2.13 Consider two subspaces, U_1 and U_2, where U_1 is the solution space of the homogeneous equation system $\mathbf{A}_1\mathbf{x} = \mathbf{0}$ and U_2 is the solution space of the homogeneous equation system $\mathbf{A}_2\mathbf{x} = \mathbf{0}$ with

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \qquad \mathbf{A}_2 = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix}.$$

a. Determine the dimension of U_1, U_2.

Solution.

To obtain U_1, we apply Gauss-Jordan reduction to obtain reduced row echelon form of $[\mathbf{A}_1|\mathbf{0}]$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & -2 & -1 & 0 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ -R_1 \\ -2R_1 \\ -R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \cdot -\frac{1}{2} \\ +\frac{1}{2}R_2 \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\implies x_1 + x_3 = 0, x_2 + x_3 = 0, x_3 = x_3$$

$$\implies x_1 = -x_3 = 0, x_2 = -x_3 = 0, x_3 = x_3$$

$$\implies \mathbf{x} = \lambda \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad |\lambda \in \mathbb{R}$$

$$\implies dim(U_1) = 1$$

To obtain U_2, we apply Gauss-Jordan reduction to obtain reduced row echelon form of $[\mathbf{A}_2|\mathbf{0}]$

$$\left[\begin{array}{ccc|c} 3 & -3 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 7 & -5 & 2 & 0 \\ 3 & -1 & 2 & 0 \end{array}\right] \quad \begin{array}{c} \cdot\frac{1}{3} \\ \\ -7R_2 \\ -R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & -19 & -19 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right] \begin{array}{c} \\ -R_1 \\ \cdot -\frac{1}{19} \\ \cdot\frac{1}{2} \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{c} \\ \cdot\frac{1}{3} \\ -\frac{1}{3}R_2 \\ -\frac{1}{3}R_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} +R_2 \\ \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is the same result as we got for U_1.

$$\Longrightarrow \mathbf{x} = \lambda \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad |\lambda \in \mathbb{R}$$

$$\implies dim(U_2) = 1$$

b. Determine bases of U_1 and U_2.

Solution.

$$\text{Basis of } U_1 = \text{ Basis of } U_2 = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

c. Determine a basis of $U_1 \cap U_2$.

Solution.

Both U_1 and U_2 are spanned by the same basis vector.

Therefore, $U_1 \cap U_2 = \text{span }[\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}]$