2.11 Write

$$\mathbf{y} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$$

as linear combination of

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \mathbf{x}_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Solution.

To solve this, we need to find a,b,c such that

$$a\mathbf{x}_1 + b\mathbf{x}_2 + c\mathbf{x}_3 = \mathbf{y} \quad | \quad a, b, c \in \mathbb{R}$$

$$\implies \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$$

We can create an augmented matrix and perform Gauss-Jordan elimination to obtain reduced row-echelon form to solve this heterogeneous system of equations.

$$
\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{array}\right] \begin{array}{l} \\ -R_1 \\ -R_1 \end{array}
$$

$$
\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{array}\right] \begin{array}{l} \\ \\ -2R_2 \end{array}
$$

$$
\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 5 & 10 \end{array}\right] \begin{array}{l} -R_2 \\ \\ \cdot\frac{1}{5} \end{array}
$$

$$
\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} -5R_3 \\ +3R_3 \\ \\ \end{array}
$$

$$
\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]
$$

$$
\implies a = -6, \quad b = 3, \quad c = 2.
$$