Math 472 Homework Assignment 5

Problem 5.1.1. Let $\{a_n\}$ be a sequence of real numbers. Hence, we can also say that $\{a_n\}$ is a sequence of constant (degenerate) random variables. Let a be a real number. Show that $a_n \to a$ is equivalent to $a_n \xrightarrow{P} a$.

Solution 5.1.1. Recall that $a_n \to a$ means that for each $\epsilon > 0$ there is a positive integer N such that $n > N$ implies $|a_n - a| < \epsilon$ or, equivalently, $|a_n - a| \geq \epsilon$ implies $n \leq N$. Said another way, for each $\epsilon > 0$ only finitely many terms of the sequence $\{a_n\}$ satisfy the condition $|a_n - a| \geq \epsilon$.

Let $\epsilon > 0$ be given. Then for each positive integer n, $P(|a_n - a| \geq \epsilon) = 1$ if and only if $|a_n - a| \geq \epsilon$ and $P(|a_n - a| \geq \epsilon) = 0$ if and only if $|a_n - a| < \epsilon$.

By definition, $a_n \xrightarrow{P} a$ if and only if $\lim_{n\to\infty} P(|a_n - a| \geq \epsilon) = 0$. Since the only possible values of $P(|a_n - a| \geq \epsilon)$ are 0 and 1, the limit is 0 if and only if at most finitely many of these probabilities are equal to 1, and this is equivalent to the condition that only finitely many terms of the sequence $\{a_n\}$ satisfy $|a_n - a| \geq \epsilon$. By definition, this means that $a_n \to a$. Thus, $a_n \xrightarrow{P} a$ is equivalent to $a_n \to a$.

Problem 5.1.2. Let the random variable Y_n have a distribution that is $b(n, p)$.

(a) Prove that Y_n/n converges in probability to p. This result is one form of the weak law of large numbers.
(b) Prove that $1 - Y_n/n$ converges in probability to $1 - p$.
(c) Prove that $(Y_n/n)(1 - Y_n/n)$ converges in probability to $p(1 - p)$.

Solution 5.1.2.

(a) Let $X_1, \ldots, X_n$ be iid random variables where the common distribution is a Bernoulli distribution with parameter p. We know that the expected value of the Bernoulli distribution is p and the variance of a Bernoulli distribution is $p(1 - p)$, which is finite. Therefore, by the weak law of large numbers, $\overline{X}_n \xrightarrow{P} p$. Since $\sum_{i=1}^n X_i$ has a $b(n, p)$ distribution, which is the same distribution as Y_n, we see that Y_n/n has the same distribution as $\overline{X}_n = \sum_{i=1}^n X_i/n$. Therefore $Y_n/n \xrightarrow{P} p$.

(b) Let $g(x) = 1-x$. Since g is a continuous function and since $Y_n/n \xrightarrow{P} p$, Theorem 5.1.4 shows that $1 - Y_n/n = g(Y_n/n) \xrightarrow{P} g(p) = 1 - p$.

(c) Since $Y_n/n \xrightarrow{P} p$ and $1 - Y_n/n \xrightarrow{P} 1 - p$, Theorem 5.1.5 shows that $(Y_n/n)(1 - Y_n/n) \xrightarrow{P} p(1 - p)$.

Problem 5.1.3. Let W_n denote a random variable with mean μ and variance b/n^p, where $p > 0$, μ, and b are constants (not functions of n). Prove that W_n converges in probability to μ. *Hint:* Use Chebyshev's inequality.

Solution 5.1.3. Let $\epsilon > 0$ be given. By Chebyshev's inequality

$$\begin{aligned} P(|W_n - \mu| \geq \epsilon) &= P(|W_n - E[W_n]| \geq \epsilon) \\ &\leq \frac{1}{\epsilon^2} \operatorname{Var}[W_n] = \frac{b}{\epsilon^2 n^p}. \end{aligned}$$

Since $p > 0$, $\lim_{n\to\infty} b/(\epsilon^2 n^p) = 0$ and therefore $\lim_{n\to\infty} P(|W_n - \mu| \geq \epsilon) = 0$, which shows that $W_n \xrightarrow{P} \mu$.

Problem 5.1.5. Let $X_1, \ldots, X_n$ be iid random variables with common pdf

$$f(x) = \begin{cases} e^{-(x-\theta)} & x > \theta, -\infty < \theta < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

This pdf is called the **shifted exponential**. Let $Y_n = \min\{X_1, \ldots, X_n\}$. Prove that $Y_n \to \theta$ in probability, by first obtaining the cdf of Y_n.

Solution 5.1.5. Let X by a random variable whose pdf is the above shifted exponential. Then for any $a > \theta$, $P(X \geq a) = \int_a^\infty e^{-(x-\theta)}\, dx = e^{-(a-\theta)}$. From this we see that

$$\begin{aligned} P(Y_n \geq a) &= P(X_1 \geq a, \ldots, X_n \geq a) \\ &= P(X_1 \geq a) \cdots P(X_n \geq a) \\ &= \left[e^{-(a-\theta)} \right]^n \\ &= e^{-n(a-\theta)}. \end{aligned}$$

Let $\epsilon > 0$ be given. Then since the support of Y_n is the interval (θ, ∞) we know $Y_n > \theta$ and

$$P(|Y_n - \theta| \geq \epsilon) = P(Y_n - \theta \geq \epsilon) = P(Y_n \geq \epsilon + \theta) = e^{-n\epsilon}.$$

Therefore $\lim_{n\to\infty} P(|Y_n - \theta| \geq \epsilon) = \lim_{n\to\infty} e^{-n\epsilon} = 0$, which shows that $Y_n \xrightarrow{P} \theta$.

Problem 5.1.6. Using the assumptions behind the confidence interval given in expression (4.2.9), show that

$$\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}} \Big/ \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \xrightarrow{P} 1.$$

Solution 5.1.6. Example 5.1.1 on page 292 shows that the sample variance S^2 converges in probability to the variance σ^2. Applying this result to each of the random variables $\overline{X}$ and $\overline{Y}$ from page 217 shows that S_1^2 converges in probability to σ_1^2 and S_2^2 converges in probability to σ_2^2. Applying Theorems 5.1.2 and 5.1.3 we conclude that

$$\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2} \xrightarrow{P} \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}. \tag{1}$$

Finally, applying Theorem 5.1.4 to (1), with $g(x) = \sqrt{x}/\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}$, proves

$$\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}} \Bigg/ \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \xrightarrow{P} 1.$$

Problem 5.1.7. For Exercise 5.1.5, obtain the mean of Y_n. Is Y_n an unbiased estimator of θ. Obtain an unbiased estimator of θ based on Y_n.

Solution 5.1.7. From problem 5.1.5 we see that $F_n(y) = P(Y_n \leq y) = 1 - e^{-n(y-\theta)}$ for $y > \theta$, zero elsewhere. A little calculation shows that

$$E[Y_n] = \int_\theta^\infty y\, F_n'(y)\, dy = \int_\theta^\infty ny\, e^{-n(y-\theta)}\, dy = \theta + \frac{1}{n}.$$

From this we see that Y_n is a biased estimator for θ, but $Y_n - 1/n$ is an unbiased estimator for θ since $E[Y_n - 1/n] = \theta + 1/n - 1/n = \theta$.