Statistical Modeling - 24DS636 (2024-25)

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1 Course Introduction

The web page contains course contents of the course titled “Statistical Modeling” offered by Abhijith M S, PhD to Masters students pursuing M.Tech in Data Science, during the even semester of the academic year 2024-25.

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2 Parameter Estimation

2.1 Point Estimation: Maximum Likelihood Estimators

2.2 Interval Estimates

- Consider a sample $X_1, X_2, \ldots, X_n$ drawn from a known distribution with an unknown mean μ.
- It is established that the sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ serves as the maximum likelihood estimator for μ.
- However, the sample mean $\bar{X}$ is not expected to be exactly equal to μ, but rather close to it.
- Therefore, instead of providing a single point estimate, it is often more useful to specify an interval within which we are confident that μ lies.
- To determine such an interval estimator, we utilize the probability distribution of the point estimator.

2.2.1 Confidence Intervals for the Mean of a normal population with known Variance

- Consider a sample $X_1, X_2, \ldots, X_n$ drawn from a normal distribution with an unknown mean μ and a known variance σ^2.
- The point estimator $\bar{X}$ is normal with mean μ and variance σ^2/n.
- Therefore, $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ follows a standard normal distribution.

> **What to do**
>
> Consider that I want to find an interval around $\bar{X}$ such that the actual population mean μ falls within the interval, say 95 % of the times.

Tip

- For finding such an interval, I can use the Z-table. From the Z-table I can find:

$$P\left(-1.96 < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < 1.96\right) = 0.9750 - 0.0250 = 0.95$$

- Rewriting the above equation:

$$P\left(-1.96\frac{\sigma}{\sqrt{n}} < \bar{X}-\mu < 1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95$$
$$P\left(1.96\frac{\sigma}{\sqrt{n}} > \mu-\bar{X} > -1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95$$
$$P\left(-1.96\frac{\sigma}{\sqrt{n}} < \mu-\bar{X} < 1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95$$
$$P\left(\bar{X}-1.96\frac{\sigma}{\sqrt{n}} < \mu < \bar{X}+1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95$$

- We have P(Z < -1.96) = 0.025, similarly P(Z > 1.96) = 0.025. Usually 1.96 is represented generally as $z_{0.025}$. Thus, P(Z < -$z_{0.025}$) = 0.025 and P(Z > $z_{0.025}$) = 0.025.

- Hence, 100(1-0.05) percent confidence interval for the mean of a normal population with known variance is:

$$P\left(\bar{X}-z_{0.025}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X}+z_{0.025}\frac{\sigma}{\sqrt{n}}\right) = 0.95$$
$$P\left(\bar{X}-z_{0.05/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X}+z_{0.05/2}\frac{\sigma}{\sqrt{n}}\right) = (1-0.05)$$

- For a confidence level of $100(1-\alpha)$ percent, the corresponding critical value from the standard normal distribution is $z_{\alpha/2}$.
- The $100(1-\alpha)$ percent confidence interval for μ is given by:

$$\mu \in \left(\bar{X}-z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X}+z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \tag{2.1}$$

- The interval as given in Equation 2.1 is called a two-sided confidence interval.

- Also the term $z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ is called the margin of error.

What if !?

What if we are interested in one sided confidence intervals !!?

Solution

- To determine such an interval, for a standard normal random variable Z, we have;

$$P\left(Z < 1.645\right) = 0.95$$

- Thus,

$$P\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < 1.645\right) = 0.95$$

$$P\left(\mu - \bar{X} > -1.645\frac{\sigma}{\sqrt{n}}\right) = 0.95$$

$$P\left(\mu > \bar{X} - 1.645\frac{\sigma}{\sqrt{n}}\right) = 0.95$$

- Thus a 95 percent one-sided upper confidence interval for μ is

$$\mu \in \left(\bar{X} - 1.645\frac{\sigma}{\sqrt{n}}, \infty\right)$$

or in other words; 100(1-0.05) percent one-sided upper confidence interval for μ is

$$\mu \in \left(\bar{X} - z_{0.05}\frac{\sigma}{\sqrt{n}}, \infty\right)$$

Oneside interval!

Can you think of another one sided confidence interval?

Note

- We have

$$P\left(Z > -1.645\right) = 0.95$$

- Proceed just like in the previous case and you will find a 100(1-0.05) percent one-sided lower confidence interval for μ as;

$$\mu \in \left(-\infty, \bar{X} + z_{0.05}\frac{\sigma}{\sqrt{n}}\right)$$

- In general, 100(1-α) percent one-sided upper confidence interval for μ is given in Equation 2.2.

$$\mu \in \left(\bar{X} - z_{\alpha}\frac{\sigma}{\sqrt{n}}, \infty\right) \tag{2.2}$$

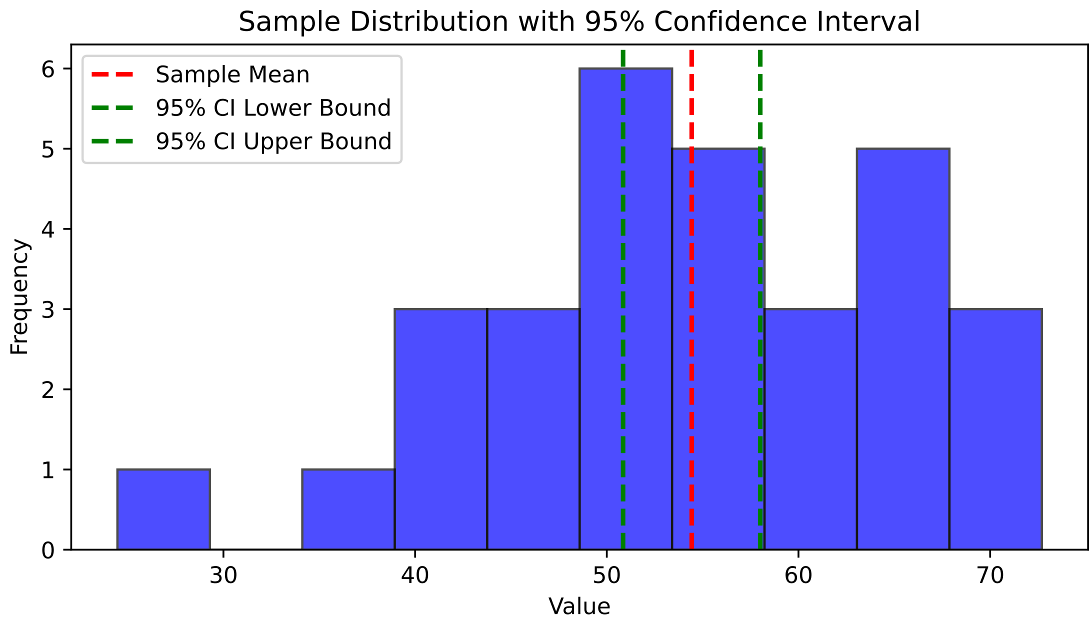
- Also, 100(1-α)percent one-sided lower confidence interval for μ is given in Equation 2.3.

$$\mu \in \left(-\infty, \bar{X} + z_{\alpha}\frac{\sigma}{\sqrt{n}}\right) \tag{2.3}$$

- The python code below creates a sample and find 95% confidence interval for the mean if the population standard deviation is assumed to be 10. Other values are specified in the code.

```
import numpy as np
import matplotlib.pyplot as plt

# Parameters
mu = 50  # true mean
sigma = 10  # known standard deviation
n = 30  # sample size
alpha = 0.05  # significance level

# Generate a sample
np.random.seed(0)
sample = np.random.normal(mu, sigma, n)
sample_mean = np.mean(sample)

# Calculate the confidence interval
z = 1.96  # z-value for 95% confidence
margin_of_error = z * (sigma / np.sqrt(n))
confidence_interval = (sample_mean - margin_of_error, sample_mean + margin_of_error)

# Plot the sample and confidence interval
plt.figure(figsize=(8, 4))
plt.hist(sample, bins=10, alpha=0.7, color='blue', edgecolor='black')
```

```
plt.axvline(sample_mean, color='red', linestyle='dashed', linewidth=2, label='Sample Mean')
plt.axvline(confidence_interval[0], color='green', linestyle='dashed', linewidth=2, label='95% CI Lower Bound')
plt.axvline(confidence_interval[1], color='green', linestyle='dashed', linewidth=2, label='95% CI Upper Bound')
plt.title('Sample Distribution with 95% Confidence Interval')
plt.xlabel('Value')
plt.ylabel('Frequency')
plt.legend()
plt.show()

print(f"Sample Mean: {sample_mean}")
print(f"95% Confidence Interval: {confidence_interval}")
```



Sample Mean: 54.42856447263174
95% Confidence Interval: (50.85011043026466, 58.007018514998826)

2.2.2 Confidence Intervals for the Mean of a normal population with unknown Variance

- If you recollect the discussion we had about the sample mean from a normal population with unknown variance we saw that variable t_{n-1} given by:

$$t_{n-1} = \sqrt{n}\frac{\bar{X} - \mu}{S}$$

has a t-distribution with n-1 degrees of freedom.

- Because of the symmetry of the t-distribution we can write for any $\alpha \in (0,\ 1/2)$;

$$P\left(-t_{\alpha/2,n-1} < \sqrt{n}\frac{\bar{X}-\mu}{S} < t_{\alpha/2,n-1}\right) = 1-\alpha$$

$$P\left(-\bar{X} - t_{\alpha/2,n-1}\frac{\sqrt{n}}{S} < -\mu < -\bar{X} + t_{\alpha/2,n-1}\frac{\sqrt{n}}{S}\right) = 1-\alpha$$

$$P\left(\bar{X} + t_{\alpha/2,n-1}\frac{\sqrt{n}}{S} > \mu > \bar{X} - t_{\alpha/2,n-1}\frac{\sqrt{n}}{S}\right) = 1-\alpha$$

$$P\left(\bar{X} - t_{\alpha/2,n-1}\frac{S}{\sqrt{n}} < \mu < \bar{X} + t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}\right) = 1-\alpha$$

- If the sample mean is $\bar{X}$ and sample standard deviation S, then we can say that with 100(1-α) percent confidence that

$$\mu \in \left(\bar{X} - t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}, \bar{X} + t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}\right)$$

- In this case 100(1-α) percent one-sided upper confidence interval can be obtained from the fact that:

$$P\left(\sqrt{n}\frac{(\bar{X}-\mu)}{S} < t_{\alpha,n-1}\right) = 1-\alpha$$

$$P\left(\mu > \bar{X} - \frac{S}{\sqrt{n}}t_{\alpha,n-1}\right) = 1-\alpha$$

- Thus $100(1-\alpha)$ percent one-sided upper confidence interval for the mean in this case is given by;

$$\mu \in \left(\bar{X} - \frac{S}{\sqrt{n}}t_{\alpha,n-1}, \infty\right)$$

- Thus $100(1-\alpha)$ percent one-sided lower confidence interval for the mean in this case is given by;

$$\mu \in \left(-\infty, \bar{X} + \frac{S}{\sqrt{n}}t_{\alpha,n-1}\right)$$

2.2.3 Confidence Intervals for the Variance of a Normal Distribution

- If we are sampling from a normal distribution with unknown mean and unknown variance then;

$$(n-1)\frac{S^2}{\sigma^2} \sim \chi^2_{n-1}$$

follows a chi-squared distribution.

- We have

$$P\left(\chi^2_{1-\alpha/2,n-1} \leq (n-1)\frac{S^2}{\sigma^2} \leq \chi^2_{\alpha/2,n-1}\right) = 1-\alpha$$

$$P\left(\chi^2_{1-\alpha/2,n-1} \leq (n-1)\frac{S^2}{\sigma^2} \leq \chi^2_{\alpha/2,n-1}\right) = 1-\alpha$$

$$P\left(\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}\right) = 1-\alpha$$

- Hence, 100(1-α) percent two-sided confidence interval for the variance in this case;

$$\sigma^2 \in \left(\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}}, \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}\right)$$

- The 100(1-α) percent one-sided upper and lower confidence intervals in this case will be respectively;

$$\left(\frac{(n-1)S^2}{\chi^2_{\alpha,n-1}}, \infty\right)$$

and

$$\left(0, \frac{(n-1)S^2}{\chi^2_{1-\alpha,n-1}}\right)$$

3 Hypothesis Testing

3.0.1 Introduction

- A statistical hypothesis is typically a statement regarding a set of parameters of a population distribution. It is termed a hypothesis because its truth value is unknown. The main challenge is to devise a method to determine whether the values of a random sample from this population align with the hypothesis. For example, consider a normally distributed population with an unknown mean value and a known variance of 1. The statement " is less than 1" is a statistical hypothesis that we can test by observing a random sample from this population. If the random sample is consistent with the hypothesis, we say the hypothesis is "accepted"; otherwise, it is "rejected."

4 Summary

In summary, this book has no content whatsoever.

References