7.1 Consider the univariate function

$$f(x) = x^3 + 6x^2 - 3x - 5.$$

Find its stationary points and indicate whether they are maximum, minimum, or saddle points.

Solution.

To find stationary points, we calculate the derivative and set it to 0, and solve for x:

$$\frac{df(x)}{dx} = 3x^2 + 12x - 3 = 0$$

$$\implies x^2 + 4x - 1 = 0$$

$$\implies x = \frac{-4 \pm \sqrt{16 - 4 * 1 * (-1)}}{2 * 1}$$

$$\implies x = \frac{-4 \pm \sqrt{20}}{2}$$

$$\implies x = -2 \pm \sqrt{5}$$

To find whether they are maximum, minimum, or saddle points, we calculate the second derivative, and substitute the values of x:

$$\frac{d^2 f(x)}{dx^2} = 6x + 12$$

$$\text{At } -2 - \sqrt{5}, \frac{d^2 f(x)}{dx^2} = 6(-2 - \sqrt{5}) + 12 = -6\sqrt{5}.$$

Since the second derivative is negative,$x = -2 - \sqrt{5}$ is a maxima.

$$\text{At } -2+\sqrt{5}, \frac{d^2 f(x)}{dx^2} = 6(-2+\sqrt{5}) + 12 = 6\sqrt{5}.$$

Since the second derivative is positive, $x = -2 + \sqrt{5}$ is a minima.