3.1 Show that $\langle \cdot, \cdot \rangle$ defined for all $\mathbf{x} = [x_1, x_2]^T \in \mathbb{R}^2$ and $\mathbf{y} = [y_1, y_2]^T \in \mathbb{R}^2$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle \coloneqq x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2(x_2 y_2)$$

is an inner product.

Solution. To prove that $\langle \cdot, \cdot \rangle$ is an inner product, we need to prove three properties:

(i) Bilinearity

Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$, where $\mathbf{x} = [x_1, x_2]^T$, $\mathbf{y} = [y_1, y_2]^T$ and $\mathbf{z} = [z_1, z_2]^T$.

We need to prove that $\langle \lambda \mathbf{x} + \psi \mathbf{y}, \mathbf{z} \rangle = \lambda \langle \mathbf{x}, \mathbf{z} \rangle + \psi \langle \mathbf{y}, \mathbf{z} \rangle \forall \lambda, \psi \in \mathbb{R}$.

Before solving, let's remember that x_1, x_2, y_1, y_2 are scalars and scalar multiplication and addition are commutative.

L.H.S $= (\lambda x_1 + \psi y_1) z_1 - ((\lambda x_1 + \psi y_1) z_2 + (\lambda x_2 + \psi y_2) z_1) + 2((\lambda x_2 + \psi y_2) z_2)$

$= \lambda x_1 z_1 + \psi y_1 z_1 - \lambda x_1 z_2 - \psi y_1 z_2 - \lambda x_2 z_1 - \psi y_2 z_1 + 2\lambda x_2 z_2 + 2\psi y_2 z_2$

$= \lambda(x_1 z_1 - (x_1 z_2 + x_2 z_1) + 2 x_2 z_2) + \psi(y_1 z_1 - (y_1 z_2 + y_2 z_1) + 2 y_2 z_2)$

$= \lambda \langle \mathbf{x}, \mathbf{z} \rangle + \psi \langle \mathbf{y}, \mathbf{z} \rangle$

=R.H.S

We also need to prove that $\langle \mathbf{x}, \lambda \mathbf{y} + \psi \mathbf{z} \rangle = \lambda \langle \mathbf{x}, \mathbf{y} \rangle + \psi \langle \mathbf{x}, \mathbf{z} \rangle \forall \lambda, \psi \in \mathbb{R}$.

L.H.S $= x_1(\lambda y_1 + \psi z_1) - (x_1(\lambda y_2 + \psi z_2) + x_2(\lambda y_1 + \psi z_1)) + 2(x_2(\lambda y_2 + \psi z_2))$

$= \lambda x_1 y_1 + \psi x_1 z_1 - \lambda x_1 y_2 - \psi x_1 z_2 - \lambda x_2 y_1 - \psi x_2 z_1 + 2\lambda x_2 y_2 + 2\psi x_2 z_2$

$= \lambda(x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2 x_2 y_2) + \psi(x_1 z_1 - (x_1 z_2 + x_2 z_1) + 2 x_2 z_2)$

$= \lambda \langle \mathbf{x}, \mathbf{y} \rangle + \psi \langle \mathbf{x}, \mathbf{z} \rangle$

=R.H.S

Bilinearity proved.

(ii) Symmetry

We need to prove that $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$

L.H.S $= x_1y_1 - (x_1y_2 + x_2y_1) + 2(x_2y_2)$

$= y_1x_1 - (y_1x_2 + y_2x_1) + 2y_2x_2 = \langle \mathbf{y}, \mathbf{x} \rangle$

= R.H.S.

Symmetry proved.

(iii) Positive definiteness

We need to prove that $\forall \mathbf{x} \in \mathbb{R}^2 \backslash \{\mathbf{0}\} : \langle \mathbf{x}, \mathbf{x} \rangle > 0, \langle \mathbf{0}, \mathbf{0} \rangle = 0$.

$\langle \mathbf{x}, \mathbf{x} \rangle = x_1x_1 - (x_1x_2 + x_2x_1) + 2(x_2x_2)$

$= x_1^2 - 2x_1x_2 + 2x_2^2$

$= (x_1 - x_2)^2 + x_2^2$

This expressions is always greater than or equal to 0, and it can only be 0 if $x_1 = x_2 = 0$.

Therefore, it holds that $\forall \mathbf{x} \in \mathbb{R}^2 \backslash \{\mathbf{0}\} : \langle \mathbf{x}, \mathbf{x} \rangle > 0$.

Also, $\langle \mathbf{0}, \mathbf{0} \rangle = 0.0 - (0.0 + 0.0) + 2(0.0) = 0$.

Therefore, positive definiteness proved.

So, we can say that $\langle \cdot, \cdot \rangle$ is an inner product.