2.12 Consider the two subspaces of $\mathbb{R}^4$:

$$U_1 = \text{span}[\begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}], \qquad U_2 = \text{span}[\begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ -2 \\ -1 \end{bmatrix}].$$

Determine a basis of $U_1 \cap U_2$.

Solution.

First we find the basis of U_1 and U_2.

For U_1, we perform Gauss-Jordan reduction on the matrix formed by it's spanning vectors:

$$\begin{array}{cc} \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ -3 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} & \begin{array}{c} \\ -R_1 \\ +3R_1 \\ -R_1 \end{array} \end{array}$$

$$\rightsquigarrow \begin{array}{cc} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 2 \\ 0 & 6 & -4 \\ 0 & -3 & 2 \end{bmatrix} & \begin{array}{c} \\ \\ +2R_2 \\ -R_2 \end{array} \end{array}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \cdot -\frac{1}{3} \\ \\ \\ \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} -2R_2 \\ \\ \\ \\ \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Longrightarrow U_1 = \text{span}\left[\begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix} \right]$$

We do the same for U_2.

$$\begin{bmatrix} -1 & 2 & -3 \\ -2 & -2 & 6 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \begin{matrix} \cdot -1 \\ -2R_1 \\ +2R_1 \\ -\frac{1}{2}R_3 \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & -6 & 12 \\ 0 & 4 & -8 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} +R_3 \\ \cdot -\frac{1}{6} \\ \cdot\frac{1}{4} \\ \\ \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ -R_2 \\ \\ \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\implies U_2 = \text{span }[\begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}]$$

Now, $\forall \mathbf{v} \in U_1 \cap U_2, \mathbf{v}$ can be represented as a linear combination of the basis vectors of U_1 and U_2 respectively.

$$\implies \mathbf{v} = a \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix} = c \begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix} + d \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix} \quad | \quad a, b, c, d \in \mathbb{R}$$

$$\implies \mathbf{v} = a\begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix} + b\begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix} - c\begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix} - d\begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} | a, b, c, d \in \mathbb{R}$$

$$\implies \begin{bmatrix} 1 & 2 & -1 & 2 \\ 1 & -1 & -2 & -2 \\ -3 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ -c \\ -d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

To solve for a,b,c,d, we perform Gauss-Jordan elimination on the augmented matrix:

$$\begin{array}{c} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 0 \\ 1 & -1 & -2 & -2 & 0 \\ -3 & 0 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ -R_1 \\ +3R_1 \\ -R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 0 \\ 0 & -3 & -1 & -4 & 0 \\ 0 & 6 & -1 & 6 & 0 \\ 0 & -3 & 2 & -2 & 0 \end{array}\right] \begin{array}{c} \\ \\ +2R_2 \\ -R_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 0 \\ 0 & -3 & -1 & -4 & 0 \\ 0 & 0 & -3 & -2 & 0 \\ 0 & 0 & 3 & 2 & 0 \end{array}\right] \begin{array}{l} \\ \cdot -\frac{1}{3} \\ \cdot -\frac{1}{3} \\ +R_3 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{4}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} -2R_2 \\ -\frac{1}{3}R_3 \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & -\frac{5}{3} & -\frac{2}{3} & 0 \\ 0 & 1 & 0 & \frac{10}{9} & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} +\frac{5}{3}R_3 \\ \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{4}{9} & 0 \\ 0 & 1 & 0 & \frac{10}{9} & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\implies a - \frac{4}{9}d = 0, \quad b - \frac{10}{9}d = 0, \quad -c - \frac{2}{3}d = 0$$

$$\implies a = \frac{4}{9}d, \quad b = \frac{10}{9}d, \quad c = -\frac{2}{3}d, \quad d = d$$

We know that

$$\mathbf{v} = a\begin{bmatrix}1\\1\\-3\\1\end{bmatrix} + b\begin{bmatrix}2\\-1\\0\\-1\end{bmatrix}$$

$$\implies \mathbf{v} = \frac{4}{9}d\begin{bmatrix}1\\1\\-3\\1\end{bmatrix} + \frac{10}{9}d\begin{bmatrix}2\\-1\\0\\-1\end{bmatrix} \forall d \in \mathbb{R}$$

$$= \frac{d}{9}\left(\begin{bmatrix}4\\4\\-12\\4\end{bmatrix} + \begin{bmatrix}20\\-10\\0\\-10\end{bmatrix}\right) \forall d \in \mathbb{R}$$

$$= \frac{d}{9}\left(\begin{bmatrix}24\\-6\\-12\\-6\end{bmatrix}\right) \forall d \in \mathbb{R}$$

$$= \frac{6d}{9}\left(\begin{bmatrix}4\\-1\\-2\\-1\end{bmatrix}\right) \forall d \in \mathbb{R}$$

$$\Longrightarrow U_1 \cap U_2 = \text{ span } [\begin{bmatrix} 4 \\ -1 \\ -2 \\ -1 \end{bmatrix}]$$