The University of Sydney

MOOC Introduction to Calculus

Notes for 'Sign diagrams, solutions sets and intervals'

Important Ideas and Useful Facts:

(i) The sign of a real number: The *sign* of a real number x refers to whether x is positive or x is negative. We say that two numbers have the *same sign* if they are both positive or they are both negative. We say that two numbers have *different signs* if one is positive and the other is negative.

Consider real numbers a and b. The product ab is positive if and only if a and b have the same sign. The product ab is negative if and only if a and b have different signs.

(ii) Sign diagrams: A *sign diagram* is a table with two rows, the first row indicating values of some variable, x say, and the second row indicating the sign (indicated by a plus sign $+$ for positive and a minus sign $-$ for negative) of some particular expression involving x. The line in the table between the rows represents the real line, and important points for x are marked off, typically corresponding to x for which the expression evaluates to zero (and this is usually obvious when the expression has been factorised).

For example, here is the sign diagram for the expression $(x-1)(x-2)$, which is zero, when $x = 1$ and $x = 2$, positive when $x < 1$ and when $x > 2$, and negative when $1 < x < 2$:

x		1		2	
$(x-1)(x-2)$	$+$	0	$-$	0	$+$

Here is the sign diagram for the expression $x(x^2 - 1) = x(x + 1)(x - 1)$, which is zero when $x = 0$, $x = 1$ and $x = -1$, positive when $-1 < x < 0$ and when $x > 1$, and negative when $x < -1$ and when $0 < x < 1$:

x		-1		0		1	
$x(x^2-1)$	$-$	0	$+$	0	$-$	0	$+$

One can also construct sign diagrams when an expression becomes undefined, especially when fractions are involved where the variable appears in the denominator, in which case the letter "u" is used in the second row, as an abbreviation for "undefined".

Here is the sign diagram for the expression $\frac{x^2-1}{x} = \frac{(x+1)(x-1)}{x}$, which is identical to the previous example, except for noting that the expression is undefined for $x = 0$:

x		-1		0		1	
$\frac{x^2-1}{x}$	$-$	0	$+$	u	$-$	0	$+$

For another variation, here is the sign diagram for the expression $\frac{x}{x^2-1} = \frac{x}{(x+1)(x-1)}$, where now the expression is undefined for $x = -1$ and $x = 1$:

x		-1		0		1	
$\frac{x}{x^2-1}$	$-$	u	$+$	0	$-$	u	$+$

(iii) Empty set: The *empty* set is the set without any elements and denoted by $\emptyset$. If A is any set then $A \cup \emptyset = A$, $A \cap \emptyset = \emptyset$ and $A \backslash A = \emptyset$.

(iv) Solution sets: The *solution set* (within the real number system $\mathbb{R}$) of an equation or an inequality involving a variable x is the set of all real numbers x that satisfy the given equation or inequality. If no real numbers satisfy the given equation or inequality then the solution set is empty.

For example, the solution set of the equation $2x + 1 = 7$ is the set $\{3\}$, since $x = 3$ is the only solution.

The solution set of the equation $(x-1)(x-2) = 0$ is the set $\{1, 2\}$, since the solutions are precisely $x = 1$ and $x = 2$.

The solution set of the inequality $2x + 1 < 7$ is the set $\{x \in \mathbb{R} \mid x < 3\}$.

The solution set of the inequality $2x + 1 \geq 7$ is the set $\{x \in \mathbb{R} \mid x \geq 3\}$.

The solution set of the inequality $(x-1)(x-2) < 0$ is the set $\{x \in \mathbb{R} \mid 1 < x < 2\}$.

The solution set of the inequality $(x-1)(x-2) \geq 0$ is the set $\{x \in \mathbb{R} \mid x \leq 1 \text{ or } x \geq 2\}$.

The solution set of the equation $x + 1 = x + 2$ is the empty set $\emptyset$, since if the solution set contained some x then, taking x away from both sides of the equation would produce the absurdity $1 = 2$.

The solution set of the inequality $x^2 + 1 < 0$ is also the empty set $\emptyset$, since squares of real numbers are always nonnegative: if the solution set were to contain some x then, taking 1 away from both sides, one would conclude that $x^2 < -1$, which is negative, contradicting that x^2 is nonnegative.

(v) Interval notation: If $a, b \in \mathbb{R}$ and $a < b$ then

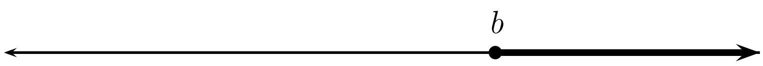
(a) $[a, b] = \{\, x \in \mathbb{R} \mid a \leq x \leq b \,\}$, called a *closed* interval.

(b) $(a, b) = \{\, x \in \mathbb{R} \mid a < x < b \,\}$, called an *open* interval.

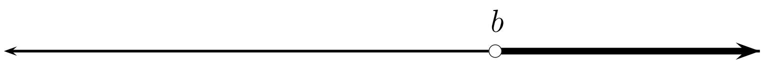
(c) $[a, b) = \{\, x \in \mathbb{R} \mid a \leq x < b \,\}$, called *half-open* or *half-closed*.

(d) $(a, b] = \{\, x \in \mathbb{R} \mid a < x \leq b \,\}$, also called *half-open* or *half-closed*.

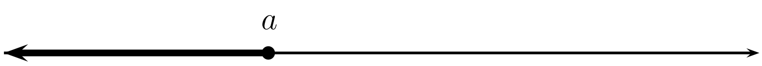
(e) $[b, \infty) = \{\, x \in \mathbb{R} \mid b \leq x \,\}$.



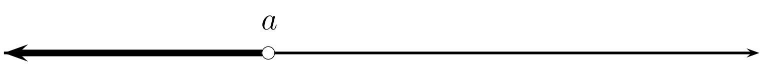
(f) $(b, \infty) = \{ x \in \mathbb{R} \mid b < x \}$.



(g) $(-\infty, a] = \{ x \in \mathbb{R} \mid x \leq a \}$.



(h) $(-\infty, a) = \{ x \in \mathbb{R} \mid x < a, \}$.



It is also common to write $\mathbb{R} = (-\infty, \infty)$, the interval that covers the entire real line.

Examples:

1. From the sign diagram for $(x-1)(x-2)$ above, the solution set for the inequality

$$(x-1)(x-2) \ < \ 0$$

is the open interval $(1, 2)$. By contrast, the solution set for the inequality

$$(x-1)(x-2) \ \leq \ 0$$

is the closed interval $[1, 2]$.

2. From the sign diagram for $(x-1)(x-2)$ above, the solution set for the inequality

$$(x-1)(x-2) \ > \ 0$$

is $(-\infty, 1) \cup (2, \infty)$, the union of two intervals. By contrast, the solution set for the inequality

$$(x-1)(x-2) \ \geq \ 0$$

is $(-\infty, 1] \cup [2, \infty)$, again the union of two intervals, but now including 1 and 2.

3. From the sign diagram for $x(x^2 - 1)$ above, the solution set for the inequality

$$x(x^2-1) \ > \ 0$$

is $(-1, 0) \cup (1, \infty)$. The solution set for the inequality

$$x(x^2-1) \ \leq \ 0$$

is $(-\infty, -1] \cup [0, 1]$.

4. From the sign diagram for $\frac{x^2-1}{x}$ above, the solution set for the inequality

$$\frac{x^2-1}{x} \ < \ 0$$

is $(-\infty, -1) \cup (0, 1)$. The solution set for the inequality

$$\frac{x^2-1}{x} \ \geq \ 0$$

is $[-1, 0) \cup [1, \infty)$.

5. From the sign diagram for $\frac{x}{x^2-1}$ above, the solution set for the inequality

$$\frac{x}{x^2-1} \; > \; 0$$

is $(-1,0) \cup (1,\infty)$. The solution set for the inequality

$$\frac{x}{x^2-1} \; \leq \; 0$$

is $(-\infty,-1) \cup [0,1)$.

6. Consider the expression

$$f(x) \;=\; \frac{x^2(x-1)}{(x+1)(x-3)^2} \,.$$

Then $f(x) = 0$ precisely when the numerator $x^2(x-1) = 0$, that is, when $x = 0$ and $x = 1$. Also $f(x)$ is undefined precisely when the denominator $(x+1)(x-3)^2 = 0$, that is, when $x = -1$ and $x = 3$.

Thinking about the combinations of signs of factors, as one moves along the real line, crossing over these important values for x, one gets the following sign diagram:

x		-1		0		1		3	
$f(x)$	$+$	u	$-$	0	$-$	0	$+$	u	$+$

From the sign diagram, one can quick read off the following solution sets:

$$\{x \in \mathbb{R} \mid f(x) = 0\} \;=\; \{0,1\} \,,$$

$$\{x \in \mathbb{R} \mid f(x) > 0\} \;=\; (-\infty,-1) \cup (1,3) \cup (3,\infty) \,,$$

$$\{x \in \mathbb{R} \mid f(x) < 0\} \;=\; (-1,0) \cup (0,1) \,,$$

$$\{x \in \mathbb{R} \mid f(x) \geq 0\} \;=\; (-\infty,-1) \cup \{0\} \cup [1,3) \cup (3,\infty) \,,$$

$$\{x \in \mathbb{R} \mid f(x) \leq 0\} \;=\; (-1,1] \,.$$