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UNIT - 1

Mathematical Logic & Set Theory

* **Proposition**: Declerative sentence that is either T or F
  Ex: Delhi is capital of India T
  Good Morning X → Neither True
  $x + 3 = 8$ X → can be not False
  $2 + 3 = 8$ F

* **Propositional Variables**: Denoted by variables in small letters.
  Ex: p - Delhi....
  q - Goa is a state

* **Compound prop^ns**: Comb^n of 2 or more prop^ns.

* **Logic operators**:
  → Negation (¬ | ~ | -)
  → Conjunction (∧)
  → Disjunction (∨) (Inclusive OR)
  → Exclusive OR (⊕)
  → Implicative | Cond^nal operator (→ | ⇒)
  → Biimpl^n | Bicond^nal (↔ | ⇔)

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* Negation :

Ex: Akshay's PC runs Linux - p

¬p: Akshay's PC don't runs Linux

⇒ ¬p: This is not the case...

T.T.

p	¬p
0	1
1	0

* Conjunction:

Ex: I bought the lottery ticket - p

q -: I won the jackpot

$p \wedge q$: I bought the lottery ticket and I won the jackpot

p	q	ans
0	0	0
0	1	0
1	0	0
1	1	1

* Disjunⁿ :

Ex: p: Students who have taken 'calculus' can attend this class.

q: Students who have taken 'computer' can attend this class.

* Exclusive OR :

Is true when exactly 1 of p or q are true.

T. T

p	q	ans
0	0	0
0	1	1
1	0	1
1	1	0

* Conditionals | Implicⁿ

$p \rightarrow q$ is false when p is True and q is false

Ex: p: I am elected

q: I will lower the taxes

$p \rightarrow q$: If ---- then ----

Followings are the ways to show condⁿᵃˡ statements

→ If p then q.

→ If p, q.

→ p is sufficient for q.

→ q if p

→ q when p

→ p implies q.

→ p only if q.

* Biimplication | Bicondⁿᵃˡ

$p \leftrightarrow q$ is true when p and q have same truth value.

p	q	ans
0	0	1
0	1	0
1	0	0
1	1	1

Ways to express

→ p is necessary and sufficient condⁿ for q.

→ if p then q.

→ if q then q.

→ p iff q.

Ques: p: It is below freezing

q: It is snowing

(a) It is below freezing and snowing : $p \wedge q$

(b) It is below freezing but not snowing $p \wedge \sim q$

(c) It is not snowing and not bel. frez. $\sim p \wedge \sim q$

(d) It is either snowing or freezing ; $p \vee q$.

(e) If it is below freezing it is also snowing : $p \rightarrow q$.

Que Determine as True or False.

i) If $1+1=2$, then $2+2=5$ $\Rightarrow$ F
(T, F)

ii) If $1+1=3$ then $2+2=4$ $\Rightarrow$ T
(F, T)

iii) If $1+1=3$ then $2+2=5$ $\Rightarrow$ T
(F, F)

iv) If monkey can fly then $1+1=3$ $\Rightarrow$ T
(F, F)

Que For each of these state OR ($\vee$) $\rightarrow$ inclusive OR or Exclusive OR ($\oplus$)

i) To take discrete mathematics you must have taken calculus or course in computers.
$\longrightarrow$ Inclusive OR (only 1 needed, both will also work)

ii) When you buy a new car you will get 25K as a cashback or a 2% car loan.
$\longrightarrow$ Exclusive OR (Only 1 can be opted not both)

iii) Dinner for 2 includes 2 items from column A or 3 items from column B
$\longrightarrow$ Exclusive OR.

iv) School is closed if more than 2 feets of snow falls or if wind is too high
$\longrightarrow$ Inclusive OR.

Ques: Construct the truth table $(p \vee q) \rightarrow (p \oplus q)$

p	q	$p \vee q$	$p \oplus q$	$p \vee q \rightarrow p \oplus q$
0	0	0	0	1
0	1	1	1	1
1	0	1	1	1
1	1	1	0	0

Precedency $\neg$ $\wedge$ $\vee$ $\rightarrow$ $\leftrightarrow$
1 2 3 4 5

Que Find OR AND XOR
(a) 1011110 , 0100001

OR
1011110
0100001
1111111

AND
1011110
0100001
0000000

XOR
1011110
0100001
1111111

Proof Methods :-

Direct (Mathematically)

Indirect
i) Contrapositive
ii) Contradictive.

Ex: If n is an odd integer then n^2 is also an odd integer.
Let $n = 2k+1$
$\therefore n^2 = (2k+1)^2$
$= 4k^2 + 4k + 1$
$= 2k(2k+2)+1$
$= 2K+1$ (odd) HP

* Contrative: if $p \rightarrow q$ is logically equivalent prove $\neg q \rightarrow \neg p$.
$\hookrightarrow$ Using mathematical Induction

* **Contradiction**: If $p \to q$ then consider p and $\neg q$ are true and prove that $\neg q$ is false.

* Principle of Mathematical Induction

To prove $P(n)$ is true for $n \in N$ where $P(n)$ is a +ve prepositional func^n.

→ Step 1: Basic step - We verify basic step by verifying at $n = 0$ or 1 or 2.

→ Step 2: Inductive step - We show that the conditional statement $P(k) \longrightarrow P(k+1)$ is true for $k \in N$.

To prove this we assume that $P(k)$ is true and under that assumption we prove that $P(k+1)$ must be true.

Ex: If n is a +ve integer prove

$$P(n) = 1 + 2 + 3 + \dots n = \frac{n(n+1)}{2}$$

Basic step:

$P(1) \Rightarrow LHS = 1$.

$RHS = \frac{1(1+1)}{2} = 1$ ∵ It is true at $n = 1$.

$LHS = RHS$

Induction step:

Let it is true for $P(k)$ i.e at $n = k$,

$$\therefore P(k) = 1 + 2 + 3 + \dots k = \frac{k(k+1)}{2}$$

∴ at $k+1$ we get

$$P(k+1) = 1 + 2 + 3 + \dots k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$\frac{n(n+1)}{2} \longleftarrow = \frac{(k+1)(k+2)}{2} \quad \text{HP}$$

where $n = k+1$

Eg: $1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

BS:

$P(1) \Rightarrow LHS = 1$

$RHS = \left(\frac{1(1+1)}{2}\right)^2 = 1$

IS:

Let...

$$P(k) = 1^3 + 2^3 + 3^3 + \dots k^3 = \left(\frac{k(k+1)}{2}\right)^2$$

∴ ...

$$P(k+1) = 1^3 + 2^3 + 3^3 + \dots k^3 + (k+1)^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

$$= \frac{k(k+1)^2 + 2^2(k+1)^3}{2^2}$$

$$= \frac{(k+1)^2(k^2 + 2^2(k+1))}{2^2}$$

$$= \frac{(k+1)^2(k+2)^2}{2^2}$$

$$= \left(\frac{(k+1)(k+2)}{2}\right)^2 \quad \text{H.P.}$$

Eg:- $1+2+2^2+ \cdots 2^n = 2^{n+1}-1$

BS:

$P(0)$: $LHS = 1$

$RHS = 2^{0+1}-1 = 2^1 - 1 = 1$

$LHS = RHS$

IS:

$P(k) = 1+2+2^2+\cdots 2^k = 2^{k+1}-1$

$\therefore$

$P(k+1) = 1+2+2^2+\cdots 2^k + 2^{k+1} = 2^{k+1}-1 + 2^{k+1}$

$= 2(2^{k+1})-1$

$= 2^{k+2}-1$

H.P

Ex: $P(n) = 1+2^2+3^2+4^2+\cdots n^2 = \dfrac{n(n+1)(2n+1)}{6}$

$P(1) \Rightarrow LHS = 1$

$RHS = \dfrac{1(1+1)(2(1)+1)}{6} = \dfrac{(2)(3)}{6} = 1$

$\therefore$

$P(k) = 1+2^2+3^2+\cdots k^2 + \cancel{(k+1)^2} = \dfrac{k(k+1)(2k+1)}{6}$

$\therefore$

$P(k+1) = 1+2^2+3^2+\cdots k^2+(k+1)^2 = \dfrac{k(k+1)(2k+1)}{6}+(k+1)^2$

$= \dfrac{k(k+1)(2k+1)+6(k+1)^2}{6}$

$= \dfrac{(k+1)(2k^2+k+6k+6)}{6}$

$= \dfrac{(k+1)(2k^2+7k+6)}{6}$

$= \dfrac{(k+1)(k+2)(k+3)}{6}$ H.P

Que $1+3+3^2+\cdots 3^{n-1} = \dfrac{(3^n-1)}{2}$

$P(1) \Rightarrow LHS = 1$

$RHS = \left(\dfrac{3^1-1}{2}\right) = 1$ $LHS = RHS$

$\therefore P(k) = 1+3+3^2+\cdots 3^{k-1} = \dfrac{(3^k-1)}{2}$

$\therefore P(k+1) = 1+3+3^2+\cdots 3^{k-1} + 3^k = \left(\dfrac{3^k-1}{2}\right)+3^k$

$= \dfrac{3^k-1+2(3^k)}{2}$

$= \dfrac{3(3^k)-1}{2} = \dfrac{3^{k+1}-1}{2}$ H.P.

* Part - 2.

Sets

- Well defined
- Distinct element
- Unordered

Sets

Roster | Tabular

$A = \{1,2,3,4,5\}$

Set Builder

$A = \{x \mid x$ is a prime no less than $100\}$.

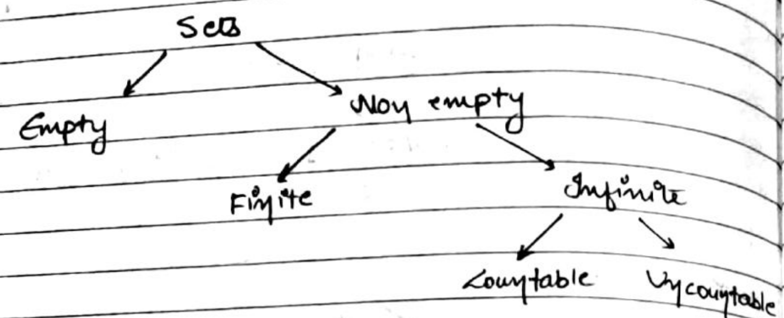
* Cardinality : Gives size of the set

$|A| = 5$

* Finite sets and Infinite sets
* Singleton set → Only 1 element
* Empty sets [A = { } or A = ϕ]

  ~~A = {ϕ}~~

* Equal sets
* Equivalent Sets i.e Cardinality must be same.



* Countable Infinite sets :
If there is one to one mapping b/w elements of sets and natural nos. then it is called as C.I.S.

* Uncountable Infinite Sets :
If there can not be one to one ... U.I.S.

N = +1, +2, + ... ∞

Z = −∞ . . . 0, 1, ... ∞ → Mapping [all ele. can be mapped]

R = −∞ . . . . 0, 1 . . . ∞

↳ Infinite no. of ele b/w 2 nos.

A : Complement of A = A' or A^c

→ Union, Intersection, = U − A

Difference, Symmetric diff

* Power Sets : P(A) Power set of A, be the finite set then the set of all subset of A is called as power set of A.

Eg: A = {1, 2, 3}

P(A) = {ϕ, {1}, {2}, {3}, {1,2}, {2,3}, {1,3}, {1, 2, 3}}.

* A ⊆ P(A) ⇒ False

↳ Subset

∵ P does not contains 1, 2, 3 elements

It contain {1}, ... {1, 2, 3}

A ∈ P(A) ⇒ True

↳ Belongs to

A is present in A { A = {1, 2, 3} }

Redd : $\phi \subseteq P(A)$ $\phi \subseteq A$

$\phi \in P(A)$ $\phi \in A$

Ques A = {5, {6}, {7}}

P(A) = {{5}, {{6}}, {{7}}, {5, {6}}, {{6}, {7}}, {5, {7}}, {5, {6}, {7}}}

↳ 8 elements = $2^n = 2^3$

Ques P(P(P(ϕ))) = ?

P(ϕ) = {ϕ}

P(P(ϕ)) = {ϕ, {ϕ}}

P(P(P(ϕ))) = {ϕ, {ϕ}, {{ϕ}}, {ϕ, {ϕ}}}

* Principle of Inclusion - Exclusion

It says that for any no of finite sets $A_1 \dots A_i$

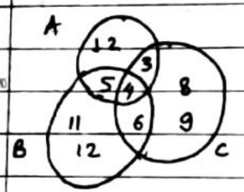
Union of sets is given by = sum of sizes of all single sets - Sum of all sets intersection + sum of all 3 sets intersection - sum of all 4 set intersection + ... $(-1)^{i-1}$ sum of all i set intersection

$|A \cup B| = |A| + |B| - |A \cap B|$ (formula for 2 sets)

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap C| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

In General

$|A_1 \cup A_2 \cup A_3 \dots \cup A_n| = \sum_1^m |A_i| - \sum |A_i \cap A_j| + \dots + (-1)^{i-1} |A_1 \cap A_2 \cap \dots A_n|$



LHS :
$|A \cup B \cup C| = 10$

RHS :
$|A| + |B| + |C| - |A \cap C| \dots$
$= 5 + 5 + 5 - 2 - 2 - 2 + 10$
$= 10$

Ques In a survey of group of 80 people it is found that 60 people like apple and 30 people's like oranges. Find the no of peoples like both apple and orange.

Given $|A \cup B| = 80$ $|A| = 60$ $|B| = 30$

$|A \cup B| = |A| + |B| - |A \cap B|$

$80 = 60 + 30 - x$

$\Rightarrow x = 10$ Ans

* Multisets :

It is an unordered set of elements in ~~with~~ which multiplicity of the element may be 1 or more than 1, or 0 (zero).

no. of times element is repeated

Eg: $A = \{l, l, l, m, n, o, o, p\}$
$B = \{a, a, a, a, b, b, c\}$.

* Union of multi-sets :

It is a multiset such that multiplicity of an element is equal to the max. of multiplicity of elements in the sets.

Eg:
$A = \{1, 1, 2, 2, 2, 3, 3, 3, 3\}$
$B = \{1, 2, 2, 2, 2, 2, 2, 3, 3\}$

$A \cup B = \{1, 1, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3\}$

* Intersection : Common elements with min multiplicity

Eg: $A = \{1, 2, 2, 3, 5, 6, 6, 6\}$
$B = \{1, 2, 4, 6\}$

$A \cap B = \{1, 2, 6\}$

* Difference of sets :

For A-B, take elements from A only which are +ve.

$A = \{1, 1, 2, 2, 2, 3, 3, 5\}$
$B = \{1, 2, 2, 3, 3, 3, 4, 4, 5, 5, 5\}$

from A only

$A - B = \{1, 2\}$.
$B - A = \{3, 4, 4, 5, 5\}$.

$A+B$ = All elements combined.

$A = \{1,3,4,4,5,5\}$

$B = \{1,1,2,2,3,3,4,5\}$ $A+B = \{\ \}$

* Cardinality of multisets

No of distinct element without considering multiplicity of any element.

Ex: $A = \{1,1,1,2,2,3,3,3,3\}$

$|A| = 3$.

PART 3

* Application of Propositional Logic

p: You are not allowed to drive vehicle

q: Age is less than 18.

r: You have age proof.

$(\neg r \vee q) \longrightarrow p$

* System Specification:

→ Consistent SS → All props are true at a time

→ Inconsistent SS → else.

Eg: i> $p \vee q$ → The diagnostic message is stored in buffer or ~~retransmitted~~.

ii> $\neg p$ → The ... not stored in buffer

iii> $p \to q$ → The ... buffer, then it is trans.

P: diagnostic message is stored in buffer

q: diagnostic message is retransmitted

T. T.

p	q	$p \vee q$	$\neg p$	$p \to q$
T	T	T	F	T
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

⇒ All true Consistent

If we add

iv> The diagnostic message is not transmitted

$\neg q$

F

T

F

T

→ All not true at same time

⇒ In Consistent.

Ques i> P if and only if it is operating normally

ii> q, then r

iii> $\neg r$ or the s

iv> $\neg p$ then it is interrupt mode

v> The system is not in interrupt mode

P: the system is in multiuser state

q: the system is operating

r: the kernel is functioning

s: the system is in interrupt mode.

i> $p \leftrightarrow q_F$ → T

ii> $q_F \longrightarrow r_F$ → T

iii> $\neg r_F \vee s_F$ → T

iv> $\neg p_T \to s_F$ → T

v> $\neg s \longrightarrow$ T

F

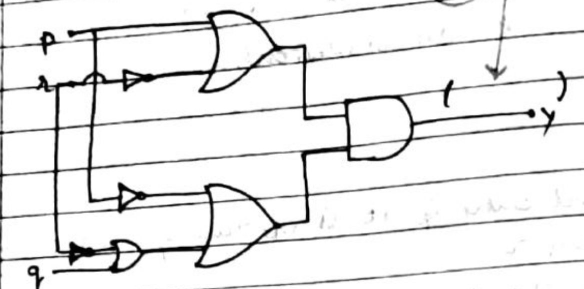
Try making all True with some value (T|F) in p, q, r, s

↳ To make ... T

∴ Inconsistent

Ques Build digital circuit

$(p \vee \neg r) \wedge (\neg p \vee (q \vee \neg r))$



* Propositional Equivalence :

The compound prop. p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology

Denoted by $\equiv$

Classific'n of compd prop are done in 3 parts

Tautology	Contradiction	Contigency
↓	↓	↓
Comp prop which is always True	Always False	Tautology Neigther Taut. ~~False~~ nor Contra.
	$p \wedge \neg p$	

Eg: $p, \neg p \Rightarrow p \vee \neg p$

Ques Show that $\neg(p \vee q)$ and $p \wedge \neg q$ are log. equivalent

p	q	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	F	F	F	F
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

$\Rightarrow$ Tautology.

* Predicates and Quantifiers.

H.W. prev. topic ques

Ques : Prove the following log. prop eq.

i) D'morgan's Law.

ii) Associative Law.

iii) Commutative

iv) Distributive Law.

v) Idempotent.

Ques : $(p \to q)$ and $(p \to r) \equiv p \to (q \wedge r)$

ii) $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$.

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* Quantifiers :

Universal

($\forall$)

↳ For all or For every

Eg $x+1 > x$

For all x, $x+1 > x$

Essential

($\exists$)

↳ There exist an element x in domain such that $P(x)$. or for some...

Eg: $\exists x P(x)$: for the essential quantification of $P(x)$

TYPE-1

Ques Translate to english

$P(x)$: x spends more than 5 hrs every week days in class.

Domain: All students.

i) $\exists x P(x)$: There are some student who spends ... class.

ii) $\forall x P(x)$: All students spends ... class

iii) $\exists x \neg P(x)$: there are some student who does not spend ... class.

iv) $\forall x \neg P(x)$: No student spends ... class.

TYPE 2

Ques $P(x)$: x can speak Russian

$Q(x)$: x knows computer language. C++.

Domain: All students at your school.

i) There is a student...who knows ... and speak ~~at~~ russian.

$\exists x \, (P(x) \wedge Q(x))$

ii) Every student at your school either .... or knows ... C++.

$\forall x \, (P(x) \vee Q(x))$

Ques $P(x) = x = x^2$

Domain : integers

Find truth value

i) $P(0)$: True

ii) $P(1)$: True

iii) $P(2)$: False.

iv) $P(-1)$: False.

v) $\exists x P(x)$: True

vi) $\forall x P(x)$: False.

* $\neg(\forall x P(x))$ is equivalent to $\exists x \neg P(x)$

$\neg(\exists x P(x))$ is equivalent to $\forall x \neg P(x)$.

↳ Proof: $\forall x P(x) \equiv P(x_1) \wedge P(x_2) \wedge \dots \wedge P(x_m)$

$\neg(\forall x P(x)) \equiv \neg(P(x_1) \wedge P(x_2) \dots \wedge P(x_m))$

Apply D'morgan's Law.

$= \neg P(x_1) \vee \neg P(x_2) \vee P(x_3) \dots \vee \neg P(x_m)$

$= \exists x \neg P(x)$

TYPE 3

Eg: Some old dogs can learn new tricks

$P(x)$: x can learn new tricks.

Domain: Old dogs.

Original: $\exists x P(x)$

Negation $\neg(\exists x P(x))$

$= \forall x \neg P(x)$

All old dogs can't learn new tricks.

Que No rabbit knows calculus.
$P(x)$: x knows calculus.
Domain: All rabbits.
Orignal: $\forall x \neg P(x)$
Negⁿ: $\neg(\forall x \neg P(x))$
$= \exists x P(x)$
There exist a rabbit who knows calculus.

Que Every birds can fly
$P(x)$: x can fly
Domain: All birds
Orignal: $\forall x P(x)$.
Neg: $\neg(\forall x P(x))$
$= \exists x \neg P(x)$
There exist a bird who can't fly.

* Nested Quantifiers:

$\forall x \exists y (x+y=0)$
→ For every real no x there is a real no. y such that $x+y=0$.

TYPE-2

Eg: $Q(x,y)$
x has sent an email to y domain for both x and y = All students in your class.

i) $\exists x \exists y Q(x,y)$
Some students in the class has sent email to some students in class.

ii) $\exists x \forall y Q(x,y)$
Some students in class has sent email to every student in class.

iii) $\forall x \exists y Q(x,y)$
All students in class has sent email to some student in class.

iv) $\forall x \forall y Q(x,y)$
All students in class has sent email to every student in class.

* Rules of Inference:
It provides a template for constructing valid arguments.

i) Modus Pones:
→ If $p \rightarrow q$ is true
and p is true
then q has to be true

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline q \end{array}$$

$((p \rightarrow q) \wedge p) \rightarrow q$.

ii) Modus Tollens

$$\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \neg p \end{array}$$

or $[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$.

iii) Hypothetical Syllogism

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline p \rightarrow r \end{array}$$

or $[(p \rightarrow q) \wedge (p \rightarrow r) \rightarrow (p \rightarrow r)]$

iv) Disjunctive syllogism

$p \vee q$
$\neg p$ or $[(p \vee q) \wedge \neg p] \rightarrow q$
q

v) Addⁿ

p — If p is true
$p \vee q$ — p OR q has to be true.

vi) Simplification

$p \wedge q$ — $(p \wedge q) \rightarrow p$ — If $p \wedge q$ is true
$\therefore p$ — or $(p \wedge q) \rightarrow q$ — p has to be true. IIIy for q.

vii) Conjunction

p — p is true and
q — q is true
$p \wedge q$ — p AND q has to be true

viii) Resolution

$p \vee q$
$\neg p \vee r$
$q \vee r$

* How to build arguments:

Eg: → "Raja works hard", "If Raja works hard, then he is a dull boy" and "If Raja is a dull boy then he will not get job" ↳ $D \rightarrow \neg J$

→ Select any 2 at a time only.

Conclusion will be given in question.

H - Raja works hard.
D - Raja is a dull boy
J - Raja will get job.

i) Solve H and $H \rightarrow D$
⇒ D i.e Raja is a dull boy

ii) D and $D \rightarrow \neg J$
⇒ $\neg J$
↳ Raja will not get job

Ques "If it does not rain or its not foggy then sailing race will be held and life saving demo will go on"
"If the sailing race is held, then the trophy will be awarded."
"The Trophy was ~~was~~ not awarded.

Conclusion : It rained

R : It is raining
F : It is foggy
S : Sailing race will be held.
D : Live saving demo will go.
T : Trophy will be awarded

i) $(\neg R \vee \neg F) \rightarrow (S \wedge D)$ (given)
ii) $S \rightarrow T$ (given)
iii) $\neg T$ (given).

(a) $S \rightarrow T$
$\neg T$
$\neg S$ ⇒ Sailing race not held.

(b) $\neg S$

$\underline{\neg S \vee \neg D}$

$\neg (S \wedge D)$

(c) $(\neg R \vee \neg F) \longrightarrow (S \wedge D)$

$\underline{\neg (S \wedge D)}$

$\neg (\neg R \vee \neg F)$

$= R \wedge F$

(d) $\underline{R \wedge F}$

R

$\longrightarrow$ It is raining.