4.6 Compute the eigenspaces of the following transformation matrices. Are they diagonalizable?

a. For

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution.

To find eigenvalues, we set $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$

Let $\mathbf{B} = \mathbf{A} - \lambda \mathbf{I}$

$$\mathbf{B} = \begin{bmatrix} 2-\lambda & 3 & 0 \\ 1 & 4-\lambda & 3 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$det(\mathbf{B}) = \begin{vmatrix} 2-\lambda & 3 & 0 \\ 1 & 4-\lambda & 3 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

Using Laplace expansion along third row,

$$\begin{vmatrix} 2-\lambda & 3 & 0 \\ 1 & 4-\lambda & 3 \\ 0 & 0 & 1-\lambda \end{vmatrix} = \Sigma_{k=1}^{n} (-1)^{k+3} a_{3k} det(\mathbf{A}_{3,k})$$

$$= 0 + 0 + (-1)^{3+3} a_{33} det(\mathbf{A}_{3,3})$$

$$= (1-\lambda)((2-\lambda)(4-\lambda) - 3 * 1)$$

$$= (1-\lambda)(8+\lambda^2-6\lambda-3)$$

$$= (1-\lambda)(\lambda^2-6\lambda+5)$$

$$= (1-\lambda)(\lambda-1)(\lambda-5)$$

Therefore, $\lambda = 1, 1, 5$

Next, to find eigenvectors, we solve the following for $\mathbf{x}$:

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = 0$$

For $\lambda = 1$,

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = 0 \quad\Longrightarrow\quad (\mathbf{A}-1\cdot\mathbf{I})\mathbf{x} = 0$$

$$\Longrightarrow \begin{bmatrix} 2-1 & 3 & 0 \\ 1 & 4-1 & 3 \\ 0 & 0 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 3 & 0 \\ 1 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow x_1+3x_2 = 0, x_1+3x_2+3x_3 = 0$$

$$\Longrightarrow x_3 = 0, x_1 = -3x_2$$

$$\Longrightarrow E_1 = \text{Span}\left[\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}\right]$$

For $\lambda = 5$,

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$$

$$\implies (\mathbf{A} - 5 \cdot \mathbf{I})\mathbf{x} = 0$$

$$\implies \begin{bmatrix} 2-5 & 3 & 0 \\ 1 & 4-5 & 3 \\ 0 & 0 & 1-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} -3 & 3 & 0 \\ 1 & -1 & 3 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies -3x_1 + 3x_2 = 0, x_1 - x_2 + 3x_3 = 0, -4x_3 = 0$$

$$\implies x_3 = 0, x_1 = x_2$$

$$\implies E_5 = \text{Span}\ [\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}]$$

Max Number of linearly independent eigenvectors = 2, which is less than 3, so it is a defective matrix and cannot be diagonalized.

b. For

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution.

To find eigenvalues, we set $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

Let $\mathbf{B} = \mathbf{A} - \lambda\mathbf{I}$

$$\mathbf{B} = \begin{bmatrix} 1-\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix}$$

$$det(\mathbf{B}) = \begin{vmatrix} 1-\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix}$$

$$= (1-\lambda)(-\lambda)(-\lambda)(-\lambda)$$

$$\implies \lambda = 1, 0, 0, 0$$

Next, to find eigenvectors, we solve the following for $\mathbf{x}$:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$$

For $\lambda = 1$,

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0 \quad \implies \quad (\mathbf{A} - 1 \cdot \mathbf{I})\mathbf{x} = 0$$

$$\implies \begin{bmatrix} 1-\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1-1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies x_1 = x_1, x_2 = 0, x_3 = 0, x_4 = 0$$

$$\implies E_1 = \text{Span } [\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}]$$

For $\lambda = 0$,

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$$

$$\implies (\mathbf{A} - 0 \cdot \mathbf{I})\mathbf{x} = 0$$

$$\implies \begin{bmatrix} 1-\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1-0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies x_1 + x_2 = 0, x_3 = x_3, x_4 = x_4$$

$$\implies x_1 = -x_2, x_3 = x_3, x_4 = x_4$$

$$\implies E_0 = \text{Span}\ [\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}]$$

Total number of eigenvectors = 3+1 = 4. Therefore, it is diagonalizable.