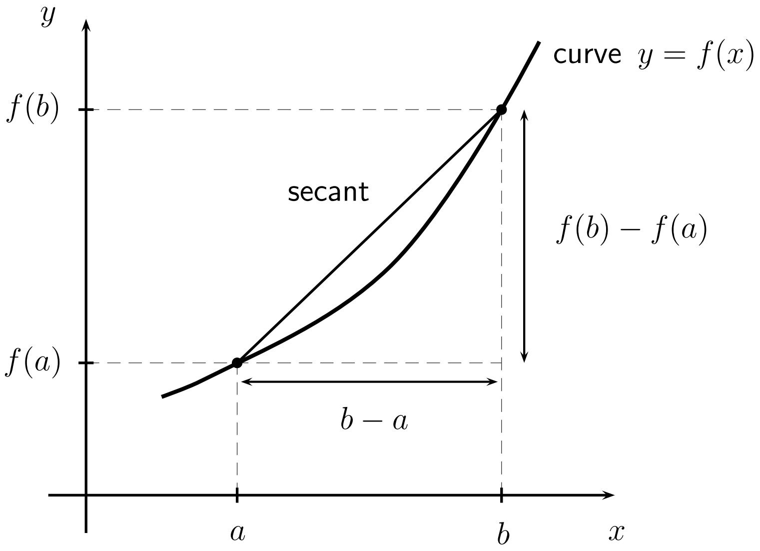
Notes for 'Slopes and average rates of change'

Important Ideas and Useful Facts:

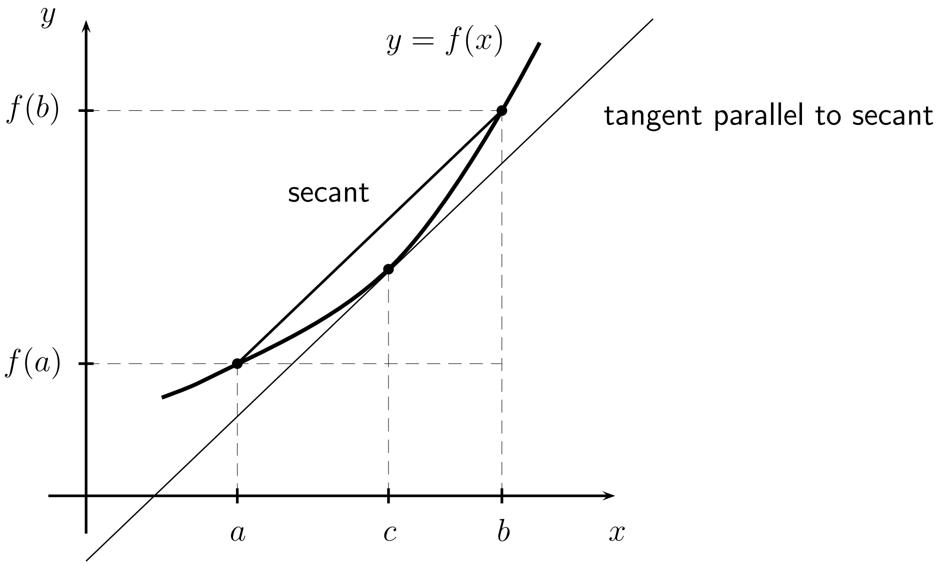
(i) Average rate of change of a function: If a function $y = f(x)$ is defined for x such that $a \leq x \leq b$ then the quotient

$$\frac{f(b) - f(a)}{b - a},$$

represents the *average rate of change* of $f(x)$ as x moves from a to b. If $f(x)$ denotes displacement of an object at time x, then this is the average speed of the object over that time interval. This is also the slope of the secant joining endpoints of the curve.



(ii) The Mean Value Theorem: This result is usually taught in advanced courses in calculus. The *Mean Value Theorem* says that the average rate of change for a smooth function is achieved as an "instantaneous rate of change" at some point within the interval. We make sense of "instantaneous rate of change" formally later when we discuss *derivatives*.



This says that the secant whose slope is the average rate of change over the entire interval, has the same slope as the tangent line to the curve at some point within the interval.

It is easy to see visually why this should be true. You can imagine moving the secant parallel to itself until it becomes the segment of a tangent line, just glancing the curve at some point. In the above diagram, this occurs when $x = c$.

Speed cameras used by road traffic authorities use this principle: they calculate the average speed of a vehicle over some time interval. If the average is more than the speed limit, then this theorem guarantees that the speedometer of the vehicle (which in principle displays the instantaneous speed from moment to moment) was indicating that at some point of time the driver exceeded the speed limit.

Examples:

1. The distance from Town A to Town B is 37 km. The speed limit is 70 kph (kilometres per hour) throughout, along the road between the two towns.

A person drives from Town A to Town B in the morning, leaving Town A at 10:32 am and arriving in Town B at 11:06 am.

The person returns in the afternoon, leaving Town B at 2:11 pm, arriving back in Town A at 2:40 pm.

The problem is to determine the average speed for each of the journeys, and to make any relevant deductions about the driver exceeding the speed limit.

Solution: The forward journey took 34 minutes, which is $\frac{34}{60}$ hours. The average speed for that leg was

$$\frac{37}{34/60} = \frac{37 \times 60}{34} \approx 65 \text{ kph} ,$$

to the nearest unit, which is safely below the speed limit. However, that does not guarantee the driver did not exceed the speed limit at some moment in the forward journey.

The return journey took 29 minutes, which is $\frac{29}{60}$ hours. The average speed for that leg was

$$\frac{37}{29/60} = \frac{37 \times 60}{29} \approx 77 \text{ kph} ,$$

to the nearest unit, which is well above the speed limit. This appears to guarantee that at some moment during the return journey the driver exceeded the speed limit.

With regard to accuracy, assuming the measurements are correct to the indicated levels of accuracy, the most time that could have been taken on the return journey would be just under 30 minutes (if the clock indicating the departure from Town B had only just shown 2:11 pm and the clock indicating arrival back at Town A was at the moment of becoming 2:41 pm). If the true distance between the towns was close to 36.5 km (which rounds up to 37 km), then the average speed for the return trip would be bounded below by

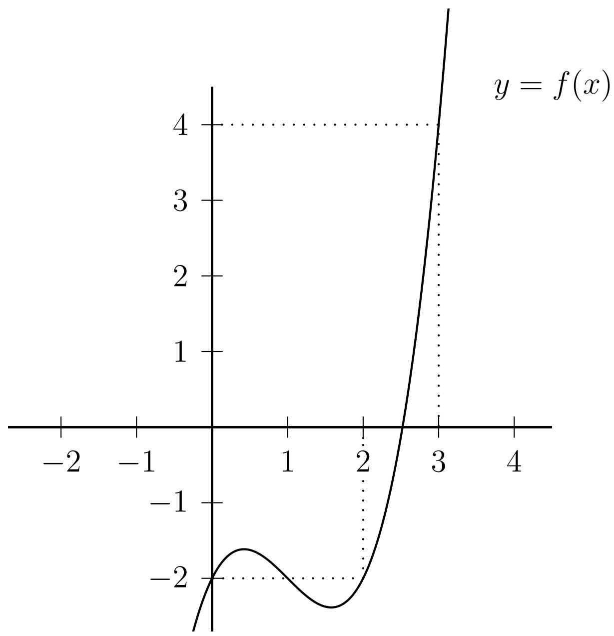
$$\frac{36.5}{30/60} = \frac{36.5 \times 60}{30} = 73 \text{ kph} .$$

Even taking into account tolerances in the accuracy of measurements, the unavoidable conclusion is that the driver exceeded the speed limit at some moment on the return journey.

2. Consider the function f with rule

$$f(x) \;=\; x^3 - 3x^2 + 2x - 2 \;,$$

whose graph appears below.



Find the average rate of change over the following intervals:

(a) $[0, 3]$ (b) $[1, 3]$ (c) $[2, 3]$ (d) $[0, 2]$

Solution: Observe that

$$f(0) \;=\; f(1) \;=\; f(2) \;=\; -2 \qquad \text{and} \qquad f(3) \;=\; 4\;.$$

For part (a), the average rate of change over the interval $[0, 3]$ is

$$\frac{4-(-2)}{3-0} \;=\; \frac{6}{3} \;=\; 2\;.$$

For part (b), the average rate of change over the interval $[1, 3]$ is

$$\frac{4-(-2)}{3-1} \;=\; \frac{6}{2} \;=\; 3\;.$$

For part (c), the average rate of change over the interval $[2, 3]$ is

$$\frac{4-(-2)}{3-2} \;=\; \frac{6}{1} \;=\; 6\;.$$

For part (d), the average rate of change over the interval $[0, 2]$ is

$$\frac{(-2)-(-2)}{2-0} \;=\; \frac{0}{2} \;=\; 0\;.$$