2.10 Are the following sets of vectors linearly independent?

a.

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} 3 \\ -3 \\ 8 \end{bmatrix}$$

Solution. We can create a matrix from these vectors by treating the vectors as the column vectors of the matrix. Then we perform Gaussian elimination to convert the matrix into row echelon form and find the number of pivot columns. if the number of pivot columns is equal to the number of columns, the vectors are linearly independent.

$$\text{Let } \mathbf{A} = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & -3 \\ 3 & -2 & 8 \end{bmatrix}$$

Performing Gaussian elimination:

$$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & -3 \\ 3 & -2 & 8 \end{bmatrix} \begin{matrix} \\ +\frac{1}{2}R_1 \\ +3R_2 \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 2 & 1 & 3 \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & 1 & -1 \end{bmatrix} \begin{matrix} \\ \cdot\frac{2}{3} \\ -\frac{3}{2}R_2 \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Only two of the three columns are pivot columns, so the three vectors are not linearly independent.

b.

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Solution.

$$\text{Let } \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Performing Gaussian elimination:

$$\begin{array}{c} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ -2R_1 \\ -R_1 \\ \\ -R_4 \end{array}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{c} \\ \\ -R_2 \\ +R_3 \\ \\ \end{array}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{c} \\ \cdot -1 \\ \\ \\ \\ \end{array}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

All three columns are pivot columns, so the vectors are linearly independent.