7. More examples of order statistics

Example 3. **Let X_1, X_2, X_3 be a random sample from a distribution of the continuous type having pdf f(x)=2x, 0<x<1, zero elsewhere.**
(a) compute the probability that the smallest of X_1, X_2, X_3 exceeds the median of the distribution.
(b) If $Y_1 \le Y_2 \le Y_3$ are the order statistics, find the correlation between Y_2 and Y_3.

Answer:
(a)

$$F(x) = P(X_i < x) = x^2;$$

$$\int_0^t 2x dx = \frac{1}{2}; t = \frac{\sqrt{2}}{2}$$

$$P(\min(X_1, X_2, X_3) > t) = P(X_1 > t, X_2 > t, X_3 > t) = P(X_1 > t)P(X_2 > t)P(X_3 > t)$$

$$= [1 - F(t)]^3 = (1 - t^2)^3 = \frac{1}{8}$$

(b)
Please refer to the textbook/notes for the order statistics pdf and joint pdf formula. We have

$$f_{Y_3}(x) = 6 * x^5; 0 < x < 1$$

$$f_{Y_2}(x) = 12 * (x^3 - x^5); 0 < x < 1$$

$$E(Y_3) = 6/7,$$

$$E(Y_2) = 24/35;$$

$$f_{Y_2,Y_3}(y_2, y_3) = 24 * (y_2)^3 * y_3; 0 < y_2 \le y_3 < 1$$

$$E(Y_2 Y_3) = \int_0^1 \left[\int_0^{y_3} y_2 * y_3 * 24 * (y_2)^3 * y_3 \, dy_2 \right] dy_3 = \frac{3}{5};$$

$$var(Y_3) = \frac{6}{8} - \left(\frac{6}{7}\right)^2 = \frac{6}{392};$$

$$var(Y_2) = \frac{1}{2} - \left(\frac{24}{35}\right)^2;$$

$$corr(Y_2, Y_3) = \frac{E(Y_2 Y_3) - E(Y_2)E(Y_3)}{\sqrt{var(Y_2)var(Y_3)}} = 0.57$$

Example 4. **Let $Y_1 \le Y_2 \le Y_3$ denote the order statistics of a random sample of size 3 from a distribution with pdf f(x) = 1, 0 < x < 1, zero elsewhere. Let Z = $(Y_1+Y_3)/2$ be the midrange of the sample. Find the pdf of Z.**