4.8 Find the SVD of the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

Solution.

To obtain SVD, we need to compute the right singular vectors $\mathbf{v}_j$, the singular values σ_k, and the left-singular vectors $\mathbf{u}_i$.

Step 1: Right Singular vectors as the eigenbasis of $\mathbf{A}^T\mathbf{A}$

We start by computing

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

We compute the singular values and right-singular vectors $\mathbf{v}_j$ through the eigenvalue decomposition of $\mathbf{A}^T\mathbf{A}$. To calculate eigenvalues of $\mathbf{A}^T\mathbf{A}$, we set

$$det(\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I}) = 0$$

$$\implies \begin{vmatrix} 13-\lambda & 12 & 2 \\ 12 & 13-\lambda & -2 \\ 2 & -2 & 8-\lambda \end{vmatrix} = 0$$

Using Sarrus rule, we get characteristic polynomial and obtain eigenvalues,

$\implies ((13-\lambda)*(13-\lambda)*(8-\lambda)) + (12*(-2)*2) + (2*12*(-2)) - (2*(13-\lambda)*2) - ((-2)*(-2)*(13-\lambda)) - (8-\lambda)*12*12) = 0$

$\implies (-\lambda^3 + 34\lambda^2 - 377\lambda + 1352) + (-48) + (-48) - (-4\lambda + 52) - (-4\lambda +$

$52) - (-144\lambda + 1152) = 0$

$$\implies -\lambda^3 + 34\lambda^2 - 225\lambda = 0$$

$$\implies (\lambda - 9)(\lambda - 25)\lambda = 0$$

$$\implies \lambda = 9, 25, 0$$

Now, to obtain eigenvectors, we solve for $(\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I}_3)\mathbf{x} = 0$

For $\lambda = 9$,

$$(\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I}_3)\mathbf{x} = 0$$

$$\implies (\mathbf{A}^T\mathbf{A} - 9\mathbf{I}_3)\mathbf{x} = 0$$

$$\implies \begin{bmatrix} 13-9 & 12 & 2 \\ 12 & 13-9 & -2 \\ 2 & -2 & 8-9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To solve this, we convert the following augmented matrix into Reduced row echelon form

$$\left[\begin{array}{ccc|c} 4 & 12 & 2 & 0 \\ 12 & 4 & -2 & 0 \\ 2 & -2 & -1 & 0 \end{array}\right] \begin{array}{l} \cdot\frac{1}{2} \\ \cdot\frac{1}{2} \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 2 & 6 & 1 & 0 \\ 6 & 2 & -1 & 0 \\ 2 & -2 & -1 & 0 \end{array}\right] \begin{array}{l} \\ -3R_1 \\ -R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 2 & 6 & 1 & 0 \\ 0 & -16 & -4 & 0 \\ 0 & -8 & -2 & 0 \end{array}\right] \begin{array}{l} \cdot\frac{1}{2} \\ \cdot-\frac{1}{2} \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 3 & \frac{1}{2} & 0 \\ 0 & -8 & -2 & 0 \\ 0 & -8 & -2 & 0 \end{array}\right] \begin{array}{l} \\ \\ -R_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 3 & \frac{1}{2} & 0 \\ 0 & -8 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \cdot-\frac{1}{8} \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 3 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} -3R_2 \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Longrightarrow x_1 - \frac{x_3}{4} = 0, x_2 + \frac{x_3}{4} = 0$$

$$\implies x_1 = \frac{x_3}{4}, x_2 = -\frac{x_3}{4}, x_3 = x_3$$

$$\text{Eigenvector is } \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

$$\text{Normalized form : } \begin{bmatrix} \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \\ \frac{2\sqrt{2}}{3} \end{bmatrix}$$

For $\lambda = 25$,

$$(\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I}_3)\mathbf{x} = 0$$

$$\implies (\mathbf{A}^T\mathbf{A} - 25\mathbf{I}_3)\mathbf{x} = 0$$

$$\implies \begin{bmatrix} 13-25 & 12 & 2 \\ 12 & 13-25 & -2 \\ 2 & -2 & 8-25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To solve this, we convert the following augmented matrix into Reduced row echelon form

$$\left[\begin{array}{ccc|c} -12 & 12 & 2 & 0 \\ 12 & -12 & -2 & 0 \\ 2 & -2 & -17 & 0 \end{array}\right] \begin{array}{l} \\ +R_1 \\ \\ \end{array}$$

$$\Longrightarrow \left[\begin{array}{ccc|c} -12 & 12 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & -2 & -17 & 0 \end{array}\right] \begin{array}{l} \cdot -\frac{1}{12} \\ \\ \cdot\frac{1}{2} \end{array}$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & -\frac{17}{2} & 0 \end{array}\right] \begin{array}{l} \\ \\ -R_1 \end{array}$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{50}{6} & 0 \end{array}\right] \begin{array}{l} \\ \\ \cdot -\frac{6}{50} \end{array}$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} +\frac{1}{6}R_3 \\ \\ \\ \end{array}$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Longrightarrow x_1 - x_2 = 0, x_3 = 0$$

$$\Longrightarrow x_1 = x_2, x_3 = 0$$

$$\text{Eigenvector is } \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{Normalized form : } \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

For $\lambda = 0$,

$$(\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I}_3)\mathbf{x} = 0$$

$$\implies (\mathbf{A}^T\mathbf{A} - 0\mathbf{I}_3)\mathbf{x} = 0$$

$$\implies \begin{bmatrix} 13-0 & 12 & 2 \\ 12 & 13-0 & -2 \\ 2 & -2 & 8-0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To solve this, we convert the following augmented matrix into Reduced row echelon form

$$\left[\begin{array}{ccc|c} 13 & 12 & 2 & 0 \\ 12 & 13 & -2 & 0 \\ 2 & -2 & 8 & 0 \end{array}\right] \begin{array}{c} -R_2 \\ \\ \cdot\frac{1}{2} \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 12 & 13 & -2 & 0 \\ 1 & -1 & 4 & 0 \end{array}\right] \begin{array}{c} \\ -12R_1 \\ -R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & -4 & 0 \\ 0 & 25 & -50 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ \cdot\frac{1}{25} \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} +R_2 \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\implies x_1 + 2x_3 = 0, x_2 - 2x_3 = 0$$

$$\implies x_1 = -2x_3, x_2 = 2x_3, x_3 = x_3$$

$$\text{Eigenvalue is : } \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{In normalized form : } \begin{bmatrix} \frac{-2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\text{Therefore, } \mathbf{V} = \begin{bmatrix} \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{-2}{3} \\ -\frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{2\sqrt{2}}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

Step 2: Singular-value matrix

Singular values are square roots of the eigenvalues of $\mathbf{A}^T\mathbf{A}$. The only nonzero singular values are $\sigma_1 = \sqrt{9} = 3$, and $\sigma_2 = \sqrt{25} = 5$.

$$\text{Therefore, } \mathbf{\Sigma} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$$

Step 3: Left-singular vectors as the normalized image of the right-singular vectors

$$\mathbf{u}_1 = \frac{1}{\sigma_1}\mathbf{A}\mathbf{v}_1 = \frac{1}{3}\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}\begin{bmatrix} \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \\ \frac{2\sqrt{2}}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\sigma_2}\mathbf{A}\mathbf{v}_2 = \frac{1}{5}\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Longrightarrow \mathbf{U} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Longrightarrow \mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{-2}{3} \\ -\frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{2\sqrt{2}}{3} & 0 & \frac{1}{3} \end{bmatrix}^T$$