3.5 Consider the Euclidean vector space $\mathbb{R}^5$ with the dot product. A subspace $U \subseteq \mathbb{R}^5$ and $\mathbf{x} \in \mathbb{R}^5$ are given by

$$U = span[\begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 5 \\ 0 \\ 7 \end{bmatrix}], \qquad \mathbf{x} = \begin{bmatrix} -1 \\ -9 \\ -1 \\ 4 \\ 1 \end{bmatrix}$$

a. Determine the orthogonal projection π_U of $\mathbf{x}$ onto U

Solution.

First, we have to find the basis of π_U by performing Gauss-Jordan elimination on

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ -1 & -3 & 4 & -3 \\ 2 & 1 & 1 & 5 \\ 0 & -1 & 2 & 0 \\ 2 & 2 & 1 & 7 \end{bmatrix} \begin{array}{l} \\ \\ +2R_2 \\ \\ +2R_2 \end{array}$$

$$\rightsquigarrow \begin{bmatrix} 0 & 1 & -3 & -1 \\ -1 & -3 & 4 & -3 \\ 0 & -5 & 9 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -4 & 9 & 1 \end{bmatrix} \begin{array}{l} \\ +3R_1 \\ +5R_1 \\ +R_1 \\ +4R_1 \end{array}$$

$$
\rightsquigarrow \begin{bmatrix} 0 & 1 & -3 & -1 \\ -1 & 0 & -5 & -6 \\ 0 & 0 & -6 & -6 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -3 & -3 \end{bmatrix} \begin{array}{l} \\ \cdot -1 \\ \cdot -\frac{1}{6} \\ \cdot -1 \\ \cdot -\frac{1}{3} \end{array}
$$

$$
\rightsquigarrow \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 5 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{array}{l} +3R_3 \\ -5R_3 \\ \\ -R_3 \\ -R_3 \end{array}
$$

$$
\rightsquigarrow \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \text{Swap with } R_2 \\ \text{Swap with } R_1 \\ \\ \\ \\ \end{array}
$$

$$
\rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \text{Swap with } R_2 \\ \text{Swap with } R_1 \\ \\ \\ \\ \end{array}
$$

The first three columns are pivot columns, so they form a basis of U.

Now let $\mathbf{B} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & -3 & 4 \\ 2 & 1 & 1 \\ 0 & -1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

We know that $\pi_U(\mathbf{x}) = \mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{x}$

$$\mathbf{B}^T\mathbf{B} = \begin{bmatrix} 0 & -1 & 2 & 0 & 2 \\ 1 & -3 & 1 & -1 & 2 \\ -3 & 4 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -3 \\ -1 & -3 & 4 \\ 2 & 1 & 1 \\ 0 & -1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 9 & 0 \\ 9 & 16 & -14 \\ 0 & -14 & 31 \end{bmatrix}$$

$$(\mathbf{B}^T\mathbf{B})^{-1} = \begin{bmatrix} 100/63 & -31/21 & -2/3 \\ -31/21 & 31/21 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

$$\mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & -3 & 4 \\ 2 & 1 & 1 \\ 0 & -1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 100/63 & -31/21 & -2/3 \\ -31/21 & 31/21 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 11/2 & -11/2 & -1/3 \\ 11/63 & -2/7 & 0 \\ 65/63 & -17/21 & -1/3 \\ 1/7 & -1/7 & 0 \\ -4/9 & 2/3 & 1/3 \end{bmatrix}$$

$$\mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T = \begin{bmatrix} 11/2 & -11/2 & -1/3 \\ 11/63 & -2/7 & 0 \\ 65/63 & -17/21 & -1/3 \\ 1/7 & -1/7 & 0 \\ -4/9 & 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & -1 & 2 & 0 & 2 \\ 1 & -3 & 1 & -1 & 2 \\ -3 & 4 & 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10/21 & -2/7 & 4/21 & -1/7 & -1/3 \\ -2/7 & 43/63 & 4/63 & 2/7 & -2/9 \\ 4/21 & 4/63 & 58/63 & 1/7 & 1/9 \\ -1/7 & 2/7 & 1/7 & 1/7 & 0 \\ -1/3 & -2/9 & 1/9 & 0 & 7/9 \end{bmatrix}$$

$$\text{Projection} = \pi_U(\mathbf{x}) = \mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{x}$$

$$= \begin{bmatrix} 10/21 & -2/7 & 4/21 & -1/7 & -1/3 \\ -2/7 & 43/63 & 4/63 & 2/7 & -2/9 \\ 4/21 & 4/63 & 58/63 & 1/7 & 1/9 \\ -1/7 & 2/7 & 1/7 & 1/7 & 0 \\ -1/3 & -2/9 & 1/9 & 0 & 7/9 \end{bmatrix} \begin{bmatrix} -1 \\ -9 \\ -1 \\ 4 \\ 1 \end{bmatrix}$$

$$\text{Projection} = \begin{bmatrix} 1 \\ -5 \\ -1 \\ -2 \\ 3 \end{bmatrix}$$

b. Determine the distance $d(\mathbf{x}, U)$

Solution.

$$d(\mathbf{x}, U) = \|\mathbf{x} - \pi_U(\mathbf{x})\|$$

$$\mathbf{x} - \pi_U(\mathbf{x}) = \begin{bmatrix} -1 \\ -9 \\ -1 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -5 \\ -1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 0 \\ 6 \\ -2 \end{bmatrix}$$

$$d(\mathbf{x}, U) = \sqrt{4 + 16 + 0 + 36 + 4} = \sqrt{60}$$