{u} = \boldsymbol{\mathcal{S}}\boldsymbol{m}$, and let[1] $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{S}}$, then:

$$\mathcal{J}(\boldsymbol{m}) = \frac{1}{2}\|\boldsymbol{\mathcal{F}}\boldsymbol{m} - \mathbf{u}^{\mathrm{obs}}\|^2 + \frac{1}{2}\|\mathcal{R}\boldsymbol{m}\|^2$$

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# Bayesian Inversion in Infinite Dimensions

## Theorem (Bayes Theorem in Infinite Dimensions)

$$\frac{d\mu_{\text{post}}^{\mathbf{u}^{\text{obs}}}}{d\mu_{\text{pr}}} \propto \pi_{\text{like}}(\mathbf{u}^{\text{obs}}|\boldsymbol{m})$$

- Gaussian Prior $\boldsymbol{m} \sim \mathcal{N}(\boldsymbol{m}_{\text{pr}}, \boldsymbol{\mathcal{C}}_0)$.
- Additive Gaussian noise

$$\mathbf{u}^{\text{obs}} = \boldsymbol{\mathcal{F}}\boldsymbol{m} + \boldsymbol{\eta}, \quad \boldsymbol{\eta} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Gamma}_{\text{noise}})$$

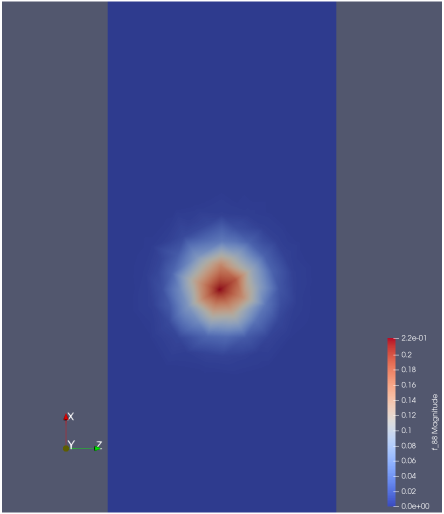
- Gaussian posterior:

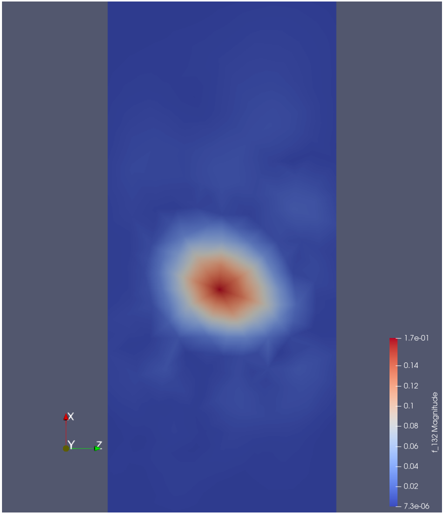
$$\mu_{\text{post}}^{\mathbf{u}^{\text{obs}}} = \mathcal{N}(\boldsymbol{m}_{\text{post}}, \boldsymbol{\mathcal{C}}_{\text{post}})$$

where:

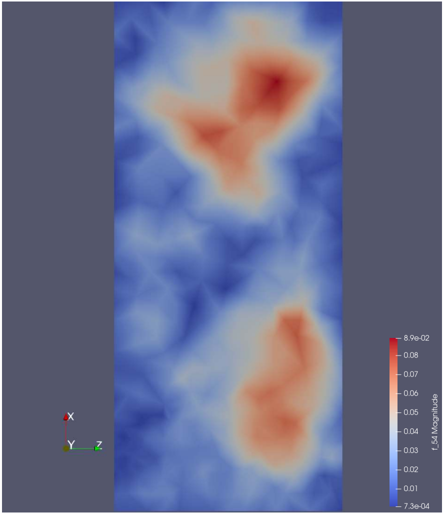
$$\boldsymbol{m}_{\text{post}} = \boldsymbol{\mathcal{C}}_{\text{post}} \left( \boldsymbol{\mathcal{F}}^* \boldsymbol{\Gamma}_{\text{noise}}^{-1} \mathbf{u}^{\text{obs}} + \boldsymbol{\mathcal{C}}_{\text{pr}}^{-1} \boldsymbol{m}_{\text{pr}} \right)$$
$$\boldsymbol{\mathcal{C}}_{\text{post}} = \left( \boldsymbol{\mathcal{F}}^* \boldsymbol{\Gamma}_{\text{noise}}^{-1} \boldsymbol{\mathcal{F}} + \boldsymbol{\mathcal{C}}_{\text{pr}}^{-1} \right)^{-1}$$

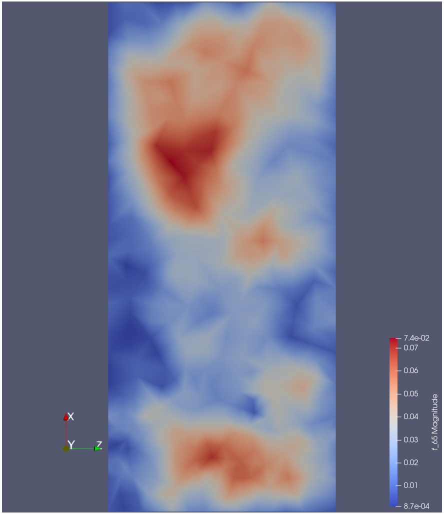
# MAP Estimation vs. True Parameter



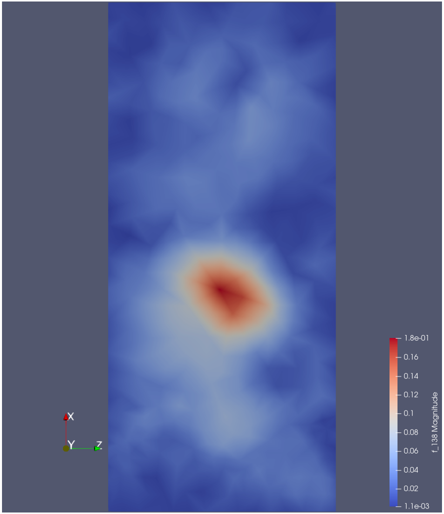



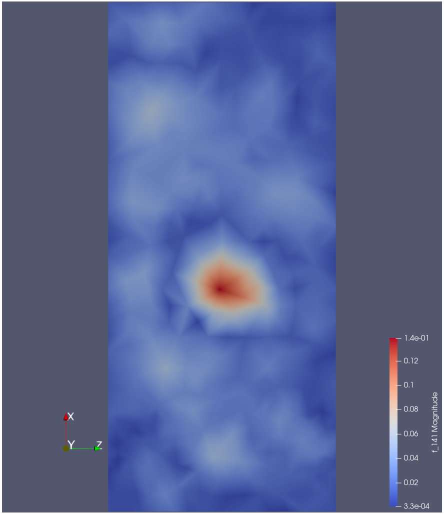
# Prior Samples



# Posterior Samples



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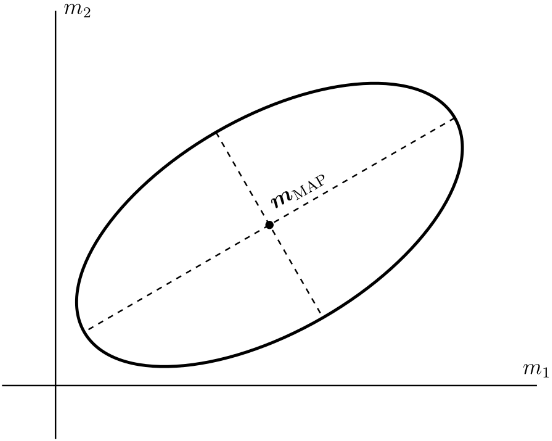


Figure: The confidence region corresponding to a linear inverse problem with a bivariate Gaussian posterior.

# Data-Misfit Hessian

## Definition (Data-Misfit Hessian)

The action of the data-misfit Hessian is given by:

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**Definition (Prior-Preconditioned Data-Misfit Hessian)**

$$\tilde{\boldsymbol{\mathcal{H}}} = \boldsymbol{\mathcal{C}}_{\text{prior}}^{1/2} \boldsymbol{\mathcal{H}} \boldsymbol{\mathcal{C}}_{\text{prior}}^{1/2}$$

Let $\{(\lambda_i, \psi_i)\}_{i=1}^k$ be the dominant eigenpairs of $\tilde{\boldsymbol{\mathcal{H}}}$, then:

$$\tilde{\boldsymbol{\mathcal{H}}} v \approx \sum_{i=1}^{k} \lambda_i \langle \psi_i, v \rangle \psi_i$$

is a low rank approximation of $\tilde{\boldsymbol{\mathcal{H}}}$.

# A Measure of Posterior Uncertainty

## Definition (Kullback-Leibler Divergence)

The KL-Divergence from the posterior to the prior quantifies the information gain and in the context of Bayesian linear inverse problems has the form:

$$\begin{aligned}\Phi_{\mathrm{KL}} &= \frac{1}{2}\left[\log\det\left(\tilde{\boldsymbol{\mathcal{H}}} + \boldsymbol{\mathcal{I}}\right) - \mathrm{trace}\left(\tilde{\boldsymbol{\mathcal{H}}}(\boldsymbol{\mathcal{I}} + \tilde{\boldsymbol{\mathcal{H}}})^{-1}\right) + \left\|\boldsymbol{m}_{\mathrm{MAP}} - \boldsymbol{m}_{\mathrm{pr}}\right\|^2_{\boldsymbol{\Gamma}^{-1}_{\mathrm{noise}}}\right] \\ &\approx \sum_{j=1}^{k}\frac{1}{2}\left[\log\left(\lambda_j + 1\right) - \frac{\lambda_j}{1+\lambda_j} + \left\|\boldsymbol{m}_{\mathrm{MAP}} - \boldsymbol{m}_{\mathrm{pr}}\right\|^2_{\boldsymbol{\Gamma}^{-1}_{\mathrm{noise}}}\right]\end{aligned}$$

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To assess the sensitivity of a Bayesian inverse problem with respect to modeling uncertaintities, we compute partial derivative of $\Phi_{\mathrm{KL}}$ with respect to said model parameters.

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$$\Phi_{\mathrm{KL}} =\approx \sum_{j=1}^{k} \frac{1}{2} \left[ \underbrace{\log\left(\lambda_j + 1\right) - \frac{\lambda_j}{1+\lambda_j}}_{\text{Eigenvalue Sensitivities}} + \underbrace{\left\| \boldsymbol{m}_{\mathrm{MAP}} - \boldsymbol{m}_{\mathrm{pr}} \right\|^2_{\boldsymbol{\Gamma}^{-1}_{\mathrm{noise}}}}_{\text{HDSA}} \right]$$

Isaac Sunseri et al. "Hyper-differential sensitivity analysis for inverse problems constrained by partial differential equations". In: *Inverse Problems* 36.12 (Dec. 2020), p. 125001. DOI: 10.1088/1361-6420/abaf63

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# Simple Finite Dimensional Example

Consider an implicitly defined matrix $\mathbf{H}$ whose action is defined by:

$$\mathbf{H}\mathbf{v} := \mathbf{C}^{\mathrm{T}}\mathbf{p} \tag{1a}$$

where

$$\mathbf{A}(\theta)\mathbf{u} + \mathbf{C}\mathbf{v} = \mathbf{0} \tag{1b}$$

$$\mathbf{A}(\theta)^{\mathrm{T}}\mathbf{p} + \mathbf{B}^{\mathrm{T}}\mathbf{B}\mathbf{u} = \mathbf{0} \tag{1c}$$

Furthermore, consider $\tilde{\mathbf{H}} = \mathbf{\Gamma}^{1/2}\mathbf{H}\mathbf{\Gamma}^{1/2}$ with $\mathbf{\Gamma}$ symmetric positive definite.

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To illustrate eigenvalue sensitivity analysis, we consider sensitivity of the largest eigenvalue of $\tilde{\mathbf{H}}$ with respect to $\theta$.

**Note:**

Let $(\lambda, \mathbf{w})$ be the dominant eigenpair of $\tilde{\mathbf{H}}$, then $\mathbf{H}\mathbf{v} = \lambda\mathbf{\Gamma}^{-1}\mathbf{v}$ with $\mathbf{v} = \mathbf{\Gamma}^{1/2}\mathbf{w}$.

# Derivation

We can define $\lambda$ as a function of uncertain model parameters implicitly:

$$\lambda(\theta) = \mathbf{v}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{p}$$

where

$$\mathbf{A}(\theta)\mathbf{u} + \mathbf{C}\mathbf{v} = \mathbf{0}$$

$$\mathbf{A}(\theta)^{\top}\mathbf{p} + \mathbf{B}^{\mathrm{T}}\mathbf{B}\mathbf{u} = \mathbf{0}$$

$$\mathbf{v}^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}\mathbf{v} - 1 = 0$$

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&\mathbf{v}^{\mathrm{T}}\mathbf{\Gamma}^{-1}\mathbf{v} - 1 = 0
\end{aligned}
$$

## Definition

To facilitate derivative computation, we consider the "meta-Lagrangian":

$$
\mathcal{L} = \langle \mathbf{v}, \mathbf{C}^{\mathrm{T}}\mathbf{p} \rangle + \langle \mathbf{p}^*, \mathbf{A}(\theta)\mathbf{u} + \mathbf{C}\mathbf{v} \rangle + \langle \mathbf{u}^*, \mathbf{A}(\theta)^{\mathrm{T}}\mathbf{p} - \mathbf{B}^{\mathrm{T}}\mathbf{B}\mathbf{u} \rangle + \lambda^* \left(1 - \mathbf{v}^{\top}\mathbf{\Gamma}^{-1}\mathbf{v}\right)
$$

# Solving for Lagrange Multipliers

$$\mathcal{L} = \langle \mathbf{v}, \mathbf{C}^{\mathrm{T}}\mathbf{p} \rangle + \langle \mathbf{p}^*, \mathbf{A}(\theta)\mathbf{u} + \mathbf{C}\mathbf{v} \rangle + \langle \mathbf{u}^*, \mathbf{A}(\theta)^{\mathrm{T}}\mathbf{p} - \mathbf{B}^{\mathrm{T}}\mathbf{B}\mathbf{u} \rangle + \lambda^* \left(1 - \mathbf{v}^\top \mathbf{\Gamma}^{-1}\mathbf{v}\right)$$

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\mathcal{L}_{\mathbf{u}} &= \mathbf{A}^\top \mathbf{p}^* + \mathbf{B}^\top \mathbf{B}\mathbf{u}^* = 0 \\
\mathcal{L}_{\mathbf{p}} &= \mathbf{A}\mathbf{u}^* + \mathbf{C}\mathbf{v} = 0 \\
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$$\mathcal{L}_{\mathbf{u}} = \mathbf{A}^{\top}\mathbf{p}^* + \mathbf{B}^{\top}\mathbf{B}\mathbf{u}^* = 0$$
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$$\mathcal{L}_{\mathbf{v}} = 2\mathbf{C}^{\top}\mathbf{p} - 2\lambda^*\mathbf{\Gamma}^{-1}\mathbf{v} = 0$$

That is, $\mathbf{u}^* = \mathbf{u}$ and $\mathbf{p}^* = \mathbf{p}$.

$$\lambda(\theta) = \mathbf{v}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{p}$$

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$$\mathbf{A}(\theta)\mathbf{u} + \mathbf{C}\mathbf{v} = \mathbf{0}$$
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\end{aligned}$$

That is, $\mathbf{u}^* = \mathbf{u}$ and $\mathbf{p}^* = \mathbf{p}$. Regarding $\lambda^*$:

$$0 = 2\mathbf{p}^{\mathrm{T}}\mathbf{C}^\top \mathbf{p} - 2\lambda^* \mathbf{p}^{\mathrm{T}}\mathbf{\Gamma}^{-1}\mathbf{v} = 2\lambda - 2\lambda^* \implies \lambda = \lambda^*$$

# Eigenvalue Sensitivity

$$\mathcal{L} = \langle \mathbf{v}, \mathbf{C}^{\mathrm{T}}\mathbf{p} \rangle + \langle \mathbf{p}^*, \mathbf{A}(\theta)\mathbf{u} + \mathbf{C}\mathbf{v} \rangle + \langle \mathbf{u}^*, \mathbf{A}(\theta)^{\mathrm{T}}\mathbf{p} - \mathbf{B}^{\mathrm{T}}\mathbf{B}\mathbf{u} \rangle + \lambda^* \left(1 - \mathbf{v}^{\top}\mathbf{\Gamma}^{-1}\mathbf{v}\right)$$

Finally, $\mathcal{L}_{\theta_j}$ simplifies to

$$\mathcal{L}_{\theta_j} = \langle \mathbf{p}^*, [\partial_j \mathbf{A}]\mathbf{u} \rangle + \langle \mathbf{u}^*, [\partial_j \mathbf{A}]^{\top}\mathbf{p} \rangle = 2\langle \mathbf{p}, [\partial_j \mathbf{A}]\mathbf{u} \rangle$$

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$$\mathcal{L}_{\theta_j} = \langle \mathbf{p}^*, [\partial_j \mathbf{A}]\mathbf{u} \rangle + \langle \mathbf{u}^*, [\partial_j \mathbf{A}]^\top \mathbf{p} \rangle = 2\langle \mathbf{p}, [\partial_j \mathbf{A}]\mathbf{u} \rangle$$

Therefore, to compute the sensitivity of $\lambda$ with respect to $\theta_j$, we have the following algorithm:

- solve the eigenproblem for eigenvector $\mathbf{v}$

# Eigenvalue Sensitivity

$$\mathcal{L} = \langle \mathbf{v}, \mathbf{C}^{\mathrm{T}}\mathbf{p} \rangle + \langle \mathbf{p}^*, \mathbf{A}(\theta)\mathbf{u} + \mathbf{C}\mathbf{v} \rangle + \langle \mathbf{u}^*, \mathbf{A}(\theta)^{\mathrm{T}}\mathbf{p} - \mathbf{B}^{\mathrm{T}}\mathbf{B}\mathbf{u} \rangle + \lambda^* \left(1 - \mathbf{v}^\top \mathbf{\Gamma}^{-1}\mathbf{v}\right)$$

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Therefore, to compute the sensitivity of $\lambda$ with respect to $\theta_j$, we have the following algorithm:

- solve the eigenproblem for eigenvector $\mathbf{v}$
- solve $\mathbf{A}\mathbf{u} + \mathbf{C}\mathbf{v} = \mathbf{0}$ for $\mathbf{u}$
- solve $\mathbf{A}^{\mathrm{T}}\mathbf{p} + \mathbf{B}^{\mathrm{T}}\mathbf{B}\mathbf{u} = \mathbf{0}$ for $\mathbf{p}$
- evaluate eigenvalue sensitivities $\partial_j \lambda(\theta) = 2\langle \mathbf{p}, [\partial_j \mathbf{A}]\mathbf{u}\rangle, \quad j \in \{1, \dots, N\}$

# Simple Numerical Example

$$\mathbf{A} = \begin{bmatrix} -20(1+a_2\theta_2) & 3(1+a_3\theta_3) & 0 \\ 1+a_1\theta_1 & -20(1+a_2\theta_2) & 3(1+a_3\theta_3) \\ 0 & 1+a_1\theta_1 & -20(1+a_2\theta_2) \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} 0.08 & 0.06 & 0.09 \end{bmatrix}^\top$$

Also, the remaining matrices for the present example are as follows:

$$\mathbf{B} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad \mathbf{\Gamma} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In this case, the matrix **H** is a $2 \times 2$ matrix.

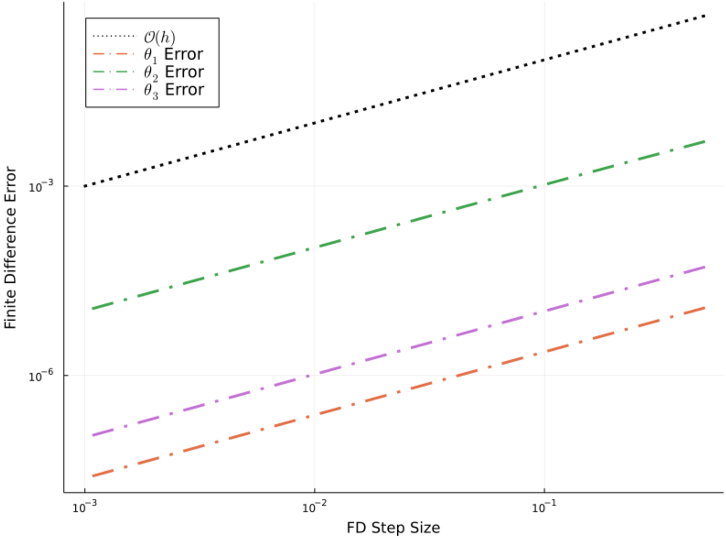


Figure: Forward finite difference error vs. exact

# Table of Contents

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For a domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$, consider the elliptic PDE:

$$\begin{aligned} -\nabla^2 u + cu &= f, \quad \text{in } \Omega \\ \boldsymbol{\nabla} u \cdot n &= g, \quad \text{in } \partial\Omega \end{aligned}$$

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Using measurements of $u$ on $\Omega$, we seek to estimate $f$.

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## Bayesian Linear Inverse Problem

Using measurements of $u$ on $\Omega$, we seek to estimate $f$.

## Goal:

Study the sensitivity of the largest eigenvalue of the data-misfit Hessian for the inversion with respect to constant $c$.

$$\mathcal{L}(u, p, f) = \frac{1}{2}\left\|\mathcal{B}u(f) - \mathbf{u}^{\text{obs}}\right\|^2_{\boldsymbol{\Gamma}^{-1}_{\text{noise}}} + \frac{1}{2}\left\|f - f_{\text{pr}}\right\|^2_{\mathcal{C}_0^{-1}}$$

$$\int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} p \, \mathrm{d}x + c\int_\Omega up \, \mathrm{d}x - \int_\Omega fp \, \mathrm{d}x - \int_{\partial\Omega} gp \, \mathrm{d}s$$

# The adjoint based expression the data-misfit Hessian action

Taking variations in $u, p$ and $f$ twice, we find the data-misfit Hessian:

$$\mathcal{L}_f^H(\tilde{f}) = \underbrace{-\int_\Omega \tilde{f}\,\hat{p}\,\mathrm{d}x}_{\mathcal{H}(f)(\hat{f},\tilde{f})} + \langle \tilde{f}, \hat{f} \rangle_{\mathcal{C}_0^{-1}}$$

which, to evaluate in direction $\hat{f}$, we need to solve the incremental system:

$$\begin{aligned}
\mathcal{L}_v^H(\tilde{p}) &= \int_\Omega \nabla \hat{u} \cdot \nabla \tilde{p}\,\mathrm{d}x + c\int_\Omega \hat{u}\tilde{p}\,\mathrm{d}x - \int_\Omega \hat{f}\,\tilde{p}\,\mathrm{d}x &&= 0, \quad \forall \tilde{p} \in V \\
\mathcal{L}_u^H(\tilde{u}) &= \int_\Omega \nabla \tilde{u} \cdot \nabla \hat{p}\,\mathrm{d}x + c\int_\Omega \tilde{u}\hat{p}\,\mathrm{d}x + \langle \mathcal{B}\tilde{u}, \mathcal{B}\hat{u} \rangle_{\boldsymbol{\Gamma}_{\mathrm{noise}}^{-1}} &&= 0, \quad \forall \tilde{u} \in V
\end{aligned}$$

# Eigenproblem Constraint

$$
\begin{aligned}
\int_\Omega \nabla \hat{u} \cdot \nabla \tilde{p} \,\mathrm{d}x + c \int_\Omega \hat{u}\tilde{p} \,\mathrm{d}x - \int_\Omega \psi \tilde{p} \,\mathrm{d}x &= 0, && \forall \tilde{p} \in V \\
\int_\Omega \nabla \tilde{u} \cdot \nabla \hat{p} \,\mathrm{d}x + c \int_\Omega \tilde{u}\hat{p} \,\mathrm{d}x + \langle \mathcal{B}\tilde{u}, \mathcal{B}\hat{u} \rangle_{\Gamma_{\mathrm{noise}}^{-1}} &= 0, && \forall \tilde{u} \in V \\
- \int_\Omega \phi \hat{p} \,\mathrm{d}x &= \lambda \langle \phi, \psi \rangle_{\mathcal{C}_0^{-1}}, && \forall \phi \in V \\
1 - \langle \psi, \psi \rangle_{\mathcal{C}_0^{-1}} &= 0
\end{aligned}
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\end{aligned}$$

$$\begin{aligned}
\mathcal{L}^E(\hat{u}, \hat{p}, \psi, \hat{u}^*, \hat{p}^*, \lambda^*) = &\left[ - \int_\Omega \psi \hat{p} \,\mathrm{d}x \right] + \lambda^* \left[ 1 - \langle \psi, \psi \rangle_{\mathcal{C}_0^{-1}} \right] \\
&+ \left[ \int_\Omega \nabla \hat{u} \cdot \nabla \hat{p}^* \,\mathrm{d}x + c \int_\Omega \hat{u}\hat{p}^* \,\mathrm{d}x - \int_\Omega \psi \hat{p}^* \,\mathrm{d}x \right] \\
&+ \left[ \int_\Omega \nabla \hat{u}^* \cdot \nabla \hat{p} \,\mathrm{d}x + c \int_\Omega \hat{u}^* \hat{p} \,\mathrm{d}x + \langle \mathcal{B}\hat{u}^*, \mathcal{B}\hat{u} \rangle_{\Gamma_{\text{noise}}^{-1}} \right]
\end{aligned}$$

# Solving for Lagrange Multipliers

$$
\begin{aligned}
\mathcal{L}^E_{\hat{p}}(\tilde{p}) &= \int_\Omega \nabla \hat{u}^* \cdot \nabla \tilde{p}\,\mathrm{d}x + c\int_\Omega \hat{u}^* \tilde{p}\,\mathrm{d}x - \int_\Omega \psi \tilde{p}\,\mathrm{d}x && = 0, \forall \tilde{p} \in V \\
\mathcal{L}^E_{\hat{u}}(\tilde{u}) &= \int_\Omega \nabla \tilde{u} \cdot \nabla \hat{p}^*\,\mathrm{d}x + c\int_\Omega \tilde{u}\hat{p}^*\,\mathrm{d}x + \langle \mathcal{B}\hat{u}^*, \mathcal{B}\tilde{u}\rangle_{\Gamma^{-1}_{\mathrm{noise}}} && = 0, \forall \tilde{u} \in V \\
\mathcal{L}^E_{\psi}(\tilde{\psi}) &= -2\int_\Omega \tilde{\psi}\hat{p}\,\mathrm{d}x - 2\lambda^* \langle \psi, \tilde{\psi}\rangle_{\mathcal{C}_0^{-1}} && = 0, \forall \tilde{\psi} \in V
\end{aligned}
$$

# Solving for Lagrange Multipliers: Relating it Back

$$\mathcal{L}^E_{\hat{p}}(\tilde{p}) = \underbrace{\int_\Omega \boldsymbol{\nabla}\hat{u}^* \cdot \boldsymbol{\nabla}\tilde{p}\,\mathrm{d}x + c\int_\Omega \hat{u}^*\tilde{p}\,\mathrm{d}x}_{\tilde{\mathbf{p}}^{\mathrm{T}}\mathbf{A}\hat{\mathbf{u}}^*} \underbrace{- \int_\Omega \psi\tilde{p}\,\mathrm{d}x}_{\tilde{\mathbf{p}}^{\mathrm{T}}\mathbf{C}\psi} \qquad = 0, \forall \tilde{p} \in V$$

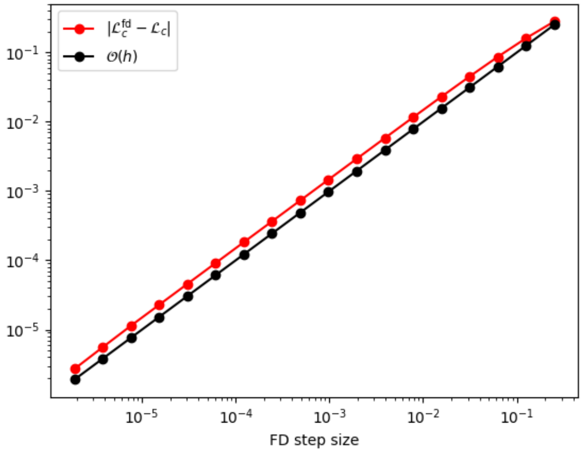
$$\mathcal{L}^E_{\hat{u}}(\tilde{u}) = \underbrace{\int_\Omega \boldsymbol{\nabla}\tilde{u} \cdot \boldsymbol{\nabla}\hat{p}^*\,\mathrm{d}x + c\int_\Omega \tilde{u}\hat{p}^*\,\mathrm{d}x}_{\tilde{\mathbf{u}}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\hat{\mathbf{p}}^*} + \langle \mathcal{B}\hat{u}^*, \mathcal{B}\tilde{u}\rangle_{\boldsymbol{\Gamma}^{-1}_{\mathrm{noise}}} \qquad = 0, \forall \tilde{u} \in V$$

$$\mathcal{L}^E_{\psi}(\tilde{\psi}) = -2\int_\Omega \tilde{\psi}\hat{p}\,\mathrm{d}x - 2\lambda^*\langle \psi, \tilde{\psi}\rangle_{\mathcal{C}_0^{-1}} \qquad = 0, \forall \tilde{\psi} \in V$$

Like in the finite dimensional example, we find $\hat{u}^* = \hat{u}, \hat{p}^* = \hat{p}$ and $\lambda^* = \lambda$. Hence:

$$\mathcal{L}^E_c = \int_\Omega \hat{u}\hat{p}^*\,\mathrm{d}x + \int_\Omega \hat{u}^*\hat{p}\,\mathrm{d}x = 2\int_\Omega \hat{u}\hat{p}\,\mathrm{d}x$$

# Forward finite difference error vs. exact



# Conclusion

In summary,

- Eigenvalue Sensitivities is a scalable and exact method of leveraging existing structures to compute derivatives of eigenvalue based expressions.
- Avoids difficulties in finite-difference methods resolving and solves half the puzzle of using the information gain as a sensitivity objective.

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- Eigenvalue Sensitivities is a scalable and exact method of leveraging existing structures to compute derivatives of eigenvalue based expressions.
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## Future Work

Wrap up sensitivity analysis of the Costa-Rica problem via the KL-Divergence.

- Complete HDSA portion of KL Divergence sensitivities
- Interpret Sensitivities in terms of real-world model