THE UNIVERSITY OF SYDNEY

MOOC INTRODUCTION TO CALCULUS

Notes for 'Limits and continuity'

Important Ideas and Useful Facts:

(i) Continuity and drawing the graph without lifting the pen: In this course, a function $y = f(x)$ involving real inputs x and outputs y is called *continuous* if the graph of the function is a single connected curve in the plane that can, in principle, be drawn without lifting the pen off the page. A function is called *discontinuous*, therefore, if this fails, that is, if the graph cannot be drawn, in principle, without lifting the pen off the page. This is an informal definition (relying on intuition about what it means to draw a graph), but useful for our purposes.

Examples of continuous functions include all polynomial functions, both the circular functions $y = \sin x$ and $y = \cos x$, the natural exponential and logarithmic functions $y = e^x$ and $y = \ln x$ (noting, in the latter case, that the domain is restricted to $(0, \infty)$), rational functions (ratios of polynomial functions) with restricted domains producing one connected branch, and the tangent function $y = \tan x$ restricted to the domain $(-\frac{\pi}{2}, \frac{\pi}{2})$.

(There are more formal definitions of continuity in higher mathematics, that are too technical to be useful for our purposes, which allow a function to be regarded as continuous even if the graph is comprised of disconnected pieces. In such cases, if we want to also regard the function as continuous, we restrict the domain so that we could focus on just one of the connected pieces of the graph, which in principle could then be drawn without lifting the pen.)

In the case that f is continuous, and the real number a lies inside a neighbourhood (an interval) entirely within the domain of the function f, the fact that the graph can be drawn without lifting the pen implies that there is no "hole" or "gap" at $x = a$, and so

$$\lim_{x \to a} f(x) \;=\; f(a) \;.$$

A lot of information is implicit in this statement, including the fact that the limit exists, that it is sensible for x to approach a from either side (which is why we require that a lies in an interval contained in the domain of f), and that the values of the limit and the function coincide.

Therefore, if one knows in advance that a function is continuous, then finding limits amounts to simple evaluation of the function at the point of interest. This occurs, for example, for all polynomial functions and rational functions (restricted to a connected branch containing the point of interest) and the circular functions, facts that we have relied on implicitly previously in evaluating certain limits.

The most common obstructions to this notion of continuity are problems with zero denominators in fractions, or when a function is defined piecewise and there are gaps in the pieces, both of which can "disconnect" the graph, so that it is unable to be then drawn without lifting the pen.

(ii) Removable discontinuities: Suppose that a function $y = f(x)$ is defined in a neighbourhood (interval) containing the real number a, but $f(a)$ is not defined (so a is not in the domain of f). Then the function must be discontinuous as there is a "hole" in the graph formed by the fact that a is missing from the domain. We say that the *discontinuity at a is removable* if the limit

$$\lim_{x\to a} f(x) = L$$

exists for some real number L, for then we can expand the definition of f to include a in the domain and define $f(a) = L$. If this missing point turns out to be the only obstruction to drawing the curve without lifting the pen, then defining $f(a)$ in this way will create a continuous function.

For example, consider the functions f and g with rules

$$f(x) = \frac{x^2-1}{x-1} \qquad \text{and} \qquad g(x) = \frac{x^2-2}{x-1},$$

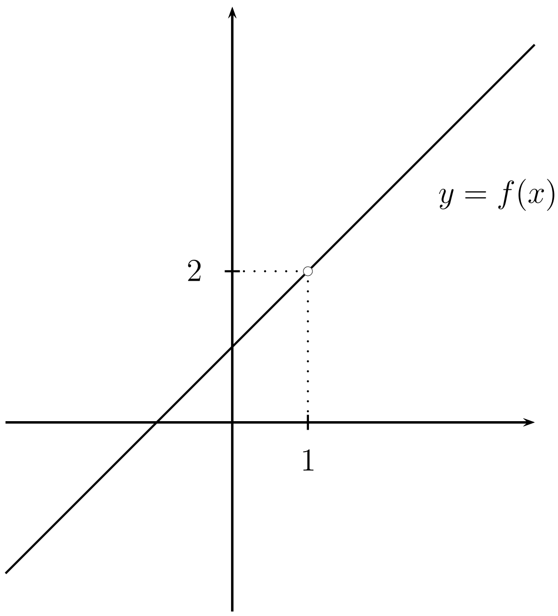
both of which have domain $\mathbb{R}\backslash\{1\}$. Then, we have seen previously,

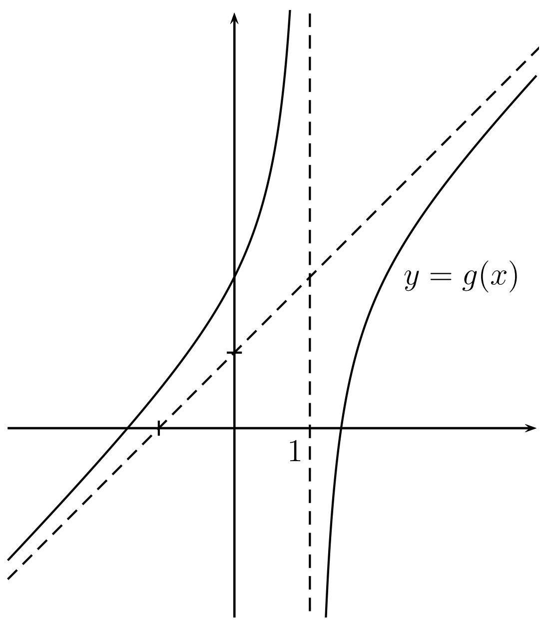
$$\lim_{x\to 1} f(x) = 2$$

exists so that the discontinuity is removable and defining $f(1) = 2$ makes f become a continuous function. However, we have seen previously that

$$\lim_{x\to 1^+} g(x) = -\infty \qquad \text{and} \qquad \lim_{x\to 1^-} g(x) = \infty,$$

so that $\lim\limits_{x\to 1} g(x)$ does not exist and the discontinuity is not removable. The contrasting behaviour can be seen in the graphs for f and g, where using the limit to define $f(1) = 2$ fills in the hole and creates a continuous line. There is no real number we can assign to $g(1)$ that will connect the graph of g so that it can be drawn without lifting the pen.



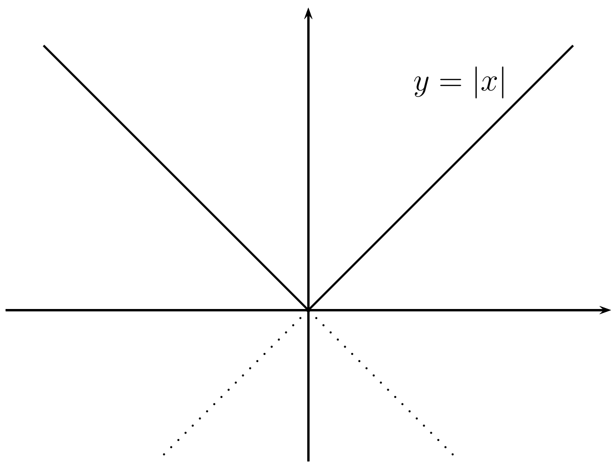


Examples:

1. Consider the function f with the rule

$$f(x) \;=\; |x| \;=\; \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0, \end{cases}$$

which one may think of as defined in two pieces, the line $y = x$ for $x \geq 0$ and the line $y = -x$ for $x < 0$.



Each of these line fragments are continuous, and in fact join up at the origin $(0, 0)$, so there are no gaps and the overall graph can be drawn without lifting the pen. Hence f is continuous.

2. Find the value of the constant k such that the following function is continuous over the whole real line:

$$f(x) \;=\; \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\ k & \text{if } x = 2. \end{cases}$$

Solution: For this function to be continuous we need

$$k \;=\; f(2) \;=\; \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \;=\; \lim_{x \to 2} \frac{(x-2)(x+2)}{x - 2} \;=\; \lim_{x \to 2}(x + 2) \;=\; 2 + 2 \;=\; 4\;.$$

3. Find the constant k such that the function f with the following rule is continuous everywhere:

$$f(x) = \begin{cases} k & \text{if } x \geq 1 \\ 3x + 2 & \text{if } x < 1. \end{cases}$$

Solution: To be able to draw the graph without lifting the pen, we need the constant k to match the limiting value of $3x + 2$ as x approaches 1 from below, that is,

$$k \;=\; \lim_{x \to 1^-} (3x + 2) \;=\; 3(1) + 2 \;=\; 5\;.$$

4. Find the constant k such that the function f with the following rule is continuous everywhere:

$$f(x) = \begin{cases} 2x+3 & \text{if } x \geq 3 \\ k-5x & \text{if } x < 3. \end{cases}$$

Solution: To be able to draw the graph without lifting the pen, we need the constant k such that the endpoints of the two pieces match up, that is

$$\lim_{x\to 3^-} f(x) \;=\; \lim_{x\to\ 3^+} f(x) \;.$$

But

$$\lim_{x\to 3^-} f(x) \;=\; \lim_{x\to 3^-} (k-5x) \;=\; k-5(3) \;=\; k-15 \;,$$

and

$$\lim_{x\to\ 3^+} f(x) \;=\; \lim_{x\to 3^+} (2x+3) \;=\; 2(3)+3 \;=\; 9 \;,$$

so we require that

$$k - 15 \;=\; 9 \;,$$

that is, $k = 15 + 9 = 24$.

5. Find the constant k such that the function f with the following rule is continuous everywhere:

$$f(x) = \begin{cases} kx^2+13 & \text{if } x \geq 4 \\ x^3-k & \text{if } x < 4. \end{cases}$$

Solution: To be able to draw the graph without lifting the pen, we need the constant k such that the endpoints of the two pieces match up, that is

$$\lim_{x\to 4^-} f(x) \;=\; \lim_{x\to\ 4^+} f(x) \;.$$

But

$$\lim_{x\to 4^-} f(x) \;=\; \lim_{x\to 4^-} (x^3-k) \;=\; 64-k \;,$$

and

$$\lim_{x\to\ 4^+} f(x) \;=\; \lim_{x\to 4^+} (kx^2+13) \;=\; 16k+13 \;,$$

so we require that

$$64 - k \;=\; 16k + 13 \;.$$

Hence $17k = 51$, so that $k = 3$.

6. Consider the function f with the rule

$$f(x) \;=\; \frac{x}{|x|} \;=\; \begin{cases} \frac{x}{x} = 1 & \text{if } x \geq 0 \\ \frac{x}{-x} = -1 & \text{if } x < 0 \\ \text{undefined} & \text{if } x = 0, \end{cases}$$

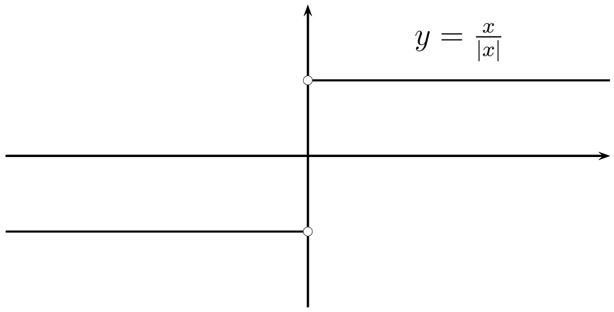
which one may think of as defined in two pieces, the constant function $y = 1$ for $x > 0$ and the constant function $y = -1$ for $x < 0$, with a "hole" or "gap" at $x = 0$. In fact, the discontinuity at $x = 0$ is not removable: the left-hand limit is

$$\lim_{x\to 0^-} f(x) \;=\; \lim_{x\to 0^-} -1 \;=\; -1 \;,$$

whilst the right-hand limit is

$$\lim_{x \to 0^+} f(x) \;=\; \lim_{x \to 0^+} 1 \;=\; 1\,,$$

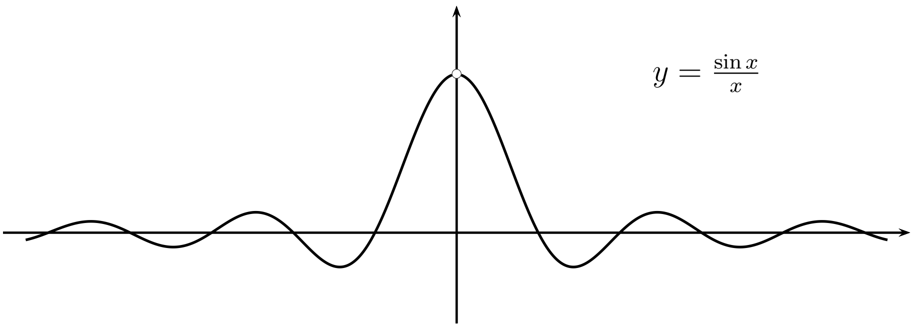
which are unequal, so the two-sided limit $\lim\limits_{x \to 0} f(x)$ does not exist.



7. Consider the function f with the rule

$$f(x) \;=\; \frac{\sin x}{x}\,,$$

which has domain $\mathbb{R}\backslash\{0\}$, so there is a discontinuity at $x = 0$.



In fact, from earlier work,

$$\lim_{x \to 0} f(x) \;=\; \lim_{x \to 0} \frac{\sin x}{x} \;=\; 1\,,$$

so the discontinuity at $x = 0$ is removable by defining $f(0) = 1$, that is

$$f(x) \;=\; \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$