7.9 Consider the negative entropy of $\mathbf{x} \in \mathbb{R}^D$,

$$f(\mathbf{x}) = \sum_{d=1}^{D} x_d \log x_d.$$

Derive the convex conjugate function $f^\star(s)$, by assuming the standard dot product.

Hint: Take the gradient of an appropriate function and set the gradient to zero.

Solution.

From Definition 7.4,

$$f^*(\mathbf{s}) = \sup_{\mathbf{x} \in \mathbb{R}^D} (\langle \mathbf{s}, \mathbf{x} \rangle) - f(\mathbf{x})$$

$$= \sup_{\mathbf{x} \in \mathbb{R}^D} \left(\sum_{d=1}^{D} x_d s_d - \sum_{d=1}^{D} x_d \log x_d \right)$$

$$= \sup_{\mathbf{x} \in \mathbb{R}^D} \left(\sum_{d=1}^{D} x_d s_d - x_d \log x_d \right)$$

$$= \sum_{d=1}^{D} \sup_{\mathbf{x}_d \in \mathbb{R}} (x_d s_d - x_d \log x_d)$$

Taking the derivative of $x_d s_d - x_d \log x_d$ w.r.t $\mathbf{x}_d$ and setting it to 0, we get:

$$s_d - (1 + log(x_d)) = 0$$

$$\implies x_d = \exp(s_d - 1)$$

Substituting the value of $\mathbf{x}_d$, we get,

$$f^*(\mathbf{s}) = \sum_{d=1}^{D} s_d \exp(s_d - 1) - \exp(s_d - 1) \log(\exp(s_d - 1))$$

$$\implies f^*(\mathbf{s}) = \sum_{d=1}^{D} s_d \exp(s_d - 1) - (s_d - 1) \exp(s_d - 1)$$

$$\implies f^*(\mathbf{s}) = \sum_{d=1}^{D} s_d \exp(s_d - 1) - s_d \exp(s_d - 1) + \exp(s_d - 1)$$

$$\implies f^*(\mathbf{s}) = \sum_{d=1}^{D} \exp(s_d - 1).$$