MODULE 3

LINEAR INEQUALITIES

LINEAR PROGRAMMING MODELS

MAXIMIZATION MODELS

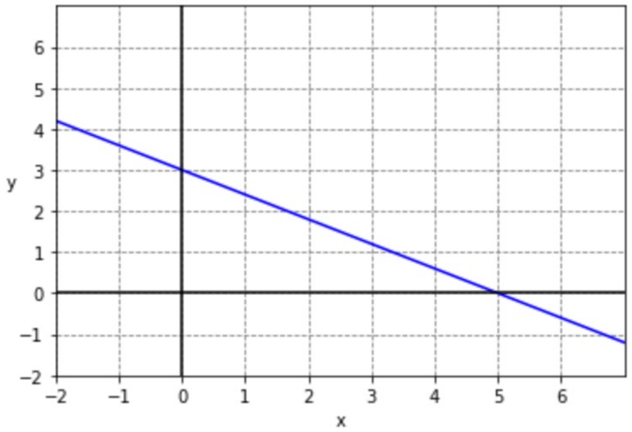
MINIMIZATION MODELS

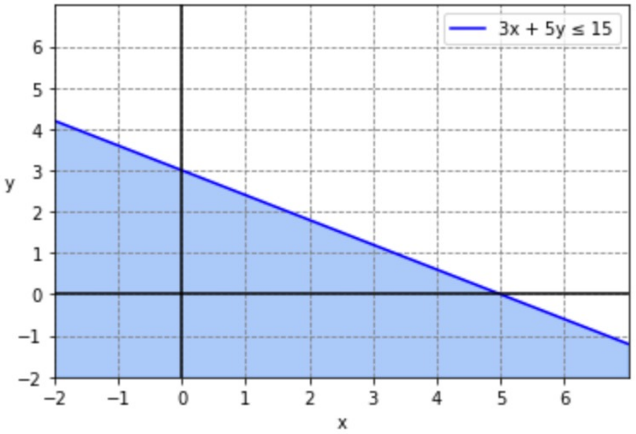
MODULE 3

LINEAR INEQUALITIES

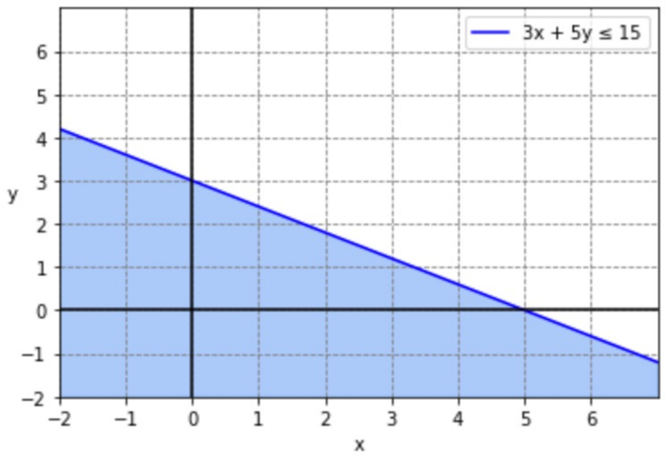
GRAPHING LINEAR INEQUALITIES

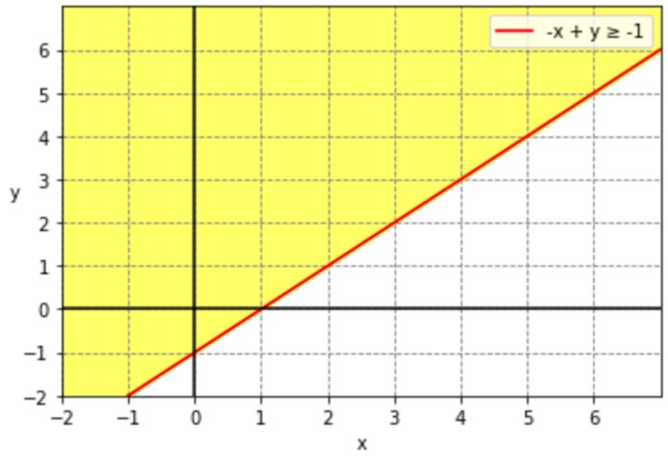
- Change the inequality to an equation
  - Example: Change $3x + 5y \leq 15$ to $3x + 5y = 15$
- Graph the linear equation
- Choose a point on either side of the line and test
  - Choose $(0, 0)$ to test: $3(0) + 5(0) = 0 \leq 15$ ✓
  - Note: If we chose say $(2, 3)$ to test, the inequality fails since $3(2) + 5(3) = 21 \nleq 15$
- If the inequality holds true, shade that side of the line
  - Shade the side of the line that contains $(0, 0)$
- Otherwise, shade the other side of the line



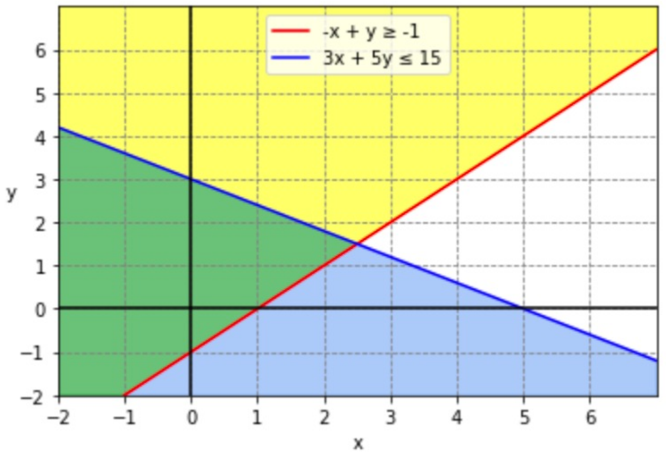


TWO LINEAR INEQUALITIES





Common region in green



MODULE 3

LINEAR PROGRAMMING MODELS

DEFINITIONS

- Decision variables
  - Quantities controlled by the decision maker and represented by mathematical symbols
  - Can take on any set of values
- Objective function
  - Defines the criterion for evaluating the solution
  - Specifies the type of optimization, either maximize or minimize
  - An optimal solution for the model is the best solution as measured by the criterion
- Constraints
  - Set of equations or inequalities that represent restrictions on numerical values assigned to decision variables
  - All possible solutions to the model must satisfy all constraints

MODULE 3

MAXIMIZATION MODELS

STANDARD MAXIMUM FORM

Example:

Maximize $z = 3x_1 + 2x_2 + x_3$
subject to: $2x_1 + x_2 + x_3 \leq 150$
$2x_1 + 2x_2 + 8x_3 \leq 200$
$2x_1 + 3x_2 + x_3 \leq 320$
with $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

Characteristics:

- The objective function is to be maximized.
- All variables are nonnegative ($x_i \geq 0$).
- All remaining constraints are stated in the form
  $a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b$, with $b \geq 0$.

Methods to solve:

- Graphing method
- Simplex method

GRAPHING METHOD

Solve the following LP model graphically:

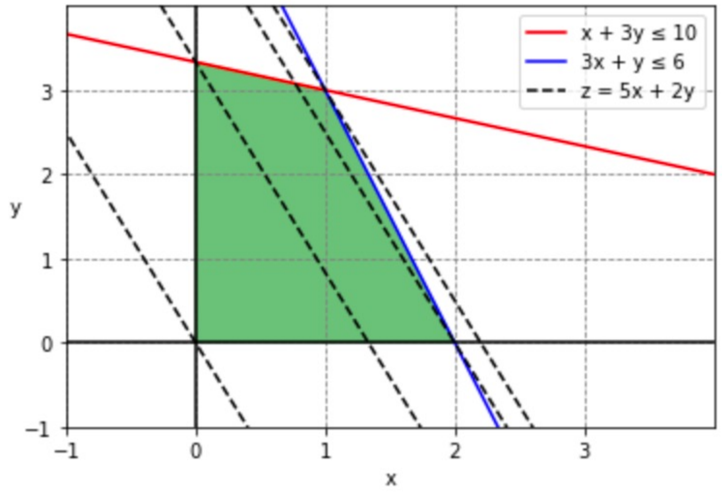
Maximize $z = 5x + 2y$
subject to $x + 3y \leq 10$
$3x + y \leq 6$
with $x \geq 0, y \geq 0$

Step 1: If the LP model is not provided, define variables, state the objective, and write the constraints

Step 2: Convert each constraint to an equation and graph

Step 3: Shade the feasible region

Feasible region is in green



Step 4: Test the corner points:

$(0,0) \rightarrow 5(0) + 2(0) = 0$

$(2,0) \rightarrow 5(2) + 2(0) = 10$

$(1,3) \rightarrow 5(1) + 2(3) = 11$ ← Produces the maximum value

$(0,\frac{10}{3}) \rightarrow 5(0) + 2(\frac{10}{3}) = \frac{20}{3}$

The optimal solution occurs when $x = 1$ and $y = 3$

SIMPLEX METHOD

$$
\begin{array}{ll}
\text{Maximize} & z = 3x_1 + 2x_2 + x_3 \\
\text{subject to:} & 2x_1 + \ \ x_2 + \ \ x_3 \leq 150 \\
 & 2x_1 + 2x_2 + 8x_3 \leq 200 \\
 & 2x_1 + 3x_2 + \ \ x_3 \leq 320 \\
\text{with} & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0
\end{array}
$$

Step 1: If the LP model is not provided, define variables, state the objective, and write all constraints.

SLACK VARIABLES

Step 2: Convert constraints to equations by adding a nonnegative variable to each

For example, $x_1 + 4x_2 \leq 15$ would become $x_1 + 4x_2 + s_1 = 15$, where $s_1 \geq 0$

Restate the following problem by using slack variables:

$$
\begin{array}{ll}
\text{Maximize} & z = 3x_1 + 2x_2 + x_3 \\
\text{subject to:} & 2x_1 + x_2 + x_3 \leq 150 \\
 & 2x_1 + 2x_2 + 8x_3 \leq 200 \\
 & 2x_1 + 3x_2 + x_3 \leq 320 \\
\text{with} & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0
\end{array}
$$

$$
\begin{array}{ll}
\text{Maximize} & z = 3x_1 + 2x_2 + x_3 \\
\text{subject to:} & 2x_1 + x_2 + x_3 + s_1 = 150 \\
 & 2x_1 + 2x_2 + 8x_3 + s_2 = 200 \\
 & 2x_1 + 3x_2 + x_3 + s_3 = 320 \\
\text{with} & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0
\end{array}
$$

INITIAL SIMPLEX TABLEAU

Maximize $z = 3x_1 + 2x_2 + x_3$

subject to: $2x_1 + x_2 + x_3 + s_1 = 150$

$2x_1 + 2x_2 + 8x_3 + s_2 = 200$

$2x_1 + 3x_2 + x_3 + s_3 = 320$

with $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$

Each equation should have all variables on the left

Convert $z = 3x_1 + 2x_2 + x_3$ to:

$-3x_1 - 2x_2 - x_3 + z = 0$

<u>Step 3</u>: Set up the initial tableau

$$\left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 & 150 \\ 2 & 2 & 8 & 0 & 1 & 0 & 0 & 200 \\ 2 & 3 & 1 & 0 & 0 & 1 & 0 & 320 \\ \hline -3 & -2 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

FINDING THE PIVOT

<u>Step 5</u>: Determine the pivot

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \end{array} \\ \left[\begin{array}{ccccccc|c} \boxed{2} & 1 & 1 & 1 & 0 & 0 & 0 & 150 \\ 2 & 2 & 8 & 0 & 1 & 0 & 0 & 200 \\ 2 & 3 & 1 & 0 & 0 & 1 & 0 & 320 \\ \hline -3 & -2 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$150/2 = 75$

$200/2 = 100$

$320/2 = 160$

<u>Step 4</u>: Locate the most negative value in the bottom row

Calculate quotients of farthest right column with corresponding values in the pivot column

Disregard any quotients with 0 or a negative number in the denominator.

If all quotients must be disregarded, no maximum solution exists.

If two quotients are both equal and smallest, choose the pivot in the row nearest the top of the matrix.

The smallest nonnegative quotient gives the location of the pivot.

ROW OPERATIONS

<u>Step 6</u>: Use row operations to change all other numbers in the pivot column to zero.

$$
\begin{array}{c}
\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \end{array}\\
\left[\begin{array}{ccccccc|c}
\boxed{2} & 1 & 1 & 1 & 0 & 0 & 0 & 150\\
2 & 2 & 8 & 0 & 1 & 0 & 0 & 200\\
2 & 3 & 1 & 0 & 0 & 1 & 0 & 320\\
\hline
-3 & -2 & -1 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
$$

$$
\begin{array}{r}
\\
\\
-R_1 + R_2 \rightarrow R_2 \Rightarrow\\
-R_1 + R_3 \rightarrow R_3 \Rightarrow\\
3R_1 + 2R_4 \rightarrow R_4 \Rightarrow
\end{array}
\begin{array}{c}
\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \end{array}\\
\left[\begin{array}{ccccccc|c}
2 & 1 & 1 & 1 & 0 & 0 & 0 & 150\\
0 & \boxed{1} & 7 & -1 & 1 & 0 & 0 & 50\\
0 & 2 & 0 & -1 & 0 & 1 & 0 & 170\\
\hline
0 & -1 & 1 & 3 & 0 & 0 & 2 & 450
\end{array}\right]
\end{array}
$$

Never take a negative multiple of the row being changed

$$
\begin{array}{r}
\\
-R_2 + R_1 \rightarrow R_1 \Rightarrow\\
\\
-2R_2 + R_3 \rightarrow R_3 \Rightarrow\\
R_2 + R_4 \rightarrow R_4 \Rightarrow
\end{array}
\begin{array}{c}
\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \end{array}\\
\left[\begin{array}{ccccccc|c}
2 & 0 & -6 & 2 & -1 & 0 & 0 & 100\\
0 & 1 & 7 & -1 & 1 & 0 & 0 & 50\\
0 & 0 & -14 & 1 & -2 & 1 & 0 & 70\\
\hline
0 & 0 & 8 & 2 & 1 & 0 & 2 & 500
\end{array}\right]
\end{array}
$$

Final tableau

<u>Step 7</u>: If the numbers in the bottom row are all positive or 0, this is the final tableau.

If not determine the next pivot and perform row operations accordingly until a tableau is obtained with no negative values in the bottom row.

READ THE SOLUTION

<u>Step 8</u>: Read the solution from the final tableau

Solution is determined from basic variables

A *basic variable* is a variable whose column has all zeros except for one nonzero entry

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \end{array} \\ \left[\begin{array}{ccccccc|c} \boxed{2} & 0 & -6 & 2 & -1 & 0 & 0 & \boxed{100} \\ 0 & \boxed{1} & 7 & -1 & 1 & 0 & 0 & \boxed{50} \\ 0 & 0 & -14 & 1 & -2 & \boxed{1} & 0 & \boxed{70} \\ \hline 0 & 0 & 8 & 2 & 1 & 0 & \boxed{2} & \boxed{500} \end{array}\right] \end{array}$$

The basic variables are x_1, x_2, and s_3

$$x_1 = 100/2 = 50$$

$$x_2 = 50/1 = 50$$

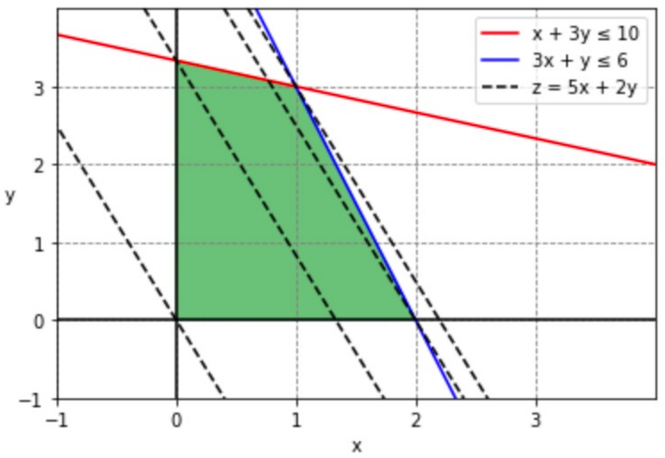
$$s_3 = 70/1 = 70$$

$$z = 500/2 = 250$$

All other variables are zero

COMPARISON

Maximize	$z = 5x + 2y$
subject to	$x + 3y \leq 10$
	$3x + y \leq 6$
with	$x \geq 0, y \geq 0$



Corner points: (0, 0), (2, 0), (1, 3), (0, 10/3)

Optimal value is 11 when x = 1 and y = 3

Method 1

$$\begin{array}{c} \\ \\ \end{array}\left[\begin{array}{ccccc|c} x & y & s_1 & s_2 & z & \\ 1 & 3 & 1 & 0 & 0 & 10 \\ \boxed{3} & 1 & 0 & 1 & 0 & 6 \\ \hline -5 & -2 & 0 & 0 & 1 & 0 \end{array}\right]$$

x and y are not basic
x = 0, y = 0, z = 0

$$\begin{array}{r} -R_2 + 3R_1 \to R_1 \\ \\ 5R_2 + 3R_3 \to R_3 \end{array} \left[\begin{array}{ccccc|c} x & y & s_1 & s_2 & z & \\ 0 & \boxed{8} & 3 & -1 & 0 & 24 \\ 3 & 1 & 0 & 1 & 0 & 6 \\ \hline 0 & -1 & 0 & 5 & 3 & 30 \end{array}\right]$$

x is basic and y is not
x = 6/3 = 2, y = 0, z = 30/3 = 10

$$\begin{array}{r} -R_1 + 8R_2 \to R_2 \\ R_1 + 8R_3 \to R_3 \end{array} \left[\begin{array}{ccccc|c} x & y & s_1 & s_2 & z & \\ 0 & 8 & 3 & -1 & 0 & 24 \\ 24 & 0 & -3 & 9 & 0 & 24 \\ \hline 0 & 0 & 3 & 39 & 24 & 264 \end{array}\right]$$

x and y are both basic
x = 24/24 =1, y = 24/8 = 3, z =264/24 = 11

Method 2

$$\tfrac{1}{3}R_2 \to R_2 \left[\begin{array}{ccccc|c} x & y & s_1 & s_2 & z & \\ 1 & 3 & 1 & 0 & 0 & 10 \\ \boxed{1} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 2 \\ \hline -5 & -2 & 0 & 0 & 1 & 0 \end{array}\right]$$

x and y are not basic
x = 0, y = 0, z = 0

$$\begin{array}{r} -R_2 + R_1 \to R_1 \\ \\ 5R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccccc|c} x & y & s_1 & s_2 & z & \\ 0 & \boxed{\frac{8}{3}} & 1 & -\frac{1}{3} & 0 & 8 \\ 1 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 2 \\ \hline 0 & -\frac{1}{3} & 0 & \frac{5}{3} & 1 & 10 \end{array}\right]$$

x is basic and y is not
x = 2, y = 0, z = 10

$$\tfrac{3}{8}R_1 \to R_1 \left[\begin{array}{ccccc|c} x & y & s_1 & s_2 & z & \\ 0 & \boxed{1} & \frac{3}{8} & -\frac{1}{8} & 0 & 3 \\ 1 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 2 \\ \hline 0 & -\frac{1}{3} & 0 & \frac{5}{3} & 1 & 10 \end{array}\right]$$

$$\begin{array}{r} -\frac{1}{3}R_1 + R_2 \to R_2 \\ \frac{1}{3}R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccccc|c} x & y & s_1 & s_2 & z & \\ 0 & 1 & \frac{3}{8} & -\frac{1}{8} & 0 & 3 \\ 1 & 0 & -\frac{1}{8} & \frac{3}{8} & 0 & 1 \\ \hline 0 & 0 & \frac{1}{8} & \frac{39}{24} & 1 & 11 \end{array}\right]$$

Both x and y are basic
x = 1, y = 3, z = 11

PYTHON

```python
from pulp import LpVariable, LpProblem, LpMaximize, LpStatus, value, LpMinimize

# declare your variables
x = LpVariable("x", 0, None) # x>=0
y = LpVariable("y", 0, None) # y>=0

# defines the problem
prob = LpProblem("problem", LpMaximize)

# defines the constraints
prob += x + 3*y <= 10
prob += 3*x + y <= 6

# defines the objective function to maximize
prob += 5*x + 2*y

# solve the problem
status = prob.solve()
LpStatus[status]

# print the results
print(value(x))
print(value(y))
print(5*value(x) + 2*value(y))
```

```
1.0
3.0
11.0
```

A company manufactures three types of lamps labeled A, B, and C. Each lamp is processed in two departments, I and II. Total available work-hours per day for departments I and II are 400 and 600, respectively. No additional labor is available. Time requirements and profit per unit for each lamp are shown in the table.

	A	B	C
Work-hours in I	2	3	1
Work-hours in II	4	2	3
Profit per unit	\$5	\$4	\$3

Determine the numbers of the three types of lamps that should be produced to maximize profits and the maximum profit

Let A = # of lamps of type A to produce, B = # of lamps of type B to produce, and C = # of lamps of type C to produce

Maximize $z = 5A + 4B + 3C$
subject to: $2A + 3B + C \leq 400$
$4A + 2B + 3C \leq 600$
with $A, B, C \geq 0$

Add slack variables:

Maximize $z = 5A + 4B + 3C$
subject to: $2A + 3B + C + s_1 = 400$
$4A + 2B + 3C + s_2 = 600$

Convert the objective function:

$-5A - 4B - 3C + z = 0$

Your turn: Solve the problem

Set up initial tableau

$$\left[\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 2 & 3 & 1 & 1 & 0 & 0 & 400 \\ 4 & 2 & 3 & 0 & 1 & 0 & 600 \\ \hline -5 & -4 & -3 & 0 & 0 & 1 & 0 \end{array}\right]$$

Determine pivot

$$\left[\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 2 & 3 & 1 & 1 & 0 & 0 & 400 \\ \boxed{4} & 2 & 3 & 0 & 1 & 0 & 600 \\ \hline -5 & -4 & -3 & 0 & 0 & 1 & 0 \end{array}\right]$$

Answer: The company should produce 125 lamps of type A, 50 lamps of type B and no lamps of type C for a maximum profit of \$825.

PYTHON

Maximize	$z = 5A + 4B + 3C$
subject to:	$2A + 3B + \quad C \leq 400$
	$4A + 2B + 3C \leq 600$
with	$A, B, C \geq 0$

```python
from pulp import LpVariable, LpProblem, LpMaximize, LpStatus, value, LpMinimize

# declare your variables
A = LpVariable("A", 0, None) # A>=0
B = LpVariable("B", 0, None) # B>=0
C = LpVariable("C", 0, None) # C>=0

# defines the problem
prob = LpProblem("problem", LpMaximize)

# defines the constraints
prob += 2*A + 3*B + C <= 400
prob += 4*A + 2*B + 3*C <= 600

# defines the objective function to maximize
prob += 5*A + 4*B+ 3*C

# solve the problem
status = prob.solve()
LpStatus[status]

# print the results
print(value(A))
print(value(B))
print(value(C))
print(5*value(A) + 4*value(B) + 3*value(C))
```

```
125.0
50.0
0.0
825.0
```

MODULE 3

MINIMIZATION MODELS

STANDARD MINIMUM FORM

Example:

Minimize $w = 8y_1 + 16y_2$
subject to: $y_1 + 5y_2 \geq 9$
$2y_1 + 2y_2 \geq 10$
with $y_1 \geq 0, y_2 \geq 0$

Characteristics:

- The objective function is to be minimized.
- All variables are nonnegative ($x_i \geq 0$).
- All remaining constraints are stated in the form $a_1y_1 + a_2y_2 + \cdots + a_ny_n \geq b$, with $b \geq 0$.

Methods to solve:

- Graphing method
- Solve the dual

DUALITY

Minimize	$w = 8y_1 + 16y_2$
subject to:	$y_1 + 5y_2 \geq 9$
	$2y_1 + 2y_2 \geq 10$
with	$y_1 \geq 0, y_2 \geq 0$

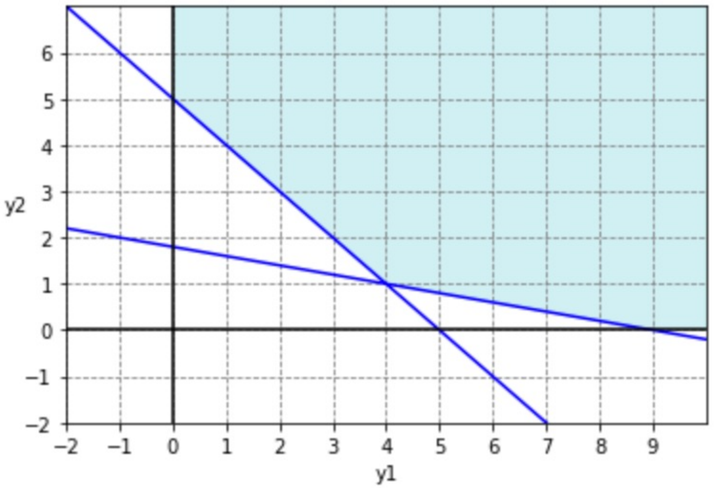
$$\left[\begin{array}{cc|c} 1 & 5 & 9 \\ 2 & 2 & 10 \\ \hline 8 & 16 & 0 \end{array}\right]$$

Transpose of a matrix:

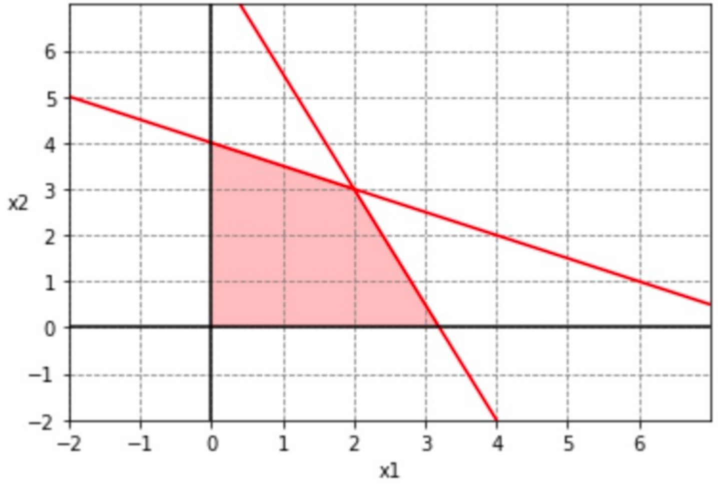
*Exchange rows and columns
*An *m* x *n* matrix becomes an *n* x *m* matrix

Maximize	$z = 9x_1 + 10x_2$
subject to:	$x_1 + 2x_2 \leq 8$
	$5x_1 + 2x_2 \leq 16$
with	$x_1 \geq 0, x_2 \geq 0$

$$\left[\begin{array}{cc|c} 1 & 2 & 8 \\ 5 & 2 & 16 \\ \hline 9 & 10 & 0 \end{array}\right]$$



Minimum
occurs at (4, 1)
with w = 48



Maximum
occurs at (2, 3)
with z = 48

DUALITY

Minimize $w = 8y_1 + 16y_2$
subject to: $y_1 + 5y_2 \geq 9$
$2y_1 + 2y_2 \geq 10$
with $y_1 \geq 0, y_2 \geq 0$

Minimum occurs at (4, 1) with w = 48

Maximize $z = 9x_1 + 10x_2$
subject to: $x_1 + 2x_2 \leq 8$
$5x_1 + 2x_2 \leq 16$
with $x_1 \geq 0, x_2 \geq 0$

Maximum occurs at (2, 3) with z = 48

Maximization problem

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & \boxed{2} & 1 & 0 & 0 & 8 \\ 5 & 2 & 0 & 1 & 0 & 16 \\ \hline -9 & -10 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$-R_1 + R_2 \to R_2$

$5R_1 + R_3 \to R_3$

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & 8 \\ \boxed{4} & 0 & -1 & 1 & 0 & 8 \\ \hline -4 & 0 & 5 & 0 & 1 & 40 \end{array}\right] \end{array}$$

$-R_2 + 4R_1 \to R_1$

$R_2 + R_3 \to R_3$

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 8 & 5 & -1 & 0 & 24 \\ 4 & 0 & -1 & 1 & 0 & 8 \\ \hline 0 & 0 & \boxed{4} & \boxed{1} & 1 & \boxed{48} \end{array}\right] \end{array}$$

For the maximization problem:

$x_1 = 8/4 = 2$

$x_2 = 24/8 = 3$

$z = 48$

However, minimization solution is found in the bottom row:

$s_1 = 4, s_2 = 1, \mathrm{w} = 48$

Caution: If the coefficient of z is not 1, divide the bottom row by the coefficient of z first.

SOLVING THE DUAL

- Find the dual standard maximization problem.
- Solve the maximization problem using the Simplex method.
- The minimum value of the objective function w is the maximum value of the objective function z.
- The optimum solution to the minimization problem is given by the entries in the bottom row of the columns corresponding to the slack variables, so long as the entry in the z column is equal to 1.

An animal food must provide at least 54 units of vitamins and 60 calories per serving. One gram of soybean meal provides 2.5 units of vitamins and 5 calories. One gram of meat byproducts provides 4.5 units of vitamins and 3 calories. One gram of grain provides 5 units of vitamins and 10 calories. A gram of soybean meal costs $0.08, a gram of meat byproducts $0.09, and a gram of grain $0.10.

What mixture of these three ingredients will provide the required vitamins and calories at a minimum cost and what is the minimum cost?

Let y_1 = grams of soybean meal, y_2 = grams of meat byproducts, and y_3 = grams of grain.

Minimize $w = 8y_1 + 9y_2 + 10y_3$

subject to: $2.5y_1 + 4.5y_2 + 5y_3 \geq 54$

$5y_1 + 3y_2 + 10y_3 \geq 60$

with $y_1, y_2, y_3 \geq 0$

Write the augmented matrix

$$\left[\begin{array}{ccc|c} 2.5 & 4.5 & 5 & 54 \\ 5 & 3 & 10 & 60 \\ \hline 8 & 9 & 10 & 0 \end{array}\right]$$

Write the transpose matrix

$$\left[\begin{array}{cc|c} 2.5 & 5 & 8 \\ 4.5 & 3 & 9 \\ 5 & 10 & 10 \\ \hline 54 & 60 & 0 \end{array}\right]$$

Write the initial tableau

$$\left[\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 2.5 & 5 & 1 & 0 & 0 & 0 & 8 \\ 4.5 & 3 & 0 & 1 & 0 & 0 & 9 \\ 5 & 10 & 0 & 0 & 1 & 0 & 10 \\ \hline -54 & -60 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Multiply the first two rows by 2 to eliminate decimals

$$\left[\begin{array}{cccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & \\ 5 & 10 & 2 & 0 & 0 & 0 & 16 \\ 9 & 6 & 0 & 2 & 0 & 0 & 18 \\ 5 & 10 & 0 & 0 & 1 & 0 & 10 \\ \hline -54 & -60 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Determine the pivot

$$
\begin{array}{cccccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & z & \\
5 & 10 & 2 & 0 & 0 & 0 & 16 \\
9 & 6 & 0 & 2 & 0 & 0 & 18 \\
5 & \boxed{10} & 0 & 0 & 1 & 0 & 10 \\
\hline
-54 & -60 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

$-R_3 + R_1 \to R_1$

$-3R_3 + 5R_2 \to R_2$

$6R_3 + R_4 \to R_4$

$$
\begin{array}{cccccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & z & \\
0 & 0 & 2 & 0 & -1 & 0 & 6 \\
\boxed{30} & 0 & 0 & 10 & -3 & 0 & 60 \\
\boxed{5} & 10 & 0 & 0 & 1 & 0 & 10 \\
\hline
-24 & 0 & 0 & 0 & 6 & 1 & 60
\end{array}
$$

Pivoting on the 5 in row 3 will produce a different result

$-R_2 + 6R_3 \to R_3$

$4R_2 + 5R_4 \to R_4$

$$
\begin{array}{cccccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & z & \\
0 & 0 & 2 & 0 & -1 & 0 & 6 \\
30 & 0 & 0 & 10 & -3 & 0 & 60 \\
0 & 60 & 0 & -10 & 9 & 0 & 0 \\
\hline
0 & 0 & 0 & 40 & 18 & 5 & 540
\end{array}
$$

$\frac{1}{5}R_4 \to R_4$

$$
\begin{array}{cccccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & z & \\
0 & 0 & 2 & 0 & -1 & 0 & 6 \\
30 & 0 & 0 & 10 & -3 & 0 & 60 \\
0 & 60 & 0 & -10 & 9 & 0 & 0 \\
\hline
0 & 0 & 0 & 8 & 3.6 & 1 & 108
\end{array}
$$

The optimal mixture is:
0 grams of soybean meal
8 grams of meat byproducts
3.6 grams of of grain
Minimum cost of $1.08.

A second optimal solution produces the same minimum cost.

Your Turn: Find the other optimal basic solution by pivoting on the third row

Answer:

0 grams of soybean meal
0 grams of meat byproducts
10.8 grams of grain
Minimum cost of $1.08

PYTHON

Minimize	$w = 8y_1 + 9y_2 + 10y_3$
subject to:	$2.5y_1 + 4.5y_2 + 5y_3 \geq 54$
	$5y_1 + 3y_2 + 10y_3 \geq 60$
with	$y_1, y_2, y_3 \geq 0$

```
from pulp import LpVariable, LpProblem, LpMaximize, LpStatus, value, LpMinimize

# declare your variables
y1 = LpVariable("y1", 0, None) # y1>=0
y2 = LpVariable("y2", 0, None) # y2>=0
y3 = LpVariable("y3", 0, None) # y3>=0

# defines the problem
prob = LpProblem("problem", LpMinimize)

# defines the constraints
prob += 2.5*y1 + 4.5*y2 + 5*y3 >= 54
prob += 5*y1 + 3*y2 + 10*y3 >= 60

# defines the objective function to minimize
prob += 8*y1 + 9*y2+ 10*y3

# solve the problem
status = prob.solve()
LpStatus[status]

# print the results
print(value(y1))
print(value(y2))
print(value(y3))
print(0.08*value(y1) + 0.09*value(y2) + 0.1*value(y3))
```

```
0.0
8.0
3.6
1.08
```

MODULE 3

NONSTANDARD MODELS

CHARACTERISTICS

- Can be maximization or minimization
- Mixed constraints ≤, ≥, and =
- Methods to solve:
  - Graphing method
  - Simplex method
  - Solve the dual in the case of minimization

NONSTANDARD ALGORITHM

- If necessary, convert the problem to a maximization problem. (Rewrite the objective function w as $z = -w$)
- Add slack variables and subtract surplus variables as needed. Equations have no slack/surplus variable.
- Write the initial simplex tableau.
- If any basic variable has a negative value, note what row it is in.
- In the row located in the previous step, find the positive entry that is farthest to the left, and note what column it is in.
- In the column found in the previous step, choose a pivot by investigating quotients.
- Use row operations to change the other numbers in the pivot column to 0.
- Continue the previous 4 steps until all basic variables are nonnegative. If impossible to continue, then the problem has no feasible solution.
- Once a feasible solution has been found, continue using the Simplex method until the optimal solution is found.

EXAMPLE OF NONSTANDARD

Minimize $w = 3y_1 + 2y_2$
subject to: $y_1 + 3y_2 \leq 6$
$2y_1 + y_2 \geq 3$
with $y_1, y_2 \geq 0$

Change to maximization by letting $z = -w$

Maximize $z = -w = -3y_1 - 2y_2$
subject to: $y_1 + 3y_2 \leq 6$
$2y_1 + y_2 \geq 3$
with $y_1, y_2 \geq 0$

Add slack variables and subtract surplus variables

$$\begin{aligned} y_1 + 3y_2 + s_1 \quad &= 6 \\ 2y_1 + y_2 \quad - s_2 \quad &= 3 \\ 3y_1 + 2y_2 \quad + z &= 0 \end{aligned}$$

Set up the initial tableau

$$\begin{array}{cccccc} y_1 & y_2 & s_1 & s_2 & z & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 3 & 1 & 0 & 0 & 6 \\ \boxed{2} & 1 & 0 & -1 & 0 & 3 \\ \hline 3 & 2 & 0 & 0 & 1 & 0 \end{array}\right]$$

The basic variables are s_1 and s_2. Since s_2 is negative and in row 2, we go to the positive value farthest to the left in row 2 to determine the pivot

$-R_2 + 2R_1 \rightarrow R_1$

$-3R_2 + 2R_3 \rightarrow R_3$

$$\begin{array}{cccccc} y_1 & y_2 & s_1 & s_2 & z & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 0 & 5 & 2 & 1 & 0 & 9 \\ 2 & 1 & 0 & -1 & 0 & 3 \\ \hline 0 & 1 & 0 & 3 & 2 & -9 \end{array}\right]$$

Now s_2 is not a basic variable, i.e. $s_2 = 0$

All basic variables are nonnegative, so the solution is optimal.

The solution is $y_1 = 3/2$, $y_2 = 0$, and $w = -(-9/2) = 9/2$

PYTHON

Minimize	$w = 3y_1 + 2y_2$
subject to:	$y_1 + 3y_2 \leq 6$
	$2y_1 + y_2 \geq 3$
with	$y_1, y_2 \geq 0$

```python
from pulp import LpVariable, LpProblem, LpMaximize, LpStatus, value, LpMinimize

# declare your variables
y1 = LpVariable("y1", 0, None) # y1>=0
y2 = LpVariable("y2", 0, None) # y2>=0

# defines the problem
prob = LpProblem("problem", LpMinimize)

# defines the constraints
prob += y1 + 3*y2 <= 6
prob += 2*y1 + y2 >= 3

# defines the objective function to minimize
prob += 3*y1 + 2*y2

# solve the problem
status = prob.solve()
LpStatus[status]

# print the results
print(value(y1))
print(value(y2))
print(3*value(y1) + 2*value(y2))
```

```
1.5
0.0
4.5
```

YOUR TURN

Minimize $w = 6y_1 + 4y_2$
subject to: $3y_1 + 4y_2 \geq 10$
$9y_1 + 7y_2 \leq 18$
with $y_1, y_2 \geq 0$

Initial tableau

$$\begin{array}{c} \begin{array}{ccccc|c} y_1 & y_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 3 & 4 & -1 & 0 & 0 & 10 \\ 9 & 7 & 0 & 1 & 0 & 18 \\ \hline 6 & 4 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Answer: $y_1 = 0, y_2 = 5/2$, and $w = 10$

QUESTIONS?