2.18 Let E be a vector space. Let f and g be two automorphisms on E such that $f \circ g = id_E$ (i.e., $f \circ g$ is the identity mapping id_E). Show that $ker(f) = ker(g \circ f)$, $Im(g) = Im(g \circ f)$ and that $ker(f) \cap Im(g) = \{\mathbf{0}_E\}$.

Solution. $f(\mathbf{0}_E) = \mathbf{0}_E$, as automorphism implies linearity.

Automorphism also implies injectivity, so $ker(f) = \{\mathbf{0}_E\}$.

Same holds for g: $g(\mathbf{0}_E) = \mathbf{0}_E$ and $ker(g) = \{\mathbf{0}_E\}$.

$g(f(\mathbf{0}_E)) = g(\mathbf{0}_E) = \mathbf{0}_E$.

$\mathbf{0}_E$ is the only value for which $g(f(\mathbf{x})) = \mathbf{0}_E$, because $g(\mathbf{x}) = \mathbf{0}_E$ only when $f(\mathbf{x}) = \mathbf{0}_E$, and $f(\mathbf{x}) = \mathbf{0}_E$ only when $\mathbf{x} = \mathbf{0}_E$.

Therefore, $ker(g \circ f) = \{\mathbf{0}_E\}$.

$\Longrightarrow ker(f) = ker(g \circ f) = \{\mathbf{0}_E\}$

Let $\mathbf{y} \in Im(g \circ f)$ and let $\mathbf{x} \in E$ such that $g(f(\mathbf{x})) = \mathbf{y}$.

Automorphism of f implies surjectivity, which implies that $f(\mathbf{x}) \in E \, \forall \mathbf{x} \in E$.

Let $f(\mathbf{x}) = \mathbf{z}$. Now $\mathbf{y} = g(f(\mathbf{x})) = g(\mathbf{z}) | \mathbf{z} \in E$.

Since each $\mathbf{y}$ can be represented as $g(\mathbf{z}), Im(g \circ f) \subseteq Im(g)$.

Let $\mathbf{y} \in Im(g)$, and let $\mathbf{x} \in E$ such that $g(\mathbf{x}) = \mathbf{y}$. Since $f(g(\mathbf{x})) = \mathbf{x}$, $\mathbf{y} = g(\mathbf{x}) = g(f(g(\mathbf{x}))) = g \circ f(g(\mathbf{x}))$.

Automorphism of g implies surjectivity, which implies that $g(\mathbf{x}) \in E \, \forall \mathbf{x} \in E$.

Let $g(\mathbf{x}) = \mathbf{z}$. Now $\mathbf{y} = g(\mathbf{x}) = g(f(g(\mathbf{x}))) = g \circ f(g(\mathbf{x})) = g \circ f(\mathbf{z}) | \mathbf{z} \in E$.

Since each $\mathbf{y}$ can be represented as $g \circ f(\mathbf{z}), Im(g) \subseteq Im(g \circ f)$.

$Im(g \circ f) \subseteq Im(g)$ and $Im(g) \subseteq Im(g \circ f) \Longrightarrow Im(g \circ f) = Im(g)$.