1. (a) $L = \{M : M \text{ has an useless state } q\ \}$
   Language L is undecidable. Suppose L was decidable then we can use that as a sub routine to solve $\overline{HP}$ as follows:
   - Consider a new machine $\grave{M}$ which has two states - start and accept state. It writes its input y on a seperate tape.
   - It then runs M on input x . If M halts on x it accepts its input y.

   The machine $\grave{M}$ can be described as :

$$\grave{M} = \begin{cases} \grave{M} \textit{ has no useless state} & \text{if } M \text{ halts on } x \\ \grave{M} \textit{ has an useless accept state} & \text{if } M \text{ does not halt on } x \end{cases}$$

   Since our assumption is that L is decidable implies there exists a total turing machine K which accpets L . Giving $\grave{M}$ as input to K , we can totally compute if M halts on x . But we know there is no such total turing machine. Hence, language L is not decidable.

   (b)

   (c) $L = \{(M1,M2)\text{: } L(M1) = \overline{L(M2)}\}$
   L is undecidable. If L was decidable we can use the total turing machine K accepting L as a sub routine to solve HP as follows:
   Consider two machines $M1$ and $M2$ which does the following:
   - Both machines simulate M on x . If M halts on x then machine $M1$ accepts its input y , while machine $M2$ rejects input y.

   So the language of machine $M1$ can be described as :

$$L(M1) = \begin{cases} \Sigma^* & \text{if } M \text{ halts on } x \\ \varnothing & \text{if } M \text{ does not halt on } x \end{cases}$$

   Similarly the language accpeted by machine $M2$ can be described as:

$$L(M2) = \begin{cases} \varnothing & \text{if } M \text{ halts on } x \\ \varnothing & \text{if } M \text{ does not halt on } x \end{cases}$$

   Giving machines ($M1$, $M2$) as input to machine K can help in totally computing if machine M halts on input x. Since, there is no such total turing machine, hence no such total turing machine K exists.

   (d) $L = \{M : \exists x \in \Sigma^* \text{ s.t. } M \text{ runs forever on input } x\}$
   This language is undecidable . If this language was decidable by a total turing machine K , then we can use it as a sub routine to solve $\overline{HP}$ as follows:
   Consider machine $\grave{M}$ which does the following:

- $\grave{M}$ writes its input y on a seperate track.
- It runs machine M on input x. If M halts on x , then it accepts its input y.

The machine $\grave{M}$ can be described as follows:

$$\grave{M} = \begin{cases} For\ all\ inputs\ \grave{M}\ doesnt\ run\ forever & \text{if } M \text{ halts on } x \\ There\ exists\ an\ input\ for\ which\ \grave{M}\ runs\ forever & \text{if } M \text{ does not halt on } x \end{cases}$$

Giving $\grave{M}$ as input to machine K can help in totally computing if machine M halts on input x. Since, there is no such total machine, hence no such total turing machine K exists.

(e) $L = \{M\text{: } M \text{ accepts atleast one plaindromic string}\}$
L is undecidable. If L was decidable we can use the total turing machine K accepting L as a sub routine to solve HP as follows:
Consider machine $\grave{M}$ which does the following:

- Machine $\grave{M}$ simulates M on input x.
- If M halts on x, then $\grave{M}$ accepts its input y

So the language of machine M can be described as :

$$L(M) = \begin{cases} \Sigma^* & \text{if } M \text{ halts on } x \\ \varnothing & \text{if } M \text{ does not halt on } x \end{cases}$$

Since, Σ^* contains atleast one palindromic string, Giving machine M as input to machine K can help in totally computing if machine M halts on input x. Since, there is no such total turing machine, hence no such total turing machine K exists.

(f) $L = \{M\text{: } M \text{ accepts only plaindromic strings}\}$
L is undecidable. If L was decidable we can use the total turing machine K accepting L as a sub routine to solve HP as follows:
Consider machine $\grave{M}$ which does the following:

- Machine $\grave{M}$ simulates M on input x.
- If M halts on x, then $\grave{M}$ checks if its input y is a plaindrome. If yes, it accepts y , else rejects y.

So the language of machine M can be described as :

$$L(M) = \begin{cases} P & \text{if } M \text{ halts on } x \\ \varnothing & \text{if } M \text{ does not halt on } x \end{cases}$$

Where P is the set of all palindromes.
Giving machine M as input to machine K can help in totally computing if machine M halts on input x. Since, there is no such total turing machine, hence no such total turing machine K exists.

(g) $L = \{(M1,M2)\text{: } L(M1) \bigcap L(M2) \neq \varnothing \}$
L is undecidable. If L was decidable we can use the total turing machine K accepting L as a sub routine to solve HP as follows:
Consider two machines $M1$ and $M2$ which does the following:

- Both machines simulate M on x . If M halts on x then both machines $M1$ and $M2$ accepts their inputs $y1$,$y2$ respectively.

So the language of machine $M1$ can be described as :

$$L(M1) = \begin{cases} \Sigma^* & \text{if } M \text{ halts on } x \\ \varnothing & \text{if } M \text{ does not halt on } x \end{cases}$$

Similarly the language accpeted by machine $M2$ can be described as:

$$L(M2) = \begin{cases} \Sigma^* & \text{if } M \text{ halts on } x \\ \varnothing & \text{if } M \text{ does not halt on } x \end{cases}$$

And the Language $L(M1) \bigcap L(M2)$ can be described as :

$$L(M1) \bigcap L(M2) = \begin{cases} \Sigma^* & \text{if } M \text{ halts on } x \\ \varnothing & \text{if } M \text{ does not halt on } x \end{cases}$$

Giving machines ($M1$, $M2$) as input to machine K can help in totally computing if machine M halts on input x. Since, there is no such total turing machine, hence no such total turing machine K exists.

2.

3.

4. (a)

   (b) L is decidable => L $\leq_m$ 1*0*

   To show this we have to find a $\sigma : \Sigma^* \rightarrow \Sigma^*$:

   - σ is totally computable
   - $\forall x, x \in L <=> \sigma(x) \in$ 1*0*

   Since, L is decidable , there exists a total turing machine K such that, $L(K) = L$. Let σ be defined as follows :

$$\sigma(x) = \begin{cases} 10 & \text{if } K \text{ accepts } x \\ 01 & \text{if } K \text{ rejects } x \end{cases}$$

   σ is totally computable since K is total.
   $x \in L <=> K\ accepts\ x <=> (\sigma(x) = 10 \in 1^*0^*)$

   Therefore, L $\leq_m$ 1*0*

   L is decidable <= L $\leq_m$ 1*0*

RHS implies there exists a totally computable σ such that $\forall x, x \in L <=> \sigma(x) \in$ 1*0* .
For any input x , calculating $\sigma(x) \in$ 1*0* , can be done by a total turing machine.(Keep 4 states. when on state 1 implies seeing 1's. First time a 0 is seen, move to state 2. Or if end of input is seen, move to state 3 and accept. In state 2 ,if end of input is reached go to state 3 and accept. If a 1 is seen, move to state 4 and reject). Hence, $x \in L$ can be totally computed . Therefore , L is decidable.

5.