6.3 You have written a computer program that sometimes compiles and sometimes does not (code does not change). You decide to model the apparent stochasticity (success vs. no success) x of the compiler using a Bernoulli distribution with parameter μ:

$$p(x|\mu) = \mu^x (1-\mu)^{1-x}, \qquad x \in \{0, 1\}.$$

Choose a conjugate prior for the Bernoulli likelihood and compute the posterior distribution $p(\mu|x_1, \ldots, x_N)$.

Solution. From table 6.2, the only conjugate prior for Bernoulli likelihood is the Beta distribution, so let's use that.

$$\implies p(\mu) = \frac{\tau(\alpha+\beta)}{\tau(\alpha)\tau(\beta)} \mu^{\alpha-1}(1-\mu)^{\beta-1}$$

$$\text{Posterior } p(\mu|x_1, \ldots, x_N) = \frac{p(x_1, \ldots, x_N|\mu)p(\mu)}{p(x_1, \ldots, x_N)}$$

Since all the trials are independent, $p(x_1, \ldots, x_N|\mu) = p(x_1|\mu) \ldots p(x_N|\mu)$

Similarly, $p(x_1, \ldots, x_N) = p(x_1)p(x_2) \ldots p(x_N)$

$$\implies p(\mu|x_1, \ldots, x_N) = \frac{p(x_1|\mu) \ldots p(x_N|\mu) * p(\mu)}{p(x_1)p(x_2) \ldots p(x_N)}$$

$$= \frac{p(\mu)\Pi_{i=1}^N p(x_i|\mu)}{\Pi_{i=1}^N p(x_i)}$$

$$= \frac{\frac{\tau(\alpha+\beta)}{\tau(\alpha)\tau(\beta)}\mu^{\alpha-1}(1-\mu)^{\beta-1} * \Pi_{i=1}^{N}\mu^{x_i}(1-\mu)^{1-x_i}}{\Pi_{i=1}^{N}p(x_i)}$$

$$= \frac{\tau(\alpha+\beta)\mu^{\alpha-1}(1-\mu)^{\beta-1}\left(\mu^{\Sigma_{i=1}^{N}x_i}\right)\left((1-\mu)^{\Sigma_{i=1}^{N}x_i}\right)}{\tau(\alpha)\tau(\beta)\Pi_{i=1}^{N}p(x_i)}$$

$$= \frac{\tau(\alpha+\beta)\left(\mu^{\alpha-1+\Sigma_{i=1}^{N}x_i}\right)\left((1-\mu)^{\beta-1+\Sigma_{i=1}^{N}x_i}\right)}{\tau(\alpha)\tau(\beta)\Pi_{i=1}^{N}p(x_i)}$$