ANLA ASSIGNMENT - 4

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NAME: ABHIK SARKAR

MATRIKELNUMMER: 23149662

IDM Id : lo88xide

EMAIL: abhik.sarkar@fau.de / abhik.sarkar.718@gmail.com

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Exercise 1 – Uniqueness of QR factorization:

Let $A \in \mathbb{C}^{m\times m}$ is non-singular.

QR factorisation: $A = QR$ $Q \to$ unitary, $R \to$ upper triangular.

To prove: QR factorisation is unique when demanding $r_{jj} > 0 \ \forall j$.

Proof: $A = QR$

$$A = [q_1 \,|\, q_2 \,|\, \cdots \,|\, q_m] \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1m} \\ 0 & r_{22} & \cdots & & r_{2m} \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & & r_{mm} \end{bmatrix}$$

So, let us assume, $A = Q_1 R_1 = Q_2 R_2$. ... (1).

$Q_i \in \mathbb{C}^{m\times m}$ and $R_i \in \mathbb{C}^{m\times n}$,

$Q_i \to$ orthogonal matrix $\to$ a real square matrix whose columns and rows are orthonormal.

Therefore, $Q_i^* \cdot Q_i = I$ ... (2)

Now, from eq^n(1), we can say that, $Q_1 R_1 = Q_2 R_2$

$\to \quad Q_1 \cdot R_1 \cdot R_2^{-1} = Q_2$.

by multiplying Q_1^* on both sides, we get,

$\to \quad Q_2 Q_1^* = \cancel{Q_1 \cdot Q_1^*} \cdot R_1 \cdot R_2^{-1}$.

$\to \quad Q_2 \cdot Q_1^* = R_1 \cdot R_2^{-1}$ ... (3).

Here, the above equation yields a lower triangular matrix for $Q_2 \cdot Q_1^*$ and an upper triangular matrix $R_1 R_2^{-1}$.

Since, upper triangular matrix is equal to lower triangular matrix, so we can say it is a diagonal matrix.

Now, let us assume the diagonal matrix as D.

$$D = Q_1 \cdot Q_2^* = R_1 \cdot R_2^{-1}.$$

Now, if we multiply R_2^{-1} in eq^n (1), we get,

$Q_1 R_1 \cdot R_2^{-1} = Q_2 R_2 \cdot R_2^{-1}$.

$Q_1 \cdot D = Q_2$ $[\because R_2 \cdot R_2^{-1} = I, \ R_1 \cdot R_2^{-1} = D]$. ... (4)

Now, as we know from eq^n (2),

$Q_2^* \cdot Q_2 = I$

$\to (Q_1 \cdot D)^* \cdot Q_1 D = I$.

$\to D^* Q_1^* \cdot Q_1 D = I$.

$\to D^* D = I$ [As $Q_1^* \cdot Q_1 = I$].

$\to D^2 = I$ [Since D is diagonal matrix, so $D^* = D$].

$\to D = I$. ... (5)

Therefore, from eq^n(4) & eq^n(5), we can say,

$Q_1 = Q_2$. ... (6).

$\therefore$ Q is unique.

Since, $Q_1 R_1 = Q_2 R_2$ (from eq^n 1).

$\to R_1 = R_2$ $(\because Q_1 = Q_2)$. ... (7).

$\therefore$ R is also unique.

$\therefore$ From the above proof, we can state that the QR factorisation of $A \in \mathbb{C}^{m\times m}$ is unique.

Exercise 2 – LU Decomposition of Banded Matrices.

Given: $A \in \mathbb{C}^{m\times m}$ is non-singular banded matrix with bandwidth $2p+1$ i.e., $a_{ij} = 0$ for $|i-j| > p$.

LU factorization with pivoting:

$A = LU$.

To prove: With their triangular shape, L and U also have same bandwidth $2p+1$.

Proof: $A = LU$.

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,m} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ l_{2,1} & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ l_{m,1} & \cdots & l_{m,m-1} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,m} \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & u_{m,m} \end{bmatrix}$$

Case I: For Base Case:

$\to$ When $m = 1$, $A \in \mathbb{C}^{1\times 1}$

$\to A_1 = [a_{1,1}]$

$\therefore$ The bandwidth of A_1 is 0 and, the unique LU factorisation of the matrix A is given by

$A_1 = L_1 U_1$, where

$L_1 = I$ and $U_1 = A_1$.

$\therefore$ L_1 and U_1 has the same bandwidth and sparcity.

Case II: for K^{th} Sub-matrix:

$\to$ When $m = K$, $A \in \mathbb{C}^{K\times K}$

$$A_K = \begin{bmatrix} a_{1,1} & \cdots & a_{1,K} \\ \vdots & \ddots & \vdots \\ a_{K,1} & \cdots & a_{K,K} \end{bmatrix}$$

$\therefore$ The unique LU factorisation of the matrix A_K is given by:

$A_K = L_K \cdot U_K$.

such that, we assume L_K and U_K has the same structure, sparsity and bandwidth $(2p+1)$ (with respect to their triangular shape).

(By Induction).

Case III: for $(K+1)^{th}$ sub-matrix,

$\to m = K+1$ and $A = \mathbb{C}^{(K+1)\times(K+1)}$

Now, let us assume,

$b_K \in \mathbb{C}^{K,1}$, $c_K \in \mathbb{C}^{K,1}$, $d_K \in \mathbb{C}$, such that,

$$A_{K+1} = \begin{bmatrix} A_K & b_K \\ c_K^T & d_K \end{bmatrix}$$

Now, let us assume, that

$$L_{K+1} = \begin{bmatrix} L_K & 0 \\ l_K^T & 1 \end{bmatrix}, \text{ where } l_K \in \mathbb{C}^K$$

$$U_{K+1} = \begin{bmatrix} U_K & u_K \\ 0 & e_K \end{bmatrix}, \text{ where } u_K \in \mathbb{C}^K \text{ and } e_K \in \mathbb{C}$$

such that, the unique LU factorisation of the matrix A_{K+1} is given by:

$\to \quad A_{K+1} = L_{K+1} \cdot U_{K+1}$

$$\to \begin{bmatrix} A_K & b_K \\ c_K & d_K \end{bmatrix} = \begin{bmatrix} L_K & 0 \\ l_K^T & 1 \end{bmatrix} \begin{bmatrix} U_K & u_K \\ 0 & e_K \end{bmatrix} = \begin{bmatrix} L_K U_K & L_K u_K \\ l_K^T U_K & l_K^T u_K + e_K \end{bmatrix}.$$

$\therefore$ $A_K = L_K U_K$ ... which is true.

$b_K = L_K u_K$.

$c_K^T = l_K^T U_K$. $\to$ $c_K = U_K^T \cdot l_K$.

$d_K = l_K^T u_K + e_K$.

Since, A_K has bandwidth $2p+1$, so, for A_{K+1} to have similar sparsity pattern and bandwidth, the structure of b_K and c_K must be:

$$b_K = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_{K+1-p,K+1} \\ \vdots \\ a_{K,K+1} \end{bmatrix}, \quad c_K = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_{K+1,K+1-p} \\ \vdots \\ a_{K+1,K} \end{bmatrix}, \quad d_K = [a_{K+1,K+1}].$$

Now, we know that,

$b_K = L_K u_K$

Since, L_K has same bandwidth as A_K, so, we can say that, u_K must have the same structure as b_K.

$$\therefore \quad u_K = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ u_{K+1-p,K+1} \\ \vdots \\ u_{K,K+1} \end{bmatrix}$$

Similarly, we know that,

$c_K = U_K^T \cdot l_K$

Now, again U_K has the same bandwidth as A_K, so, l_K must have the same structure as c_K.

$$\therefore \quad l_K = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ l_{K+1,K+1-p} \\ \vdots \\ l_{K+1,K} \end{bmatrix}$$

$$\therefore \quad L_{K+1} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ l_{2,1} & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ l_{p+1,1} & & & \ddots & 0 \\ 0 & \ddots & & & \vdots \\ 0 & \cdots\ 0 & l_{K+1,K+1-p} & \cdots\ l_{K+1,K} & 1 \end{bmatrix}$$

$$\to \quad U_{K+1} = \begin{bmatrix} u_{1,1} & \cdots & u_{1,p+1} & 0 & \cdots & 0 \\ 0 & \ddots & & \ddots & & 0 \\ \vdots & & \ddots & & & u_{K+1-p,K+1} \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & u_{K,K+1} \\ 0 & \cdots & & & 0 & u_{K+1,K+1} \end{bmatrix}$$

$\therefore$ Now, we can say A_{K+1} has the same bandwidth as U_{K+1} & L_{K+1} with respect to their triangular shape.

Therefore, by proof of induction, we can say that, if $m = K$ holds, then it is also true for $m = K+1$.

$\therefore$ for a non-singular banded matrix $A \in \mathbb{C}^{m\times m}$ with bandwidth $2p+1$ has an LU factorization without pivoting such that $A = LU$, where L and U also have same bandwidth $2p+1$ w.r.t its triangular shape.