Notes for 'Differentials and applications'

Important Ideas and Useful Facts:

(i) Interpreting the equation linking differentials: Consider $y = f(x)$ with derivative $y' = f'(x)$. In Leibniz notation we have

$$\frac{dy}{dx} = y' = f'(x) ,$$

which can be rewritten as an equation using differentials:

$$dy = f'(x)dx ,$$

which may be regarded as an 'idealisation' of the following approximation:

$$\Delta y \approx f'(x)\Delta x ,$$

where Δx is a small change in x and Δy is the corresponding small change in y. We thus get the following approximation:

$$y + \Delta y \approx y + f'(x)\Delta x .$$

This approximation improves (towards equality 'in the limit') as Δx approaches 0. The reason is that the tangent line to the curve is a good approximation to the curve at a given point of interest.

(ii) The equation of the tangent line to the natural exponential function at the y-intercept: Euler's number e is defined so that the slope of the tangent line to the curve $y = e^x$ at the y-intercept has slope 1. It follows that the equation of this tangent line is

$$y = x + 1 ,$$

and it becomes a good approximation to the curve near the y-intercept. A consequence is that, for small real numbers Δ, we have

$$1 + \Delta \approx e^{\Delta} ,$$

which is often used to simplify formulae in applications, especially when Δ represents some small periodic percentage increase.

(iii) The equation of the tangent line to the natural logarithm function at the x-intercept: The slope of the tangent line to the curve $y = \ln x$ at the x-intercept has slope 1. It follows that the equation of this tangent line is

$$y = x - 1 ,$$

and it becomes a good approximation to the curve near the x-intercept. A consequence is that, for small real numbers Δ, we have

$$\ln(1 + \Delta) \approx 1 + \Delta - 1 = \Delta .$$

An application is the *Rule of Seventy*, which says that the number of years needed for the principal to double in value, when invested at compound annual interest rate $i\%$, is approximately 70 divided by i. For example, if the interest rate is 2% then one expects the principal to take about $70/2 = 35$ years to double.

Examples and derivations:

1. Use the Rule of Seventy to estimate how long it takes for the principal to double when invested at 1%, 5% and 10% compound annual interest rates respectively.

   Solution: We estimate that it takes $70/1 = 70$, $70/5 = 14$ and $70/10 = 7$ years respectively for the principal to double.

2. Estimate the annual interest rate $i\%$ (to the nearest tenth of one per cent) required that a given principal should double within about 25 years.

   Solution: We want i such that $25 \approx 70/i$, that is,

   $$i \approx \frac{70}{25} = 2.8 .$$

   Thus we estimate that, using an interest rate of about 2.8%, the principal should double in about 25 years.

3. We explain where the Rule of Seventy comes from. Let P be the principal that has been invested initially at the compound interest rate of $i\%$ per annum. Let $y = y(x)$ be the value of the investment after x years, so $P = y(0)$ and

   $$y = y(x) = P\left(1 + \frac{i}{100}\right)^x .$$

   We want to estimate the number of years x such that $y(x) = 2P$, that is,

   $$2P = P\left(1 + \frac{i}{100}\right)^x ,$$

   which becomes, after cancelling P from both sides,

   $$2 = \left(1 + \frac{i}{100}\right)^x .$$

   Taking natural logarithms of both sides, we get

   $$\ln 2 = x \ln\left(1 + \frac{i}{100}\right) .$$

   Rearranging, and using the approximation $\ln(1 + \Delta) \approx \Delta$ noted earlier, taking $\Delta = \frac{i}{100}$, we get

   $$x = \frac{\ln 2}{\ln\left(1 + \frac{i}{100}\right)} \approx \frac{\ln 2}{i/100} = \frac{100 \ln 2}{i} ,$$

   which we expect to be a good approximation for small i. But the numerator can be approximated by 69, 70 and 72:

   $$100 \ln 2 \approx 69.3 \approx 69 \approx 70 \approx 72 ,$$

   leading to the Rule of Seventy, by dividing i into 70.

   It also leads to the Rule of Sixty-nine (slightly more accurate), by dividing i into 69, and the Rule of Seventy-two (the least accurate), by dividing i into 72.

   The Rule of Seventy-two is a convenient rule of thumb if the interest rate happens to exactly divide 72. For example, if the principal is invested at 6% per annum, then this rule estimates that it takes about $72/6 = 12$ years to double. It is slightly awkward to divide 6 into 69 or 70, and in both cases the answer rounds up to 12.

4. Find an equation relating the differentials dy and dx, and an associated approximation relating small changes Δy and Δx, when $y = x^2$.

Use this approximation formula to estimate the change in area of a square paddock of side length 100 metres, if the side length is extended by 10 metres, 5 metres, 1 metre and 1 cm respectively.

Compare the answers with the exact changes in area in each case.

Solution: We have $dy/dx = y' = 2x$, so that

$$dy \;=\; 2xdx \qquad \text{and} \qquad \Delta y \;\approx\; 2x\Delta x \; .$$

Let y square metres denote the area of the paddock of side length x metres, so $y = x^2$. We are considering making small changes to $x = 100$ and $y = 100^2 = 10,000$.

In the first case, $\Delta x = 10$, and, by our formula,

$$\Delta y \;\approx\; 2x\Delta x \;=\; 2 \times 100 \times 10 \;=\; 2,000 \; ,$$

so we expect an increase of about $2,000$ square metres. But

$$y(110) - y(100) \;=\; 110^2 - 100^2 \;=\; 2,100 \; ,$$

so the true change in area is $2,100$ square metres, which agrees with our estimate to within 100 square metres.

In the second case, $\Delta x = 5$, and, by our formula,

$$\Delta y \;\approx\; 2x\Delta x \;=\; 2 \times 100 \times 5 \;=\; 1,000 \; ,$$

so we expect an increase of about $1,000$ square metres. But

$$y(105) - y(100) \;=\; 105^2 - 100^2 \;=\; 1,025 \; ,$$

so the true change in area is $1,025$ square metres, which agrees with our estimate to within 25 square metres.

In the third case, $\Delta x = 1$, and, by our formula,

$$\Delta y \;\approx\; 2x\Delta x \;=\; 2 \times 100 \times 1 \;=\; 200 \; ,$$

so we expect an increase of about 200 square metres. But

$$y(101) - y(100) \;=\; 101^2 - 100^2 \;=\; 201 \; ,$$

so the true change in area is 201 square metres, which agrees with our estimate to within 1 square metre.

In the fourth case, $\Delta x = 0.01$ (since 1 cm equals 0.01 m), and, by our formula,

$$\Delta y \;\approx\; 2x\Delta x \;=\; 2 \times 100 \times 0.01 \;=\; 2 \; ,$$

so we expect an increase of about 2 square metres. But

$$y(100.01) - y(100) \;=\; 100.01^2 - 100^2 \;=\; 2.0001 \; ,$$

so the true change in area is 2.0001 square metres, which agrees with our estimate to within 0.0001 square metres.

5. Find the equation of the tangent line to the curve $y = x^2$ at the point $(3, 9)$ and use it to estimate 3.01^2, and compare the estimate with the true value.

Solution: We have $y' = 2x$, so the slope of the tangent line is $2 \times 3 = 6$. Hence the tangent line must have equation

$$y \;=\; 6x + k \;,$$

for some constant k. But the point $(3, 9)$ lies on this line, so $9 = 6(3) + k = 18 + k$, so that $k = -9$. Hence the equation of the tangent line is

$$y \;=\; 6x - 9 \;.$$

Using the tangent line, with input $x = 3.01$, gives the estimate

$$3.01^2 \;\approx\; 6(3.01) - 9 \;=\; 9.06 \;.$$

In fact, $3.01^2 = 9.0601$, which agrees with the estimate to three decimal places.

6. Find the equation of the tangent line to the curve $y = \sqrt{x}$ at the point $(9, 3)$ and use it to estimate $\sqrt{9.5}$ and $\sqrt{8.5}$, and compare with the true values.

Solution: We have $y' = \frac{1}{2\sqrt{x}}$, so the slope of the tangent line is $\frac{1}{2\sqrt{9}} = \frac{1}{6}$. Hence the tangent line must have equation

$$y \;=\; \frac{x}{6} + k \;,$$

for some constant k. But the point $(9, 3)$ lies on this line, so $3 = \frac{9}{6} + k = \frac{3}{2} + k$, so that $k = \frac{3}{2}$. Hence the equation of the tangent line is

$$y \;=\; \frac{x}{6} + \frac{3}{2} \;=\; \frac{x+9}{6} \;.$$

Using the tangent line, with input $x = 9.5$, gives the estimate

$$\sqrt{9.5} \;\approx\; \frac{9.5+9}{6} \;=\; 3.08\dot{3} \;.$$

In fact, $\sqrt{9.5} \approx 3.0822$, to four decimal places, which agrees with the estimate to two decimal places.

Using the tangent line, with input $x = 8.5$, gives the estimate

$$\sqrt{8.5} \;\approx\; \frac{8.5+9}{6} \;=\; 2.91\dot{6} \;.$$

In fact, $\sqrt{8.5} \approx 2.9155$, to four decimal places, which again agrees with the estimate to two decimal places.

7. Find the equation of the tangent line to the curve $y = \sqrt[3]{x} = x^{1/3}$ at the point $(64, 4)$ and use it to estimate $\sqrt[3]{70}$, $\sqrt[3]{65}$ and $\sqrt[3]{63}$, and compare with the true values.

Solution: We have $y' = \frac{1}{3}x^{-2/3}$, so the slope of the tangent line is $\frac{1}{3} \times 64^{-2/3} = \frac{1}{48}$. Hence the tangent line must have equation

$$y \;=\; \frac{x}{48} + k \;,$$

for some constant k. But the point $(64, 4)$ lies on this line, so $4 = \frac{64}{48} + k = \frac{4}{3} + k$, so that $k = \frac{8}{3}$. Hence the equation of the tangent line is

$$y = \frac{x}{48} + \frac{8}{3} = \frac{x + 128}{48} .$$

Using the tangent line, with input $x = 70$, gives the estimate

$$\sqrt[3]{70} \approx \frac{70 + 128}{48} = 4.125 .$$

In fact, $\sqrt[3]{70} \approx 4.1213$, to four decimal places, which agrees with the estimate to almost two decimal places.

Using the tangent line, with input $x = 65$, gives the estimate

$$\sqrt[3]{65} \approx \frac{65 + 128}{48} = 4.0208\dot{3} .$$

In fact, $\sqrt[3]{65} \approx 4.0207$, to four decimal places, which agrees with the estimate to almost four decimal places.

Using the tangent line, with input $x = 63$, gives the estimate

$$\sqrt[3]{63} \approx \frac{63 + 128}{48} = 3.9791\dot{6} .$$

In fact, $\sqrt[3]{63} \approx 3.9791$, to four decimal places, which again agrees with the estimate to almost four decimal places.

8. Imagine that you are manufacturing metal ball-bearings in the shape of spheres with radius 5 mm, with up to 2% error in the radius. The problem is to use differentials to estimate the percentage error in volume.

Solution: The volume $V = V(r)$ of a sphere of radius r is given by the formula

$$V = V(r) = \frac{4\pi r^3}{3} .$$

Hence $dV/dr = 4\pi r^2$, so that $dV = 4\pi r^2 dr$, becoming the following approximation, relating small changes ΔV in the volume to small changes Δr in the radius:

$$\Delta V \approx 4\pi r^2 \Delta r .$$

In our application, $r = 5$. We are not sure whether the fluctuations in the radius cause it to be smaller or greater than $r = 5$. To capture all possibilities, we use the magnitude of Δr, and express the given information in terms of a bound on the true proportion of r that this error in the radius represents:

$$\frac{|\Delta r|}{r} \leq 0.02 .$$

Working now with the proportion of V that the magnitude of ΔV represents, we have

$$\frac{|\Delta V|}{V} \approx \frac{4\pi r^2 \Delta r}{4\pi r^3/3} = \frac{3\Delta r}{r} \leq 3 \times 0.02 = 0.06 .$$

Hence we estimate an error in the volume of the ball-bearings of up to about 6%.

Notice that the final answer is independent of the value of the radius r.