MATH 472 HOMEWORK ASSIGNMENT 6

Problem 5.2.1. Let $\overline{X}_n$ denote the mean of a random sample of size n from a distribution that is $N(\mu, \sigma^2)$. Find the limiting distribution of $\overline{X}_n$.

Solution 5.2.1. Since the random sample is taken from a distribution with finite mean μ and finite variance σ^2, we may apply the weak law of large numbers to conclude that $\{\overline{X}_n\}$ converges to μ in probability. Theorem 5.2.1 says that if a sequence of random variables $\{Y_n\}$ converges to the random variable Y in probability, then $\{Y_n\}$ converges to Y in distribution. Therefore, since $\{\overline{X}_n\}$ converges to μ in probability we conclude that $\{\overline{X}_n\}$ converges to μ in distribution. Hence the limiting distribution for $\overline{X}_n$ is the distribution of the degenerate random variable μ,

$$F_\mu(x) = \begin{cases} 0, & x < \mu \\ 1, & x \geq \mu \end{cases}.$$

Problem 5.2.2. Let Y_1 denote the minimum of a random sample of size n from a distribution that has pdf $f(x) = e^{-(x-\theta)}$, $\theta < x < \infty$, zero elsewhere. Let $Z_n = n(Y_1 - \theta)$. Investigate the limiting distribution of Z_n.

Solution 5.2.2. A routine calculation shows that $P(Y_1 > y) = e^{-n(y-\theta)}$ for $y > \theta$, and $P(Y_1 > y) = 1$ for $y \leq \theta$. Since the support of Y_1 is the interval (θ, ∞) it follows that the support of $Z_n = n(Y_1 - \theta)$ is the interval $(0, \infty)$ and therefore $F_{Z_n}(t) = P(Z_n \leq t) = 0$ for all $t \leq 0$. Let $t > 0$, then

$$\begin{aligned} F_{Z_n}(t) &= P(Z_n \leq t) \\ &= P(n(Y_1 - \theta) \leq t) \\ &= P(Y_1 \leq \frac{t}{n} + \theta) \\ &= 1 - P(Y_1 > \frac{t}{n} + \theta) \\ &= 1 - e^{-t}. \end{aligned}$$

We see that, for every natural number n, F_{Z_n} is the cdf for the exponential distribution with mean $\mu = 1$. Therefore, $\{Z_n\}$ converges in distribution to an exponential distribution with mean $\mu = 1$.

Problem 5.2.5. Let the pmf of Y_n be $p_n(y) = 1$, $y = n$, zero elsewhere. Show that Y_n does not have a limiting distribution. (In this case, the probability has "escaped" to infinity.)

Solution 5.2.5. The cdf for the degenerate random variable Y_n is

$$F_{Y_n}(y) = \begin{cases} 0, & y < n \\ 1, & y \geq n \end{cases}.$$

For all y, $\lim_{n\to\infty} F_{Y_n}(y) = 0$. To show that Y_n does not have a limiting distribution, we must show that there does not exist a distribution function

F with the property that $F(y) = 0$ for every $y \in C(F)$. Equivalently, we must show that if F is a distribution function then there exists a point y such that F is continuous at y and $F(y) \neq 0$.

Let F be a distribution function. Since $\lim_{y\to\infty} F(y) = 1$, there exists a real number y_0 such that $F(y) > 1/2$ for all $y > y_0$. Recall that the set of discontinuous points for a distribution is always a countable set. Since the interval (y_0, ∞) is an uncountable set, there must be at least one point $y \in (y_0, \infty)$ such that F is continuous at y. Since $y > y_0$, we also have that $F(y) > 1/2 > 0$. Therefore every distribution function F has at least one point y such that F is continuous at y and $F(y) \neq 0$. We conclude that Y_n does not have a limiting distribution.

Problem 5.2.7. Let X_n have a gamma distribution with parameter $\alpha = n$ and β, where β is not a function of n. Let $Y_n = X_n/n$. Find the limiting distribution of Y_n.

Solution 5.2.7. From Appendix D on page 667 we see that $E[X_n] = n\beta$ and $\mathrm{Var}[X_n] = n\beta^2$, and therefore $E[Y_n] = E[X_n/n] = \beta$ and $\mathrm{Var}[Y_n] = \mathrm{Var}[X_n/n] = \beta^2/n$. Since the expected value of Y_n does not depend on n and since the variance of Y_n is a constant divided by n, Exercise 5.1.3 on page 293 shows that $\{Y_n\}$ converges to β in probability. Applying Theorem 5.2.1 on page 298 shows that $\{Y_n\}$ converges to β in distribution. We conclude that the limiting distribution of Y_n is the distribution for the degenerate random variable β,

$$F_\beta(t) = \begin{cases} 0, & t < \beta \\ 1, & t \geq \beta \end{cases}.$$

Problem 5.2.12. Prove Theorem 5.2.3.

Solution 5.2.3.

Theorem. *Suppose X_n converges to X in distribution and Y_n converges in probability to 0. Then $X_n + Y_n$ converges to X in distribution.*

Proof. Let x be a continuous point for F_X. We want to show that $\lim_{n\to\infty} F_{X_n+Y_n}(x) = F_X(x)$.

Let $\epsilon > 0$ be given. Suppose that $x+\epsilon$ and $x-\epsilon$ are also continuous points for F_X. Observe that $X_n + Y_n \leq x$ and $|Y_n| \leq \epsilon$ imply that $X_n \leq x + \epsilon$. Thus,

$$\begin{aligned} F_{X_n+Y_n}(x) &= P(X_n + Y_n \leq x) \\ &= P(X_n + Y_n \leq x, |Y_n| < \epsilon) + P(X_n + Y_n \leq x, |Y_n| \geq \epsilon) \\ &\leq P(X_n + Y_n \leq x, |Y_n| \leq \epsilon) + P(X_n + Y_n \leq x, |Y_n| \geq \epsilon) \\ &\leq P(X_n \leq x + \epsilon) + P(|Y_n| \geq \epsilon) \\ &= F_{X_n}(x + \epsilon) + P(|Y_n| \geq \epsilon). \end{aligned} \tag{1}$$

Since $X_n \xrightarrow{D} X$ and since $x+\epsilon$ is a continuity point for F_X we have that

$$\lim_{n\to\infty} F_{X_n}(x+\epsilon) = F_X(x+\epsilon),$$

and since $Y_n \xrightarrow{P} 0$ we have that

$$\lim_{n\to\infty} P(|Y_n| \geq \epsilon) = 0.$$

The inequality in (1), together with these last two limits, shows that

$$\limsup_{n\to\infty} F_{X_n+Y_n}(x) \leq F_X(x+\epsilon). \tag{2}$$

Now, observe that $X_n + Y_n > x$ and $|Y_n| < \epsilon$ imply that $X_n > x - \epsilon$. Thus,

$$\begin{aligned} 1 - F_{X_n+Y_n}(x) &= P(X_n + Y_n > x) \\ &= P(X_n + Y_n > x, |Y_n| < \epsilon) + P(X_n + Y_n > x, |Y_n| \geq \epsilon) \\ &\leq P(X_n > x - \epsilon) + P(|Y_n| \geq \epsilon) \\ &= 1 - F_{X_n}(x+\epsilon) + P(|Y_n| \geq \epsilon). \end{aligned}$$

Rearranging the terms of this inequality yields

$$F_{X_n}(x-\epsilon) - P(|Y_n| \geq \epsilon) \leq F_{X_n+Y_n}(x). \tag{3}$$

Since $X_n \xrightarrow{D} X$ and since $x-\epsilon$ is a continuity point for F_X we have that

$$\lim_{n\to\infty} F_{X_n}(x-\epsilon) = F_X(x-\epsilon),$$

and since $Y_n \xrightarrow{P} 0$ we have that

$$\lim_{n\to\infty} P(|Y_n| \geq \epsilon) = 0.$$

The inequality in (3), together with these last two limits, shows that

$$F_X(x-\epsilon) \leq \liminf_{n\to\infty} F_{X_n+Y_n}(x). \tag{4}$$

Combining inequalities (2) and (4) we have

$$F_X(x-\epsilon) \leq \liminf_{n\to\infty} F_{X_n+Y_n}(x) \leq \limsup_{n\to\infty} F_{X_n+Y_n}(x) \leq F_X(x+\epsilon). \tag{5}$$

The inequalities in (5) require that $x-\epsilon$ and $x+\epsilon$ be continuity points for F_X. We claim that there exists a sequence of positive numbers $\{\epsilon_k\}$ such that $\lim_{k\to\infty} \epsilon_k = 0$ and for all natural numbers k the points $x \pm \epsilon_k$ are continuity points for F_X. This claim is an immediate consequence of the fact that the set of discontinuous points of F_X is countable.

Choose $\{\epsilon_k\}$ as described in the previous paragraph. Since F_X is continuous at x, $\lim_{k\to\infty} F_X(x \pm \epsilon_k) = F_X(x)$. Since F_X is continuous at the points $x \pm \epsilon_k$ for every natural number k,

$$F_X(x-\epsilon_k) \leq \liminf_{n\to\infty} F_{X_n+Y_n}(x) \leq \limsup_{n\to\infty} F_{X_n+Y_n}(x) \leq F_X(x+\epsilon_k). \tag{6}$$

Finally, letting $k \to \infty$ in (7) shows that $\lim_{n\to\infty} F_{X_n+Y_n}(x) = F_X(x)$, and we conclude that $X_n + Y_n \xrightarrow{D} X$. $\square$

Problem 5.2.20. Use Stirling's formula, (5.2.2), to show that the first limit in Example 5.2.3 is 1.

Solution 5.2.20. Let us introduce three sequences,

$$a_n = \frac{\Gamma\left(\frac{n-1}{2}+1\right)}{\sqrt{2\pi}\left(\frac{n-1}{2}\right)^{n/2} e^{-(n-1)/2}} \tag{7}$$

$$b_n = \frac{\sqrt{2\pi}\left(\frac{n-2}{2}\right)^{(n-1)/2} e^{-(n-2)/2}}{\Gamma\left(\frac{n-2}{2}+1\right)} \tag{8}$$

$$c_n = e^{-1/2}\sqrt{1-\frac{2}{n}}\left(\frac{n-1}{n-2}\right)^{n/2}. \tag{9}$$

Stirling's formula states that

$$\lim_{k\to\infty}\frac{\Gamma(k+1)}{\sqrt{2\pi}k^{k+\frac{1}{2}}e^{-k}} = 1.$$

We may apply Sterling's formula to a_n with $k = (n-1)/2$ to conclude that $\lim_{n\to\infty} a_n = 1$. In a similar way we may apply Steling's formula to b_n with $k = (n-2)/2$ to conclude that $\lim_{n\to\infty} b_n = 1$.

Notice that $\lim_{n\to\infty}[(n-1)/(n-2)]^{n/2}$ is a 1^∞ indeterminant form. Applying Calculus II methods that are taken from the section on L'Hôpital's Rule we are able to evaluate that limit. We leave it as an exercise to confirm that $\lim_{n\to\infty}[(n-1)/(n-2)]^{n/2} = e^{1/2}$. Knowing this, we see that $\lim_{n\to\infty} c_n = 1$.

After some simplification, which we leave to the student, we find that

$$a_n \cdot b_n \cdot c_n = \frac{\Gamma[(n+1)/2]}{\sqrt{n/2}\,\Gamma(n/2)}.$$

We therefore conclude that

$$\lim_{n\to\infty}\frac{\Gamma[(n+1)/2]}{\sqrt{n/2}\,\Gamma(n/2)} = \lim_{n\to\infty} a_n \cdot b_n \cdot c_n = 1\cdot 1\cdot 1 = 1.$$