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ASSIGNMENT - I

Exercise 1 → Induced Matrix Norms

→ matrix, $A \in \mathbb{C}^{m \times m}$

→ Spectral Radius of A, $\rho_A = \max_{i=1}^{m} |\lambda_i|$.

for any eigen value λ, there is a corresponding eigen vector,

$$Ax = \lambda x \qquad \{x \rightarrow \text{eigen vector}\}.$$

$$\|Ax\| = \|\lambda x\|$$

$$\|Ax\| = |\lambda| \cdot \|x\| \qquad \{\because \lambda \text{ is a scalar}\}$$

Now, by submultiplicative property of matrix Norm, we say,

$$\|Ax\| \leq \|A\| \cdot \|x\|.$$

Therefore $|\lambda| \|x\| \leq \|A\| \|x\|$

$$|\lambda| \leq \frac{\|A\| \|x\|}{\|x\|}$$

$$|\lambda| \leq \|A\|$$

$$|\lambda|_{max} \leq \|A\|.$$

Therefore, from the above equation, we say that the spectral radius $\rho(A)$ is bounded from above by $\|A\|$.

Exercise 2 → Unitary Matrices

(a) Let Q be unitary,

$Q \in \mathbb{C}^{m \times m}$

Euclidian Norm → $\|Qx\|_2 = \|x\|_2$

Proof → $\|Qx\|_2^2 = (Qx)^* \cdot Qx$

$= x^* Q^* \cdot Q x$ [$\because Q^* \cdot Q$ → Identity Matrix]

$= x^* \cdot x$

$= \|x\|_2^2$

Therefore, $\|Qx\|_2 = \|x\|_2$ [Q.E.D].

(b) Eigen values of Q.

Let λ be the eigen values of Q.

As Q is unitary matrix, .. $\forall\, x \in \mathbb{C}^m$.

$QX = \lambda X$. ... eq$^{\underline{n}}$ (1)

→ $(QX)^* = (\lambda X)^*$

→ $X^* Q^* = X^* \bar{\lambda}$ .... eq$^{\underline{n}}$ (2)

Multiply eq$^{\underline{n}}$ (1) with eq$^{\underline{n}}$(2),

→ $X^* Q^* \cdot Q \cdot X = X^* \cdot \bar{\lambda} \cdot \lambda \cdot X$.

→ $X^* \cdot I \cdot X = X^* \cdot \bar{\lambda} \cdot \lambda \cdot X$ ... [$\because Q^* \cdot Q = 1$].

→ $I \cdot X^* \cdot X = \bar{\lambda} \cdot \lambda \cdot X^* \cdot X$.

→ $(I - \bar{\lambda} \cdot \lambda)\, X^* \cdot X = 0$.

As $X^* \cdot X$ can't be equal to 0,

So, $I - \bar{\lambda} \cdot \lambda = 0$.

$\bar{\lambda} \cdot \lambda = I$.

Let assume $\lambda = a + ib$, $\bar{\lambda} = a - ib$,

$(a+ib)(a-ib) = 1$.

$a^2 + b^2 = 1$

$|\lambda|^2 = 1$

$\lambda = 1$

(c) To find : Determinant of unitary matrix Q.

$$Q \cdot Q^* = I$$

$$|Q \cdot Q^*| = |I|$$

$$|Q| \cdot |Q^*| = 1 .$$

$$|Q| \cdot |\bar{Q}| = 1 \qquad [\because \bar{Q} = (\bar{Q})^T]$$

Let us assume Q as $(a+ib) \rightarrow \bar{Q} = (a-ib)$

$$(a+ib)(a-ib) = 1$$

$$a^2 + b^2 = 1$$

$$\therefore \quad |Q| = 1 .$$

Exercise 3 : Hermitian Matrices

Preposition : $A \in \mathbb{C}^{m \times m}$ is hermitian iff it is both unitarily diagonisable and all it's eigen values are real.

Proof : Let us assume A is a hermitian matrix -

$$A \in \mathbb{C}^{m \times m}$$

1 → Suppose, let us assume A is unitarily diagonalizable

$\rightarrow A = Q D Q^*$ [∀ Q is unitary matrix
D is diagonal matrix].

$\rightarrow A^* = Q D^* Q^*$

$\rightarrow A^* = Q \bar{D} Q^*$ [$\because D^* = \bar{D}$].

Calculation for $A^*.A$ -

$\rightarrow A^* A = Q \bar{D} Q^* \cdot Q D Q^*$

$\rightarrow A \cdot A = Q \bar{D} \cdot D \cdot Q^*$ [$\because A^* = A$ (hermitian matrix)].

Now, for calculating for $A^* A$.

$\rightarrow A \cdot A^* = Q D Q^* \cdot Q \bar{D} Q^*$

$A \cdot A = Q D \bar{D} Q^*$ [$\because A = A^*$].

$A \cdot A = Q \bar{D} \cdot D Q^*$ [$\because \bar{D} = D$ as diagonal matrix is commutative].

$\therefore A^* A = A \cdot A^*$

Therefore, A is a unitarily diagonalizable matrix.

2 → Let λ be the eigen value of the hermitian matrix A.

$A X = \lambda X$ → $X \in \mathbb{C}^m$ [Eigen vector].

Multiplying x^* on both sides -

$x^* A X = x^* \lambda X$.

$\rightarrow \lambda \|x\|^2 = x^* A \cdot X$.

$$\rightarrow \lambda \|x\|^2 = x^* A^* x \qquad [\text{since } A = A^*].$$

$$= (Ax)^* x \qquad [\because x^* A^* = (Ax)^*].$$

$$= (\lambda x)^* x \qquad [\because Ax = \lambda x].$$

$$= x^* \lambda^* x$$

$$= \lambda^* \|x\|^2.$$

This implies, $\lambda \|x\|^2 = \lambda^* \|x\|^2$

$\therefore \lambda = \lambda^*$ for $\|x\| \neq 0$.

Therefore, we can state that for $A \in \mathbb{C}^{m \times m}$ is hermitian iff it is unitarily diagonalizable and all it's eigen values are real.

Exercise 4 : Rank one Perturbations.

for $u, v \in \mathbb{C}^m \setminus \{0\}$,

matrix $A = I + uv^*$ — Rank one pertubation of the Identity.

(a) To prove : under which condition on u and v, A is singular.

To find : null (A).

Proof : Let us assume 'A' as a singular matrix such that,

$Ax = 0$, for some non-zero vector space $x \in \mathbb{C}^m \setminus \{0\}$.

$\rightarrow (I + uv^*) x = 0$ ... eq^n (4-1).

$\rightarrow x + uv^* x = 0$.

$\rightarrow x = -uv^* x$

$\therefore$ 'x' is a multiple of u.

$\rightarrow x = C_1 u$ [$C_1 \rightarrow$ non zero scalar value]

Now, substituting the value of x in equation (4.1), we get,

$\rightarrow (I + uv^*) C_1 u = 0$

$\rightarrow I + uv^* = 0$

$v^* u = -1$

$\rightarrow$

$v^* u \neq 0$.

$\therefore$ Matrix A is singular as

$\therefore$ Null (A) $= \{ C_1 u, \ C_1 \in \mathbb{C}$ and $u \neq 0\}$.

$\rightarrow$ Null (A) $= \text{span}\{u\}$

Therefore, $A(C_1 u) = 0$.

(b) Now, for A is a non-singular matrix.

and $A^{-1} = I + \alpha uv^*$.

To find: The value of α

Solution : Now, $AA^{-1} = I$.

$(I + uv^*)(I + \alpha uv^*) = I$.

$\rightarrow \cancel{I} + uv^* + \alpha uv^* + uv^* \alpha uv^* = \cancel{I}$.

Now, $uv^* (1 + \alpha + \alpha . v^* u) = 0$.

If $uv^* = 0$, $A = I$ (This is a normal or trivial case)

[Hence, $(1 + \alpha + \alpha \cdot v^* u)$ should be equal to zero],

$$1 + \alpha + \alpha . v^* u = 0$$

$$\alpha . (1 + v^* u) = -1$$

$$\alpha = -(1 + v^* u)^{-1}.$$

4. c. $A = I + w u u^*$

$A^* = (I + w u u^*)^*$

$= I + u u^* w^*$.

$A^* A = I$ [for A is unitary].

$\rightarrow (I + u u^* w^*)(I + w u u^*) = I$.

$\rightarrow \cancel{I} + u u^* w^* + w u u^* + u u^* w^* w u u^* = \cancel{I}$

$\rightarrow u u^* (w^* + w + w^* . w . u^* u) = 0$.

$\rightarrow$ since, $u u^* \neq 0$ as $u u^* \in \mathbb{C}^{m \times m}$.

Therefore, $w^* + w + w^* . w . u^* . u = 0$.

Now, $w \in \mathbb{C}$,

So, Let us assume w as a complex number $x + iy$.

$\rightarrow w^* = x - iy$.

$\rightarrow x - iy + x + iy + (x - iy)(x + iy) u^* . u = 0$.

we Know that, $u \in \mathbb{C}^m \setminus \{0\}$.

so, $u^* . u$ is a scalar value,

$\therefore u^* . u = s$ ... (Assume)

$\rightarrow x - \cancel{iy} + x + \cancel{iy} + (x - iy)(x + iy) s = 0$.

$\rightarrow 2x + (x^2 + y^2) s = 0$.

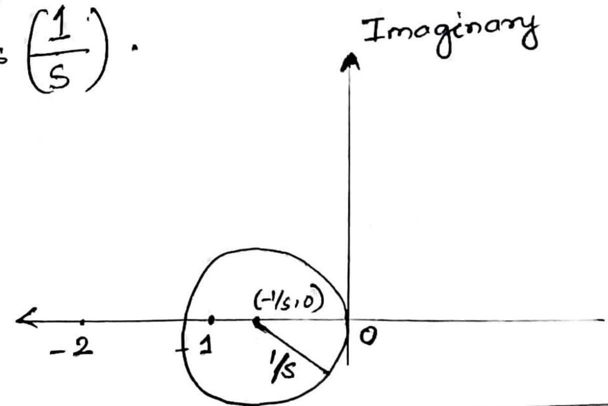
$\rightarrow x^2 + y^2 = -\frac{2x}{s}$.

$\rightarrow x^2 + y^2 + \frac{2x}{s} = 0$.

$\rightarrow x^2 + 2 . x . \frac{1}{s} + \left(\frac{1}{s}\right)^2 - \left(\frac{1}{s}\right)^2 + y^2 = 0$.

$\rightarrow \left(x + \frac{1}{s}\right)^2 + y^2 = \left(\frac{1}{s}\right)^2$.

$\therefore$ This gives the equation of a circle whose center is at $\left(-\frac{1}{s}, 0\right)$ and Radius is $\left(\frac{1}{s}\right)$.



(d) since, $A^* = I + uu^* w^*$.

$\rightarrow I + \alpha uv^* = I + uu^* w^*$.

$\rightarrow \quad \alpha uv^* = uu^* w^*$.

$\rightarrow$ As u is a scalar Quantity so let us assume, $u = vx$.

$\rightarrow \quad \alpha vx \cdot v^* = (vx)(vx)^* w^*$

$\rightarrow \quad \alpha vx \cdot v^* - vx \cdot x^* v^* w^* = 0$.

$\rightarrow \quad vx\,[\alpha v^* - x^* v^* w^*] = 0$.

since, $vx \neq 0$ as $u \neq 0$,

$\rightarrow \quad \alpha v^* = x^* w^* v^*$.

$\rightarrow \quad \alpha = x^* \cdot w^*$.

since, $w \in \mathbb{C}$, so,

α is also a complex number.

$\alpha \in \mathbb{C}$.

Exercise 5:

$$A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}.$$

We know that, SVD is $A = U \Sigma V^*$

$$\therefore A^T A = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\det |A^T A - \lambda I| = \det \begin{vmatrix} 9-\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} = 0.$$

$$(9-\lambda)(4-\lambda) = 0$$

$$\lambda = 9, 4.$$

$\therefore$ Eigen Values $\rightarrow \lambda_1 = 9, \quad \lambda_2 = 4.$

$\rightarrow$ Singular values $\rightarrow \sigma_1 = 3, \quad \sigma_2 = 2.$

$$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \qquad |\Sigma| = 6$$

$$\Sigma^{-1} = \frac{1}{6} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}$$

Now, for $\lambda = 9$,

$$\rightarrow A^T A - 9I = \begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix}, \text{ eigen vector, } V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

for $\lambda = 4$,

$$A^T . A - 4I = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

$$U = A.V.\Sigma^{-1}.$$

$$\rightarrow U = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

$$\rightarrow U = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$\rightarrow U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Since, $A = U \Sigma V^*$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \rightarrow B^T = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\therefore B^T \cdot B = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}.$$

$$\det |B^T \cdot B - \lambda I|$$

$$\rightarrow \det \begin{vmatrix} 4-\lambda & 0 \\ 0 & 9-\lambda \end{vmatrix} = 0$$

$$\rightarrow (4-\lambda)(9-\lambda) = 0$$

$$\lambda = 4, 9.$$

Eigen values, $\lambda_1 = 4$; $\lambda_2 = 9$.

Singular values → $\sigma_1 = 2$, $\sigma_2 = 3$.

$$\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\Sigma^{-1} = \frac{1}{6} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

Now, for $\lambda = 4$,

$$B^T \cdot B - 4I = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$, Eigen vector, $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

For $\lambda = 9$,

$$B^T \cdot B - 9I = \begin{bmatrix} -5 & 0 \\ 0 & 0 \end{bmatrix}$$, Eigen vector $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$U = BV\Sigma^{-1}.$$

$$\rightarrow U = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B = U \Sigma V^*$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. $\rightarrow$ $C^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$$C = U \Sigma V^*$$

$U \rightarrow$ eigen vectors of $C \cdot C^T$

$$C \cdot C^T = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

eigen values $\rightarrow$ $|C \cdot C^T - \lambda I| = 0$

$$\rightarrow \begin{vmatrix} 2-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0 .$$

$$\lambda = 2, 0 .$$

Non-zero eigen value $= 2$.

singular value $\rightarrow$ $\sigma_1 = \sqrt{2}$.

singular matrix $\rightarrow$ $\Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$

for U,

case I $\rightarrow$ for $\lambda = 2$,

$$[C \cdot C^T - 2I] \quad (X_1) = 0 .$$

$$\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 .$$

$$-2y = 0 \rightarrow y = 0 , \quad x = 1 .$$

$$X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

case II $\rightarrow$ for $\lambda = 0$,

$$[C \cdot C^T - 0 \cdot I] \; (X_2) = 0 .$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 .$$

$$2x = 0, \quad x = 0, \; y = 1 .$$

$$X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

$$U = [X_1 \cdot X_2] .$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

$V \rightarrow$ Eigen vectors of $C^T \cdot C$.

$$C^T \cdot C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Eigen values $\rightarrow |C^T \cdot C - \lambda I| = 0$.

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0.$$

$$(1-\lambda)^2 - 1 = 0.$$

$$\lambda^2 - 2\lambda = 0.$$

$$\lambda = 2, 0.$$

Case I $\rightarrow$ for $\lambda = 2$,

$$(C^T \cdot C - 2I)(Y_1) = 0$$

$\rightarrow$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

$$-x + y = 0.$$

$$x = y$$

$Y_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow$ unit Eigen vector, $Y_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

Case II $\rightarrow$ for $\lambda = 0$,

$$(C^T \cdot C - 0 \cdot I) \quad (Y_2) = 0.$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

$$x + y = 0.$$

$$x = -y.$$

$Y_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow$ unit Eigen vector, $Y_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

$$V = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

$$C = U \Sigma V^T$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$