Assn Assignment - 2
Mathematics - II

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SOLO ASSIGNMENT - 2

Exercise 1. Singular Value Decomposition.

Given the spectral decomposition of positive definite matrix $A \in \mathbb{C}^{n \times n}$ then $\exists$ an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ of $\mathbb{C}^n$ where v_i is an eigen-vector corresponding to λ_i respectively.

To prove: Singular decomposition is unique when descending λ_i's are used.

Proof by Induction:
Base case: $n = 1$

$\Rightarrow$ A has just a number, $a \in \mathbb{C}^{1 \times 1}$ where

$a = \lambda_1 \cdot 1 \cdot 1 \cdot \bar{1} = \lambda_1$ and $\lambda_1 > 0$ (positive definite)

$\Rightarrow$ $\lambda_1 = \lambda_1 \cdot 1 \cdot 1 \cdot \bar{1}$ is the only decomposition given

$\Rightarrow$ $\lambda_1 = \lambda_1 \cdot \begin{bmatrix} 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \end{bmatrix} \cdot \begin{bmatrix} \bar{1} \end{bmatrix}$ and $\lambda_1 > 0$

Since the above equation, λ_1 can have at most one positive & negative value, but, we prefered a quotient, λ_1 there is positive and which remains the decomposition is unique.

Now for induction step we assume the decomposition is unique. Let $n \in \mathbb{N}$

$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_n \end{bmatrix}$ where $\lambda_1 > \lambda_2 > \cdots > \lambda_n$

In the next step we will show that the uniqueness of the decomposition was valid for $n+1$.

$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_{n+1} \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{n+1} \end{bmatrix} \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_{n+1} \end{bmatrix}$

Now, $\lambda_1 > \lambda_2 > \cdots > \lambda_{n+1}$ is dependent on the λ_i's in n uniquely.

$\Rightarrow \lambda_1 > \lambda_2 > \cdots > \lambda_n$

$\Rightarrow \lambda_{n+1} < \lambda_n$

Now, λ_{n+1} is dependent on the and the λ_i's in n uniquely.

Now, $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_{n+1} \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{n+1} \end{bmatrix} \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_{n+1} \end{bmatrix}$

Now as n is induction and positive definite,

$\Rightarrow \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_n \end{bmatrix}$ is unique

$\Rightarrow \begin{bmatrix} v_1 & v_2 & \cdots & v_n & v_{n+1} \end{bmatrix} \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_n & \\ & & & & \lambda_{n+1} \end{bmatrix} \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_n \\ \bar{v}_{n+1} \end{bmatrix}$ is unique

$\Rightarrow A = \begin{bmatrix} v_1 & v_2 & \cdots & v_{n+1} \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{n+1} \end{bmatrix} \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_{n+1} \end{bmatrix}$ is unique

Now, $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_{n+1} \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{n+1} \end{bmatrix} \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_{n+1} \end{bmatrix}$

$\Rightarrow \begin{bmatrix} v_1 & v_2 & \cdots & v_{n+1} \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{n+1} \end{bmatrix} \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_{n+1} \end{bmatrix}$ is unique quantity

$\Rightarrow \begin{bmatrix} v_1 & v_2 & \cdots & v_{n+1} \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{n+1} \end{bmatrix} \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_{n+1} \end{bmatrix}$ is unique (quantity)

$\Rightarrow A = \begin{bmatrix} v_1 & v_2 & \cdots & v_{n+1} \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{n+1} \end{bmatrix} \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_{n+1} \end{bmatrix}$ is unique (quantity)

Exercise 2. Eigen Value Properties.

Given $A \in \mathbb{C}^{n \times n}$ and $\lambda \in \mathbb{C}$ is an eigen value of A then $\exists v \in \mathbb{C}^n$.

To Show statement is false.

Let us consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (1.1.1)

Let the eigenvalue corresponding eigenvector be $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (1.1.2)

$\Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for $\lambda = 1$

Eigen value is only 1 and not 0.

(a) If A is real and λ is an eigen value of A then $\lambda \in \mathbb{R}$.

Approach 1: Let v be an eigen vector corresponding to eigen value λ.

$\Rightarrow Av = \lambda v$ (1.1.3)

$\Rightarrow \bar{A}\bar{v} = \bar{\lambda}\bar{v}$ (Taking conjugate)

$\Rightarrow A = \bar{A}$ real matrix, so $\bar{A} = A$

$\Rightarrow Av = \bar{\lambda}\bar{v}$ (1.1.4)

From the above equations we see that

$\bar{\lambda}$ is an eigen value of A corresponding to eigen vector $\bar{v}$.

Statement is true.

Approach 2: Let λ be an characteristic polynomial root of A.

$\Rightarrow$ Roots of characteristic polynomial are the eigenvalues of A. (1.1.5)

$\Rightarrow$ A is real matrix, characteristic polynomial is real and a polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ (1.1.6)

Then given

$f(\lambda) = 0$ (1.1.7)

$\Rightarrow \bar{f}(\bar{\lambda}) = 0$ (1.1.8)

$\Rightarrow \bar{\lambda}$ is also a root of $f(x)$ hence $\bar{\lambda}$ is an eigenvalue of A.

Statement is true.

(b) If λ is an eigen value of A and B is non-singular then λ is an eigen value of $B^{-1}AB$.

Proof: Let λ be an eigen value corresponding to eigen vector v.

$\Rightarrow Av = \lambda v$ (1.1.9)

Multiplying B^{-1} on both sides

$B^{-1}Av = B^{-1}\lambda v$

Multiplying B on both sides to get (1.1.10)

$\Rightarrow BB^{-1}Av = B\lambda v$

$\Rightarrow Av = \lambda Bv$

$\Rightarrow \lambda Bv = Av$ [Multiplying B^{-1}] (1.1.11)

$\Rightarrow \lambda Bv = BAv$

$\Rightarrow \lambda v = BAv$

$\Rightarrow \lambda v = BAB^{-1}Bv$

$\Rightarrow \lambda v = BAB^{-1}v$ [Multiplying B^{-1}] (1.1.12)

Eigen values of $B^{-1}AB$ are λ.

Eigenvector corresponding to Bv. (1.1.13)

The above statement is true, λ is an eigen value of A.

(c) If λ is invertible and λ is an eigen value of A then λ^{-1} is an eigen value of A^{-1}.

Proof: Since A is invertible, $\det A \neq 0$.

Let λ be an eigen value corresponding to eigen vector v.

$\Rightarrow Av = \lambda v$ (1.1.14)

Multiplying A^{-1} on both sides

$A^{-1}Av = A^{-1}\lambda v$

$\Rightarrow v = \lambda A^{-1}v$

$\Rightarrow \lambda^{-1}v = A^{-1}v$ [Since $\lambda \neq 0$] (1.1.15)

$\Rightarrow \lambda^{-1}v = A^{-1}v$ [Since $\det A \neq 0$]

Eigen values of A^{-1} are λ^{-1}.

Eigenvector remains v. (1.1.16)

The above statement is true, λ^{-1} is an eigenvalue of A^{-1}.

(d) If λ is an eigen value of A then λ^2 is an eigen value of A^2.

Proof: Let λ be an eigen value corresponding to eigen vector v.

$\Rightarrow Av = \lambda v$ (1.1.17)

Now for $\det(A^2 - \lambda^2 I)$

$= (A - \lambda I)(A + \lambda I)$

$= (A - \lambda I)(A - (-\lambda)I)$

$= \det(A - \lambda I)\det(A - (-\lambda)I)$

$= \det(A - \lambda I)\det(A - \lambda I)$

$= (\det(A - \lambda I))^2$

$\Rightarrow \det(A^2 - \lambda^2 I) = 0$ (1.1.18)

Since all the eigen values of A are equal, so we can say that the eigen value for the new matrix A^2 has eigenvalues λ^2.

$\Rightarrow \lambda = \pm\lambda$

Considering value of λ to equation (1.1.17), we get,

$Av = \pm\lambda v$

$\Rightarrow Av = \pm(Av)$

$\Rightarrow Av = Av$ (Negative having λ is the root which was the eigenvalue)

The above statement is true.

(e) The absolute value of the determinant of A is the product of its eigen values.

$|\det(A)| = \prod_{i=1}^{n} \lambda_i$

Let us consider the matrix A has $\det$ as

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (1.1.19)

Now finding determinant we both sides we get,

$\det(A) = ad - bc$

$\Rightarrow |\det(A)| = |ad - bc|$

$\Rightarrow$ A is an orthonormal matrix.

$\Rightarrow |\det(A)| \text{ and } |\det(A)|$

$\Rightarrow |\det(A)| = |\det(A)| = \prod_{i=1}^{n} \lambda_i$

As A is a diagonal matrix of an eigenvalue matrix, so the determinant is

$\det A = a - b = \lambda_1$

$\det A = c - d = \lambda_2$

$\Rightarrow |\det(A)| = \lambda_1 - \lambda_2$

The above statement is true.

Exercise 3. Quadratic Forms.

Theorem: $A \in \mathbb{C}^{n \times n}$ Every eigenvalue of A due to it and Hermitian and by the n distinct vectors in the complex plane with unitary λ and the vector $\{v_i\}$ Eigenvectors λ_i is about the sum of λ_i is then zero. If λ_i is about from the column of the matrix A is zero, then the matrix is equivalent to a diagonal matrix.

Proof: Let λ be an eigenvalue of Hermitian.

$\Rightarrow Av = \lambda v$ (1.1.20)

Now $\langle Av, v \rangle = \langle \lambda v, v \rangle$ (1.1.21)

Since $\langle Av, v \rangle$ is real, so $\langle \lambda v, v \rangle$ is real and $\langle Av, v \rangle \in \mathbb{R}$.

Now we know the λ is equivalent to the above equation

Hermitian matrix λ is equivalent to

$\Rightarrow \langle \lambda v, v \rangle = \lambda$ (1.1.22)

Hence, the above λ is real.

Now for the eigenvalue λ and μ are distinct, we can show that the eigenvalues for the real matrix A has the eigenvalues λ and μ.

$\Rightarrow \lambda = \pm\lambda$

Considering value of λ to equation (1.1.17), we get,

$Av = \pm\lambda v$

$\Rightarrow Av = \pm(Av)$

$\Rightarrow Av = Av$ (Negative having λ is the root which was the eigenvalue)

The above statement is true.

(a) The absolute value of the determinant of A is the product of its eigen values.

$|\det(A)| = \prod_{i=1}^{n} \lambda_i$

Let us consider the matrix A has $\det$ as

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (1.1.23)

Now finding determinant we both sides we get,

$\det(A) = ad - bc$

$\Rightarrow |\det(A)| = |ad - bc|$

$\Rightarrow$ A is an orthonormal matrix.

$\Rightarrow |\det(A)| \text{ and } |\det(A)|$

$\Rightarrow |\det(A)| = |\det(A)| = \prod_{i=1}^{n} \lambda_i$

As A is a diagonal matrix of an eigenvalue matrix, so the determinant is

$\det A = a - b = \lambda_1$

$\det A = c - d = \lambda_2$

$\Rightarrow |\det(A)| = \lambda_1 - \lambda_2$

The above statement is true.

Exercise 4. Quadratic Forms.

Theorem: $A \in \mathbb{C}^{n \times n}$ Every eigenvalue of A due to it and Hermitian and by the n distinct vectors in the complex plane with unitary λ and the vector $\{v_i\}$ Eigenvectors λ_i is about the sum of λ_i is then zero. If λ_i is about from the column of the matrix A is zero, then the matrix is equivalent to a diagonal matrix.

Proof: Let λ be an eigenvalue of Hermitian.

$\Rightarrow Av = \lambda v$ (1.1.24)

Now $\langle Av, v \rangle = \langle \lambda v, v \rangle$ (1.1.25)

Since $\langle Av, v \rangle$ is real, so $\langle \lambda v, v \rangle$ is real and $\langle Av, v \rangle \in \mathbb{R}$.

Now we know the λ is equivalent to the above equation

Hermitian matrix λ is equivalent to

$\Rightarrow \langle \lambda v, v \rangle = \lambda$ (1.1.26)

Hence, the above λ is real.

Now for the eigenvalue λ and μ are distinct, we can show that the eigenvalues for the real matrix A has the eigenvalues λ and μ.

$\Rightarrow \lambda = \pm\lambda$

Considering value of λ to equation (1.1.17), we get,

$Av = \pm\lambda v$

$\Rightarrow Av = \pm(Av)$

$\Rightarrow Av = Av$ (Negative having λ is the root which was the eigenvalue)

The above statement is true.

Exercise 5. Quadratic Forms.

Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$ (1.1.27)

Input A Real Symmetric

Solution:

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -5 \\ 3 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -5 \\ 0 & -5 & -9 \end{bmatrix}$ (1.1.28)

Characteristic polynomial

$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -\lambda & 1 \\ 3 & 1 & -\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & -4-\lambda & -5 \\ 0 & -5 & -9-\lambda \end{vmatrix}$ (1.1.29)

$\Rightarrow \det(A - \lambda I) = (1-\lambda)(-\lambda(-\lambda) - 1) + 2(-2\lambda - 3) + 3(-2 - 3\lambda)$ (1.1.30)

$\Rightarrow \det(A - \lambda I) = (1-\lambda)(\lambda^2 - 1) - 4\lambda - 6 - 6 - 9\lambda$ (1.1.31)

$\Rightarrow \det(A - \lambda I) = \lambda^3 - \lambda - \lambda^2 - 1 - 4\lambda - 6 - 6 - 9\lambda$ (1.1.32)

$\Rightarrow \det(A - \lambda I) = \lambda^3 - \lambda^2 - 14\lambda - 13$ (1.1.33)

Eigenvalues are the roots of the characteristic polynomial of the matrix A (1.1.33) so they can be approximated as the eigenvalues (precision) of A.

$\Rightarrow$ eigenvalues are $\lambda_1, \lambda_2, \lambda_3$.

$\Rightarrow$ since the eigenvalues are pairwise unity we can diagonalize.

$\Rightarrow$ Centers of the Gershgorin's disk are the eigenvalues and the center and radius of each disk is the sum of the λ.

$\Rightarrow$ sum for $\lambda_1 = \lambda_2(1 - \lambda_3)$.

$\Rightarrow$ λ_1 is a terminal, the Gershgorin's disk of λ_1 is also not because in radius with respect to λ_2.

Therefore the eigen value will be only accordingly.

Now given the final point by the Theorem, we can state, that the eigenvalues are belonging to the center of the n-dimensional space in the complex plane.

$\Rightarrow \{\lambda = \alpha + i\beta \in \mathbb{C} \mid (\alpha - 1)^2 + (\beta - 0)^2 \leq 1\}$ (1.1.34)

Gershgorin $G_1(A)$ is a closed Gershgorin's complex disk with radius $1 \in \mathbb{C}$.

Since the above center is equivalent to the sum of the eigenvalues of the Gershgorin's disk radius is equivalent to $G_1(A) = G_2(A) = G_3(A)$. (1.1.35)

Hence, we can, decomposed the above matrix $\{1, 2, 3\}$ into four parts which consist of the $\{1, 2\}$ and $\{1, 3\}$ and $\{2, 3\}$.

$G_{12}(A) \cup G_{13}(A) \cup G_{23}(A)$ (1.1.36)

Hence, radius $\{1, 2, 3\}$.

$\Rightarrow$ $\{G_1(A) \cup G_2(A) \cup G_3(A)\}$

$\Rightarrow$ sum of eigen values in the column $\{1, 2, 3\}$ will be equal to a real eigen value in the column $\{1, 2, 3\}$ and be equal to zero.

Thus we can conclude that eigen λ corresponds interestingly to the sum of eigen values zero present in the column.

(1) To find: Gershgorin Radius: $\lambda_1 = 1.83, 1.0, \lambda_3$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$ (1.1.37)

Input A Real Symmetric

Solution:

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -5 \\ 3 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -5 \\ 0 & -5 & -9 \end{bmatrix}$ (1.1.38)

Characteristic polynomial

The above matrix is symmetric, therefore we can be positive. The condition necessary is diagonal is.

$\Rightarrow \lambda_1 = \max_{1 \leq i \leq n} \left(\frac{\sum_{j=1}^{n} |a_{ij}|}{|a_{ii}|} \right)$ $\max_{1 \leq i \leq n} \left(\frac{\sum_{j=1}^{n} |a_{ij}|}{\text{the condition number of a Hermitian}} \right)$

$\Rightarrow \lambda_1 = \max_{1 \leq i \leq n} \left(\frac{\sum_{j=1}^{n} |a_{ij}|}{\text{the condition number of a Hermitian}} \right)$ $\left(\frac{\sum_{j=1}^{n} |a_{ij}|}{\text{the condition number of a Hermitian}} \right)$

Now, you have Theorem 8.9 of the text Gershgorin's Bound, we can state

$\Rightarrow$ According to Gershgorin's Theorem, the eigenvalues are about from the eigen value in the row and the eigen value in the column of A and the.

$\Rightarrow \lambda = \left(\frac{\sum_{j=1}^{n} |a_{ij}|}{\sum_{j=1}^{n} |a_{ji}|} \right)$ $\left(\frac{\sum_{j=1}^{n} |a_{ij}| + \sum_{j=1}^{n} |a_{ji}|}{\sum_{j=1}^{n} |a_{ij}| + \sum_{j=1}^{n} |a_{ji}|} \right)$

For the above diagonal, there are 3 Gershgorin's disks, where is the center of the disk is the sum of the λ and the.

Hence, from the diagonal, the absolute eigen value is maximum in the range of the real estimation in the range of the center of λ_1 is 1.83.

$\Rightarrow$ the maximum eigen value

$\Rightarrow \left(\frac{\sum_{j=1}^{n} |a_{ij}|}{\sum_{j=1}^{n} |a_{ji}|} \right)$

$\Rightarrow$ the minimum eigen value

$\Rightarrow \left(\frac{\sum_{j=1}^{n} |a_{ij}|}{\sum_{j=1}^{n} |a_{ji}|} \right)$

$\Rightarrow$ $\lambda_1 = 1.83$

$\Rightarrow$ $\lambda_2 = 1.0$

$\Rightarrow$ $\lambda_3 = \lambda_3$

So for condition number κ, the upper bound should be

$\Rightarrow \kappa = \frac{\sum_{j=1}^{n} |a_{ij}|}{\sum_{j=1}^{n} |a_{ji}|} = \frac{\lambda_1}{\lambda_3} \neq 0$

and the lower bound should be,

$\Rightarrow \kappa = \frac{\sum_{j=1}^{n} |a_{ij}|}{\sum_{j=1}^{n} |a_{ji}|} = \frac{\lambda_2}{\lambda_3} \neq 0$

$\Rightarrow \lambda_1 \leq \lambda \leq \lambda_2$