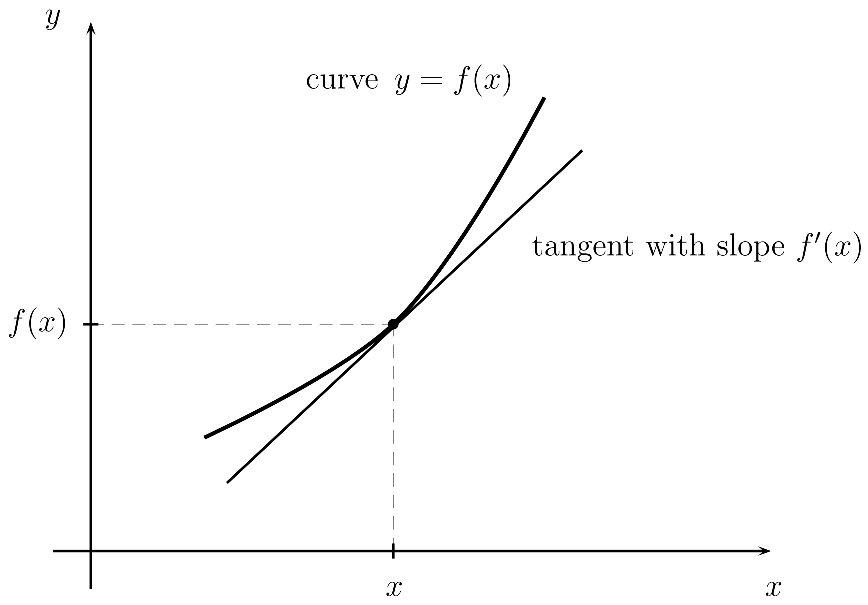
Notes for 'The derivative as a limit'

Important Ideas and Useful Facts:

(i) Derivative of a function: Consider a function $y = f(x)$. We define the *derivative* of f at x to be the following limit, if it exists:

$$f'(x) \;=\; \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \,.$$

Then $f'(x)$ represents the slope of the tangent line to the curve at the point $(x, f(x))$.



If $f(x)$ denotes displacement of an object at time x, then $f'(x)$ is the instantaneous velocity of the object at time x.

(ii) Derivative of the square function: If $y = f(x) = x^2$, the rule for the square function, then $f'(x) = 2x$. For example, $f'(6) = 12$ and $f'(-7) = -14$.

(iii) Derivative of the cube function: If $y = f(x) = x^3$, the rule for the cube function, then $f'(x) = 3x^2$. For example, $f'(6) = 3(6)^2 = 108$ and $f'(-7) = 3(-7)^2 = 147$.

(iv) Derivative of the power function: If $y = f(x) = x^n$, the rule for the power function with fixed exponent n then

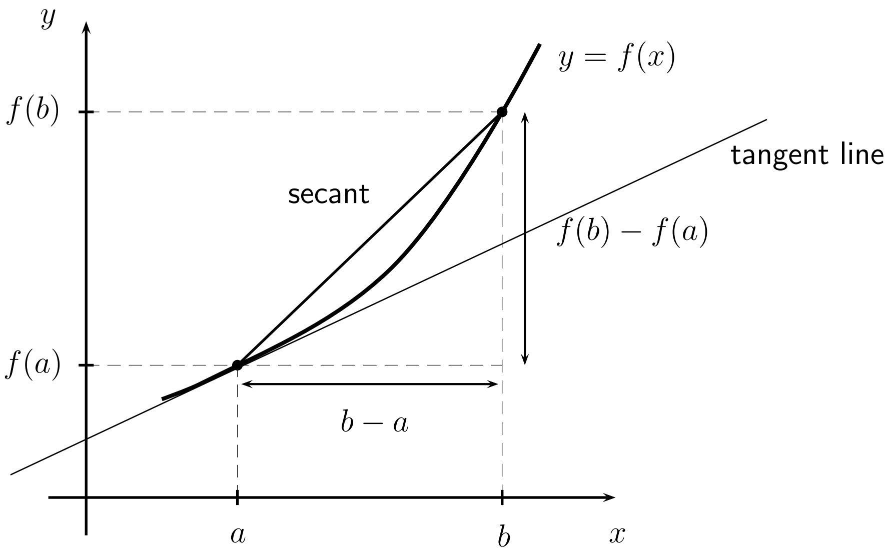
$$f'(x) \;=\; nx^{n-1} \,,$$

that is, the exponent falls down to the front, and the new exponent is obtained by subtracting one. The derivatives of the square and cube functions are special cases when $n = 2$ and 3.

For example if $f(x) = x^4$, then $f'(x) = 4x^3$. If $f(x) = x^{-3}$ then $f'(x) = -3x^{-4}$. If $f(x) = x^{1/3}$ then $f'(x) = \frac{1}{3}x^{-2/3}$.

This rule works generally for all real exponents (though there may be certain restrictions such as requiring x to being positive). An explanation why this works so generally will be given in the next module.

(v) Alternative limit definition of the derivative: An alternative, but equivalent, approach to finding an expression for the slope of the tangent line to the curve $y = f(x)$ is to take the average rate of change over an interval $[a, b]$, where $a < b$, and allow b to approach a.



The slope of the tangent line at $x = a$ then becomes the derivative

$$f'(a) \;=\; \lim_{b \to a} \frac{f(b) - f(a)}{b - a} \;.$$

To check algebraically that the limit definitions really are equivalent, put $h = b - a$, so that $b = a + h = x + h$, and then this becomes

$$f'(x) \;=\; f'(a) \;=\; \lim_{b \to a} \frac{f(b) - f(a)}{b - a} \;=\; \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \;,$$

noting that $h \to 0$ as $b \to a$.

Suppose that $f(x) = x^3$. We verify that $f'(x) = 3x^2$, using both limit definitions.

Using the original definition, we have

$$\begin{aligned} f'(x) \;&=\; \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \;=\; \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \;=\; \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &=\; \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \;=\; \lim_{h \to 0} \frac{(3x^2 + 3xh + h^2)h}{h} \;=\; \lim_{h \to 0} \,(3x^2 + 3xh + h^2) \\ &=\; 3x^2 + 3x(0) + (0)^2 \;=\; 3x^2 \;, \end{aligned}$$

as required. Alternatively, to practise the second definition, based on the limit of the average rate of change, we have, using a difference of two cubes formula in the middle step,

$$\begin{aligned} f'(a) \;&=\; \lim_{b \to a} \frac{f(b) - f(a)}{b - a} \;=\; \lim_{b \to a} \frac{b^3 - a^3}{b - a} \;=\; \lim_{b \to a} \frac{(b - a)(b^2 + ba + a^2)}{b - a} \\ &=\; \lim_{b \to a} \,(b^2 + ba + a^2) \;=\; a^2 + a^2 + a^2 \;=\; 3a^2 \;, \end{aligned}$$

again giving $f'(x) = 3x^2$, by taking $x = a$.