The University of Sydney

MOOC Introduction to Calculus

Notes for 'Different kinds of limits'

Important Ideas and Useful Facts:

(i) Two-sided finite limits: Let $y = f(x)$ be a function defined on an interval containing $x = a$, but not necessarily defined at $x = a$, and suppose that L is a real number. We say that the *limit of $f(x)$ is L as x approaches a*, and write

$$\lim_{x \to a} f(x) \;=\; L \;,$$

if $f(x)$ gets closer and closer to L as x gets closer and closer to a. There are no restrictions about how x approaches a, from below or above, but we avoid allowing x to become equal to a.

The arrow notation $x \to a$ is a symbolic translation of "x approaches a".

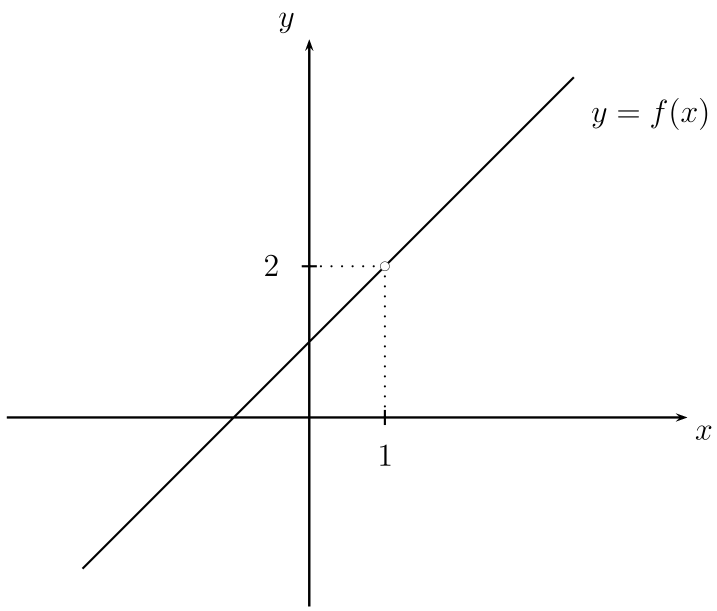
For example, consider the function f given by the rule

$$f(x) \;=\; \frac{x^2 - 1}{x - 1} \;.$$

Then $f(1)$ is not defined and the domain of f is $\mathbb{R}\backslash\{1\}$. Nevertheless, we can investigate the behaviour of $f(x)$ as x gets closer and closer to 1 from either side, and discover that

$$\lim_{x \to 1} f(x) \;=\; \lim_{x \to 1} \frac{x^2 - 1}{x - 1} \;=\; \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1} \;=\; \lim_{x \to 1} (x + 1) \;=\; 2 \;.$$

In fact the rule for f simplifies and just becomes $f(x) = x + 1$, provided $x \neq 1$, and the graph of f is just the line $y = x + 1$ with a "hole" at the missing point $(1, 2)$.



(ii) One-sided finite limits: We may investigate limiting behaviour of $f(x)$ as x approaches a from one particular side, to get *one-sided* limits

$$\lim_{x \to a^+} f(x) = L \qquad \text{and} \qquad \lim_{x \to a^-} f(x) = L$$

from above and below a respectively, where L is some fixed real number. Here

$$x \to a^+ \qquad \text{and} \qquad x \to a^-$$

are translations of "x approaches a from above (the positive or right-hand side)" and "x approaches a from below (the negative or left-hand side)" respectively. These are particularly useful if $y = f(x)$ is defined on an interval and we need to understand behaviour near an endpoint.

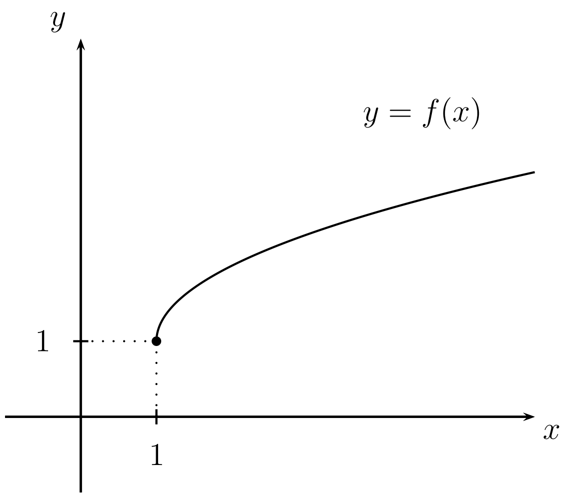
For example, consider the function f given by the rule

$$f(x) = 1 + \sqrt{x-1}\,.$$

Then $f(x)$ is not defined for $x < 1$ and the domain of f is the interval $[1, \infty)$. We cannot consider the behaviour of $f(x)$ as x approaches 1 from below, as the function is not defined there. It is meaningful, however, to consider the limit as x approaches 1 from above:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (1 + \sqrt{x-1}) = 1 + 0 = 1\,,$$

since $\sqrt{x-1}$ approaches $\sqrt{1-1} = \sqrt{0} = 0$ as x approaches 1 from the right. (We are implicitly invoking *limit laws* that are discussed later, but this particular limit behaviour is intuitively clear, for example, from the diagram below.)



(iii) Use of the infinity symbols: The *infinity symbols* ∞ and $-\infty$ and the arrow notation

$$x \to \infty \qquad \text{and} \qquad x \to -\infty$$

are used to indicate when some quantity x is getting arbitrarily large and positive, in the first case, and arbitrarily large (in magnitude) and negative, in the second case.

These are useful when combined with limit notation in the study and description of asymptotic behaviour (see below).

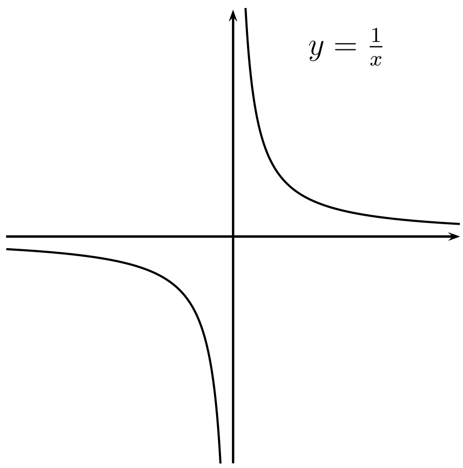
(iv) Horizontal asymptotes: We may investigate the limiting behaviour of $f(x)$ as x gets arbitrarily large and positive (written $x \to \infty$) or arbitrarily large and negative (written $x \to -\infty$). If $f(x)$ approaches some real number L in one or both of these cases, we may write

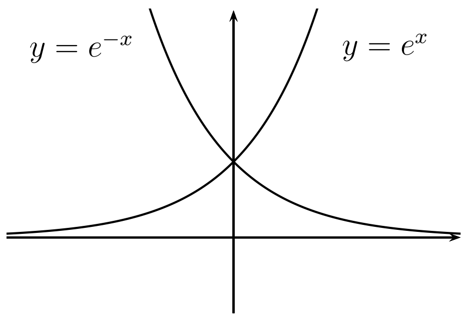
$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L ,$$

and we say that the line $y = L$ is a *horizontal asymptote* of the curve $y = f(x)$.

We have mentioned asymptotes before in discussing the hyperbola $y = \frac{1}{x}$, and noted that the x-axis is a horizontal asymptote, which we can express as follows in terms of limits:

$$\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to-\infty} \frac{1}{x} = 0 .$$





Horizontal asymptotic behaviour can occur on one side only. For example

$$\lim_{x\to\infty} e^{-x} = \lim_{x\to-\infty} e^{x} = 0 ,$$

so that the negative half of the x-axis acts as an asymptote for the curve $y = e^x$, whilst the positive half acts as an asymptote for the curve $y = e^{-x}$.

(iv) Vertical asymptotes: When the value of $f(x)$ gets arbitrarily large and positive, or arbitrarily large and negative, as x approaches some real number a then the line $x = a$ becomes a *vertical asymptote*. This behaviour can be described using limit notation and one of the infinity symbols, and is typically one-sided:

$$\lim_{x\to a^+} f(x) = \infty , \quad \lim_{x\to a^-} f(x) = \infty , \quad \lim_{x\to a^+} f(x) = -\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = -\infty .$$

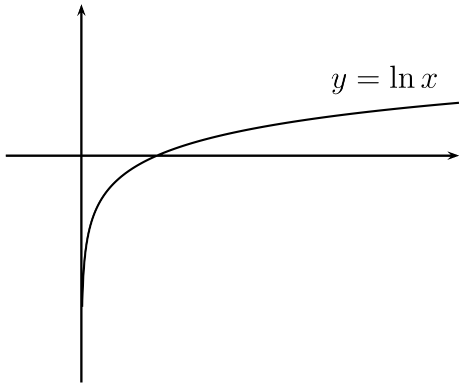
Thus, for the hyperbola $y = \frac{1}{x}$, pictured above, the y-axis is a vertical asymptote:

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \qquad \text{and} \qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty .$$

In the case of the natural logarithm function, we have

$$\lim_{x\to 0^+} \ln x = -\infty ,$$

that is, the negative y-axis becomes a vertical asymptote.



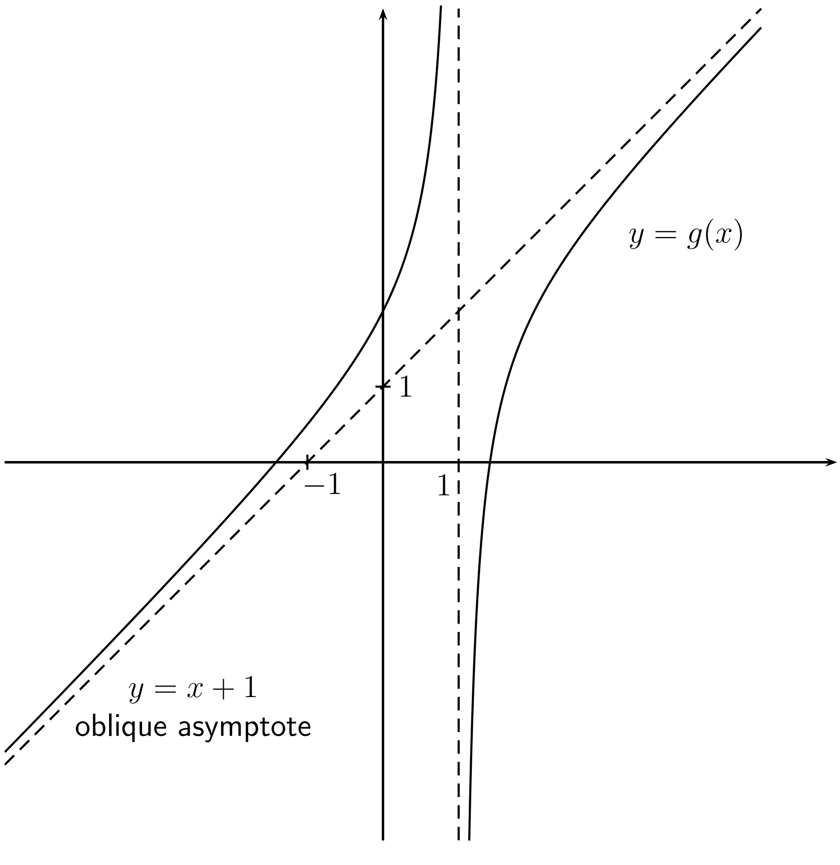
(v) Oblique asymptotes: A line approached by a curve $y = f(x)$ that is neither horizontal nor vertical is called an *oblique asymptote*. For example, consider the function g with rule

$$g(x) \;=\; \frac{x^2-2}{x-1}\;.$$

Then g is a *rational function*, that is, its rule is a ratio of polynomials, and we can easily see (for example, by long division) that

$$g(x) \;=\; x+1-\frac{1}{x-1} \;=\; x+1+\frac{1}{1-x}\;.$$

But, as x lets very large in magnitude, the fraction $\frac{1}{1-x}$ becomes vanishingly small, so that the curve $y = g(x)$ approaches the line $y = x+1$, which becomes an oblique asymptote.



As x gets large and positive, the fraction $\frac{1}{1-x}$ approaches zero, but from the negative side, so the curve $y = g(x)$ approaches the oblique asymptote from below.

By contrast, as x gets large and negative, the fraction $\frac{1}{1-x}$ approaches zero, but now from the positive side, so the curve $y = g(x)$ approaches the oblique asymptote from above.

Notice also that

$$\lim_{x\to 1^+} g(x) \;=\; \lim_{x\to 1+} \frac{x^2-2}{x-1} \;=\; -\infty \;,$$

since $x^2 - 2$ gets close to -1 as x gets close to 1, whilst $x - 1$ gets close to 0 from the positive side, and dividing a negative number close to -1 by a very small positive number produces a very large negative number.

By contrast,

$$\lim_{x\to 1^-} g(x) \;=\; \lim_{x\to 1-} \frac{x^2-2}{x-1} \;=\; \infty \;,$$

since $x^2 - 2$ still gets close to -1 as x gets close to 1, but $x - 1$ gets close to 0 from the negative side, and dividing a negative number close to -1 by a very small negative number produces a very large positive number.

Hence the line $x = 1$ becomes a vertical asymptote, and the behaviour on either side is captured succinctly by these two one-sided limits using the infinity symbols.

These remarks explain the vertical and oblique asymptotic behaviour of the graph of $y = g(x)$ above. The other behaviour of the curve is not obvious, relying on techniques from curve sketching, and will be explained in detail in the next module.