4.3 Compute the eigenspaces of

a.

$$\mathbf{A} := \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Solution.

We set $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$$\implies \begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix}$$

$$\implies (1-\lambda)^2 = 0$$

The only eigenvalue is 1.

To obtain eigenvectors, we solve the following for $\mathbf{x}$:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$$

$$\implies (\mathbf{A} - 1 \cdot \mathbf{I})\mathbf{x} = 0$$

$$\implies (\mathbf{A} - \mathbf{I})\mathbf{x} = 0$$

$$\implies \begin{bmatrix} 1-1 & 0-0 \\ 1-0 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies x_1 = 0, x_2 = x_2$$

Therefore, Eigenspace $E_1 = \text{Span}\ [\begin{bmatrix} 0 \\ 1 \end{bmatrix}]$

b.

$$\mathbf{A} \coloneqq \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$$

Solution.

We set $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$$\implies \begin{vmatrix} -2-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}$$

$$\implies (-2-\lambda)(1-\lambda) - 4 = 0$$

$$\implies \lambda^2 + \lambda - 2 - 4 = 0$$

$$\implies \lambda^2 + \lambda - 6 = 0$$

$$\implies (\lambda + 3)(\lambda - 2) = 0$$

$$\implies \lambda = -3, 2$$

To obtain eigenvectors, we solve the following for $\mathbf{x}$:

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$$

For $\lambda = -3$,

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$$

$$\implies (\mathbf{A} + 3 \cdot \mathbf{I})\mathbf{x} = 0$$

$$\implies \begin{bmatrix} -2+3 & 2 \\ 2 & 1+3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies 2 \cdot \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies x_1 + 2x_2 = 0 \implies x_1 = -2x_2$$

$$\implies E_{-3} = \text{Span}\ [\begin{bmatrix} -2 \\ 1 \end{bmatrix}]$$

For $\lambda = 2$,

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$$

$$\implies (\mathbf{A} - 2 \cdot \mathbf{I})\mathbf{x} = 0$$

$$\implies \begin{bmatrix} -2-2 & 2 \\ 2 & 1-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies -2 \cdot \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies 2x_1 - x_2 = 0 \implies x_2 = 2x_1$$

$$\implies E_2 = \text{Span}\left[\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right]$$