IITM-CS6014 : Advanced Theory of Computation — Given on: Sep 20, 5pm

Problem Set #3 — Due on : Oct 3, 5pm

- Rules of the game remain the same. Submissions must be single file in LaTeX format at the upload link set up in cse moodle page of the course. Note that we do not accept any other format (no pdfs !).

1. (10 points) Recall
$$K = \{x \mid M_x \text{ halts on } x\}$$
Prove the following.
   (a) K is SD-complete.
   (b) Let $K^X = \{x \mid M_x^X \text{ is an oracle TM with oracle } X \text{ and } M_x^X \text{ halts on } x\}$. Show that K^K is Σ_2-complete.

2. (10 points) A set P is *partially productive* if there is a partial recursive function(i.e. computed by a Turing machine N - which need not be total) ψ called the *productive function* such that:
$$\forall x \quad (\mathcal{W}_x \subseteq P \implies N \text{ halts on } x \ \& \ \ \psi(x) \in P \setminus \mathcal{W}_x)$$
Show that any productive set P has an injective recursive productive function.
(Hint : First prove that it can be made recursive and then attempt on making it injective).

3. (5 points) Prove that there must exist a recursive function such that $\{\mathcal{W}_{f(n)}\}_{n\in\mathbb{N}}$ consists of precisely decidable sets.

4. (5 points) A simple set is *effectively simple* if there is a recursive function f such that:
$$\forall n \in \mathbb{N} : \mathcal{W}_n \subseteq \overline{A} \implies |\mathcal{W}_n| \leq f(n)$$
Show that Post's simple set is effectively simple.

   (Extra Credit) If a set A is effectively simple, argue that $K \leq_T A$. This justifies why Friedberg-Muchnik had to do a different construction. (Hint: Normal homework rules does not apply to this question. You can look up anywhere, but cite your sources. !).

5. (10 points) Consider the following computational problems :
   (a) Given a Turing machine M, test if $L(M)$ is Productive.
   (b) Given a Turing machine M, test if $L(M)$ is Simple.

   Are they decidable? semi-decidable?

   (Extra Credit) Place them in the arithmetic hierarchy.