In differential calculus we learn the concept of the derivative of a function and learn techniques to calculate it. The process of finding the derivative $f'(x)$ of a function $f(x)$ is called **differentiating** (not "deriving"!).

Antiderivatives (Dr. Davis' Notes 4.3, Apex 5.1)

It turns out that a lot of the subject of **integral calculus** involves the reverse process: given a function $f(x)$, what function is it the derivative of? Can you find a function $F(x)$ such that $F'(x) = f(x)$? We say that $F(x)$ is an **antiderivative** of $f(x)$, and the process of finding an antiderivative is called **antidifferentiation**.

$$\begin{tikzcd} F(x) \arrow[r, bend left, "\text{differentiation}"] & f(x) = F'(x) \arrow[l, bend left, "\text{antidifferentiation}"] \end{tikzcd}$$

Since antidifferentiation is the reverse of differentiation, it is important that you are familiar with the basic differentiation formulas and rules to help you find antiderivatives.

Example: Find an antiderivative of the function $f(x) = x^2$.

> This means we want a function $F(x)$ such that $F'(x) = f(x)$. Based on the power rule for derivatives, we might try x^3; but
>
> $$\frac{d}{dx}\left(x^3\right) = 3x^2,$$
>
> which has an extra factor of 3. So instead we try $F(x) = \frac{1}{3}x^3$, and then $F'(x) = \frac{1}{3}(3x^2) = x^2$.
>
> Thus $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$.

Why do we just say *an* (not *the*) antiderivative? Because it is not unique—there are others. For example, $F(x) = \frac{1}{3}x^3 + 2$ is also an antiderivative of x^2, and in general $F(x) = \frac{1}{3}x^3 + C$ is an antiderivative for any constant C, because the derivative of a constant is 0.

Exercise: What is an antiderivative for the constant function $f(x) = 1$? What about the constant function $f(x) = k$?

The basic derivative formulas can be reversed to give you some basic antiderivatives.

Examples

$f(x)$	$F(x) =$ antiderivative of $f(x)$
e^x	e^x
$\dfrac{1}{x}$	$\ln(x) \quad (\text{if } x > 0)$
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$
$\dfrac{1}{1+x^2}$	$\arctan(x) \quad (\text{or } \tan^{-1}(x))$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin(x) \quad (\text{or } \sin^{-1}(x))$

In each of these cases, you could get another antiderivative of $f(x)$ by adding a constant to $F(x)$.

Notice that the antiderivatives of $\sin(x)$ and $\cos(x)$ are the *opposites* of their derivative formulas. You will need to be careful with minus signs when working with these functions.

Most functions will not be direct derivatives of one of the basic functions, so antidifferentiation can be very challenging. A lot of this course will be about techniques for finding antiderivatives.

But it is much easier to check *if* a function $F(x)$ is an antiderivative of $f(x)$; just calculate the derivative of $F(x)$ and see if you get $f(x)$.

Example: Check that $F(x) = \frac{1}{2}\sin^2 x$ and $G(x) = -\frac{1}{2}\cos^2 x$ are antiderivatives of $f(x) = \sin x \cos x$.

> To check that these are antiderivatives, we differentiate $F(x)$ and $G(x)$.
> $F'(x) = \frac{1}{2}(2\sin x)\cos x = \sin x \cos x = f(x)$ (using the Chain Rule).
>
> And similarly, $G'(x) = -\frac{1}{2}(2\cos x)(-\sin x) = \cos x \sin x = f(x)$.

But how many different antiderivatives can a function have? It turns out that the only other possible antiderivatives for $f(x)$ (on an interval $a < x < b$) are functions of the form $F(x) + C$, where C is a constant (that is, does not depend on x).

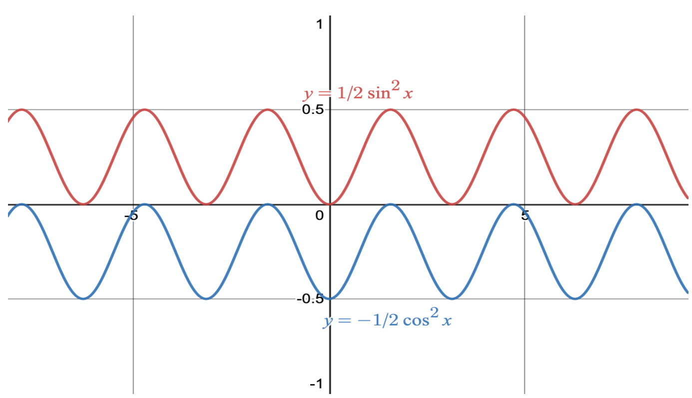
Note: If $F(x)$ is an antiderivative of $f(x)$ (so $F'(x) = f(x)$), then for any constant C it is clear that $F(x) + C$ is also an antiderivative, since

$$\frac{d}{dx}\left(F(x) + C\right) = F'(x) + 0 = F'(x) = f(x).$$

Graphically, adding a constant to $F(x)$ just moves the graph of $y = F(x)$ vertically up or down, which does not change its slope anywhere, so its derivative doesn't change.

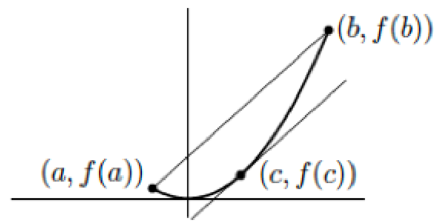
What is not so obvious is why this is the *only* way to get another antiderivative.

> For example, we saw above that $F(x) = \frac{1}{2}\sin^2 x$ and $G(x) = -\frac{1}{2}\cos^2 x$ are two antiderivatives of $f(x) = \sin x \cos x$. Does this violate the claim that the only antiderivatives of $f(x)$ have the form $F(x) + C$?
>
> No! Observe that
>
> $$F(x) = \frac{1}{2}\sin^2 x = \frac{1}{2}\left(1 - \cos^2 x\right) = \frac{1}{2} - \frac{1}{2}\cos^2 x = \frac{1}{2} + G(x)$$
>
> or $G(x) = F(x) - \dfrac{1}{2}$, which has the form $F(x) + C$ with $C = -\dfrac{1}{2}$.



Theorem: If $F(x)$ is an antiderivative of a function $f(x)$ on an interval, then *every* antiderivative of $f(x)$ on that interval must be given by $F(x) + C$ for some constant C. In other words, any two antiderivatives of $f(x)$ must differ by a constant.

The proof of this theorem uses the **Mean Value Theorem** (MVT) for derivatives.



The idea is that if $F(x)$ and $G(x)$ are two antiderivatives of $f(x)$, then

$$F'(x) = G'(x) = f(x) \implies G'(x) - F'(x) = 0$$

for all x in the interval. So by the MVT the slope of the function $G(x) - F(x)$ *between* any two points on the interval must also be 0, which means the value of $G(x) - F(x)$ must be the same at any two points—in other words, $G(x) - F(x)$ has a constant value C.

If $F(x)$ is an antiderivative of $f(x)$ on an interval, then we say that $F(x) + C$ is the **general antiderivative** of $f(x)$. (This is actually a family of functions.)

We can find a **particular antiderivative** if we have additional information, such as the value of $F(c)$ at some point $x = c$, to determine a specific value of the constant C.

Exercise: Find the particular antiderivative $F(x)$ of the function

$$f(x) = 4x^3 - 2e^x + \sin(x)$$

that satisfies $F(0) = 5$.

Indefinite Integrals

Definition and Notation: If $F(x)$ is any antiderivative of $f(x)$ on an interval, then we write

$$\boxed{\int f(x)\,dx = F(x) + C}$$

where C is constant. We call this the **indefinite integral** of $f(x)$ and it represents the general antiderivative of $f(x)$ on that interval. Technically, it is a *collection of functions*.

For example, $\displaystyle\int \sin(x)\cos(x)\,dx = \frac{1}{2}\sin^2 x + C$ and $\displaystyle\int x^2\,dx = \frac{1}{3}x^3 + C$.

Some Basic Indefinite Integrals

We can find some formulas for indefinite integrals by reversing known derivative formulas.

For example, since $\dfrac{d}{dx}\left(x^{n+1}\right) = (n+1)x^n$, we can divide both sides by the constant $n+1$ (if $n \neq -1$) to give $\dfrac{d}{dx}\left(\dfrac{1}{n+1}x^{n+1}\right) = x^n$. Therefore

$$\boxed{\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C} \quad \text{if } n \neq -1$$

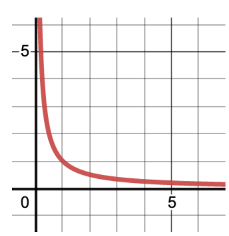
Think of this as "add 1 to the exponent and divide by the new exponent".

Example:

$$\int x^{2/3}\,dx = \frac{1}{\frac{2}{3}+1}x^{2/3+1} + C = \frac{3}{5}x^{5/3} + C$$

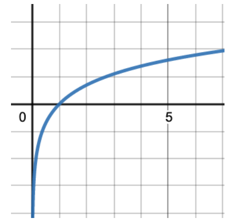
We can't use the power-rule formula when $n = -1$.

So what is $\displaystyle\int x^{-1}\,dx = \int \frac{dx}{x}$?

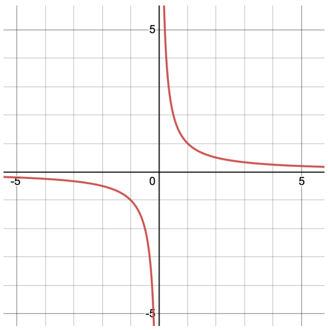


We know that $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ (for $x > 0$), so we have

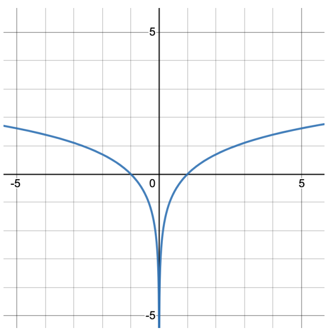
$$\boxed{\int \frac{1}{x}\,dx = \ln(x) + C} \quad \text{if } x > 0$$



But the function $f(x) = 1/x$ is also defined for $x < 0$, though $\ln(x)$ is not. So what is the antiderivative then?



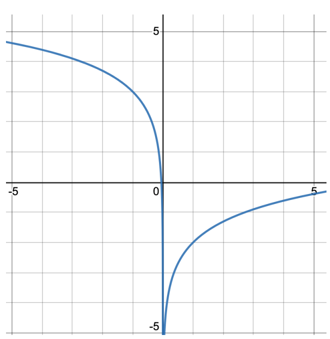
We see that the value of $f(x) = \frac{1}{x}$ for $x < 0$ is exactly the opposite of the value of $\frac{1}{|x|}$. So what does that tell us about the slopes of the antiderivative?



The antiderivative should just be the mirror image of $\ln(x)$, with opposite slopes for $x < 0$ and $x > 0$. This function is $\ln(|x|)$.

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C} \quad \text{for } x \neq 0$$

Technically, the constant C could be different for $x < 0$ and $x > 0$, since those are separate intervals:



Some other easy antiderivative formulas: $\dfrac{d}{dx}(e^x) = e^x$, so $\boxed{\int e^x\,dx = e^x + C}$

$$\frac{d}{dx}(\tan x) = \sec^2 x \implies \boxed{\int \sec^2 x\,dx = \tan x + C}$$

Here is a table of basic antiderivatives that you should memorize!

Theorem 5.1.2 Derivatives and Antiderivatives

Common Differentiation Rules	Common Indefinite Integral Rules
1. $\frac{d}{dx}\big(cf(x)\big) = c \cdot f'(x)$	1. $\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx$
2. $\frac{d}{dx}\big(f(x) \pm g(x)\big) = f'(x) \pm g'(x)$	2. $\int \big(f(x) \pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
3. $\frac{d}{dx}\big(C\big) = 0$	3. $\int 0\,dx = C$
4. $\frac{d}{dx}\big(x\big) = 1$	4. $\int 1\,dx = \int dx = x + C$
5. $\frac{d}{dx}\big(x^n\big) = n \cdot x^{n-1}$	5. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$
6. $\frac{d}{dx}\big(\sin x\big) = \cos x$	6. $\int \cos x\,dx = \sin x + C$
7. $\frac{d}{dx}\big(\cos x\big) = -\sin x$	7. $\int \sin x\,dx = -\cos x + C$
8. $\frac{d}{dx}\big(\tan x\big) = \sec^2 x$	8. $\int \sec^2 x\,dx = \tan x + C$
9. $\frac{d}{dx}\big(\csc x\big) = -\csc x \cot x$	9. $\int \csc x \cot x\,dx = -\csc x + C$
10. $\frac{d}{dx}\big(\sec x\big) = \sec x \tan x$	10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\frac{d}{dx}\big(\cot x\big) = -\csc^2 x$	11. $\int \csc^2 x\,dx = -\cot x + C$
12. $\frac{d}{dx}\big(e^x\big) = e^x$	12. $\int e^x\,dx = e^x + C$
13. $\frac{d}{dx}\big(a^x\big) = \ln a \cdot a^x$	13. $\int a^x\,dx = \frac{1}{\ln a} \cdot a^x + C$
14. $\frac{d}{dx}\big(\ln x\big) = \frac{1}{x}$	14. $\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$

Here is a more complicated example: if k is a constant, what is $\int e^{kx}\,dx$?

We might guess that an antiderivative is just e^{kx}, but we can check that by taking its derivative. And since kx is a function of x, we need the Chain Rule for the derivative! Note that $\frac{d}{dx}(kx) = k$.

$$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx} \implies \frac{d}{dx}\left(\frac{1}{k}e^{kx}\right) = e^{kx}$$

So

$$\boxed{\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C} \qquad \text{if } k \text{ is constant}$$

Some Indefinite Integral Properties

Derivative rules can often be turned around into rules for indefinite integrals (antiderivatives).

$\frac{d}{dx}(F(x) \pm G(x)) = F'(x) \pm G'(x)$ and $\frac{d}{dx}(CF(x)) = CF'(x)$ become the following sum, difference and constant multiple rules for integrals:

(a) $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

(b) $\int (f(x) - g(x))\,dx = \int f(x)\,dx - \int g(x)\,dx$

(c) $\int Cf(x)\,dx = C\int f(x)\,dx$ (where C is constant)

Warning! There is no general formula for $\int f(x)g(x)\,dx$ or $\int \frac{f(x)}{g(x)}\,dx$.
(No "product" or "quotient" rule for integrals.)

For example, $\int x\cos(x)\,dx \neq \frac{1}{2}x^2\sin(x) + C$