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Black Hole Perturbations and Quasi Normal Modes

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Abstract

This project presents a comprehensive analysis of quasinormal modes (QNMs) for Schwarzschild and Reissner-Nordström black holes under various perturbations - gravitational, electromagnetic, and scalar. Using the WKB approximation method, we systematically investigate the behavior of these perturbations and their corresponding frequencies. The study includes an error analysis to determine the optimal WKB order for accurate QNM calculations, addressing the balance between precision and numerical stability. We explore how the lapse function and effective potential vary with fundamental parameters such as mass, radius, and charge, visualized through extensive graphical analysis. Special attention is given to comparing the behavior of different perturbation types and their dependencies on black hole parameters. The results provide insights into the stability and radiative properties of these black hole spacetimes, contributing to our understanding of black hole physics in both charged and uncharged scenarios.

Contents

1 Introduction

Black holes, as predicted by Einstein's General Theory of Relativity, represent one of the most fascinating objects in our universe. Their behavior under perturbations provides crucial insights into their stability and observable characteristics, particularly through gravitational wave emissions. This project focuses on analyzing the perturbations of two classical black hole solutions: the Schwarzschild and Reissner-Nordström black holes.

1.1 Black Holes

Black holes are among the most enigmatic and extreme objects in the universe, formed from the gravitational collapse of massive stars or other dense matter. Characterized by their immense gravitational pull, black holes prevent even light from escaping beyond a certain boundary called the event horizon. Within this boundary lies a singularity, a point where density and spacetime curvature theoretically become infinite. Classical black holes are generally described by their mass, charge, and angular momentum, resulting in various well-known solutions to Einstein's field equations, such as the Schwarzschild, Kerr, and Reissner-Nordström black holes.
These solutions offer insight into how black holes interact with surrounding spacetime. For instance, rotating black holes like the Kerr black hole exhibit frame-dragging effects, while charged black holes such as the Reissner-Nordström black hole involve additional electromagnetic fields. Each type of black hole solution shapes our understanding of spacetime under extreme gravitational conditions, where the geometry is altered significantly and spacetime becomes dynamic and curved.

1.2 Perturbation Theory

Perturbations in black holes refer to small disturbances or fluctuations in the black hole's spacetime geometry caused by external influences, such as infalling matter, surrounding electromagnetic fields, or gravitational waves from nearby astrophysical events. These perturbations play a vital role in studying black holes, as they offer insight into the stability of black hole solutions and provide a way to explore the nature of spacetime. When a black hole is perturbed, it tends to oscillate in specific modes known as quasinormal modes (QNMs). These modes decay over time, radiating energy in the form of gravitational waves. The frequency and decay rate of these oscillations depend on the black hole's characteristics, such as mass, charge, and angular momentum.
Analyzing black hole perturbations and the resulting QNMs has become especially important with the advent of gravitational wave astronomy. By observing the QNMs in gravitational waves, scientists can infer properties of black holes and test predictions of general relativity in strong-field regimes. There are several types of perturbations that can affect black holes, including gravitational, electromagnetic, and scalar perturbations, each interacting with the black hole's geometry in unique ways.

1.2.1 Gravitational Perturbations

Gravitational perturbations are disturbances in the curvature of spacetime itself, often arising from gravitational waves produced by nearby massive objects or binary mergers. These perturbations lead to the deformation of the black hole's horizon and spacetime geometry. In the framework of linearized perturbation theory, gravitational perturbations around a black hole are often described using the Regge-Wheeler and Zerilli equations for non-rotating black holes, or the Teukolsky equation for rotating black holes. These equations simplify the complex dynamics of gravitational perturbations, allowing researchers to study their evolution and the emission of gravitational waves.

The resulting oscillations from gravitational perturbations are reflected in the QNMs of the black hole, which decay over time due to energy loss through gravitational radiation. Gravitational perturbations are instrumental in understanding black hole stability, as the decay of these modes indicates that the black hole returns to its original state, demonstrating stability under perturbations. The frequencies of these modes, directly related to the black hole's mass and angular momentum, can be detected as gravitational waves, providing a way to measure the parameters of distant black holes.

1.2.2 Electromagentic Perturbations

Electromagnetic perturbations describe disturbances caused by electromagnetic fields in the vicinity of a black hole, particularly relevant for charged black holes or those in environments with strong magnetic fields. These perturbations influence the surrounding spacetime and interact with the black hole's charge, leading to oscillatory behaviors similar to gravitational perturbations. For charged black holes like the Reissner-Nordström solution, electromagnetic perturbations are coupled with gravitational perturbations, resulting in complex dynamics governed by the Newman-Penrose formalism or the Teukolsky equation.

Electromagnetic perturbations around black holes are important for understanding how charged particles and fields behave near these extreme objects, as well as for studying the stability of charged black hole solutions. These perturbations also provide valuable information about the interaction between black holes and external magnetic fields, which is crucial for understanding astrophysical phenomena such as jets and accretion processes in active galactic nuclei and quasars.

1.2.3 Scalar Perturbations

Scalar perturbations involve disturbances in scalar fields, which are simpler field representations compared to vector (electromagnetic) or tensor (gravitational) fields. Scalar fields are characterized by a single value at each point in space and time and can represent hypothetical particles such as the Higgs boson or even dark matter fields. Scalar perturbations around black holes provide a simpler model for studying black hole stability, often acting as a first step before tackling more complex gravitational or electromagnetic perturbations.

Scalar perturbations are particularly useful in exploring theories beyond standard general relativity, such as those involving scalar-tensor theories of gravity or models with extra dimensions. When scalar fields interact with a black hole, they produce QNMs similar to those generated by gravitational perturbations. By studying these oscillations, researchers can test various theoretical predictions and explore how black holes might behave under modified gravitational theories. Scalar perturbations are valuable for probing the nature of black holes in alternative models and can potentially provide insights into new physics beyond the standard gravitational framework.

1.3 Quasi Normal Modes and Roll-off Phase

Quasinormal modes (QNMs) are characteristic oscillations of black holes that occur when the black hole is perturbed by external forces or disturbances. These oscillations are a direct result of the black hole's response to gravitational, electromagnetic, or scalar perturbations. Unlike normal modes that continue indefinitely, QNMs are damped oscillations that decay over time, emitting radiation in the form of gravitational waves, electromagnetic waves, or other field perturbations.

QNMs provide crucial information about the black hole's intrinsic properties, such as its mass, charge, and angular momentum. They are determined by solving perturbation equations around a black hole's background metric (such as Schwarzschild, Kerr, etc.), and they depend on the boundary conditions at the event horizon and asymptotic infinity. The frequency and damping time of QNMs depend on the black hole's parameters:

Real part: The frequency of oscillation, which is related to the natural frequency of the black hole's vibrational modes. Imaginary part: The decay rate, indicating how quickly the oscillations dampen out due to energy emission (e.g., gravitational radiation). QNMs are crucial for understanding the stability of black holes and for interpreting astrophysical observations of black holes, particularly in gravitational wave astronomy. When two black holes merge, the resulting waveform can be matched with the theoretical QNM frequencies to extract information about the black holes involved. The roll-off phase refers to the stage at which the energy radiated from the black hole through QNMs diminishes over time. After a black hole is perturbed, it oscillates with specific QNM frequencies, and this oscillation gradually fades away due to energy loss in the form of radiation. The roll-off phase marks the point at which the damping becomes significant, and the gravitational wave signal (or other waveforms) decreases exponentially.

In the context of gravitational waves from black hole mergers, the roll-off phase corresponds to the tail end of the waveform after the initial burst and ringdown phase. The exponential decay of the ringdown signal is dominated by the QNMs, and the roll-off phase is critical for measuring the black hole's properties such as mass and spin.

The roll-off phase is a key feature in gravitational wave observations because it allows for precise measurements of the black hole's QNMs. The time evolution of this phase can be used to extract the black hole's mass, spin, and other characteristics.

1.4 WKB Approach to Find QNMs

The Wentzel–Kramers–Brillouin (WKB) approximation is a powerful semi-analytic method used to calculate the QNMs of black holes, especially for perturbations of the black hole's spacetime. The WKB approach is used to find the complex frequencies of QNMs by solving the perturbation equations in a manner that approximates the solution in terms of classical trajectories and quantum mechanics.

The WKB method involves the following steps:

Linearizing the Perturbation Equation: First, the perturbation equation for the given black hole background is linearized. For example, in the case of gravitational perturbations, this involves solving the Einstein field equations for the perturbation in the form of a wave equation.

Approximating the Wave Equation: The WKB method approximates the wave equation as a classical wave equation by assuming that the solution to the perturbation is of the form of a rapidly oscillating function. This approach essentially treats the wave as if it behaves like a particle in a potential, allowing us to apply quantum mechanics' semi-classical methods to find the frequency of the perturbations.

Finding the QNM Frequencies: By solving the WKB approximation for the boundary conditions (typically at the event horizon and at infinity), we can find the complex frequencies ω of the QNMs. These frequencies will have both real and imaginary parts, corresponding to the oscillatory and damping behavior of the modes, respectively.

In practice, the real part of the frequency represents the oscillation frequency of the QNM, while the imaginary part relates to how quickly the perturbation dies out (the damping rate).

Higher-order WKB Corrections: The basic WKB approximation can be improved by including higher-order corrections. These corrections account for finer details of the perturbation's behavior in different regions of spacetime (near the event horizon and at infinity). The inclusion of these terms leads to more accurate estimates of the QNM frequencies.

1.4.1 Applications of the WKB Approach

The WKB method is widely used because it provides a simple and efficient way to compute QNMs for a variety of black hole models, including those with gravitational, electromagnetic, and scalar perturbations. It is particularly useful for solving problems where the exact solution is difficult to obtain. The WKB method has been applied to many different black hole solutions, such as Schwarzschild, Kerr, and Reissner-Nordström black holes, to study their stability and the resulting QNMs. In recent years, it has also been employed to calculate QNMs in more exotic black holes, such as those in modified gravity theories.

2 Schwarzschild Black Holes

The Schwarzschild black hole is the simplest solution to Einstein's field equations of general relativity, representing an uncharged, non-rotating black hole with spherical symmetry. Discovered by Karl Schwarzschild in 1916, this solution describes the spacetime geometry around a massive, static object where gravitational effects dominate and curvature significantly warps spacetime. Schwarzschild black holes are characterized solely by their mass, making them the most straightforward model in black hole studies.

The Schwarzschild metric, given in spherical coordinates (t,r,θ,ϕ), reveals key features of these black holes, such as the event horizon, defined at r=2M (where M is the mass of the black hole), and the singularity at r=0. The event horizon is the boundary beyond which no information can escape, marking the "point of no return" for particles and radiation. Schwarzschild black holes are sometimes called "eternal black holes," as they are stable, vacuum solutions and are expected to persist indefinitely without losing mass through processes like accretion or Hawking radiation.

Schwarzschild black holes play a foundational role in the study of black hole perturbations. Understanding perturbations in the Schwarzschild spacetime provides insight into gravitational waves, stability analysis, and resonant modes known as quasinormal modes (QNMs). These QNMs describe the characteristic "ringdown" phase of a black hole's response to perturbations, where spacetime oscillates and gravitational waves are emitted as it settles back to equilibrium. Studying Schwarzschild black hole perturbations is crucial for understanding the dynamics of gravitational waves, a field significantly advanced by recent detections through observatories like LIGO and Virgo.

2.1 Schwarzschild Metric

The Schwarzschild metric for a non-rotating, uncharged black hole with mass M in spherical coordinates (t, r, θ, ϕ) is given by:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

For a Schwarzschild black hole, the metric components $g_{\mu\nu}$ in spherical coordinates (t, r, θ, ϕ) are given by:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2M}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

2.2 GravitationalPerturbations in Schwarzschild Black Holes

In Regge-Wheeler formalism, the space-time is perturbed by adding perturbation in the bakcground metric [g$\mu\nu^0$]$.Mathematicallywecanrepresentitas:$

Since Regge-Wheeler formalism has been taken for this analysis, so spherically symmetric perturbations are divided into odd (axial perturbations) and even waves (polar perturbations). The axial Perturbations in RW canonical gauge are defined by this matrix:

[h$\mu\nu$]=

$$\begin{bmatrix} 0 & 0 & 0 & h_0(r)e^{-i\omega t}\sin(\theta)\frac{d}{d\theta}P_l(\theta) \\ 0 & 0 & 0 & h_1(r)e^{-i\omega t}\sin(\theta)\frac{d}{d\theta}P_l(\theta) \\ 0 & 0 & 0 & 0 \\ h_0(r)e^{-i\omega t}\sin(\theta)\frac{d}{d\theta}P_l(\theta) & h_1(r)e^{-i\omega t}\sin(\theta)\frac{d}{d\theta}P_l(\theta) & 0 & 0 \end{bmatrix}$$

Adding this perturbation matrix to the background matrix gives the following matrix:

$$\begin{bmatrix} -e^{\nu(r)} & 0 & 0 & \varepsilon h_0(r)e^{-i\omega t}\sin(\theta)\frac{d}{d\theta}P_l(\theta) \\ 0 & e^{-\nu(r)} & 0 & \varepsilon h_1(r)e^{-i\omega t}\sin(\theta)\frac{d}{d\theta}P_l(\theta) \\ 0 & 0 & r^2 & 0 \\ \varepsilon h_0(r)e^{-i\omega t}\sin(\theta)\frac{d}{d\theta}P_l(\theta) & \varepsilon h_1(r)e^{-i\omega t}\sin(\theta)\frac{d}{d\theta}P_l(\theta) & 0 & r^2\sin^2(\theta) \end{bmatrix}$$

Using this background metric to find ricci tensor R$\mu\nu$ gives only three non-zero components which satisfy the Einstein's field equations. This ricci tensor if the combination of ricci tensor corresponding to unperturbed background metric and first order pertubations of Ricci tensor. The three components are:
dR03=

$$\left(\frac{L(L+1)h_0(r)}{2r^2} - \frac{i\omega\left(\frac{d}{dr}h_1(r) + \frac{h_1(r)e^{\nu(r)}}{r}\right)e^{\nu(r)}}{2} - \frac{e^{\nu(r)}\frac{d^2}{dr^2}h_0(r)}{2} - \frac{h_0(r)e^{\nu(r)}\frac{d}{dr}\nu(r)}{r}\right)e^{-i\omega t}\sin(\theta)\frac{d}{d\theta}P_l(\theta)$$

dR13=

$$\left(\frac{i\omega\left(\frac{d}{dr}h_0(r) - \frac{2h_0(r)}{r}\right)e^{-\nu(r)}}{2} + \frac{\left(\frac{L(L+1)}{r^2} - \omega^2 e^{-\nu(r)} + \frac{\left(-2\frac{d}{dr}\nu(r) - \frac{2}{r}\right)e^{\nu(r)}}{r}\right)h_1(r)}{2}\right)e^{-i\omega t}\sin(\theta)\frac{d}{d\theta}P_l(\theta)$$

dR23=

$$\frac{\left(\sin(\theta)\frac{d^2}{d\theta^2}P_l(\theta) - \cos(\theta)\frac{d}{d\theta}P_l(\theta)\right)\left(i\omega h_0(r) + h_1(r)e^{2\nu(r)}\frac{d}{dr}\nu(r) + e^{2\nu(r)}\frac{d}{dr}h_1(r)\right)e^{-i\omega t}e^{-\nu(r)}}{2}$$

2.3 Finding Effective Potential of the Perturbation

We find the effective potential from the Schrondinger type final diffential equation that we get after finding h0 and h1 in dR03, dR13, dR23. By putting R$_{23}^{(1)} = 0, wegeth0intermsofh1whichisexpressedbelow:$
h1(r)=

$$\frac{i\left(h_1(r)\frac{d}{dr}\nu(r)+\frac{d}{dr}h_1(r)\right)e^{2\nu(r)}}{\omega}$$

After finding h0, redefine the h1 as $x_1.\psi.e^{-v} and plugging it back into the equation we get from R_{13}^{(1)} = 0$:

$$\frac{d^2\psi(r)}{dr^2}+\frac{d\nu(r)}{dr}\frac{d\psi(r)}{dr}+\left(\omega^2e^{-2\nu(r)}+\frac{3}{r}\frac{d\nu(r)}{dr}-\frac{l(l+1)e^{-\nu(r)}}{r^2}\right)\psi(r)=0$$

This is the standard Schrondinger equation:

$$\frac{d^2\psi}{dr_*^2}+\left(\omega^2-V(r)\right)\psi=0,$$

where ψ represents the perturbation, ω is the complex frequency of the perturbation, and $V(r)$ is the effective potential. On converting radial coordinate to tortoise coordinate, first order derivatives are removed from the equation and we can find V(r) by comparing the differential equation with standard Schrondinger equation:

$$V(r)=\left(1-\frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2}+\frac{6M}{r^3}\right).$$

This equation describes how gravitational waves propagate around the black hole, with the QNMs (quasinormal modes) of the black hole arising from the solutions to this equation.

2.4 Analyzing Gravitational Perturbations of the Schwarzschild Black Hole using WKB Approximation Method

Gravitational perturbations of the Schwarzschild black hole, first studied by Regge and Wheeler (1957) and Zerilli (1970), represent small deviations from the spherically symmetric background geometry. These perturbations can be classified into two types based on their behavior under parity transformations:

- **Odd (Axial) Perturbations:** These transform as $(-1)^{l+1}$ under parity and are described by the Regge-Wheeler equation:

$$\frac{d^2\Psi}{dr_*^2}+(\omega^2-V_{RW})\Psi=0 \tag{1}$$

  where the Regge-Wheeler potential is:

$$V_{RW}=f(r)\left[\frac{l(l+1)}{r^2}-\frac{6M}{r^3}\right] \tag{2}$$

- **Even (Polar) Perturbations:** These transform as $(-1)^l$ under parity and are governed by the Zerilli equation with a more complex potential:

$$V_Z=f(r)\frac{2n^2(n+1)r^3+6n^2Mr^2+18nM^2r+18M^3}{r^3(nr+3M)^2} \tag{3}$$

  where $n=\frac{(l-1)(l+2)}{2}$

Despite their different forms, these potentials are known to be isospectral, meaning they yield the same quasinormal mode frequencies. This remarkable property, known as the Chandrasekhar transformation, demonstrates a deep connection between the two types of perturbations.
The study of these perturbations is crucial for understanding:

- Stability of black holes
- Gravitational wave emission
- Ring-down phase of black hole mergers
- Tests of General Relativity in strong-field regime

2.4.1 Applying WKB approximation method

The WKB (Wentzel-Kramers-Brillouin) approximation is a method for finding approximate solutions to linear differential equations with spatially varying coefficients. In the context of black hole physics, it's particularly useful for calculating quasinormal mode frequencies using the effective potential barrier.

2.4.2 Basic Formulation

For a wave equation in the standard form:

$$\frac{d^2\psi}{dr_*^2} + Q(r_*)\psi = 0 \tag{4}$$

where $Q(r_*) = \omega^2 - V(r_*)$, the WKB approximation provides a solution in the form:

$$\psi(r_*) \approx Q^{-1/4}(r_*) \exp\left(\pm i \int \sqrt{Q(r_*)} dr_*\right) \tag{5}$$

2.4.3 Error Propagation

The WKB method can be extended to higher orders for improved accuracy, but error propagation becomes significant:

- **First Order:** Provides basic approximation with errors of order $\mathcal{O}(\hbar)$
- **Third Order:** Includes correction terms but may show instabilities near turning points
- **Sixth Order:** While theoretically more accurate, numerical instabilities can arise due to:
  - Accumulation of roundoff errors in higher derivatives
  - Increased sensitivity to potential function irregularities
  - Growth of error terms near singular points

For practical calculations of quasinormal modes, the 6th order WKB formula provides a balance between accuracy and computational stability, particularly for large l modes where the potential barrier is more pronounced, however the optimal order WKB approximation can be chosen from all the order by comparing their error estimations. Below are given different tables containing WKB order, Quasinormal Frequency, and Error Estimation for differ l values. (The spin parameter for gravitational perturbations is 2).

2.4.4 Finding Quasinormal Modes

For l=2:

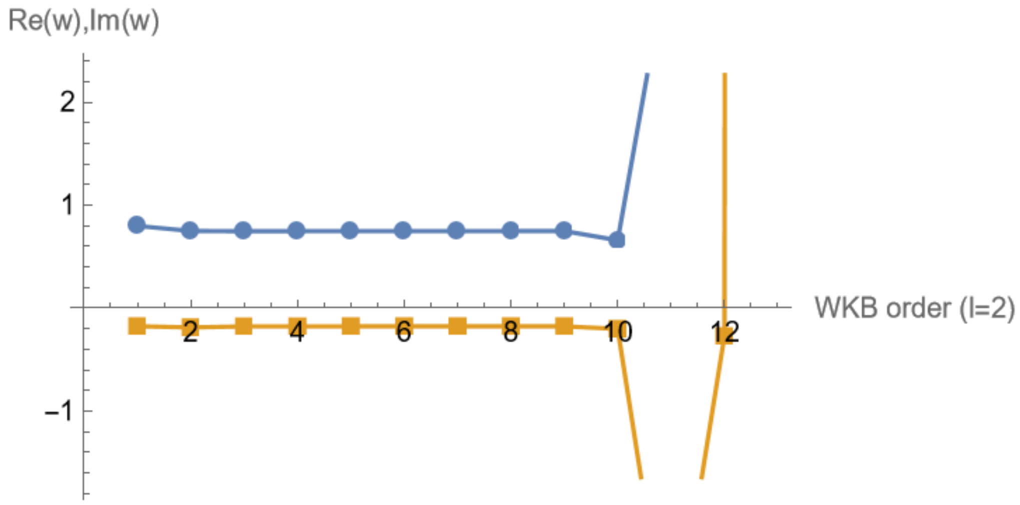

Figure 2: Enter Caption

WKB order	quasinormal frequency	error estimation
12	$0.747826 - 0.180069i$	0.00383226
11	$0.748934 - 0.179803i$	0.00121956
10	$0.748434 - 0.177707i$	0.00112774
9	$0.747788 - 0.17786i$	0.000382123
8	$0.7477 - 0.177492i$	0.000320131
7	$0.747198 - 0.177611i$	0.000272642
6	$0.747239 - 0.177782i$	0.000147252
5	$0.747009 - 0.177837i$	0.000242172
4	$0.747107 - 0.178248i$	0.000454565
3	$0.746324 - 0.178435i$	0.0050031

From the table, it is clearly seen that 6th order WKB approximation is optimal with the least error estimation of 0.000147252, so **the QNM for l=2 is 0.747239 0.177782i.**

For l=3:

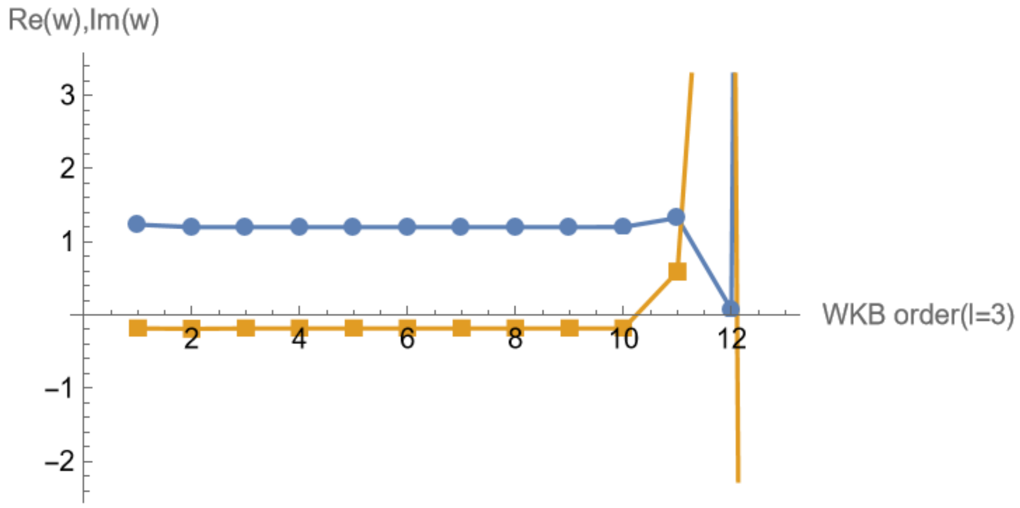

Figure 3: Enter Caption

WKB order	quasinormal frequency	error estimation
12	$768.592 - 1.98119i$	1165.56
11	$38.9662 - 39.0781i$	384.283
10	$0.0426603 + 2.95611i$	27.4613
9	$0.590905 + 0.213416i$	1.53659
8	$0.555953 - 0.0739014i$	0.144868
7	$0.554774 - 0.0740586i$	0.000595123
6	$0.554773 - 0.0740537i$	7.9693038*^-6
5	$0.554788 - 0.0740517i$	0.0000551452
4	$0.554774 - 0.0739434i$	0.000254611
3	$0.554281 - 0.0740092i$	0.0012896

From the table, it is clearly seen that 6th order WKB approximation is optimal with the least error estimation of 0.00000796, so **the QNM for l=3 is 0.554773-0.0740537i.**

For l=4:

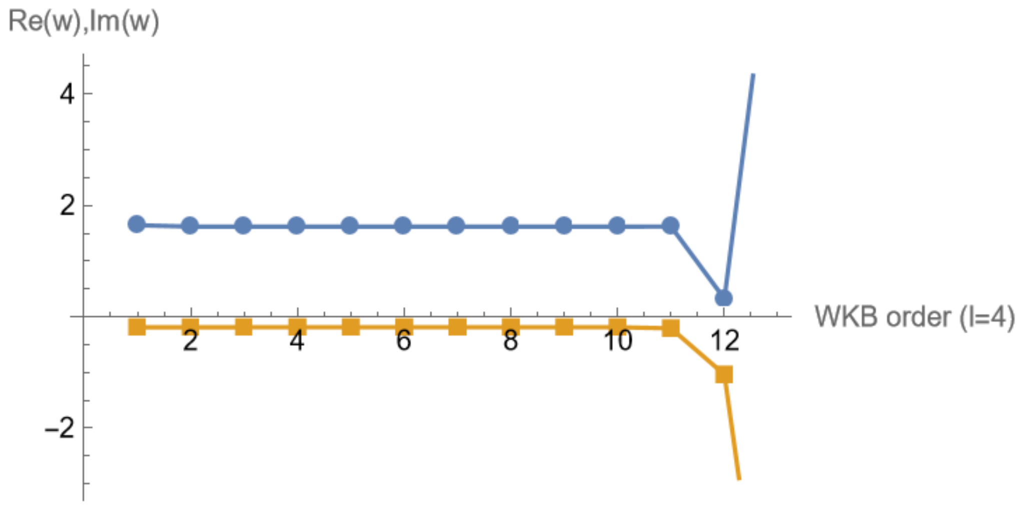

Figure 4: Enter Caption

WKB order	quasinormal frequency	error estimation
12	$2.5551 \times 10^8 + 0.i$	1.67605×10^9
11	$5.52948 \times 10^6 + 0.i$	1.27755×10^8
10	$0. + 234936.i$	2.76165×10^6
9	$6173.39 + 0.i$	117468.
8	$135.795 + 0.i$	3083.52
7	$6.35328 + 0.i$	67.8988
6	$0. + 0.88977i$	3.20567
5	$0. + 0.860835i$	0.0145312
4	$0. + 0.860708i$	0.0000635004
3	$0. + 0.860708i$	0.0000153305

From the table, it is clearly seen that 3rd order WKB approximation is optimal with the least error estimation of 0.0000153305, so the **QNM for l=4 is 0. + 0.860708i.**

2.5 Analyzing Scalar Perturbations of the Schwarzschild Black Holes using WKB Apprroximation method

2.5.1 Finding Quasinormal Modes

FOr l=2:

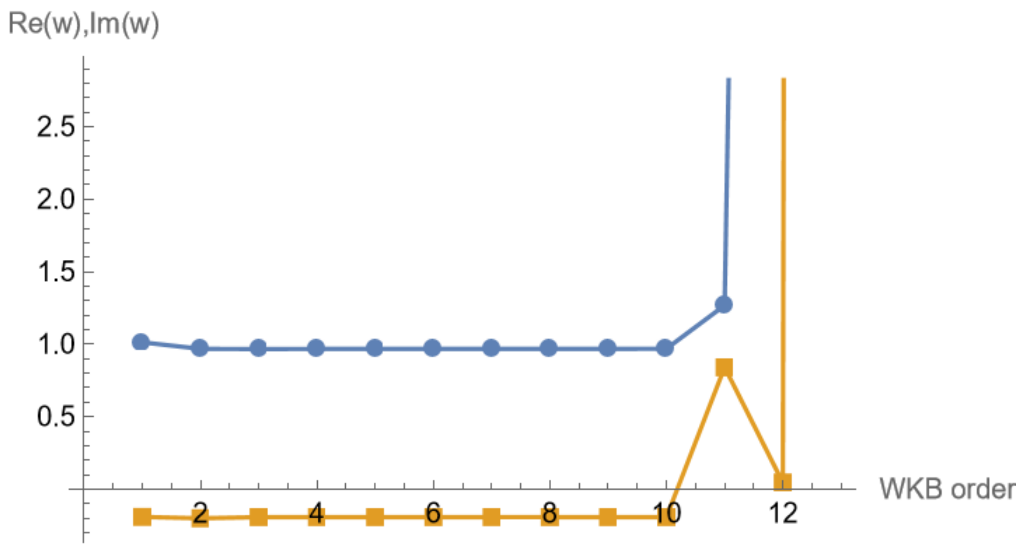

Figure 5: QNM vs WKB order for l=2

WKB order	quasinormal frequency	error estimation
13	$2.00787 - 0.161723i$	4.41461
12	$1.73413 - 0.187252i$	0.137447
11	$1.73474 - 0.192795i$	0.00278846
10	$1.73483 - 0.192784i$	0.0000467895
9	$1.73483 - 0.192783i$	5.566414464839634*^-7
8	$1.73483 - 0.192783i$	2.82766178566507*^-7
7	$1.73483 - 0.192784i$	6.845162729805434*^-7
6	$1.73483 - 0.192784i$	3.969185145674324*^-6
5	$1.73483 - 0.192776i$	0.000076372
4	$1.73468 - 0.192793i$	0.00119543

From the table, it is clearly seen that 8th order WKB approximation is optimal with the least error estimation of 0.0000002827, so the **QNM for l=2 is 1.734830.192783i.**

For l=3:

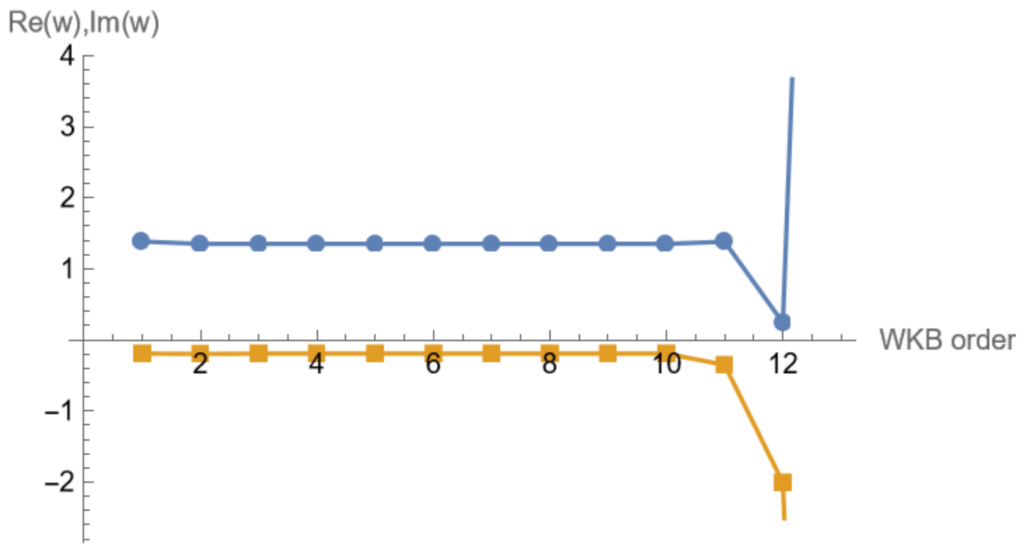

Figure 6: QNM vs WKB for l=3

WKB order	quasinormal frequency	error estimation
13	$0.242202 - 2.00704i$	14.7008
12	$1.38161 - 0.351843i$	1.06271
11	$1.34994 - 0.193107i$	0.080911
10	$1.35073 - 0.192994i$	0.000399617
9	$1.35073 - 0.192999i$	2.645548588822223*^-6
8	$1.35073 - 0.192999i$	1.038565958048954*^-6
7	$1.35073 - 0.193001i$	2.665901929076742*^-6
6	$1.35074 - 0.193i$	0.000011478
5	$1.35073 - 0.192978i$	0.000162715
4	$1.35041 - 0.193024i$	0.00197435

From the table, it is clearly seen that 8th order WKB approximation is optimal with the least error estimation of 0.000001038, so the **QNM for l=4 is 1.35073 - 0.19299 i.**

For l=4:

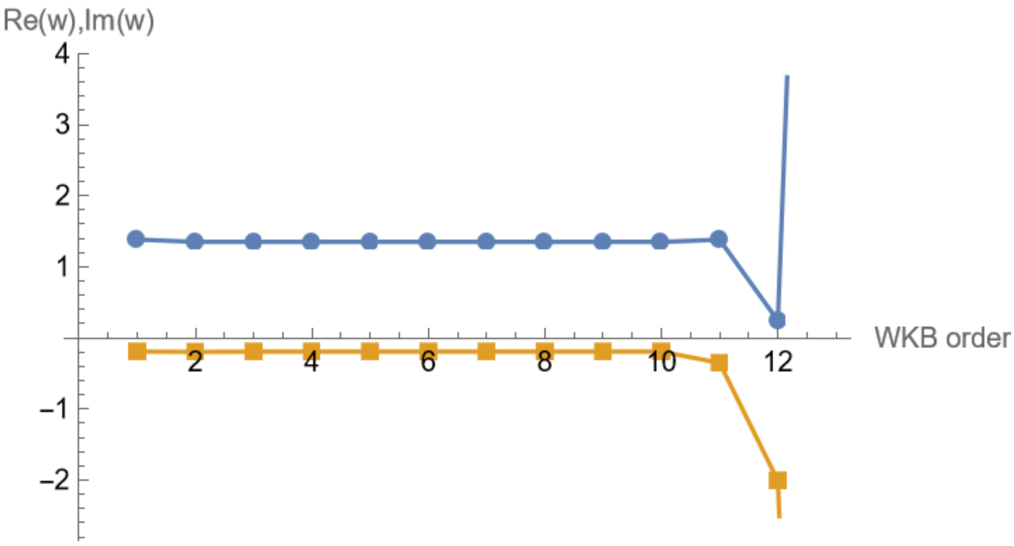

Figure 7: QNM vs WKB order for l=4

$$\begin{pmatrix} 13 & 0.242202-2.00704i & 14.7008 \\ 12 & 1.38161-0.351843i & 1.06271 \\ 11 & 1.34994-0.193107i & 0.080911 \\ 10 & 1.35073-0.192994i & 0.000399617 \\ 9 & 1.35073-0.192999i & \text{2.645548588822223*^-6} \\ 8 & 1.35073-0.192999i & \text{1.038565958048954*^-6} \\ 7 & 1.35073-0.193001i & \text{2.665901929076742*^-6} \\ 6 & 1.35074-0.193i & 0.000011478 \\ 5 & 1.35073-0.192978i & 0.000162715 \\ 4 & 1.35041-0.193024i & 0.00197435 \end{pmatrix}$$

From the table, it is clearly seen that 8th order WKB approximation is optimal with the least error estimation of 0.000001038, so the **QNM for l=4 is 1.350730.192999i.**

2.6 Trends Analysis in Mathematica

2.6.1 Trends in Lapse function

The lapse function, typically denoted as $f(r)$, is a fundamental component in the Schwarzschild metric that characterizes how proper time relates to coordinate time at different radial distances from a black hole. For a Schwarzschild black hole of mass M, the lapse function takes the form:

$$f(r) = N(r) = \sqrt{-g_{tt}} = \sqrt{1 - \frac{2M}{r}} \tag{6}$$

Lapse function radial dependence is shown in the images.
This function plays several crucial physical roles:

- At spatial infinity ($r \to \infty$), $f(r) \to 1$, indicating that proper time flows at the same rate as coordinate time for distant observers.
- At $r = 2M$ (the event horizon), $f(r) = 0$, signifying the complete freezing of proper time from the perspective of an external observer.

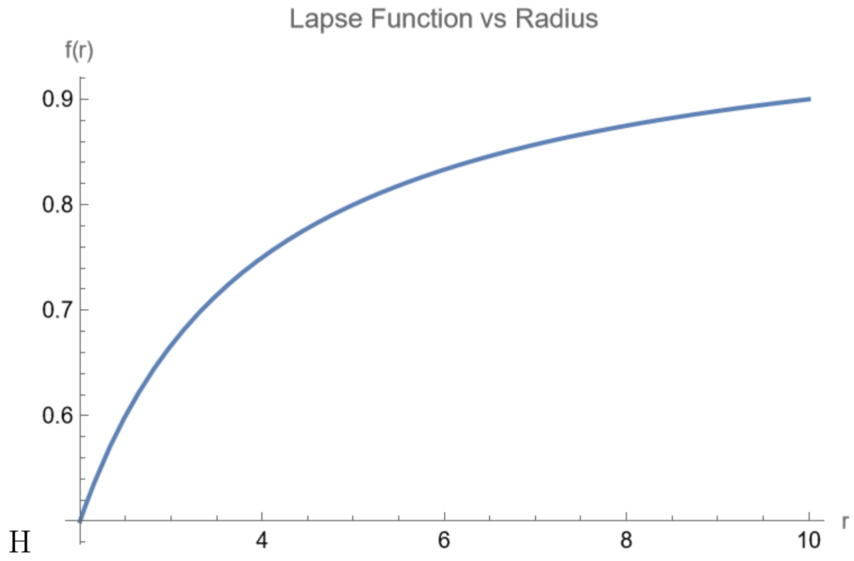

Figure 8: Radial dependence of Lapse function

- For $r < 2M$, the function becomes imaginary, reflecting the transition to a region where r becomes timelike and t becomes spacelike.

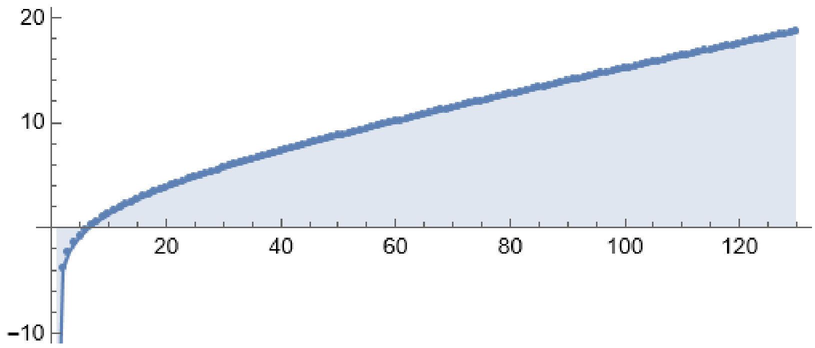

Figure 9: Radial coordinate vs Tortoise coordinate

The radial dependence of lapse function has been shown in the image.

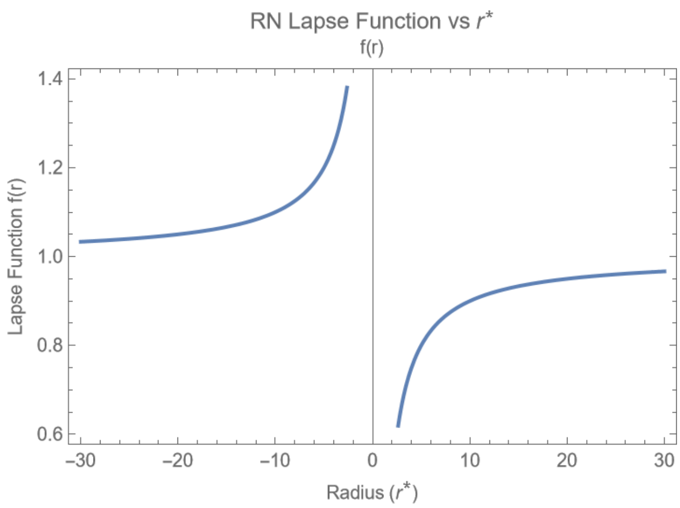

Figure 10: Lapse function variation with tortoise coordinate

The lapse function is essential in understanding various physical phenomena, including gravitational time dilation and the causal structure of spacetime near a black hole. It appears in the Schwarzschild metric:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{7}$$

2.6.2 Trends in Effective Potential

The effective potential V(r) (derived for perturbations in the previous section) acts as a barrier that scatters incoming waves and is responsible for characteristic oscillations known as Quasinormal Modes(QNMs), which describe how black holes respond to perturbations.

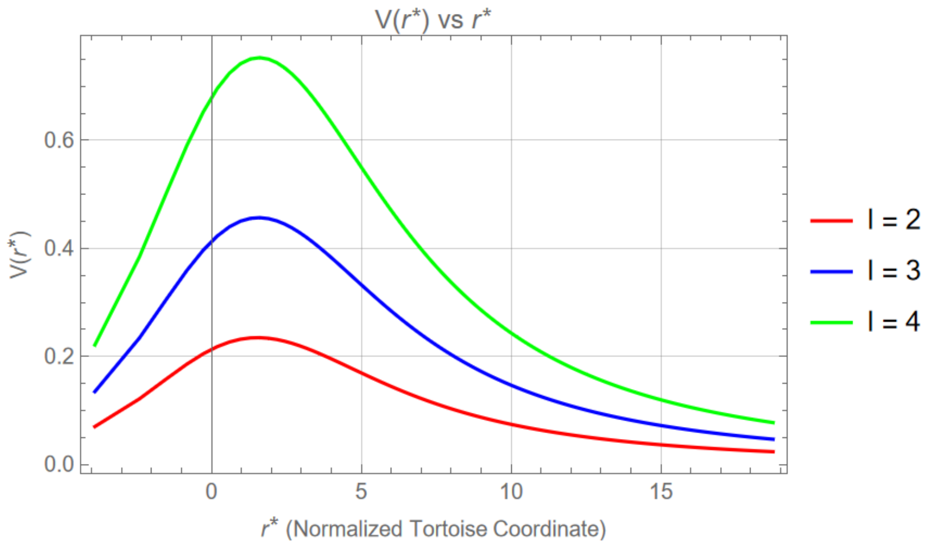

Figure 11: Effective Potential vs Tortoise coordinate with different l values

2.6.3 Trends in QNM frequency with mass and radius

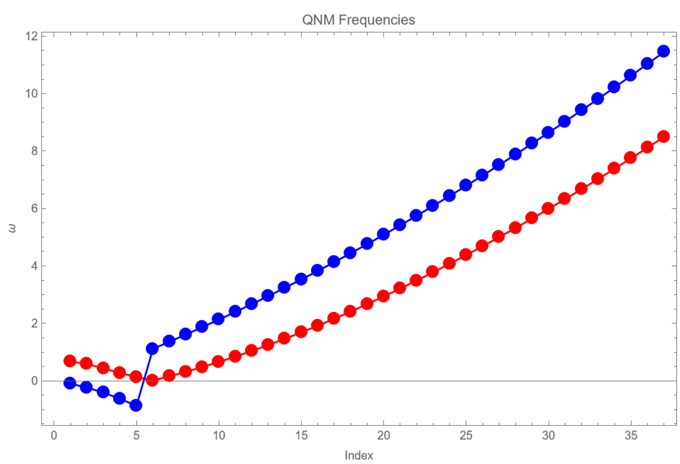

Figure 12: QNM vs Mass for l=2. The blue dot and line shows the real part of the QNM while the red one shows the imaginary part.

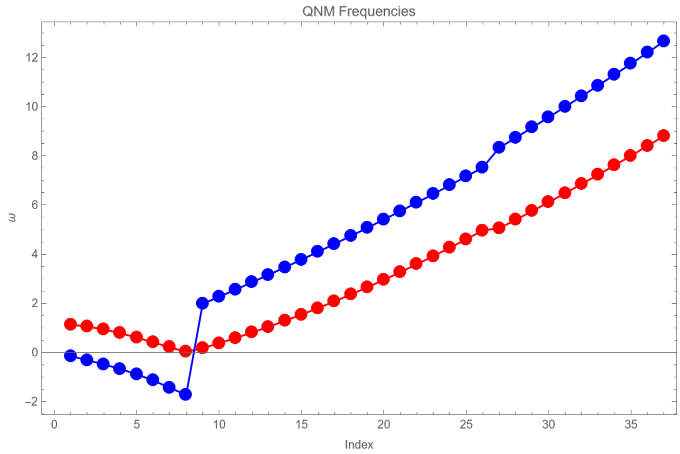

Figure 13: QNM vs Mass for l=3. The blue dot and line shows the real part of the QNM while the red one shows the imaginary part.

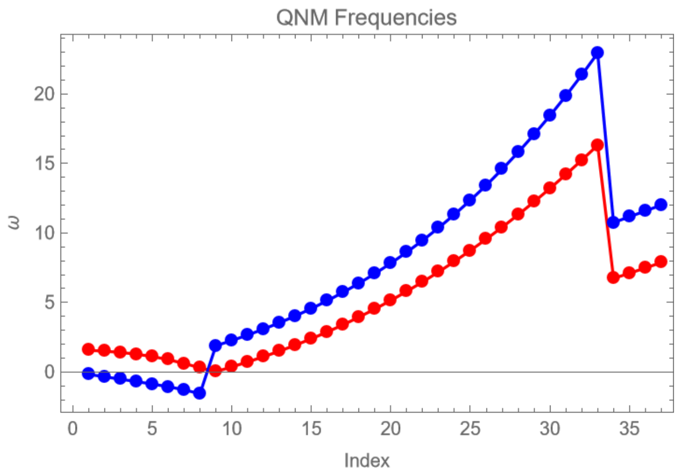

Figure 14: QNM vs Mass for l=4.The blue dot and line shows the real part of the QNM while the red one shows the imaginary part.

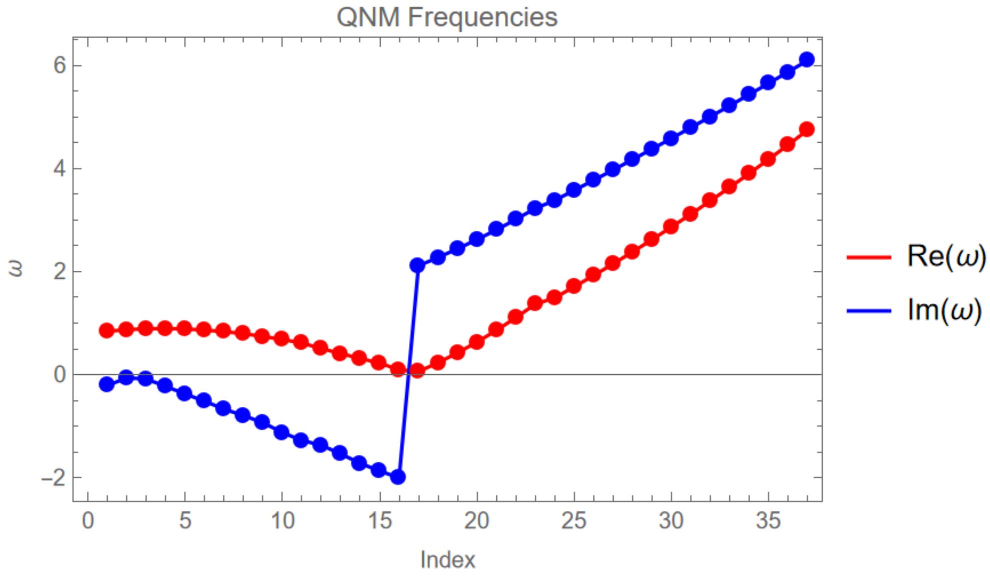

Figure 15: The radial dependence of QNM has been shown for l=2. This plot remains same for all other values of l.

From the above graphs it has been analyzed that **the QNMs for a scalar perturbations remains unchanged even if angular momentum paramter is changed.**

3 Reissner-Nordström Black Hole

The Reissner-Nordström black hole is a static solution to Einstein's field equations that describes a spherically symmetric, charged black hole. Originally developed by Hans Reissner and Gunnar Nordström in the early 20th century, this solution generalizes the Schwarzschild metric by introducing an electric charge Q. Unlike Schwarzschild black holes, Reissner-Nordström black holes possess both gravitational and electrostatic fields. The resulting metric describes the spacetime around a charged, non-rotating black hole, where the electric charge produces a repulsive force in addition to the gravitational attraction.

The Reissner-Nordström solution has two horizons: the outer event horizon and an inner Cauchy horizon. The presence of these horizons is tied to the charge-to-mass ratio of the black hole. When the charge is below a certain threshold, the black hole remains regular with two distinct horizons. If the charge reaches a critical value, the two horizons merge, forming an extremal Reissner-Nordström black hole with zero surface gravity. Beyond this critical value, no event horizon exists, leading to a naked singularity where the central singularity is exposed to the external universe, violating the cosmic censorship conjecture.

Reissner-Nordström black holes are idealized models since, in nature, black holes would typically lose charge due to interactions with surrounding matter. However, they remain significant in theoretical physics for exploring black hole thermodynamics, stability, and potential connections to quantum field theory. The Reissner-Nordström solution also serves as a foundation for studying black holes in more complex scenarios, such as those with both charge and rotation (Kerr-Newman black holes), further enriching our understanding of black hole physics.

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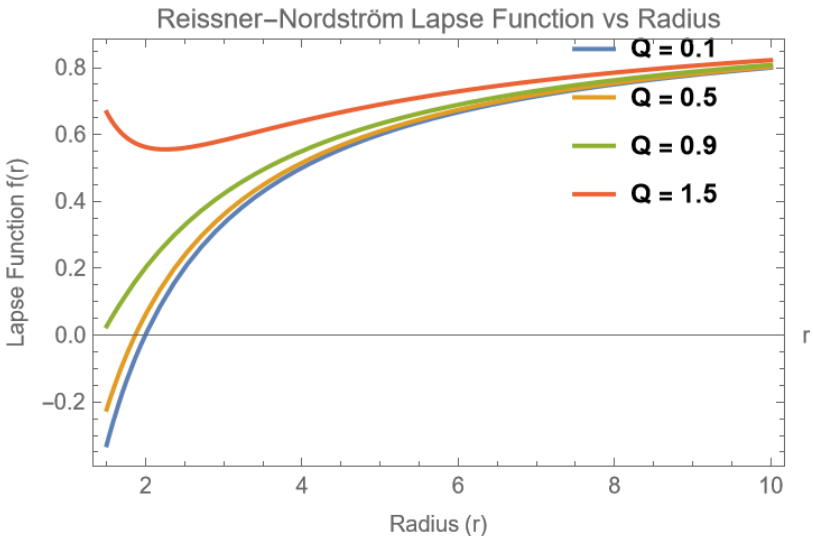

charged, non-rotating black hole, where the electric charge produces a repulsive force in addition to the gravitational attraction.

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3.1 Reissner-Nordström Metric

The Reissner-Nordström metric for a non-rotating, charged black hole with mass M and charge Q in spherical coordinates is:

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

For a Reissner-Nordström black hole with mass M and charge Q, the metric components $g_{\mu\nu}$ in spherical coordinates are:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

3.2 Analysis of EM Perturbations in Reissner-Nordström black hole

For a Reissner-Nordström black hole (a charged black hole), the equation governing electromagnetic perturbations can be written as:

$$\frac{d^2\Phi}{dr_*^2} + \left(\omega^2 - V_{\rm EM}(r)\right)\Phi = 0,$$

where Φ represents the perturbation in the electromagnetic field, ω is the frequency, and $V_{\rm EM}(r)$ is the effective potential for electromagnetic perturbations, which is generally more complex than the gravitational potential and depends on the charge of the black hole.

$$V_{\rm EM}(r) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\left(\frac{l(l+1)}{r^2} + \frac{6M}{r^3}\right),$$

where Q is the charge of the black hole.

3.2.1 Finding Quasinormal Modes

For l=2:

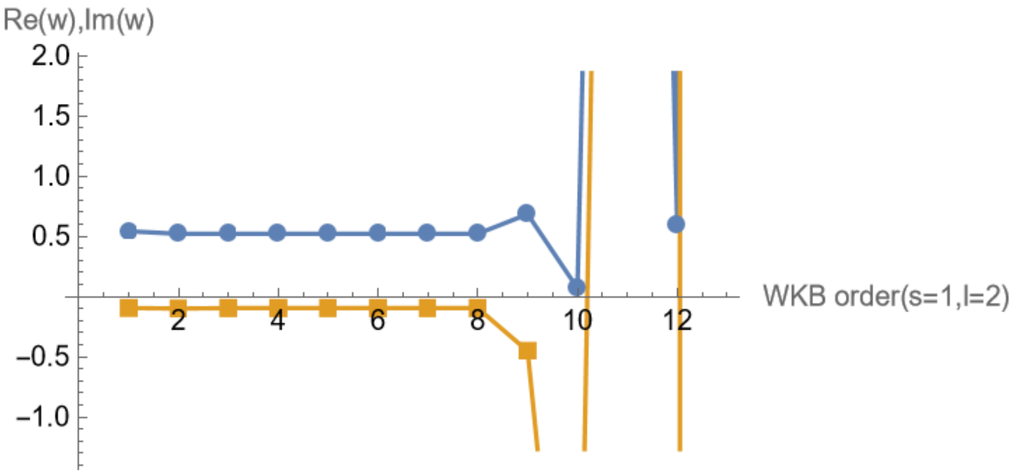

Figure 16: QNM freqs vs WKB order for l=2

WKB order	quasinormal frequency	error estimation
13	$0.595199 + 551.131i$	6447.71
12	$17.8308 + 18.3969i$	277.83
11	$0.0682701 - 4.52928i$	12.7414
10	$0.682799 - 0.452863i$	2.22767
9	$0.520116 - 0.0969072i$	0.195748
8	$0.519604 - 0.0970028i$	0.000260603
7	$0.519604 - 0.0970061i$	0.0000135588
6	$0.519631 - 0.0970012i$	0.0000406504
5	$0.519617 - 0.0969258i$	0.000225607
4	$0.51918 - 0.0970074i$	0.00173554

From the table, it is clearly seen that 7th order WKB approximation is optimal with the least error estimation of 0.0000135588, so the **QNM for l=2 is 0.519604 0.0970061i.**

For l=3:

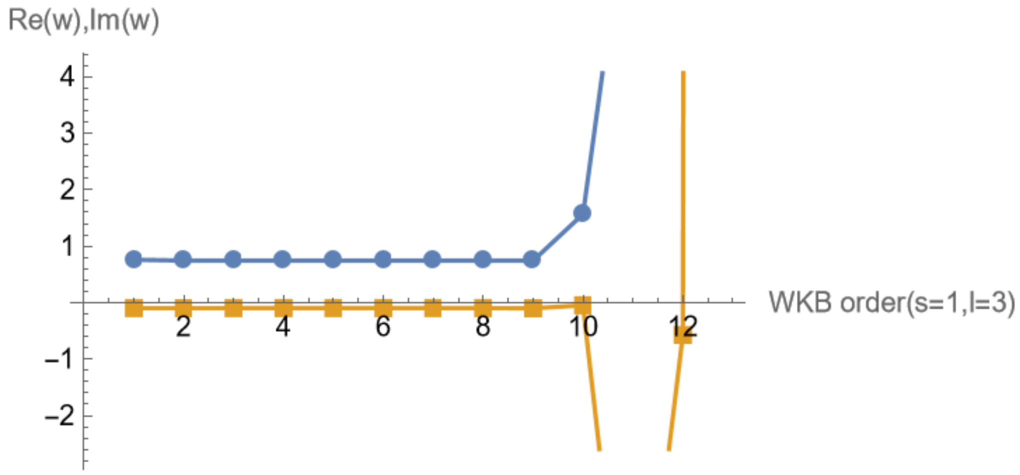

Figure 17: QNM freqs vs WKB orders for l=3

WKB order	quasinormal frequency	error estimation
13	$108.39 - 0.577006i$	341.982
12	$7.98616 - 7.83124i$	53.4124
11	$1.56613 - 0.0476224i$	5.29667
10	$0.743866 - 0.100264i$	0.412067
9	$0.743505 - 0.097548i$	0.00137258
8	$0.743685 - 0.0975244i$	0.0000910409
7	$0.743686 - 0.0975249i$	2.3870208592289587*^-6
6	$0.74369 - 0.0975243i$	9.847202572393437*^-6
5	$0.743688 - 0.0975054i$	0.0000788969
4	$0.743532 - 0.0975257i$	0.000861814

From the table, it is clearly seen that 7th order WKB approximation is optimal with the least error estimation of 0.000002387, so the **QNM for l=3 is 0.743686 0.0975249i.**

For l=4:

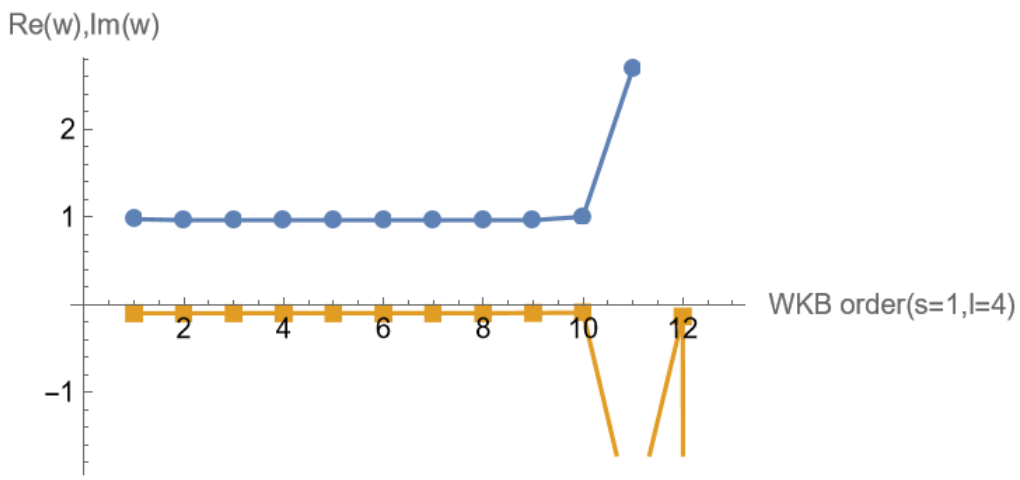

Figure 18: QNM freqs vs WKB order for l=4

WKB order	quasinormal frequency	error estimation
13	$108.39 - 0.577006i$	341.982
12	$7.98616 - 7.83124i$	53.4124
11	$1.56613 - 0.0476224i$	5.29667
10	$0.743866 - 0.100264i$	0.412067
9	$0.743505 - 0.097548i$	0.00137258
8	$0.743685 - 0.0975244i$	0.0000910409
7	$0.743686 - 0.0975249i$	2.3870208592289587*^-6
6	$0.74369 - 0.0975243i$	9.847202572393437*^-6
5	$0.743688 - 0.0975054i$	0.0000788969
4	$0.743532 - 0.0975257i$	0.000861814

From the table, it is clearly seen that 8th order WKB approximation is optimal with the least error estimation of 0.000001038, so the **QNM for l=4 is 1.350730.192999i.**

3.3 Trends Analysis in Mathematica

3.3.1 Trends in Lapse Function with Radius, Mass and Charge

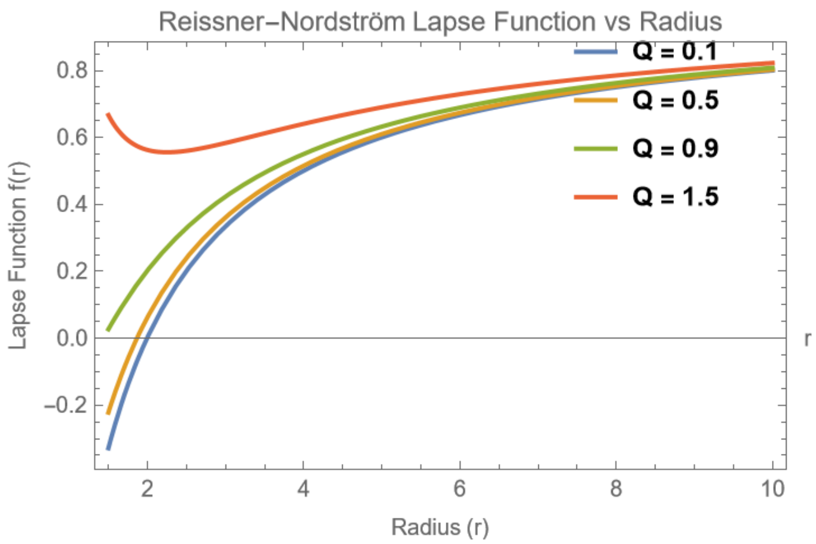

Figure 19: RN Lapse Function vs Radius for different Q values

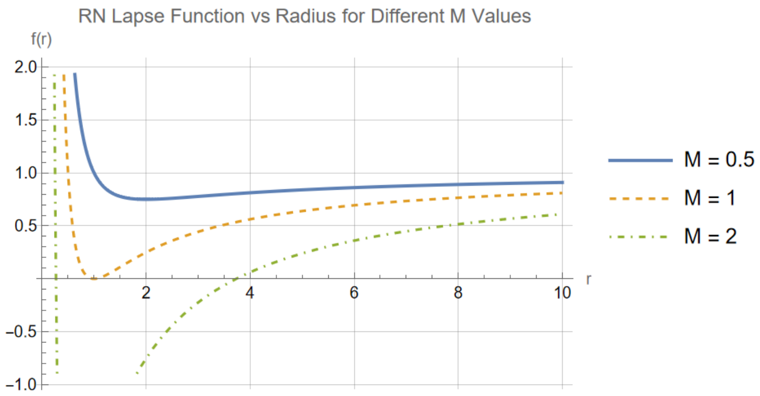

Figure 22: RN Lapse Function vs Radius for different M values

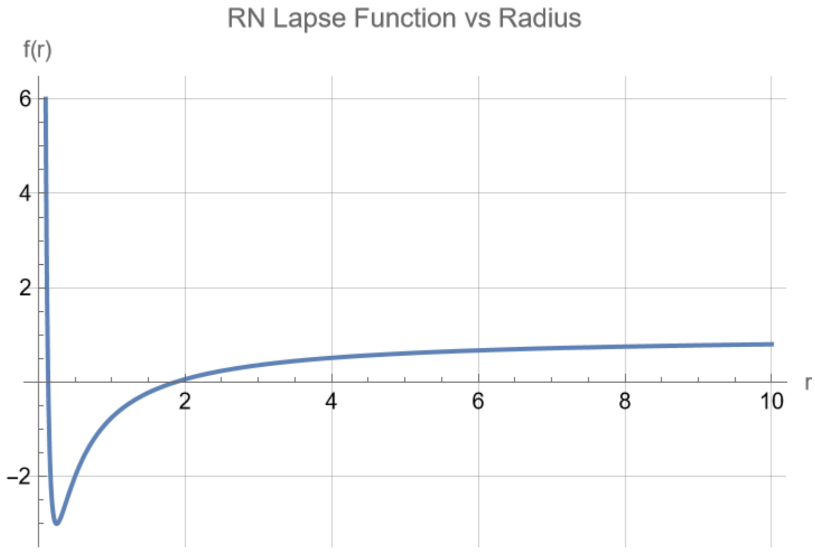

Figure 20: RN Lapse Function vs Radius with M and Q constant.

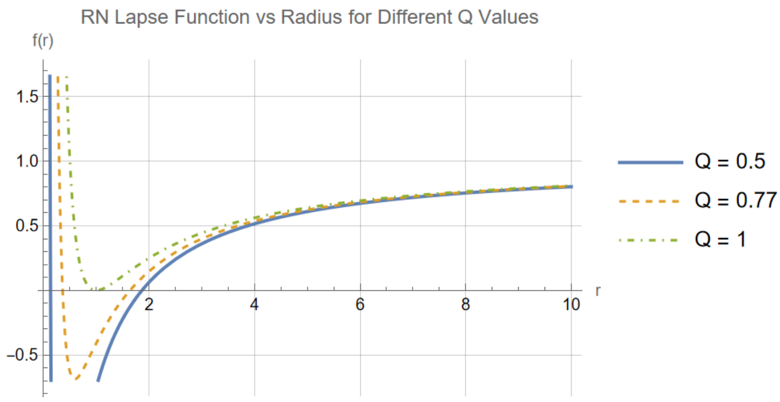

Figure 21: RN Lapse Function vs Radius for different Q values

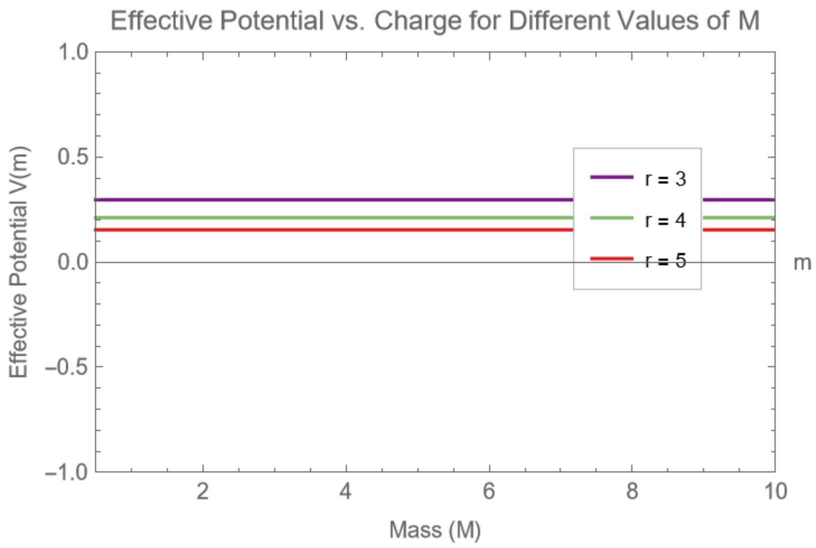

Figure 23: Effective Potential vs Radius for constant M and Q.

3.3.2 Trends in QNM frequency with Mass, Charge and Radius

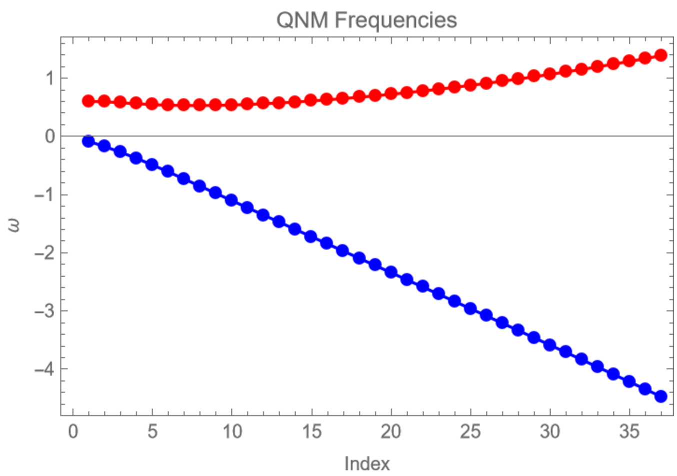

Figure 24: Effective Potential vs Mass

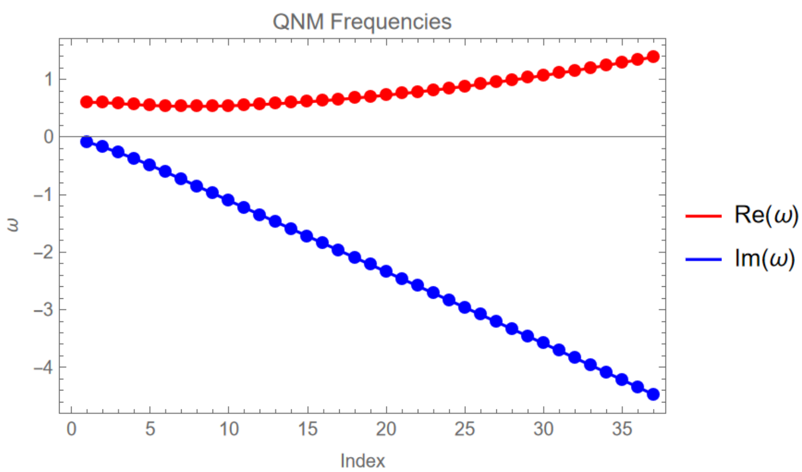

Figure 25: Effective Potential vs Charge

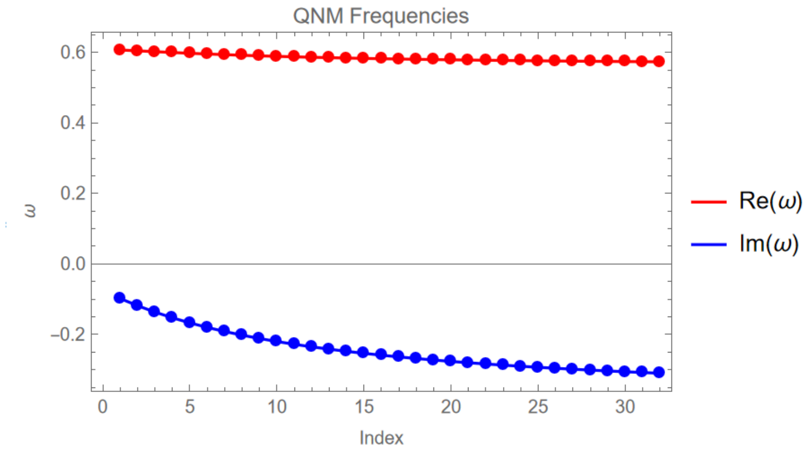

Figure 26: Effective Potential vs Radius

It has been observed that RN metric converges for s=0 and s=2.

4 Conclusion

Through this comprehensive study of quasinormal modes in Schwarzschild and Reissner-Nordström black holes, we have developed a detailed understanding of how different types of perturbations behave in these spacetimes. Our analysis reveals several key findings:

- The WKB approximation's accuracy varies significantly with order, with an optimal order identified through careful error analysis for each perturbation type.
- The lapse function shows distinct behavior between the two black hole types, with the Reissner-Nordström case exhibiting more complex dependencies due to the charge parameter.
- The effective potential profiles demonstrate characteristic differences across gravitational, electromagnetic, and scalar perturbations, particularly in their peak heights and positions.
- For the Reissner-Nordström black hole, increasing charge leads to systematic shifts in QNM frequencies, reflecting the modified spacetime structure.