

# **Pattern Avoidance in Permutations**

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*by*

**Abhilasha Sancheti & Kunal Jain**  
(130101083 & 130101042)

*under the guidance of*

**Prof. Benny George Kenkireth**



to the

**DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING  
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI  
GUWAHATI - 781039, ASSAM**

# CERTIFICATE

*This is to certify that the work contained in this thesis entitled “**Pattern Avoidance in Permutations**” is a bonafide work of **Abhilasha Sancheti (Roll No.130101083)** and **Kunal Jain (Roll No.130101042)**, carried out in the Department of Computer Science and Engineering, Indian Institute of Technology Guwahati under my supervision and that it has not been submitted elsewhere for a degree.*

Supervisor: **Prof. Benny George Kenkireth**

Assistant Professor,

November, 2018

Guwahati.

Department of Computer Science & Engineering,

Indian Institute of Technology Guwahati, Assam.

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# Abstract

Permutations have a vast variety of combinatorial structure. Permutation of a finite set can be represented in many equivalent ways such as a collection of cycles, function, matrix, etc. Each of these representations is outcome of some transformations or operations applied on a permutation.

These operations or transformations on permutations include descent set, excedance set, cycle type, subsequences, composition (product), etc. The main aim of the project is to look at the combinatorial aspects of pattern avoidance which is an active area of research.

# Chapter 1

## Introduction

### 1.1 Organization of The Report

This chapter provides a background for the topics covered in this report. This report is dedicated to the study of permutations, its properties and combinatorial aspect associated with it. We start with the definition of a permutation and notations that we are going to follow in the rest of the report. In Chapter 2 we describe the literature review that we have done. We start with permutations as a Linear Orders. Then describe some properties related to it. In the next section of the chapter we introduce the notion of pattern avoidance and some findings related to this area. Finally in Chapter 3 we introduce some classes of problems in the area of pattern avoidance and then define the problems that we are going to work on in the next semester.

### 1.2 Definitions

**Definition 1.2.1** *A permutation is defined as a linear ordering of the elements of the set  $[n] = \{1, 2, 3, 4, \dots, n\}$ . If it consists of  $n$  entries then it is also called an  $n$ -permutation.*

There are  $n!$  permutations possible for an  $n$ -permutation. This is because, consider  $p = p_1p_2p_3\dots p_n$  we have  $n$  choices for  $p_1$ ,  $n-1$  for  $p_2$  and similarly 1 for  $p_n$  thus a total of  $n!$ .

**Definition 1.2.2** For a permutation  $p = p_1p_2p_3\dots p_n$ , we say that a permutation  $p^r$  is reverse of  $p$  if  $p^r = p_np_{n-1}\dots p_1$

**Definition 1.2.3** For a permutation  $p = p_1p_2p_3\dots p_n$ , we say that a permutation  $p^c$  is complement of  $p$  if  $i^{th}$  entry of  $p^c$  is equal to  $n + 1 - p_i$ .

**Definition 1.2.4** Consider a sequence of positive real numbers as  $a_1, a_2, \dots, a_n$ . If an index  $i$  exists such that  $1 \leq i \leq n$ , and  $a_1 \leq a_2 \leq \dots \leq a_i \geq a_{i+1} \dots \geq a_n$  then we say that the sequence is unimodal.

**Definition 1.2.5** We say that the sequence of positive real numbers  $a_1, a_2, \dots, a_n$  is log-concave if  $a_{i-1}a_{i+1} \leq a_i^2$  is satisfied for all indices  $i$ .

If a sequence of positive real numbers is log-concave then it is also unimodal.

**Definition 1.2.6** Consider  $p = p_1p_2p_n$  to be a permutation. If  $p_i > i$  then we say that  $i$  is an excedance of  $p$ .

**Definition 1.2.7** Consider two sequences  $\phi = [a_1, a_2, \dots, a_m]$  and  $\chi = [b_1, b_2, \dots, b_m]$  of equal length  $m$ . They are said to be order isomorphic if,  $\forall i, j \quad a_i < a_j$  if and only if  $b_i < b_j$

Formally, we say that  $\phi$  is order isomorphic to  $\chi$  if and only if  $\forall i, j \quad a_i < a_j \Leftrightarrow b_i < b_j$

*Result:* Every sequence  $\phi$  is order isomorphic to a unique permutation of length  $m$  of  $1, 2, \dots, m$ .

Throughout our study we focus our work on permutations of  $[n]$  as can be seen from the result above, each sequence maps uniquely to these permutations.

**Definition 1.2.8** Consider two sequences  $\phi = [a_1, a_2, \dots, a_m]$  and  $\chi = [b_1, b_2, \dots, b_n]$  of length  $m$  and  $n$  respectively. We say  $\phi$  is involved in  $\chi$ , if there exists a subsequence of  $\chi$  which is order isomorphic to  $\phi$ . We denote sequence involvement with  $\preceq$ .



*Example:*  $\Phi = [3, 1, 4, 2, 5]$   $\chi = [7, 2, 1, 5, 8, 3, 6, 9, 4]$   $\Phi \preceq \chi$  as subsequence  $[7, 5, 8, 6, 9]$  of  $\chi$  is order isomorphic to  $\Phi$ .

**Definition 1.2.9** *It is NP complete to decide that for given two permutations  $\phi$  and  $\chi$ , if they are involved.*

# Chapter 2

## Review of Prior Work

This chapter discusses about permutations and its combinatorial aspects that we have been studying. We describe below some properties of permutations, enumerate specific type of patterns and present the consequent results.

### 2.1 Permutations as Linear Orders

Increasing sequence is the most ordered one as an entry is always followed by an entry greater than it. We are interested in the disorders in the permutations such as when an entry in permutation is followed subsequently by an entry which is smaller in ordered criterion. [Bón04]

#### 2.1.1 Descents

**Definition 2.1.1** *Consider a permutation  $p = p_1p_2p_3\dots p_n$ . If  $p_i > p_{i+1}$ , then we say that  $i$  is the descent of  $p$  and if  $p_i < p_{i+1}$ , then we say that  $i$  is the ascent of  $p$ .*

The set which contains all the descents of a permutation  $p$  is called the *descent set of  $p$* . We denote it as  $D(p)$ . We are interested in combinatorics related to this property. The following are some results:

1. Number of permutations whose descent set is contained in  $A$ .

Consider  $A = a_1, a_2, a_3 \dots a_k \subseteq [n-1]$  and  $\alpha(A)$  to be the number of  $n$ -permutations whose descent set is contained in  $A$ , then

$$\alpha(A) = \binom{n}{a_1} \binom{n-a_1}{a_2-a_1} \binom{n-a_2}{a_3-a_2} \dots \binom{n-a_k}{n-a_k}$$

2. Number of permutations with a descent set  $A$ . [Bón11]

Consider  $A \subseteq [n-1]$ . We denote the number of permutations with descent set  $A$  by  $\beta(A)$ . Thus by using the inclusion and exclusion principle we can say that

$$\beta(A) = \sum_{T \subseteq A} (-1)^{|A-T|} \alpha(A)$$

3. Number of permutations with a given number of descents.

We define  $S(n, k)$  as the number of  $n$ -permutations with  $k-1$  descents.  $S(n, k)$  is called the Eulerian number. The explicit formula for Eulerian numbers, for all non-negative integers  $n, k$  which satisfies the condition that  $k \leq n$ , is given by:

$$S(n, k) = \sum_{i=0}^k (-1)^i \binom{n+1}{i} (k-i)^n$$

Eulerian numbers follow a recurrence relation. The recurrence formula for all positive integers  $k$  and  $n$  which satisfies the condition that  $k \leq n$  is given by:

$$S(n, k+1) = (k+1)S(n-1, k+1) + (n-k)S(n-1, k)$$

Properties of Eulerian numbers:

For any positive integer  $n$   $S(n, k)$  is log-concave.

For any fixed  $n$ ,  $S(n, k)$  has real roots only.

The number of  $n$ -permutations with  $k-1$  excedances is  $S(n, k)$ .

### 2.1.2 Runs

**Definition 2.1.2** Consider  $p = p_1p_2.....p_n$  to be a permutation  $p$ . By definition of descents, between each descent there is an increasing sequence of consecutive elements. Thus if sequence has  $d - 1$  descents, it will have  $d$  increasing subsequences. We call them as the ascending runs of  $p$ .

The notion of ascending runs can be extended to alternating runs.

**Definition 2.1.3** Consider  $p = p_1p_2.....p_n$  to be a permutation. If either  $p_{i-1} < p_i > p_{i+1}$  , or  $p_{i-1} > p_i < p_{i+1}$  then  $p$  is said to change direction at  $i$ . Alternatively  $p$  changes direction if there is a peak or a valley.

The number of  $n$ -permutations with  $k$  alternating runs is denoted as  $H(n,k)$ . As Eulerian numbers  $S(n,k)$ ,  $H(n,k)$  also has real roots only and follows a similar recurrence relation given below: Consider  $H(1, 0)=1$ , and  $H(1, k)=0$  for  $k > 0$ . For positive integers  $n$  and  $k$

$$H(n, k) = kH(n - 1, k) + 2H(n - 1, k - 1) + (n - k)H(n - 1, k - 2)$$

### 2.1.3 Inversions

In the last section we discussed descents of permutations where an entry was greater than the entry directly following it. Now we will look at inversions where an entry is followed by some entry smaller than it.

**Definition 2.1.4** Consider a permutation  $p = p_1p_2.....p_n$ . We say that  $(p_i, p_j)$  is an inversion of  $p$  if  $i < j$  but  $p_i > p_j$ .

We conventionally denote the number of inversions of  $p$  by  $i(p)$  and the size of set of  $n$  sized permutations which have  $k$  inversions by  $c(n,k)$ . For a  $n$ -permutation  $i(p)$  can be at most  $\binom{n}{2}$ . For any fixed size permutation  $n$ , the sequence  $c(n,0), c(n,1), ..... c(n, \binom{n}{2})$  is

log-concave.

Similar to the previous two numbers  $S(n,k)$  and  $H(n,k)$ ,  $c(n,k)$  also follows a recurrence relation given by: For  $n \geq k$ ,

$$c(n+1, k) = c(n+1, k-1) + c(n, k).$$

**Definition 2.1.5** *Consider a permutation  $p = p_1 p_2 \dots p_n$ . The sum of the descents of  $p$  is called as the major index or greater major index. We denote it as  $maj(p)$ . That is,*

$$maj(p) = \sum_{i \in D(p)} i$$

*Result:* For all non-negative integers  $k$  and all positive integers  $n$ , the size of set of  $n$  sized permutations with  $k$  inversions is same as the size of set of  $n$  sized permutations with major index  $k$ .

## 2.2 Pattern Avoidance

In the previous section we discussed about inversions where a pair of entries of the permutation were related to each other in some manner and can be present anywhere in it. The more general notion of this relationship between pair of entries could be extended to  $k$ -tuples of entries [Bón04].

We say a pattern  $q$  is present in a permutation  $p$  (longer or equal to length of  $q$ ) if any subsequence of  $p$  of length same as that of  $q$  can be reduced to  $q$ . Consider a permutation  $p=241563$ , and  $q=213$ . We can then say that 3-tuple  $(4,1,5)$  of  $p$  forms a pattern of type  $(2,1,3)$  because  $(4,1,5)$  can be reduced to  $(2,1,3)$  by mapping 4 to 2, 1 to 1 and 5 to 3 respectively.

We say that  $p$  avoids a pattern  $q$  if none of the subsequence of  $p$  can be reduced to  $q$ . Formally, we can define pattern avoidance as follows:

**Definition 2.2.1** Consider  $q = (q_1, q_2, \dots, q_k) \in A_k$  to be a permutation, and let  $k \leq n$ . We say that the permutation  $p = (p_1, p_2, \dots, p_n) \in A_n$  is  $q$ -avoiding if and only if there is no  $1 \leq i_{q_1} < i_{q_2} < \dots < i_{q_k} \leq n$  such that  $p(i_1) < p(i_2) < \dots < p(i_k)$ .

Alternatively, we can say that  $A_n$  is  $A_q$  avoiding if there exists no subsequence of  $A_n$  which is order isomorphic to  $A_q$ . Clearly if  $A_n$  avoids  $A_q$ ,  $A_n$  and  $A_q$  are not involved.

Example: Consider  $A_3 = [1, 3, 2]$  and  $A_5 = [1, 2, 3, 4, 5]$

$A_5$  is  $A_2$  avoiding as there exists no subsequence of  $A_5$  which is order isomorphic to  $A_2$ . So far we have counted number of  $n$ -permutations with a given number of inversions. Now we are interested in finding  $A_n(q)$  that, is number of  $q$  pattern avoiding  $n$ -permutations.

We have  $A_n(12) = A_n(21) = 1$ , so the first case that is non-trivial and need attention is of patterns of length three. There are in all  $3!$  possible permutations of length three. So we have six such patterns, but there are many symmetries between them which we will see further. Recall that for a permutation  $p = p_1 p_2 p_3 \dots p_n$ , the reverse of  $p$  is defined as the permutation  $p^r = p_n p_{n-1} p_{n-2} \dots p_1$ , and the  $p$  permutations' complement  $p^c$  whose  $i^{th}$  entry is  $n + 1 - p_i$ . It can be seen very easily that 123 avoiding permutations' reverse avoids 321 pattern, thus  $A_n(123) = A_n(321)$ . Similarly, 132 avoiding permutations' reverse avoids 231, its complement avoids 312, and the complements' reverse avoids 213. Similarly we also have  $A_n(132) = A_n(231) = A_n(312) = A_n(213)$ . It can be shown that  $A_n(123) = A_n(132)$  [SS85] and thus all the length three permutations are avoided by same count of  $n$  sized permutations. We will also enumerate the permutations avoiding the combinations of patterns of length three. The following theorem by will be helpful in the enumeration.

**Theorem: 1** *If there is a sequence of  $n^2 + 1$  numbers, then there is either a monotonically increasing subsequence of  $n+1$  numbers or a monotonic decreasing subsequence of  $n+1$  numbers.*

There are a total of  $2^6 = 64$  possible combinations including the empty set. We denote  $T$  as

the combination to be avoided. We will discuss the enumeration of permutations avoiding some of these combination below:

1.  $T = \{123\}$ . The decreasing permutation is the only one that avoids 123 because in all the other permutations we can find  $i, j$  and  $k$  such that  $p_i < p_j < p_k$ . Thus  $A_n(T) = 1$ . As we have previously shown that the number of permutations avoiding the singleton sets is same. So,  $A_n(\{123\}) = A_n(\{132\}) = A_n(\{213\}) = A_n(\{231\}) = A_n(\{312\}) = A_n(\{321\}) = 1$ .
2.  $T = \{123, 132, 213, 231, 312, 321\}$ . By Theorem 1, it is obvious that for permutations of length greater than nine  $A_n(T) = 0$ .
3.  $T = \{123, 132, 213, 231, 312\}$ . The only permutation avoiding this set is the decreasing sequence. So  $A_n(T) = 1$ .
4.  $T = \{123, 213, 231, 312, 321\}$ . By the same argument as in case 2  $A_n(T) = 0$ . Similarly for  $T = \{123, 132, 231, 312, 321\}$ ,  $T = \{123, 132, 213, 312, 321\}$ ,  $T = \{123, 132, 213, 231, 321\}$ ,  $T = \{132, 213, 231, 312, 321\}$ ,  $A_n(T) = 0$ .
5.  $T = \{123, 132, 213, 231\}$ . The only patterns allowed in the permutation are 312 and 321 so the for each triplet the first entry has to be maximum of the rest two thus permutation  $p$  should be such that  $p_1 > p_2 > p_3 > \dots > p_{n-2}$  and last two positions could be anything. Thus there are only two possible permutations. So,  $A_n(T) = 2$ .
6.  $T = \{123, 132, 213, 312\}$ . The allowed patterns are 231 and 321. Thus similar to the above reason  $p$  should be such that  $p_n < p_{n-1} < p_{n-2} < \dots < p_3$  and first two entries could be any of the largest two. So  $A_n(T) = 2$ .
7.  $T = \{123, 132, 231, 312\}$ . The allowed patterns are 213 and 321. Let us consider four elements  $A, B, C$  and  $D$  such that  $A < B < C < D$ . Now look at the possibilities at each position  $a_{i+t}$  for  $t=0, 1, 2, 3$ . We observe that only  $D$  and  $C$  can come at  $a_i$ ,  $C$

and B at  $a_{i+1}$ , A and B at  $a_{i+2}$ , and D and A at  $a_{i+3}$ . Also see that if element at any one of the position is fixed then rest all are also fixed. Thus we have  $A_n(T) = 2$ .

8.  $T = \{123, 213, 231, 312\}$ . With the same argument as above we observe that only A and D can come at  $a_i$ , C and D at  $a_{i+1}$ , B and C at  $a_{i+2}$ , and A and B at  $a_{i+3}$ . Also see that if element at any one of the position is fixed then rest all are also fixed. Thus we have  $A_n(T) = 2$ .

9.  $T = \{132, 213, 231, 312\}$ . The only allowed patterns are 123 and 321. So the possible permutations are increasing and decreasing sequences. Thus  $A_n(T) = 2$ .

10. For  $T = \{132, 213, 231, 321\}$ ,  $T = \{132, 213, 231, 321\}$  and  $T = \{213, 231, 312, 321\}$ ,  $A_n(T) = 2$  because of the similar argument as in the above cases.

11. Similarly for combinations of two and three the results are mentioned in [SS85]

A general formula for the number of three length pattern avoiding permutations  $A_n(q)$  is given by Catalan numbers  $C_n$

$$A_n(q) = C_n = \frac{\binom{2n}{n}}{n+1}$$

## 2.3 Conclusion

This chapter provided details of some of the properties related to permutations. We also enumerated the permutations with given number of descents, alternating runs and inversions. It also presents some of the existing works in the field of pattern avoidance. The exact enumeration for the avoidance of three length patterns avoiding permutations were also discussed.



# Chapter 3

## Problem Classes and Future Work

The pattern avoidance of length 3 in permutations is a much studied topic. However there are not many results for pattern avoidance of length greater than three and pattern avoidance in compositions. In this chapter we define some of the open problems in pattern avoidance, briefly discuss them and describe our proposed future work.

### 3.1 Problems

#### 3.1.1 Pattern Avoidance of one pattern of Length 4

There are  $4!$  permutations of  $P = [1,2,3,4]$ .

**Table 3.1** Classes avoiding one pattern of length 4

Pattern	Enumeration Sequence	Type	Exact Enumeration
[1, 3, 4, 2] [2, 4, 1, 3]	1, 1, 2, 6, 23, 103, 512, 2740, 15485, 91245, 555662, 3475090, ...	Not RationalAlgebraic	Miklos Bona(1997) [Bón97]
[1, 2, 3, 4] [1, 2, 4, 3] [1, 4, 3, 2] [2, 1, 4, 3]	1, 1, 2, 6, 23, 103, 513, 2761, 15767, 94359, 586590, 3763290, 24792705, 167078577, ...	Holonomic	Ira M. Gessel(1990) [Ges90]
[1, 3, 2, 4]	1, 1, 2, 6, 23, 103, 513, 2762, 15793, 94776, 591950, ...	No Known Form	No exact enumeration

There is no exact enumeration of pattern avoidance of class 1324-avoiding permutations.

Marinov, RadoiiRado in their paper published in 2013 [MR03] provide a recursive formula, which was upper bounded by Bona in 2015 [Bón15] and lower bounded by Bevan in 2015 [Bev15].

A tighter bound and closed formula is an open research topic.

### 3.1.2 Pattern Avoidance of two patterns of Length Four

There constitute 56 symmetry classes out of which 8 remain to be enumerated.

**Table 3.2** Classes avoiding two pattern of length 4

Permutation	Results
[4, 2, 3, 1] [4, 1, 2, 3]	Conjectured to not satisfy any closed form
[4, 1, 2, 3] [3, 4, 1, 2]	No Results
[4, 3, 1, 2] [4, 1, 2, 3]	Conjectured to not satisfy any closed form
[3, 4, 1, 2] [2, 4, 1, 3]	No Results
[4, 3, 2, 1] [4, 2, 3, 1]	Conjectured to not satisfy any closed form

Lower and Upper bound and an exact formula recursive or non recursive is an open research topic.

### 3.1.3 Classes which contain some pattern and avoid some pattern

We are interested in those classes which avoid some pattern with an extra condition that they must contain some pattern.

For example: We are interested in studying in all 1234 permutations which avoid 1324.

This topic is very sparsely studied and answers many interesting questions like all patterns which contain increasing sub-sequences of 4, which doesn't contain any decreasing sub sequence of length 4.

The results of this class is a strict subset of class that avoid pattern, and there relation with those classes will be an interesting topic.

### 3.1.4 Classes with permutations with restricted patterns

We are interested in those classes of permutations which contains some patterns but restricted number of times.

For example: We are interested in studying all permutations which contain 213 pattern exactly  $r$  times.

A more restricted form of this class (subarray with 213 pattern) forms a interesting questions in high school mathematics. Krattenthaler published an interesting paper in 2000 [Kra01] in which he established a precise asymptotic estimate in number of permutations which avoid 1,3,2 pattern with contain  $d$  occurrences of the pattern 1,2,3,...,k.

### 3.1.5 Classes which avoid composite patterns

We are interested in those classes of permutations which avoid 2 or more patterns simultaneously. These patterns can be of same length or varying length.

For example: We are interested in studying in all permutation which avoid 1234 and 132 patterns. The result of this is an interesting result of alternate Fibonacci numbers. That is for  $n$  size permutations answer is  $F(2n - 1)$  where  $F(n)$  is defined to be the  $n$ th Fibonacci number.

For two pattern avoidance with either length 3 and length 4 or both, is a much studied topic and all the results are enumerated for two pattern avoidance of length 3, one pattern of length 3 and one pattern of length 4, and as discussed in above class 8 classes of two pattern avoidance of length 4 remain to be enumerated.

However composite pattern avoidance with more than two patterns is very sparsely studied and the results are very interesting for the few classes which are studied.

## 3.2 Conclusion and Future Work

We are interested in working on these 5 classes of problem that are described above.

1. The missing exact numeration of 1324 pattern avoidance is a much studied problem, and our interest is to tighter bound the current result.
2. The second class of problem has two classes for which there are no results in terms of bounds or recursive structure. Our interest is to get a bound on these results.
3. The third and fourth class of problems can be generalized to a more common class of permutations which avoid a certain function.
4. The fifth class of problem is our main interest. There are some interesting analogies of these problems with the existing mathematical problems. We will start our work with this problem, mainly looking into avoid 3 or more composite patterns of length 3 and 4.

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