MATH 728 Homework №3

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<u>4.5.8</u> Let us say the life of a tire in miles, say X, is normally distributed with mean θ and standard deviation 5000. Past experience indicates that $\theta = 30000$. The manufacturer claims that the tires made by a new process have mean $\theta > 30000$. It is possible that $\theta = 35000$. Check his claim by testing $H_0 : \theta = 30000$ against $H_1 : \theta > 30000$. We shall observe n independent values of X, say $x_1, \dots, x_n$, and we shall reject H_0 (thus accept H_1) if and only if $\overline{x} \geq c$. Determine n and c so that the power function $\gamma(\theta)$ of the test has the values $\gamma(30000) = 0.01$ and $\gamma(35000) = 0.98$.

Solution. To determine n and c we use given conditions for the power function and critical values of normal distribution:

$$\frac{c - 30000}{5000/\sqrt{n}} = z_{0.01} = 2.326 \quad \text{and} \quad \frac{c - 35000}{5000/\sqrt{n}} = -z_{0.02} = -2.054.$$

Solving the system we obtain $c = 32655.3$ and $n = 19.184 \approx 19$.

Then, we reject H_0 (thus accept H_1) if and only if $\overline{x} \geq 32655.3$.

<u>4.5.9</u> Let X have a Poisson distribution with mean θ. Consider the simple hypothesis $H_0 : \theta = 1/2$ and the alternative composite hypothesis $H_1 : \theta < 1/2$. Thus $\Omega = \{\theta : 0 < \theta \leq 1/2\}$. Let $X_1, \dots, X_{12}$ denote a random sample of size 12 from this distribution. We reject H_0 if and only if the observed value of $Y = X_1 + \ldots + X_{12} \leq 2$. If $\gamma(\theta)$ is a power function of the test, find the powers $\gamma(1/2),\ \gamma(1/3),\ \gamma(1/4),\ \gamma(1/6)$, and $\gamma(1/12)$. Sketch the graph of $\gamma(\theta)$. What is the significance level of the test?

Solution. If random variables $X_1, \dots, X_{12} \sim \text{Poisson}(\theta)$, then $Y = X_1 + \ldots + X_{12} \sim \text{Poisson}(12 \cdot \theta)$.

Hence, for $\theta = 1/2$ $Y \sim \text{Poisson}(6)$ and

$$\gamma(1/2) = P\left(Y \leq 2\right) = \sum_{k=0}^{2} \frac{6^k}{k!} e^{-6} = e^{-6}\left(1 + 6 + \frac{6^2}{2}\right) = 0.0619.$$

For $\theta = 1/3$ $Y \sim \text{Poisson}(4)$ and

$$\gamma(1/3) = P\left(Y \leq 2\right) = \sum_{k=0}^{2} \frac{4^k}{k!} e^{-4} = e^{-4}\left(1 + 4 + \frac{4^2}{2}\right) = 0.238.$$

For $\theta = 1/4$ $Y \sim \text{Poisson}(3)$ and

$$\gamma(1/4) = P\left(Y \leq 2\right) = \sum_{k=0}^{2} \frac{3^k}{k!} e^{-3} = e^{-3}\left(1 + 3 + \frac{3^2}{2}\right) = 0.423.$$

For $\theta = 1/6$ $Y \sim \text{Poisson}(2)$ and

$$\gamma(1/6) = P(Y \le 2) = \sum_{k=0}^{2} \frac{2^k}{k!} e^{-2} = e^{-2}\left(1 + 2 + \frac{2^2}{2}\right) = 0.677.$$

For $\theta = 1/12$ $Y \sim \text{Poisson}(1)$ and

$$\gamma(1/12) = P(Y \le 2) = \sum_{k=0}^{2} \frac{1^k}{k!} e^{-1} = e^{-1}\left(1 + 1 + \frac{1}{2}\right) = 0.919.$$

Significance level of the test is $\gamma(1/2) = 0.0619 = 6.2\%$.

4.5.10 Let Y have a binomial distribution with parameters n and p. We reject $H_0:\ p = 1/2$ and accept $H_1:\ p > 1/2$ if $Y \ge c$. Find n and c to give a power function $\gamma(p)$ which is such that $\gamma(1/2) = 0.1$ and $\gamma(2/3) = 0.95$, approximately.

Solution. To determine n and c we use given conditions for the power function and critical values of normal distribution:

$$\frac{c - np}{\sqrt{np(1-p)}} = \frac{c - n/2}{\sqrt{n}/2} = z_{0.1} = 1.282$$

and

$$\frac{c - 2n/3}{\sqrt{2n}/3} = -z_{0.05} = -1.645.$$

Solving the system we obtain $c = 41.562$ and $n = 72.225 \approx 72$. Then power function of the test is

$$\gamma(p) = P(Y \ge c) = P\left(\frac{Y - np}{\sqrt{np(1-p)}} \ge \frac{c - np}{\sqrt{np(1-p)}}\right) =$$

$$= P\left(\frac{Y - 72p}{\sqrt{72p(1-p)}} \ge \frac{41.562 - 72p}{\sqrt{72p(1-p)}}\right), \quad 1/2 \le p < 1.$$

<u>4.5.13</u> Let p denote the probability that, for a particular tennis player, the first serve is good. Since $p = 0.4$, this player decided to take lessons in order to increase p. When the lessons are completed, the hypothesis $H_0: \ p = 0.4$ will be tested against $H_1: \ p > 0.4$ based on $n = 25$ trials. Let y equal the number of first serves that are good, and let the critical region be defined by $C = \{y: \ y \geq 13\}$.

(a) Determine $\alpha = P(Y \geq 13; p = 0.4)$.

Solution.

$$\alpha = P(Y \geq 13; p = 0.4) = \sum_{k=13}^{n} \binom{n}{k} p^k (1-p)^{n-k} =$$

$$= \sum_{k=13}^{25} \binom{25}{k} (0.4)^k (0.6)^{25-k} = 0.1537.$$

(b) Find $\beta = P(Y < 13)$ when $p = 0.6$; that is, $\beta = P(Y \leq 12; \ p = 0.6)$ so that $1 - \beta$ is the power at $p = 0.6$.

$$\beta = P(Y \leq 12; \ p = 0.6) = \sum_{k=1}^{12} \binom{n}{k} p^k (1-p)^{n-k} =$$

$$= \sum_{k=1}^{12} \binom{25}{k} (0.6)^k (0.4)^{25-k} = 0.1537.$$

<u>4.6.4</u> Consider the one-sided t-test for $H_0: \ \mu = \mu_0$ versus $H_{A1}: \ \mu > \mu_0$ constructed in Ex. 5.5.4 and the two-sided t-test for $H_0: \ \mu = \mu_0$ versus $H_0: \ \mu \neq \mu_0$ given in (5.6.9). Assume that both tests are of size α. Show that for $\mu > \mu_0$, the power function of the one-sided test is larger than the power function of the two-sided test.

Solution. For both tests $X_n \sim N(\mu, \sigma^2)$. The first test's hypotheses are $H_0 : \mu = \mu_0$ and $H_1 : \mu > \mu_0$, and we reject H_0 if $\frac{\mu - \mu_0}{\sigma/\sqrt{n}} \geq t_{\alpha,n-1}$. The second test's hypotheses are $H_0 : \mu = \mu_0$ and $H_1 : \mu \neq \mu_0$, and we reject H_0 if $\left|\frac{\mu-\mu_0}{\sigma/\sqrt{n}}\right| \geq t_{\alpha/2,n-1}$. Then, since $t_{\alpha,n-1} < t_{\alpha/2,n-1}$, for $\mu > \mu_0$

$$\gamma_2(\mu) = P\left(\left|\frac{\mu - \mu_0}{\sigma/\sqrt{n}}\right| \geq t_{\alpha/2,n-1}\right) = P\left(\frac{\mu - \mu_0}{\sigma/\sqrt{n}} \geq t_{\alpha/2,n-1}\right) \leq$$

$$\leq P\left(\frac{\mu - \mu_0}{\sigma/\sqrt{n}} \geq t_{\alpha,n-1}\right) = \gamma_1(\mu).$$

4.6.5 Assume that the weight of cereal in a "10-ounce box"is $N(\mu, \sigma^2)$. To test $H_0:\ \mu = 10.1$ against $H_1:\ \mu > 10.1$, we take a random sample of size $n = 16$ and observe that $\overline{x} = 10.4$ and $s = 0.4$.

(a) Do we accept or reject H_0 at the 5% significance level?

Solution.

$$t_{obs} = \frac{\overline{x} - \mu_0}{s/\sqrt{n}} = \frac{10.4 - 10.1}{0.4/4} = 3.$$

Then, since $t_{obs} = 3 > t_{0.05,15} = 1.753$, we reject H_0 and accept H_1 at the 5% significance level.

(b) What is the approximate p-value of this test?

To determine p-value of the test we have to find α such that $t_{\alpha,15} \approx 3$. Using table with critical values of t distribution, p-value$= \alpha = 0.005$.

4.6.7 Among the data collected for the World Health Organization air quality monitoring project is a measure of suspended particles in $\mu g/m^3$. Let X and Y equal the concentration of suspended particles in $\mu g/m^3$ in the city center (commercial district) for Melbourne and Houston, respectively. Using $n = 13$ observations of X and $m = 16$ observations of Y, we shall test $H_0:\ \mu_X = \mu_Y$ against $H_1:\ \mu_X < \mu_Y$.

(a) Define the test statistic and critical region, assuming that the unknown variances are equal. Let $\alpha = 0.05$.

Solution. The test statistic and the critical region will be

$$T = \frac{\overline{Y} - \overline{X}}{S_p\sqrt{1/n_1 + 1/n_2}}, \quad \text{where } S_p = \sqrt{\frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}},$$

and

$$C = \{(X_1, \ldots, X_{13}, Y_1, \ldots, Y_{16}):\ T \geq t_{0.05,27} = 1.703\}.$$

(b) If $\overline{x} = 72.9$, $s_x = 25.6$, $\overline{y} = 81.7$, and $s_y = 28.3$, calculate the value of the test statistic and state your conclusion.

$$S_p = \sqrt{\frac{12(25.6)^2 + 15(28.3)^2}{27}} = 27.133,$$

and

$$T_{obs} = \frac{81.7 - 72.9}{27.133\sqrt{1/13 + 1/16}} = 0.8685.$$

Then, since $T_{abs} < t_{0.05,27} = 1.703$, we do not reject H_0 at the significance level $\alpha = 0.05$.

4.6.8 Let p equal the proportion of drivers who use a seat belt in a state that does not have a mandatory seat belt law. It was claimed that $p = 0.14$. An advertising campaign was conducted to increase this proportion. Two months after the campaign, $y = 104$ out of a random sample of $n = 590$ drivers were wearing their seat belts. Was the campaign successful?

(a) Define the null and alternative hypotheses.

Solution. Hypotheses are

$$H_0: \; p = 0.14 \quad \text{against} \quad H_1: \; p > 0.14.$$

(b) Define a critical region with an $\alpha = 0.01$ significance level.

The critical region is

$$C = \left\{ y: \; \frac{\widehat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} \geq z_{0.01} = 2.326 \right\}, \quad \text{where } \widehat{p} = \frac{y}{n}.$$

(c) Determine the approximate p-value and state your conclusion.

$$z_{obs} = \frac{\widehat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} = \frac{0.1763 - 0.14}{\sqrt{0.14(0.86)/590}} = 2.539.$$

Then, using the table with critical values of normal distribution, correspondent p-value$= 0.0055$. And, since $0.0055 < \alpha = 0.01$, we have to reject H_0 at the significance level $\alpha = 0.01$.

4.6.9 In exercise 5.4.14 we found a confidence interval for the variance σ^2 using the variance S^2 of a random sample of size n arising from $N(\mu, \sigma^2)$, where the mean μ is unknown. In testing $H_0: \; \sigma^2 = \sigma_0^2$ against $H_1: \; \sigma^2 > \sigma_0^2$, use the critical region defined by $(n-1)S^2/\sigma_0^2 \geq c$. That is, reject H_0 and accept H_1 if $S^2 \geq c\sigma_0^2(n-1)$. If $n = 13$ and the significance level $\alpha = 0.025$, determine c.

Solution. Since $\frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2(n-1)$, given mentioned conditions,

$$c = \chi^2_{0.025}(12) = 4.404.$$

4.6.10 In exercise 5.4.25, in finding a confidence interval for the ratio of the variances of two normal distributions, we used a statistic S_1^2/S_2^2 which has an F-distribution when those two variances are equal. If we denote that statistic by F, we can test $H_0:\ \sigma_1^2 = \sigma_2^2$ against $H_1:\ \sigma_1^2 > \sigma_2^2$ using the critical region $F \geq c$. If $n = 13,\ m = 11$, and $\alpha = 0.05$, find c.

Solution. Since $S_1^2/S_2^2 \sim F_\alpha(n-1, m-1)$, given mentioned conditions,

$$c = F(12, 10; 0.05) = 2.91.$$