NAME : ABHIK SARKAR

MATRIKELNUMMER : 23149662

Id : lo88xide

EMAIL : abhik.sarkar@fau.de / abhik.sarkar.718@gmail.com

1. Let $A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}$

(a) Determine real SVD of A in form of $A = U\Sigma V^T$

$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}$ $A^T = \begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix}$.

$U \rightarrow$ Eigen vectors of $A \cdot A^T$.

$A \cdot A^T = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} \cdot \begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix}$.

$= \begin{bmatrix} 125 & 75 \\ 75 & 125 \end{bmatrix}$.

$|A \cdot A^T - \lambda I| = \begin{bmatrix} 125-\lambda & 75 \\ 75 & 125-\lambda \end{bmatrix} = 0$

$(125-\lambda)^2 - (75)^2 = 0$.

$\lambda^2 - 250\lambda + 10000 = 0$.

$\lambda_1 = 200$, $\lambda_2 = 50$.

$\therefore$ Singular values, $\sigma_1 = \sqrt{200} = 10\sqrt{2}$

$\sigma_2 = \sqrt{50} = 5\sqrt{2}$.

Singular Value Matrix, $\Sigma = \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}$

For Eigen vector, U –

Case I : when $\lambda_1 = 200$,

$A \cdot A^T - \lambda I = AA^T - 200I = \begin{bmatrix} -75 & 75 \\ 75 & -75 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$.

$-75x + 75y = 0$.

$x = y$

$\therefore X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow$ unit eigen vector $\rightarrow \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

Case II : for $\lambda = 50$,

$$|A \cdot A^T - \lambda I| = \begin{bmatrix} 75 & 75 \\ 75 & 75 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 .$$

$$75x + 75y = 0 .$$

$$x = -y .$$

$X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ unit eigen vector $= \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

$$U = [X_1 \quad X_2]$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

$V \rightarrow$ $\quad A V_1 = \sigma_1 U_1$

$$\begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 10\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

$\rightarrow$ $-2v_1 + 11v_2 = 10\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 10$.

$\rightarrow$ $-2v_1 + 11v_2 = 10$ ... equation no. (1) .

$\rightarrow$ $-10v_1 + 5v_2 = 10$ ... equation no. (2) .

$\rightarrow$ This gives,

$v_1 = -0.6 = -\frac{3}{5}$, $v_2 = 0.8 = \frac{4}{5}$

Similarly, $A v_2 = \sigma_2 u_2$.

$$\begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 5\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}.$$

$\rightarrow$ $-2v_1 + 11v_2 = 5$

$\rightarrow$ $-10v_1 + 5v_2 = -5$.

$v_1 = 0.8 = \frac{4}{5}$, $v_2 = \frac{3}{5} = 0.6$.

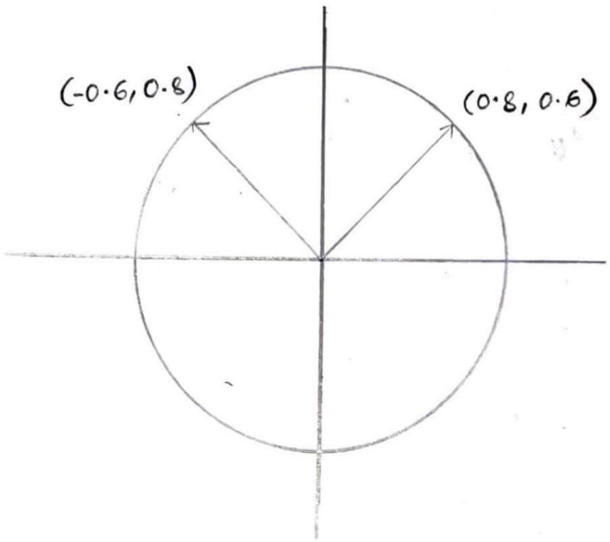
$$\therefore V = \begin{bmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}.$$

$$\therefore \quad A = U \Sigma V^T$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix} \begin{bmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix}.$$

(b) Singular values of A are $10\sqrt{2}$ and $5\sqrt{2}$.

Right Singular Vectors - $\pm \begin{bmatrix} -0.6 \\ 0.8 \end{bmatrix}$ and $\pm \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$.



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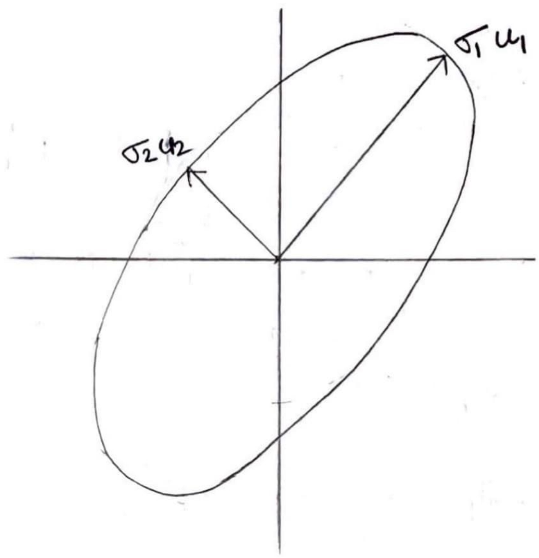

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(c) $\rightarrow \|A\|_1$ = maximum column sum of $|A|$

$$= \max \left[\left| \begin{matrix} -2 \\ -10 \end{matrix} \right| , \left| \begin{matrix} 11 \\ 5 \end{matrix} \right| \right]$$

$$= \max (12, 16)$$

$$= 16$$

$\therefore \|A\|_1 = 16$

$\rightarrow \|A\|_2$ = maximum singular value $= \sigma_1 = 10\sqrt{2}$

$\therefore \|A\|_2 = 10\sqrt{2}$.

$\rightarrow \|A\|_\infty$ = maximum row sum of $|A|$

$$= \max \left[\left| -2 \quad 11 \right| , \left| -10 \quad 5 \right| \right]$$

$$= \max [13, 15)$$

$$= 15$$

$\therefore \|A\|_\infty = 15$.

$\rightarrow \|A\|_F = \sqrt{\sigma_1^2 + \sigma_2^2}$

$$= \sqrt{(10\sqrt{2})^2 + (5\sqrt{2})^2} = \sqrt{250} = 5\sqrt{10} = 15.8113$$

$\|A\|_F = 5\sqrt{10} = 15.8113$

(d) $A^{-1} = (U \Sigma V^T)^{-1}$

$$= V \Sigma^{-1} U^T$$

$$= \begin{bmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}^{-1} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix} \cdot \frac{1}{100} \begin{bmatrix} 5\sqrt{2} & 0 \\ 0 & 10\sqrt{2} \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

$$= \frac{1}{100} \begin{bmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

$$= \frac{1}{500} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 5 \\ 10 & -10 \end{bmatrix}.$$

$$= \frac{1}{100} \begin{bmatrix} 5 & -11 \\ 10 & -2 \end{bmatrix} = \begin{bmatrix} 1/20 & -11/100 \\ 1/10 & -1/50 \end{bmatrix}$$

Exercise 2:

$A \in \mathbb{C}^{m \times n}$ with $m \geq n$.

To prove: $A^* A$ is non-singular iff A has full rank.

Proof: Let us assume A has full rank

$\text{Rank}(A) = n$ (Since $m \geq n$).

$\rightarrow$ No. of non-zero singular values of A is n as it has full rank. Then, it indicates that $A^* A$ will also have n- non zero eigen values.

$\therefore$ $A^* A$ is non-singular.

Exercise 3:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\rightarrow a_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $a_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

For the orthogonal matrix -

$$Q = [q_1 \quad q_2].$$

$\rightarrow q_1 = \dfrac{a_1}{\| a_1 \|}$

$\rightarrow \quad = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}.$

$\rightarrow q_2 = \dfrac{a_2^{\perp}}{\| a_2^{\perp} \|}$

$\rightarrow a_2^{\perp} = a_2 - \langle a_2, q_1 \rangle \cdot q_1$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \left[\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right] \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \left[\frac{1}{\cancel{\sqrt{2}}} \cdot \cancel{[2]} \right] \cdot \frac{1}{\cancel{\sqrt{2}}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

$\rightarrow q_2 = \dfrac{a_2^{\perp}}{\| a_2^{\perp} \|}$

$$= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}.$$

$$\rightarrow Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix}$$

Let $R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$.

$\rightarrow \quad r_{11} = \| a_1 \|$

$= \sqrt{2}$

$\rightarrow \quad r_{12} = \dfrac{\langle q_1 , a_2 \rangle}{\langle q_1 , q_1 \rangle} = \dfrac{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$.

$= \dfrac{\frac{1}{\sqrt{2}} \cdot 2}{\frac{1}{\sqrt{2} \cdot \sqrt{2}} \cdot 2} = \sqrt{2}$.

$\rightarrow r_{12} = \sqrt{2}$.

$\rightarrow \quad r_{22} = \| a_2^{\perp} \|$

$= \sqrt{3}$

$\rightarrow \quad r_{21} = 0$

$R = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$.

Reduced QR factorization :

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}.$$

Full Rank $\rightarrow \quad q_3 = \begin{bmatrix} q_{13} \\ q_{23} \\ q_{33} \end{bmatrix}$.

where, $\langle q_1 , q_3 \rangle = 0$

$\langle q_2 , q_3 \rangle = 0$

$\langle q_3 , q_3 \rangle = 1$.

$\rightarrow \langle q_1 , q_3 \rangle = 0$

$\rightarrow \frac{1}{\sqrt{2}} q_{13} + 0 + \frac{1}{\sqrt{2}} \cdot q_{33} = 0$

$\rightarrow \quad q_{13} + q_{33} = 0 \quad \dots (3.1)$.

$\rightarrow \langle q_2, q_3 \rangle = 0$.

$\rightarrow \frac{1}{\sqrt{3}} q_{13} + \frac{1}{\sqrt{3}} q_{23} - \frac{1}{\sqrt{3}} q_{33} = 0$.

$$q_{13} + q_{23} - q_{33} = 0 \quad \ldots (3.2).$$

Substituting value of a_{33} from (3.1) to (3.2),

$$q_{13} + q_{23} + q_{13} = 0.$$

$$q_{23} = -2q_{13},$$

$\therefore \quad q_3 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \rightarrow$ Unit matrix $= \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$

$\therefore$ Full Rank of QR factorization :

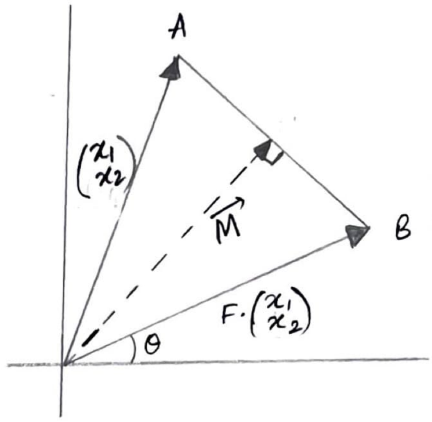
$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$$

Exercise 4:

Orthogonal matrices, $F, J \in \mathbb{R}^{m\times m}$.

$$F_\theta = \begin{bmatrix} -c & s \\ s & c \end{bmatrix}, \quad J_\theta = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

$s = \sin\theta$, $c = \cos\theta$



Let us take a vector $\vec{x}$,

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \forall \vec{x} \in \mathbb{R}^2$$

Now, we apply left multiplication by F on $\vec{x}$,

$$F.\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} -c & s \\ s & c \end{bmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

$$= \begin{pmatrix} -cx_1 + sx_2 \\ sx_1 + cx_2 \end{pmatrix}$$

The midpoint of $\vec{x}$ and $\overrightarrow{F(x)}$ is

$$\vec{M} = \frac{\vec{x} + \overrightarrow{F(x)}}{2} = \frac{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -cx_1 + sx_2 \\ sx_1 + cx_2 \end{pmatrix}}{2}.$$

$$= \frac{1}{2}\begin{bmatrix} (1-c)x_1 + sx_2 \\ sx_1 + (1+c)x_2 \end{bmatrix}.$$

$\therefore$ Midpoints of $\vec{x}$ and $\overrightarrow{F(x)}$ are $\begin{bmatrix} \left(\frac{1-c}{2}\right)x_1 + \frac{sx_2}{2} \\ \frac{sx_1}{2} + \left(\frac{1+c}{2}\right)x_2 \end{bmatrix}$.

$\Rightarrow \quad \vec{M}^* \cdot [\vec{F(x)} - \vec{x}]$

$$\rightarrow \frac{1}{2}\begin{bmatrix} (1-c)x_1 + sx_2 \\ sx_1 + (1+c)x_2 \end{bmatrix}^* \cdot \begin{bmatrix} -cx_1 + sx_2 - x_1 \\ sx_1 + cx_2 - x_2 \end{bmatrix}.$$

$$\rightarrow \frac{1}{2}\begin{bmatrix} (1-c)x_1 + sx_2 & sx_1 + (1+c)x_2 \end{bmatrix} \begin{bmatrix} sx_2 - (1+c)x_1 \\ sx_1 - (1-c)x_2 \end{bmatrix}.$$

$$\rightarrow \frac{1}{2}\left[(sx_2 + (1-c)x_1) \cdot (sx_2 - (1+c)x_1) + (sx_1 + (1+c)x_2) \cdot (sx_1 - (1-c)x_2)\right]$$

$$\rightarrow \frac{1}{2}\Big[s^2x_2^2 + \cancel{s \cdot (1-c)x_1x_2} - \cancel{s \cdot (1+c)x_1x_2} - (1-c)(1+c)x_1^2 + s^2x_1^2 - \cancel{s \cdot (1-c)x_1x_2} + \cancel{s \cdot (1+c)x_1x_2} - (1+c)(1-c)x_2^2\Big].$$

$$\rightarrow \frac{1}{2}\left[s^2x_2^2 - x_1^2 + c^2x_1^2 + s^2x_2^2 - x_2^2 + c^2x_2^2\right].$$

$$\rightarrow \frac{1}{2}\left[(s^2 + c^2)x_2^2 - x_2^2 + (s^2+c^2)x_1^2 - x_1^2\right].$$

[since $s = \sin\theta$ and $c = \cos\theta$ so, $(s^2+c^2) = 1$]

$$\rightarrow \frac{1}{2}\left[\cancel{x_2^2} - \cancel{x_2^2} + \cancel{x_1^2} - \cancel{x_1^2}\right] = 0.$$

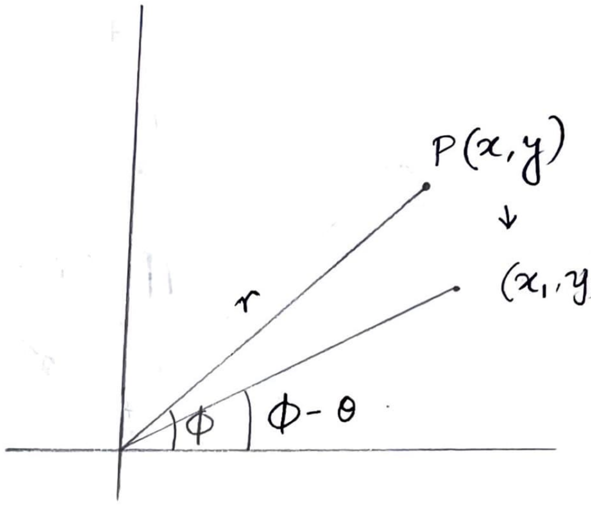
→ Hence, the midpoint M is the perpendicular bisector of the vector joining $\begin{pmatrix} x \\ y \end{pmatrix}$ and $F\begin{pmatrix} x \\ y \end{pmatrix}$.

→ Therefore, F reflect $\vec{x}$ along the direction of $\vec{M}$

→ This proofs F is a reflection.

For J, let us take a vector $\vec{x} = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}$

$$\rightarrow \quad J \cdot \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}$$



$$\rightarrow \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}$$

$$\rightarrow \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}$$

$$\rightarrow r \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi \\ -\sin\theta\cos\phi + \cos\theta\sin\phi \end{bmatrix}$$

$$\rightarrow r \begin{bmatrix} \cos(\phi-\theta) \\ \sin(\phi-\theta) \end{bmatrix}$$

$\therefore$ J represents a rotation in a clockwise direction at an angle θ.

4.(b) According to the Householder triangularization, a matrix $A \in \mathbb{C}^{m\times n}$ is multiplied with a householder reflector at every step such that at the end, A gets converted to a triangular matrix.

∴ Algorithm of QR factorisation using Givens Rotation:

```
Q = eye(m)
for i = 1 to n,
    for j = i+1 to m,
        A = eye(m)
        [c, s] = givensrotation(R(j-1, i), R(j, i))
        A([j-1, j], [j-1, j]) = [c -s ; s c].
        R = A^T * R
        Q = Q * A
    end
end
```