MODULE 2

SYSTEMS OF LINEAR EQUATIONS

MATRICES

CRAMER'S RULE

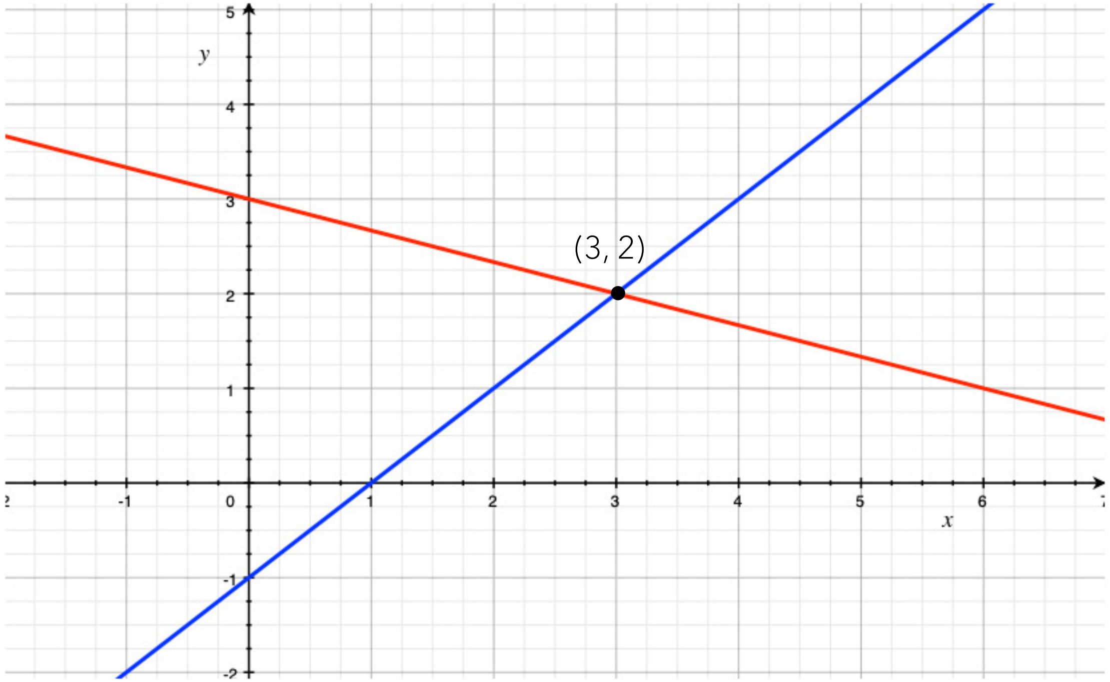
MODULE 2

SYSTEMS OF LINEAR EQUATIONS

TYPES OF SYSTEMS

Independent: One solution

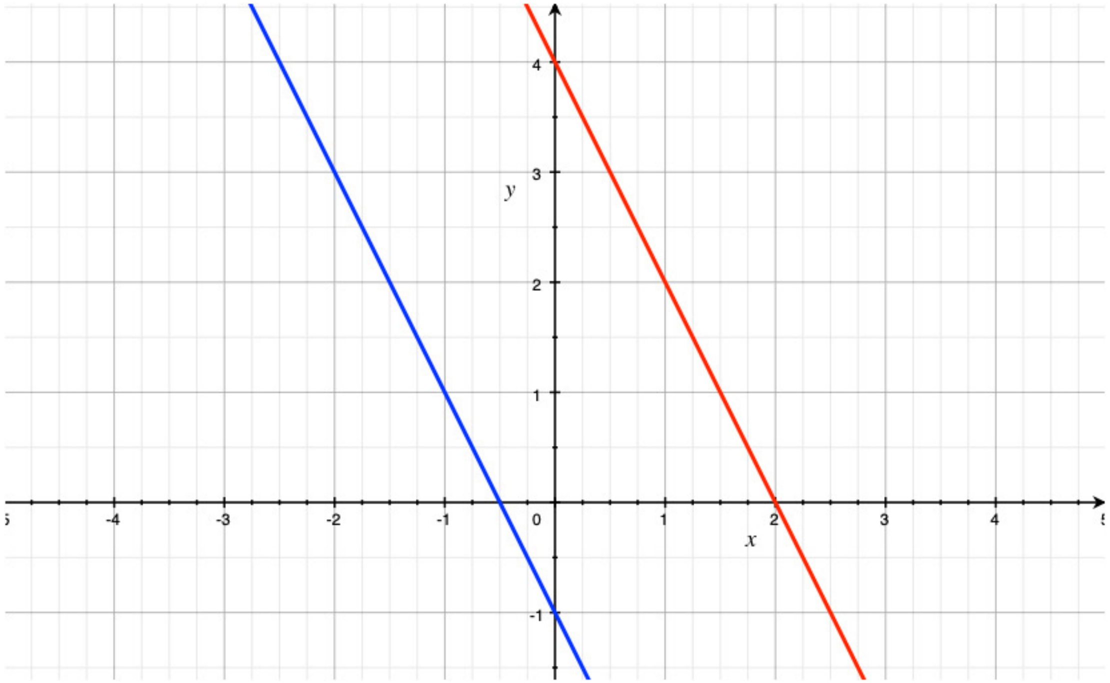
$x - y = 1$
$x + 3y = 9$



TYPES OF SYSTEMS

Inconsistent: No solutions

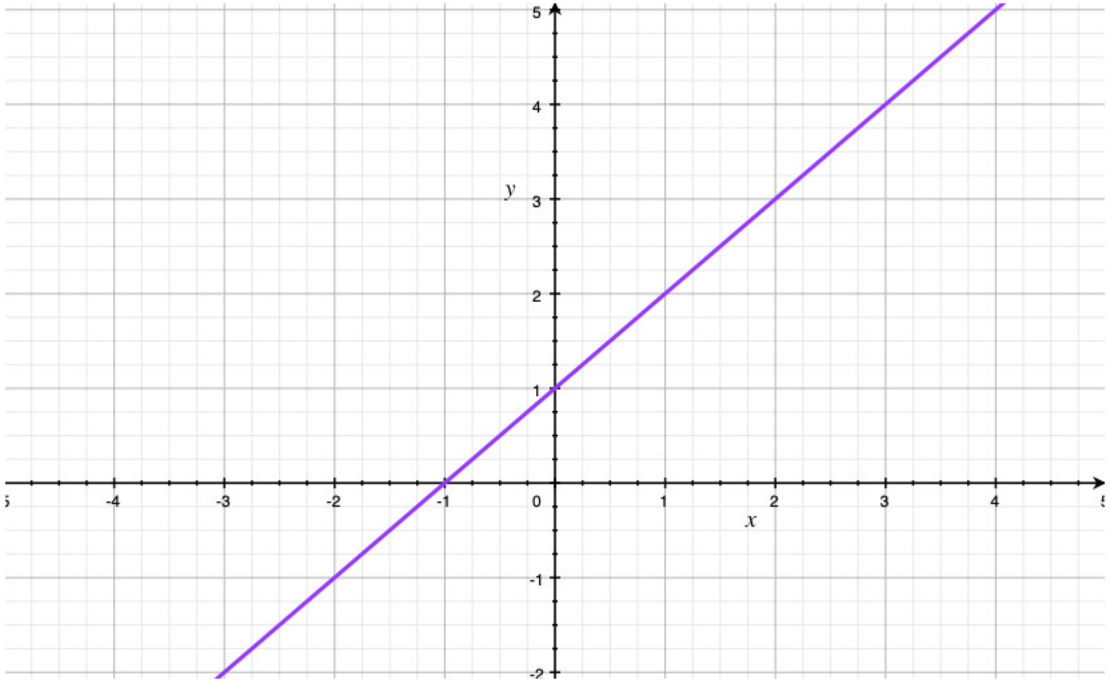
$2x + y = -1$
$2x + y = 4$



TYPES OF SYSTEMS

Dependent: Infinite solutions

$-x + y = 1$
$2x - 2y = -2$



METHODS TO SOLVE

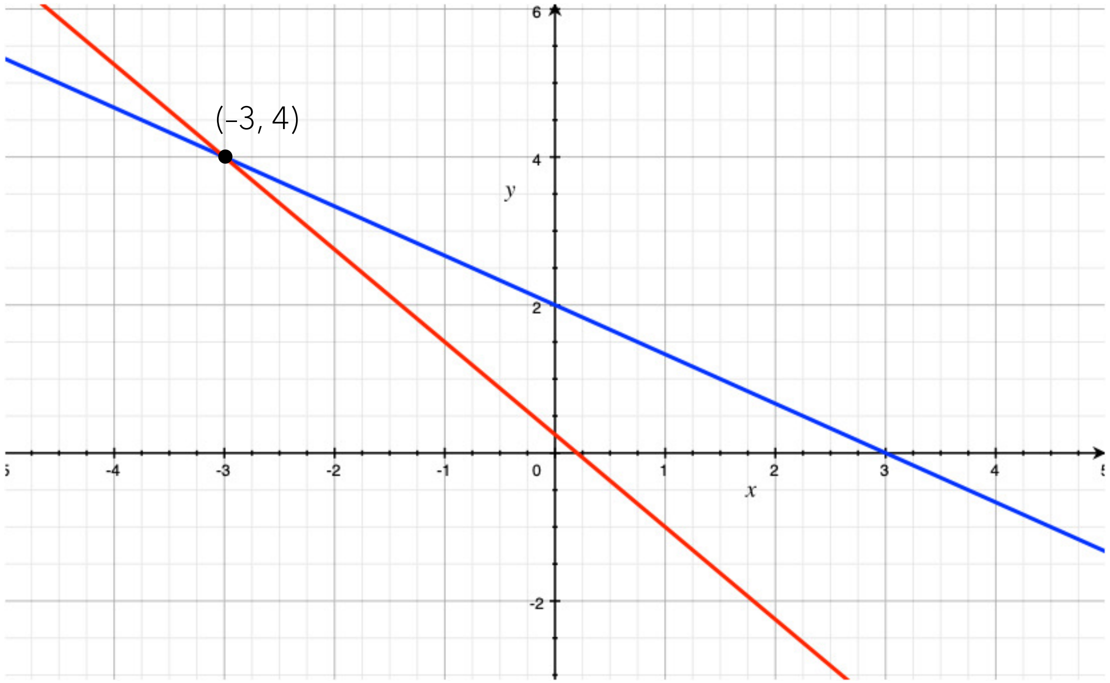
- Graphing Method
- Substitution Method
- Elimination (Addition) Method

GRAPHING METHOD

$$\begin{cases} \frac{x}{9} + \frac{y}{6} = \frac{1}{3} \\ 2x + \frac{8y}{5} = \frac{2}{5} \end{cases} \longrightarrow \begin{cases} 2x + 3y = 6 \\ 10x + 8y = 2 \end{cases} \longrightarrow \begin{cases} 2x + 3y = 6 \\ 5x + 4y = 1 \end{cases}$$

$2x + 3(0) = 6 \rightarrow x = 3$
$2(0) + 3y = 6 \rightarrow y = 2$
$(3,0)$ and $(0,2)$

$5x + 4(0) = 1 \rightarrow x = \frac{1}{5}$
$5(0) + 4x = 1 \rightarrow y = \frac{1}{4}$
$\left(\frac{1}{5}, 0\right)$ and $\left(0, \frac{1}{4}\right)$



SUBSTITUTION METHOD

$$\begin{cases} -3x + y = 4 \\ x - 2y = 4 \end{cases} \longrightarrow y = 3x + 4 \longrightarrow x - 2(3x + 4) = 4$$

$$x - 6x - 8 = 4$$

$$-5x = 12$$

$$x = -\frac{12}{5}$$

$$y = 3\left(-\frac{12}{5}\right) + 4$$

$$y = -\frac{36}{5} + \frac{20}{5} = -\frac{16}{5}$$

$$\left(-\frac{12}{5}, -\frac{16}{5}\right)$$

<u>Your Turn</u>:

Check the solution. Make sure both equations balance with $x = -\frac{12}{5}$ and $y = -\frac{16}{5}$

ELIMINATION METHOD EXAMPLE 1

$$5p + 11q = -7$$
$$3p - 8q = 25$$

$-3R_1 + 5R_2 \rightarrow R_2$ $\Longrightarrow$

$$\begin{aligned} -15p - 33q &= 21 \\ \underline{+(15p - 40q = 125)} \\ -73q &= 146 \end{aligned}$$

$\Longrightarrow$

$$5p + 11q = -7$$
$$-73q = 146$$

$$q = -2$$

$\Longleftarrow$

$$5p + 11(-2) = -7$$

$$5p = 15$$
$$p = 3$$

$$(3, -2)$$

Your Turn:

Check the solution. Make sure both equations balance with $p = 3$ and $q = -2$

ELIMINATION METHOD EXAMPLE 2

$$\begin{gathered} 3x - 6y + 3z = 11 \\ 2x + y - z = 2 \\ 5x - 5y + 2z = 6 \end{gathered}$$

$$\begin{gathered} -2R_1 + 3R_2 \to R_2 \\ -5R_1 + 3R_3 \to R_3 \end{gathered} \quad \Longrightarrow \quad \begin{gathered} 3x - 6y + 3z = 11 \\ 15y - 9z = -16 \\ 15y - 9z = -37 \end{gathered}$$

$$-R_2 + R_3 \to R_3 \quad \Longrightarrow \quad \begin{gathered} 3x - 6y + 3z = 11 \\ 15y - 9z = -16 \\ 0 = -21 \end{gathered}$$

No solution

ELIMINATION METHOD EXAMPLE 3

$$x - z = -3$$
$$y + z = 9$$
$$-2x + 3y + 5z = 33$$

$2R_1 + R_3 \to R_3$ ⟹

$$x - z = -3$$
$$y + z = 9$$
$$3y + 3z = 27$$

$-3R_2 + R_3 \to R_3$ ⟹

$$x - z = -3$$
$$y + z = 9$$
$$0 = 0$$

<u>Infinitely many solutions</u>

Let z be the parameter

$$x = z - 3$$
$$y = -z + 9$$

General solution:
$(z - 3, 9 - z, z)$

<u>Your Turn</u>:
Let x be the parameter

Answer: $(x, -x + 6, x + 3)$

Alaina's furniture factory has 1950 machine hours available each week in the cutting department, 1490 hours in the assembly department, and 2160 hours in the finishing department. Manufacturing a chair requires 0.2 hours of cutting, 0.3 hours of assembly, and 0.1 hours of finishing. A cabinet requires 0.5 hours of cutting, 0.4 hours of assembly, and 0.6 hours of finishing. A buffet requires 0.3 hours of cutting, 0.1 hours of assembly, and 0.4 hours of finishing. How many chairs, cabinets, and buffets should be produced in order to use all the available production capacity?

Let x = number of chairs to produce, y = number of cabinets to produce, and z = number of buffets to produce.

$0.2x + 0.5y + 0.3z = 1950$ ← Cutting hours

$0.3x + 0.4y + 0.1z = 1490$ ← Assembly hours

$0.1x + 0.6y + 0.4z = 2160$ ← Finishing hours

$$\begin{aligned} 2x + 5y + 3z &= 19500 \\ 3x + 4y + z &= 14900 \\ x + 6y + 4z &= 21600 \end{aligned}$$

← Eliminate decimals

$$\begin{aligned} 2x + 5y + 3z &= 19500 \\ -7y - 7z &= -28700 \quad \leftarrow -3R_1 + 2R_2 \to R_2 \\ 7y + 5z &= 23700 \quad \leftarrow -R_1 + 2R_3 \to R_3 \end{aligned}$$

$$\begin{aligned} 2x + 5y + 3z &= 19500 \\ -7y - 7z &= -28700 \\ -2z &= -5000 \quad \leftarrow R_2 + R_3 \to R_3 \end{aligned}$$

$$z = 2500$$

$$\begin{aligned} -7y - 7(2500) &= -28700 \\ -7y &= -11200 \\ y &= 1600 \end{aligned}$$

$$\begin{aligned} 2x + 5(1600) + 3(2500) &= 19500 \\ 2x &= 4000 \\ x &= 2000 \end{aligned}$$

The factory needs to produce 2000 chairs, 1600 cabinets, and 2500 buffets

<u>Your Turn</u>: Check the solution in all 3 equations

PYTHON

```
import sympy as sym

sym.init_printing()

x,y,z = sym.symbols('x,y,z')

solns = sym.solve([
    0.2*x + 0.5*y + 0.3*z - 1950,
    0.3*x + 0.4*y + 0.1*z - 1490,
    0.1*x + 0.6*y + 0.4*z - 2160],
    [x, y, z])

chairs = round(solns[x],0)
cabinets = round(solns[y],0)
buffets = round(solns[z],0)

print(f"The solution is",chairs, "chairs,",cabinets, "cabinets, and", buffets,"buffets.")
```

```
The solution is 2000.00000000000 chairs, 1600.00000000000 cabinets, and 2500.00000000000 buffets.
```

MODULE 2

MATRICES

MATRIX SIZE

Number of rows by number of columns:

$\begin{bmatrix} -3 & 5 \\ 2 & 0 \\ 5 & -1 \end{bmatrix}$ is a 3 x 2 matrix

$\begin{bmatrix} 0.5 & 8 & 0.9 \\ 0 & 5.1 & -3 \\ -4 & 0 & 5 \end{bmatrix}$ is a 3 x 3 matrix

$\begin{bmatrix} 1 & 6 & 5 & -2 & 5 \end{bmatrix}$ is a 1 x 5 matrix

$\begin{bmatrix} 3 \\ -5 \\ 0 \\ 2 \end{bmatrix}$ is a 4 x 1 matrix

MATRIX OPERATIONS

- Addition and subtraction
- Scalar multiplication
- Multiplication
- Inverse

EXAMPLE

Perform the following operations, if possible:

$$(3)\begin{bmatrix}2 & -1\\ 0 & 13\end{bmatrix}-\begin{bmatrix}4 & 8\\ -5 & 7\end{bmatrix}+\begin{bmatrix}12 & 7\\ 5 & 3\end{bmatrix}$$

$$=\begin{bmatrix}3(2) & 3(-1)\\ 3(0) & 3(13)\end{bmatrix}-\begin{bmatrix}4 & 8\\ -5 & 7\end{bmatrix}+\begin{bmatrix}12 & 7\\ 5 & 3\end{bmatrix}=\begin{bmatrix}6 & -3\\ 0 & 39\end{bmatrix}-\begin{bmatrix}4 & 8\\ -5 & 7\end{bmatrix}+\begin{bmatrix}12 & 7\\ 5 & 3\end{bmatrix}$$

$$=\begin{bmatrix}6-4 & -3-8\\ 0-(-5) & 39-7\end{bmatrix}+\begin{bmatrix}12 & 7\\ 5 & 3\end{bmatrix}=\begin{bmatrix}2 & -11\\ 5 & 32\end{bmatrix}+\begin{bmatrix}12 & 7\\ 5 & 3\end{bmatrix}$$

$$=\begin{bmatrix}2+12 & -11+7\\ 5+5 & 32+3\end{bmatrix}=\begin{bmatrix}14 & -4\\ 10 & 35\end{bmatrix}$$

```python
import numpy as np

a = np.array([[2,-1],[0,13]])
b = np.array([[4,8],[-5,7]])
c = np.array([[12,7],[5,3]])

result = 3*a - b + c
print(result)
```

```
[[14 -4]
 [10 35]]
```

Sam’s Shoes and Frank’s Footwear both have outlets in California and Arizona. Sam’s sells shoes for \$80, sandals for \$40, and boots for \$120. Frank’s prices are \$60, \$30, and \$150, respectively. Half of all sales in California stores are shoes, 1/4 are sandals, and1/4 are boots. In Arizona, the fractions are 1/5 shoes, 1/5 sandals, and 3/5 boots.

- (a) Write a 2 x 3 matrix called P representing prices for the two stores and the three types of footwear.
- (b) Write a 3 x 2 matrix called F representing the fraction of each type of footwear sold in each state.
- (c) Calculate the product PF and describe what the entries represent.
- (d) From the answer to part (c), what is the average price for a pair of footwear at an outlet of Frank’s in Arizona?

$$P = \begin{array}{c} \begin{array}{ccc} \textbf{Sh} & \textbf{Sa} & \textbf{B} \end{array} \\ \begin{bmatrix} 80 & 40 & 120 \\ 60 & 30 & 150 \end{bmatrix} \end{array} \begin{array}{l} \\ \textbf{Sam's} \\ \textbf{Frank's} \end{array} \qquad F = \begin{array}{c} \begin{array}{cc} \textbf{CA} & \textbf{AR} \end{array} \\ \begin{bmatrix} 1/2 & 1/5 \\ 1/4 & 1/5 \\ 1/4 & 3/5 \end{bmatrix} \end{array} \begin{array}{l} \\ \textbf{Sh} \\ \textbf{Sa} \\ \textbf{B} \end{array}$$

$$PF = \begin{bmatrix} 80 & 40 & 120 \\ 60 & 30 & 150 \end{bmatrix} \begin{bmatrix} 1/2 & 1/5 \\ 1/4 & 1/5 \\ 1/4 & 3/5 \end{bmatrix} = \begin{bmatrix} 80\left(\frac{1}{2}\right) + 40\left(\frac{1}{4}\right) + 120\left(\frac{1}{4}\right) & 80\left(\frac{1}{5}\right) + 40\left(\frac{1}{5}\right) + 120\left(\frac{3}{5}\right) \\ 60\left(\frac{1}{2}\right) + 30\left(\frac{1}{4}\right) + 150\left(\frac{1}{4}\right) & 60\left(\frac{1}{5}\right) + 30\left(\frac{1}{5}\right) + 150\left(\frac{3}{5}\right) \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{cc} \textbf{CA} & \textbf{AR} \end{array} \\ \begin{bmatrix} 80 & 96 \\ 75 & 108 \end{bmatrix} \end{array} \begin{array}{l} \\ \textbf{Sam's} \\ \textbf{Frank's} \end{array}$$

The rows give the average price per pair of footwear sold by each store and the columns give the state.

The entry in the 2nd row, 2nd column represents the average price of footwear at an outlet of Frank’s in Arizona: \$108

PYTHON

```
import numpy as np

A = np.array([[80,40,120],[60,30,150]])
B = np.array([[1/2,1/5],[1/4,1/5],[1/4,3/5]])
np.dot(A,B)
```

```
array([[ 80.,  96.],
       [ 75., 108.]])
```

MULTIPLICATIVE INVERSES

2 x 2 Identity matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If A^{-1} exists, then $AA^{-1} = A^{-1}A = I$

Find A^{-1} if $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$

Form the augmented matrix $[A|I]$:

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right]$$

Perform row operations to transform A to I:

$$-2R_1 + R_2 \rightarrow R_2 \quad \Longrightarrow \quad \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 2 & -2 & 1 \end{array}\right]$$

$$R_2 + 2R_1 \rightarrow R_1 \quad \Longrightarrow \quad \left[\begin{array}{cc|cc} 2 & 0 & 0 & 1 \\ 0 & 2 & -2 & 1 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{2}R_1 \rightarrow R_1 \quad \Longrightarrow \\ \frac{1}{2}R_2 \rightarrow R_2 \quad \Longrightarrow \end{array} \left[\begin{array}{cc|cc} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & -1 & \frac{1}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$$

Your turn: Check $AA^{-1} = A^{-1}A = I$

ALTERNATIVE METHOD

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc \neq 0$, then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Find A^{-1} for $A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}$, if it exists

$$2(5) - 1(-3) = 13$$

$$A^{-1} = \frac{1}{13}\begin{bmatrix} 5 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix}$$

Check:

$$AA^{-1} = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix} = \begin{bmatrix} 2\left(\frac{5}{13}\right) + (-3)\left(-\frac{1}{13}\right) & 2\left(\frac{3}{13}\right) + (-3)\left(\frac{2}{13}\right) \\ 1\left(\frac{5}{13}\right) + 5\left(-\frac{1}{13}\right) & 1\left(\frac{3}{13}\right) + 5\left(\frac{2}{13}\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

<u>Your turn</u>: Verify $A^{-1}A = I$

INVERSE DOES NOT EXIST

Find A^{-1} for $A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$, if it exists

$$ad - bc = 2(-2) - (-4)(1) = -4 + 4 = 0$$

Inverse does not exist

Form the augmented matrix $[A|I]$

$$\left[\begin{array}{cc|cc} 2 & -4 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{array}\right]$$

Perform row operations to transform A to I:

$$-R_1 + 2R_2 \rightarrow R_2 \quad \Longrightarrow \quad \left[\begin{array}{cc|cc} 2 & -4 & 1 & 0 \\ 0 & 0 & -1 & 2 \end{array}\right]$$

No way to complete the process. A^{-1} does not exist.

EXAMPLE 3X3

Find A^{-1} if $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

Form the augmented matrix $[A|I]$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1 + R_2 \to R_2 \Rightarrow \\ -3R_1 + R_3 \to R_3 \Rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right]$$

Column 2 has zeros in required positions

$$\begin{array}{r} R_3 + 3R_1 \to R_1 \Rightarrow \\ -R_3 + R_2 \to R_2 \Rightarrow \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{3}R_1 \to R_1 \Rightarrow \\ -\frac{1}{2}R_2 \to R_2 \Rightarrow \\ -\frac{1}{3}R_3 \to R_3 \Rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right]$$

<u>Your turn</u>:

Verify $AA^{-1} = A^{-1}A = I$

$$A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

PYTHON

```
import numpy as np
from scipy import linalg
a = np.array([[1,0,1],[2,-2,-1],[3,0,0]])
linalg.inv(a)
```

```
array([[-0.        ,  0.        ,  0.33333333],
       [-0.5       , -0.5       ,  0.5       ],
       [ 1.        ,  0.        , -0.33333333]])
```

USING INVERSE TO SOLVE SYSTEM

- Suppose we represent a system of linear equations by $AX = B$
- A is the square matrix of coefficients and A^{-1} exists
- X is the matrix of variables
- B is the matrix of constants
- Then $X = A^{-1}B$ is the solution to the system

Three brands of fertilizer are available to provide nitrogen, phosphoric acid, and potash. One bag of each brand provides the units of each nutrient shown in the table.

For ideal growth, the soil on a Michigan farm needs 18 units of nitrogen, 23 units of phosphoric acid, and 13 units of potash per acre. How many bags of each brand of fertilizer should be used per acre for ideal growth on the farm?

		Brand		
		Veg Health	Grow Big	Nutriplant
Nutrient	Nitrogen	1	2	3
	Phosphoric Acid	3	1	2
	Potash	2	0	1

Let x = bags fertilizer from Veg Health, y = bags of fertilizer from Grow Big, and z = bags of fertilizer from Nutriplant

$$\begin{aligned} x + 2y + 3z &= 18 \\ 3x + y + 2z &= 23 \\ 2x + z &= 13 \end{aligned}$$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad B = \begin{bmatrix} 18 \\ 23 \\ 13 \end{bmatrix}$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$-3R_1 + R_2 \to R_2$
$-2R_1 + R_3 \to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -7 & -3 & 1 & 0 \\ 0 & -4 & -5 & -2 & 0 & 1 \end{array}\right]$$

$2R_2 + 5R_1 \to R_1$
$-4R_2 + 5R_3 \to R_3$

$$\left[\begin{array}{ccc|ccc} 5 & 0 & 1 & -1 & 2 & 0 \\ 0 & -5 & -7 & -3 & 1 & 0 \\ 0 & 0 & 3 & 2 & -4 & 5 \end{array}\right]$$

$-R_3 + 3R_1 \to R_1$
$7R_3 + 3R_2 \to R_2$

$$\left[\begin{array}{ccc|ccc} 15 & 0 & 0 & -5 & 10 & -5 \\ 0 & -15 & 0 & 5 & -25 & 35 \\ 0 & 0 & 3 & 2 & -4 & 5 \end{array}\right]$$

$\frac{1}{15}R_1 \to R_1$
$-\frac{1}{15}R_2 \to R_2$
$\frac{1}{3}R_3 \to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{5}{3} & -\frac{7}{3} \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \end{array}\right]$$

$$X = A^{-1}B = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{5}{3} & -\frac{7}{3} \\ \frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \end{bmatrix} \begin{bmatrix} 18 \\ 23 \\ 13 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$$

Use 5 bags of Veg Health, 2 bags of Grow Big, and 3 bags of Nutriplant

PYTHON

```
import numpy as np
from scipy import linalg
a = np.array([[1,2,3],[3,1,2],[2,0,1]])
b = np.array([18,23,13])

np.dot(linalg.inv(a),b)
```

```
array([5., 2., 3.])
```

MODULE 2

CRAMER'S RULE

DETERMINANT: 2X2

Determinant of $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$:

$$\det(A) = |A| = a_{11}a_{22} - a_{12}a_{21}$$

Calculate the determinant of $\mathrm{A} = \begin{bmatrix} -5 & 3 \\ 7 & -1 \end{bmatrix}$

$$|A| = (-5)(-1) - (3)(7) = 5 - 21 = -16$$

Note: This is a square matrix with determinant ≠ 0, so the inverse exists

<u>Your turn</u>: Find A^{-1}. Use $A^{-1} = \frac{1}{|A|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

DETERMINANT: 3X3

Determinant of $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$:

Find $|A|$ for $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 7 & -3 \\ 3 & 8 & -5 \end{bmatrix}$

$$\det(A) = |A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Augment A with the first two columns of A

$$\left|\begin{array}{ccc|cc} 1 & 3 & -2 & 1 & 3 \\ 2 & 7 & -3 & 2 & 7 \\ 3 & 8 & -5 & 3 & 8 \end{array}\right|$$

$$|A| = (1)(7)(-5) + (3)(-3)(3) + (-2)(2)(8) - (-2)(7)(3) - (1)(-3)(8) - (3)(2)(-5)$$

$$= -35 - 27 - 32 + 42 + 24 + 30 = 2$$

PYTHON

```python
import numpy as np

A = np.array([[1,3,-2],[2,7,-3],[3,8,-5]])
D = np.linalg.det(A)
print(D)
```

```
2.0000000000000004
```

CRAMER'S RULE: 2X2

Consider the system of equations:

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

Let D be the determinant of the coefficient matrix and assume $D \neq 0$

Define

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \text{ and } D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

The solution to the system is given by

$$x = \frac{D_x}{D}, \mathrm{y} = \frac{D_y}{D}$$

$$2x + 5y = 15$$
$$x + 4y = 9$$

$$D = \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = (2)(4) - (5)(1) = 3$$

$$D_x = \begin{vmatrix} 15 & 5 \\ 9 & 4 \end{vmatrix} = (15)(4) - (5)(9) = 15$$

$$D_y = \begin{vmatrix} 2 & 15 \\ 1 & 9 \end{vmatrix} = (2)(9) - (15)(1) = 3$$

$$x = \frac{D_x}{D} = \frac{15}{3} = 5 \qquad y = \frac{D_y}{D} = \frac{3}{3} = 1$$

<u>Your turn</u>: Check the solution

CRAMER'S RULE: 3X3

Consider the system of linear equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Let D be the determinant of the coefficient matrix and assume $D \neq 0$

Define

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

The solution to the system is given by

$$x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}$$

EXAMPLE

Recall the fertilizer example

$$\begin{aligned} x + 2y + 3z &= 18 \\ 3x + y + 2z &= 23 \\ 2x + z &= 13 \end{aligned}$$

The solution is $(5, 2, 3)$

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = -3$$

$$D_x = \begin{vmatrix} 18 & 2 & 3 \\ 23 & 1 & 2 \\ 13 & 0 & 1 \end{vmatrix} = -15 \qquad x = \frac{-15}{-3} = 5$$

$$D_y = \begin{vmatrix} 1 & 18 & 3 \\ 3 & 23 & 2 \\ 2 & 13 & 1 \end{vmatrix} = -6 \qquad y = \frac{-6}{-3} = 2$$

$$D_z = \begin{vmatrix} 1 & 2 & 18 \\ 3 & 1 & 23 \\ 2 & 0 & 13 \end{vmatrix} = -9 \qquad z = \frac{-9}{-3} = 3$$

PYTHON

```python
import numpy as np

D = np.array([[1,2,3],[3,1,2],[2,0,1]])
Dx = np.array([[18,2,3],[23,1,2],[13,0,1]])
Dy = np.array([[1,18,3],[3,23,2],[2,13,1]])
Dz = np.array([[1,2,18],[3,1,23],[2,0,13]])

print('D = ', np.linalg.det(D))
print('D_x = ', np.linalg.det(Dx))
print('D_y = ', np.linalg.det(Dy))
print('D_z = ', np.linalg.det(Dz))
```

```
D =  -3.0000000000000001
D_x =  -15.0
D_y =  -5.9999999999999997
D_z =  -9.0000000000000005
```

QUESTIONS?