The University of Sydney

MOOC Introduction to Calculus

Notes for 'Limit laws'

Important Ideas and Useful Facts:

(i) Limits respect arithmetic operations: Assuming the relevant limits exist, then we can bring limits inside arithmetic operations as follows:

$$\lim \big(f(x) \pm g(x)\big) \;=\; \big(\lim f(x)\big) \pm \big(\lim g(x)\big) \,,$$

that is, *limits respect addition and subtraction*;

$$\lim \big(f(x)g(x)\big) \;=\; \big(\lim f(x)\big)\big(\lim g(x)\big) \,,$$

that is, *limits respect taking products*, and, as a special case,

$$\lim \big(kg(x)\big) \;=\; k \lim g(x) \,,$$

where k is a constant, and we often say "constants come out the front"; and

$$\lim \big(f(x)/g(x)\big) \;=\; \big(\lim f(x)\big)/\big(\lim g(x)\big) \,,$$

that is, *limits respect taking quotients*, provided the denominator on the right-hand side is nonzero.

Here, to be as general as possible, the nature of the limit is unspecified, but these can involve two-sided limits, $x \to a$, one-sided limits, $x \to a^+$ or $x \to a^-$, or use the infinity symbols, $x \to \infty$ or $x \to -\infty$.

(ii) Simple limits: The following limits are immediate in themselves, and can be used, in conjunction with the limit laws, to evaluate quite difficult limits:

$$\lim_{x\to a} x \;=\; a \,, \qquad \lim_{x\to a} k \;=\; k \quad (k \text{ constant}) \,, \qquad \lim_{x\to\infty} \frac{1}{x} \;=\; \lim_{x\to-\infty} \frac{1}{x} \;=\; 0 \,.$$

(iii) Evaluating polynomials or rational expressions: If $p(x)$ and $q(x)$ are polynomials and $a \in \mathbb{R}$ then

$$\lim_{x\to a} p(x) \;=\; p(a) \qquad \text{and} \qquad \lim_{x\to a} \frac{p(x)}{q(x)} \;=\; \frac{p(a)}{q(a)} \,,$$

provided that $q(a) \neq 0$. This means that, provided the denominator is nonzero in the second case, finding limits of polynomial or rational expressions involving x is simply a matter of evaluating the expressions using ordinary arithmetic after substituting a for x.

This follows from the first two simple limits above and the fact that limits respect arithmetic operations, since polynomial and rational expressions are built from the variable x and constants using arithmetic operations.

(iv) Important limit involving the sine function: The following limit is of central importance in the development of calculus related to trigonometric functions, and a derivation is given towards the end of these notes:

$$\lim_{x\to 0} \frac{\sin x}{x} \;=\; 1\,.$$

(v) The squeeze limit law: Suppose that $g(x) \le f(x) \le h(x)$ for all x near a, and that

$$\lim_{x\to a} g(x) \;=\; \lim_{x\to a} h(x) \;=\; L\,,$$

for some fixed real number L. Then

$$\lim_{x\to a} f(x) \;=\; L\,,$$

that is all three limits are equal to L. We think of $f(x)$ as "squeezed" or "sandwiched" in between $g(x)$ and $h(x)$, so that there is nowhere else that $f(x)$ can go than to tend also to the same common limit as x approaches a.

Examples and derivations:

1. Find $\lim\limits_{x\to 3} \left(x^2 + 2x - 1\right)$.

Solution: We can solve this by simply evaluating the quadratic at $x = 3$ to get the limit $3^2 + 2(3) - 1 = 14$. To illustrate why this evaluation technique works, we can use the fact that limits respect arithmetic operations, and write this out in more detail (though one does not do this in practice):

$$\begin{aligned}\lim_{x\to 3} \left(x^2 + 2x - 1\right) \;&=\; \left(\lim_{x\to 3} x\right)^2 + \left(\lim_{x\to 3} 2\right)\left(\lim_{x\to 3} x\right) - \left(\lim_{x\to 3} 1\right)\\ &=\; 3^2 + (2)(3) - 1 \;=\; 14\,.\end{aligned}$$

2. Find $\lim\limits_{x\to -1} \left(5x^3 + 4x^2 + 7x - 9\right)$.

Solution: We solve this by evaluating the cubic at $x = -1$:

$$\lim_{x\to -1} \left(5x^3 + 4x^2 + 7x - 9\right) \;=\; 5(-1)^3 + 4(-1)^2 + 7(-1) - 9 \;=\; -17\,.$$

3. Find $\lim\limits_{x\to -1} \left(\dfrac{5x^3 + 4x^2 + 7x - 9}{7x^3 + 2x^2 - 8x + 5}\right)$.

Solution: We solve this by evaluating the rational expression at $x = -1$, noting that the denominator is nonzero:

$$\lim_{x\to -1} \left(\frac{5x^3 + 4x^2 + 7x - 9}{7x^3 + 2x^2 - 8x + 5}\right) \;=\; \frac{5(-1)^3 + 4(-1)^2 + 7(-1) - 9}{7(-1)^3 + 2(-1)^2 - 8(-1) + 5} \;=\; -\frac{17}{8}\,.$$

4. Find $\lim\limits_{x\to \infty} \left(\dfrac{5x^3 - 4x^2 + 7x - 9}{7x^3 + 2x^2 - 8x + 5}\right)$.

Solution: Intuitively, one expects the limit to be $\frac{5}{7}$, since, for large x, the numerator is dominated by $5x^3$ and the denominator by $7x^3$, and the other terms become insignificant.

To confirm our intuition, and make formal use of the limit laws, we divide the numerator and the denominator by x^3:

$$\lim_{x\to\infty}\left(\frac{5x^3-4x^2+7x-9}{7x^3+2x^2-8x+5}\right) \;=\; \lim_{x\to\infty}\left(\frac{5-\frac{4}{x}+\frac{7}{x^2}-\frac{9}{x^3}}{7+\frac{2}{x}-\frac{8}{x^2}+\frac{5}{x^3}}\right)$$

$$= \frac{\left(\lim_{x\to\infty}5\right)-\left(\lim_{x\to\infty}\frac{4}{x}\right)+\left(\lim_{x\to\infty}\frac{7}{x^2}\right)-\left(\lim_{x\to\infty}\frac{9}{x^3}\right)}{\left(\lim_{x\to\infty}7\right)+\left(\lim_{x\to\infty}\frac{2}{x}\right)-\left(\lim_{x\to\infty}\frac{8}{x^2}\right)+\left(\lim_{x\to\infty}\frac{5}{x^3}\right)}$$

$$= \frac{5-0+0-0}{7+0-0+0} \;=\; \frac{5}{7}\,,$$

as expected. This tells us, in fact, that the curve whose rule is the rational expression inside the limit above has a horizontal asymptote $y=\frac{5}{7}$. The same calculation works as $x\to-\infty$, so the horizontal asymptote is approached both to the right and to the left.

5. Find $\lim\limits_{x\to\pi/4}\big(\sin x-\cos x\big)$.

Solution: Both $y=\sin x$ and $y=\cos x$ are *continuous* functions, which means that their graphs can be drawn without lifting the pen off the page (a concept explored in more detail later), so that the individual limits are just the respective values of these functions, that is,

$$\lim_{x\to\pi/4}\sin x \;=\; \sin\left(\frac{\pi}{4}\right) \;=\; \frac{1}{\sqrt{2}} \qquad \text{and} \qquad \lim_{x\to\pi/4}\cos x \;=\; \cos\left(\frac{\pi}{4}\right) \;=\; \frac{1}{\sqrt{2}}\,.$$

Hence, using the fact that limits respect subtraction, we get

$$\lim_{x\to\pi/4}\big(\sin x-\cos x\big) \;=\; \Big(\lim_{x\to\pi/4}\sin x\Big)-\Big(\lim_{x\to\pi/4}\cos x\Big) \;=\; \frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}} \;=\; 0\,.$$

6. Find, if possible, $\lim\limits_{x\to3^+}\dfrac{|x-3|}{x-3}$ and $\lim\limits_{x\to3^-}\dfrac{|x-3|}{x-3}$.

Solution: First note that

$$\frac{|x-3|}{x-3} \;=\; \begin{cases}\frac{x-3}{x-3}=1 & \text{if } x>3\\ \frac{3-x}{x-3}=-1 & \text{if } x<3\\ \text{undefined} & \text{if } x=3\,.\end{cases}$$

If $x\to3^+$, then x remains always greater than 3, so the first alternative applies, so that

$$\lim_{x\to3^+}\frac{|x-3|}{x-3} \;=\; \lim_{x\to3^+}1 \;=\; 1\,.$$

However, if $x\to3^-$, then x remains always less than 3, so the second alternative applies, so that

$$\lim_{x\to3^-}\frac{|x-3|}{x-3} \;=\; \lim_{x\to3^+}-1 \;=\; -1\,.$$

Thus both of these one-sided limits exist and equal 1 and -1 respectively. Note that since these values are different, the two-sided limit

$$\lim_{x\to3}\frac{|x-3|}{x-3}$$

does not exist.

7. Find $\lim\limits_{x\to 0} \dfrac{\sin 5x}{x}$.

Solution: The trick is to rewrite the expression to resemble the usual limit involving the sine function. We begin by multiplying the top and bottom by 5, bring the constant 5 out the front, and then make a substitution of, say, y for $5x$, and note that y tends to zero as x tends to zero:

$$\lim_{x\to 0} \frac{\sin 5x}{x} \;=\; \lim_{x\to 0} \frac{5\sin 5x}{5x} \;=\; 5\lim_{x\to 0} \frac{\sin 5x}{5x} \;=\; 5\lim_{y\to 0} \frac{\sin y}{y} \;=\; 5\times 1 \;=\; 5\,.$$

8. Find $\lim\limits_{x\to 0} \dfrac{x}{\sin 7x}$.

Solution: This uses the same technique as the previous exercise, now using the substitution y for $7x$, and also the fact that limits respects arithmetic operations involving fractions:

$$\begin{aligned}
\lim_{x\to 0} \frac{x}{\sin 7x} \;&=\; \lim_{x\to 0} \frac{7x}{7\sin 7x} \;=\; \frac{1}{7}\lim_{x\to 0} \frac{7x}{\sin 7x} \;=\; \frac{1}{7}\lim_{y\to 0} \frac{y}{\sin y} \\
&=\; \frac{1}{7}\lim_{y\to 0} \frac{1}{\sin y/y} \;=\; \frac{1}{7}\,\frac{\lim_{y\to 0} 1}{\lim_{y\to 0} \frac{\sin y}{y}} \\
&=\; \frac{1}{7}\times\frac{1}{1} \;=\; \frac{1}{7}\,.
\end{aligned}$$

9. Find $\lim\limits_{x\to 0} \dfrac{\tan x}{x}$.

Solution: We begin by expanding the definition of tan, regroup the expression, and then use the fact that limits respect arithmetic operations, noting that, at the last step, each of the three limits evaluate to 1:

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan x}{x} \;&=\; \lim_{x\to 0} \frac{\sin x/\cos x}{x} \;=\; \lim_{x\to 0} \frac{\sin x}{x\cos x} \;=\; \lim_{x\to 0} \left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right) \\
&=\; \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{1}{\cos x}\right) \;=\; \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\frac{\lim_{x\to 0} 1}{\lim_{x\to 0} \cos x}\right) \\
&=\; 1\times\frac{1}{1} \;=\; 1\,.
\end{aligned}$$

10. Find $\lim\limits_{x\to 0} \dfrac{\tan 5x}{\sin 7x}$.

Solution: The new trick is to introduce $\frac{x}{x}$ inside the expression, rearrange the components and use the results of previous exercises and the fact that limits respect products:

$$
\begin{aligned}
\lim_{x\to 0} \frac{\tan 5x}{\sin 7x} &= \lim_{x\to 0} \frac{\sin 5x/\cos 5x}{\sin 7x} = \lim_{x\to 0} \frac{x \sin 5x}{x \sin 7x \cos 5x} \\
&= \lim_{x\to 0} \left(\frac{\sin 5x}{x}\right)\left(\frac{x}{\sin 7x}\right)\left(\frac{1}{\cos 5x}\right) \\
&= \left(\lim_{x\to 0} \frac{\sin 5x}{x}\right)\left(\lim_{x\to 0} \frac{x}{\sin 7x}\right)\left(\frac{\lim_{x\to 0} 1}{\lim_{x\to 0} \cos 5x}\right) \\
&= 5 \times \frac{1}{7} \times \frac{1}{1} = \frac{5}{7} .
\end{aligned}
$$

11. Find $\displaystyle\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$.

Solution: As $x \to 0$, $1/x$ diverges, getting arbitrarily large and positive or negative, so the term $\sin\left(\frac{1}{x}\right)$ will oscillate backwards and forwards between 1 and -1 infinitely often. At first sight, this might make our problem seem intractable. But we can set up the use the squeeze law, since we have, for all nonzero x,

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1 .$$

Multiplying through by x when $x > 0$ retains the inequalities:

$$-x \le x \sin\left(\frac{1}{x}\right) \le x .$$

Multiplying through by x when $x < 0$ reverses the inequalities:

$$-x \ge x \sin\left(\frac{1}{x}\right) \ge x .$$

These can be combined into the following single inequality, by using the magnitude of x:

$$-|x| \le x \sin\left(\frac{1}{x}\right) \le |x| .$$

Clearly $\displaystyle\lim_{x\to 0} (-|x|) = \lim_{x\to 0} |x| = 0$, so, by the squeeze law,

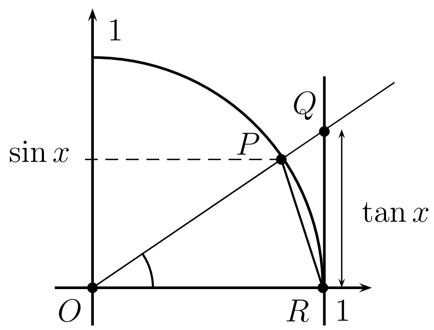
$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0 .$$

12. We explain how to derive the following important limit, by using facts about areas associated with triangles and the unit circle and an application of the squeeze law:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 .$$

Proof: It suffices to prove the result for $x \to 0^+$, where the angle x is always positive. The proof for $x \to 0^-$ follows by multiplying through by minus one throughout, and reversing inequalities.

It suffices also to assume x is an acute angle, so the following diagram, involving the first quadrant only, is enough to guide the mathematics. The real number x is the arc length along the unit circle, representing the angle as one moves along the circle from point R to point P. As usual O represents the origin. The point Q marks the intersection of the line extending the radius with the tangent line to the circle at the horizontal intercept. Thus the distance between R and Q is $\tan x$.



The area of the full unit circle is $\pi r^2 = \pi$ (since the radius r is 1). The area of the sector of the circle subtended by x is just the proportion of this represented by the sector, which is therefore

$$\frac{x}{2\pi} \times \pi \;=\; \frac{x}{2}\;.$$

This sector contains the triangle with vertices O, P and R, which has area $\dfrac{\sin x}{2}$, and is contained in the triangle with vertices O, Q and R, which has area $\dfrac{\tan x}{2}$.

We therefore have the following chain of inequalities (which could be written using $<$, but we don't need that fact for the application of the squeeze law below):

$$\frac{\sin x}{2} \;\le\; \frac{x}{2} \;\le\; \frac{\tan x}{2} \;=\; \frac{\sin x}{2\cos x}\;.$$

Dividing by $\sin x$ and multiplying by 2, which are both positive, retains the inequalities, so that we get

$$1 \;\le\; \frac{x}{\sin x} \;\le\; \frac{1}{\cos x}\;.$$

We can now reciprocate everything, reversing the inequalities, since all numbers are positive, to get

$$1 \;\ge\; \frac{\sin x}{x} \;\ge\; \cos x\;.$$

But

$$\lim_{x\to 0^+} 1 \;=\; \lim_{x\to 0^+} \cos x \;=\; 1\;,$$

so, by the squeeze law,

$$\lim_{x\to 0^+} \frac{\sin x}{x} \;=\; 1\;,$$

completing the proof, as this implies the two-sided limit also, by remarks at the beginning of this proof.