Sample Theory Lecture

- Focuses on Temporal Logics
  - A specification language for describing properties of reactive systems
  - Describes syntax and semantics with examples and intuition
  - Exercises are given during lecture.
    - Slides 16-17, 29-32.
  - Clears common misconceptions among students.
    - Slides 33-35.

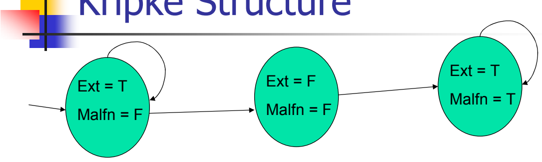
Temporal Logics

Abhik Roychoudhury
CS 6214

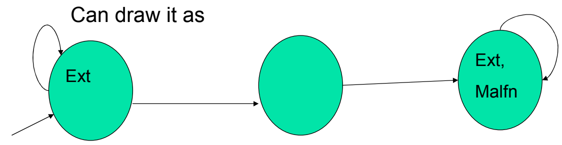
Models

- Kripke Structure
  - Transition System (States and Transitions)
    - Can depict the control flow in a program.
  - Set of Propositions Prop
    - Abstraction of the data variables.
  - Truth/Falsehood of each proposition of Prop in each state of the transition system.
    - Abstraction of data flow in the program.

Kripke Structure



Can draw it as



Kripke Structure

- M = (S, S0, R, L)
  - S = set of states
  - S0 = set of initial states
  - R = Transition Relation
  - L = Labeling function
    - Label states with propositions – these are the propositions which are true in the state.

Model Checking

- Generate models (Kripke Structure) from the program
  - Involves predicate abstraction
  - Represent the control flow explicitly (Control Flow Graph or variants).
  - Abstract data store via predicate abstraction.
    - Implicitly blows up the CFG.
  - State space search now involves traversing the blown up graph (note: symbolic search).

Model Checking

- Specify property in temporal logic
  - Clock time is not explicitly represented.
  - Temporal modalities to capture dynamic program behavior
- Verify property automatically via search
  - Return "yes" if true
  - Return counterexample "evidence" if false.

Temporal Logic

- Propositional / First-order logic formulae
  - Capture properties of states
  - Cannot capture properties of state changes
- Temporal Logic formulae
  - Capture properties of evolution of states (program behavior)
  - Possible program behaviors can be
    - Execution traces OR Execution Tree

What is temporal ?

- Consider ordering of events in system execution.
  - Describes properties of such orderings
    - Whenever V = 0, U is not 0
    - Always V = 0
    - Whenever V=0, eventually U = 0
  - Exact time is not represented e.g.
    - V =0 will happen at t = 22 secs.

Purpose of TL

- Describe properties of program behavior
- What are the units of behavior
  - Computation Tree of the program
  - Set of computation traces of the program
- Linear and branching time logics.
- TL can specify behavior of
  - Terminating and non-terminating programs.
  - Sequential as well as concurrent programs.

LTL: What ?

- Linear-time Temporal Logic
- An LTL formula is interpreted over infinite execution traces.
- LTL can capture properties of
  - Usual terminating programs
  - Non-terminating reactive programs which are continuously interacting with environment
    - Example: Controller software

LTL syntax

- An LTL property φ is true of a program model iff all its traces satisfy φ
- φ = Xφ | Gφ | Fφ | φ U φ | φ R φ | $\neg\varphi$ | $\varphi \wedge \varphi$ | Prop
- Prop denotes the set of Propositions.
- X, G, F, U, R are temporal operators.
- Building a temporal logic above propositional logic (included above)

LTL semantics

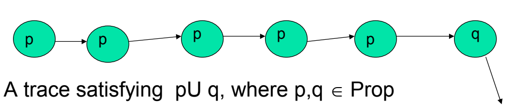
- $M,\pi \models \varphi$
  - Path $\pi = s_0,s_1,s_2,\ldots$ in model M satisfies property φ
- $M,\pi^k \models \varphi$
  - Notation for
    Path s_k , s_{k+1} , … in model M satisfies the property φ
- Semantics for different temporal operators are now given.

Semantics of LTL operators

- $M,\pi \models X\varphi$ iff $M,\pi^1 \models \varphi$
  - Path starting from next state satisfies φ
- $M,\pi \models F\varphi$ iff $\exists k \geq 0\ M,\pi^k \models \varphi$
  - Path starting from an eventually reached state satisfies φ
- $M,\pi \models G\varphi$ iff $\forall k \geq 0\ M,\pi^k \models \varphi$
  - Path always satisfies φ (all suffixes of the path satisfy φ

Semantics of LTL operators

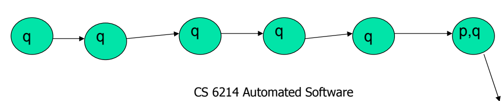
- $M,\pi \models \varphi1\ U\ \varphi2$ iff $\exists k \geq 0$ such that
  - $M,\pi^k \models \varphi2$, and
  - $\forall 0 \leq j < k\ M, \pi^j \models \varphi1$



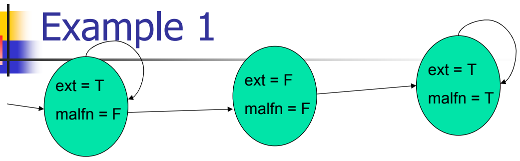
A trace satisfying pU q, where p,q $\in$ Prop

Semantics of LTL operators

- $M,\pi \models \varphi1\ R\ \varphi2$ iff
  - Either $\forall k \geq 0\ M,\pi^k \models \varphi2$
  - OR both of the following hold
    - $\exists k \geq 0\ M,\pi^k \models \varphi1$
    - $\forall 0 \leq j \leq k\ \ M,\pi^j \models \varphi2$
- $\varphi1$ releases the req. for $\varphi2$ to hold.



Example 1

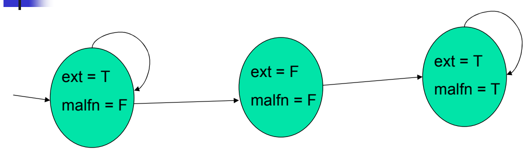


Property: G ext

M |= G ext iff for traces π in M we have M, π |= G ext

Any trace which does not satisfy G ext is a counter-example. How many counterexamples can you find in the above ??

Example 2



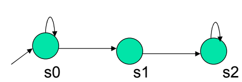
What about the formulae: FG ext, GF ext ??

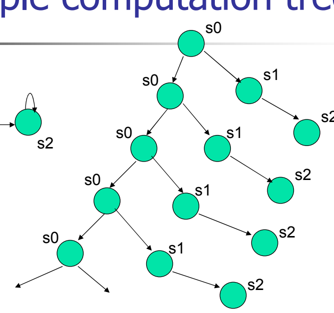
How many counter-examples for each ?

Computation traces/trees

- LTL describes properties of computation traces (a property holds in a sys. if all traces satisfy it).
- Alternate way to characterize system dynamics
  - Computation tree
  - Start from an initial state and unfold the Kripke structure to construct an infinite tree (in general).
- Logics to describe properties of such trees
  - Existential / Universal quantification over paths
  - Temporal operators to specify properties of a path.

Example computation tree





A logic for trees

- s = Prop | ¬ s | s /\ s | **A** p | **E** p
- p = **X** s | **G** s | **F** s | s **U** s | s **R** s

- p denotes formulae of paths.
- s denotes formulae of states.
- The temporal operators are as before.
- Computation Tree logic (CTL).
  - All state formulae as defined above.

CTL semantics

- A model satisfies a CTL formula iff its initial states satisfy the formula.
- A state s satisfies the formula A p iff all outgoing paths from s satisfy the formula p
  - Note that p must be a path formula
- A state s satisfies the formula E p iff there exists an outgoing path from s satisfying the formula p
- What about the temporal operators ?

CTL semantics

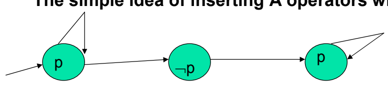
- Semantics of temporal operators (X,F,G,U,R)
  - Minimally modified to handle …
  - A path satisfies a state formula iff its first state satisfies the formula.
- Exercise : Do it Now !
  - Meaning of AG, EG, AF, EF (intuitively)
  - Duality of (R, U), (F, G), (A, E)
  - Express F in terms of U
  - Minimal set of temporal & path operators

CTL formulae

- AG f (invariant property)
  - M |= AG f (CTL formula) if and only if M satisfies the LTL formula G f
- AG (f ⇒ EF g)
  - what does it mean ?
  - Involves both path quantifiers
  - Not expressible in LTL
- Does that mean CTL is strictly more powerful than LTL ?

CTL and LTL

The simple idea of inserting A operators will not work.



Satisfies the LTL formula FG p

What about the CTL formula AFAG p ?

The issues of Temporal Logic expressivity are much deeper and somewhat outside the scope of this course.

CTL*

- s = Prop | ¬s | s ∧ s | A p | E p
- p = s | ¬p | p ∧ p | Xp | Fp | Gp |
- p U p | p R p
  - s denotes formulae interpreted over states
  - p denotes formulae interpreted over paths
  - CTL* is the set of all state formulae above
  - A p is a state formula not expressed in prop. Logic
  - Covers both CTL and LTL.

CTL and CTL*

- CTL is a sublogic of CTL*
  - Temporal operators in CTL*: X,F,G,U,R
  - Path Quantifiers: A, E
  - CTL enforces the occurrence of a temporal operator to be immediately preceded by a path quantifier.
  - AGFφ is not allowed.

Some common CTL formulae

- AG p
  - Invariant: always p.
- EF p
  - Reachability: of a state where p holds.
- AF p
  - Inevatibility of reaching a state where p holds.
- AG EF p
  - Recovery: from any state we can reach a state where p holds.
- AG (p ⇒ AF q)
  - Non-starvation : ***p*** request is always provided ***q*** response.

Exercise

Consider a resource allocation protocol where n processes $P_1, \ldots, P_n$ are contending for exclusive access of a shared resource. Access to the shared resource is controlled by an arbiter process. The atomic proposition req_i is true only when P_i explicitly sends an access request to the arbiter. The atomic proposition gnt_i is true only when the arbiter grants access to P_i. Now suppose that the following LTL formula holds for our resource allocation protocol.

- **$G\ (req_i \Rightarrow (req_i\ U\ gnt_i\ ))$**

Exercise

- Explain in English what the property means.
- Is this a desirable property of the protocol ?
- Suppose that the resource allocation protocol has a distributed implementation so that each process is implemented in a different site. Does the LTL property affect the communication overheads among the processes in any way ?

More Exercises (1)

- Express each of the following properties (stated in English) as an LTL formula. Assume that p, q and r are atomic propositions.
  - If p occurs, q never occurs in the future.
  - Always if p occurs, then eventually q occurs followed immediately by r.
  - Any occurrence of p is followed eventually by an occurrence of q. Furthermore, r never occurs between p and q.

More Exercises (2)

- Consider the LTL formula GFp and the CTL formula AGEFp where p is an atomic proposition. Give an example of a Kripke Structure which satisfies AGEFp but does not satisfy GFp. You may assume that p is the only atomic proposition for constructing the labeling function.

Satisfaction

- A CTL formula is satisfiable if some state of some Kripke structure satisfies it.
  - Otherwise unsatisfiable. Examples ??
  - Similarly for LTL formula .
- A CTL formula is valid if all states of all Kripke structures satisfy it.

Formula Equivalence

- Two temporal properties are equivalent iff they are satisified by exactly the same states of any Kripke structure.
  - EF p and E(true U p)
- Where does model checking stand ??
  - Is it checking for satisfiability of a temporal property ? Is it checking for validity ?

Model Checking

- … is not checking for satisifiability / validity.
- It is checking for satisfaction of a temporal property for a given Kripke structure.
  - This is a very different problem from traditional satisfiability checking !!
- We will discuss MC in next class.
  - But some warm up for now !
  - This is the more formal part of our discussion. Use it more for your own understanding of TL.

Fixed point characterizations

- An alternate semantic understanding of temporal formulae such as CTL properties.
- Yields a procedure for model checking these properties directly
  - Correct by construction model checking algorithm.

Intuition -- (1)

- Give semantics of CTL formulae by associating a formula with the states in a Kripke structure in which the formula will be true.
- A set of states drawn from a Kripke Structure forms a predicate.
- Define functions on sets of states
  - F: 2^S $\rightarrow$ 2^S
  - S is the set of all states drawn from Kripke structure
  - Such functions are called predicate transformers.

Intuition – (2)

- Define a CTL formula as the set of states obtained by
  - Repeated application of a predicate transformer
  - Starting from null set or set of all states.
  - Ending when one more application of the transformer does not change the result
    - Fixed point is reached.
  - For a predicate transformer, there can be several fixed-points. It is possible to show that for predicate transformers with certain properties

Intuition – (3)

  - Fixed point reached by starting from null set is the least fixed point
  - Fixed point reached by starting from set of all states is the greatest fixed point
- This gives a characterization of CTL formulae as least or greatest fixed points of transformers.
  - It also gives a computational mechanism for verifying these CTL formulae.
  - Given a Kripke structure M and a formula f, apply the predicate transformer for f until fixed point reached
  - Check whether the initial states of M are in the fixed point.

Predicates

- M = (S, S0, R, L)
  - Assume S is finite
  - R (transition relation)
  - L (Labeling function)
- $x \in 2^S$
  - x is a predicate over S
- Powerset 2^S comes with a natural partial order
  - Set inclusion

Predicate Transformers

- A function $f : 2^S \rightarrow 2^S$
  - Monotonic: $X \subseteq Y \Rightarrow f(X) \subseteq f(Y)$
  - Fixed point: $f(X) = X$
  - X, Y are subsets of S (set of states)
- Continuous functions
  - $f(X \cup Y) = f(X) \cup f(Y)$
  - $f(X \cap Y) = f(X) \cap f(Y)$

Example

- $S = \{s0, s1\}$
- $f : 2^S \rightarrow 2^S$
  - $f(X) = X \cup \{s0\}$
  - Show that f is monotonic.
  - What are the fixed points of f ?
- $f : 2^S \rightarrow 2^S$
  - Exercise: Give an example of non-monotone function.

Why such functions are imp ?

- For monotonic, continuous functions we can define
  - Least fixed point, greatest fixed point
  - $\text{lfp } f = \cap \{ x \mid f(x) = x \}$
  - $\text{gfp } f = \cup \{ x \mid f(x) = x\}$
  - Exercise: Convince yourself that the above definitions produce fixed points.

Why fixed points are imp ?

- Meaning of CTL operators can be expressed as lfp, gfp of monotone functions.
  - EG, EU, AG, AF, ….
- These lfp, gfp can be easily computed.
- Leads to a straightforward model checking algorithm for CTL formulae.

Fixed points

- If S is finite, then for a monotone continuous function $f : 2^S \rightarrow 2^S$
  - $\exists$ I $\in$nat lfp f = $f^I(\phi)$
  - $\exists$ I $\in$nat gfp f = $f^I(S)$
- Class Exercise:
  - Let us prove this theorem now.

LFP computation procedure

- Function lfp (f : PredicateTransformer): Predicate
- Begin
- Q := ϕ; // Null-set
- Q1 := f(Q);
- while (Q1 $\neq$ Q) do
- Q := Q1; Q1 := f(Q)
- endwhile;
- return(Q)
- End.

GFP computation procedure

- Function gfp (f : PredicateTransformer): Predicate
- Begin
- Q := S; // All states in the Kripke Structure
- Q1 := f(Q);
- while (Q1 $\neq$ Q) do
- Q := Q1; Q1 := f(Q)
- endwhile;
- return(Q)
- End.

Defining CTL operators

- M = (S, R, L) a finite Kripke structure
- φ is a CTL formula
- $[\varphi] \subseteq S$ denotes the set of states satisfying φ
- Now, define a predicate transformer
  - $f_\varphi(Y) = [\varphi] \cap \{s \mid \exists s' \; s \, R \, s' \text{ and } s' \in Y\}$

Defining EG

- $\{s \mid \exists s' \;\; s\,R\,s'$ and $s' \in Y\}$
  - Stands for [EX Y], the set of states in M which satisfy EX Y.
  - For convenience we will avoid the [...]
- $f_{\varphi}(Y) = \varphi \cap \boldsymbol{EX}\,Y$
  - Show that this transformer is monotonic.
  - You will need the definition of EX

Defining EG

- Show that EGφ is a fixed point of
  - $f_{\varphi}(Y) = \varphi \cap \boldsymbol{EX}\,Y$.
  - Any state satisfying EGφ satisfies φ and there is an outgoing state satisfying EGφ
  - The other direction of the proof …
- Show that EGφ is the gfp of
  - $f_{\varphi}(Y) = \varphi \cap \boldsymbol{EX}\,Y$
  - Any state in a fixed point of f_{φ} satisfies EG φ
    - Complete the proof exploiting this intuition.

Defining EU

- E(φ U ψ) is the least fixed point of
  - $f_{\varphi,\psi}(Y) = \psi \cup (\varphi \cap \text{EX } Y)$
- Cast the EU checking algo in terms of the LFP computation procedure outlined earlier
- Full discussion on CTL MC in next class.

Property Equivalences

- EG φ = $\varphi \wedge$ EXEGφ
- E(φ U ψ) = $\psi \vee (\varphi \wedge$ EX E(φ U ψ))
- We can derive similar equivalences using the fixed point characterization of other CTL properties.

Readings

- Chapter 3 of "Model Checking"
  - Clarke, Grumberg, Peled
  - See E-reserves of IVLE.
- Chapter 3 of Logic in Computer Science (Huth and Ryan) QA76.9 Log.Hu
  - RBR in Science Library
  - Chapter 3.9 contains discussion on fixed point characterizations.
- Other journal/survey articles covering this topic available from IVLE lesson plan.