Black Holes Perturbations and QNMs Project-Report 1

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1 Abstract

This report explores the foundational concepts leading to the study of black hole, focusing on the their formation and the Schwarzschild geometry that describes non-rotating black holes. The transition from Newtonian mechanics to Einstein's General Relativity is discussed, highlighting the failures of classical mechanics in accurately describing gravitational phenomena, such as the perihelion precession of Mercury and the gravitational bending of light. These shortcomings necessitated the development of a more robust framework, ultimately leading to General Relativity and the modern understanding of spacetime. The report aims to provide a concise understanding of the basic principles governing black holes, setting the stage for more advanced studies in black hole perturbations and quasi normal modes.

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3 Introduction

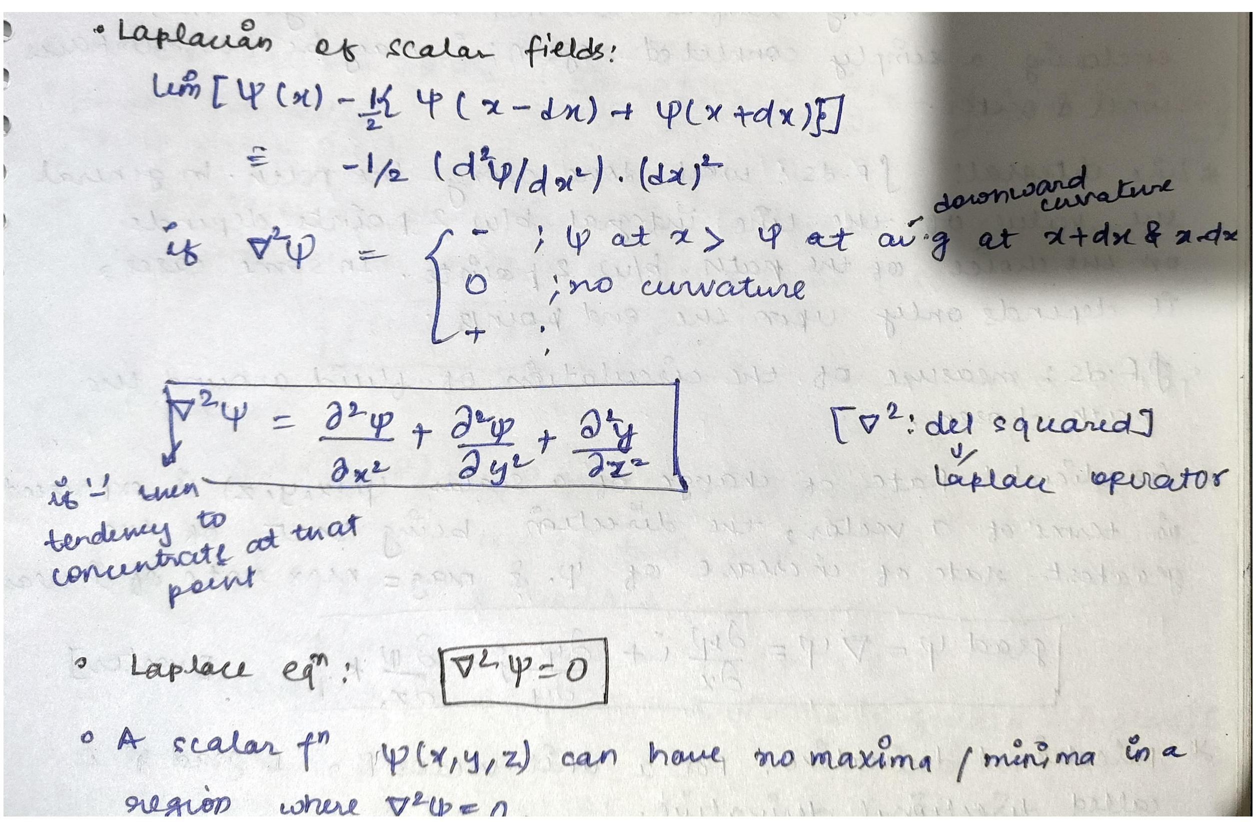
The study of black holes stands at the frontier of modern astrophysics, offering profound insights into the nature of gravity, spacetime, and the fundamental laws governing the universe. Newtonian mechanics, once the cornerstone of our understanding of gravitational phenomena, has been found inadequate in extreme environments, such as those near black holes. This inadequacy, evidenced by phenomena like the unexplained precession of Mercury's orbit and the bending of light by gravity, spurred the development of General Relativity. Einstein's theory revolutionized our understanding by describing gravity not as a force but as the curvature of spacetime itself.

One of the simplest and most significant solutions to Einstein's field equations is the Schwarzschild geometry, which describes the spacetime around a non-rotating, uncharged black hole. This geometry provides the foundation for understanding black hole formation, event horizons, and the singularity, where traditional physics breaks down. As we delve into the complexities of black hole perturbations, it is crucial to first grasp these fundamental concepts, which not only underpin the modern theory of gravity but also guide us in exploring the mysteries of the universe's most enigmatic objects.

4 Basic Terms

Fields: A field is a set of of functions of the coordinates of a point in space. In another words , a field is some convenient mathematical idealization of a physical situation in which extension is an essential element, i.e., which cannot be analysed in terms of the positions of a finite number of particles.

Scalar Fields: When the field under consideration turns out to be simple number, a single function of space and time, it is called a scalar field. They are invariant under coordinate transformations.


• Laplacian of scalar fields:

$$\lim \left[\psi(x) - \tfrac{1}{2}\{\psi(x-dx) + \psi(x+dx)\}\right]$$

$$\cong -1/2\, (d^2\psi/dx^2)\cdot(dx)^2$$

if $\nabla^2\psi = \begin{cases} - \;;\ \psi \text{ at } x > \psi \text{ at avg at } x+dx \;\&\; x-dx & \text{downward curvature} \\ 0 \;;\ \text{no curvature} \\ + \;; \end{cases}$

$$\boxed{\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}}$$

[∇^2: del squared]
$\Downarrow$
Laplace operator

if $-$ then tendency to concentrate at that point

• Laplace eqn : $\boxed{\nabla^2\psi = 0}$

• A scalar fn $\psi(x,y,z)$ can have no maxima / minima in a region where $\nabla^2\psi = 0$

Vector Fields: Fields which require a magnitude and direction to be given at each point for complete characterization, are called vector fields.

Proper Time: When an object is moved in space by amounts dx,dy, and dz in a time dt, with respect to observer A, the time as measured by observer B, moving with the object is dτ_b, where equation of dτ_b can be given as:

$$d\tau = \sqrt{g_{\mu\nu}\, dx^\mu\, dx^\nu} \tag{1}$$

As long as the velocities dx/dt, etc. are small compared to c, the proper time d tau an and $d\tau_b$ will differ by little values; but if the relative velocities nearly equals c, the time intervals may differ considerably.

Coordinate Systems: Coordinates are a systematic way of labeling the points of space-time. The choice of coordinates is arbitrary as long as they supply a unique set of labels for each point in the region they cover. For a particular problem one coordinate system may be more useful than another. In Newtonian Mechanics:

- Cartesian coordinates (x, y, z): The most common system, using three perpendicular axes.
- Spherical coordinates (r, θ, ϕ): Useful for problems with spherical symmetry.
- Cylindrical coordinates (r, ϕ, z): Helpful for problems with cylindrical symmetry.
- Inertial reference frames: These are frames where Newton's laws hold without modification. They move at constant velocity relative to each other.
- Non-inertial reference frames: These are accelerating frames where fictitious forces (like centrifugal force) appear.

  In General Relativity:

- Spacetime coordinates (t, x, y, z): GR uses a 4-dimensional coordinate system, including time as the fourth dimension.
- Manifolds: Spacetime is treated as a 4D manifold, which can be curved and may require multiple coordinate patches to describe fully.
- Metric tensor: This fundamental object $g_{\mu\nu}$ relates coordinate differences to proper distances and times in spacetime.
- Coordinate invariance: The laws of physics in GR are invariant under arbitrary smooth coordinate transformations, a key principle of the theory.
- Schwarzschild coordinates (t, r, θ, ϕ): Used to describe the spacetime around a non-rotating spherical mass.
- Kruskal-Szekeres coordinates: An extension of Schwarzschild coordinates that covers the entire spacetime manifold of a black hole.
- Eddington-Finkelstein coordinates: Another coordinate system used to study black holes, particularly useful for describing in-falling matter.

Christoffel's Symbol Christoffel symbols are mathematical objects used in General Relativity and differential geometry to describe how vectors change as they move along curved surfaces or spacetime. They represent the connection coefficients that account for the curvature of the space or spacetime.

In essence, Christoffel symbols help define how the components of a vector change when parallel transported along a curve in a curved space. They are not tensors themselves but are derived from the metric tensor, which encodes the geometry of the space. The symbols are crucial for computing the covariant derivative, which generalizes the concept of differentiation to curved spaces, ensuring that the derivative of a vector remains valid in curved geometries. There are two types of Christoffel symbols, denoted as Γ

- First kind Γ_{ijk}: symmetric in the lower indices.
- Second kind Γ^{i}_{jk}: used in the geodesic equation, which describes the path of free-falling objects in curved spacetime.

The Christoffel symbols are essential in Einstein's field equations, where they describe the curvature effects due to gravity.

5 Newtonian Mechanics: Description of space-time and gravity

Newtonian theory rests on the assumption that there exists an absolute time, which is the same for every observer, so that t'=t.

If we consider two events A and B that have coordinates (t_A, x_A, y_A, z_A) and (t_B, x_B, y_B, z_B) respectively, it is straightforward to show that both the time difference $\Delta t = t_B - t_A$ and the quantity

$$\boxed{\Delta r^2 = \Delta x^2 + \Delta y^2 + \Delta z^2}$$

are *separately invariant* under any Galilean transformation. This leads us to consider space and time as separate entities. Moreover, the invariance of Δr^2 suggests that it is a geometric property of space itself. Of course, we recognise Δr^2 as the square of the distance between the events in a three-dimensional Euclidean space. This defines the *geometry* of space and time in the Newtonian picture.

5.1 The principle of Relativity

"Identical experiments carried out in different inertial frames give identical results." A principle of relativity relating the form of the laws of physics in inertial frames differing by displacements and rotations is possible only because the geometry of Euclidean space shares those symmetries. The laws of physics would not be invariant under displacements and rotations if the geometry of space were curved like the surface of a potato is in two dimension.

We can verify these symmetries of Euclidean geometry mechanically by examining how its line element changes under displacements and rotation, which will ultimately results in depicting that the form of the line element is invariant under these transformations; so, therefore is Euclidean geometry.

5.2 Newtonian Gravity: Gravitational and Inertial Mass

Newton's law of gravity specifies the gravitational force F that a point mass A with mass M exerts on another point mass B with mass m at a distance r away. The force is an example of central force, means it is attractive, directed along the line between the masses and inversely proportional to r^2.

$$m\vec{a} = -m\vec{\nabla}\Phi$$

It has been clear that all the bodies fall with the same acceleration in a gravitational field independently of their mass or composition. Mass plays two distinguished roles: on the left-hand side of the above equation governs the inertial properties of the body and in this role it is called inertial mass m_1 of the body, while mass on the right-hand side measures the strength of the gravitational force between bodies and is called the body's gravitational mass, m_g.

A similarity between gravitational force law and the electrostatic law can be observed. The analogy can be observed in the table below:

TABLE 3.1 Newtonian Gravity and Electrostatics

	Newtonian Gravity	Electrostatics
Force between two sources	$\vec{F}_{\text{grav}} = -\frac{GmM}{r^2}\vec{e}_r$	$\vec{F}_{\text{elec}} = +\frac{qQ}{4\pi\epsilon_0 r^2}\vec{e}_r$
Force derived from potential	$\vec{F}_{\text{grav}} = -m\vec{\nabla}\Phi(\vec{x}_B)$	$\vec{F}_{\text{elec}} = -q\vec{\nabla}\Phi_{\text{elec}}(\vec{x}_B)$
Potential outside a spherical source	$\Phi = -\frac{GM}{r}$	$\Phi_{\text{elec}} = \frac{Q}{4\pi\epsilon_0 r}$
Field equation for potential	$\nabla^2\Phi = 4\pi G\mu$	$\nabla^2\Phi_{\text{elec}} = -\rho_{\text{elec}}/\epsilon_0$

5.3 Variational Principle for Newtonian Mechanics

Variational Principle for Newtonian Mechanics.

- The laws of Newtonian Mechanics can be formulated in terms of variational principle called principle of extreme action. Extention of this principle will be the route to formulating the E.O.M.
- An action is a scalar (numerical value).

$$F[f(x)] \rightarrow \text{action/scalar/numerical value.}$$

- This action is for a path taken by the particle.
- Action is written as:

$$\boxed{S[x(t)] = \int_{t_a}^{t_b} dt.\, L[\dot{x}(t), x(t)]}$$

- First order derivative / Extrema can be written as:

$$\delta S[x(t)] = \int_{t_A}^{t_B} dt. \left[\frac{\partial L}{\partial \dot{x}(t)} . \delta\dot{x}(t) + \frac{\partial L}{\partial x(t)} . \delta x(t)\right]$$

$$= \frac{\partial L}{\partial \dot{x}(t)} . \delta x(t) \Big|_{t_A}^{t_B} + \int_{t_A}^{t_B} dt \left[-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}(t)}\right) + \left(\frac{\partial L}{\partial x(t)}\right)\right] \delta x(t) \quad \text{--- (1)}$$

- Variations of the path that connects x_A to x_B at t_A & t_B necessarily vanish at the endpoints – $\delta x(t_A) = \delta x(t_B) = 0$
- First term in eq^n (1) vanishes at endpoints and the remaining term has to be 0 for any arbitary $\delta x(t)$ to satisfy these conditions for $\delta S[x(t)]$ to vanish. It is possible only when:

$$\boxed{-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial x} = 0} \text{ — Satisfying Lagrange's equation.}$$

- For multidimensional:

$$\boxed{-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^{\mu}}\right) + \frac{\partial L}{\partial x^{\mu}} = 0} \quad \text{where } \mu = 0, 1, 2 \cdots n$$

- Path of extremal action occurs at $\boxed{dS/dx = 0}$

Variational Principle for N.M.:

A particle moves b/w a point in space at one time and another point in space at a later time so as to extremize the action in between.

- Newton's theorem:

The gravitational field outside a spherically symmetric mass distribution depends only on its total mass.

6 Transition to Theory of Relativity

The rise of General Relativity stemmed from key failures of Newtonian mechanics:

- **Perihelion Precession of Mercury:** Newtonian gravity couldn't fully explain Mercury's orbit precession, which General Relativity resolved by considering spacetime curvature.
- **Inconsistency with Special Relativity:** Newton's concept of instantaneous gravity conflicted with Special Relativity's speed of light limit, while General Relativity described gravity as spacetime curvature propagating at light speed.
- **Gravitational Time Dilation:** Newtonian mechanics treated time as absolute, ignoring how gravity affects time, a concept introduced by General Relativity.
- **Gravitational Redshift:** Newtonian theory lacked an explanation for the redshift of light in a gravitational field, which General Relativity accounted for.
- **Equivalence Principle:** Newtonian mechanics treated gravitational and inertial mass separately, whereas General Relativity unified them under the equivalence principle.
- **Light Bending by Gravity:** Newtonian predictions for light bending were inaccurate; General Relativity matched observations.
- **Strong Gravitational Fields:** Newtonian mechanics failed under strong gravity, whereas General Relativity successfully predicted phenomena like black holes.

These shortcomings led to the development of General Relativity, revolutionizing our understanding of gravity.

7 Theory of Relativity and its mechanics

In special relativity, Einstein abandoned the postulate of an absolute time and replaced it by the postulate that the speed of light is the same in all inertial frames. Minkowski advocated that space and time are united in four-dimensional continuum called spacetime, whose geometry is characterized by interval(squared):-

$$\boxed{\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2}$$

This quantity is Lorentz invariant. Also it is to be noted that the spacetime in special relativity is non-Euclidean, because of minus sign in above equation, which is often called the pseudo-Euclidean or Minkowski geometry.

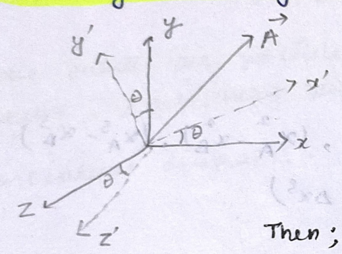
7.1 Minkowski Spacetime: Concept of Four-Vectors

A four-vector is defined as a directed line segment in four dimensional flat space-time in the same way as a three-dimensional vector can be defined as a directed line segment in three-dimensional Euclidean space.

Four-vectors can be multiplied by numbers, added, and subtracted according to the usual rules for vectors.

Four Vectors :-

There are 3 vectors and 4 vectors. 3 vectors are normal vectors we use in our daily lives; they have 3 components for ex:- $\vec{A} = \vec{a_x} + \vec{a_y} + \vec{a_z}$. But a 4th dimensional representation of this 3 vector is known as 4 vectors. If we rotate the co-ordinate axis by some angle; then the vector remains unchanged but its components undergo changes definitely



$$\begin{bmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{bmatrix} = \begin{bmatrix} \hat{x}\cdot\hat{x}' & \hat{y}\,\hat{x}' & \hat{z}\,\hat{x}' \\ \hat{x}\,\hat{y}' & \hat{y}\,\hat{y}' & \hat{z}\,\hat{y}' \\ \hat{x}\,\hat{z}' & \hat{y}\,\hat{z}' & \hat{z}\,\hat{z}' \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Then; 4 vector that has 4 components (a^0, a^1, a^2, a^3) are a set of 4 quantities that under a Lorentz transformation transforms a/c to a similar expression.

$$\boxed{\bar{a}^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\ \nu} a^{\nu}}$$

i.e. $\bar{a}^{\mu}$ and a^{μ} will have same expression a/c to the above eqn.

[Lorentz transf. is a condition for the existence of 4 vectors]

Scalar product :- $\boxed{a^{\mu} \cdot b^{\mu} = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3.}$

As the scalar product of 2 vectors in 3D is invariant in the same way, the scalar product of 2 vectors in 4D is also invariant.

Since we consider 4-dimensions in Minkowski spacetime, we can redefine displacement, velocity and acceleration as displacement four-vector, four-velocity, and four-acceleration.

Four Vectors

$$a = \sum_{\alpha=0}^{3} a^{\alpha} e_{\alpha}$$

- Scalar Product : $a \cdot b = (a^{\alpha} e_{\alpha}) \cdot (b^{\beta} e_{\beta})$

$$\boxed{a \cdot b = (a^{\alpha} b^{\beta}) \cdot (\underbrace{e_{\alpha} e_{\beta}}_{\eta_{\alpha\beta}})}$$

$\eta_{\alpha\beta}$ (metric of flat spacetime)

- $\boxed{ds^2 = \eta_{\alpha\beta}\, dx^{\alpha} dx^{\beta}}$
- $\boxed{a \cdot b = -a^t b^t + a^x b^x + a^y b^y + a^z b^z}$
- **Four Velocity**

$$\boxed{u^{\alpha} = \frac{dx^{\alpha}}{d\tau}}$$

here, $\boxed{u^x = \frac{dx}{d\tau} = \frac{dx}{dt} \cdot \frac{dt}{d\tau} = \frac{v^x}{\sqrt{1-\bar{v}^2}}}$

and $\boxed{u^t = \frac{dt}{d\tau} = \frac{1}{\sqrt{1-\bar{v}}}}$

thus, $\boxed{u^{\alpha} = (\gamma, \gamma\bar{v})}$ where $\vec{v} = d\bar{x}/dt$

- **Normalization of Four velocity**

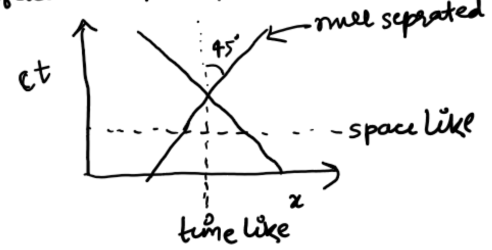
$$\boxed{u \cdot u = \eta_{\alpha\beta} \frac{dx^{\alpha}}{d\tau} \cdot \frac{dx^{\beta}}{d\tau} = -1}$$

7.2 Light cones and World Lines

- Light Cones

$$(\Delta s)^2 = \begin{cases} >0 & \text{(space like separated)} :- [\Delta t = 0 \text{ but } \Delta x \neq 0] \\ =0 & \text{(null separated)} :- [\Delta x = c\Delta t \text{ but } \Delta y = \Delta z] \text{ also light like} \\ <0 & \text{(timelike separated)} :- [\Delta x = \Delta y = \Delta z = 0] \text{ but } \Delta t \neq 0 \end{cases}$$

- The locus of points that are null separated from a point P in spacetime is its light cone. The light cone of P is 3-d surface in 4-d spacetime.



- Particles with non-zero rest mass move along timelike world line that are always inside the light cone of any point along their trajectory. That way their velocity is always $< c$ at that point.
- Spacelike : tachyons [hypothetical particles]

Light rays move along null curves in spacetime along which $ds^2=0$. The family of null directions emerging from, or converging on, a point P span the local future and past light cones at P.

Particles move on timelike world lines which can be specified parametrically by four functions $x^\alpha(\tau)$ of the distance τ along them, just as it can in special relativity (Section 5.2). In curved spacetime the distance between a point A and a point B along a timelike world line is given by the curved spacetime generalization of (4.13),

$$\tau_{AB} = \int_A^B \left[-g_{\alpha\beta}(x)dx^\alpha dx^\beta\right]^{1/2}, \tag{7.19}$$

where the integral is along the world line. A clock carried along this curve mea-

The local light cone structure of general relativity is the same as that of flat spacetime.

7.3 Gravity in General Relativity

Newtonian gravity can be expressed completely in geometric terms in the curved spacetime by following equation:

$$\boxed{ds^2 = -\left(1 + \frac{2\Phi(x^i)}{c^2}\right)(cdt)^2 + \left(1 - \frac{2\Phi(x^i)}{c^2}\right)(dx^2 + dy^2 + dz^2),}$$

Rather than saying the presence of mass produces a gravitational potential Φ, which determines particle motion, one can say the presence of mass produces spacetime curvature described by above equation, and particles move in this geometry along paths of extremal proper time. ***Concepts of force and affects on clocks have been replaced by geometric ideas.*** In a sense, the equality of gravitational and inertial mass has been explained because the idea of mass never enters into the description of motion of a particle moving under the influence of a curvature-producing mass. The law of motion is the same as that of a free particle, but in a curved spacetime. In flat spacetime the straight line path between two points is also a curve of longest proper time.

TABLE 6.1 Newtonian and Geometric Formulations of Gravity Compared

	Newtonian	Geometric Newtonian	General Relativity
What a mass does	Produces a field Φ causing a force on other masses $\vec{F} = -m\vec{\nabla}\Phi$	Curves spacetime $ds^2 = -\left(1+\frac{2\Phi}{c^2}\right)(c\,dt)^2 + \left(1-\frac{2\Phi}{c^2}\right)(dx^2+dy^2+dz^2)$	Curves spacetime
Motion of a particle	$m\vec{a} = \vec{F}$	Curve of extremal proper time (first order in $1/c^2$)	Curve of extremal proper time
Field equation	$\nabla^2\Phi = +4\pi G\mu$	$\nabla^2\Phi = +4\pi G\mu$	Einstein's equation

7.4 Curved Spacetime

There are actually two distinguishable ways in which geometry eneters the theory of general relativity. One is the geometry of lengths and angles in four-dimensional spacetime, which is inherited from the metric structure ds^2 of special relativity. Schild's argument shows that the special relativistic ideas of length and angle must be modified. The modified ideas of metric structure lead to Riemannian geometry. However, geometry also enters into general relativity because of the equivalence principle, which can already be stated within Newtonian gravitational theory, in which no concepts of a spacetime metric enters but only the Euclidean metric structure of three-dimensional space. The equivalnce-princple view of Newtonian theory again insists that the local standard of reference be freely falling particles. this requirement leads to the study of a spacetime geometry in which the curved world lines of freely falling particles are defined to be locally straight. They lay the role in a curved spacetime geometry that straight lines play in flat spacetime.

7.5 Geodesics

Geodesics is already defined in the previous sections. Here we will dive deeper into this concept and deal with its mathematical framework.

The equations governing the motion of test particles and light rays in a general curved spacetime are derived and analysed in this section. Only test particles are from any influences other than the curvature of spacetime are considered. Such particles are called free or **freely falling** in general relativity. In general relativity, free means free from any influences besides the curvature of spacetime.

Variational Principle for Free Test Particle Motion: The world line of a free test particle between two timelike points extremizes the proper time between them and this extremal proper time world lines are called geodesics, and the equations of motion that determine them comprise the geodesic equation. There are two properties of geodesics:

- First, its tangent vector always points in the same direction along the line.
- Secong, it is the curve of shortest length between two points.

The Geodesic Equation can be written as:-

$$\boxed{\frac{d^2x^a}{du^2} + \Gamma^a{}_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0.}$$

7.5.1 Null Geodesics

The paths followed by free particles through curved spacetime has been explored, these are the timelike geodesics. Light rays move along null world lines for which $ds^2 = 0$, i.e.,

$$\mathbf{u}\cdot\mathbf{u} = g_{\alpha\beta}(x)\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0.$$

If the geodesic is timelike, then it is a possible world line for a freely falling particle, and its uniformly ticking parameter λ (called 'affine parameter') is a multiple of the particle's proper time, $\lambda = a\tau + b$. (Principle of equivalence: test particles move on straight lines in local Lorentz frames, and each particle's clock ticks at a uniform rate as measured by an Lorentz observer).

7.6 Gravitational Lensing

The gravitational attraction of mass deflects light. Because of this bending there can be multiple pathways for light to use in traveling from a source to an observer. An intervening mass can, therefore , produce multiple images of a distant source. Acting in this way, the concentration of mass is called a gravitational lens.

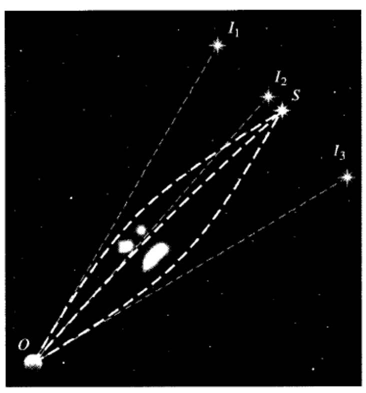


FIGURE 11.1 The idea behind a gravitational lens. Intervening mass can bend light from a distant source S to produce multiple pathways for light to travel from it to an observer O. The observer sees these as multiple images of the source. The diagram illustrates how images of one source S could be produced at angular locations I_1, I_2, and I_3. Almost everything about the diagram is exaggerated for clarity. Realistically, the size of the lens is tiny compared to the distances involved, the bending angles are minute, and the images unlikely to line up in a plane.

(a) First subfigure

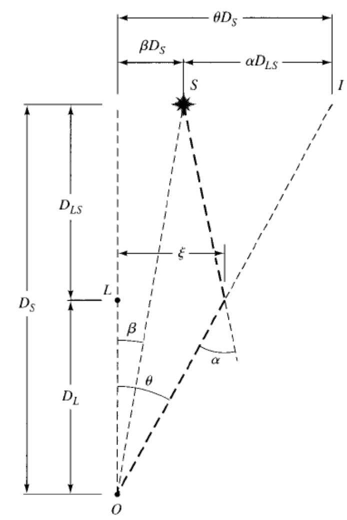


FIGURE 11.2 The geometry of a gravitational lens in the thin lens approximation. O is the observer. L is the location of the lensing mass at a distance D_L from the observer. S is the source located a distance D_S from the observer and D_{LS} from the lens. The figure shows the source-lens-observer plane. The heavy dashed line shows the path of a light ray from source to observer. The ray passes by the lens with an impact parameter that differs negligibly from the distance ξ and is deflected by an angle $\alpha = 4GM/(c^2\xi)$, where M is the mass of the lens. In the thin lens approximation, the lens is treated as a point and all the deflection takes place in a transverse plane at the position of the lens, L. An image of the source, I, appears at an angle θ from the observer-lens axis rather than its true direction, β. The transverse distances in this diagram are all greatly exaggerated. Were they drawn to true scale it would not be possible to distinguish any of the lines in the figure. The relationship between the transverse distances at the top of the figure constitutes the lens equation.

(b) Second subfigure

Lens Geometry and Image Position

The deflection angle α for a light ray passing by a mass M at an impact parameter $b \gg M$ is given by (9.83) and is

$$\alpha = \frac{4GM}{c^2 b} \equiv \frac{2R_S}{b}. \tag{11.1}$$

The relationship between the transverse distances at the top of Figure 11.2 is

$$\boxed{\theta D_S = \beta D_S + \alpha D_{LS}} \tag{11.3}$$

and is called the *lens equation*. Because $b \approx \xi$ and $\xi \approx \theta D_L$ in the small-angle approximation, the lens equation can be written using (11.1) as

$$\theta = \beta + \frac{\theta_E^2}{\theta}, \tag{11.4}$$

where

$$\boxed{\theta_E \equiv \left[2R_S\left(\frac{D_{LS}}{D_S D_L}\right)\right]^{1/2}} \tag{11.5}$$

is called the *Einstein angle*. The solutions of (11.4) determine the angular position of the images on the sky.

$$\Delta\theta_\pm = \frac{1}{2}\left[1 \pm \frac{\beta}{\left(\beta^2 + 4\theta_E^2\right)^{1/2}}\right]\Delta\beta.$$

The images of the galaxy are thus elongated and distorted.

From this discussion it follows that the ratio of the brightness of the images $I_\pm$ at the positions $\theta_\pm$ to the unlensed brightness I_* will be the ratio of the solid angles $\Delta\Omega_\pm$ that the images subtend when the lens is present to the value $\Delta\Omega_*$ they would subtend were it not. Using the familiar expression for an element of solid angle in polar coordinates, this is

$$\frac{I_\pm}{I_*} = \frac{\Delta\Omega_\pm}{\Delta\Omega_*} = \left|\frac{\theta_\pm \Delta\theta_\pm \Delta\phi}{\beta \Delta\beta \Delta\phi}\right|. \tag{11.9}$$

Since $\Delta\phi$ is preserved, the magnification is

$$\frac{I_\pm}{I_*} = \left|\left(\frac{\theta_\pm}{\beta}\right)\left(\frac{d\theta_\pm}{d\beta}\right)\right| = \frac{1}{4}\left(\frac{\beta}{\left(\beta^2 + 4\theta_E^2\right)^{1/2}} + \frac{\left(\beta^2 + 4\theta_E^2\right)^{1/2}}{\beta} \pm 2\right) \tag{11.10}$$

from (11.6) and (11.7). Since $x + 1/x \geq 2$ for any x, the expression in brackets is always positive. The image outside the Einstein ring is brighter, and the one inside is dimmer.

For microlensing by stars where the images cannot be resolved, the *total* magnification is of interest:

$$\boxed{\frac{I_{\text{tot}}}{I_*} \equiv \frac{I_+ + I_-}{I_*} = \frac{1}{2}\left(\frac{\beta}{\left(\beta^2 + 4\theta_E^2\right)^{1/2}} + \frac{\left(\beta^2 + 4\theta_E^2\right)^{1/2}}{\beta}\right).} \tag{11.11}$$

This function of β is always *greater* than unity. The gravitational lens therefore always enhances total brightness, and if the source is close to the observer-lens axis so that β is small, this enhancement can be substantial. As we will see shortly, this enhancement is the reason that gravitational lenses can be detected and used even when the individual images cannot be resolved.

7.7 Schwarzschild Geometry

The simplest curved spacetimes of general relativity are the ones with the most symmetry, and the most useful of these is the geometry of empty space outside a spherically symmetric source of curvature, for example, a spherical star. This is called the **Schwarzschild geometry**.

In a particularly suitable set of coordinates, the line element summarizing the Schwarzschild geometry is given by ($c \neq 1$ units)

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)(c\,dt)^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{9.1}$$

The metric corresponding to Schwarzschild coordinates is called Schwarzchild metric, which has following properties:

- **Time Independent:** There is a killing vector ξ associated with this symmetry under displacements in the coordinate time t which has the components:

$$\xi^\alpha = (1, 0, 0, 0)$$

- **Spherically Symmetric:** The Schwarzschild geometry has the symmetries of a sphere with regard to changes in the angle θ and ϕ. The killing vector associated with this symmetry is:

$$\eta^\alpha = (0, 0, 0, 1).$$

- **Mass M:** If GM/c^2 is small, the coefficient of dr^2 in the line element (eqn 9.1) can be expanded to give:

$$ds^2 \approx -\left(1 - \frac{2GM}{c^2 r}\right)(c\,dt)^2 + \left(1 + \frac{2GM}{c^2 r}\right)dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{9.6}$$

This is exactly the form of the static, weak field metric (6.20) with a Newtonian gravitational potential Φ given by

$$\Phi = -\frac{GM}{r}. \tag{9.7}$$

This leads to the identification of the constant M in the Schwarzschild metric (9.1) with the *total mass* of the source of curvature.

- **Schwarzschild Radius:** Some unusual things happen at r=0 and $r = 2GM/c^2$, the later is known as Schwarzchild Radius and is the characteristic length scale for the curvature in the Schwarzschild geometry.

In geometrized units the Schwarzschild line element has the form

$$\boxed{ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right).} \tag{9.9}$$

Explicitly the metric $g_{\alpha\beta}$ is

$$g_{\alpha\beta} = \begin{array}{c} \\ t \\ r \\ \theta \\ \phi \end{array} \begin{array}{c} \begin{array}{cccc} t & r & \theta & \phi \end{array} \\ \begin{pmatrix} -(1-2M/r) & 0 & 0 & 0 \\ 0 & (1-2M/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \end{array}. \tag{9.10}$$

7.7.1 Geodesics in the Schwarzschild geometry

Lagrangian L for a Schwarzschild geomtery can be given by:

$$L = c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right).$$

Substituting this in the Euler-Lagrange equations, we find the four resulting geodesic equations(for μ=0,1,2,3):

$$\left(1-\frac{2\mu}{r}\right)\dot{t} = k, \tag{9.17}$$

$$\left(1-\frac{2\mu}{r}\right)^{-1}\ddot{r} + \frac{\mu c^2}{r^2}\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-2}\frac{\mu}{r^2}\dot{r}^2 - r\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) = 0, \tag{9.18}$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0, \tag{9.19}$$

$$r^2\sin^2\theta\,\dot{\phi} = h. \tag{9.20}$$

Here, the quantities k and h are constants and $\mu = 2GM/c^2$.Because of the spherical symmetry of the Schwarzschild metric, we can therefore, with no loss of generality, confine our attention to particles moving in the 'equatorial plane' given by $\theta = \pi/2$. In this case our set of geodesic equations:

$$\left(1-\frac{2\mu}{r}\right)\dot{t} = k,$$

$$\left(1-\frac{2\mu}{r}\right)^{-1}\ddot{r} + \frac{\mu c^2}{r^2}\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-2}\frac{\mu}{r^2}\dot{r}^2 - r\dot{\phi}^2 = 0,$$

$$r^2\dot{\phi} = h.$$

7.7.2 Trajectories of massive particles

Trajectory of a massive particle is a timelike geodesic. Since we are considering a timelike geodesic, we can choose our affine parameter sigma to be proper time τ along the path. Thus we find that the worldline $x_\mu(\tau)$ of a massive particle moving in the equatorial plane of the Schwarzschild geometry must satisfy the equations:

$$\left(1-\frac{2\mu}{r}\right)\dot{t}=k, \qquad (9.29)$$

$$c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2-\left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2-r^2\dot{\phi}^2=c^2, \qquad (9.30)$$

$$r^2\dot{\phi}=h. \qquad (9.31)$$

We shall use the following 'energy' equation for the r-coordinate to discuss radial free fall and stability of orbits.

$$\boxed{\dot{r}^2+\frac{h^2}{r^2}\left(1-\frac{2GM}{c^2r}\right)-\frac{2GM}{r}=c^2(k^2-1),}$$

we can find the expression for k as $k=E/(m_0c^2)$ by letting r tending to infinity and $\dot{r}$ equals to 0 in eqn 9.32. A second useful equation, which enables us to determine the shape of a particle orbit (i.e. r as a function of ϕ) may be found by using $h=r^2\dot{\phi}^2$ to express $\dot{r}$ in the energy equation 9.32 as:

the energy equation (9.32) as

$$\frac{dr}{d\tau}=\frac{dr}{d\phi}\frac{d\phi}{d\tau}=\frac{h}{r^2}\frac{dr}{d\phi}.$$

We thus obtain

$$\left(\frac{h}{r^2}\frac{dr}{d\phi}\right)^2+\frac{h^2}{r^2}=c^2(k^2-1)+\frac{2GM}{r}+\frac{2GMh^2}{c^2r^3}.$$

If we make the substitution $u\equiv 1/r$ that is usually employed in Newtonian orbit calculations, we find that

$$\left(\frac{du}{d\phi}\right)^2+u^2=\frac{c^2}{h^2}(k^2-1)+\frac{2GMu}{h^2}+\frac{2GMu^3}{c^2}.$$

We now differentiate this equation with respect to ϕ to obtain finally

$$\boxed{\frac{d^2u}{d\phi^2}+u=\frac{GM}{h^2}+\frac{3GM}{c^2}u^2.} \qquad (9.33)$$

There are two interesting special cases of massive-particle orbits, namely radial motion and motion in a circle.(Though they are not covered in this report)

7.7.3 Trajectories of mass-less particles:photons

The trajectory is a photon or any other mass-less particle(rest mass is zero) is a null geodesic. We cannot use the proper time τ as parameter, so instead we use some affine parameter sigma along the geodesic. Applying null geodesic condition in the above geodesic equations, in the equatorial plane we can have:

$$\left(1-\frac{2\mu}{r}\right)\dot{t}=k, \qquad (9.45)$$

$$c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2-\left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2-r^2\dot{\phi}^2=0, \qquad (9.46)$$

$$r^2\dot{\phi}=h. \qquad (9.47)$$

For photon trajectories, an analogue of the energy equation (9.32) can again be obtained by substituting (9.45) and (9.47) into (9.46), which gives

$$\boxed{\dot{r}^2 + \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right) = c^2k^2.} \tag{9.48}$$

Similarly, the analogue for photons of the shape equation (9.33) is obtained by substituting $h = r^2\dot{\phi}$ into (9.46) and using the fact that

$$\frac{dr}{d\sigma} = \frac{dr}{d\phi}\frac{d\phi}{d\sigma} = \frac{h}{r^2}\frac{dr}{d\phi}.$$

Making the usual substitution $u \equiv 1/r$ and differentiating with respect to ϕ we find

$$\boxed{\frac{d^2u}{d\phi^2} + u = \frac{3GM}{c^2}u^2.} \tag{9.49}$$

7.7.4 Stability of Circular Orbits: Massive Particles

Stable circular orbits occur at the radii $r = r_{min}$ of the minima of the effective potential. In Newtonian dynamics, a finite angular momentum provides an angular momentum barrier preventing a particle reaching r=0. This is not true in general relativity.

In Newtonian dynamics the equation of motion of a particle in a central potential can be written

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + V_{\text{eff}}(r) = E,$$

where $V_{\text{eff}}(r)$ is the effective potential and E is the total energy of the particle per unit mass. For an orbit around a spherical mass M, the effective potential is

$$V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{h^2}{2r^2}, \tag{9.43}$$

where h is the specific angular momentum of the particle. This effective potential is shown below:

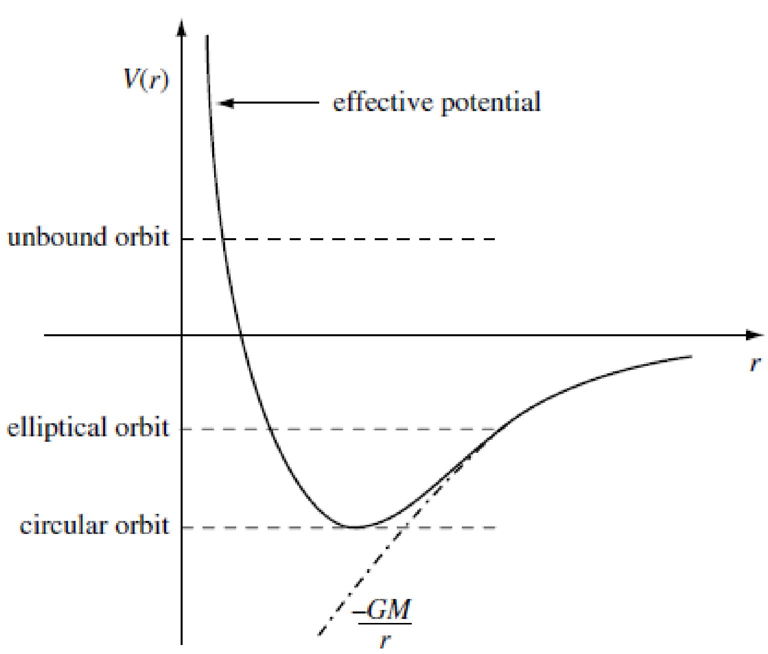


Figure 9.3 The Newtonian effective potential for $h \neq 0$, showing how an angular momentum barrier prevents particles reaching $r = 0$.

In general relativity we identify the effective potential per unit mass as:

$$V_{\text{eff}}(r) = -\frac{\mu c^2}{r} + \frac{h^2}{2r^2} - \frac{\mu h^2}{r^3},$$

This reduces to the Newtonian form when the non-relativistic limit c$\rightarrow \infty$.

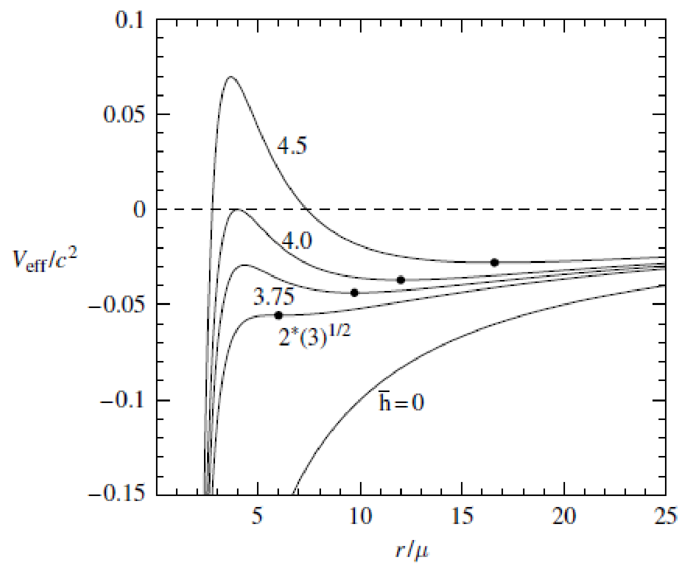


Figure 9.4 The general relativistic effective potential plotted for several values of the angular momentum parameter $\bar{h}$.

This can be helpful to calculate the inner most stable orbit which comes out to be:

$$r_{\min} = 6\mu = \frac{6GM}{c^2}.$$

Also, we can find the angular velocity in stable circular orbit through this expression:

$$\boxed{\Omega^2 = \frac{M}{r^3} \qquad \text{(circular orbits).}}$$

7.7.5 Stability of Orbits: Photons

We can rewrite the ‘energy’ equation (9.48) for photon orbits as

$$\frac{\dot{r}^2}{h^2} + V_{\text{eff}}(r) = \frac{1}{b^2}, \tag{9.51}$$

where we have defined the quantity $b = h/(ck)$ and the effective potental

$$V_{\text{eff}}(r) = \frac{1}{r^2}\left(1 - \frac{2\mu}{r}\right).$$

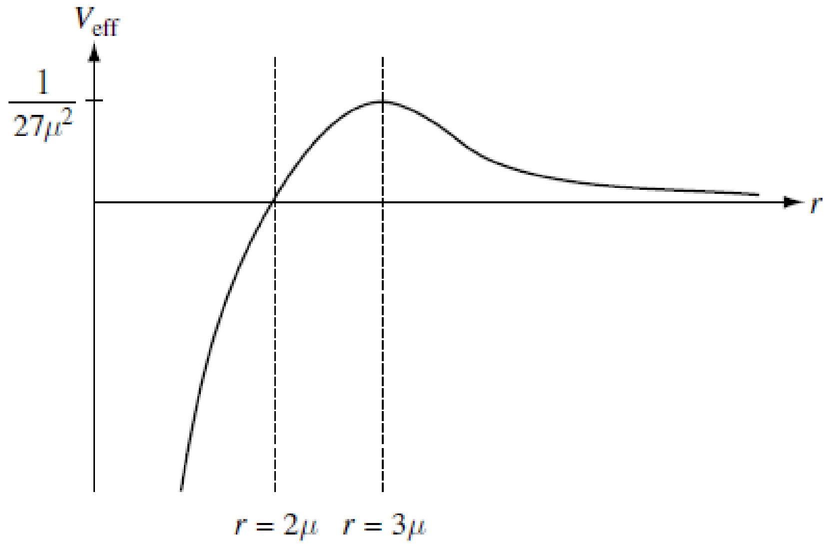


Figure 9.7 The effective potential for photon orbits.

It is straightfoward to show that if a photon is emitted from within the region r $= 2\mu$ to r $= 3\mu$ then the opening angle alpha from the radial direction for the photon to escape varies from $\alpha = 0$ at r $= 2\mu$ to r $= \pi/2$ at r $= 3\mu$.

7.8 Gravitational Collapse and Formation of a Black Hole

When a star runs out of thermonuclear fuel, there are two possibilities: Either the end state is an equilibrium star, supported against the force of gravity by a non thermal source of pressure, or the star never reaches equilibrium and the end state is ongoing gravitational collapse. In 1930, Chandrasekhar realized that the more massive a white dwarf, the denser it must be and so the stronger the gravitational field. For white dwarfs over a critical mass of about 1.4 $M_\odot$ (now called the Chandrasekhar limit), gravity would overwhelm the degeneracy pressure and no stable solution would be possible.Thus, the gravitational collapse of the object must continue. There is a pressure because the Pauli exclusion principle forbids two electrons from being in the same quantum state. This is called **electron Fermi pressure**. White dwarfs and neutron stars are able to balance this pressure. The state of ongoing gravitational collapse leads to a black hole.

7.8.1 The Schwarzschild Black Hole

Eddington-Finkelstein Coordinates Due to collapse, we have to face singularity at r=2M and r=0. The singularity in the Schwarzschild metric at r=2M turns out not to be a singularity in the geometry of spacetime, but a singularity in Schwarzschild coordinates.Using Eddington-Finkelstein Coordinates, we can understand why the Schwarzschild geometry is a black hole. To introduce EF coordinate, trade the Schwarzschild time coordinate t for a new coordinate v defined for by

$$\boxed{t = v - r - 2M \log\left|\frac{r}{2M} - 1\right|.} \tag{12.1}$$

Starting from either $r < 2M$ or $r > 2M$ and transforming t to v in the line element (9.9) gives the same result (Problem 3):

$$\boxed{ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dv\,dr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).} \tag{12.2}$$

At r=2M, we will see nothing special about the local spacetime and at larger r, the metric approaches a flat metric- the usual flat metric with t replaced by v-r.

Light Cones of the Schwarzschild Geometry

The key to understanding the Schwarzschild geometry as a black hole is the behavior of radial light rays. These move along world lines for which $d\theta = d\phi = 0$ (radial) and $ds^2 = 0$ (null), i.e., from (12.2), those for which

$$-\left(1 - \frac{2M}{r}\right) dv^2 + 2dv\,dr = 0. \tag{12.3}$$

An immediate consequence is that some radial light rays move along the curves

$$\boxed{v = \text{const.} \qquad \text{(ingoing radial light rays).}} \tag{12.4}$$

From (12.1) we see that these are *ingoing* light rays because as t increases, r must decrease to keep v constant. The other possible solution to (12.3) is

$$-\left(1 - \frac{2M}{r}\right) dv + 2dr = 0. \tag{12.5}$$

This can be solved for dv/dr and the result integrated to find that these radial light rays move on the curves

$$\boxed{v - 2\left(r + 2M \log\left|\frac{r}{2M} - 1\right|\right) = \text{const.} \qquad \left(\begin{array}{l}\text{radial light rays} \\ \text{outgoing } r > 2M \\ \text{ingoing } r < 2M\end{array}\right).} \tag{12.6}$$

8 Conclusion

The exploration of black holes and their underlying geometry offers a profound glimpse into the fabric of our universe, challenging and expanding our understanding of gravity and spacetime. This report has traced the journey from Newtonian mechanics, with its limitations in describing gravitational phenomena, to the revolutionary framework of General Relativity, which redefined gravity as the curvature of spacetime. Through the study of Schwarzschild geometry, we have gained insight into the nature of black holes, understanding them as regions where spacetime is warped to an extreme degree, leading to phenomena like event horizons and singularities.

As we advance towards the study of black hole perturbations, the groundwork laid by these fundamental concepts becomes increasingly critical. The perturbation theory will allow us to examine the stability of black holes and the gravitational waves emitted in dynamic processes, opening new avenues for testing the predictions of General Relativity. In conclusion, the study of black holes not only illuminates the limits of classical physics but also paves the way for deeper inquiries into the very nature of reality, continuing the quest to unlock the secrets of the universe.

9 References

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