MODULE 7 PART I

INTEGRATION

AREA AND THE DEFINITE INTEGRAL

THE FUNDAMENTAL THEOREM OF CALCULUS

MODULE 7 PART I

INTEGRATION

ANTIDERIVATIVE

If $F'(x) = f(x)$, then $F(x)$ is an antiderivative of $f(x)$.

If $F(x) = 10x$, then $F'(x) = 10$, so $F(x) = 10x$ is an antiderivative of $f(x) = 10$.

For $F(x) = x^2$, $F'(x) = 2x$, making $F(x) = x^2$ an antiderivative of $f(x) = 2x$.

Find an antiderivative of $F(x) = 8x^7$

Recall that the derivative of x^n is nx^{n-1}

If $nx^{n-1} = 8x^7$, then $n - 1 = 7$ and $n = 8$

So, x^8 is an antiderivative of $8x^7$

FAMILY OF ALL ANTIDERIVATIVES

Note: $x^8 + 15$ and $x^8 - 9$ are also antiderivatives of $8x^7$

The family of all antiderivatives can only differ by a constant

If $F(x)$ and $G(x)$ are both antiderivatives of a function $f(x)$ on an interval, then there is a constant C such that $F(x) - G(x) = C$.

The family of all antiderivatives of the function f is indicated by

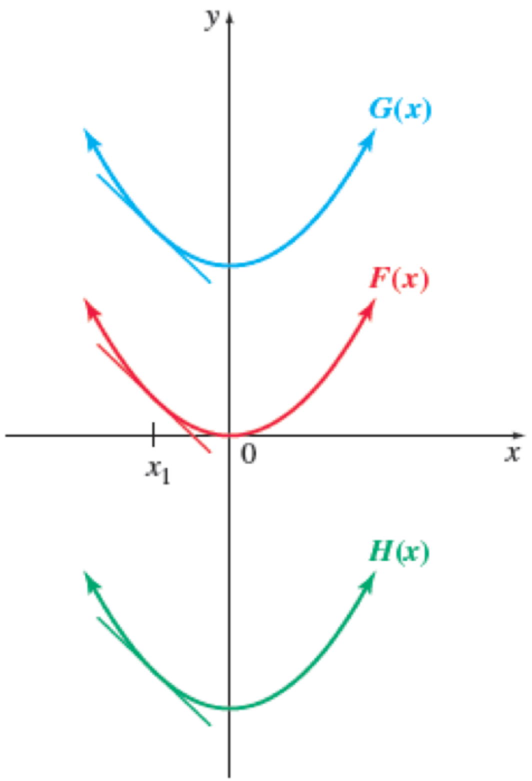
$$\int f(x)dx = F(x) + C$$

Indefinite Integral

If $F'(x) = f(x)$, then

$$\int f(x)dx = F(x) + C$$

for any real number C



RULES OF INTEGRATION

Power Rule

For any real number $n \neq -1$,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

Find each indefinite integral

$\int t^3 dt$ $\quad$ $\int t^3\, dt = \frac{t^{3+1}}{3+1} + C = \frac{t^4}{4} + C$ $\quad$ Check that the derivative of $t^4/4 + C$ is t^3

$\int \frac{1}{t^2} dt$ $\quad$ $\int \frac{1}{t^2} dt = \int t^{-2}\, dt = \frac{t^{-2+1}}{-2+1} + C = \frac{t^{-1}}{-1} + C = -\frac{1}{t} + C$ $\quad$ Check that the derivative of $-1/t + C$ is $1/t^2$

$\int \sqrt{u}\, du$ $\quad$ $\int \sqrt{u}\, du = \int u^{1/2}\, du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3} u^{3/2} + C$ $\quad$ Check that the derivative of $(2/3)u^{3/2} + C$ is $\sqrt{u}$

RULES OF INTEGRATION

<u>Constant Multiple Rule</u>

$$\int kf(x)dx = k\int f(x)dx$$

for any real number k

<u>Sum or Difference Rule</u>

$$\int [f(x) \pm g(x)] = \int f(x)dx \pm \int g(x)dx$$

Find each indefinite integral

$$\int 2v^3\,dv \qquad \int 2v^3\,dv = 2\int v^3\,dv = 2\left(\frac{v^4}{4}\right) + C = \frac{v^4}{2} + C$$

$$\int \frac{12}{z^5}dz \qquad \int \frac{12}{z^5}dz = 12\int z^{-5}\,dz = 12\left(\frac{z^{-4}}{-4}\right) + C = -3z^{-4} + C = -\frac{3}{z^4} + C$$

$$\int (3z^2 - 4z + 5)\,dz \qquad \int (3z^2 - 4z + 5)\,dz = 3\int z^2\,dz - 4\int z\,dz + 5\int dz$$

$$= 3\left(\frac{z^3}{3}\right) - 4\left(\frac{z^2}{2}\right) + 5z + C = z^3 - 2z^2 + 5z + C$$

TECHNIQUES

Find each indefinite integral

$$\int \frac{x^2+1}{\sqrt{x}}dx$$

$$\int \frac{x^2+1}{\sqrt{x}}dx = \int \left(\frac{x^2}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right)dx = \int \left(x^{3/2} + x^{-1/2}\right)dx$$

$$= \int x^{3/2}\,dx + \int x^{-1/2}\,dx$$

$$= \frac{x^{5/2}}{5/2} + \frac{x^{1/2}}{1/2} + C$$

$$= \frac{2}{5}x^{5/2} + 2x^{1/2} + C$$

$$\int (x^2-1)^2\,dx$$

$$\int (x^2-1)^2\,dx = \int (x^4 - 2x^2 + 1)\,dx$$

$$= \frac{x^5}{5} - \frac{2x^3}{3} + x + C$$

EXPONENTIAL FUNCTIONS

$$\int e^x\,dx = e^x + C$$

$$\int e^{kx}\,dx = \frac{e^{kx}}{k} + C, \qquad k \neq 0$$

For $a > 0, a \neq 1$:

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int a^{kx}\,dx = \frac{a^{kx}}{k(\ln a)} + C$$

$$\int 9e^t\,dt = 9\int e^t\,dt = 9e^t + C$$

$$\int e^{9t}\,dt = \frac{e^{9t}}{9} + C$$

$$\int 3e^{4u}\,du = 3\left(\frac{e^{4u}}{4}\right) + C = \frac{3}{4}e^{4u} + C$$

$$\int 2^{-5x}\,dx = \frac{2^{-5x}}{-5(\ln 2)} + C = -\frac{2^{-5x}}{5(\ln 2)} + C$$

INDEFINITE INTEGRAL OF $1/x$

Recall: The power rule did not apply for x^{-1}

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int \frac{4}{x} dx = 4\ln|x| + C$$

$$\int \left(-\frac{5}{x} + e^{-2x}\right) dx = -5\ln|x| - \frac{1}{2}e^{-2x} + C$$

INTEGRATION BY SUBSTITUTION

If $u = f(x)$, the differential, $du = f'(x)dx$

For example, if $u = 2x^3 + 1$, then $du = 6x^2dx$

Find the following indefinite integral

$$\int (2x^3 + 1)^4 6x^2\, dx$$

Let $u = 2x^3 + 1$, then $du = 6x^2dx$

$$\int (2x^3 + 1)^4 6x^2\, dx = \int u^4 du$$

$$= \frac{u^5}{5} + C = \frac{(2x^3 + 1)^5}{5} + C$$

Check by finding derivative:

$$\frac{d}{dx}\left[\frac{(2x^3 + 5)^5}{5} + C\right] = \frac{1}{5}(5)(2x^3 + 1)^4(6x^2) + 0 = (2x^3 + 1)^4 6x^2$$

EXAMPLE

Find $\int x^2\sqrt{x^3+1}\,dx$

Method 1

Let $u = x^3 + 1$, then $du = 3x^2dx$

Integrand does not contain 3, so multiply by $3/3$ and place 3 in the integrand and $1/3$ outside

$$\int x^2\sqrt{x^3+1}\,dx = \frac{1}{3}\int 3x^2\sqrt{x^3+1}\,dx = \frac{1}{3}\int \sqrt{x^3+1}\,(3x^2dx)$$

$$= \frac{1}{3}\int \sqrt{u}\,du = \frac{1}{3}\int u^{1/2}\,du$$

$$= \left(\frac{1}{3}\right)\left(\frac{u^{3/2}}{3/2}\right) + C = \frac{2}{9}u^{3/2} + C$$

$$= \frac{2}{9}(x^3+1)^{3/2} + C$$

Method 2

Again, let $u = x^3 + 1$, then $du = 3x^2dx$

Solve for x^2dx:

$$\frac{1}{3}du = x^2dx$$

$$\int \sqrt{x^3+1}\,(x^2dx) = \int \frac{1}{3}\sqrt{u}\,du = \frac{1}{3}\int u^{1/2}\,du$$

$$\ldots = \frac{2}{9}(x^3+1)^{3/2} + C$$

MORE EXAMPLES

Find the following integrals

$$\int \frac{x+3}{(x^2+6x)^2}\,dx$$

Let $u = x^2 + 6x$, then $du = (2x+6)dx$

$$du = 2(x+3)dx$$
$$\frac{1}{2}du = (x+3)dx$$

$$\int \frac{x+3}{(x^2+6x)^2}\,dx = \frac{1}{2}\int \frac{1}{u^2}\,du = \frac{1}{2}\int u^{-2}\,du$$

$$\left(\frac{1}{2}\right)\left(\frac{u^{-1}}{-1}\right) + C = -\frac{1}{2u} + C$$

$$= -\frac{1}{2(x^2+6x)} + C$$

$$\int \frac{2x-3}{x^2-3x}\,dx$$

Let $u = x^2 - 3x$, then $du = (2x-3)dx$

$$\int \frac{2x-3}{x^2-3x}\,dx = \int \frac{1}{u}\,du$$

$$= \ln|u| + C = \ln|x^2-3x| + C$$

$$\int x^2 e^{x^3}\,dx$$

Let $u = x^3$, then $du = 3x^2dx$

$$\frac{1}{3}du = x^2dx$$

$$\int x^2 e^{x^3}\,dx = \frac{1}{3}\int e^u\,du$$

$$= \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C$$

$$\int x\sqrt{1-x}\,dx$$

Let $u = 1 - x$, then $du = -dx$

Note: $-du = dx$ and $x = 1 - u$

$$\int x\sqrt{1-x}\,dx = -\int (1-u)\sqrt{u}\,du$$

$$= -\int \left(u^{1/2} - u^{3/2}\right)du$$

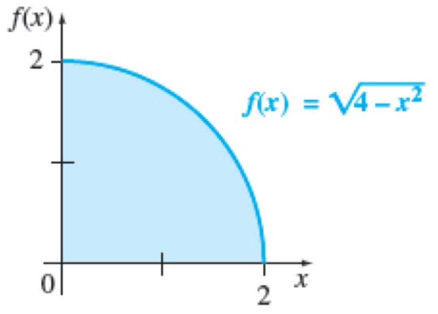
$$= -\left(\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right) + C$$

$$= \frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2} + C$$

MODULE 7 PART I

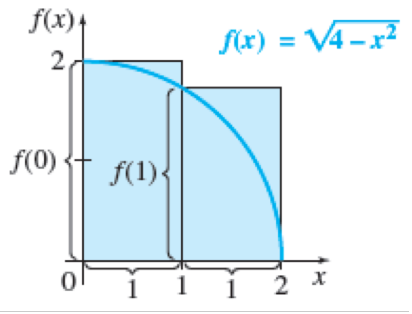
AREA AND THE DEFINITE INTEGRAL

APPROXIMATION OF AREA



Consider the region bounded by the y-axis, the x-axis, and the graph of $f(x) = \sqrt{4 - x^2}$

Notice this is the graph of $1/4$ of a circle with radius 2. The exact area is π



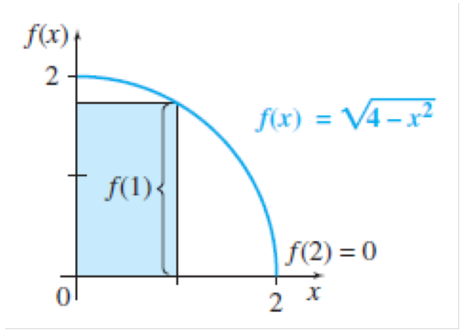
Approximate the area of the region using two rectangles and left endpoints

Each rectangle has width 1

The height of the 1st rectangle is $f(0) = 2$

The height of the 2nd rectangle is $f(1) = \sqrt{3}$

The total area of the two rectangles is
$1 \cdot f(0) + 1 \cdot f(1) = 2 + \sqrt{3} \approx 3.7321$ square units



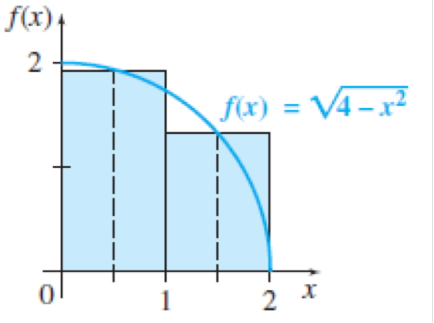
Approximate the area using right endpoints. Note: the 2nd rectangle has height 0 since $f(2) = 0$

Using right endpoints, the area is
$1 \cdot f(1) + 1 \cdot f(2) = \sqrt{3} + 0 \approx 1.7321$ square units

Since using left endpoints overestimates the area and right endpoints underestimates the area, we could average the two to get 2.7321 square units

This is known as the trapezoidal rule

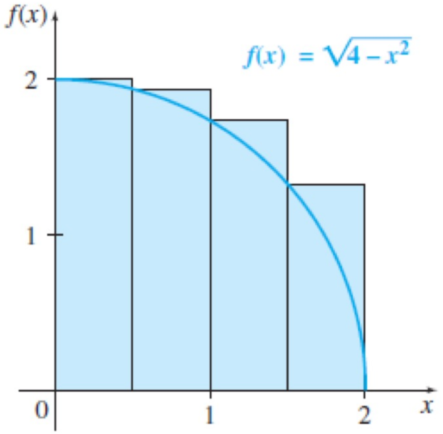
BETTER APPROXIMATIONS



Suppose instead we use the midpoint of each rectangle to determine the height

$$1 \cdot f\left(\frac{1}{2}\right) + 1 \cdot f\left(\frac{3}{2}\right) = \frac{\sqrt{15}}{2} + \frac{\sqrt{7}}{2}$$

≈ 3.2594 square units

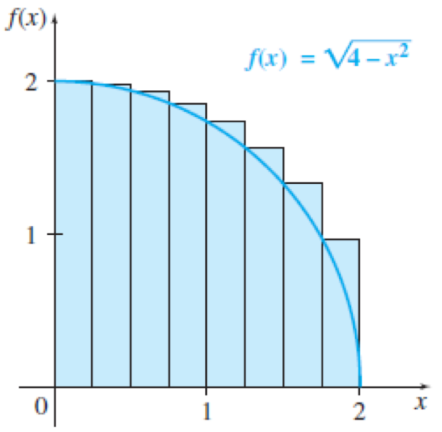


Use left endpoints and 4 rectangles

$$\frac{1}{2} \cdot f(0) + \frac{1}{2} \cdot f\left(\frac{1}{2}\right) + \frac{1}{2} \cdot f(1) + \frac{1}{2} \cdot f\left(\frac{3}{2}\right)$$

$$= \frac{1}{2}(2) + \frac{1}{2}\left(\frac{\sqrt{15}}{2}\right) + \frac{1}{2}(\sqrt{3}) + \frac{1}{2}\left(\frac{\sqrt{7}}{2}\right)$$

≈ 3.4957 square units

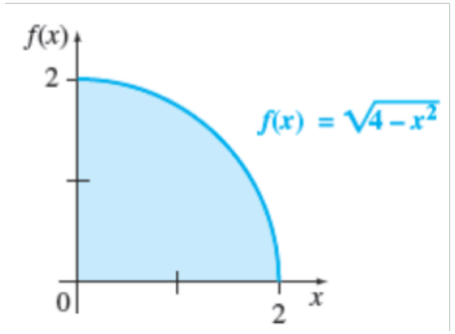


Use left endpoints and 8 rectangles

$$\frac{1}{4} \cdot f(0) + \frac{1}{4} \cdot f\left(\frac{1}{4}\right) + \frac{1}{4} \cdot f\left(\frac{1}{2}\right) + \frac{1}{4} \cdot f\left(\frac{3}{4}\right)$$

$$+ \frac{1}{4} \cdot f(1) + \frac{1}{4} \cdot f\left(\frac{5}{4}\right) + \frac{1}{4} \cdot f\left(\frac{3}{2}\right) + \frac{1}{4} \cdot f\left(\frac{7}{4}\right)$$

≈ 3.3398 square units



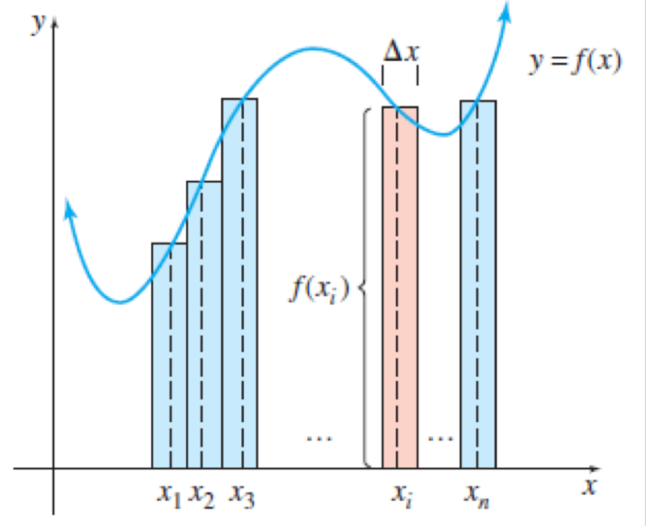
Recall: The exact area of this region is π

	Approximations to the Area			
n	Left Sum	Right Sum	Trapezoidal	Midpoint
2	3.7321	1.7321	2.7321	3.2594
4	3.4957	2.4957	2.9957	3.1839
8	3.3398	2.8398	3.0898	3.1567
10	3.3045	2.9045	3.1045	3.1524
20	3.2285	3.0285	3.1285	3.1454
50	3.1783	3.0983	3.1383	3.1426
100	3.1604	3.1204	3.1404	3.1419
500	3.1455	3.1375	3.1415	3.1416

As the number of rectangles increases, we get closer and closer to $\pi \approx 3.1416$

THE DEFINITE INTEGRAL

$$\Delta x = \frac{b-a}{n}$$



n rectangles of equal width

x_i is an arbitrary point in the i^{th} interval

The pink rectangle is an arbitrary rectangle called the i^{th} rectangle.

Area of the i^{th} rectangle = $f(x_i) \cdot \Delta x$

<u>Area of all n rectangles</u>

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

<u>Exact area</u>

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

<u>The Definite Integral</u>

If f is defined on the interval $[a, b]$, the definite integral of f from a to b is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

provided the limit exists, where $\Delta x = (b-a)/n$ and x_i is any value of x in the i^{th} interval.

The sum in the definition is a Riemann sum

<u>Your turn</u>: Approximate the the area of the region under the graph of $f(x) = 2x$ above the x-axis and between $x = 0$ and $x = 4$ by using four rectangles of equal width whose heights are the midpoint of each subinterval.

<u>Answer</u>: 16

TOTAL CHANGE

Let $f(x) = x^2 + 20$ be the marginal cost of some item

$f(2) = 24$ represents the rate of change of cost (marginal cost) at $x = 2$

A unit change in x at this point will produce a change of 24 units in the cost function

Similarly, $f(3) = 29$ means that each unit of change in x when $x = 3$ will produce a change of 29 units in the cost function.

The total change in cost from $x = 2$ to $x = 3$ can be found using the definite integral, i.e. the area under the curve between $x = 2$ and $x = 3$

If $f(x)$ gives the rate of change of $F(x)$ for x in $[a, b]$, then the total change in $F(x)$ as x goes from a to b is given by

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\, dx$$

MODULE 7 PART I

THE FUNDAMENTAL THEOREM OF CALCULUS

FUNDAMENTAL THEOREM OF CALCULUS

Recall, if $f(x) \geq 0$,

$$\int_a^b f(x)\,dx$$

gives the area between the graph of $f(x)$ and the x-axis from $x = a$ to $x = b$

Also, if $f(x)$ gives the rate of change of $F(x)$, the definite integral

$$\int_a^b f(x)\,dx$$

gives the total change of $F(x)$ as x changes from a to b

If $f(x)$ gives the rate of change of $F(x)$, then $F(x)$ is an antiderivative of $f(x)$

If we write the total change in $F(x)$ from $x = a$ to $x = b$ as $F(b) - F(a)$, then we connect antiderivatives and definite integrals

<u>Fundamental Theorem of Calculus</u>

Let f be continuous on an interval $[a, b]$, and let F be any antiderivative of f. Then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

EXAMPLE

First find $\int 4t^3\,dt$ and then find $\int_1^2 4t^3\,dt$

By the power rule,

$$\int 4t^3\,dt = t^4 + C$$

By the Fundamental Theorem of Calculus, no constant C is required:

$$\int_1^2 4t^3\,dt = t^4\Big|_1^2 = 2^4 - 1^4 = 15$$

A definite integral is a real number, and an indefinite integral is a family of functions

Note: No constant C is required for the definite integral. If we add it to the antiderivative, it will be eliminated in the final answer:

$$\int_a^b f(x)\,dx = (F(x) + C)\Big|_a^b = (F(b) + C) - (F(a) + C) = F(b) - F(a)$$

PROPERTIES OF DEFINITE INTEGRALS

Assume all indicated integrals in each property exist

For any real constant k,

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

The integral of a constant times a function is the constant times the integral of the function

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

The integral of a sum (difference) of functions is the sum (difference) of the integrals of each function

$$\int_a^a f(x)\,dx = 0$$

The integral at a single point is $\mathbf{0}$

For any real number c

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

The integral of a function on an interval $[a, b]$ is the sum of the integrals on a finite number of subintervals of $[a, b]$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

If the limits of integration are reversed, we negate the integral

EXAMPLES

Find the following integrals

$$\int_2^5 (6x^2 - 3x + 5)\, dx$$

$$\int_1^2 \frac{dy}{y}$$

$$\int_0^5 x\sqrt{25 - x^2}\, dx$$

$$\int_2^5 (6x^2 - 3x + 5)\, dx$$

$$= 6\int_2^5 x^2\, dx - 3\int_2^5 x\, dx + 5\int_2^5 dx$$

$$= 2x^3 \Big|_2^5 - \frac{3}{2}x^2 \Big|_2^5 + 5x \Big|_2^5$$

$$= 2(5^3 - 2^3) - \frac{3}{2}(5^2 - 2^2) + 5(5 - 2)$$

$$= 234 - \frac{63}{2} + 15 = \frac{435}{2}$$

$$\int_1^2 \frac{dy}{y} = \int_1^2 \frac{1}{y} dy = \ln|y| \Big|_1^2$$

$$= \ln 2 - \ln 1 \approx 0.6931 - 0 = 0.6931$$

Let $u = 25 - x^2$, then $du = -2xdx$ and $-\frac{1}{2}du = xdx$

If $x = 0$, then $u = 25 - 0^2 = 25$
If $x = 5$, then $u = 25 - 5^2 = 0$

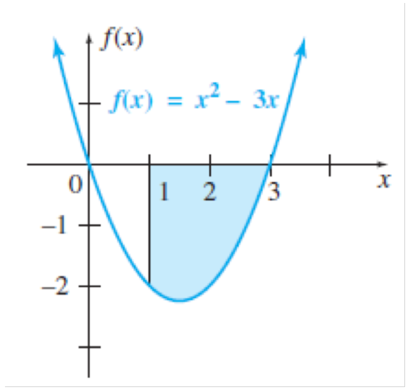
$$\int_0^5 x\sqrt{25 - x^2}\, dx = -\frac{1}{2}\int_{25}^0 \sqrt{u}\, du$$

$$= -\frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right)\Big|_{25}^0$$

$$= \left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(0^{3/2} - 25^{3/2}\right) = -\frac{1}{3}(-125) = \frac{125}{3}$$

AREA

Find the area of the region between the x-axis and the graph of $f(x) = x^2 - 3x$ from $x = 1$ to $x = 3$



Since the entire region is below the x-axis, the area is given by

$$\left| \int_1^3 (x^2 - 3x)\, dx \right|$$

By the FTC,

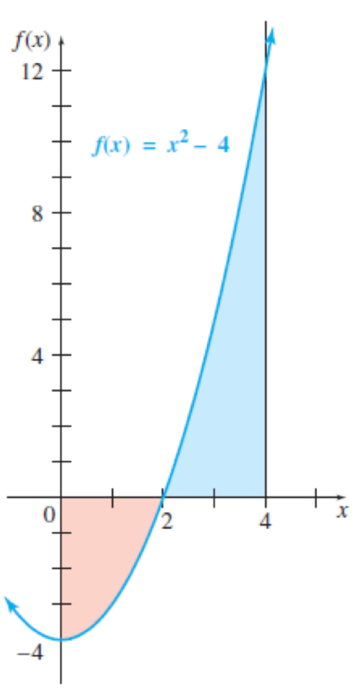
$$\int_1^3 (x^2 - 3x)\, dx = \left(\frac{x^3}{3} - \frac{3x^2}{2} \right) \Big|_1^3 = \left(\frac{27}{3} - \frac{27}{2} \right) - \left(\frac{1}{3} - \frac{3}{2} \right) = -\frac{10}{3}$$

So the area is

$$\left| -\frac{10}{3} \right| = \frac{10}{3}$$

<u>Your turn</u>: Find the area between the x-axis and the graph of $f(x) = x^2 - 4$ from $x = 0$ to $x = 4$

<u>Answer</u>: 16



INTEGRATION BY PARTS

If u and v are both differentiable functions, then uv is also differentiable

By the product rule for derivatives

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$d(uv) = u\,dv + v\,du$$

$$\int d(uv) = \int u\,dv + \int v\,du$$

or

$$uv = \int u\,dv + \int v\,du$$

If u and v are differentiable functions, then

$$\int u\,dv = uv - \int v\,du$$

EXAMPLE

Find $\int_0^1 xe^{5x}\,dx$

First find the indefinite integral using integration by parts

Write $xe^{5x}dx$ as a product of two functions u and dv in such a way that $\int dv$ can be found

Choose $dv = e^{5x}dx$ and $u = x$, then $du = dx$

$$v = \int dv = \int e^{5x}\,dx = \frac{e^{5x}}{5}$$

$$uv - \int v\,du = x\left(\frac{e^{5x}}{5}\right) - \int \frac{e^{5x}}{5}\,dx$$

$$= \frac{xe^{5x}}{5} - \frac{e^{5x}}{25} + C = \frac{e^{5x}}{25}(5x-1) + C$$

$$\int_0^1 xe^{5x}\,dx = \frac{e^{5x}}{25}(5x-1)\Big|_0^1$$

$$= \frac{e^{5}}{25}(4) - \frac{e^{0}}{25}(-1) \approx 23.7861$$

QUESTIONS?