MODULE 5

LIMITS

CONTINUITY

RATES OF CHANGE

DERIVATIVES

DIFFERENTIATION RULES

MODULE 5

LIMITS

MOTIVATION

What happens to $f(x) = x^2$ when x is a number *very close* to (but not equal to) 2?

x approaches 2 from the left → | ← x approaches 2 from the right

$\boldsymbol{x}$	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1
$\boldsymbol{f(x)}$	3.61	3.9601	3.96001	3.99960001	4	4.00040001	4.004001	4.0401	4.41

$f(x)$ approaches 4 → | ← $f(x)$ approaches 4

The limit of $f(x)$ as x approaches 2 from the left is written

$$\lim_{x \to 2^-} f(x) = 4$$

The limit of $f(x)$ as x approaches 2 is 4

$$\lim_{x \to 2} f(x) = 4$$

The limit of $f(x)$ as x approaches 2 from the right is written

$$\lim_{x \to 2^+} f(x) = 4$$

A two-sided limit such as this
exists only if both one-sided limits exist and are equal

LIMIT OF A FUNCTION

Let f be a function and let a and L be real numbers. If

1. as x takes values closer and closer (but not equal) to a on both sides of a, the corresponding values of $f(x)$ get closer and closer (and perhaps equal) to L; and

2. the value of $f(x)$ can be made as close to L as desired by taking values of x close enough to a;

then L is the limit of $f(x)$ as x approaches a, written

$$\lim_{x \to a} f(x) = L$$

The definition of a limit describes what happens to $f(x)$ when x is near, but not equal to a.

The definition of a limit is not affected by how (or even whether) $f(a)$ is defined.

The definition of a limit implies that the function values cannot approach two different numbers, so that if a limit exists, it is unique.

Find $\lim_{x \to 2} g(x)$, where

$$g(x) = \frac{x^3 - 2x^2}{x - 2}$$

Note: the function $g(x)$ is undefined when $x = 2$

Method 1

x	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1
$f(x)$	3.61	3.9601	3.96001	3.99960001	undefined	4.00040001	4.004001	4.0401	4.41

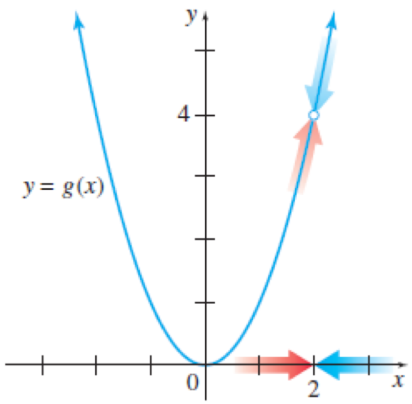
Table suggests

$$\lim_{x \to 2} g(x) = 4$$

Method 2

$$g(x) = \frac{x^3 - 2x^2}{x - 2} = \frac{x^2(x-2)}{x - 2} = x^2$$

provided $x \neq 2$



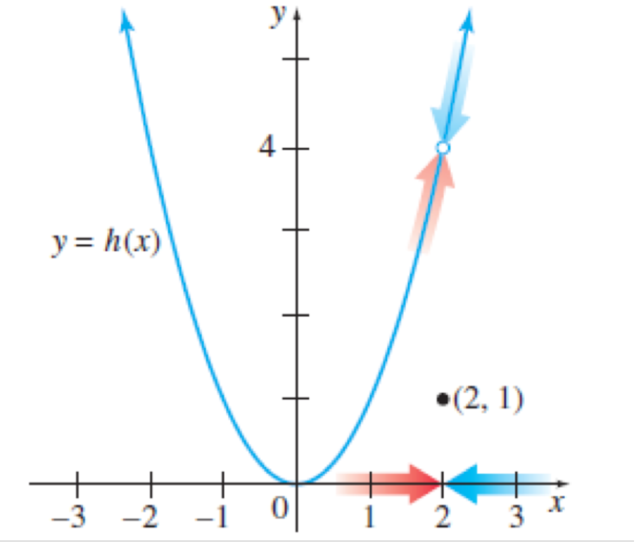
Look at values of x close to but not equal to 2

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} x^2 = 4$$

EXAMPLE

Determine $\lim_{x \to 2} h(x)$ for the function h defined by

$$h(x) = \begin{cases} x^2, & if\ x \neq 2 \\ 1, & if\ x = 2 \end{cases}$$



A function defined by two or more cases is called a *piecewise function*

$h(2) = 1$, but $h(x) = x^2$ when $x \neq 2$

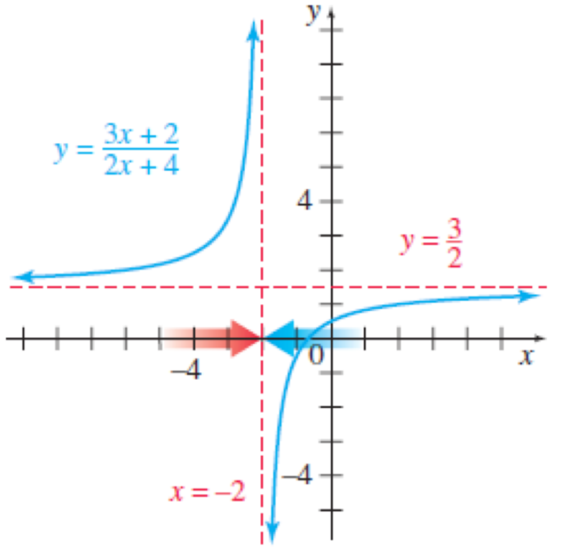
We only care about values of $h(x)$ when x is close to 2, but not equal to 2

$$\lim_{x \to 2} h(x) = \lim_{x \to 2} x^2 = 4$$

EXAMPLE

Find $\lim_{x \to -2} f(x)$, where

$$f(x) = \frac{3x+2}{2x+4}$$



x approaches −2 from left

x approaches −2 from right

x	−2.1	−2.01	−2.001	−2.0001	−1.9999	−1.999	−1.99	−1.9
$f(x)$	21.5	201.5	2001.5	20,001.5	−19,998.5	−1998.5	−198.5	−18.5

As x approaches -2 from the left, $f(x)$ becomes very large without bound

This is because as x approaches -2, the denominator approaches 0 and the numerator approaches -4

We write

$$\lim_{x \to -2^-} f(x) = \infty$$

Similarly,

$$\lim_{x \to -2^+} f(x) = -\infty$$

$$\lim_{x \to 2} \frac{3x+2}{2x+4} \quad \text{does not exist}$$

EXISTENCE OF LIMITS

The limit of f as x approaches a may not exist

If $f(x)$ becomes infinitely large in magnitude (positive or negative) as x approaches a from either side, we write

$$\lim_{x \to a} f(x) = \infty$$

or

$$\lim_{x \to a} f(x) = -\infty$$

Note: This is simply a description of the behavior of the function near $x = a$. This does not mean the limit exists.

If $f(x)$ becomes infinitely large in magnitude (positive) as x approaches a from one side and infinitely large in magnitude (negative) as x approaches a from the other side, then

$$\lim_{x \to a} f(x) \text{ does not exist}$$

If $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = M$, and $L \neq M$, then $\lim_{x \to a} f(x)$ does not exist

RULES FOR LIMITS

Let $a, A,$ and B be real numbers, and let f and g be functions such that

$$\lim_{x\to a} f(x) = A \quad \text{and} \quad \lim_{x\to a} g(x) = B$$

If k is a constant, then

$$\lim_{x\to a} k = k \quad \text{and} \quad \lim_{x\to a}[kf(x)] = k\lim_{x\to a} f(x) = kA$$

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = A \pm B$$

$$\lim_{x\to a}[f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right] = AB$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{A}{B} \quad \text{if } B \neq 0$$

If you need a review of exponential and logarithmic functions, read *Section 1.5* from *Calculus Volume I*

If $p(x)$ is a polynomial, then

$$\lim_{x\to a} p(x) = p(a)$$

For any real number k,

$$\lim_{x\to a}[f(x)]^k = \left[\lim_{x\to a} f(x)\right]^k = A^k$$

For example, this limit does not exist when $A < 0$ and $k = 1/2$ or when $A = 0$ and $k \leq 0$

provided this limit exists

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) \quad \text{if} \quad f(x) = g(x) \quad \text{for all} \quad x \neq a$$

For any real number $b > 0$,

$$\lim_{x\to a}\left[b^{f(x)}\right] = b^{\left[\lim_{x\to a} f(x)\right]} = b^A$$

For any real number b such that $0 < b < 1$ or $b > 1$,

$$\lim_{x\to a}[\log_b f(x)] = \log_b\left[\lim_{x\to a} f(x)\right] = \log_b A \quad \text{if} \quad A > 0$$

TECHNIQUES

Find

$$\lim_{x\to 3}\frac{x^2-x-1}{\sqrt{x+1}}$$

$$\lim_{x\to 3}\frac{x^2-x-1}{\sqrt{x+1}}=\frac{\lim_{x\to 3}(x^2-x-1)}{\lim_{x\to 3}\sqrt{x+1}}$$

$$=\frac{\lim_{x\to 3}(x^2-x-1)}{\sqrt{\lim_{x\to 3}(x+1)}}$$

$$=\frac{3^2-3-1}{\sqrt{3+1}}=\frac{5}{\sqrt{4}}=\frac{5}{2}$$

Find

$$\lim_{x\to 2}\frac{x^2+x-6}{x-2}$$

$$\lim_{x\to 2}\frac{x^2+x-6}{x-2}=\lim_{x\to 2}\frac{(x+3)(x-2)}{x-2}$$

$$=\lim_{x\to 2}(x+3)=2+3=5$$

Your turn: Simplify

$$\left(\frac{\sqrt{x}-2}{x-4}\right)\left(\frac{\sqrt{x}+2}{\sqrt{x}+2}\right)$$

then find the limit as x approaches 4

Find

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$$

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}=\lim_{x\to 4}\left(\frac{\sqrt{x}-2}{(\sqrt{x})^2-2^2}\right)$$

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{(\sqrt{x}+2)(\sqrt{x}-2)}=\lim_{x\to 4}\frac{1}{\sqrt{x}+2}$$

$$=\frac{1}{\sqrt{4}+2}=\frac{1}{2+2}=\frac{1}{4}$$

Alternatively, rationalize the numerator

LIMITS AT INFINITY

For any positive number n,

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

If x is negative, x^n doesn't exist for certain values of n, so the 2nd limit is undefined for those values of n

<u>Finding limits at infinity</u>

If $f(x) = p(x)/q(x)$, for polynomials $p(x)$ and $q(x)$ with $q(x) \neq 0$, then

$$\lim_{x\to\infty} f(x) \quad and \quad \lim_{x\to-\infty} f(x)$$

can be found as follows:

Step 1: Divide $p(x)$ and $q(x)$ by the highest power of x in $q(x)$

Step 2: Use the rules of limits including the rule above to find the limit of the result from Step 1

FIND EACH LIMIT

$$\lim_{x\to\infty}\frac{8x+6}{3x-1}$$

$$\lim_{x\to\infty}\frac{8x+6}{3x-1}=\lim_{x\to\infty}\frac{8+\frac{6}{x}}{3-\frac{1}{x}}$$

$$=\lim_{x\to\infty}\frac{8+6\left(\frac{1}{x}\right)}{3-\frac{1}{x}}=\frac{8+0}{3-0}=\frac{8}{3}$$

$$\lim_{x\to\infty}\frac{3x+2}{4x^3-1}$$

$$\lim_{x\to\infty}\frac{3x+2}{4x^3-1}=\lim_{x\to\infty}\frac{\frac{3}{x^2}+\frac{2}{x^3}}{4-\frac{1}{x^3}}$$

$$=\frac{0+0}{4-0}=0$$

$$\lim_{x\to\infty}\frac{3x^2+2}{4x-3}$$

$$\lim_{x\to\infty}\frac{3x^2+2}{4x-3}=\lim_{x\to\infty}\frac{3x+\frac{2}{x}}{4-\frac{3}{x}}$$

$$=\lim_{x\to\infty}\frac{3x}{4}=\infty$$

$$\lim_{x\to\infty}\frac{5x^2-4x^3}{3x^2+2x-1}$$

$$\lim_{x\to\infty}\frac{5x^2-4x^3}{3x^2+2x-1}=\lim_{x\to\infty}\frac{5-4x}{3+\frac{2}{x}-\frac{1}{x^2}}$$

$$=\lim_{x\to\infty}\frac{5-4x}{3}=-\infty$$

MODULE 5

CONTINUITY

CONTINUITY AT A SINGLE VALUE

A function f is continuous at $x = c$ if the following three conditions are satisfied

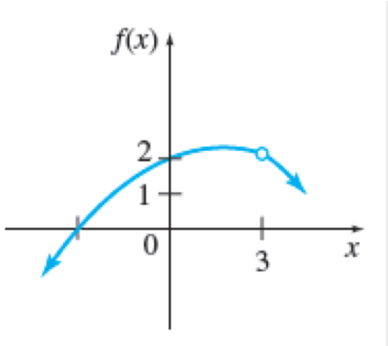
1. $f(c)$ is defined

2. $\lim_{x \to c} f(x)$ exists, and

3. $\lim_{x \to c} f(x) = f(c)$

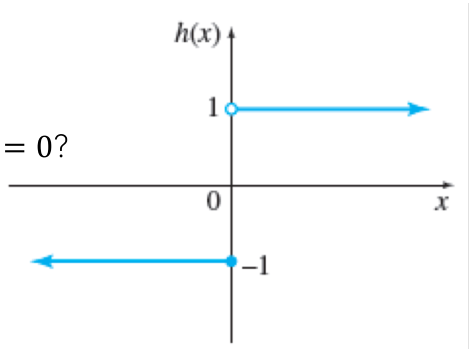
If f is not continuous at c, then f is discontinuous there

Is $f(x)$ continuous at $x = 3$?



No: $f(x)$ does not exist at $x = 3$

Is $h(x)$ continuous at $x = 0$?

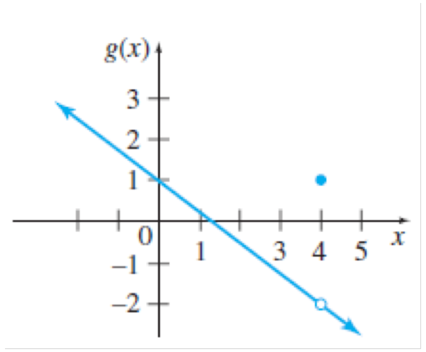


$h(x)$ exists at $x = 0$ and is equal to -1

However, the limit as x approaches -1 does not exist

MORE EXAMPLES

Is $g(x)$ continuous at $x = 4$?



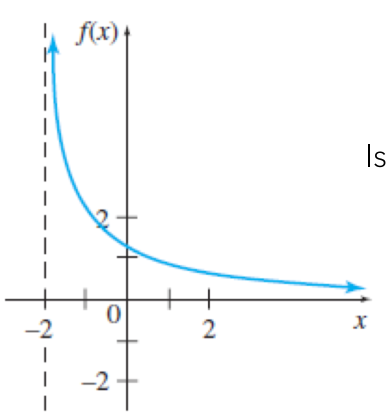
$g(x)$ is defined at $x = 4$ and equals 1

The limit exists at $x = 4$ and

$$\lim_{x \to 4} g(x) = -2$$

However,

$$g(4) \neq \lim_{x \to 4} g(x)$$



Is $f(x)$ continuous at $x = -2$

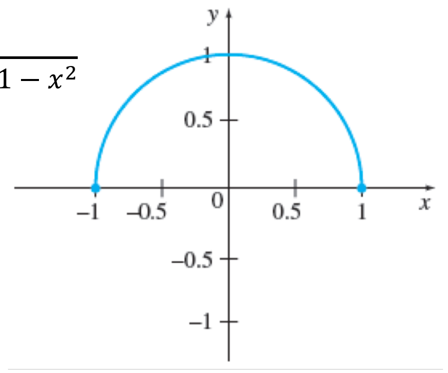
No: The function is not defined at $x = -2$

CONTINUITY ON AN INTERVAL

A function is continuous on a closed interval $[a, b]$ if

1. it is continuous on the open interval (a, b),
2. it is continuous from the right at $x = a$, and
3. it is continuous from the left at $x = b$

$f(x) = \sqrt{1 - x^2}$



f is continuous on the closed interval $[-1,1]$

We do not need to worry about the fact that $\sqrt{1 - x^2}$ does not exist to the left of $x = -1$ or to the right of $x = 1$

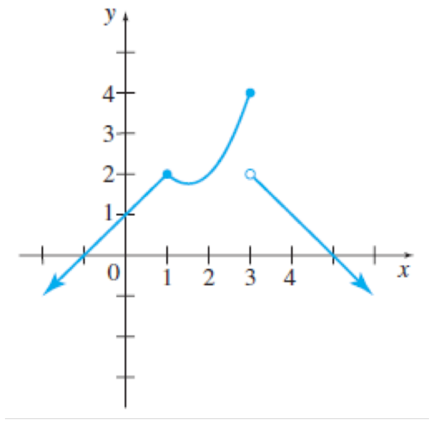
CONTINUOUS FUNCTIONS

- Polynomials – continuous for all real numbers
- Rational functions – continuous for all real numbers where the denominator is ≠ 0
- Square root functions ($y = \sqrt{ax + b}$) – continuous for all x where ax + b ≥ 0
- Exponential functions ($y = a^x, a > 0$) – continuous for all real numbers
- Logarithmic functions ($\mathrm{y} = \log_a x$) – continuous for all x > 0

EXAMPLE

Find all values of x where the following function is discontinuous

$$f(x) = \begin{cases} x+1 & \text{if } x < 1 \\ x^2 - 3x + 4 & \text{if } 1 \leq x \leq 3 \\ 5 - x & \text{if } x > 3 \end{cases}$$



Each piece is a polynomial, so the only possible points of discontinuity occur at $x = 1$ and $x = 3$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+1) = 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 - 3x + 4) = 2$$

Furthermore, $f(1) = 1^2 - 3 + 4 = 2$, so

$$\lim_{x \to 1} f(x) = 2$$

Therefore, f is continuous at $x = 1$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 - 3x + 4) = 4$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (5 - x) = 2$$

The one-sided limits exist, but are not equal so the two-sided limit at $x = 3$ does not exist

Therefore, f is discontinuous at $x = 3$

MODULE 5

RATES OF CHANGE

AVERAGE RATE OF CHANGE

The average rate of change of $f(x)$ with respect to x as x changes from a to b is

$$\frac{f(b) - f(a)}{b - a}$$

Based on population projections for 2000 to 2050, the projected Hispanic population (in millions) for a certain country can be modeled by the exponential function

$$H(t) = 37.791(1.021)^t$$

where $t = 0$ corresponds to 2000 and $0 \le t \le 50$. Use H to estimate the average rate of change in the Hispanic population from 2000 to 2010.

The years 2000 and 2010 correspond to $t = 0$ and $t = 10$, respectively

Tip: Use technology

```
(37.791*1.021**10-37.791*1.021**0)/10
0.8729653294860398
```

$$\frac{H(10) - H(0)}{10 - 0} = \frac{37.791(1.021)^{10} - 37.791(1.021)^0}{10}$$

$$\approx \frac{8.73}{10} = 0.873$$

Never round until the last step

Based on this model, the Hispanic population increased at an average rate of approximately 873,000 people per year between 2000 and 2010

INSTANTANEOUS RATE OF CHANGE

Suppose a car is stopped at a traffic light. When the light turns green, the car begins to move along a straight road. Assume that the distance traveled by the car is given by $s(t) = 3t^2$, for $0 \leq t \leq 15$ where t is time in seconds and $s(t)$ is distance traveled in feet.

How do we find the exact velocity of the car at say, $t = 10$?

Interval	Average velocity
$t = 10$ to $t = 10.1$	$\frac{s(10.1) - s(10)}{10.1 - 10} = \frac{306.03 - 300}{0.1} = 60.3$
$t = 10$ to $t = 10.01$	$\frac{s(10.01) - s(10)}{10.01 - 10} = \frac{300.6003 - 300}{0.01} = 60.03$
$t = 10$ to $t = 10.001$	$\frac{s(10.001) - s(10)}{10.001 - 10} = \frac{300.060003 - 300}{0.001} = 60.003$

Table suggests that the velocity at $t = 10$ is 60 ft/sec.

Consider the following where h is small but not 0

$$\frac{s(10+h) - s(10)}{(10+h) - 10} = \frac{s(10+h) - s(10)}{h}$$

Velocity represents both how fast something is moving and its direction, so *velocity can be negative.*

$$\frac{s(10+h) - s(10)}{h} = \frac{3(10+h)^2 - 3(10)^2}{h}$$

$$= \frac{3(100 + 20h + h^2) - 300}{h}$$

$$= \frac{300 + 60h + 3h^2 - 300}{h}$$

$$= \frac{60h + 3h^2}{h} = \frac{h(60 + 3h)}{h} = 60 + 3h$$

$$\lim_{h \to 0} \frac{s(10+h) - s(10)}{h} = \lim_{h \to 0} (60 + 3h) = 60 \text{ ft/sec}$$

INSTANTANEOUS RATE OF CHANGE

The instantaneous rate of change for a function f when $x = a$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists

<u>Difference Quotient</u>

$$\frac{f(a+h) - f(a)}{h}$$

Alternate Form

The instantaneous rate of change for a function f when $x = a$ can be written as

$$\lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

provided this limit exists

EXAMPLE

Suppose the total profit in hundreds of dollars from selling x items is given by $P(x) = 2x^2 - 5x + 6$. Find and interpret the following:

(a) The average rate of change of profit from $x = 2$ to $x = 4$
(b) The average rate of change of profit from $x = 2$ to $x = 3$
(c) The instantaneous rate of change of profit with respect to the number produced when $x = 2$

$$\frac{P(4) - P(2)}{4 - 2} = \frac{(2(4)^2 - 5(4) + 6) - (2(2)^2 - 5(2) + 6)}{2}$$

$$= \frac{18 - 4}{2} = 7$$

The average rate of change of profit from $x = 2$ to $x = 4$ is $\$700$ per item

$$\frac{P(3) - P(2)}{3 - 2} = \frac{(2(3)^2 - 5(3) + 6) - (2(2)^2 - 5(2) + 6)}{1}$$

$$= 9 - 4 = 5$$

The average rate of change of profit from $x = 2$ to $x = 3$ is $\$500$ per item

$$\lim_{h \to 0} \frac{P(2 + h) - P(2)}{h} = \lim_{h \to 0} \frac{(2(2 + h)^2 - 5(2 + h) + 6) - 4}{h}$$

$$= \lim_{h \to 0} \frac{(8 + 8h + 2h^2 - 10 - 5h + 6) - 4}{h}$$

$$= \lim_{h \to 0} \frac{2h^2 + 3h}{h}$$

$$= \lim_{h \to 0} (2h + 3) = 3$$

The instantaneous rate of change of profit with respect to the number of items produced when $x = 2$ is $\$300$ per item

MODULE 5

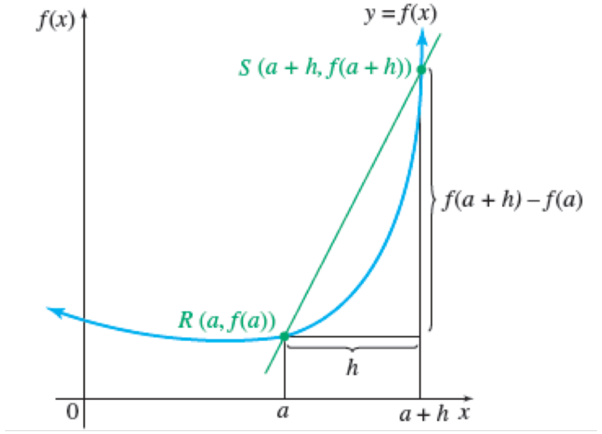
DERIVATIVES

SECANT AND TANGENT LINES

The slope of the secant line of the graph of $y = f(x)$ containing the points $(a, f(a))$ and $(a+h, f(a+h))$ is given by

$$\frac{f(a+h) - f(a)}{h}$$

Slope of secant line = average rate of change

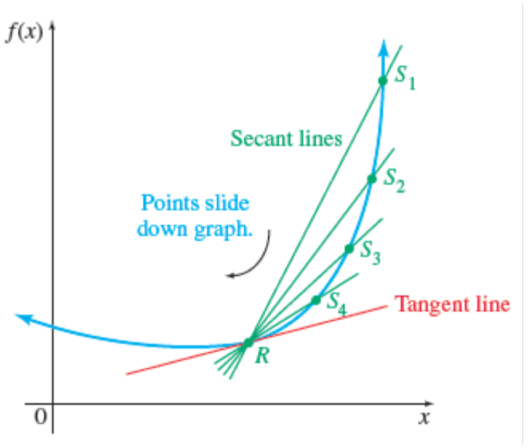


The slope of the tangent line of the graph of $y = f(x)$ at the point $(a, f(a))$ is given by

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists. If this limit does not exist, then there is no tangent at the point.

Slope of tangent line = instantaneous rate of change



DEFINITION OF THE DERIVATIVE

The derivative of the function f at x is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The function $f'(x)$ represents the instantaneous rate of change of $y = f(x)$ with respect to x

The function $f'(x)$ represents the slope of the graph at any point x

If $f'(x)$ is evaluated at the point $x = a$, then it represents the slope of the curve, or the slope of the tangent line at that point

EXAMPLE

Let $f(x) = 4/x$. Find $f'(x)$.

$$f(x+h) = \frac{4}{x+h}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{4}{x+h} - \frac{4}{x}}{h}$$

$$= \frac{\frac{4x}{x(x+h)} - \frac{4(x+h)}{x(x+h)}}{h}$$

$$= \left(\frac{-4h}{x(x+h)}\right)\left(\frac{1}{h}\right) = \frac{-4}{x(x+h)}$$

$$f'(x) = \lim_{h\to 0} \frac{-4}{x(x+h)} = -\frac{4}{x^2}$$

Find the equation of the tangent line to the graph of $f(x) = 4/x$ at $x = 2$.

$$f(2) = \frac{4}{2} = 2$$

Slope of the tangent line at $x = 2$ is $f'(2)$

$$f'(2) = -\frac{4}{2^2} = -1$$

Use $(2,2)$ and $m = -1$:

$$2 = (-1)(2) + b$$
$$b = 4$$

The equation of the tangent line to the graph of $f(x) = 4/x$ at $x = 2$ is

$$y = -x + 4$$

EXISTENCE OF THE DERIVATIVE

The derivative of a function f at a point exists when f satisfies ***all*** of the following conditions:

1. f is continuous,
2. f is smooth, and
3. f does not have a vertical tangent line

The derivative does not exist at a point when ***any*** of the following conditions are true:

1. f is discontinuous,
2. f has a sharp corner, or
3. f has a vertical tangent line

MODULE 5

DIFFERENTIATION RULES

TECHNIQUES

If $f(x) = k$ for any real number k, then $f'(x) = 0$

If $f(x) = 9$, then $f'(x) = 0$
If $H(t) = -3$, then $H'(t) = 0$

If $f(x) = x^n$ for any real number n, then $f'(x) = nx^{n-1}$

If $f(x) = x^6$, then $f'(x) = 6x^{6-1} = 6x^5$

If $f(x) = 1/x^3$, rewrite as $f(x) = x^{-3}$ and $f'(x) = -3x^{-3-1} = -3x^{-4}$

If $f(z) = \sqrt{z}$, rewrite as $f(z) = z^{1/2}$ and $f'(z) = \frac{1}{2}z^{-1/2}$

Let k be any real number. If $g'(x)$ exists and $f(x) = kg(x)$, then $f'(x) = kg'(x)$

If $D(p) = 10p^{3/2}$, then $D'(p) = 10\left(\frac{3}{2}p^{1/2}\right) = 15p^{1/2}$

If $g(t) = 6/t$, rewrite as $g(t) = 6t^{-1}$ and $g'(t) = -6t^{-2} = -6/t^2$

If $f(x) = u(x) \pm v(x)$ and if $u'(x)$ and $v'(x)$ exist, then $f'(x) = u'(x) \pm v'(x)$

If $h(x) = 6x^3 + 15x^2$, then $h'(x) = 18x^2 + 30x$

Suppose

$$f(x) = \frac{x^3 + 3\sqrt{x}}{x}$$

Rewrite as

$$f(x) = \frac{x^3}{x} + \frac{3\sqrt{x}}{x} = x^2 + 3x^{-1/2}$$

$$f'(x) = 2x - \frac{3}{2}x^{-3/2}$$

PRODUCTS AND QUOTIENTS

If $f(x) = u(x)v(x)$ and if $u'(x)$ and $v'(x)$ both exist, then

$$f'(x) = u(x)v'(x) + u'(x)v(x)$$

Find the derivative of $f(x) = (\sqrt{x} + 3)(x^2 - 5x)$

Let $u(x) = \sqrt{x} + 3$, then $u'(x) = \frac{1}{2}x^{-1/2}$

Let $v(x) = x^2 - 5x$, then $v'(x) = 2x - 5$

$$f'(x) = u(x)v'(x) + u'(x)v(x)$$

$$= (\sqrt{x} + 3)(2x - 5) + \left(\frac{1}{2}x^{-1/2}\right)(x^2 - 5x)$$

If $f(x) = u(x)/v(x)$, with $v(x) \neq 0$ and $u'(x)$ and $v'(x)$ both exist, then

$$f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}$$

Find $f'(x)$ if

$$f(x) = \frac{2x - 1}{4x + 3}$$

$u(x) = 2x - 1$ and $u'(x) = 2$

$v(x) = 4x + 3$ and $v'(x) = 4$

$$f'(x) = \frac{(4x + 3)(2) - (2x - 1)(4)}{(4x + 3)^2}$$

$$= \frac{8x + 6 - 8x + 4}{(4x + 3)^2}$$

$$= \frac{10}{(4x + 3)^2}$$

CHAIN RULE

If $y = f(g(x))$, then $y' = f'\big(g(x)\big)g'(x)$

Find y' if $y = (3x^2 - 5x)^{1/2}$

Apply the power rule to the outer most function, then multiply by the derivative of the innermost function

$$y' = \frac{1}{2}(3x^2 - 5x)^{-1/2}(6x - 5)$$

Find the derivative of $y = 4x(3x + 5)^5$

Use the product rule for y and the chain rule for $(3x + 5)^5$

$$y' = 4(3x + 5)^5 + 4x[5(3x + 5)^4(3)]$$

<u>Your turn</u>:

Find the derivative of $p(t) = 4t^2(t^2 + 1)^{5/4}$

<u>Answer</u>: $p'(t) = 8t(t^2 + 1)^{5/4} + 8t^3(t^2 + 1)^{1/4}$

EXPONENTIAL & LOGARITHMIC FUNCTIONS

If $f(x) = e^x$, then $f'(x) = e^x$

If $f(x) = e^{g(x)}$, then $f'(x) = e^{g(x)} g'(x)$

If $f(x) = a^x$ for $a > 0$ and $a \neq 1$, then $f'(x) = (\ln a) a^x$

If $f(x) = a^{g(x)}$, then $f'(x) = (\ln a)\, a^{g(x)} g'(x)$

If $f(x) = \ln x$, then $f'(x) = 1/x$

If $f(x) = \ln |g(x)|$, then $f'(x) = g'(x)/g(x)$

If $f(x) = \log_a x$, then

$$f'(x) = \frac{1}{(\ln a)\, x}$$

If $f(x) = \log_a |g(x)|$, then

$$f'(x) = \left(\frac{1}{\ln a}\right)\left(\frac{g'(x)}{g(x)}\right)$$

The amount in grams in a sample of uranium-239 after t years is given by

$$A(t) = 100e^{-0.362t}$$

Find the rate of change of the amount present after 3 years

$$A'(t) = 100(e^{-0.362t})(-0.362) = -36.2e^{-0.362t}$$

After 3 years, the rate of change is

$$A'(3) = -36.2e^{-0.362(3)} = -36.2e^{-1.086}$$
≈ -12.2 grams per year

Based on projections, the resale value of a certain 2014 vehicle can be approximated by the following function

$$f(t) = 30781 - 24277 \ln(0.46t + 1)$$

where t is the number of years since 2014. Find and interpret $f'(4)$.

$$f'(t) = \frac{(-24277)(0.46)}{0.46t + 1}$$

so $f'(4) \approx -4692$. This means in 2018, the average resale value of the vehicle is decreasing by $\$4{,}692$ per year.

TRIG FUNCTIONS

Add derivatives of trig functions

QUESTIONS?