NAME: ABHIK SARKAR
MATRIKELNUMMER: 23149662
IDM Id : lo88xide
EMAIL: abhik.sarkar@fau.de / abhik.sarkar.718@gmail.com

ANLA ASSIGNMENT - 3

Exercise 1 - Big-O Notation

(a) $\sin x = O(1)$ as $x \to \infty$

We can say, $|\sin x| \leq 1 \quad \forall \, x \in \mathbb{R}$

$\rightarrow \quad |\sin x| \leq 1 \quad \forall \, x \geq 0$

$\therefore \; |\sin x| \leq c \cdot (1) \quad \forall \, x \geq t_0$ where $c = 1$ and $t_0 = 0$.

— TRUE

(b) $\sin x = O(1)$ as $x \to 0$

$\rightarrow |\sin x| \leq 1 \quad \forall \, x \in [0, \pi/2]$

$\rightarrow |\sin x| \leq c \cdot (1) \quad \forall \, x \leq t_0$ where $c = 1$ and $t_0 = \pi/2$.

— TRUE

(c) $\ln x = O(x^{1/100})$ as $x \to \infty$.

$\rightarrow |\ln x| \leq c \cdot x^{1/100}$

$\rightarrow \frac{1}{c} \cdot |\ln x| \leq x^{1/100}$.

$\rightarrow |\ln x^{1/c}| \leq x^{1/100}$.

Let us assume $C = 100$,

$\rightarrow |\ln x^{1/100}| \leq x^{1/100}$.

Since, we know that $(|\ln K| \leq K \quad \forall\ K \geq 1)$.

$\therefore |\ln x| \leq c \cdot x^{1/100} \quad \forall\ x \geq t_0$ where $C = 100$ and $t_0 = 1$.

— TRUE

(d) $n! = O\left(\left(\frac{n}{e}\right)^n\right)$ as $n \to \infty$ (Hint: Stirling's Approximation).

$\rightarrow n! \leq c \cdot \left(\left(\frac{n}{e}\right)^n\right)$.

We know from Stirling's Equation :

$$e^{\frac{1}{12n+1}} \leq \frac{n!}{\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n} \leq e^{\frac{1}{12n}}.$$

$$\rightarrow \frac{n!}{\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n} \leq e^{\frac{1}{12n}}.$$

$$\rightarrow n! \leq \underbrace{e^{\frac{1}{12n}} \cdot \sqrt{2\pi n}}_{c} \cdot \underbrace{\left(\frac{n}{e}\right)^n}_{g(x)}.$$

Here, $c = e^{1/12n} \cdot \sqrt{2\pi n}$

$\rightarrow$ c is directly proportional to n, so, c is not a constant. $c \longrightarrow \infty$ when $n \rightarrow \infty$

— FALSE

(e) $fl(\pi) - \pi = O(\varepsilon_m)$ as $\varepsilon_m \rightarrow 0$

$\rightarrow \pi(1+\varepsilon) - \pi \leq c(\varepsilon_m).$

$\rightarrow \pi\varepsilon \leq c(\varepsilon_m).$ $\quad (\because \varepsilon < \varepsilon_m)$

$\rightarrow$ $c = \pi$. will satisfy the above equation where $\varepsilon < \varepsilon_m$. where thresold $t_0 = 100$.

— TRUE

(f) $fl(n\pi) - n\pi = O(\varepsilon_m)$ as $\varepsilon_m \to 0$ [uniformly for all integers n]

$\rightarrow n\pi(1+\varepsilon) - n\pi \leq c \cdot \varepsilon_m$.

$\rightarrow n\pi\varepsilon \leq c \cdot \varepsilon_m$. $\quad [\because \varepsilon < \varepsilon_m]$.

$\rightarrow$ $c = n\pi$ will satisfy the above equation.

$\rightarrow$ Here c is directly proportional to the variable n, so, c is not a constant.

$\therefore c \to \infty$ when $n \to \infty$

— FALSE.

Exercise 2 — Stability:

(a) Data : $x \in \mathbb{C}$. Solution: $2x$, computed as $x \oplus x$.

$$\tilde{f}(x) = x \oplus x$$
$$= fl(x) + fl(x)$$
$$= [x(1+\varepsilon) + x(1+\varepsilon)](1+\varepsilon_0)$$
$$= [2x(1+\varepsilon)](1+\varepsilon_0) \qquad [\text{As } \varepsilon_0 \ll 1 \text{ so, } 1+\varepsilon_0 \simeq 1].$$
$$= 2\tilde{x} \qquad [\because \tilde{x} = x \cdot (1+\varepsilon)]$$

Now, checking for accuracy :

Condition to satisfy the accuracy is :

$$\rightarrow \quad \frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} = O(\varepsilon_{machine})$$

$$\text{LHS} = \frac{\| 2x(1+\varepsilon) - (x+x) \|}{\|(x+x)\|}$$

$$= \frac{\| 2x + 2x\varepsilon - 2x \|}{\| 2x \|}$$

$$= \frac{\| 2x\varepsilon \|}{\| 2x \|} = \|\varepsilon\| = O(\varepsilon_{machine}).$$

$\therefore$ The above algorithm is accurate.

Now, checking for backward stability :

Condition to be satisfied for backward stability,

$\tilde{f}(x) = f(\tilde{x})$ for some $\tilde{x}$ with $\frac{\|\tilde{x} - x\|}{\|x\|} = O(\varepsilon_{machine})$

$$\text{LHS} \rightarrow \tilde{f}(x) = 2\tilde{x}.$$

$$\text{RHS} \rightarrow f(\tilde{x}) = \tilde{x} + \tilde{x} = 2\tilde{x}.$$

$$\because \tilde{f}(x) = f(\tilde{x})$$

$\therefore$ The above algorithm is backward stable.

(b) Data : $x \in \mathbb{C}$. Solution : x^2, computed as $x \otimes x$.

$$\tilde{f}(x) = x \otimes x$$
$$= fl(x) \otimes fl(x)$$
$$= x(1+\varepsilon) \otimes x(1+\varepsilon).$$
$$= [x(1+\varepsilon)]^2 \cdot (1+\varepsilon_0) \quad [\text{As } \varepsilon_0 \text{ is very small, } 1+\varepsilon_0 \approx 1]$$
$$= (\tilde{x})^2. \quad [\because \tilde{x} = x(1+\varepsilon)].$$

First, we are checking for the accuracy :

Condition to satisfy accuracy,

$$\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} = O(\varepsilon_{machine})$$

$$\text{LHS} = \frac{\| [x(1+\varepsilon)]^2 - (x \times x) \|}{\|(x \times x)\|}$$
$$= \frac{\| x^2 + x^2\varepsilon^2 + 2\varepsilon x^2 - x^2 \|}{\| x^2 \|}$$
$$= | \varepsilon^2 - 2\varepsilon x^2 |$$

$$= |\varepsilon(\varepsilon - 2x^2)| = O(\varepsilon_{machine}).$$

$\therefore$ The above algorithm is accurate.

Now, we are checking for backward stability :

Condition to be satisfied for backward stability,

$$\tilde{f}(x) = f(\tilde{x}) \text{ for some } \tilde{x} \text{ with } \frac{\|\tilde{x} - x\|}{\|x\|} = O(\varepsilon_{machine})$$

LHS $\rightarrow$ $\tilde{f}(x) = (\tilde{x})^2.$

RHS $\rightarrow$ $f(\tilde{x}) = (\tilde{x} \times \tilde{x}) = (\tilde{x})^2$

$$\because \tilde{f}(x) = f(\tilde{x})$$

$\therefore$ The above algorithm is backward stable

(e) Data : $x \in \mathbb{C} \setminus \{0\}$. Solution : 1, computed as $x \oslash x$.

$$\tilde{f}(x) = x \oslash x.$$
$$= fl(x) \, / \, fl(x).$$

$$= 1 (1 + \varepsilon_0)$$

Now, we are checking for accuracy,

Condition to satisfy accuracy,

$$\rightarrow \frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} = O(\varepsilon_{machine})$$

$$\text{LHS} \rightarrow \frac{\|1 + \varepsilon_0 - (x/x)\|}{\|(x/x)\|}$$

$$\rightarrow \frac{|1 + \varepsilon_0 - 1|}{|1|}$$

$$\rightarrow \varepsilon_0 = O(\varepsilon_{machine}).$$

∴ The above algorithm is accurate.

Now, we are checking for backward stability

Condition to be satisfied for backward stability,

$\tilde{f}(x) = f(\tilde{x})$ for some $\tilde{x}$ with $\dfrac{\|\tilde{x} - x\|}{\|x\|} = O(\varepsilon_{machine})$

$$\text{LHS} \rightarrow \tilde{f}(x) = 1 + \varepsilon_0.$$

RHS $\rightarrow$ $f(\tilde{x}) = \tilde{x} / \tilde{x} = 1.$

$$\because \quad \tilde{f}(x) \neq f(\tilde{x})$$

$\therefore$ The above algorithm is not backward stable.

Now, let us check for Stability,

condition to be satisfied for Stability,

$$\frac{\| \tilde{f}(x) - f(\tilde{x}) \|}{\| f(\tilde{x}) \|} = O(\varepsilon_{machine}).$$

$$\text{LHS} = \frac{\| 1 + \varepsilon_0 - 1 \|}{\| 1 \|}$$

$$= | \varepsilon_0 | = O(\varepsilon_{machine}).$$

$\therefore$ The above algorithm is stable.

(d) Data : $x \in \mathbb{C}$. Solution $= 0$, computed as $x \ominus x$.

$$\tilde{f}(x) = x \ominus x.$$

$$= fl(x) - fl(x)$$
$$= [x(1+\varepsilon) - x(1+\varepsilon)](1-\varepsilon_0)$$
$$= 0$$

Now, we are checking for accuracy,
Condition to satisfy accuracy,

$$\rightarrow \frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} = O(\varepsilon_{machine})$$

$$\rightarrow \frac{\|0 - (x-x)\|}{\|x-x\|}$$

$$\rightarrow \frac{0}{0} \neq O(\varepsilon_{machine}).$$

$\therefore$ The above algorithm is not accurate.

Now, we are checking for backward stability

Condition to be satisfied for backward stability,

$\tilde{f}(x) = f(\tilde{x})$ for some $\tilde{x}$ with $\frac{\|\tilde{x} - x\|}{\|x\|} = O(\varepsilon_{machine})$

LHS $\rightarrow \tilde{f}(x) = 0$

RHS $\rightarrow$ $f(\tilde{x}) = \tilde{x} - \tilde{x} = 0$

$\therefore$ $\tilde{f}(x) = f(\tilde{x})$.

$\therefore$ The above algorithm is backward stable.

Exercise 3 – Least Squares:

Given : $A \in \mathbb{C}^{m \times n}$, rank $= n \rightarrow m > n$

$b \in \mathbb{C}^m$

$$\begin{pmatrix} I & A \\ A^* & 0 \end{pmatrix} \begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}.$$

$I \rightarrow m \times m$ identity.

To show : System has unique solution $(r, x)^T$ and vectors r & x are residual and the solution for least square problem.

$$\begin{pmatrix} I & A \\ A^* & 0 \end{pmatrix} \begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} Ir + Ax \\ A^* r + 0 \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}.$$

$\rightarrow$ $r + Ax = b$, $A^* r = 0$

Now, by finding the determinant of the matrix, we could say,

$$\begin{vmatrix} I & A \\ A^* & 0 \end{vmatrix} = |I \times 0 - A^*A| = A^*A \neq 0$$

$\because$ A has full rank, so A^*A is non-singular and cannot be equal to 0.

$\therefore$ As A^*A is non-singular and A has full rank, so, it has unique solutions such as x and residual r. where $r = b - Ax$.

Exercise 4 – Reverse Engineering

```
import numpy as np

def magic (A):                                                        (A = U·S·V^T)
    U, S, V = np.linalg.svd (A) #calculating Singular Value Decomposition
    eps = np.spacing (1). # calculating machine epsilon
    tol = max(np.shape (A)) * S[0] * eps # max(m:n) * σ1 * eps (calculated abv.)
    r = sum (S > tol). # Calculating the rank
    S = np.diag (np.ones (r) / S[0 : r]) #calculating inverse of S.
    X = np.dot (V.transpose () [:, 0 : r], np.dot (S, U[:, 0:r].transpose []))
        #Calculating the value of A^-1 = (V. S^-1. U^T).
```

```
return R.  # Returning the A^-1 value as X.
```

Therefore, the above function magic(A) computes the pseudo inverse of matrix A of shape (mxn) through Singular Value decomposition.