3.9 Let $n \in \mathbb{N}$ and let $x_1, ...x_n > 0$ be n positive real numbers so that $x_1 + ... + x_n = 1$. Use the Cauchy-Schwarz inequality and show that

a. $\Sigma_{i=1}^n x_i^2 \geq 1/n$

Solution.

Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1/n \\ 1/n \\ \vdots \\ 1/n \end{bmatrix}$ Both $\mathbf{x}$ and $\mathbf{y}$ are n dimensional vectors.

Using Cauchy-Schwarz inequality, we know that

$\|\mathbf{x}\|\|\mathbf{y}\| \geq \langle \mathbf{x}, \mathbf{y} \rangle$

$\implies (\Sigma_{i=1}^n x_i^2)^{1/2} (\Sigma_{i=1}^n (1/n)^2)^{1/2} \geq \Sigma_{i=1}^n x_i/n$

$\implies (\Sigma_{i=1}^n x_i^2)^{1/2} (1/n)^{1/2} \geq 1/n$

$\implies (\Sigma_{i=1}^n x_i^2)^{1/2} \geq (1/n)^{1/2} \implies \Sigma_{i=1}^n x_i^2 \geq 1/n$

b. $\Sigma_{i=1}^n 1/x_i \geq n^2$

Solution.

Let $\mathbf{x} = \begin{bmatrix} x_1^{1/2} \\ x_2^{1/2} \\ \vdots \\ x_n^{1/2} \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} (1/x_1)^{1/2} \\ (1/x_2)^{1/2} \\ \vdots \\ (1/x_n)^{1/2} \end{bmatrix}$

Using Cauchy-Schwarz inequality, we know that

$\|\mathbf{x}\|\|\mathbf{y}\| \geq \langle \mathbf{x}, \mathbf{y} \rangle$

$\implies (\Sigma_{i=1}^n x_i)^{1/2} (\Sigma_{i=1}^n (1/x_i))^{1/2} \geq \Sigma_{i=1}^n (x_i * 1/x_i)$

$\implies (1)^{1/2} (\Sigma_{i=1}^n (1/x_i))^{1/2} \geq n$

$\implies (\Sigma_{i=1}^n (1/x_i))^{1/2} \geq n$

$\implies \Sigma_{i=1}^n (1/x_i) \geq n^2$