3.6 Consider $\mathbb{R}^3$ with the inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^T \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \mathbf{y}.$$

Futhermore, we define $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ as the standard/canonical basis in $\mathbb{R}^3$.

a. Determine the orthogonal projection $\pi_U(\mathbf{e}_2)$ of $\mathbf{e}_2$ onto

$$U = \text{span}\,[\mathbf{e}_1, \mathbf{e}_3].$$

Hint: Orthogonality is defined through the inner product.

Solution.

If $\pi_U(\mathbf{e}_2)$ is the orthogonal projection of $\mathbf{e}_2$ onto U, then we can say that

$$(\pi_U(\mathbf{e}_2) - \mathbf{e}_2) \perp U$$

This implies that $(\pi_U(\mathbf{e}_2) - \mathbf{e}_2)$ is perpendicular to all the basis vectors of U.

$$\implies \langle \pi_U(\mathbf{e}_2) - \mathbf{e}_2, \mathbf{e}_1 \rangle = 0 \qquad \text{and} \qquad \langle \pi_U(\mathbf{e}_2) - \mathbf{e}_2, \mathbf{e}_3 \rangle = 0$$

$$\implies \langle \pi_U(\mathbf{e}_2), \mathbf{e}_1 \rangle = \langle \mathbf{e}_2, \mathbf{e}_1 \rangle \qquad \text{and} \qquad \langle \pi_U(\mathbf{e}_2), \mathbf{e}_3 \rangle = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$$

$$\langle \mathbf{e}_2, \mathbf{e}_1 \rangle = \mathbf{e}_2^T \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \mathbf{e}_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1$$

$$\langle \mathbf{e}_2, \mathbf{e}_3 \rangle = \mathbf{e}_2^T \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \mathbf{e}_3 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -1$$

Now, let $\pi_U(\mathbf{e}_2) = \lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_3$

$$= \begin{bmatrix} \lambda_1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ 0 \\ \lambda_2 \end{bmatrix} |\lambda_1, \lambda_2 \in \mathbb{R}.$$

$$\implies \langle \pi_U(\mathbf{e}_2), \mathbf{e}_1 \rangle = \begin{bmatrix} \lambda_1 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1$$

$$\implies \begin{bmatrix} 2\lambda_1 & \lambda_1 - \lambda_2 & 2\lambda_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1$$

$\implies \lambda_1 = \frac{1}{2}$

Also, $\langle \pi_U(\mathbf{e}_2), \mathbf{e}_3 \rangle = \begin{bmatrix} \lambda_1 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -1$

$\Longrightarrow \begin{bmatrix} 2\lambda_1 & \lambda_1 - \lambda_2 & 2\lambda_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -1$

$\Longrightarrow \lambda_2 = -\frac{1}{2}$

Therefore, $\pi_U(\mathbf{e}_2) = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}$

b. Compute the distance $d(\mathbf{e}_2, U)$.

Solution.

$$d(\mathbf{e}_2, U) = \|\mathbf{e}_2 - \pi_U(\mathbf{e}_2)\| = \sqrt{\langle \mathbf{e}_2 - \pi_U(\mathbf{e}_2), \mathbf{e}_2 - \pi_U(\mathbf{e}_2) \rangle}$$

$$\mathbf{e}_2 - \pi_U(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

$$d(\mathbf{e}_2, U) = \sqrt{\begin{bmatrix} -\frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix}} = \sqrt{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix}} = 1$$

c. Draw the scenario: standard basis vectors and $\pi_U(\mathbf{e}_2)$.

Solution.