2.6 Using Gaussian elimination, find all solutions of the inhomogeneous equation system $\mathbf{Ax} = \mathbf{b}$ with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Solution.

Augmented matrix:

$$\left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Applying Gaussian elimination to obtain reduced row echelon form:

$$\left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} \\ \\ -R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 \end{array}\right] \begin{array}{l} +R_3 \\ +R_3 \\ \cdot -1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

Applying minus-1 trick to get general solution, we get the following matrix:

$$\rightsquigarrow \left[\begin{array}{cccccc|c} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

Specific solution:

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Full solution:

$$S := \left\{ \mathbf{x} \in \mathbb{R}^6 : \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix} \middle| \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \right\}$$