The University of Sydney

MOOC Introduction to Calculus

Notes for 'Leibniz notation'

Important Ideas and Useful Facts:

(i) Leibniz notation and differentials: An alternative way of expressing the derivative as a limit is the following:

$$f'(x) \;=\; \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

where Δx represents a small change in the value of x and Δy as the corresponding small change in the value of $y = f(x)$. With this notation it is common and very useful to write

$$f'(x) \;=\; \frac{dy}{dx} \qquad \text{and} \qquad f'(x)dx \;=\; dy\;,$$

called *Leibniz notation*, and think of dy and dx as 'infinitesimally' small idealised values called *differentials*. We think of the Greek Δ becoming our ordinary d 'in the limit'.

(ii) Some common derivatives and properties using Leibniz notation:

(a) $\dfrac{d}{dx}(k) = 0$ for any constant k, $\dfrac{d}{dx}(x) = 1$, $\dfrac{d}{dx}(x^2) = 2x$ $\dfrac{d}{dx}(x^3) = 3x^2$.

(b) $\dfrac{d}{dx}(x^n) = nx^{n-1}$ for any exponent n (though possibly with restrictions on x).

(c) $\dfrac{d}{dx}(ky) = k\dfrac{dy}{dx}$ (constants come out the front).

(d) $\dfrac{d}{dx}(u+v) = \dfrac{du}{dx} + \dfrac{dv}{dx}$ (the derivative is additive).

(e) $\dfrac{d}{dx}(\sin x) = \cos x$, $\dfrac{d}{dx}(\cos x) = -\sin x$, $\dfrac{d}{dx}(\tan x) = \dfrac{1}{\cos^2 x} = \sec^2 x$.

(f) $\dfrac{d}{dx}(e^x) = e^x$, $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$.

(iii) Inverting the derivative using Leibniz notation: If the function $y = f(x)$ is invertible then the rule for the derivative of the inverse function f^{-1} can be deduced by 'inverting' $\frac{dy}{dx}$, as though it were an ordinary fraction, to get

$$\frac{dx}{dy} \;=\; \frac{1}{dy/dx}\;,$$

and expressing the right-hand side in terms of y. If we revert to using x as an input then we produce a rule for the derivative of the inverse function.

Examples and derivations:

1. Find the derivative $\dfrac{dy}{dx}$ when $y = 4x^3 - x^2 + 6x - 9$.

Solution: We have

$$\frac{dy}{dx} \;=\; 4\frac{d}{dy}(x^3) - \frac{d}{dx}(x^2) + 6\frac{d}{dx}(x) - \frac{d}{dx}(9) \;=\; 4(3x^2) - 2x + 6 \;=\; 12x^2 - 2x + 6\;.$$

2. Find the derivative $\dfrac{dy}{dt}$ when $y = 6e^t - 3\ln t + 4t^2$.

Solution: We have

$$\frac{dy}{dt} \;=\; 6\frac{d}{dt}(e^t) - 3\frac{d}{dt}(\ln t) + 4\frac{d}{dt}(t^2) \;=\; 6e^t - \frac{3}{t} + 8t \;.$$

3. Find the derivative $\dfrac{dy}{dx}$ when $y = 2\sin x - 3\cos x$.

Solution: We have

$$\frac{dy}{dx} \;=\; 2\frac{d}{dx}(\sin x) - 3\frac{d}{dx}(\cos x) \;=\; 2\cos x - 3(-\sin x) \;=\; 2\cos x + 3\sin x \;.$$

4. Find the derivative $\dfrac{du}{d\theta}$ when $u = \theta^2 + 7\cos\theta - 5\sin\theta$.

Solution: We have

$$\frac{du}{d\theta} \;=\; \frac{d}{d\theta}(\theta^2) + 7\frac{d}{d\theta}(\cos\theta) - 5\frac{d}{d\theta}(\sin\theta) \;=\; 2\theta - 7\sin\theta - 5\cos\theta \;.$$

5. We use Leibniz notation to explain why $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$.

Proof: Put $y = e^x$, so that $x = \ln y$ and, from earlier work,

$$\frac{dy}{dx} \;=\; y' \;=\; e^x \;=\; y \;.$$

Hence, reciprocating, we get

$$\frac{d}{dy}(\ln y) \;=\; \frac{dx}{dy} \;=\; \frac{1}{dy/dx} \;=\; \frac{1}{y} \;.$$

Reverting to using x as an input, we get

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \;,$$

as required.

6. We use Leibniz notation to explain why $\dfrac{d}{dx}\big(\sqrt{x}\big) = \dfrac{1}{2\sqrt{x}}$, for $x \geq 0$.

Proof: Put $y = x^2$, so that $x = \sqrt{y}$ and, from earlier work,

$$\frac{dy}{dx} \;=\; y' \;=\; 2x \;=\; 2\sqrt{y} \;.$$

Hence, reciprocating, we get

$$\frac{d}{dy}(\sqrt{y}) \;=\; \frac{dx}{dy} \;=\; \frac{1}{dy/dx} \;=\; \frac{1}{2\sqrt{y}} \;.$$

Reverting to using x as an input, we get

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}} \;,$$

as required.

Note that, using fractional power notation, this becomes

$$\frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2} ,$$

which is a special case of the formula $\dfrac{d}{dx}(x^n) = nx^{n-1}$, when $n = \frac{1}{2}$.

7. We use Leibniz notation to explain why $\dfrac{d}{dx}\left(\sqrt[3]{x}\right) = \dfrac{1}{3x^{2/3}}$.

Proof: Put $y = x^3$, so that $x = \sqrt[3]{y} = y^{1/3}$ and, from earlier work,

$$\frac{dy}{dx} = y' = 3x^2 = 3y^{2/3} .$$

Hence, reciprocating, we get

$$\frac{d}{dy}(\sqrt[3]{y}) = \frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{3y^{2/3}} .$$

Reverting to using x as an input, we get

$$\frac{d}{dx}(\sqrt[3]{x}) = \frac{1}{3x^{2/3}} ,$$

as required.

Note that, using fractional power notation, this becomes

$$\frac{d}{dx}\left(x^{1/3}\right) = \frac{1}{3}x^{-2/3} ,$$

which is a special case of the formula $\dfrac{d}{dx}(x^n) = nx^{n-1}$, when $n = \frac{1}{3}$.