Étale Cohomology and Rationality of the L-series of Varieties

A Thesis

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

by

Abhishek Koparde

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

April, 2022

Supervisor: Dr. Arvind Nair

Certificate

This is to certify that this dissertation entitled Étale Cohomology and Rationality of the L-series of Varietiestowards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Abhishek Koparde at Indian Institute of Science Education and Research under the supervision of Dr. Arvind Nair, Associate Professor, School of Mathematics, TIFR Mumbai , during the academic year 2021-2022.

Dr. Arvind Nair

Committee:

Dr. Arvind Nair

Dr. Amit Hogadi

This thesis is dedicated to all the people who inspired me to study Mathematics

Declaration

I hereby declare that the matter embodied in the report entitled Étale Cohomology and Rationality of the L-series of Varieties are the results of the work carried out by me at the School of Mathematics, TIFR Mumbai, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Arvind Nair and the same has not been submitted elsewhere for any other degree.

Abhishek Koparde

Acknowledgments

I have received a great deal of help and support throughout this thesis.

First and foremost, I would like to thank my advisor, Dr. Arvind Nair, for his encouragement and support over the past year. He was always very generous with his time. He would patiently explain many abstract and complex mathematical ideas in our long online meetings. Discussions with him have helped me navigate the often tumultuous mathematical arguments in this subject and have enabled me to see the broader context of where things fall in place.

I want to thank all my teachers at IISER Pune, especially Dr. Baskar Balasubramanyam, Dr. Chandrasheel Bhagwat, and Dr. Amit Hogadi, for playing a crucial role in shaping my mathematical interests.

Getting through my thesis required more than academic support. I have many people to thank for listening to and, at times, having to tolerate me over the past five years. Amey, Shubhankar, and Rushikesh have been unwavering in their support during my time in Pune. I would not have come to the other side of the application season if not for the essential help of these people. I must thank Vidyadheesh, Gautam, and everyone above for many memorable treks, lunches, and philosophical musings. I want to thank Amey and Rushikesh for setting the standard high from the start and setting me on the right track.

Finally, I would like to thank my parents and sister for the psychological support and for always being there for me, especially during the pandemic days when my interest in Algebraic Geometry flourished.

Abstract

Étale cohomology was introduced by A. Grothendieck and developed by him with the help of M. Artin and J. L. Verdier, to explain A. Weil's insight that for polynomial equations with integer coefficients, the topology of the set of complex solutions should profoundly influence the number of solutions of the equations modulo a prime number. Weil conjectured that the zeta function (introduced by Hasse for curves and by Weil in general) of a smooth projective variety over a finite field $\mathbb{F}_q$, which is a generating function that captures the growth of the number of points defined over $\mathbb{F}_{q^n}$ as n increases, is a rational function, satisfies a certain functional equation and has its zeroes at restricted places.

Since étale cohomology gives a replacement, for arbitrary schemes, of the cohomology of the space of complex points of a variety, and since it gives a sheaf theory and cohomology theory whose properties closely resemble those arising from the complex topology, it allows for the use of topological ideas over general fields, both in algebraic geometry and in many areas (number theory, representation theory, algebra) where algebraic geometry plays an essential role.

Contents

Introduction

Let X be a variety (separated, finite type, integral scheme) defined over a finite field k (for the sake of this discussion let $k = \mathbb{F}_p$ where p is a prime number). Let k' be a finite extension of the field k. By a k'-valued point of X, we mean a map $\text{Spec } k' \to X$ of schemes. Our hypothesis on X (finite type) imply that the set of all k' valued points $X(k')$ is a finite set. Moreover, given a tower $k''/k'/k$ of field extensions, we get that $X(k) \subset X(k') \subset X(k'')$. Since we are over a finite field, and all finite fields of the same size are canonically isomorphic, we get a sequence $\{a_n\}$ of integers where $a_n = \#(X(\mathbb{F}_{p^n}))$.

One can construct a generating function from this sequence, which becomes an algebraic invariant of the variety X. For example, let X be the r-dimensional projective space $\mathbb{P}^r_{\mathbb{F}_p}$ over k. It is easy to see that $\mathbb{P}^r$ has

$$\frac{(p^n)^{r+1} - 1}{p^n - 1} = p^{rn} + p^{(r-1)n} + \cdots + p^n + 1$$

many points defined over the finite field $\mathbb{F}_{p^n}$ (points on the projective space are nonzero $r+1$ tuples with values in the field modulo scalar multiplication). We define the zeta function

$$Z(\mathbb{P}^r, t) = exp(\sum_{i \geq 1} \frac{t^i}{i} \#(X(\mathbb{F}_{p^i}))).$$

Putting in the number of points calculated above, we get

$$
\begin{aligned}
Z(\mathbb{P}^r, t) &= exp(\sum_{i\geq 1} \frac{t^i}{i}(p^{ri} + p^{(r-1)i} + \cdots + p^i + 1)) \\
&= \prod_{j=0}^{r} exp(\sum_{i\geq 1} \frac{t^i}{i} p^j) \\
&= \prod_{j=0}^{r} \frac{1}{1 - p^j t}.
\end{aligned}
$$

In particular, this zeta function is a rational function.

In 1949, André Weil [Wei49] proposed the following conjectures for a nonsingular n-dimensional projective variety X defined over a finite field $\mathbb{F}_q$:

Let $\zeta(X, t) = exp(\sum_{i\geq 1} \#(X(\mathbb{F}_{q^i}))\frac{q^{-it}}{i})$ be the zeta function

1. (Rationality) $\zeta(X, t)$ is a rational function of $T = q^(-t)$. Moreover, it can be written as
$$\zeta(X, t) = \frac{P_1(T) \cdots P_{2n-1}(T)}{P_0(T) \cdots P_{2n}(T)},$$
where each P_i is an integral polynomial and $P_0(T) = 1 - T$, $P_{2n}(T) = 1 - q^n T$, and for $1 \leq i \leq 2n - 1$, $P_i(T)$ factors over $\mathbb{C}$ as $P_i(T) = \prod_j (1 - \alpha_{ij} T)$.
2. (Functional equation) The zeta function satisfies
$$\zeta(X, q^{-n}T^{-1}) = \pm q^{\frac{nE}{2}} T^E \zeta(X, T),$$
where E is the euler characteristic of X.
3. (Riemann hypothesis) $|a_{ij}| = q^{i/2}$ for all $1 \leq i \leq 2n - 1$ and all j.
4. (Betti numbers) If X is obtained by a "good reduction" mod p of a non singular variety Y defined over a number field embedded in the complex numbers, then the degree of P_i is the i^{th} betti number of the space of complex points of Y.

The first two statements were proved by Grothendieck and his collaborators by giving an interpretation of the Hasse-Weil zeta function using the machinery of étale cohomology.

The last, the analogue of the Riemann hypothesis for these zeta functions, was proved by P. Deligne by proving an analogue of the hard Lefschetz theorem in the topology of complex algebraic varieties.

0.1 Original Contribution

This thesis aims to serve as a guided tour of the theory of étale cohomology, and the result on the rationality of the L-series of varieties defined over finite fields. We also delve deep into the proofs of the base change theorem for proper morphisms and the comparison of étale cohomology with singular cohomology for varieties defined over complex numbers as the techniques involved in these proofs are of wider interest. This thesis is an attempt to give a cohesive narrative of this subject, providing statements of key theorems and either proofs or references to the proofs of those statements. Some of the results could not make it to the final draft, but the reader should be able to find a large chunk of them in the references provided.

Chapter 1

Preliminaries

In this chapter we give an account of Grothendieck topologies which is a generalization of the notion of a topological space, and Étale morphisms which are analogues of local isomorphisms in topology. Unless stated otherwise, all rings appearing in this report should be assumed commutative.

1.1 Flat Morphisms

Definition 1.1.1. *Let A be a ring. An A-module M is called flat if the functor $- \otimes_A M$ is exact. A map of rings $A \to B$ is flat if B is flat viewed as an A-module under this map. A morphism of schemes $X \to Y$ is flat if the stalk maps at all points are flat.*

Definition 1.1.2. *An A-module M is called faithfully flat if for any complex of A-modules*

$$N' \to N \to N'',$$

it is exact at N if and only if the complex

$$M \otimes_A N' \to M \otimes_A N \to M \otimes_A N''$$

is exact at $M \otimes_A N$. A morphism of schemes $X \to Y$ is called faithfully flat if it is flat and surjective.

Lemma 1.1.3 (Going-down for flat morphisms). *Let $B \to A$ be a flat ring map, π : Spec $A \to$ Spec B be the corresponding flat map of schemes. Let $\mathfrak{q}' \subset \mathfrak{q}$ be prime ideals of B. Let $\mathfrak{p}$ be a prime ideal of A such that $\pi([\mathfrak{p}]) = [\mathfrak{q}]$. Then there exists a prime ideal $\mathfrak{p}' \subset \mathfrak{p}$ such that $\pi([\mathfrak{p}']) = [\mathfrak{q}']$.*

Proof. Since $B \to A$ is flat, the map $B_\mathfrak{q} \to A_\mathfrak{p}$ of local rings is flat. We claim that this map is faithfully flat. Indeed, note that $B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q} \otimes_{B_\mathfrak{q}} A_\mathfrak{p} = B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q} \otimes_{B_\mathfrak{q}} A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p}$ is nonzero. Let N be any $B_\mathfrak{q}$ module, and suppose $N \otimes_{B_\mathfrak{q}} A_\mathfrak{p} = 0$. This implies $N \otimes_{B_\mathfrak{q}} A_\mathfrak{p} \otimes_{B_\mathfrak{q}} B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q} = 0$. Since $B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q} \otimes_{B_\mathfrak{q}} A_\mathfrak{p} = B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q} \otimes_{B_\mathfrak{q}} A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p}$ is a nonzero $B_\mathfrak{q}/\mathfrak{q}B_\mathfrak{q}$ vector space, we must have $N/\mathfrak{q}N = 0$. By Nakayama lemma, we must have $N = 0$ when N is a finitely generated module, but since any module is a filtered colimit of finitely generated modules, we get that the map $B_\mathfrak{q} \to A_\mathfrak{p}$ is faithfully flat. Consequently, the corresponding map of schemes is surjective and hence we are done. ☐

Lemma 1.1.4. *Let $\pi : X \to Y$ be a morphism of schemes that is flat and locally of finite type and Y be locally noetherian (or more generally, let π be locally of finite presentation). Then π is an open map.*

Proof. Since the question is local both on the source and target, we reduce to the case of a map of affine schemes $\pi : X = \text{Spec } A \to Y = \text{Spec } B$. By Chevalley's theorem on constructible sets, we get that $\pi(X)$ is a constructible subset of Y. By going-down theorem for flat morphisms, $\pi(X)$ is stable under generization. Therefore $\pi(X)$ is open. ☐

1.2 Unramified Morphisms

Definition 1.2.1. *A local homomorphism of local rings $\varphi : A \to B$ (i.e., $\varphi(\mathfrak{m}_A) \subset \mathfrak{m}_B$) is called unramified if*

1. *$\varphi(\mathfrak{m}_A)$ generates $\mathfrak{m}_B$, i.e., $\varphi(\mathfrak{m}_A) \cdot B = \mathfrak{m}_B$.*
2. *The map of residue fields $\kappa_A = A/\mathfrak{m}_A \to \kappa_B = B/\mathfrak{m}_B$ is separable.*

A morphism $f : X \to Y$ of schemes is called unramified at $x \in X$ if the stalk map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an unramified homomorphism of local rings.

1.3 Étale morphisms

Definition 1.3.1. *A morphism $f : X \to Y$ of schemes is called étale if*

1. *f is locally of finite presentation.*

2. *f is flat.*

3. *f is unramified.*

We state the following proposition from the Stacks Project highlighting some important properties of étale morphisms.

Proposition 1.3.2. *Facts on étale morphisms.*

1. *Let k be a field. A morphism of schemes $U \to Spec\ (k)$ is étale if and only if $U \cong \coprod_{i \in I} Spec\ (k_i)$ such that for each $i \in I$ the ring k_i is a field which is a finite separable extension of k.*

2. *Let $\varphi : U \to S$ be a morphism of schemes. The following conditions are equivalent:*

   (a) φ is étale,

   (b) φ is locally finitely presented, flat, and all its fibres are étale,

   (c) φ is flat, unramified and locally of finite presentation.

3. *A ring map $A \to B$ is étale if and only if $B \cong A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ such that $\Delta = \det\left(\frac{\partial f_i}{\partial x_j}\right)$ is invertible in B.*

4. *The base change of an étale morphism is étale.*

5. *Compositions of étale morphisms are étale.*

6. *Fibre products and products of étale morphisms are étale.*

7. *An étale morphism has relative dimension 0.*

8. *Let $Y \to X$ be an étale morphism. If X is reduced (respectively regular) then so is Y.*

9. *Étale morphisms are open.*

10. *If $X \to S$ and $Y \to S$ are étale, then any S-morphism $X \to Y$ is also étale.*

Proof. See [Sta18, Tag 03PC]. □

1.4 Grothendieck Topologies

Definition 1.4.1. *Let $\mathcal{C}$ be a category in which fibered products exist. A Grothendieck Topology τ on the category $\mathcal{C}$ is the data of the set of coverings in $\mathcal{C}$ (called $Cov(\mathcal{C})$), satisfying the following axioms:*

1. *If $U' \xrightarrow{f} U$ is an isomorphism, then $\{U' \xrightarrow{f} U\} \in Cov(\mathcal{C})$.*
2. *If $\{U_\alpha \to U\}_{\alpha \in I}$ is a covering and for each α $\{U_{\alpha,\beta} \to U_\alpha\}_{\beta \in I_\alpha}$, then the set $\{U_{\alpha,\beta} \to U\}_{\alpha \in I, \beta \in I_\alpha}$ of composite morphisms is a covering.*
3. *If $\{U_\alpha \to U\}_{\alpha \in I}$ is a covering and $V \to U$ is any morphism in $\mathcal{C}$, then $\{V \times_U U_\alpha \to V\}_{\alpha \in I}$ is a covering.*

The data of a category $\mathcal{C}$ together with a Grothendieck topology τ is called a **Site**.

Example 1.4.2. *Let X be a topological space. Let $\mathcal{C}_X$ be the category whose objects are open subsets of X and the morphisms are inclusion maps. We let $\{U_i \xrightarrow{\varphi_i} U\}_{i \in I}$ be a covering if $\bigcup_{i \in I} \varphi_i(U_i) = U$. It is easy to see that the category $\mathcal{C}_X$ together with the coverings described is a site.*

1.4.1 Faithfully-Flat descent

We show that if $A \to B$ is a faithfully flat morphism, then given any B-module N satisfying some descent conditions, there exists an A-module M such that, $N \cong M \otimes_A B$ as a B-module.

Lemma 1.4.3. *Suppose $\phi : A \to B$ is a faithfully flat map of rings. Then*

$$0 \to A \xrightarrow{\phi} B \xrightarrow{d_0} B \otimes_A B$$

is an exact sequence of A-modules, where $d_0(b) = b \otimes 1 - 1 \otimes b$.

Proof. The following proof is due to Grothendieck. We proceed in three stages.

1. First suppose that the map ϕ has a section $\psi : B \to A$. Then clearly ϕ is injective, and we have a decomposition of B as $B = A \oplus I$, where $I = Ker(\psi)$. Now, it is easy to see that $d_0(b) = d_0(b - \psi(B))$, and $b - \psi(b) \in I$. We have,

$$B \otimes_A B = (A \otimes_A A) \oplus (I \otimes_A A) \oplus (A \otimes_A I) \oplus (I \otimes_A I),$$

so $d_0(b) = d_0(b - \psi(b)) = 0$ implies $b = \psi(b)$ because of the above direct sum. Thus, $Ker(d_0) = A$ and the sequence is exact.

2. Suppose $A \to C$ is a faithfully flat extension. Then, we have

$$(B \otimes_A B) \otimes_A C = (B \otimes_A C) \otimes_A (B \otimes_A C).$$

It suffices to prove that

$$0 \to C \to B \otimes_A C \to (B \otimes_A C) \otimes_A (B \otimes_A C)$$

is exact because of faithful flatness.

3. Consider the faithfully flat ring map $A \to B$. By step (2), we get the ring map

$$B \to B \otimes_A B, \ b \mapsto 1 \otimes b.$$

This ring map clearly has a section (namely, $b \otimes c \mapsto bc$), and hence the base changed sequence is exact by step (1). Since $A \to B$ is faithfully flat, the original sequence is exact and we are done.

$\square$

Lemma 1.4.4. *Let $\phi : A \to B$ be a faithfully flat map of rings and let M be any A-module. Then*

$$0 \to M \to M \otimes_A B \xrightarrow{d_0} M \otimes_A B^{\otimes 2} \to \cdots \xrightarrow{d_{r-2}} M \otimes_A B^{\otimes r}$$

is an exact sequence of A modules, where

$$B^{\otimes r} = B \otimes_A B \otimes_A \cdots \otimes_A B, r \text{ times}$$

$$d_{r-1}(b) = \sum_i (-1)^i e_i$$

$$e_i(b_0 \otimes \cdots \otimes b_{r-1}) = b_0 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_{r-1}.$$

Proof. It is easy to see that the composition of two maps is 0. To show the exactness, we reduce to the case where there exists a section $\psi : B \to A$. Then consider the morphism $k_r : B^{\otimes r+2} \to B^{\otimes r+1}$ defined as

$$k_r(b_0 \otimes \cdots \otimes b_r) \mapsto g(b_0)b_1 \otimes \cdots \otimes b_r.$$

From this it follows that $d_r \circ k_r + k_{r+1} \circ d_{r+1} = 1$ for all $r \geq -1$. Now suppose there exists some $a \in M \otimes_A B^{\otimes r+2}, r \geq -1$ such that $d_{r+1}(a) = 0$. Then, applying $d_r \circ k_r + k_{r+1} \circ d_{r+1} = 1$ to a, we get $a = d_r(k_r(a))$, so $a \in Im(d_r)$ which proves exactness at all $M \otimes_A B^{\otimes r}, r \geq 1$. To show the injectivity of the map $M \to M \otimes_A B$, suppose there exists $m \in M$, such that $m \otimes 1 = 0 \in M \otimes_A B$, but then $m = k_{-1}(m \otimes 1) = 0$, which proves injectivity and we are done. □

Let $\phi : A \to B$ be a morphism of commutative rings. Let N be a B-module. Then N becomes a $B^{\otimes 2}$ module in two different ways, as $N \otimes_A B$ and as $B \otimes_A N$. Analogously, a B-module N becomes a $B^{\otimes 3}$ module in three different ways, as $B \otimes_A B \otimes_A N$, $B \otimes_A N \otimes_A B$, and as $N \otimes_A B \otimes_A B$. Lets assume that we have a homomorphism $\psi : N \otimes_A B \to B \otimes_A N$ of $B \otimes_A B$ modules. Then there are three associated homomorphisms of $B^{\otimes 3}$ modules

$$\psi_1 : N \otimes_A B \otimes_A B \to B \otimes_A N \otimes_A B,$$
$$\psi_2 : B \otimes_A N \otimes_A B \to B \otimes_A B \otimes_A N,$$
$$\psi_3 : N \otimes_A B \otimes_A B \to B \otimes_A B \otimes_A N$$

where $\psi_1 = \psi \otimes id$, $\psi_2 = id \otimes \psi$ and $\psi_3(x_1 \otimes x_2 \otimes x_3) = \sum y_i \otimes x_2 \otimes z_i$ where $\psi(x_1 \otimes x_3) = \sum y_i \otimes z_i$.

Definition 1.4.5 (Descent data). *Let $\phi : A \to B$ be a ring map.*

1. *A descent datum (N, ψ) for modules with respect to ϕ is given by a B-module N where $\psi : N \otimes_A B \to B \otimes_A N$ is an isomorphism of $B \otimes_A B$ modules, such that we have the*

following commutative diagram of $B \otimes_A B \otimes_A B$ module isomorphisms:

$$\begin{tikzcd} N \otimes_A B \otimes_A B \arrow[rr, "\psi_3"] \arrow[dr, "\psi_1"'] & & B \otimes_A B \otimes_A N \\ & B \otimes_A N \otimes_A B \arrow[ur, "\psi_2"'] & \end{tikzcd}$$

We will call this the cocycle condition.

2. *A morphism $(N, \psi) \to (N', \psi')$ of descent data is a map $\alpha : N \to N'$ of B-modules such that the diagram of $B \otimes_A B$-modules*

$$\begin{tikzcd} N \otimes_A B \arrow[r, "\psi"] \arrow[d, "\alpha\otimes id"'] & B \otimes_A N \arrow[d, "id\otimes\alpha"] \\ N' \otimes_A B \arrow[r, "\psi'"] & B \otimes_A N' \end{tikzcd}$$

is commutative.

Definition 1.4.6. *Let $\phi : A \to B$ be a ring map. We say that a descent datum (N, ψ) is effective if there exists an A-module M and an isomorphism of descent data $(B \otimes_A M, id \otimes i_M)$ and (N, ψ), where*

$$id \otimes i_M : B \otimes_A M \otimes_A B \to B \otimes_A B \otimes_A M$$

sends $(b_1 \otimes m) \otimes b_2 \mapsto b_1 \otimes (b_2 \otimes m)$.

We have following theorem:

Theorem 1.4.7. *Let $\phi : A \to B$ be a faithfully flat ring map. Then any descent datum (N, ψ) with respect to $\phi : A \to B$ is effective.*

Proof. For the descent data (N, ψ), define $\overline{N} = \{n \in N | \psi(n \otimes 1) = 1 \otimes n\}$, which is clearly an A-module. The usual rule $\theta : B \otimes_A \overline{N} \to N$, $b \otimes n \mapsto bn$ defines a homomorphism of B-modules. Consider the diagram:

$$\begin{tikzcd} B \otimes_A B \otimes_A \overline{N} \arrow[r, "id\otimes\theta"] & B \otimes_A N \\ B \otimes_A \overline{N} \otimes_A B \arrow[r, "\theta\otimes id"] \arrow[u, "id\otimes i_{\overline{N}}"] & N \otimes_A B \arrow[u, "\psi"'] \end{tikzcd}$$

We have

$$\begin{aligned}
\psi(\theta \otimes id)(b \otimes n \otimes b') &= \psi(bn \otimes b') \\
&= \psi((b \otimes b')(n \otimes 1)) \\
&= (b \otimes b')\psi(n \otimes 1) \\
&= (b \otimes b')(1 \otimes n) \\
&= (b \otimes b'n) \\
&= (id \otimes \theta)(b \otimes b' \otimes n) \\
&= (id \otimes \theta)(id \otimes i_{\overline{N}})(b \otimes n \otimes b').
\end{aligned}$$

Therefore the diagram commutes and we have a morphism of descent data $(B \otimes_A \overline{N}, id \otimes i_{\overline{N}}) \to (N, \psi)$.

Consider morphisms $\alpha, \beta : N \to B \otimes_A N$ defined by $\alpha(n) = 1 \otimes n$ and $\beta(n) = \psi(n \otimes 1)$. By definition, $\overline{N}$ is the kernel of $\alpha - \beta$ and because of faithful flatness of the ring map $A \to B$, we have the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \overline{N} \otimes_A B \arrow[r, "i \otimes id"] \arrow[d, "\theta \circ i"'] & N \otimes_A B \arrow[r, "(\alpha-\beta)\otimes id"] \arrow[d, "\psi"] & B \otimes_A N \otimes_A B \arrow[d, "\psi_2"] \\
0 \arrow[r] & N \arrow[r, "\gamma"] & B \otimes_A N \arrow[r, "(e_2 - e_1)\otimes id_N"] & B \otimes_A B \otimes_A N
\end{tikzcd}$$

Here, $e_1(b) = b \otimes 1,\ e_2(b) = 1 \otimes b,\ \gamma(n) = 1 \otimes n.$ Let us show that this diagram is commutative. For the first square, we have

$$\begin{aligned}
\psi(i \otimes id)(n \otimes b) &= \psi((1 \otimes b)(n \otimes 1)) \\
&= (1 \otimes b)(1 \otimes n) \\
&= (1 \otimes bn) \\
&= \gamma(bn) \\
&= \gamma(\theta \otimes i)(n \otimes b)
\end{aligned}$$

for any $n \otimes b \in \overline{N} \otimes_A B$, so first square commutes. Now for the second square, suppose

$n \otimes b \in N \otimes_A B$, and suppose $\psi(n \otimes b) = \sum b_i \otimes n_i$. We then have

$$
\begin{aligned}
((e_2 - e_1) \otimes id_N)\psi(n \otimes b) &= (e_2 - e_1) \otimes id_N(\sum b_i \otimes n_i) \\
&= \sum (1 \otimes b_i - b_i \otimes 1) \otimes n_i \\
&= \psi_2(\alpha \otimes id)(n \otimes b) - \psi_2\psi_1(n \otimes 1 \otimes b) \\
&= \psi_2(\alpha \otimes id)(n \otimes b) - \psi_2(\psi(n \otimes 1) \otimes b) \\
&= \psi_2(\alpha \otimes id)(n \otimes b) - \psi_2(\beta(n) \otimes b) \\
&= \psi_2((\alpha - \beta) \otimes id)(n \otimes b).
\end{aligned}
$$

Therefore, the second square also commutes. Now the middle and right vertical arrows are isomorphism, so the left vertical arrow is also an isomorphism and that finishes the proof. $\square$

1.5 The FPQC topology

Fpqc translates to faithfully-flat quasi-compact in English. We will study the fpqc site before the etale site as it has nice properties for quasi-coherent sheaves.

Definition 1.5.1. *Let T be a scheme. An fpqc covering of T is a family $\{\phi_i : T_i \to T\}_{i \in I}$ such that*

1. *each ϕ_i is a flat morphism and $\bigcup_{i \in I} \phi_i(T_i) = T$, and*
2. *for each affine open $U \subset T$, there exists a finite set K and a map $i : K \to I$ and affine opens $U_{i(k)} \subset T_{i(k)}$ such that $\bigcup_{k \in K} \phi_{i(k)}(U_{i(k)}) = U$.*

Definition 1.5.2. *Let T be an affine scheme. A standard fpqc covering of T is a family $\{f_i : T_i \to T\}_{i=1,\dots,n}$ with each T_i affine, flat over T such that $T = \bigcup f_i(T_i)$.*

Example 1.5.3. *Some fpqc coverings:*

1. *Suppose $A \to B$ is a faithfully flat morphism of rings. Then the corresponding morphism $\{Spec\ B \to Spec\ A\}$ is an fpqc covering.*
2. *Any zariski open covering is an fpqc covering.*

3. *If $f : X \to Y$ is a flat, surjective and quasi-compact morphism of schemes, then $\{f : X \to Y\}$ is an fpqc covering. The converse is false in general as we can take X to be an infinite disjoint union of copies of Y. The map will be flat and surjective, but won't be quasi-compact.*

Since flat morphisms of schemes are stable under base change and composition, fpqc coverings as defined above satisfy the axioms of a Grothendieck topology.

Lemma 1.5.4. *Let S be a scheme. Let $\mathcal{F}$ be a presheaf over Sch/S. Then $\mathcal{F}$ satisfies the sheaf condition for the fpqc topology if and only if*

1. *$\mathcal{F}$ satisfies the sheaf condition for zariski open coverings.*
2. *For every standard fpqc covering $\{U_i \to U\}_{i\in\{1,\dots,n\}}$, $U = Spec\ A, U_i = Spec\ A_i$, the sheaf condition holds.*
3. *For every faithfully flat morphism $Spec\ B \to Spec\ A$ of affine schemes over S, the sheaf condition holds for the covering $\{Spec\ B \to Spec\ A\}$.*

Moreover, (2) and (3) are equivalent in presence of (1).

Proof. Suppose (1) and (2) hold. Let $\{T_i \to T\}$ be an fpqc covering. Let $s_i \in \mathcal{F}(T_i)$ be a family of elements such that $s_i|_{T_i \times T_j} = s_j|_{T_i \times T_j}$. Let $W \subset T$ be the maximal open subset such that there exists unique $s \in \mathcal{F}(W)$ such that $s|_{f_i^{-1}(W)} = s_i|_{f_i^{-1}(W)}$. Such an open set exists because of the sheaf condition for zariski covers. We will show that $W = T$. Let $t \in T$ be a point. Let $t \in U \subset T$ be an affine open. We claim that there exists unique $s \in \mathcal{F}(U)$ with $s|_{f_i^{-1}(U)} = s_i|_{f_i^{-1}(U)}$.

Since we are in the fpqc topology, we can find a standard fpqc covering $\{U_i \to U\}_{i\in\{1,\dots,n\}}$ refining $\{T_i \times U \to U\}$ say by morphisms $h_j : U_j \to T_{i_j}$. By (2), we obtain a unique element $s \in \mathcal{F}(U)$ such that $s|_{U_j} = \mathcal{F}(h_j)(s_{i_j})$. Now for any scheme $V \to U$, there exists a unique section $s_V \in \mathcal{F}(V)$ such that $s_V|_{V \times_U U_j} = \mathcal{F}(h_j \circ pr_2)(s_{i_j})$ for $j = \{1, \dots, n\}$. This holds when V is affine by (2) and in general by gluing such sections over affines because of (1). In particular, $s_V = s|_V$ Note that this statement is basically what we want to prove, but done for finite coverings. Now, suppose $V = T_i \times_T U$. Then, $V \times_U U_j = (T_i \times_T U) \times_U U_j = T_i \times_T U_j$, and we have the following diagram:

$$\begin{tikzcd}
V \times_U U_j \arrow[r, equal] & T_i \times_T U_j \arrow[r] \arrow[d] & U_j \arrow[d] \\
 & T_i \times_T T_{i_j} \arrow[r] \arrow[d] & T_{i_j} \arrow[d] \\
 & T_i \arrow[r] & T.
\end{tikzcd}$$

Using $s_i|_{T_i \times_T T_{i_j}} = s_{i_j}|_{T_i \times_T T_{i_j}}$, we get that $\mathcal{F}(h_j \circ pr_2)(s_{i_j}) = s_i|_{T_i \times_T U_j}$ on $V \times_U U_j (= T_i \times U_j)$ because of the above diagram. So, these sections glue to give $s_i|_{U \times_T T_i}$. Summing this up, we get $s|_{U \times_T T_i} = s_V = s_i|_{U \times_T T_i}$, which is what we wanted to show!

To show equivalence of (2) and (3) in presence of (1), suppose $\{T_i \to T\}$ be a standard fpqc covering, then $\coprod T_i \to T$ is a faithfully flat morphism of affine schemes. In presence of (1), $\mathcal{F}(\coprod T_i) = \prod \mathcal{F}(T_i)$ and similarly, $\mathcal{F}((\coprod T_i) \times_T (\coprod T_j)) = \prod \mathcal{F}(T_i \times_T T_j)$. Thus the sheaf condition for $\{T_i \to T\}$ is tantamount to the sheaf condition for $\{\coprod T_i \to T\}$. $\square$

1.6 Descending Morphisms

Let S be a scheme and let $\{U_i \to S\}$ be an fpqc covering. A descent datum relative to the given covering is the data (X_i, φ_{ij}) where each X_i is a scheme over U_i,

$$\varphi_{ij} : X_i \times_S U_j \to U_i \times_S X_j$$

is an isomorphism for every pair (i, j), and for every triple (i, j, k) the diagram

$$\begin{tikzcd}
X_i \times_S U_j \times_S U_k \arrow[rr, "pr_{02}^* \varphi_{ik}"] \arrow[dr, "pr_{01}^* \varphi_{ij}"'] & & U_i \times_S U_j \times_S X_k \\
 & U_i \times_S X_j \times_S U_k \arrow[ur, "pr_{12}^* \varphi_{jk}"'] &
\end{tikzcd}$$

of schemes over $X_i \times X_j \times X_k$ commutes. A morphism $\psi : (X_i, \varphi_{ij}) \to (Y_i, \varphi'_{ij})$ is given by a family of morphisms $(\psi_i : X_i \to Y_i)$ such that for each pair (i, j), the diagram

$$\begin{tikzcd} X_i \times_S U_j \arrow[r, "\varphi_{ij}"] \arrow[d, "\psi_i \times id"'] & U_i \times_S X_j \arrow[d, "id \times \psi_j"] \\ Y_i \times_S U_j \arrow[r, "\varphi'_{ij}"] & U_i \times_S Y_j \end{tikzcd}$$

commutes.

Lemma 1.6.1. *Let $\mathcal{P}$ be a property of morphisms of schemes over a base. Suppose that*

1. *$\mathcal{P}$ is stable under base change,*
2. *if $Y_j \to V_j$, $j = 1, \ldots, m$ have $\mathcal{P}$, then so does $\coprod Y_j \to \coprod V_j$,*
3. *for any surjective map of affine schemes $X \to S$ which is flat, any descent datum (V, φ) relative to X over S such that $\mathcal{P}$ holds for $V \to X$ is effective.*

Then, morphisms of type $\mathcal{P}$ satisfy descent for fpqc coverings.

Proof. [Stacks, tag 02W3]. □

Lemma 1.6.2. *Let S be a scheme. Let $\{U_i \to X\}$ be an fpqc covering. Let $(X_i/U_i, \varphi_{ij})$ be a descent datum relative to $\{U_i \to S\}$. If each morphism $X_i \to U_i$ is affine, then the descent datum is effective.*

Proof. The property of being affine is local on the base and is stable under base change and composition. By above lemma, it suffices to prove the statement for a single map $X \to S$ of affine schemes which is flat and surjective. Let $X = \text{Spec } A, S = \text{Spec } R$, so that $R \to A$ is a faithfully flat ring map. Let $V = \text{Spec } B$ for some A-algebra B , as V is affine over X. The isomorphism φ then corresponds to the ring map

$$\varphi^\sharp : A \otimes_R B \leftarrow B \otimes_R A.$$

The cocycle condition for φ says

$$\begin{tikzcd} B \otimes_R A \otimes_R A & & A \otimes_R A \otimes_R B \arrow[ll] \arrow[dl] \\ & A \otimes_R B \otimes_R A. \arrow[ul] & \end{tikzcd}$$

Inverting the arrows, we get the usual cocycle condition for descent of modules. By descent of modules, we know that the module kernel $C = Ker(B \to A \otimes_R B)$ is an R-module such that $B \cong A \otimes_R C$, and this isomorphism obeys the descent data. In this case, since φ is a map of algebras, C is actually an R-algebra whose base change to A is isomorphic to B compatibly with the descent data. Spec C is the scheme that we were looking for. $\square$

1.7 Sheafification

Let $\mathcal{C}$ be a site and let $\mathcal{F}$ be a presheaf on that site. Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a covering in $\mathcal{C}$. We define the zeroth Čech cohomology group of $\mathcal{F}$ with respect to the covering $\mathcal{U}$ by

$$\check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F}) = \{(s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}(U_i) \text{ such that } s_i|_{U_i \times_U U_j} = s_j|_{U_i\times_U U_j}.$$

We have the following theorem:

Theorem 1.7.1. *Let $\mathcal{C}$ be a site and let $\mathcal{F}$ be a presheaf on $\mathcal{C}$.*

1. *The map*
$$U \mapsto \mathcal{F}^+(U) := colim_{\mathcal{U} \text{ a covering of } U} \check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F})$$
forms a presheaf on $\mathcal{C}$.

2. *There is a canonical map $\mathcal{F} \to \mathcal{F}^+$ of presheaves.*

3. *$\mathcal{F}^+$ is a separated presheaf.*

4. *$\mathcal{F}^\# = (\mathcal{F}^+)^+$ is a sheaf and the functor taking a presheaf $\mathcal{F}$ to $\mathcal{F}^\#$ is left adjoint to the forgetful functor from sheaves to presheaves.*

After this point, the phrase "Constant sheaf" means the sheaf associated to the constant presheaf.

1.8 The small étale site

Let S be a scheme. Consider the full subcategory $S_{ét}$ of the category Sch/S with objects whose structure map to S is étale. It follows from the properties of étale morphisms that any morphism in the category $S_{ét}$ is automatically étale. $S_{ét}$ is called the small étale site over S. Note that we could consider another subcategory S_{Zar} with objects whose structure map is an open immersion. This category is called the Zariski site over S. We have the following theorem:

Theorem 1.8.1. *The category of abelian sheaves on a site is an abelian category with enough injectives.*

Proof. See [Sta18, Tag 01DP]. □

We define the étale cohomology of a sheaf $\mathcal{F}$ on the small étale site $S_{ét}$ as the right derived functors of the of the global sections functor:

$$H^p_{ét}(S, \mathcal{F}) := R^p\Gamma(S, \mathcal{F}) = H^p(\Gamma(U, \mathcal{I}^\bullet)),$$

where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in the category of abelian sheaves on the site $S_{ét}$.

Chapter 2

Étale Cohomology of Curves

Lemma 2.0.1. *Let X be a smooth curve over an algebraically closed field. Then*

$$H^q_{ét}(X, \mathcal{O}_X^*) = 0 \quad \textit{for all } q \geq 2.$$

Proof. See [Sta18, Tag 03RM]. □

Theorem 2.0.2. *Let X be a smooth connected projective curve of genus g over an algebraically closed field k of characteristic p. Let n be a positive integer invertible in k. We then have canonical identifications:*

1. *$H^0_{ét}(X, \mu_n) = \mu_n(k)$.*
2. *$H^1_{ét}(X, \mu_n) = {}_nPic(X)$, where*

   $$_nPic(X) = \{\mathcal{L} \in Pic(X) | \mathcal{L}^n \cong \mathcal{O}_X\}.$$

   Furthermore,

   $$_nPic(X) \cong (\mathbb{Z}/n\mathbb{Z})^{2g} \quad \textit{(noncanonically)}.$$

3. *$H^2_{ét}(X, \mu_n) = \mathbb{Z}/n\mathbb{Z}$.*
4. *$H^q_{ét}(X, \mu_n) = 0$ for $q > 2$.*

Proof. The proof uses the Kummer sequence which is a short exact sequence

$$1 \to \mu_{n,X} \to \mathcal{O}_X^* \xrightarrow{(\cdot)^n} \mathcal{O}_X^* \to 1.$$

We have the associated long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mu_n(k) \arrow[r] & H^0_{ét}(X, \mathcal{O}_X^*) \arrow[r, "(\cdot)^n"] & H^0_{ét}(X, \mathcal{O}_X^*) \arrow[dll] \\
& H^1_{ét}(X, \mu_n) \arrow[r] & H^1_{ét}(X, \mathcal{O}_X^*) \arrow[r, "(\cdot)^n"] & H^1_{ét}(X, \mathcal{O}_X^*) \arrow[dll] \\
& H^2_{ét}(X, \mu_n) \arrow[r] & H^2_{ét}(X, \mathcal{O}_X^*) \arrow[r, "(\cdot)^n"] & H^2_{ét}(X, \mathcal{O}_X^*) \cdots
\end{tikzcd} \tag{2.1}$$

The last assertion follows from the above lemma. Since the field k is algebraically closed, the map

$$H^0_{ét}(X, \mathcal{O}_X^*) \xrightarrow{(\cdot)^n} H^0_{ét}(X, \mathcal{O}_X^*)$$

is surjective. Also, we have from the above lemma we get

$$H^2_{ét}(X, \mu_n) = Coker(Pic(X) \xrightarrow{(\cdot)^n} Pic(X)),$$

$$H^1_{ét}(X, \mu_n) = Ker(Pic(X) \xrightarrow{(\cdot)^n} Pic(X)),$$

by using the identification $Pic(X) \cong H^1_{ét}(X, \mathcal{O}_X^*)$. We have the following diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & Pic^0(X) \arrow[r] \arrow[d, "(\cdot)^n"] & Pic(X) \arrow[r, "deg"] \arrow[d, "(\cdot)^n"] & \mathbb{Z} \arrow[r] \arrow[d, hook, "n"] & 0 \\
0 \arrow[r] & Pic^0(X) \arrow[r] & Pic(X) \arrow[r, "deg"] & \mathbb{Z} \arrow[r] & 0,
\end{tikzcd} \tag{2.2}$$

where $Pic^0(X)$ is the subgroup of equivalence classes of degree-0 line bundles. By the snake lemma, we get

$$Ker(Pic(X) \xrightarrow{(\cdot)^n} Pic(X)) \cong Ker(Pic^0(X) \xrightarrow{(\cdot)^n} Pic^0(X)).$$

By [Sta18, Tag 0BA0], $Pic^0(X)$ is the set of $k-$points of the group scheme $\underline{Pic}^0_{X/k}$. By the same lemma, $\underline{Pic}^0_{X/k}$ is a $g-$dimensional abelian variety. As explained in [Mum70], the group of $k-$points $\underline{Pic}^0_{X/k}(k)$ is divisible, so the left vertical arrow in the above diagram is surjective and the kernel is isomorphic (noncanonically) to $(\mathbb{Z}/n\mathbb{Z})^{2g}$. $\square$

Up to choosing a primitive root of unity, we know that $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$. By the above theorem, we know the cohomology of the sheaf $\mathbb{Z}/n\mathbb{Z}$ on a smooth projective curve over an algebraically closed field whose characteristic is relatively prime to n. We have the following general result for the cohomology of the constant sheaf $\underline{\mathbb{Z}/p\mathbb{Z}}$ on the small étale site of X, where X is a separated, finite type scheme over an algebraically closed field k of characteristic p.

Lemma 2.0.3. *Let X be a separated, finite type scheme over an algebraically closed field k of characteristic $p > 0$. Then $H^q_{\acute{e}t}(X, \underline{\mathbb{Z}/p\mathbb{Z}}) = 0$ for $q > dim(X) + 1$. Moreover, if X is proper then $H^q_{\acute{e}t}(X, \underline{\mathbb{Z}/p\mathbb{Z}})$ is a finite $\mathbb{Z}/p\mathbb{Z}$ module for all q.*

Proof. See [Sta18, Tag 0A3N, Tag 0A3P]. $\square$

Let X be an affine curve. Lemma 2.0.1 holds in this case and we get an exact sequence

$$\mathcal{O}_X^* \xrightarrow{(\cdot)^n} \mathcal{O}_X^* \to H^1_{\acute{e}t}(X, \mu_n) \to Pic(X) \xrightarrow{(\cdot)^n} Pic(X) \to H^2_{\acute{e}t}(X, \mu_n) \to 0.$$

We can embed X in a complete smooth projective curve $\overline{X}$. The elements of $\mathcal{O}_X^*(X)$ are rational functions on $\overline{X}$ whose divisors are supported on the finite set $\overline{X} - X$. Since the map $k^* \xrightarrow{(\cdot)^n} k^*$ is surjective, the group

$$Coker(\mathcal{O}_X^* \xrightarrow{(\cdot)^n} \mathcal{O}_X^*) \cong Ker(H^1_{\acute{e}t}(X, \mu_n) \to Pic(X))$$

is finite. Moreover we have an exact sequence

$$0 \to E \to Pic(\overline{X}) \to Pic(X) \to 0.$$

By the finiteness of $Ker(Pic(\overline{X}) \xrightarrow{(\cdot)^n} Pic(\overline{X}))$ and the kernel E being finitely generated, we have that $Ker(Pic(X) \xrightarrow{(\cdot)^n} Pic(X))$ is finite. From this, we conclude that $H^1_{\acute{e}t}(X, \mu_n)$ is a finite group. Now note that we can extend any divisor on X to a degree zero divisor on $\overline{X}$ as $\overline{X} - X \neq \emptyset$. Now, since $Pic^0(\overline{X}) \xrightarrow{(\cdot)^n} Pic^0(\overline{X})$ is surjective, we conclude that $H^2_{\acute{e}t}(X, \mu_n) = 0$.

Theorem 2.0.4. *Let k be a separably closed field and let X be a finitely generated $k-$scheme, $dim(X) \leq 1$. Let $\mathcal{F}$ be a constructible sheaf whose sections have order relatively prime to the characteristic of k. Then:*

1. *The cohomology groups* $H^q_{ét}(X,\mathcal{F})$ *are finite.*

2. $H^q_{ét}(X,\mathcal{F}) = 0$ *for* $q \geq 3$.

3. *If* X *is affine then* $H^q_{ét}(X,\mathcal{F}) = 0$ *for* $q \geq 2$.

Parts (2) and (3) can be extended from constructible sheaves to torsion sheaves by passage to limits.

Proof. If we have a map of sheaves $\mathcal{F} \to \mathcal{G}$ which is an isomorphism outside a finite set of points, then the assertions hold for $\mathcal{F}$ if and only if they hold for $\mathcal{G}$. Secondly, if $f : X' \to X$ is a finite morphism and $\mathcal{F}'$ is a constructible sheaf on X', then the statements hold for $(X, f_*\mathcal{F})$ if and only if they hold for $(X', \mathcal{F}')$ because we can compute the cohomology of $(X', \mathcal{F}')$ using the Leray spectral sequence and vanishing of finite higher direct images implies that the cohomology of $(X', \mathcal{F}')$ is equal to the cohomology of $(X, f_*\mathcal{F}')$.
By the topological invariance of the small étale site ([Sta18, Tag 04DY]), we can assume that k is algebraically closed and that X is reduced. Let $f : X' \to X$ be the normalization mapping. Since it is an isomorphism outside finitely many points, the adjunction map $\mathcal{F} \to f_*f^*\mathcal{F}$ is an isomorphism outside finitely many points. By the above remark, it suffices to prove the statements for the sheaf $f^*\mathcal{F}$ on X'. This reduces the problem to the case of smooth curves.

1. It suffices to show that every constructible sheaf $\mathcal{F}$ can be embedded in a constructible sheaf with finite cohomology groups as in that case we have an exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots,$$

where $\mathcal{F}_i's$ are constructible and have finite cohomology groups. We know that for a constructible sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$ modules, $\mathcal{F}$ can be embedded as a subsheaf of a finite direct sum of sheaves

$$(f_\nu)_*(\underline{\mathbb{Z}/n\mathbb{Z}}_{Y_\nu})$$

that are pushforwards of constant sheaves $\underline{\mathbb{Z}/n\mathbb{Z}}$ along finite morphisms $f_\nu : Y_\nu \to X$. We may assume that each Y_ν is normal (by further normalizing it) and we have $dim(Y_\nu) \leq dim(X) \leq 1$. We already know the cohomology

$$H^i_{ét}(X, (f_\nu)_*(\underline{\mathbb{Z}/n\mathbb{Z}})) = H^i_{ét}(Y_\nu, \underline{\mathbb{Z}/n\mathbb{Z}})$$

is finite by the above result for smooth projective curves, thus finishing the proof of (1).

2. If the stalk of $\mathcal{F}$ at a geometric point over the generic point, then there is some open set U such that restriction of $\mathcal{F}$ to U is 0. This implies that the sheaf $\mathcal{F}$ is concentrated on finitely many points, and therefore the cohomology vanishing follows by dimensional vanishing. Now let $\mathcal{F}$ be an arbitrary torsion sheaf. We have the diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "\eta"] & X \\ \operatorname{Spec} \overline{K}, \arrow[u] \arrow[ur, "\overline{\eta}"] & \end{tikzcd}$$

where $\overline{K}$ is the separable closure of the function field of X (therefore, $\eta : \operatorname{Spec} \overline{K} \to X$ is a geometric point). The projective limit of the system $(U \times_X \operatorname{Spec} K)$, where U runs through all affine étale neighborhoods of the geometric point $\overline{\eta}$, is $\operatorname{Spec} \overline{K} \times_X \operatorname{Spec} K = \operatorname{Spec} \overline{K}$. Using the result on stalks of higher direct images, we get

$$R^q\eta_*(\eta^*\mathcal{F})_{\overline{\eta}} = \begin{cases} \mathcal{F}_{\overline{\eta}}, & \text{if } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the adjunction mapping

$$\mathcal{F} \to \eta_*\eta^*\mathcal{F}$$

has isomorphic stalks at the generic point, so using the remark at the beginning, it suffices to prove the vanishing result for the sheaf $\eta_*\eta^*\mathcal{F}$, so it suffices to prove that $H^q_{ét}(X, \eta_*\eta^*\mathcal{F}) = 0$ for $q \geq 3$.

Since X is a curve, the function field K is of transcendence degree 1. By Tsen's theorem ([Sta18, Tag 03RD]) K is a C_1 field. Therefore, for any finite extension K' of K, the Brauer group vanishes by [Sta18, Tag 03RC]. By [Sta18, Tag 03R8], we conclude that $H^q(Gal(\overline{K}/K), M) = 0$ for $q \geq 2$ and M any torsion $Gal(\overline{K}/K)$ module. By [Sta18, Tag 03QU], we have $H^q_{ét}(\operatorname{Spec} K, \eta^*\mathcal{F}) = H^q(Gal(\overline{K}/K), \eta^*\mathcal{F}(\operatorname{Spec} K)) = 0$ for $q > 2$. Moreover, $R^q\eta_*(\eta^*\mathcal{F}) = 0$ when $q \geq 2$. Since the stalk at the geometric point over the generic point of the sheaf $R^q\eta_*(\eta^*\mathcal{F})$ vanishes for $q \geq 1$, it is a sheaf concentrated on finitely many points, and therefore its higher cohomology vanishes, i.e.,

$$H^p(X, R^q\eta_*(\eta^*\mathcal{F})) = 0, \quad \text{for } p, q > 0.$$

We have the Leray spectral sequence

$$H^p(X, R^q\eta_*(\eta^*\mathcal{F}) \implies H^{p+q}(\text{Spec } K, \eta^*\mathcal{F}).$$

By the preceding remarks, we get

$$H^p(X, \eta^*\eta_*\mathcal{F}) = 0$$

for $p \geq 3$.

3. Let X be an affine smooth irreducible curve over k. We may assume that $\mathcal{F}$ is a sheaf of $\Lambda = \mathbb{Z}/n\mathbb{Z}$ modules (since an arbitrary constructible sheaf satisfying the hypotheses above can be expressed as a direct sum of sheaves of $\mathbb{Z}/n_i\mathbb{Z}$ modules where n_i's are relatively prime to the characteristic of k). Since $H^q_{ét}(X, \mathcal{F}) = 0$ for $q \geq 3$, the functor $H^2_{ét}(X, -)$ is exact. We know that the sheaf $\mathcal{F}$ is a quotient sheaf of a finite direct sum of sheaves of the type $(\tilde{Y})^\Lambda$, where $\tilde{Y}$ is the sheaf (of sets) represented by the étale scheme $Y \to X$, and $(-)^\Lambda$ is the left adjoint functor to the forgetful functor from the sheaves of $\Lambda-$modules to the sheaves of sets (See [FK88] Chapter 1, Section 4). Hence it suffices to show that $H^2_{ét}(X, (\tilde{Y})^\Lambda) = 0$.

Remark. *Let $f : U \to V$ be an étale map. We have*

$$Hom_\Lambda((\tilde{U})^\Lambda, f_*(\Lambda_U)) = Hom(\tilde{U}, f_*(\Lambda_U)) = \Gamma(U, f_*(\Lambda_U) = \Gamma(U \times_V U, \Lambda_U),$$

where the middle equality is by yoneda lemma. The diagonal $\Delta_U \subset U \times_V U$ is both an open and a closed subset (we assume the map f is separated). Therefore, there is a section of Λ_U on $U \times_V U$ which when restricted to the diagonal is 1 and is 0 outside. Corresponding to this section, we get a morphism

$$\varphi : (\tilde{U})^\Lambda \to f_*(\Lambda_U).$$

In the case where V is a point, it is easy to see that the morphism φ of sheaves is an isomorphism. Moreover, this construction is compatible with base change.

In our original situation, we denote the map $Y \to X$ by q. There exists a compactification of this map as

$$\begin{tikzcd} Y \arrow[r, hook, "j"] \arrow[dr, "q"'] & \overline{Y} \arrow[d, "\overline{q}"] \\ & X. \end{tikzcd}$$

where $\overline{q}$ is a finite morphism so $\overline{Y}$ is affine. We have homomorphisms

$$(\tilde{Y})^{\Lambda} \to q_*(\Lambda_Y) \leftarrow \overline{q}_*(\Lambda_{\overline{Y}}).$$

Both homomorphisms are isomorphisms outside a finite set of points. So it suffices to show the vanishing for the sheaf $\overline{q}_*(\Lambda_{\overline{Y}})$. We have already shown that

$$H^2_{\acute{e}t}(X, \overline{q}_*(\Lambda_{\overline{Y}})) = H^2_{\acute{e}t}(\overline{Y}, \Lambda_{\overline{Y}}) = 0.$$

□

Chapter 3

Important Theorems

3.1 The base-change mapping

Let

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "g'"] & T \arrow[d, "g"] \\ Y \arrow[r, "f"] & S \end{tikzcd}$$

be a commutative diagram. Let $\mathcal{F}$ be a sheaf on Y. We have the identity map $g'^*\mathcal{F} \to g'^*\mathcal{F}$. By adjunction we have the map $\mathcal{F} \to g'_*g'^*\mathcal{F}$. Applying f_*, we get $f_*\mathcal{F} \to f_*g'_*g'^*\mathcal{F}$, using the commutativity of the diagram, we get $f_*\mathcal{F} \to g_*f'_*g'^*\mathcal{F}$, and applying adjunction one more time, we get the base change map

$$g^*f_*\mathcal{F} \to f'_*g'^*\mathcal{F}.$$

More generally, let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution, and let $g'^*(\mathcal{I}^\bullet) \to \mathcal{J}^\bullet$ be an injective resolution, that is a quasi isomorphism from the complex $g'^*(\mathcal{I}^\bullet)$ to the complex $\mathcal{J}^\bullet$ of injectives. Applying f'_*, we get a map

$$f'_*g'^*(\mathcal{I}^\bullet) \to f'_*\mathcal{J}^\bullet.$$

Composing with the map defined above, we get

$$g^* f_*(\mathcal{I}^\bullet) \to f'_* g'^*(\mathcal{I}^\bullet) \to f'_* \mathcal{J}^\bullet.$$

Taking cohomology and using the fact that g^* is an exact functor, we get the base change map

$$g^*(R^i f_* \mathcal{F}) \to R^i f'_*(g'^* \mathcal{F}).$$

Let $\mathcal{F}^\bullet$ be a complex of torsion sheaves on Y, and $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ be an injective resolution. We get the base change map at the level of complexes (in the exact same way as above):

$$g^*(Rf_* \mathcal{F}^\bullet) \to Rf'_*(g'^* \mathcal{F}^\bullet).$$

Theorem 3.1.1 (Base change for proper morphisms)**.** *Let $f : X \to S$ be a proper morphism and let*

$$\begin{array}{ccc} X_T = X \times_S T & \xrightarrow{f'} & T \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & S \end{array}$$

be a cartesian diagram. For every torsion sheaf $\mathcal{F}$ on X, the base change homomorphism

$$g^*(R^i f_* \mathcal{F}) \to R^i f'_*(g'^* \mathcal{F})$$

is an isomorphism.

Remarks. 1. *It suffices to prove this theorem in the case where $f : X \to S$ is projective, as the general case is obtained from this by Chow's lemma.*

2. *Let $\mathcal{F}^\bullet$ be a complex of sheaves on X bounded below, and suppose the base change theorem holds for all sheaves $\mathcal{F}^\nu$. Then we have*

$$g^* Rf_* \mathcal{F}^\bullet \xrightarrow{\sim} Rf'_* g'^* \mathcal{F}^\bullet \quad \textit{(in } D(T)\textit{)}.$$

3. *Let $f_1 : X \to X'$, $f_2 : X' \to S$ be proper morphisms of schemes. If the base change theorem holds for f_1, f_2, then it holds for the composition $f = f_2 \circ f_1$. Moreover, if the base change theorem holds for f and f_1, then it holds for f_2 for complexes sheaves of the type $Rf_{1*}\mathcal{F}^\bullet$.*

4. *If $T =$ Spec k is the spectrum of a separably closed field k, $g = s :$ Spec $k \to S$ then*

in this case, the base change theorem means that the stalk $(R^i f_* \mathcal{F})_s$ *is isomorphic to* $H^i_{\acute{e}t}(X_s, \mathcal{F}_s)$, *where* $X_s = X \times_S T$, $\mathcal{F}_s = \mathcal{F}|_{X_s}$.

Proof. We will prove the theorem in the case when $\mathcal{F}$ is a sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules, n invertible in S. We have to prove that the base change map on certain sheaves is an isomorphism. We know that this holds if and only if the stalks are isomorphic. We can assume that T is finitely generated as an $S-$scheme (as this problem is local on T, we can reduce to the case where S and T are affine schemes. Further, we prove the result when T is finitely generated over S and then apply the limit theorem). Since T is finitely generated over S, it suffices to show the isomorphism on stalks at geometric points that are closed in the fibers. Let

$$t : \text{Spec } k \to T$$

be a geometric point and let $T(t) = \text{Spec } \tilde{\mathcal{O}}_{T,t}$ be the strict henselization of the Zariski stalk at the underlying point (étale stalk). For a sheaf $\mathcal{G}$ on X_T, the stalk $(Rf'^i_* \mathcal{G})_t$ is equal to $H^i_{\acute{e}t}(X_T \times_T T(t), \mathcal{G}(t))$, where $\mathcal{G}(t)$ is the inverse image of $\mathcal{G}$ on $X_T \times_T T(t)$. Therefore, it suffices to prove the theorem when

1. The base change mapping is of the form $T = \text{Spec } K \to \text{Spec } k = S$, where K is a finite algebraic extension of the separably closed field k.

2. S is the spectrum of a strictly henselian ring and T is the spectrum of the residue field, $T \to S$ is the natural mapping.

Case (1) follows from the topological invariance of the étale site (see [Sta18, Tag 03SI]).
In Case (2), the theorem says for $S = \text{Spec } A$, A strictly henselian, $g = s : \text{Spec } k \to S$, $k = A/\mathfrak{m}$, we have

$$H^i_{\acute{e}t}(X, \mathcal{F}) = H^i_{\acute{e}t}(X_s, \mathcal{F}_s),$$

where $X_s = X \times_S \text{Spec } k$, and $\mathcal{F}_s = \mathcal{F}|_{X_s}$.
We first prove the case where the special fiber X_s is at most one-dimensional. We claim that for $dim(X_s) \leq 1$, the natural mapping

$$H^i_{\acute{e}t}(X, \mathbb{Z}/n\mathbb{Z}) \to H^i_{\acute{e}t}(X_s, \mathbb{Z}/n\mathbb{Z}),$$

n invertible in $\mathcal{O}_X$ is

1. bijective for $i = 0$,

2. surjective for $i > 0$.

When $\mathbf{i} > \mathbf{2}$, there is nothing to prove because of vanishing of cohomology. For $\mathbf{i} = \mathbf{0}$, the assertion follows from the Zariski connectedness theorem for a proper scheme over a henselian ring (which states that the number of connected components of X, and X_s are the same).
For $\mathbf{i} = \mathbf{1}$, it suffices to show that every $\mathbb{Z}/n\mathbb{Z}-$Galois étale covering space of the special fiber X_s can be extended to the to a Galois étale covering space of X.
The functor

$$Sh(X_i) \to Sh(X_{i-1}),$$

where $X_i = X \times_A \text{Spec } (A/\mathfrak{m}^n)$ is an equivalence of categories (by topological invariance of the étale site). So, a galois étale covering space of $X_1 = X_s$ can be extended inductively to all X_n. By the fundamental existence theorem of Grothendieck, it follows that every galois étale covering space of the special fiber X_s can be extended to a galois étale covering space of the formal completion $X \times_A \hat{A}$, where $\hat{A} = \varprojlim A/\mathfrak{m}^n$. We now want to apply Artin's comparison theorem, but in Artin's theory, only the rings that strict henselizations of finitely generated $\mathbb{Z}$-algebras are allowed, but every strictly henselian ring is a colimit of such rings and so in the view of the limit theorem, we can restrict to such rings. We have a functor on the category of (noetherian) $A-$algebras which associates an algebra B the set $F(B)$ of galois étale $\mathbb{Z}/n\mathbb{Z}$ covering spaces. This functor is limit preserving so,

$$F(\varinjlim B_\nu) = \varinjlim F(B_\nu).$$

By the Artin approximation theorem [Art69], for a given $\nu \geq 1$, there is a galois étale covering space of X that agrees with the covering space constructed for $X \times_A \text{Spec } \hat{A}$ over $X \times_A \text{Spec } A_\nu$.
For $\mathbf{i} = \mathbf{2}$, we have the commutative diagram

$$\begin{array}{ccc} Pic(X) & \longrightarrow & H^2_{\acute{e}t}(X, \mu_n) \\ \downarrow & & \downarrow \\ Pic(X_s) & \longrightarrow & H^2_{\acute{e}t}(X_s, \mu_n) \end{array}$$

The bottom horizontal arrow is surjective (in characteristic 0, $H^2_{\acute{e}t}(X, \mathcal{O}^*_X)$ vanishes and in characteristic p, it is p-torsion, and n is relatively prime to p). Since the field $k = A/\mathfrak{m}$

is separably closed, it (and therefore the henselian ring A) contains a primitive n^{th} root of unity. Up to the choice of this root, we can identify the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ with the sheaf μ_n on X and X_s. Therefore it is enough to show that every line bundle on X_s can be extended to a line bundle on X. By using the same strategy as above, it suffices to show that every line bundle on $X_n = X \times_A \operatorname{Spec} A/\mathfrak{m}^n$ can be extended to a line bundle on X_{n+1}. For this, we use the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}^*_{X_{n+1}} \to \mathcal{O}^*_{X_n} \to 1,$$

$\alpha \in \mathcal{I} \mapsto 1 + \alpha \in \mathcal{O}^*_{X_{n+1}}$, where $\mathcal{I} = \mathfrak{m}^n \mathcal{O}_X / \mathfrak{m}^{n+1} \mathcal{O}_X = Ker(\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n})$. Now, the sheaf $\mathcal{I}$ is coherent and X_s is at most 1-dimensional. So, $H^2_{ét}(X, \mathcal{I})$ vanishes, and hence the map $Pic(X_{n+1}) \to Pic(X_n)$ is surjective proving the above assertion. We now apply this result to schemes $X' \xrightarrow{p} X$ that are finite over X. By vanishing of finite higher direct images, the base change theorem holds for such maps. So, we have the map

$$H^i_{ét}(X, \mathcal{F}) \to H^i_{ét}(X_s, \mathcal{F}_s)$$

is bijective for $i = 0$, and surjective for $i > 0$, for all sheaves $\mathcal{F}$ that are isomorphic to a finite direct sum of sheaves of the form $p_*((\mathbb{Z}/n\mathbb{Z})_{X'})$. From $(*)$ we know that every constructible sheaf is a subsheaf of such a sheaf. Let $\mathcal{F}$ be a constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$ modules. Let $\mathcal{G}$ be a sheaf of the above type such that $\mathcal{F} \hookrightarrow \mathcal{G}$. From the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0_{ét}(X, \mathcal{F}) & \longrightarrow & H^0_{ét}(X, \mathcal{G}) & \longrightarrow & H^0_{ét}(X, (\mathcal{G}/\mathcal{F})) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^0_{ét}(X_s, \mathcal{F}_s) & \longrightarrow & H^0_{ét}(X_s, \mathcal{G}_s) & \longrightarrow & H^0_{ét}(X_s, (\mathcal{G}/\mathcal{F})_s),
\end{array}$$

we deduce that the map $H^0_{ét}(X, \mathcal{F}) \to H^0_{ét}(X_s, \mathcal{F}_s)$ is injective. But this holds for all constructible sheaves of $\mathbb{Z}/n\mathbb{Z}$ modules. Using the injectivity of $H^0_{ét}(X, (\mathcal{G}/\mathcal{F})) \to H^0_{ét}(X_s, (\mathcal{G}/\mathcal{F})_s)$, we get that the map $H^0_{ét}(X, \mathcal{F}) \to H^0_{ét}(X_s, \mathcal{F}_s)$ is bijective.
We now prove $H^i_{ét}(X, \mathcal{F}) \to H^i_{ét}(X_s, \mathcal{F}_s)$, where $\mathcal{F}$ is constructible, is bijective for all i by induction. Suppose the result is proved for all $i < p$. Let $\mathcal{G}$ be a torsion sheaf of $\mathbb{Z}/n\mathbb{Z}$ modules such that $\mathcal{F}$ is a subsheaf of $\mathcal{G}$ and $H^0_{ét}(X, \mathcal{G}) \to H^0_{ét}(X_s, \mathcal{G}_s)$. Let

$$0 \to \mathcal{G} \to \mathcal{I} \to \mathcal{K} \to 0$$

be a short exact sequence where the sheaf $\mathcal{I}$ is injective. In the diagram

$$\begin{tikzcd}
H^{p-1}_{\acute{e}t}(X,\mathcal{I}) \arrow[r] \arrow[d] & H^{p-1}_{\acute{e}t}(X,\mathcal{K}) \arrow[r] \arrow[d] & H^{p}_{\acute{e}t}(X,\mathcal{G}) \arrow[r] \arrow[d] & 0 \arrow[d] \\
H^{p-1}_{\acute{e}t}(X_s,\mathcal{I}_s) \arrow[r] & H^{p-1}_{\acute{e}t}(X_s,\mathcal{K}_s) \arrow[r] & H^{p}_{\acute{e}t}(X_s,\mathcal{G}_s) \arrow[r] & H^{p}_{\acute{e}t}(X_s,\mathcal{I}_s),
\end{tikzcd}$$

the first two vertical arrows are bijections by the induction hypothesis. Therefore the map $H^p_{\acute{e}t}(X,\mathcal{G}) \to H^p_{\acute{e}t}(X_s,\mathcal{G}_s)$ is surjective (proven above) and injective (from the diagram), so it is a bijection. Now consider the diagram of long exact sequence in cohomology

$$\begin{tikzcd}
H^{p-1}_{\acute{e}t}(X,\mathcal{G}) \arrow[r] \arrow[d] & H^{p-1}_{\acute{e}t}(X,\mathcal{G}/\mathcal{F}) \arrow[r] \arrow[d] & H^{p}_{\acute{e}t}(X,\mathcal{F}) \arrow[r] \arrow[d] & H^{p}_{\acute{e}t}(X,\mathcal{G}) \arrow[r] \arrow[d] & H^{p}_{\acute{e}t}(X,\mathcal{G}/\mathcal{F}) \arrow[d] \\
H^{p-1}_{\acute{e}t}(X_s,\mathcal{G}_s) \arrow[r] & H^{p-1}_{\acute{e}t}(X_s,(\mathcal{G}/\mathcal{F})_s) \arrow[r] & H^{p}_{\acute{e}t}(X_s,\mathcal{F}_s) \arrow[r] & H^{p}_{\acute{e}t}(X_s,\mathcal{G}_s) \arrow[r] & H^{p}_{\acute{e}t}(X_s,(\mathcal{G}/\mathcal{F})_s),
\end{tikzcd}$$

where the first, second and fourth vertical arrows are bijections. From this, injectivity of the map

$$(*) : H^p_{\acute{e}t}(X,\mathcal{F}) \to H^p_{\acute{e}t}(X_s,\mathcal{F}_s)$$

follows. Since $\mathcal{F}$ was arbitrary, the injectivity follows for $\mathcal{G}/\mathcal{F}$. From this we get the surjectivity of the map $(*)$. This completes the proof of the base change theorem for projective morphisms whose fibers are at most of dimension 1. So the base change theorem holds for the projection map

$$\mathbb{P}^1_S \to S,$$

and as a consequence, it holds for the n-fold product of projective lines over S. We know that the quotient by the symmetric group of the n-fold product of lines is isomorphic to $\mathbb{P}^n_S$. That is the quotient map $p : \mathbb{P}^1_S \times \cdots \times \mathbb{P}^1_S \to \mathbb{P}^n_S$ is a finite morphism of schemes. For any sheaf $\mathcal{G}$ on $\mathbb{P}^n_S$, the map

$$\mathcal{G} \to p_*p^*\mathcal{G}$$

is injective, so the sheaf $\mathcal{G}$ is quasi-isomorphic to a complex $\mathcal{F}^\bullet$, where each $\mathcal{F}^\nu$ is a pushed forward sheaf along p. The base change theorem hold for each $\mathcal{F}^\nu$, so it holds for $\mathcal{F}^\bullet$ and therefore for $\mathcal{G}$ concluding the proof. $\square$

Theorem 3.1.2 (Smooth base change)**.** *Let*

$$\begin{tikzcd} X_T = X \times_S T \arrow[r, "f'"] \arrow[d, "g'"] & T \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

be a cartesian diagram of noetherian schemes. The base change homomorphism

$$g^*(R^i f_* \mathcal{F}) \to R^i f'_*(g'^* \mathcal{F})$$

is an isomorphism if the following hypotheses are satisfied:

1. *g is smooth*
2. *$\mathcal{F}$ is a torsion sheaf, and the orders of sections of $\mathcal{F}$ are relatively prime to the residue characteristics of S.*

Proof. See [FK88] Chapter 1, section 7. □

3.2 Étale cohomology of varieties over $\mathbb{C}$

A **basic analytic variety** is a subset of a domain $U \subset \mathbb{C}^n$ defined by the zero locus of a family of analytic functions defined on U. An analytic variety can be given the structure of a locally ringed space by restricting analytic functions from U.
An analytic space $(X, \mathcal{H}_X)$ is a locally ringed topological space satisfying the following condition: X is covered by a family of open sets $\{U_i\}$ such that $(U_i, \mathcal{H}_X|_{U_i})$ is isomorphic (as a locally ringed space) to a basic analytic variety.
Let X be a locally finite type $\mathbb{C}$ scheme. Indeed X is covered by affine open sets of the form Spec $\mathbb{C}[x_1, \dots, x_n]/(g_1, \dots, g_n)$. So we can form a basic analytic space on zeroes of g_is as polynomial functions are analytic. Since the complex topology is finer than the Zariski topology, the gluing data of X as a scheme gives us the gluing data for these basic analytic varieties. We thus obtain the analytic space $(X_{an}, \mathcal{H}_{X_{an}})$ We have an obvious morphisms of locally ringed spaces

$$\varphi_X : (X_{an}, \mathcal{H}_{X_{an}}) \to (X, \mathcal{O}_X).$$

Note that X_{an} satisfies the following universal property: given an analytic space Y and a morphism of locally ringed spaces $\sigma : Y \to X$, σ factors uniquely as $Y \to X_{an} \to X$. This problem is local on the target, so we may assume that X is a finite type affine $\mathbb{C}$-scheme. Indeed, every point of Y maps to a closed point of X (since every point of Y is given by a map of locally ringed spaces Spec $\mathbb{C} \to Y$, by composition with $Y \to X$, we get a closed point of X). Now, the coordinate functions of X (and hence of X_{an}) pull back to analytic functions on Y. We know that analytic functions and its complex conjugate are continuous, so the real coordinate functions pull back to continuous functions. Thus the map

$$Y \to X_{an}$$

is continuous. Now every analytic function on X_{an} is locally given as a power series in the coordinate functions of X_{an}. Since coordinate functions pull back to analytic functions, we see that analytic functions on X_{an} pull back to analytic functions on Y. Hence the map $Y \to X_{an}$ is a morphism of locally ringed topological spaces. Because of this universal property, we observe that X_{an} is defined uniquely.

Let X_{cx} be the "étale site of X_{an}", i.e., the category whose objects are all analytic spaces $U \to X_{an}$ over X_{an} which are local isomorphisms and the morphisms are commuting triangles. We have a continuous map of sites $s : X_{cx} \to X_{an}$, and $h : X_{an} \to X_{ét}$, where the latter map is induced by analytification of étale morphisms (which gives local isomorphisms). We have functors $s_* : Sh(X_{cx}) \to Sh(X_{an})$, and $h_* : Sh(X_{cx}) \to Sh(X_{ét})$. It is easy to see that s_* is an exact functor and the functor h_* is left exact and takes flasque sheaves to flasque sheaves.

Proposition 3.2.1. *Let Λ be a finite abelian group and X a smooth $\mathbb{C}$-scheme. Then $R^q h_* \underline{\Lambda} = 0$ for $q \geq 0$.*

Proof. We first claim that when $q > 1$, and $\gamma \in H^q(X_{cx}, \mathcal{F})$ where $\mathcal{F}$ is a finite locally constant sheaf then for any $x \in X(\mathbb{C})$, there is an étale morphism $U \to X$ whose image contains x and $\gamma|_U = 0$. This claim implies the stalk $(R^q h_* \mathcal{F})_{\overline{x}} = 0$. This implies the sheaf $R^q h_* \mathcal{F} = 0$. We prove this using induction on $dim(X)$. When $dim(X) = 0$, the result follows from dimensional vanishing. Since the statement is local on the étale topology, we may assume that $\mathcal{F}$ is a constant sheaf $\underline{\Lambda}$.

Let U be a "small" open neighborhood of x such that it embeds as a dense open subset of a projective scheme Y, and let $Z = Y - U$, which we assume to be smooth. We have a Gysin sequence

$$\cdots \to H^{q-2}(Z_{an}, \Lambda) \to H^q(Y_{an}, \Lambda) \to H^q(U_{an}, \Lambda) \to \cdots .$$

Now suppose we can fiber Y over something of dimension 1 less. That is, we have the following diagram

$$\begin{tikzcd} U \arrow[r, hook, "j"] \arrow[dr, "f"'] & Y \arrow[d, "\overline{f}"] & Z \arrow[l, hook', "i"'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

over $\mathbb{C}$ and

1. $Z = Y - U$;

2. i is a closed immersion;

3. j is an open immersion;

4. S is of dimension $n-1$;

5. f is smooth of relative dimension 1;

6. g is finite étale with nonempty fibers (this implies that g is proper);

7. $\overline{f}$ is smooth projective with fibers smooth connected curves;

8. U is dense in every fiber of f.

As shown in [AGV71] Exposé 11, such a fibration always exists. Let $S' \subset S_{an}$ be a complex open set. We obtain the exact sequence (from the Gysin sequence)

$$\cdots \to H^{q-2}(g_{an}^{-1}(S'), \Lambda) \to H^q(\overline{f}_{an}^{-1}(S'), \Lambda) \to H^q(f_{an}^{-1}(S'), \Lambda) \to \cdots .$$

This gives the exact sequence of sheaves

$$\cdots \to R^{q-2}g_{an,*}\Lambda \to R^q\overline{f}_{an,*}\Lambda \to R^q f_{an,*}\Lambda \to \cdots .$$

Now for any $s \in S(\mathbb{C})$, we have the following diagram with exact rows

$$\begin{array}{ccccccccc}
(R^{q-2}g_{an,*}\Lambda)_s & \longrightarrow & (R^q\overline{f}_{an,*}\Lambda)_s & \longrightarrow & (R^q f_{an,*}\Lambda)_s & \longrightarrow & (R^{q-1}g_{an,*}\Lambda)_s & \longrightarrow & (R^{q+1}\overline{f}_{an,*}\Lambda)_s \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{q-2}((Z_{an})_s,\Lambda) & \longrightarrow & H^q((Y_{an})_s,\Lambda) & \longrightarrow & H^q((U_{an})_s,\Lambda) & \longrightarrow & H^{q-1}((Z_{an})_s,\Lambda) & \longrightarrow & H^{q+1}((Y_{an})_s,\Lambda)
\end{array}$$

Here, the bottom row is obtained from the Gysin sequence for $Z_{an} \subset Y_{an}$. By the topological proper base change theorem, all vertical arrows except the middle one are isomorphisms. By the five-lemma, the middle vertical arrow is also an isomorphism. Since $dim_{\mathbb{C}}((U_{an})_x) = 1$, we can apply dimensional vanishing and obtain that $R^q f_{an,*}\Lambda = 0$ for all $q > 1$ (since the fibers are noncompact, the top cohomology group vanishes).

Lemma 3.2.2. *Let $\varphi : X \to S$ be a proper submersive map of manifolds, and let Ω be a constant sheaf on X. Then $R^q\varphi_*\Omega$ is a finite locally constant sheaf on S for all $q \geq 0$.*

Proof. By the topological proper base change theorem, we have $(R^q\varphi_*\Omega)_s = H^q(\varphi^{-1}(s), \Omega)$. Moreover since φ is smooth and a submersion, it is a fibration and so we have a ball $V \ni s$ such that $\varphi^{-1}V = V \times \varphi^{-1}(s)$. This implies the sheaf $R^q\varphi_*\Omega|_W$ is the constant sheaf $H^q(\varphi^{-1}(s), \Omega)$. □

Corollary 3.2.3. *$R^q f_{an,*}\Lambda$ is finite locally constant for all $q \geq 0$.*

Proof. We have the exact sequence (obtained from the Gysin sequence)

$$0 \to \overline{f}_{an,*}\Lambda \to f_{an,*}\Lambda \to 0.$$

Since $\overline{f}_{an,*}\Lambda$ is finite locally constant so is $f_{an,*}\Lambda$. Apart from $R^1 f_{an,*}\Lambda$, all other higher direct images vanish as shown above. Since $\overline{f}$ is locally a fibration, so is f. By homotopy equivalence we can construct a canonical isomorphism $H^p(U_s, \Lambda) \to H^p(U_u, \Lambda)$ for u in a small disc around s. This implies local constancy. □

We have a Leray spectral sequence of U over S

$$H^p(S_{cx}, R^q f_{an,*}\underline{\Lambda}) \implies H^{p+q}(U_{cx}, \underline{\Lambda}).$$

We get the following exact sequence (using the fact that $R^2 f_{an,*}\underline{\Lambda} = 0$, and calculating the cohomology of the target using the second page of the spectral sequence)

$$\cdots \to H^p(S_{cx}, f_{an,*}\underline{\Lambda}) \to H^p(U_{cx}, \underline{\Lambda}) \to H^{p-1}(S_{cx}, R^1 f_{an,*}\underline{\Lambda}) \to \cdots .$$

Let $\gamma \in H^p(U_{cx}, \underline{\Lambda})$ map to $t \in H^{p-1}(S_{cx}, R^1 f_{an,*}\underline{\Lambda})$. Since $R^1 f_{an,*}\underline{\Lambda}$ is finite locally constant and since the dimension of S_{an} is one less than the dimension of U_{an}, we can apply the inductive hypothesis to get an étale map $W \to S$ whose image contains $f(x)$ and is such that $t|_{W_{cx}} = 0$. Apply the Leray spectral sequence argument above to the base changed map $f' : U_{cx} \times_{S_{cx}} W_{cx} \to W_{cx}$ to obtain the exact sequence

$$\cdots \to H^p(W_{cx}, f'_*\underline{\Lambda}) \to H^p(U_{cx} \times_{S_{cx}} W_{cx}, \underline{\Lambda}) \to H^{p-1}(W_{cx}, R^1 f'_*\underline{\Lambda}) \to \cdots .$$

Now $\gamma|_{U_{cx} \times_{S_{cx}} W_{cx}}$ is in the kernel of the map from H^p to H^{p-1} so it is the image of some $\theta \in H^p(W_{cx}, f'_*\underline{\Lambda})$. We now apply the inductive hypothesis to θ to get an étale map $V \to W$ with its image containing $f'(x)$ and $\theta|_V = 0$. Take $U' = U \times_S V$ and then $\gamma|_{U'} = 0$.
For the $q = 1$ case of the proposition, we consider the low degree exact sequence associated to the spectral sequence

$$0 \to H^0(X_{ét}, R^1 h_*\underline{\Lambda}) \to H^1(X_{cx}, \underline{\Lambda}) \to H^1(X_{ét}, \underline{\Lambda}) \to 0.$$

Here, the last term is 0 because it is $H^0(X_{ét}, R^2 h_*\underline{\Lambda})$, and we proved that $R^2 h_*\underline{\Lambda} = 0$. Now by the Grauert-Remmert theorem (see [GR58]) we have $H^0(X_{ét}, R^1 h_*\underline{\Lambda}) = 0$. But this statement is true for any smooth $X_{ét}$. Now if $\pi : U \to X$ is an étale morphism, then

$$H^0(U_{ét}, R^1 h_*\underline{\Lambda}) = H^0(U_{ét}, \pi^* R^1 h_*\underline{\Lambda}) = H^0(U_{ét}, R^1 h'_*\underline{\Lambda}) = 0,$$

where h' is the map of sites $h' : U_{cx} \to U_{ét}$. This concludes the proof of the proposition for all $q \geq 1$. $\square$

Theorem 3.2.4 (Comparison)**.** *Let X be a smooth $\mathbb{C}-$scheme and Λ be a finite abelian group. Then there is a natural isomorphism*

$$H^i_{ét}(X, \underline{\Lambda}) = H^i(X(\mathbb{C}), \underline{\Lambda}).$$

Proof. Since the functor h_* takes flasque sheaves to flasque sheaves (which are acyclic and have no cohomology), we can apply Grothendieck spectral sequence to get

$$H^p(X_{ét}, R^q h_* \underline{\Lambda}) \implies H^{p+q}(X_{cx}, \underline{\Lambda}).$$

Since $R^q h_* \underline{\Lambda} = 0$ by above proposition, we are done. $\square$

Chapter 4

Cohomology with compact support

4.1 The functor $f_!$

Let $f : X \to Y$ be an étale map and let $\mathcal{F}$ be a sheaf of abelian groups on $X_{ét}$. We define $f_!(\mathcal{F})$ to be the sheaf associated to the presheaf

$$\{U \xrightarrow{\alpha} Y\} \mapsto \bigoplus_{\beta:U\to X\, f\circ\beta=\alpha} \mathcal{F}(\beta).$$

Theorem 4.1.1. *Let $f : X \to Y$ be an étale mapping.*

1. *The functor $f_!$ is left adjoint to the pullback functor f^*.*

2. *Given any étale mapping $g : Y \to Z$, we have $(g \circ f)_! = g_! \circ f_!$.*

3. *Given a base change diagram*

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{f'} & Z \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y, \end{array}$$

we have $g^* f_! \mathcal{F} = f'_! g'^* \mathcal{F}$.

4. *Let $\alpha : Spec\ k \to Y$ be a geometric point with values in a separably closed field k. Let*

$$\alpha_\nu : Spec\ k \to X, \quad \nu = 1, 2, \ldots r$$

be geometric points above α. Then,

$$(f_!\mathcal{F})_\alpha = \prod_\nu \mathcal{F}_{\alpha_\nu}.$$

5. *Let $\mathcal{F}$ be a constructible sheaf. Then, $f_!\mathcal{F}$ is also constructible.*

Proof. See [FK88], Chapter 1, section 8. □

Theorem 4.1.2. *Let k be a separably closed field and let X be a finitely generated k-scheme, $dimX \leq 1$. Let $\mathcal{F}$ be a constructible sheaf whose sections have order relatively prime to the characteristic of k. Then:*

1. *The cohomology groups $H^q_{ét}(X, \mathcal{F})$ are finite.*
2. *$H^q_{ét}(X, \mathcal{F}) = 0$ for $q \geq 3$.*
3. *If X is affine, then $H^q_{ét}(X, \mathcal{F}) = 0$ for $q \geq 2$.*

4.2 Compactifications

Let $f : X \to S$ be given. A compactification of f is a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] \\ & S \end{tikzcd}$$

where j is an open embedding and $\overline{f}$ is a proper morphism. A morphism is called compactifiable if there exists such a compactification. For example, quasi-projective morphisms are compactifiable.

One says that a compactification

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & \overline{\overline{X}} \arrow[d, "\overline{\overline{f}}"] \\ & S \end{tikzcd}$$

dominates the compactification

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] \\ & S \end{tikzcd}$$

if there is a morphism $\varphi : \overline{\overline{X}} \to \overline{X}$ such that $\varphi \circ i = j$, and $\overline{\overline{f}} \circ \varphi = \overline{f}$. It is easy to see that given two compactifications of a morphism, there is always a third one that dominates them both.

Lemma 4.2.1. *Consider the commutative diagram*

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "f"] & \overline{X} \arrow[d, "\overline{f}"] \\ Y \arrow[r, hook, "j"] & \overline{Y} \end{tikzcd}$$

with i, j open embeddings and f a proper mapping. Let $\mathcal{F}$ be a sheaf on X. Then there exists a natural transformation

$$j_!(f_*\mathcal{F}) \to \overline{f}_*(i_!\mathcal{F}).$$

This induces a natural transformation on the derived functors

$$j_!Rf_*, R\overline{f}_*i_! : D_+(X, Tor) \to D_+(\overline{Y}, Tor),$$

$$(*) : j_!Rf_*\mathcal{F}^\bullet \to R\overline{f}_*i_!\mathcal{F}^\bullet.$$

The mapping $(*)$ is a quasi isomorphism when $\overline{f}$ is also proper.

Proof. See [FK88], Chapter 1, section 8. □

4.3 Higher direct images with proper support

We now define higher direct images with proper support. Suppose we have two compactifiable mappings where one dominates the other. We have a commutative diagram

$$\begin{tikzcd} & \overline{\overline{X}} \arrow[d, "g"] \\ X \arrow[ur, hook, "i"] \arrow[r, hook, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] \\ & S. \end{tikzcd}$$

Here i, j are open embeddings and $g, \overline{f}$ are proper. Let $\overline{f} \circ g = \overline{\overline{f}}$. Let $\mathcal{F}^\bullet$ be a complex of sheaves in $D_+(X, tor)$. It follows from the above lemma that $j_!(\mathcal{F}^\bullet) \xrightarrow{\sim} Rg_*(i_!\mathcal{F}^\bullet)$. Moreover the functors

$$R\overline{\overline{f}}_*, R\overline{f}_* \circ Rg_* : D_+(\overline{\overline{X}}, tor) \to D_+(S, tor)$$

are isomorphic in a natural way. We therefore conclude that the functors

$$R\overline{\overline{f}}_* i_!, R\overline{f}_* j_! : D_+(X, tor) \to D_+(S, tor)$$

are isomorphic in a natural way. Now given any two compactifications

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & \overline{\overline{X}} \arrow[d, "\overline{\overline{f}}"] \\ & S, \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] \\ & S \end{tikzcd}$$

there is a third one which dominates both of them. Therefore we get an isomorphism of functors $R\overline{\overline{f}}_* i_! \cong R\overline{f}_* j_!$. A priori, this isomorphism depends on the choice of a dominating compactification and domination mappings, but given any two such dominating compactifications, there is always a third one which further dominates them both and so the above isomorphism is independent of choices. This discussion motivates the following definition:

Definition 4.3.1. *Let $f : X \to S$ be a compactifiable mapping. Using a compactification*

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] \\ & S \end{tikzcd}$$

we define the $\delta-$functor

$$Rf_! : D_+(X, tor) \to D_+(S, tor),$$

where $Rf_! =_{def} (R\overline{f}_*) \circ j_!$. We call the cohomology sheaves

$$R^\nu f_!(\mathcal{F}^\bullet) = \mathcal{H}^\nu(Rf_!(\mathcal{F}^\bullet))$$

the higher direct images with proper support. For a sheaf $\mathcal{F}$, we have $R^0 f_!(\mathcal{F}) = \overline{f}_*(j_!\mathcal{F})$, which when f is étale gives

$$R^0 f_! \mathcal{F} = f_! \mathcal{F}$$

by Theorem 4.1.1. If $S = Spec\ K$ is the spectrum of a separably closed field, then we write

$$R\Gamma_c(\mathcal{F}^\bullet) =_{def} \Gamma(S, Rf_!(\mathcal{F}^\bullet)),$$

$$H^\nu_c(X, \mathcal{F}^\bullet) =_{def} \Gamma(S, R^\nu f_!(\mathcal{F}^\bullet)).$$

For a particular sheaf $\mathcal{F}$, we write

$$\Gamma_c(X, \mathcal{F}) = H^0_c(X, \mathcal{F}).$$

Remark. *Let $f : X \to S$ be an étale morphism. By Zariski's main theorem, there exists a compactification*

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] \\ & S \end{tikzcd}$$

where $\overline{f}$ is a finite morphism. By vanishing of finite higher direct images, we get

$$R^i\overline{f}_*(j_!\mathcal{F}) = 0 \quad \textit{for } i > 0,$$

$$R^0\overline{f}_*(j_!\mathcal{F}) = f_!\mathcal{F}.$$

We assemble some properties of higher direct images with proper supports in the following theorem:

Theorem 4.3.2. *1. Base change: Consider a cartesian diagram:*

$$\begin{tikzcd} X_T \arrow[r, "f'"] \arrow[d, "g'"] & T \arrow[d, "g"] \\ X \arrow[r, "f"] & S, \end{tikzcd}$$

where f and hence f' are compactifiable. There is a natural base change isomorphism (in $D(T)$)

$$g^* Rf_!(\mathcal{F}^\bullet) \xrightarrow{\sim} Rf'_! g'^*(\mathcal{F}^\bullet)$$

for $\mathcal{F}^\bullet \in D_+(X, tor)$.

2. Composition: Suppose we have mappings

$$X \xrightarrow{f} Y \xrightarrow{g} S, \quad g \circ f = h,$$

where g, h and hence f are compactifiable. In the derived category $D(S)$, we get the isomorphism of functors

$$Rg_! \circ Rf_! \cong Rh_!.$$

This yields a Leray type spectral sequence for computing higher direct images of the composite morphism.

3. The long exact sequence for an open subscheme: Let $f : X \to S$ be a compactifiable mapping, $i : U \hookrightarrow X$ an open embedding, $j : Z \to X$ a closed embedding, where Z is the complement of U as a set. For any torsion sheaf $\mathcal{F}$ on X, there is a functorial long exact sequence

$$0 \to (f \circ j)_!(j^*\mathcal{F}) \to f_!\mathcal{F} \to (f \circ i)_!(j^*\mathcal{F}) \to \cdots$$

$$\to R^\nu(f \circ j)_!(j^*\mathcal{F}) \to R^\nu f_!\mathcal{F} \to \cdots .$$

More abstractly, for any complex $\mathcal{F}^\bullet$ in the derived category $D_+(X, tor)$, one can assign *in a functorial way the admissible triangle*

$$\begin{tikzcd}[column sep=small] & R(f\circ i)_!(i^*(\mathcal{F}^\bullet)) \arrow[dl] & \\ R(f\circ j)_!(j^*(\mathcal{F}^\bullet)) \arrow[rr] & & Rf_!(\mathcal{F}^\bullet) \arrow[ul] \end{tikzcd}$$

in $D(S)$ such that the long exact sequence of cohomology sheaves above comes from this admissible triangle in the case of a single sheaf $\mathcal{F}$.

Proof. See [FK88], Chapter 1, section 8. □

As a typical application of the properties above we have the following vanishing result.

Theorem 4.3.3. *Let $f : X \to S$ be a compactifiable mapping. Let $dim(X/S)$ denote the maximum of all the dimensions of geometric fibers. For any torsion sheaf $\mathcal{F}$ on X, we have*

$$R^\nu f_!(\mathcal{F}) = 0 \quad \textit{for } \nu > 2dim(X/S)$$

under the condition that there exists a nonzero natural number m such that for any étale scheme $U \to X$, the restriction of $\mathcal{F}$ to U is annihilated by an appropriate power of m.

Proof. We use the method of reduction to curves. Using noetherian induction and the exact sequence of higher direct images with proper support, it suffices to prove the result when $X \to S$ is affine. By the base change theorem, it is enough to show that when $X \to S = \mathrm{Spec}\ K$ is an affine scheme over a separably closed field and $\mathcal{F}$ is a sheaf on X, we have

$$H^\nu_c(X, \mathcal{F}) = 0, \quad \nu > 2dim(X).$$

We know that f has a factorization as $f = f_n \circ \cdots \circ f_1$, where

$$f_i : X_i \to X_{i+1}$$

with $dim(X_i/X_{i+1}) \leq 1$. By Theorem 4.3.2, it suffices to prove the result for individual f_is. By further taking stalks and using the base change theorem, it is enough to prove

$$H^\nu_c(X, \mathcal{F}) = 0 \quad \text{for } \nu \geq 3.$$

where $X \to \text{Spec } K$ is an affine scheme with $dim(X) \leq 1$. We know that all separated curves are quasi projective, so there exists a projective variety $\overline{X}$ with an open embedding $X \hookrightarrow \overline{X}$ and $dim(\overline{X}) \leq 1$. Using this compactification to compute the cohomology with compact support, we get the required vanishing by the result on cohomology vanishing for curves (see Theorem 2.0.4). $\square$

We have the following two results for higher direct images along smooth proper morphisms and higher direct images with proper support along compactifiable morphisms.

Theorem 4.3.4. *Let $f : X \to S$ be a smooth proper morphism and let $\mathcal{F}$ be a constructible locally constant torsion étale sheaf on X with torsion orders invertible in S, i.e., there is an integer n, invertible on S with $n\mathcal{F} = 0$. Then all sheaves $R^\nu f_* \mathcal{F}$ are constructible locally constant.*

Theorem 4.3.5. *Let $f : X \to S$ be a compactifiable mapping. Then the sheaves $R^\nu f_!(\mathcal{F}^\bullet)$ are constructible for any complex $\mathcal{F}^\bullet \in D_+(X, tor)$ with constructible cohomology sheaves.*

4.4 The Kunneth formula

Let $\Lambda = \mathbb{Z}/n\mathbb{Z}$ with n invertible in all the schemes appearing. We will only consider the sheaves of Λ modules in the following discussion.
Let $\mathcal{F}^\bullet, \mathcal{G}^\bullet$ be two complexes of sheaves of Λ-modules. The tensor product

$$\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet = \mathcal{K}^\bullet$$

is defined as follows:

$$\mathcal{K}^n = \bigoplus_{p+q=n} \mathcal{F}^p \otimes \mathcal{G}^q,$$

$$d_{\mathcal{K}^\bullet}|_{\mathcal{F}^p \otimes \mathcal{G}^q} = d_{\mathcal{F}^\bullet} \otimes Id + (-1)^p Id \otimes d_{\mathcal{G}_\bullet}.$$

Let $\mathcal{F}^\bullet$ be complex of sheaves bounded above. One can construct a flat resolution of $\mathcal{F}$, i.e., a complex of flat sheaves $\mathcal{P}^\bullet(\mathcal{F}^\bullet)$ bounded above and a quasi isomorphism

$$\mathcal{P}^\bullet(\mathcal{F}^\bullet) \to \mathcal{F}^\bullet.$$

We then define
$$\mathcal{F}^\bullet \underline{\otimes} \mathcal{G}^\bullet = \mathcal{P}^\bullet(\mathcal{F}^\bullet) \otimes \mathcal{G}^\bullet.$$
We thus get a left derived functor in the sense of Verdier
$$-\underline{\otimes}- : D_-(X,\Lambda) \times D(X,\Lambda) \to D(X,\Lambda).$$
In the same way we get a derived functor with respect to the second factor
$$-\underline{\otimes}- : D(X,\Lambda) \times D_-(X,\Lambda) \to D(X,\Lambda).$$
One can show that these two constructions give the "same" derived functor on $D_-(X,\Lambda) \times D_-(X,\Lambda)$.

Proposition 4.4.1. *Let $f : X \to S$ be a compactifiable mapping, $\mathcal{F}^\bullet$ a complex in $D_-(S,\Lambda)$, and $\mathcal{G}^\bullet \in D(X,\Lambda)$. Then in the derived category, there is a natural isomorphism*
$$\mathcal{F}^\bullet \underline{\otimes} Rf_!(\mathcal{G}^\bullet) \xrightarrow{\sim} Rf_!(f^*(\mathcal{F}^\bullet) \underline{\otimes} \mathcal{G}^\bullet).$$

Proof. It is enough to assume f is proper. For a sheaf $\mathcal{F}$ on S and a sheaf $\mathcal{G}$ on X, we have
$$f^*(\mathcal{F} \otimes f_*(\mathcal{G})) = f^*(\mathcal{F}) \otimes f^* f_*(\mathcal{G}).$$
Composing with the adjunction mapping, we get a map
$$f^*(\mathcal{F} \otimes f_*(\mathcal{G})) \to f^*(\mathcal{F}) \otimes \mathcal{G}.$$
This induces a homomorphism
$$\mathcal{F} \otimes f_*(\mathcal{G}) \to f_*(f^*(\mathcal{F}) \otimes \mathcal{G}).$$
This homomorphism extends to the category of complexes of sheaves. Let $\mathcal{F}$ be a $\Lambda-$flat sheaf on S and G an f_* acyclic sheaf on X. We claim that

1. $R^\nu f_*(f^*(\mathcal{F}) \otimes \mathcal{G}) = 0 \quad \nu > 0,$
2. $f_*(f^*(\mathcal{F}) \otimes \mathcal{G}) = \mathcal{F} \otimes f_*(\mathcal{G}).$

By the proper base change theorem, it suffices to prove this result when $S = \text{Spec } K$ is the spectrum of a separably closed field. In this case, $\mathcal{F}$ is a constant sheaf associated to a flat $\Lambda-$module M. Since Λ is an artinian ring, M is a projective module and hence is a direct summand of a free module. Both assertions follow from the f_* acyclicity of $\mathcal{G}$ and the fact that cohomology commutes with taking direct sums of sheaves.
We can assume that $\mathcal{F}^\bullet$ is a complex of $\Lambda-$flat sheaves on S and $\mathcal{G}^\bullet$ is a complex of f_* acyclic sheaves on X. Then $f^*(\mathcal{F}^\bullet)$ is also a complex of $\Lambda-$flat sheaves. From (1) and using the compatibility of $R^\nu f_*$ with direct sums, we get that $f^*(\mathcal{F}^\bullet) \otimes \mathcal{G}^\bullet$ is a complex of f_* acyclic sheaves. Consequently, we have

$$Rf_*(f^*(\mathcal{F}^\bullet)\underline{\otimes}\mathcal{G}^\bullet) = f_*(f^*(\mathcal{F}^\bullet) \otimes \mathcal{G}^\bullet).$$

From (2) and compatibility of f_* with direct sums, we have

$$f_*(f^*(\mathcal{F}^\bullet) \otimes \mathcal{G}^\bullet) = \mathcal{F}^\bullet \otimes f_*(\mathcal{G}^\bullet) = \mathcal{F}\underline{\otimes}Rf_*(\mathcal{G}^\bullet).$$

□

Corollary 4.4.2. *Let X and Y be finitely generated schemes over K, K separably closed field. Let $\mathcal{F}$ be a sheaf of $\Lambda-$ modules on X and $\mathcal{G}$ be a sheaf of $\Lambda-$ modules on Y. Suppose one of the two sheaves is flat. Then in the derived category, there is a natural isomorphism*

$$R\Gamma_c(X, \mathcal{F})\underline{\otimes}R\Gamma_c(Y, \mathcal{G}) \cong R\Gamma_c(X \times Y, p^*(\mathcal{F}) \otimes q^*(\mathcal{G})),$$

where p, q are projections from $X \times Y$ to X and Y respectively.

Remark. *If all the cohomology groups with compact support of $\mathcal{F}$ or $\mathcal{G}$ are flat Λ-modules, then the above isomorphism implies we have*

$$\bigoplus_{i+j=n} H^i_c(X, \mathcal{F}) \otimes H^j_c(Y, \mathcal{G}) = H^n(X \times Y, p^*(\mathcal{F}) \otimes q^*(\mathcal{G})).$$

Proof. Use the proper base change theorem for base change of the map $Y \to \text{Spec } K$ by the map $X \to \text{Spec } K$ and apply the above theorem for mappings

$$X \times Y \to X, \quad \textit{and} \quad X \to \text{Spec } K.$$

□

Chapter 5

The Trace Formula

5.1 Poincaré Duality

5.1.1 Trace Map

Let $X \to \text{Spec } K$ be a smooth projective curve over an algebraically closed base field. As shown earlier, using the Kummer sequence we get the isomorphism

$$Pic(X)/nPic(X) \cong H^2(X, \mu_{n,X}).$$

In this case $Pic(X)/nPic(X)$ is canonically isomorphic to $\Lambda = \mathbb{Z}/n\mathbb{Z}$. We therefore have a map

$$tr_{X/K} : H^2(X, \mu_{n,X}) \xrightarrow{\sim} \Lambda,$$

which sends degree 1 line bundles to $1 \in \Lambda$. This is called the Chern class map. If X is not complete, we can do the same thing over a compactification to get a map

$$tr_{X/K} : H^2_c(X, \mu_{n,X}) \xrightarrow{\sim} \Lambda.$$

Locally in this case, the trace mapping is an isomorphism of sheaves

$$tr_{X/K} : R^2 f_!(\mu_{n,X}) \to \underline{\Lambda_{n,X}}.$$

In the following discussion we only consider morphisms $X \to S$ of schemes such that there exists an integer d such that all nonempty geometric fibers are of pure dimension d. We call these maps as maps of type α. We want to construct trace mappings

$$R^{2d(X/S)} f_!(\mathcal{T}_{X/S}) \to \underline{\Lambda_{n,X}}$$

for all such maps $f : X \to S$, where $\mathcal{T}_{X/S} = (\mu_{n,X})^{\otimes d}$, $d = d(X/S)$ is the relative dimension. Moreover we want it to satisfy the following properties:

1. Composition: Let $g : X \to T, \quad h : T \to S, \; f = h \circ g$ be morphisms of type α. By vanishing of higher direct images with proper support, we have

$$R^{\nu} g_!(\mathcal{F}) = 0, \quad \text{if } \nu > 2dim(X/S),$$

$$R^{\nu} h_!(\mathcal{G}) = 0, \quad \text{if } \nu > 2dim(S/T).$$

By the Leray spectral sequence, we have a natural isomorphism

$$R^{2[d(X/T)+d(T/S)]} f_!\mathcal{F} \xrightarrow{\sim} R^{2d(T/S)} h_!(R^{2d(X/T)} g_!\mathcal{F})$$

for ever torsion sheaf $\mathcal{F}$ on X. We know that for a sheaf $\mathcal{F}$ on X and a locally constant sheaf $\mathcal{G}$ on T, there is a natural isomorphism

$$R^{\nu} g_!(\mathcal{F} \otimes g^*(\mathcal{G})) \cong R^{\nu} g_!\mathcal{F} \otimes \mathcal{G}.$$

Suppose we have trace mappings

$$tr_{X/T} : R^{2d(X/T)} g_! \mathcal{T}_{X/T} \to \underline{\Lambda_T},$$

$$tr_{T/S} : R^{2d(T/S)} h_! \mathcal{T}_{T/S} \to \underline{\Lambda_S},$$

$$tr_{X/S} : R^{2[d(X/T)+d(T/S)]} f_! \mathcal{T}_{X/S} \to \underline{\Lambda_S}.$$

Then we have mappings

$$R^{2d(X/S)} f_!(\mathcal{T}_{X/S}) \xrightarrow{\sim} R^{2d(T/S)} h_!(R^{2d(X/T)} g_!(\mathcal{T}_{X/T} \otimes g^*(\mathcal{T}_{T/S})))$$

$$\xrightarrow{\sim} R^{2d(T/S)} h_!(R^{2d(X/T)} g_! \mathcal{T}_{X/T} \otimes \mathcal{T}_{T/S}) \xrightarrow{tr_{X/T}} R^{2d(T/S)} h_!(\mathcal{T}_{T/S}) \to \underline{\Lambda_S}.$$

We require that the composite of these mappings is the trace $tr_{X/S}$.

2. Base change: Let

$$\begin{tikzcd} X_T \arrow[r, "f'"] \arrow[d, "g'"] & T \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

be a cartesian diagram. Let the map f (and hence also f') be morphisms of type α. Set $d = d(X/S) = d(X_T/T)$, and we have $g^*(\mathcal{T}_{X/S}) = \mathcal{T}_{X_T/T}$. We want trace mappings $tr_{X_T/T}, tr_{X/S}$ compatible with base change. Precisely, we want the following diagram to be commutative:

$$\begin{tikzcd} g^* R^{2d} f_!(\mathcal{T}_{X/S}) \arrow[r, "\sim"] \arrow[d, "tr_{X/S}"] & R^{2d} f'_!(\mathcal{T}_{X_T/T}) \arrow[d, "tr_{X_T/T}"] \\ g^*(\underline{\Lambda_S}) \arrow[r, phantom, "="] & \underline{\Lambda_T}, \end{tikzcd}$$

where the first horizontal isomorphism is the base change theorem for higher direct images with proper support.

We have the following theorem:

Theorem 5.1.1. *One can assign to every compactifiable morphism $f : X \to S$ of type α a trace mapping*

$$tr_{X/S} : R^{2d(X/S)} f_!(\mathcal{T}_{X/S}) \to \underline{\Lambda_S}$$

satisfies the following properties and is uniquely determined by them:

1. *$tr_{X/S}$ is compatible with composition as defined above.*

2. *$tr_{X/S}$ is compatible with base change as defined above.*

3. *If $f : X \to S$ is étale (here relative dimension is 0), then the trace map*

$$tr_{X/S} : f_! f^*(\underline{\Lambda_S}) \to \underline{\Lambda_S}$$

is the adjunction mapping.

4. *If $f : X \to S$ is a smooth curve over the spectrum S of an algebraically closed field, then the trace map $tr_{X/S}$ is the one defined above using the Kummer sequence.*

Proof. See [FK88] Chapter 2, Section 1. $\square$

5.1.2 Cup Product

Let X be a finitely generated scheme over a separably closed field, and $\mathcal{F}, \mathcal{G}$ be two sheaves on X. Let

$$\mathcal{P}^\bullet \to \mathcal{F}$$

be a flat resolution bounded above. Since the functor Γ has finite homological dimension, $\mathcal{P}^\bullet$ and $\mathcal{G}$ have Γ-acyclic resolutions bounded above

$$\mathcal{P}^\bullet \to \mathcal{I}^\bullet(\mathcal{P}^\bullet), \quad \mathcal{G} \to I^\bullet(\mathcal{G}).$$

Now if $\Lambda = \mathbb{Z}/n\mathbb{Z}$, and $\mathcal{F}, \mathcal{G}$ are sheaves of $\Lambda-$modules, then we can also manage (using Godement construction) that the resolution $\mathcal{I}^\bullet(\mathcal{P}^\bullet)$ is also flat. Then $\mathcal{I}^\bullet(\mathcal{P}^\bullet) \otimes \mathcal{I}^\bullet(\mathcal{G})$ is quasi-isomorphic to $\mathcal{P}^\bullet \otimes \mathcal{G}$. Let

$$\mathcal{I}^\bullet(\mathcal{P}^\bullet) \otimes \mathcal{I}^\bullet(\mathcal{G}) \to \mathcal{J}^\bullet$$

be a bounded above Γ-acyclic resolution. Then there are natural mappings of cohomology groups

$$H^p(X, \mathcal{F}) \otimes H^q(X, \mathcal{G}) = H^p(\Gamma(X, \mathcal{I}^\bullet(\mathcal{P}^\bullet))) \otimes H^q(\Gamma(X, \mathcal{I}^\bullet(\mathcal{G})))$$
$$\to H^{p+q}(\Gamma(X, \mathcal{J}^\bullet)) = H^{p+q}(X, \mathcal{P}^\bullet \otimes \mathcal{G}) \to H^{p+q}(X, \mathcal{F} \otimes \mathcal{G}).$$

The composition of these mappings is called the cup product. A similar construction over a compactification yields cup products

$$H_c^p(X, \mathcal{F}) \otimes_\Lambda H_c^q(X, \mathcal{G}) \to H_c^{p+q}(X, \mathcal{F} \otimes_\Lambda \mathcal{G}),$$

$$H^p(X, \mathcal{F}) \otimes_\Lambda H_c^q(X, \mathcal{G}) \to H_c^{p+q}(X, \mathcal{F} \otimes_\Lambda \mathcal{G}),$$

$$Ext_\Lambda^p(\mathcal{F}, \mathcal{G}) \otimes_\Lambda H_c^q(X, \mathcal{G}) \to H_c^{p+q}(X, \mathcal{F} \otimes_\Lambda \mathcal{G}).$$

The last cup product will be important to us. We construct its derived version. Consider a compactification

$$j : X \hookrightarrow \overline{X}$$

and Λ injective resolutions

$$j_!(\mathcal{F}) \to \mathcal{I}^\bullet, \quad j_!(\mathcal{G}) \to \mathcal{J}^\bullet.$$

By restricting to X, we get

$$\mathcal{F} \to j^*(\mathcal{I}^\bullet), \quad \mathcal{G} \to j^*(\mathcal{J}^\bullet),$$

which are injective resolutions of $\mathcal{F}, \mathcal{G}$ respectively. The complexes

$$\Gamma(\overline{X}, \mathcal{I}^\bullet) \cong R\Gamma_c(X, \mathcal{F}), \quad \Gamma(\overline{X}, \mathcal{J}^\bullet) \cong R\Gamma_c(X, \mathcal{G})$$

are complexes of injective Λ-modules in $D(\Lambda)$. We have

$$RHom_\Lambda(\mathcal{F}, \mathcal{G}) \cong Hom_\Lambda(\mathcal{F}, j^*(\mathcal{J}^\bullet)) \to Hom_\Lambda(j_!(\mathcal{F}), \mathcal{J}^\bullet) \xleftarrow{\sim} Hom_\Lambda(\mathcal{I}^\bullet, \mathcal{J}^\bullet)$$

$$\to Hom_\Lambda(\Gamma(\overline{X}, \mathcal{I}^\bullet), (\Gamma(\overline{X}, \mathcal{J}^\bullet)) \cong RHom_\Lambda(R\Gamma_c(X, \mathcal{F}), R\Gamma_c(X, \mathcal{G})).$$

The composition of these maps is the map

$$RHom_\Lambda(\mathcal{F}, \mathcal{G}) \to RHom_\Lambda(R\Gamma_c(X, \mathcal{F}), R\Gamma_c(X, \mathcal{G})).$$

Let $f : X \to \text{Spec } K = S$ be a smooth algebraic variety of constant dimension d over the algebraically closed field K. We know from cohomology (with compact support) vanishing that $R\Gamma_c(X, \mathcal{T}_{X/S})$ is quasi-isomorphic to a complex $\mathcal{K}^\bullet$, where $H^\nu(\mathcal{K}^\bullet) = 0, \quad \text{for } \nu > 2d$. In the derived category of complexes of $\Lambda-$modules, we get a map

$$R\Gamma_c(X, \mathcal{T}_{X/S}) \to H^{2d}(\mathcal{K}^\bullet)[-2d] = H_c^{2d}(X, \mathcal{T}_{X/S})[-2d].$$

After shifting the above map and composing with the trace map, we get a morphism in the derived category

$$tr_{X/S} : R\Gamma_c(X, \mathcal{T}_{X/S}[2d]) \to \Lambda$$

which we denote similarly as the trace before.
For any sheaf $\mathcal{F}$, we have a natural transformation (shown above)

$$RHom(\mathcal{F}, \mathcal{T}_{X/S}[2d]) \to RHom(R\Gamma_c(X, \mathcal{F}), R\Gamma_c(X, \mathcal{T}_{X/S}[2d])).$$

Applying the above trace map $tr_{X/S}$ in the right factor, we get the map

$$RHom(\mathcal{F}, \mathcal{T}_{X/S}[2d]) \to RHom(R\Gamma_c(X, \mathcal{F}), \Lambda).$$

Since Λ is injective as a module over itself, we have $RHom(-, \Lambda) = Hom(-, \Lambda)$. We thereby obtain the duality mapping

$$\Delta_{X/S} : RHom(\mathcal{F}, \mathcal{T}_{X/S}[2d]) \to Hom(R\Gamma_c(X, \mathcal{F}), \Lambda).$$

Theorem 5.1.2 (Poincaré Duality)**.** *The duality mapping*

$$\Delta_{X/S} : RHom(\mathcal{F}, \mathcal{T}_{X/S}[2d]) \to Hom(R\Gamma_c(X, \mathcal{F}), \Lambda)$$

is a quasi isomorphism of complexes of $\Lambda-$modules for all sheaves $\mathcal{F}$ on X. The induced mapping

$$\Delta'_{X/S} : Ext^p(\mathcal{F}, \mathcal{T}_{X/S}) \to Hom_\Lambda(H_c^{2d-p}(X, \mathcal{F}), \Lambda)$$

is an isomorphism of $\delta-$functors.

Remark. *The duality mapping $\Delta'_{X/S}$ can also be described in terms of the cup product and the trace map for cohomology groups. The duality theorem states that the pairing*

$$Ext^p(\mathcal{F}, \underline{\mu_X}^{\otimes d}) \times H_c^{2d-p}(X, \mathcal{F}) \to H_c^{2d}(X, \underline{\mu_X}^{\otimes d}) \to \Lambda$$

induced by the cup product and the trace map is nondegenerate.

Proof. See [FK88] Chapter 2, Section 1. □

5.1.3 Duality for l-adic sheaves

Let l be a prime number invertible on X. Passing to limits, we obtain trace mappings

$$H_c^{2d}(X, \mathbb{Z}_l(d)) \to \mathbb{Z}_l,$$

$$H_c^{2d}(X, \mathbb{Q}_l(d)) \to \mathbb{Q}_l.$$

Let $\mathcal{F} = (\mathcal{F}_n)$ be a locally constant locally free $l-$adic sheaf (i.e., $\mathcal{F}_n$ is a locally constant locally free sheaf of $\mathbb{Z}/n\mathbb{Z}$ modules). Then

$$Hom(\mathcal{F}, \mathbb{Z}_l(d)) = (Hom(\mathcal{F}_n, \mu_{l^{n+1}}^d))_n$$

is a locally constant locally free $l-$adic sheaf, and

$$Hom(\mathcal{F} \otimes \mathbb{Q}_l, \mathbb{Q}_l(d)) = Hom(\mathcal{F}, \mathbb{Z}_l(d)) \otimes \mathbb{Q}_l$$

is a locally constant constructible sheaf of $\mathbb{Q}_l$-vector spaces. Passing to limits in the cup product and composing with the trace (in limit), we get a pairing

$$H^p(X, Hom(\mathcal{F}, \mathbb{Z}_l(d))) \times H_c^{2d-p}(X, \mathcal{F}) \to \mathbb{Z}_l.$$

It follows that the induced mapping

$$H^p(X, Hom(\mathcal{F}, \mathbb{Z}_l(d))) \to Hom(H_c^{2d-p}(X, \mathcal{F}), \mathbb{Z}_l)$$

has only torsion modules as its kernel and cokernel. In general this is not an isomorphism as $H_c^{\nu}(X, \mathcal{F}_n)$ is not a free $\mathbb{Z}/l^{n+1}\mathbb{Z}$-module. Passing to quotients we get:

Theorem 5.1.3. *For ever locally constant locally free constructible sheaf $\mathcal{F} \otimes \mathbb{Q}_l$ of $\mathbb{Q}_l-$vector spaces, the natural pairing*

$$H^p(X, Hom(\mathcal{F} \otimes \mathbb{Q}_l, \mathbb{Q}_l(d))) \times H_c^{2d-p}(X, \mathcal{F} \otimes \mathbb{Q}_l) \to \mathbb{Q}_l$$

is nondegenerate.

5.2 Cohomology Classes of Algebraic Cycles

Let X be a smooth algebraic variety of constant dimension d over $S = \mathrm{Spec}\ K$, where K is an algebraically closed field. Let n be invertible in K, $\Lambda = \mathbb{Z}/n\mathbb{Z}$. Let

$$j : Y \hookrightarrow X$$

be an irreducible reduced closed subvariety of X. We have the restriction mapping on cohomology

$$j^* : H^{2(d-s)}(X, \mu_{n,X}^{d-s}) \to H^{2(d-s)}(Y, \mu_{n,Y}^{d-s}).$$

So far we only defined the trace map for smooth maps. For a reduced algebraic variety Y of dimension t over S, the subset Y_0 of regular points is dense. So we have $dim(Y - Y_0) < t$. Therefore, the natural mapping

$$H_c^{2t}(Y_0, \mu_{n,Y_0}^t) \to H_c^{2t}(Y, \mu_{n,Y}^t)$$

is an isomorphism. Composing the inverse of this map with $tr_{Y_0/k}$, we get a trace mapping $tr_{Y/k}$. Composing this trace with the pullback map, we get a map

$$\beta : H_c^{2(d-s)}(X, \mu_{n,X}^{d-s}) \to \Lambda.$$

β is therefore an element of $Hom_\Lambda(H_c^{2(d-s)}(X, \mu_{n,X}^{d-s}), \Lambda)$, and by Poincaré duality we get a class in the cohomology group $H^{2s}(X, \mu_{n,X}^s)$.

Definition 5.2.1. *The cohomology class constructed above is called the cohomology class associated to the closed subscheme $Y \hookrightarrow X$. Extending linearly, we get a homomorphism*

$$cl_X : Z^s(X) \to H^{2s}(X, \mu_{n,X}^s)$$

from the group $Z^s(X)$ of codimension s algebraic cycles to the cohomology group $H^{2s}(X, \mu_{n,X}^s)$.

The map cl_X in the case of codimension 1 sends codimension 1 cycles (divisors) to cohomology classes in $H^2(X, \mu n, X)$. Given any divisor, we can consider the associated line bundle and take it's Chern class as defined before. We have the following theorem:

Theorem 5.2.2. *Let $D \in Z^1(X)$ be a divisor on X, and $\mathcal{L}(D)$ the associated line bundle. Let $cl(\mathcal{L}(D))$ be the first Chern class of $\mathcal{L}(D)$ defined using the Kummer sequence. We have*

$$cl_X(D) = cl(\mathcal{L}(D)).$$

Proof. See [FK88], Chapter 2, Section 2. □

Since the Chern class is compatible with taking inverse images, we have the following corollary:

Corollary 5.2.3. *The mapping cl_X is compatible with taking inverse images in the sense of intersection theory. For $f : \tilde{X} \to X$ we have*

$$f^*(cl_X(D)) = cl_{\tilde{X}}(f^*(D)).$$

We want to deduce a Lefschetz type fixed point theorem. This is possible because the map $cl_X(-)$ is compatible with taking inverse images. We compute the cohomology class of the diagonal

$$D \subset X \times X.$$

After fixing a primitive root of unity, we get isomorphisms $\mu_{n,X} \to \Lambda_X$ and $\mu_{n,X}^{\otimes s} \to \Lambda_X^{\otimes s} \cong \Lambda_X$. Using these, we make the identification

$$H_c^\nu(X, \mu_{n,X}^s) = H_c^\nu(X, \Lambda_X).$$

Let X be smooth, complete of constant dimension d. Suppose the cohomology groups $H^\nu(X, \Lambda)$ are free modules over Λ. Let $p : X \times X \to X, \quad q : X \times X \to X$ be projections onto the first and second factors respectively.

$$\Delta : X \to X \times X$$

be the diagonal mapping. We can compute $H^\bullet(X \times X, \Lambda)$ using the Künneth formula. Let $\alpha, \beta \in H^{2d}(X, \Lambda)$. We have $tr_{X\times X/S}(p^*(\alpha) \smile q^*(\beta)) = tr_{X/S}(\alpha) tr_{X/S}(\beta)$. For each $0 \leq \nu \leq 2d$, we choose a basis $e_1^\nu, e_2^\nu, \ldots$ and let $f_1^\nu, f_2^\nu, \ldots$ be a dual basis (for $H^{2d-\nu}(X, \Lambda)$). We have

$$S_X(e_i^\nu \smile f_i^\nu) = \delta_{ij}.$$

Proposition 5.2.4. *The cohomology class of the diagonal can be calculated as*

$$cl_{X\times X}(D) = \Sigma_{i,\nu} p^*(e_i^\nu) \smile q^*(f_i^\nu).$$

Proof. See [FK88], Chapter 2, Section 2. □

Let $h : X \to X$ be a morphism and Γ_h be the associated graph morphism. Let $M^\nu = (c_{ij}^\nu)$ be the matrix representation of the pullback map $h^* : H^\nu(X, \Lambda) \to H^\nu(X, \Lambda)$ for the basis

(e_i^ν). We have

$$S_{X/k}(\Gamma_h^*(cl_{X\times X}(D))) = \sum_\nu (-1)^\nu \sum S_X(h^*(e_i^\nu) \smile f_i^\nu)$$

$$\to \sum_\nu (-1)^\nu \sum_{i,j} c_{ij}^\nu \delta_{ij} = \sum_\nu (-1)^\nu Trace(M^\nu).$$

When X is a curve, we have proved (cohomology class of a cycle commutes with inverse images for the case of codimension one) that

$$\Gamma_h^*(cl_{X\times X}(D)) = cl_X(\Gamma_h^*(D)).$$

Since in the case of curves, the cycle class coincides with the Chern class of the associated line bundle, we have

$$S_{X/k}(cl_X(D)) = deg(D) \in \Lambda$$

for a divisor D. Consequently we have

Theorem 5.2.5 (Lefschetz fixed point formula)**.** *Let X be a smooth, complete curve over an algebraically closed field k and let $h : X \to X$ be an endomorphism. Then $\Gamma_h^*(D)$ (inverse image of the diagonal $D \subset X \times X$ in the sense of intersection theory ([Ful84])) satisfies*

$$S_{X/k}(\Gamma_h^*(D)) = \sum_i (-1)^i Trace(h^*|H^i(X, \Lambda)) \in \Lambda.$$

Remark. *When the map h only has isolated fixed points, then $deg(\Gamma_h^*(D)$ is equal to the number of fixed points counted with multiplicity. When h is the identity map, the above formula implies*

$$2 - 2g = \sum_\nu (-1)^i dim(H^i(X, \Lambda)).$$

If we assume that the formation of cycle class is compatible with the formation of inverse images, then we can extend the fixed point formula to smooth complete varieties of arbitrary dimension.

5.3 Rationality of L-series of l-adic sheaves

Let A be a finitely generated projective module over a commutative ring R, and $f : A \to A$ be an endomorphism. We know that there exists an R-module B such that $A \oplus B$ is a finite rank free module. We can define a map

$$f_A : A \oplus B \to A \oplus B$$

where $f_A(a, b) = (f(a), 0)$. For the map f_A, we can define the notion of trace and determinant. We define

$$tr_R(f) =_{def} tr(f_A),$$

$$det(1 - tf) =_{def} det(Id_{A\oplus B} - tf_A).$$

It can be showed that these notions are well defined. For a map $f^\bullet : A^\bullet \to A^\bullet$ of bounded complexes of projective modules, we can define

$$tr_R(f^\bullet) = \sum_i tr_R(f^i),$$

$$det(1 - tf^\bullet) = \prod_i det(1 - tf^i)^{(-1)^i}.$$

It can be showed that if a complex as above has no cohomology, then the trace and determinant are trivial. Given a commutative diagram (up to homotopy)

$$\begin{tikzcd} A^\bullet \arrow[r, "\varphi^\bullet"] \arrow[d, "f^\bullet"] & B^\bullet \arrow[d, "g^\bullet"] \\ A^\bullet \arrow[r, "\varphi^\bullet"] & B^\bullet \end{tikzcd}$$

$f^\bullet, g^\bullet$ define a map $h^\bullet$ on the mapping cone $\mathcal{C}^\bullet(\varphi^\bullet)$. It can be showed that

$$tr_R(g^\bullet) = tr_R(f^\bullet) + tr_R(h^\bullet),$$

$$det(1 - tg^\bullet) = det(1 - tf^\bullet) \cdot det(1 - th^\bullet).$$

Using these properties, it can be showed that the definition of trace and determinant makes sense for an endomorphism of a perfect complex (complexes which are quasi isomorphic to a bounded complex of finitely generated projective modules). Moreover, we have the additivity of trace and determinant in the above sense for distinguished triangles of perfect complexes in the derived category $D(R)$ of R-modules, where the endomorphism of the triangle comes from the derived category of endomorphisms of R-modules.

5.3.1 The Frobenius Homomorphism

Let k be a finite field of characteristic p with $\#(k) = q = p^s$, and let $\overline{k}$ be its algebraic closure. For any k-algebra A, we have the map

$$Fr_A : A \to A, \quad a \mapsto a^q.$$

For any $k-$scheme, we get the map $Fr_X : X \to X$ corresponding to the q^{th} power map. This map is called the frobenius morphism of X. It is easy to see that the frobenius morphism commutes with any morphism of k-schemes. Let $U \to X$ be a map of schemes. By the commutativity of the diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ \big\downarrow{\scriptstyle Fr_U} & & \big\downarrow{\scriptstyle Fr_X} \\ U & \longrightarrow & X \end{array}$$

we get a map

$$Fr_{U|X} : U \to X_{Fr_X} \times_X U = Fr_{X^{-1}}(U),$$

where X_{Fr_X} is the scheme X considered over itself via the frobenius morphism. It turns out that tha morphism $Fr_{U|X}$ is an isomorphism when the map $U \to X$ is étale. Now let $\mathcal{F}$ be a sheaf of sets on the small étale site of X. For every étale scheme $U \to X$, there is a map

$$\mathcal{F}(U) \xrightarrow{Fr_{U|X}^{-1}} \mathcal{F}(Fr_X^{-1}(U)).$$

By the compatibility properties of the frobenius, we get a map of sheaves

$$\mathcal{F} \to (Fr_X)_*\mathcal{F},$$

and we have the adjoint map

$$(Fr_X)^*\mathcal{F} \to \mathcal{F}.$$

We call this map the geometric frobenius of of $\mathcal{F}$ with respect to k.

Theorem 5.3.1. *Let U_0 be a smooth curve over κ and $\mathcal{F}_0$ a constructible sheaf of Λ modules whose stalks are projective $\Lambda = \mathbb{Z}/n\mathbb{Z}$- modules, n invertible on X. Let U be the base change of U_0 to an algebraic closure $k = \overline{\kappa}$, $\mathcal{F}$ be the inverse image of $\mathcal{F}_0$ on U, Fr_U, $Fr_{\mathcal{F}}$ are the mappings obtained from Fr_{U_0}, $Fr_{\mathcal{F}_0}$ by base change. Then $R\Gamma_c(U, \mathcal{F})$ is a perfect complex of Λ modules and*

$$Tr_\Lambda(R\Gamma_c(Fr_U^d, Fr_{\mathcal{F}}^d)) = \sum_{a \in Fix(Fr_U^d)} Tr_\Lambda(Fr_{\mathcal{F}_a}^d),$$

where $Fix(Fr_U^d)$ is the set of geometric fixed points with values in the algebraically closed field $k = \overline{\kappa}$.

Proof. See [Del77], [FK88]. □

This result can be generalized to schemes over k of arbitrary schemes over k, and moreover to the morphisms of schemes. Note that the fixed point formula above is true in the ring Λ. We need to lift this analysis to characteristic zero where we can count points!

5.3.2 l-adic sheaves

Let $\mathbb{Z}_l$ be the ring of l-adic numbers, and let V be a finitely generated $\mathbb{Z}_l$ module. Let $f : V \to V$ be a module endomorphism of V. We define the trace and determinant of f as follows:

$$Tr_{\mathbb{Z}_l}(f) = Tr_{\mathbb{Q}_l}(f \otimes \mathbb{Q}_l),$$

$$det(1 - tf) = det(1 - tf \otimes \mathbb{Q}_l).$$

For a perfect complex $\mathcal{G}^\bullet$ of $\mathbb{Z}_l$ modules, and an endomorphism $f^\bullet : G^\bullet \to \mathcal{G}^\bullet$ in the derived category, we define

$$tr_{\mathbb{Z}_l}(f^\bullet) = \sum_i (-1)^i tr_{\mathbb{Z}_l}(H^i(f^\bullet)),$$

$$det(1 - tf^\bullet) = \prod_i det(1 - tH^i(f^\bullet)),$$

where $H^i(f^\bullet)$ is the map induced on the i^{th} cohomology of $\mathcal{G}^\bullet$ from $f^\bullet$.

Let X be an algebraic scheme over a field k where k is separably closed and $\mathcal{G} = (\mathcal{G}_n)$ be a torsion free l-adic sheaf on X. The sheaves $\mathcal{G}_n$ of $\mathbb{Z}/l^{n+1}\mathbb{Z}$ modules are then constructible with flat stalks. Let

$$\varphi : X \to X$$

be a k-morphism and let $f : \varphi^*\mathcal{G} \to G$ be a morphism of l-adic sheaves. We have the following technical lemma

Lemma 5.3.2. *For the mappings*

$$R\Gamma_c(\varphi, f_\nu) : R\Gamma_c(X, \mathcal{G}_\nu) \to R\Gamma_c(X, \mathcal{G}_\nu),$$

and

$$H_c^m(\varphi, f) = \varprojlim H_c^m(\varphi, f_\nu) : H_c^m(X, \mathcal{G}) \to H_c^m(X, \mathcal{G}),$$

where $H_c^m(X, \mathcal{G}) = \varprojlim H_c^m(X, \mathcal{G}_\nu)$*, induced by the maps* φ, f*, the image of*

$$\sum_m (-1)^m tr H_c^m(\varphi, f)$$

in $\mathbb{Z}/l^{n+1}\mathbb{Z}$ *is the trace* $tr R\Gamma_c(\varphi, f_n)$*.*

Proof. See [FK88] Chapter 2, section 4. □

This Lemma allows us to extend the fixed point formula to l-adic sheaves.

Proposition 5.3.3. *Let* X_0 *be an algebraic scheme and let* $\mathcal{G}_0$ *be an* l*-adic sheaf on* X_0*. Let* X*,* $\mathcal{G}$ *be the base change to algebraic closure of* $X_0, \mathcal{G}$ *respectively as before. We have*

$$\sum_i (-1)^i tr((H_c^i(Fr_X, Fr_{\mathcal{G}}))^d) = \sum_{a \in Fix(Fr_X^d)} tr_{\mathbb{Z}_l}(Fr_{\mathcal{G},a}^d).$$

Here $Fix(Fr_X^d)$ is in canonical bijection with the set of points defined over a degree d extension $k_d \subset \overline{k}$ of k.

Proof. See [FK88], Chapter 2, Section 4. □

Let X_0 be a scheme over k and let $Cl(X_0)$ be the set of closed points of X_0. Let X be the base change of X_0 to the separable closure $\overline{k}$ of k. It is easy to see that the sets of geometric points of X_0 and X are in canonical bijection. Moreover, for any sheaf $\mathcal{G}_0$ on X_0, the stalk of $\mathcal{G}_0$ at a geometric point a is isomorphic to the stalk at the corresponding geometric point of X of the base changed sheaf $\mathcal{G}$.

There is the underlying closed point associated to every geometric point. The map taking geometric points to the underlying closed points is surjective. Let $\alpha \in Cl(X_0)$ be a closed point, $k(\alpha)$ be the residue field and let $d(\alpha) = [k(\alpha) : k]$. From Galois theory, we know that there exist $d(\alpha)$ many distinct embeddings of $k(\alpha)$ in the separable closure $\overline{k}$. Therefore there exist $d(\alpha)$ many distinct geometric points over the closed point α.

Let $\alpha \in Cl(X_0)$ be a closed point and let $Gal(\alpha)$ be the group $Gal(\overline{k}/k(\alpha))$. This group contains the frobenius automorphism

$$f_\alpha = f_a : k \to k, \quad x \mapsto x^{\#k(\alpha)},$$

where a is a geometric point over α. The stalks $\mathcal{G}_a$ of abelian sheaves $\mathcal{G}$ are $Gal(\alpha)$ modules where α is the underlying closed point of a, and the stalks at all geometric points over the closed point α are isomorphic. For the endomorphism $f_{a,\mathcal{G}_a} : \mathcal{G}_a \to \mathcal{G}_a$ induced by the map f_a on $\overline{k}$, it follows that $det(1 - tf_{a,\mathcal{G}_a}^{-1})$ depends only on the underlying closed point of a. For every closed point $\alpha \in Cl(X_0)$, we choose a closed point a above it and define $f_{\alpha,\mathcal{G}_\alpha} =_{def} f_{a,\mathcal{G}_a}$.

A geometric point a of $X = X_0 \times_k \operatorname{Spec} \overline{k}$ is a fixed point of the d^{th} power frobenius Fr_X^d if and only if $d(a)|d$ (here, $d(a)$ is the degree of the residue field of the underlying closed point of a), i.e., when the field $k(a) \subset k_d$, where k_d is a degree d extension of k. We have the formula

$$f_{a,\mathcal{G}_a}^{-1} = Fr_{\mathcal{G},a}^{d(a)}.$$

Definition 5.3.4. *The L-series of an l-adic constructible sheaf $\mathcal{G}_0$ on an algebraic variety X_0 over k is*

$$Z_{\mathcal{G}_0}(t) = \prod_{\alpha \in Cl(X_0)} \frac{1}{det(1 - f_{\alpha,\mathcal{G}_\alpha}^{-1} t^{d(\alpha)})}.$$

We want to calculate the logarithmic derivative of $Z_{\mathcal{G}_0}(t)$. For an endomorphism φ of a finite Λ-module, the logarithmic derivative of $det(1 - t\varphi)^{-1}$ is

$$\sum_\nu tr_\Lambda(\varphi^\nu) t^{\nu-1}.$$

We therefore have

$$
\begin{aligned}
\frac{Z'_{\mathcal{G}_0}}{Z_{\mathcal{G}_0}}(t) &= \sum_{\alpha \in Cl(X_0)} \sum_{\nu=1}^{\infty} d(\alpha) tr_\Lambda(f_{\alpha,\mathcal{G}_\alpha}^{-\nu}) t^{\nu d(\alpha)-1} \\
&= \sum_{a \in X(\overline{k})} \sum_{\nu=1}^{\infty} tr_\Lambda(f_{a,\mathcal{G}_a}^{-\nu}) t^{\nu d(\alpha)-1} \\
&= \sum_{a \in X(\overline{k})} \sum_{\nu=1}^{\infty} tr_\Lambda(Fr_{\mathcal{G},a}^{-\nu d(a)}) t^{\nu d(\alpha)-1} \\
&= \sum_{n=1}^{\infty} \sum_{\substack{a \in X(\overline{k}) \\ d(a)|n}} tr_\Lambda(Fr_{\mathcal{G},a}^{n}) t^{n-1} \\
&= \sum_{n=1}^{\infty} \sum_{a \in Fix(Fr_X^n)} tr_\Lambda(Fr_{\mathcal{G},a}^{n}) t^{n-1},
\end{aligned}
$$

where $\Lambda = \mathbb{Z}_l$ is the ring of l-adic integers.

From this it follows that $\frac{Z'_{\mathcal{G}_0}}{Z_{\mathcal{G}_0}}(t)$ is the logarithmic derivative of

$$\prod_{\nu} det(1 - H_c^\nu(Fr_X, Fr_{\mathcal{G}}))^{(-1)^{\nu-1}}.$$

Therefore we have

Theorem 5.3.5. *Let $\mathcal{G}_0$ be a constructible l-adic sheaf on a finitely generated scheme X_0 over a finite field k, and let $\mathcal{G}$ be the inverse image of $\mathcal{G}_0$ on $X = X_0 \times_k Spec\ \overline{k}$. Then*

$$
\begin{aligned}
Z_{\mathcal{G}_0}(t) &= \prod_{\nu} det(1 - H_c^\nu(Fr_X, Fr_{\mathcal{G}}))^{(-1)^{\nu-1}}, \\
\frac{Z'_{\mathcal{G}_0}}{Z_{\mathcal{G}_0}}(t) &= \sum_{n=1}^{\infty} \sum_{a \in Fix(Fr_X^n)} tr_\Lambda(Fr_{\mathcal{G},a}^{n}) t^{n-1}.
\end{aligned}
$$

This implies that the L-series $Z_{\mathcal{G}_0}$ is a rational function!

Bibliography

[AGV71] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier. *Theorie de Topos et Cohomologie Etale des Schemas I, II, III*. Vol. 269, 270, 305. Lecture Notes in Mathematics. Springer, 1971.

[Art69] M. Artin. "Algebraic approximation of structures over complete local rings". In: *Inst. Hautes Études Sci. Publ. Math.* 36 (1969), pp. 23–58.

[Del77] P. Deligne. *Cohomologie étale*. Vol. 569. Lecture Notes in Mathematics. Séminaire de géométrie algébrique du Bois-Marie SGA $4\frac{1}{2}$. Springer-Verlag, Berlin, 1977.

[FK88] Eberhard Freitag and Reinhardt Kiehl. *Étale cohomology and the Weil conjecture*. Vol. 13. Ergebnisse der Mathematik und ihrer Grenzgebiete (3). Springer-Verlag, Berlin, 1988.

[Ful84] William Fulton. *Intersection theory*. Vol. 2. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1984.

[GR58] Hans Grauert and Reinhold Remmert. "Komplexe Räume". In: *Math. Ann.* 136 (1958), pp. 245–318.

[Har77] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[Mil80] James S. Milne. *Étale cohomology*. Princeton Mathematical Series, No. 33. Princeton University Press, Princeton, N.J., 1980.

[Mum70] David Mumford. *Abelian varieties*. Vol. 5. Tata Institute of Fundamental Research Studies in Mathematics. Oxford University Press, London, 1970.

[Sta18] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`. 2018.

[Wei49] André Weil. "Numbers of solutions of equations in finite fields". In: *Bull. Amer. Math. Soc.* 55 (1949), pp. 497–508.