MODULE 4

INTRODUCTION TO PROBABILITY

BAYES' THEOREM

COUNTING PRINCIPLES

MODULE 4

INTRODUCTION TO PROBABILITY

TERMINOLOGY

- Experiment – An activity or occurrence with an observable result
- Trial – Each repetition of an experiment
- Outcome – The result of an experiment
- Event – Any particular outcome or group of outcomes
- Sample space – The set of all possible outcomes for an experiment

FLIPPING A COIN

Flip a coin one time

Sample space = {H, T}

Possible outcomes would be H or T

Possible events would be H or T

Flip a coin two times

Sample space = {HH, HT, TH, TT}

4 possible outcomes

Some examples of events:

- Flipping 2 heads
- Flipping heads at least once
- Flipping 2 tails

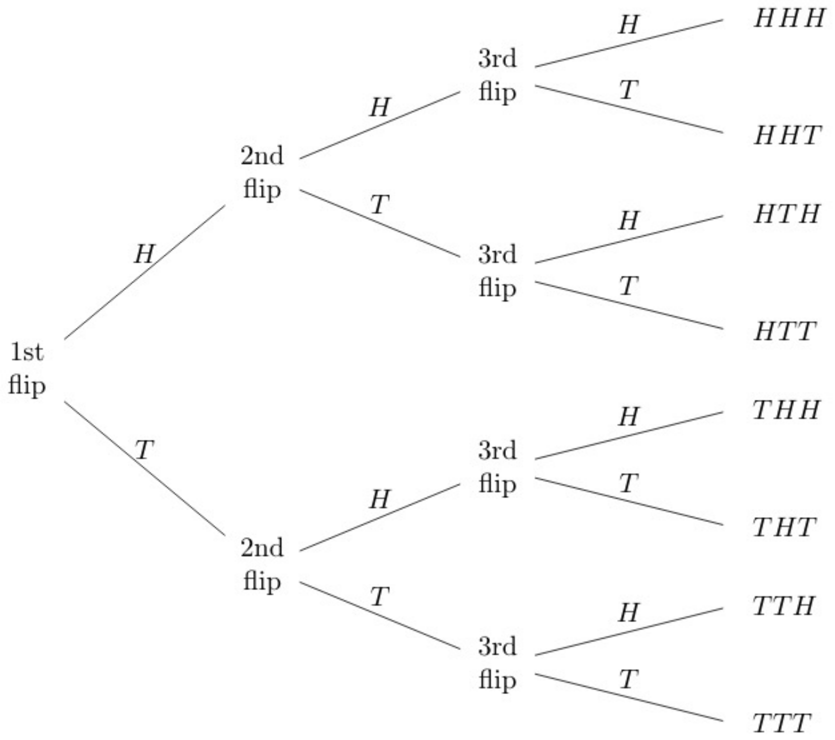
Flip a coin three times

8 possible outcomes

Sample space = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

Some examples of events:

- Flipping exactly 2 heads
- Flipping 3 tails
- Flipping at least 2 tails


1st
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TTH
TTT

BASIC PROBABILITY

$S =$ sample space of equally likely outcomes
$E =$ any event, i.e. a subset of S

$n(S) =$ the number of elements in S
$n(E) =$ the number of elements in E

The probability that event E occurs is given by:

$$P(E) = \frac{n(E)}{n(S)}$$

Flipping a coin 3 times:

$S =$ {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}
$E =$ At least 2 heads = {HHH, HHT, HTH, THH}

$n(S) = 8$
$n(E) = 4$

$$P(E) = \frac{4}{8} = 0.5$$

EXAMPLE

The table lists the estimated number of injuries in the US associated with recreation equipment.

Recreation Equipment Injuries	
Equipment	**Number of Injuries**
Bicycles	515,871
Skateboards	143,682
Trampolines	107,345
Climbing equipment	77,845
Swings or swing sets	59,144

Find the probability that a randomly selected person whose injury is associated with recreation equipment was hurt on a skateboard.

Add the number of injuries in each category from the table to get 903,887

$$P(Skateboards) = \frac{143{,}682}{903{,}887} \approx 0.15896$$

PROPERTIES OF PROBABILITY

Suppose S is a sample space consisting of n distinct outcomes, $s_1, s_2, \ldots, s_n$

Assign probabilities p_i to each outcome s_i subject to the following:

The probability of each outcome is between 0 and 1, inclusive:

$$0 \leq p_1 \leq 1, 0 \leq p_2 \leq 1, \ldots, 0 \leq p_n \leq 1$$

The sum of the probabilities of all possible outcomes is 1:

$$p_1 + p_2 + \cdots + p_n = 1$$

COMPLIMENT OF AN EVENT

The complement of an event E is typically denoted E'

Complement rule:

$P(E) = 1 - P(E')$ and $P(E') = 1 - P(E)$

A card is drawn from a standard deck of 52 cards. What is the probability that the card drawn is *not* a king?

There are 4 kings in the deck so the probability of drawing a king is 4/52 = 1/13

The probably of *not* drawing a king is 1 – 1/13 = 12/13

ADDITION RULE

If A and B are any events, P(A or B) = P(A) + P(B) – P(A and B)

Notation:
$P(A \cup B)$ ← Probability of A or B
$P(A \cap B)$ ← Probability of A and B

Suppose one card is drawn from a deck of cards. Find the probability that it will be a *red card or a face card*.

Let R be the event the card is red, and
F be the event it's a face card.

P(R) = 26/52

P(F) = 12/52

P(R ∩ F) = 6/52

P(R ∪ F) = 26/52 + 12/52 – 6/52 = 32/52 = 8/13

Tip: Because of addition and subtraction in the formula, often it's best to wait until the end to simplify your fractions.

Your turn: Find the probability the card is an ace or a club

Answer: 4/13

CONDITIONAL PROBABILITY

The table below contains the results of a survey of a firm's stockbrokers to determine if doing research to select stocks is more effective than following instincts to select stocks.

	Picked stocks that went up (A)	Didn't pick stocks that went up (A')	Totals
Used research (B)	30	15	45
Didn't use research (B')	30	25	55
Totals	60	40	100

$P(A) = \frac{60}{100} = 0.6$ Probability the broker picked stocks that went up

$P(A') = \frac{40}{100} = 0.4$ Probability the broker didn't pick stocks that went up

$P(B) = \frac{45}{100} = 0.45$ Probability the broker used research to select stocks

$P(B') = \frac{55}{100} = 0.55$ Probability the broker didn't use research to select stocks

Find the probability that a broker who uses research will pick stocks that go up

$$P(broker\ who\ uses\ research\ picks\ stocks\ that\ go\ up) = \frac{30}{45} \approx 0.6667$$

Different than the probability a broker picks stocks that go up because the sample space has been reduced with the information the broker uses research

This is a conditional probability, i.e. the probability that a broker picks stocks that go up given they used research

$$P(A|B) = \frac{30}{45}$$

$$P(A|B) = \frac{P(broker\ picks\ stocks\ that\ go\ up\ \textcolor{red}{and}\ uses\ research)}{P(broker\ uses\ research)} = \frac{30/100}{45/100} = \frac{30}{45}$$

CONDITIONAL PROBABILITY

The conditional probability of event A given B is defined by:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

with $P(B) \neq 0$

In a class with 2/5 women and 3/5 men, 10% are female business majors. Find the probability that a student chosen randomly from the class is a woman who is a business major.

$B =$ the event "business major"
$W =$ the event "woman"

Find $P(B|W)$

$P(W) = 0.4$ and $P(B \cap W) = 0.1$

$$P(B|W) = \frac{P(B \cap W)}{P(W)} = \frac{0.1}{0.4} = 0.25$$

PRODUCT RULE

If A and B are events, then

$$P(A \cap B) = P(A) \cdot P(B|A) \quad \text{or} \quad P(A \cap B) = P(B) \cdot P(A|B)$$

Recall

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

← Solve either formula for $P(A \cap B)$

PRODUCT RULE

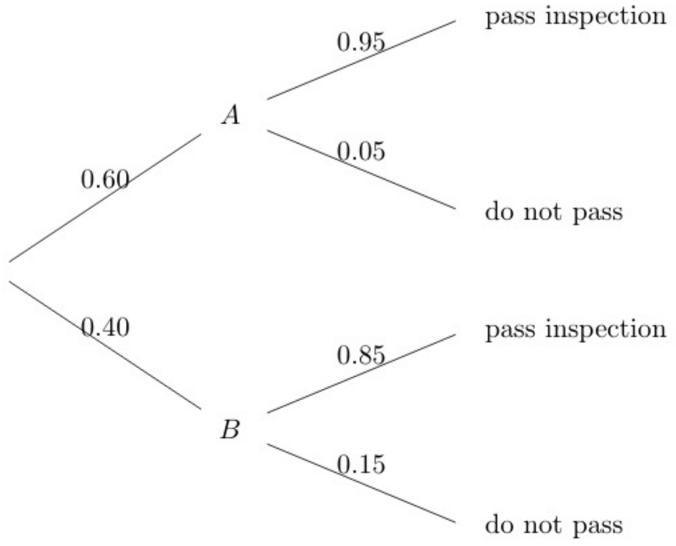
A bicycle factory has two assembly lines, A and B. Suppose 95% of products from line A pass inspection and 85% of products from line B pass inspection. Also, suppose 60% of the factory's bikes come off line A and 40% off line B.

Find the probability that one of the bikes did not pass inspection and came off line B.

$A =$ came off line A
$B =$ came off line B

$$P(B) = 0.4$$
$$P(pass|B) = 0.85$$
$$P(not\ pass|B) = 0.15$$

$$P(not\ pass \cap B) = P(B) \cdot P(not\ pass|B) = (0.4)(0.15) = 0.06$$



Your turn: Find the probability that one of the bikes did not pass inspection and came off line A.

Answer: 0.03

INDEPENDENT EVENTS

Two events A and B are independent if the probability of B occurring does not affect the probability of A occurring

Two events A and B are independent if

$P(A|B) = P(A)$ or $P(B|A) = P(B)$

Whether or not B has occurred has no bearing on P(A)

Whether or not A has occurred has no bearing on P(B)

Product rule for independent events:

Two events A and B are independent if and only if

$$P(A \cap B) = P(A) \cdot P(B)$$

Do not confuse mutually exclusive events with independent events

Mutually exclusive events cannot possibly both occur at the same time

<u>Mutually exclusive</u>:

If we flip a coin one time, the only possible outcomes are H and T

$P(H) = 0.5$
$P(T) = 0.5$

<u>Not independent</u>:

$P(H|T) = 0$
(if only one flip H cannot occur)

and

$P(H|T) \neq P(H)$

INDEPENDENT EVENTS

In a two-child family, if we assume that the probabilities of a male child and a female child are each 0.5, are the events *all children are the same sex* and *at most one male* independent?

S_1 = {MM, MF, FM, FF}

S_2 = {MF, FM, FF}

P(same sex) = 1/2

P(same sex | at most one male) = 1/3

Not independent

Are they independent for a three-child family?

S_3 = {MMM, MMF, MFM, MFF, FMM, FMF, FFM, FFF}

S_4 = {MFF, FMF, FFM, FFF}

P(same sex) = 2/8 = 1/4

P(same sex | at most one male) = 1/4

Independent

In a certain area, 15% of the population are joggers and 40% of joggers are women. If 55% of those who do not jog are women, find the probabilities that an individual from that community fits the following descriptions.

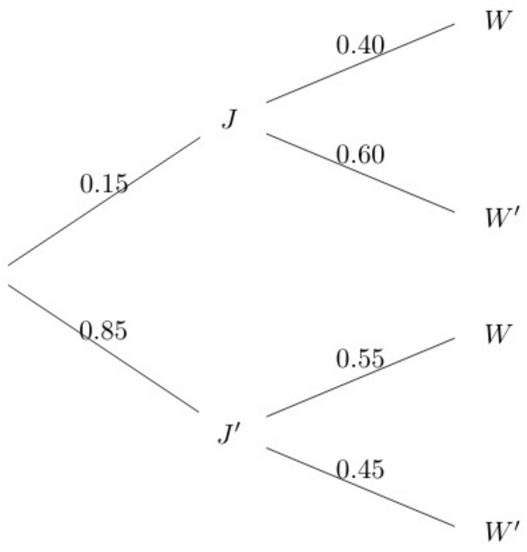
(a) A woman jogger
(b) A man who is not a jogger
(c) A woman

W = woman
J = jogger

$$P(woman\ jogger) = P(W \cap J) = P(J) \cdot P(W|J) = (0.15)(0.4) = 0.06$$

$$P(man\ who\ is\ not\ a\ jogger) = P(W' \cap J') = P(J') \cdot P(W'|J') = (0.85)(0.45) = 0.3825$$

$$P(woman) = P(W \cap J) + P(W \cap J') = (0.15)(0.4) + (0.85)(0.55) = 0.5275$$



(d) Are the events being a woman and being a jogger independent?

$$P(W \cap J) = 0.06$$

$$P(W) \cdot P(J) = (0.5275)(0.15) \approx 0.079$$

Not independent: $P(W \cap J) \neq P(W) \cdot P(J)$

Alternatively: $P(W|J) = 0.4 \neq 0.5275 = P(W)$

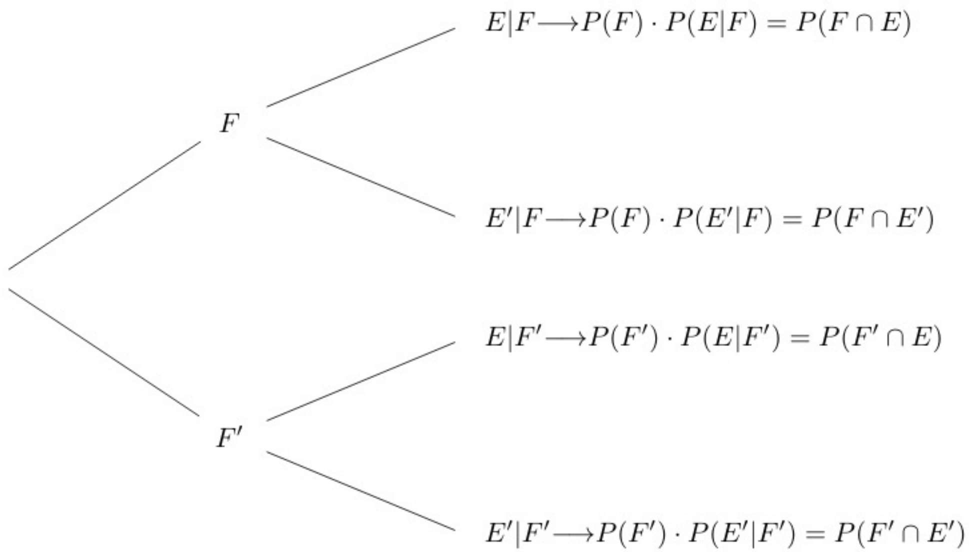
MODULE 4

BAYES' THEOREM

BAYES' THEOREM

Bayes' Theorem (Special Case)

$$P(F|E) = \frac{P(F) \cdot P(E|F)}{P(F) \cdot P(E|F) + P(F') \cdot P(E|F')}$$



Bayes' Theorem

$$P(F_i|E) = \frac{P(F_i) \cdot P(E|F_i)}{P(F_1) \cdot P(E|F_1) + P(F_2) \cdot P(E|F_2) + \cdots + P(F_n) \cdot P(E|F_n)}$$

EXAMPLE 1

For a fixed length of time, the probability of a worker error on a certain production line is 0.1, the probability that an accident will occur where there is a worker error is 0.3, and the probability that an accident will occur when there is no worker error is 0.2. Find the probability of a worker error if there is an accident.

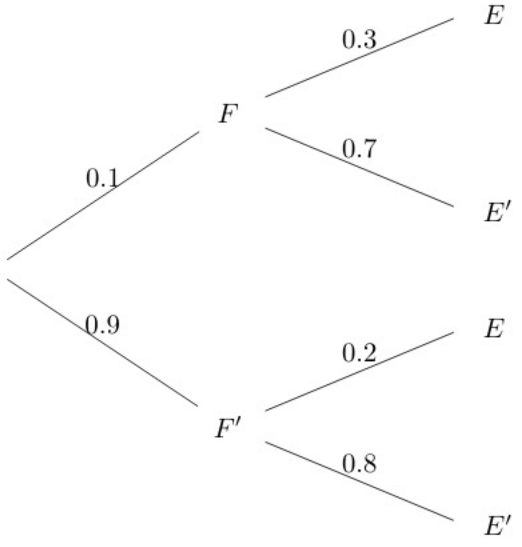
E = accident
F = worker error

Based on the information given:

$P(F) = 0.1, P(E|F) = 0.3$, and $P(E|F') = 0.2$

$$P(F|E) = \frac{P(F) \cdot P(E|F)}{P(F) \cdot P(E|F) + P(F') \cdot P(E|F')}$$

$$= \frac{(0.1)(0.3)}{(0.1)(0.3) + (0.9)(0.2)} = \frac{0.03}{0.21} \approx 0.1429$$



Probability

$P(F) \cdot P(E|F) = P(F \cap E)$

$P(F) \cdot P(E'|F) = P(F \cap E')$

$P(F') \cdot P(E|F') = P(F' \cap E)$

$P(F') \cdot P(E'|F') = P(F' \cap E')$

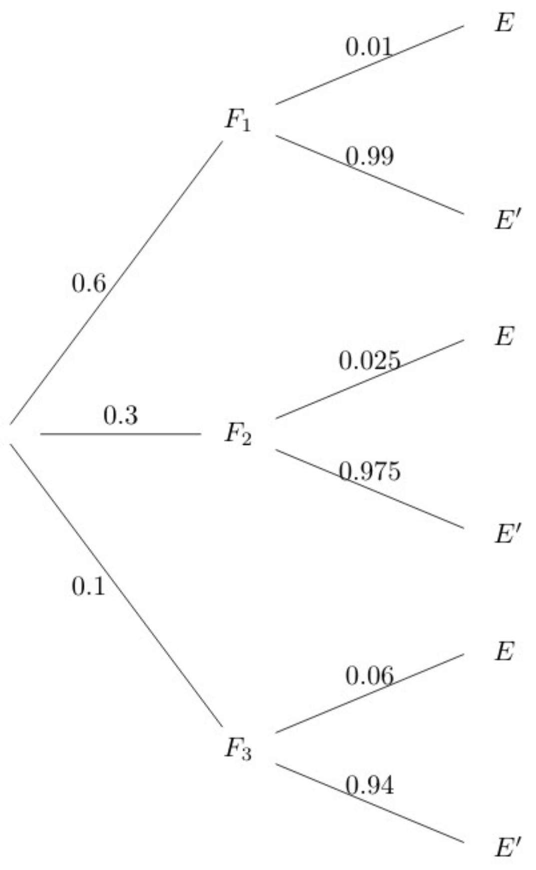
EXAMPLE 2

Based on past experience, a company knows that an experienced machine operator (one or more years of experience) will produce a defective item 1% of the time. Operators with some experience (up to one year) have a 2.5% defect rate, and new operators have a 6% defect rate. At any one time, the company has 60% experienced operators, 30% with some experience, and 10% new operators. Find the probability that a particular defective item was produced by a new operator.

E = item is defective
F_1 = item was made by an experienced operator
F_2 = item was made by an operator with some experience
F_3 = item was made by a new operator

$$P(F_3|E) = \frac{(0.1)(0.06)}{(0.6)(0.01) + (0.3)(0.025) + (0.1)(0.06)} = \frac{0.006}{0.0195} \approx 0.3077$$

Find all paths that lead to E



$P(F_1)P(E|F_1) = (0.6)(0.01)$

$P(F_2)P(E|F_2) = (0.3)(0.025)$

$P(F_3)P(E|F_3) = (0.1)(0.06)$

MODULE 4

COUNTING PRINCIPLES

MULTIPLICATION PRINCIPLE

Suppose n choices must be made, with m_1 ways to make choice 1, m_2 ways to make choice 2, …, and m_n ways to make choice n.

Then there are $m_1 \cdot m_2 \cdots m_n$ different ways to make the entire sequence of choices.

A four-digit passcode can contain any sequence of 4 single digit numbers (0 – 9). How many sequences are possible?

$10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ different sequences

How many different sequences are possible if no number is repeated?

$10 \cdot 9 \cdot 8 \cdot 7 = 5{,}040$ different sequences

PERMUTATIONS

Factorial notation

For any natural number n

$n! = n(n-1)(n-2)\ldots(3)(2)(1)$

Also, by definition

$$0! = 1$$

Permutations

If $P(n,r)$ (where $r \leq n$) is the number of permutations of n elements taken r at a time, then

$$P(n,r) = \frac{n!}{(n-r)!}$$

Order matters with permutations

Four-digit passcode

$$P(10{,}4) = \frac{10!}{6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 9 \cdot 8 \cdot 7 = 5{,}040$$

EXAMPLE

A local union chapter has 35 members. In how many different ways can the chapter select a president, a vice-president, a treasurer, and a secretary?

The number of ways to select the 4 officers is

$$P(35,4) = \frac{35!}{(35-4)!} = 35 \cdot 34 \cdot 33 \cdot 32 = 1{,}256{,}640$$

A concert to raise money for an economics prize is to consist of 5 works: 2 overtures, 2 sonatas, and 1 piano concerto.

(a) In how many ways can the program be arranged?
(b) In how many ways can the program be arranged if an overture must come first?

The number of ways the 5 works can be arranged is $P(5,5) = 5! = 120$

Pick one of the overtures to go first followed by arrangements of the 4 remaining pieces:

$$P(2,1) \cdot P(4,4) = 2 \cdot 24 = 48$$

COMBINATIONS

The number of combinations of n elements taken r at a time, where $r \leq n$ is given by

$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

Order does not matter

How many different ways can 3 lawyers be selected from a group of 30 to work on a special project?

$$C(30,3) = \frac{30!}{27!\,3!} = \frac{30 \cdot 29 \cdot 28 \cdot 27!}{27! \cdot 3 \cdot 2 \cdot 1} = \frac{30 \cdot 29 \cdot 28}{6} = 4060$$

A manager has 6 employees eligible for promotion and must select at least 4. In how may ways can she select the employees?

$$C(6,4) + C(6,5) + C(6,6) = \frac{6!}{2!\,4!} + \frac{6!}{1!\,5!} + \frac{6!}{0!\,6!} = 15 + 6 + 1 = 22$$

PROBABILITY APPLICATION

At a conference promoting excellence in education for African Americans in Detroit, special-edition books were selected to be given away in contests. There were 9 books written by Langston Hughes, 5 books by James Baldwin, and 7 books by Toni Morrison. The judges of one contest selected 6 books at random for prizes. Find the probabilities that the selection consisted of the following:

(a) 3 Hughes and 3 Morrison books
(b) Exactly 4 Baldwin books
(c) 2 Hughes, 3 Baldwin, and 1 Morrison book
(d) At least 4 Hughes books
(e) Exactly 4 books written by males (Morrison is female)
(f) No more than 2 books written by Baldwin

There are 21 books, so the number of selection of any 6 books is $C(21,6) = 54{,}264$

(a)
$$\frac{C(9,3)C(7,3)}{C(21,6)} = \frac{2940}{54{,}264} \approx 0.0542$$

(b)
$$\frac{C(5,4)C(16,2)}{C(21,6)} = \frac{600}{54{,}264} \approx 0.0111$$

(c)
$$\frac{C(9,2)C(5,3)C(7,1)}{C(21,6)} = \frac{2520}{54{,}264} \approx 0.464$$

(d)
$$\frac{C(9,4)C(12,2) + C(9,5)C(12,1) + C(9,6)C(12,0)}{C(21,6)}$$

$$= \frac{8316 + 1512 + 84}{54{,}264} \approx 0.1827$$

(e)
$$\frac{C(14,4)C(7,2)}{C(21,6)} = \frac{21{,}021}{54{,}264} \approx 0.387$$

(f)
$$\frac{C(5,0)C(16,6) + C(5,1)C(16,5) + C(5,2)C(16,4)}{C(21,6)}$$

$$\frac{8008 + 21{,}840 + 18{,}200}{54{,}264} \approx 0.8854$$

EXPECTED VALUE

Random Variable

A random variable is a function that assigns a real number to each outcome of an experiment.

Expected Value

Suppose a random variable x can take on the n values $x_1, x_2, \dots, x_n$. Also, suppose the corresponding probabilities that these values occur are $p_1, p_2, \dots, p_n$, respectively. Then the expected value of the random variable is

$$E(x) = x_1 p_1 + x_2 p_2 + \cdots + x_n p_n$$

EXAMPLE

An insurance policy on an electrical device pays a benefit of \$4000 is the device fails during the first year. The amount of the benefit decreases by \$1000 each successive year until it reaches 0. If the device has not failed by the beginning of any given year, the probability of failure during that year is 0.4. What is the expected benefit under this policy?

The probability of the device failing in year 1 is 0.4

The probability of the device failing in year 2 is the probability it did not fail in year 1 times the probability it fails in year 2

$$(0.6)(0.4) = 0.24$$

The probability of the device failing in year 3 is the probability it did not fail in years 1 and 2 times the probability it fails in year 3

$$(0.6)(0.6)(0.4) = 0.144$$

The probability of the device failing in year 4 is the probability it did not fail in years 1, 2, and 3 times the probability it fails in year 4

$$(0.6)(0.6)(0.6)(0.4) = 0.0864$$

The probability the device fails beyond year 4 is

$$1 - (0.4 + 0.24 + 0.144 + 0.0864) = 1 - 0.8704 = 0.1296$$

Payout in year 1 = \$4000
Payout in year 2 = \$3000
Payout in year 3 = \$2000
Payout in year 4 = \$1000
Payout in years > 4 = \$0

$$E(x) = 4000(0.4) + 3000(0.24) + 2000(0.144) + 1000(0.0864) + 0(0.1296) = 2694.4$$

Expected payout is \$2694.40

QUESTIONS?