CHERN CLASSES OF CONFORMAL BLOCKS

These are the notes for the final presentation based on the paper of the same title by N. Fakhruddin [Fak12].

The sheaves of conformal blocks on a smooth family of curves have a natural flat projective connection which can be lifted to a flat connection in genus zero or one. This connection has logarithmic singularities along the boundary divisors for a degenerating family of smooth curves. Calculating residues of this connection along the simple normal crossings boundary divisors of $\overline{M}_{0,n}$ and using a result of Esnault and Verdier ([EV86], Appendix B), one can compute chern classes.

THE KZ/HITCHIN/WZW CONNECTION

Let $\mathcal{C} \to \operatorname{Spec} A = S$ be a family of curves with n-sections $p_1, \ldots, p_n$. We choose a formal parameter, i.e., isomorphism $\eta_i : \hat{\mathcal{O}}_{\mathcal{C},p_i(S)} \cong A[[\xi]]$ for each i, and a suitable symmetric bidifferential ω on the family. Given this data and a vector field D on S, its action is defined as follows: First lift D to $\mathcal{C} \setminus \cup_i p_i(S)$, and write $(\eta_i^{-1})^*(D) = D_{i,hor} + D_{i,vert}$ where $D_{i,hor}(\xi) = 0$ and $D_{i,vert}$ kills A. The Sugawara operators $T(D_{i,vert})$ act on $\mathcal{H}_{\lambda_i}$ and hence on $\mathcal{H}_{\overline{\lambda}}$. The bidifferential ω gives elements $a_{\omega,i}(D_{i,vert}) \in A$ whose sum over all i is denoted by $a_\omega(D_{vert})$. The action of D on $\mathbb{V}_{\mathcal{C}}(\overline{p}, \overline{\lambda})$ is induced from the action of D on $\mathcal{H}_{\overline{\lambda}} \otimes A$ which we denote by ∇_D and is given by:

$$\nabla_D(\nu \otimes f) = D_{hor}(\nu \otimes f) + (\sum_{i=1}^{n} T(D_{i,vert})(\nu)) \otimes f - \nu \otimes a_\omega(D_{vert}) \cdot f,$$

where $\nu \in \mathcal{H}_{\overline{\lambda}}$ and $f \in A$.

When we use different indentifications η_i' we get a different action ∇_D'. The difference of the two actions $\nabla_D - \nabla_D'$ is given by the operator

$$D_{hor} - D'_{hor} + T(D_{vert}) - T'(D'_{vert}) - (a_\omega(D_{vert}) - a'_\omega(D'_{vert})).$$

It turns out that this difference in connections acts by an operator U and as a function of $\overline{\lambda}$ it only depends on $c(\lambda_i)$.

Lemma 0.0.1. *(1) For each i, $D_{i,vert} - D'_{i,vert}$ is of the form $f_i(\xi)\frac{d}{d\xi}$ with $f_i(\xi) = \sum_{j=q}^{\infty} a_j^i \xi^j$.*

(2) $\nabla_D - \nabla'_D$ is given by multiplication by

$$(2(l + \check{h}))^{-1}(\sum_i a_1^i c(\lambda_i)).$$

Now let C' be a smooth projective curve and let $q_1, q_2 \in C$ be distinct points. Let C be the curve obtained from C' by gluing q_1 and q_2. One can construct a natural smoothening $\pi : \mathcal{C} \to S = \operatorname{Spec} k[[t]]$ such that the generic fiber is smooth and the special fiber is C. Let $p_1, \ldots, p_n$ be smooth rational points on C which extend to the family, let $\overline{\lambda} \in P_l^n$. Let D be the vector field $t\frac{d}{dt}$ on S. The Sugawara action of D induces an operator on $\mathbb{V}_{\mathcal{C}}(\overline{p}, \overline{\lambda})$ which restricts to the operator on $\mathbb{V}_C(\overline{p}, \overline{\lambda})$:

Proposition 0.0.2. *Under the identification $\mathbb{V}_C(\overline{p}, \overline{\lambda}) \cong \oplus_{\mu \in P_l} \mathbb{V}_{C'}(\overline{pq}, \overline{\lambda\mu})$ given by the factorization formula, the operator acts on the summand $\mathbb{V}_{C'}(\overline{pq}, \overline{\lambda\mu})$ by multiplication by $\frac{c(\mu)}{2(l+\check{h})}$ in genus zero.*

Calculations in genus zero

We identify the moduli space $M_{0,n}$ of smooth genus zero curves with n marked points with the open subset of $\mathbb{A}^{n-3}$ given by

$$\{(z_1,\dots,z_{n-3}) \in \mathbb{A}^{n-3} | z_i \neq 0,1 \text{ for all } i \text{ and } z_i \neq z_j \text{ for } i \neq j\}.$$

The universal family over $M_{0,n}$ is given by $M_{0,n} \times \mathbb{P}^1$ with n ordered sections given by n morphisms $M_{0,n} \to \mathbb{P}^1$ sending $(z_1,\dots,z_{n-3}) \mapsto z_1, z_2, \dots, z_{n-3}, 0, 1, \infty$. If x is a coordinate on $\mathbb{P}^1$, these sections are given by $x = z_1, \dots z_{n-3}, 0, 1$ and $1/x = 0$.

Boundary divisors on the moduli space of stable genus zero curves with n marked points correspond to partions $A \sqcup B$ of the set $\{1,\dots,n\}$ of markings where $|A|, |B| \geq 2$. In coordinates above they correspond to blow ups of the following loci:

(1) For $\emptyset \neq S \subset \{1,\dots,n-3\}$, the locus in $\mathbb{A}^{n-3} \supset M_{0,n}$ given by the equations $\{z_i = 0\}_{i \in S}$.
(2) For $\emptyset \neq S \subset \{1,\dots,n-3\}$, the locus in $\mathbb{A}^{n-3} \supset M_{0,n}$ given by the equations $\{z_i = 1\}_{i \in S}$.
(3) For $\emptyset \neq S \subset \{1,\dots,n-3\}$, the locus in $(\mathbb{P}^1)^{n-3} \supset \mathbb{A}^{n-3} \supset M_{0,n}$ given by the equations $\{1/z_i = 0\}_{i \in S}$.
(4) For $S \subset \{1,\dots,n-3\}$, with $|S| \geq 2$, the locus in $\mathbb{A}^{n-3} \supset M_{0,n}$ given by the equations $\{z_i = z_j\}_{i,j \in S}$.

Boundary divisors of type 1: Assume $S = \{1,2,\dots,r\}$. In the blowup, an open set $U \cong \mathbb{A}^{n-3}$ with coordinates $(t, w_2, \dots, w_r, z_{r+1}, \dots, z_{n-3})$ and the map to $M_{0,n}$ is given by

$$(t, w_2, \dots, w_r, z_{r+1}, \dots, z_{n-3}) \mapsto (t, tw_2, \dots, tw_r, z_{r+1}, \dots, z_{n-3}),$$

with the exceptional divisor B given by the equation $t = 0$. The universal family near the exceptional divisor is given by blowing up the locus $t = x = 0$ in $U \times \mathbb{P}^1$.

Let $y = t/x$. The sections defined on $M_{0,n}$ extend to sections over this family as follows:

- Those given by equations $x = z_{r+1}, \dots, z_{n-3}$, $x = 1$ and $1/x = 0$ are given by the same equations.
- The section given by $x = 0$ is given by $1/y = 0$.
- The sections given by $x = z_i, 1 \leq i \leq r$ are given by $y = w_i^{-1}$ with $w_1 = 1$.

Replacing k with $k(w_2, \dots, w_r, z_{r+1}, \dots, z_{n-3})$ (taking generic fiber over the boundary divisor) it follows from the above proposition that the residue of the connection in above coordinates and the new equations for sections is given by the endomorphism of $\mathbb{V}_{\bar{\lambda}}|_B$ which acts on the summand $\mathbb{V}_{\bar{\lambda}'\mu} \otimes \mathbb{V}_{\bar{\lambda}''\mu^*}$ by multiplication by $\frac{c(\mu)}{2(l+\check{h})}$.

Let $D = \partial/\partial t$ and lift it to a derivation on the universal family over U with trivial action in the fiber direction. To compare the connection in new coordinates with the KZ connection, we have to calculate the functions $a_1^i, i = 1, \dots, n$, occuring in above lemma. Writing $D = D_{hor} + D_{vert} = D'_{hor} + D'_{vert}$, we get:

- for sections given by $x = z_{r+1}, \dots, z_n$, $x = 1$ and $1/x = 0$, $D_{vert} = D'_{vert} = 0$ so $a_1 = 0$.
- for the section $x = 0$, $D_{vert} = 0$ but $D'_{vert} = -(x/t)\partial_x$ so $a_1 = 1/t$.
- for the sections given by equations $x = z_i, 1 \leq i \leq r$, by substituting $z_i = tw_i$, we see that $D_{vert} = -w_i \partial_x$ whereas $(\partial_t + (x/t)\partial_x)((t/x) - w_i^{-1}) = 0$ so $D'_{vert} = -(x/t)\partial_x = -(\frac{x - z_i}{t} + w_i)\partial_x$. Therefore, $a_1 = 1/t$.

Adding all the terms, the residue of the KZ connection along this divisor (type 1) is the endomorphism of $\mathbb{V}_{\overline{\lambda}}|_B$ which acts by multiplication by

$$\frac{c(\mu)-c(\lambda_{n-2})-\sum_{i\in S}c(\lambda_i)}{2(l+\check{h})}$$

on the summand $\mathbb{V}_{\overline{\lambda}'\mu}\otimes\mathbb{V}_{\overline{\lambda}''\mu^*}$ for each $\mu\in P_l$.

Similar local coordinate calculations in the blow up of specified loci yield residues of the KZ connection along other types of boundary divisors:

Boundary divisors of type 2: The residue of the KZ connection along this divisor (type 2) is the endomorphism of $\mathbb{V}_{\overline{\lambda}}|_B$ which acts by multiplication by

$$\frac{c(\mu)-c(\lambda_{n-1})-\sum_{i\in S}c(\lambda_i)}{2(l+\check{h})}$$

on the summand $\mathbb{V}_{\overline{\lambda}'\mu}\otimes\mathbb{V}_{\overline{\lambda}''\mu^*}$ for each $\mu\in P_l$.

Boundary divisors of type 3: The residue of the KZ connection along this divisor (type 3) is the endomorphism of $\mathbb{V}_{\overline{\lambda}}|_B$ which acts by multiplication by

$$\frac{c(\mu)-c(\lambda_{n})+\sum_{i\in S}c(\lambda_i)}{2(l+\check{h})}$$

on the summand $\mathbb{V}_{\overline{\lambda}'\mu}\otimes\mathbb{V}_{\overline{\lambda}''\mu^*}$ for each $\mu\in P_l$.

Boundary divisors of type 4: The residue of the KZ connection along this divisor (type 4) is the endomorphism of $\mathbb{V}_{\overline{\lambda}}|_B$ which acts by multiplication by

$$\frac{c(\mu)-\sum_{i\in S}c(\lambda_i)}{2(l+\check{h})}$$

on the summand $\mathbb{V}_{\overline{\lambda}'\mu}\otimes\mathbb{V}_{\overline{\lambda}''\mu^*}$ for each $\mu\in P_l$.

For a smooth projective variety X, $D=\cup_i D_i$ a divisor with simple normal crossings, V a vector bundle on X and ∇ a connection on $U=X-D$ with logarithmic singularities, we have

Proposition 0.0.3 (Esnault, Verdier)**.**

$$N_p(V)=(-1)^p\sum_{\alpha_1+\cdots+\alpha_s=p}\binom{p}{\alpha}\mathrm{Tr}(\Gamma_1^{\alpha_1}\circ\cdots\circ\Gamma_s^{\alpha_s})[D_1]^{\alpha_1}\cdots[D_s]^{\alpha_s}$$

where N_p is the $p'th$ Newton polynomial in the chern roots of V, $[D_i]$ denotes the class of D_i in Hodge cohomology and Γ_i is the endomorphism of $V|_{D_i}$ given by the residue of ∇ along D_i.

The KZ connection on $\mathbb{V}_{\overline{\lambda}}$ has logarithmic singularities, so using the calculations of residues above, we get the following implicit expression of all chern classes of $\mathbb{V}_{\overline{\lambda}}$ in Hodge cohomology or rational Chow groups of $\overline{M}_{0,n}$.

Theorem 0.0.4. *Let $\mathfrak{g}$ be a simple lie algebra, $l\geq 0$ an integer and $\overline{\lambda}=(\lambda_1,\ldots,\lambda_n)\in P_l^n$. Then*

$$N_p(\mathbb{V}_{\overline{\lambda}})=(-1)^p\sum_{\alpha_1+\cdots+\alpha_s=p}\binom{p}{\alpha}\mathrm{Tr}(\Gamma_1^{\alpha_1}\circ\cdots\circ\Gamma_s^{\alpha_s})[B_1]^{\alpha_1}\cdots[B_s]^{\alpha_s}$$

in $CH^p(\overline{M}_{0,n})_{\mathbb{Q}}$, where N_p denotes $p'th$ Newton class, B_i, $i=1,\ldots,s$, are the irreducible components of $\overline{M}_{0,n}\setminus M_{0,n}$, and Γ_i denotes the residue of the KZ connection along B_i given by one of the constants determined above.

The KZ connection depends on the choice of coordinates and so do the residues and therefore the representing cycle for $c_1(\mathbb{V}_{\overline{\lambda}})=N_1(\mathbb{V}_{\overline{\lambda}})$. By averaging over all choices one obtains a canonical representative:

Corollary 0.0.5.

$$c_1(\mathbb{V}_{\overline{\lambda}}) = \frac{1}{2(l+\check{h})}\cdot$$

$$\sum_{i=2}^{\lfloor n/2 \rfloor} \epsilon_i \{ \sum_{\substack{A\subset\{1,\dots n\} \\ |A|=i}} \{ \frac{r_{\overline{\lambda}}}{(n-1)(n-2)} \{(n-i)(n-i-1) \sum_{a\in A} c(\lambda_a) + i(i-1) \sum_{a'\in A^c} c(\lambda_{a'})\}$$

$$-\{ \sum_{\mu\in P_l} c(\mu)\cdot r_{\overline{\lambda}_{A,\mu}} \cdot r_{\overline{\lambda}_{A^c,\mu^*}} \}\} \cdot [D_{A,A^c}]\}$$

in $\operatorname{Pic}(\overline{M}_{0,n})_{\mathbb{Q}}$, *where* D_{A,A^c} *is the irreducible boundary divisor corresponding to the partition* $\{1,\dots,n\} = A\cup A^c$ *and* $\epsilon_i = 1/2$ *if* $i = n/1$ *and* 1 *otherwise.*

Specializing to $n=4$ we get

Corollary 0.0.6. *Let* $\mathfrak{g}$ *be a simple lie algebra,* $l \geq 0$ *an integer and* $\overline{\lambda} = (\lambda_1,\lambda_2,\lambda_3,\lambda_4) \in P_l^4$*. Then*

$$\deg(\mathbb{V}_{\overline{\lambda}}) = \frac{1}{2(l+\check{h})} \times \{\{r_{\overline{\lambda}} \sum_{i=1}^{4} c(\lambda_i)\}$$

$$-\{\sum_{\lambda\in P_l} c(\lambda)\{r_{(\lambda_1,\lambda_2,\lambda)}\cdot r_{(\lambda_3,\lambda_4,\lambda^*)} + r_{(\lambda_1,\lambda_3,\lambda)}\cdot r_{(\lambda_2,\lambda_4,\lambda^*)} + r_{(\lambda_1,\lambda_4,\lambda)}\cdot r_{(\lambda_2,\lambda_3,\lambda^*)}\}\}\}.$$

When $\mathfrak{g} = \mathfrak{sl}_2$**,** $n = 4$**:**

Proposition 0.0.7. *Suppose* $\overline{\lambda} = \overline{\lambda} = (\lambda_1,\lambda_2,\lambda_3,\lambda_4) \in P_l^4$ *with* $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ *and* $2s := \sum_i \lambda_i$ *even. Then*

$$\deg(\mathbb{D}_{\overline{\lambda}}) = \begin{cases} \max\{0,(l+1-\lambda_4)(s-l)\} & \text{if } \lambda_1+\lambda_4 \geq \lambda_2+\lambda_3 \\ \max\{0,(l+1+\lambda_1-s)(s-l)\} & \text{if } \lambda_1+\lambda_4 \leq \lambda_2+\lambda_3. \end{cases}$$

Moreover for level 1 and any n, we have

Theorem 0.0.8. *For any* $n \geq 4$*, the set of non-trivial determinants of conformal blocks of level* $l = 1$ *for* $\mathfrak{sl}_2$ *form a basis of* $\operatorname{Pic}(\overline{M}_{0,n})$*.*

References

[EV86] Hélène Esnault and Eckart Viehweg, *Logarithmic de Rham complexes and vanishing theorems*, Invent. Math. **86** (1986), no. 1, 161–194. MR 853449

[Fak12] Najmuddin Fakhruddin, *Chern classes of conformal blocks*, Compact moduli spaces and vector bundles, Contemp. Math., vol. 564, Amer. Math. Soc., Providence, RI, 2012, pp. 145–176. MR 2894632