2.9 Which of the following sets of vectors are subspaces of $\mathbb{R}^3$?

a. $A = \{(\lambda, \lambda + \mu^3, \lambda - \mu^3) | \lambda, \mu \in \mathbb{R}\}$

Solution.

If $\lambda, \mu \in \mathbb{R}$, then it holds that $\lambda, \lambda + \mu^3, \lambda - \mu^3 \in \mathbb{R} \forall \lambda, \mu$.

So, $A \subseteq \mathbb{R}^3$.

Since $\mathbb{R}^3$ is a vector space, and $A \subseteq \mathbb{R}^3$, we need to show the following to prove that A is subspace of $\mathbb{R}^3$:

(a) $A \neq \emptyset$, in particular: $\mathbf{0} \in A$.

Proof:

For $\lambda, \mu = 0, \mathbf{x} = (\lambda, \lambda + \mu^3, \lambda - \mu^3) = (0, 0, 0)$.

$\implies \mathbf{0} \in A$ and $A \neq \emptyset$.

(b) Closure of A with respect to outer operation: $\forall \beta \in \mathbb{R}, \forall \mathbf{x} \in A$: $\beta \mathbf{x} \in A$.

Proof:

Let $\beta \in \mathbb{R}$, and $\mathbf{x} = (\lambda, \lambda + \mu^3, \lambda - \mu^3) | \lambda, \mu \in \mathbb{R}$.

$\implies \beta.\mathbf{x} = \beta.(\lambda, \lambda + \mu^3, \lambda - \mu^3) = (\beta\lambda, \beta\lambda + \beta\mu^3, \beta\lambda - \beta\mu^3)$

Let $\gamma = \beta\lambda$ and let $\psi = \beta^{\frac{1}{3}}\mu$.

$\implies \beta.\mathbf{x} = (\gamma, \gamma + \psi^3, \gamma - \psi^3)$

Since $\gamma, \psi \in \mathbb{R}, \quad \beta.\mathbf{x} \in A$.

(c) Closure of A with respect to inner operation: $\forall \mathbf{x}, \mathbf{y} \in A : \mathbf{x} + \mathbf{y} \in A$.

Proof:

Let $\mathbf{x}_1 = (\lambda_1, \lambda_1 + \mu_1^3, \lambda_1 - \mu_1^3), \mathbf{x}_2 = (\lambda_2, \lambda_2 + \mu_2^3, \lambda_2 - \mu_2^3) | \lambda_1, \mu_1, \lambda_2, \mu_2 \in \mathbb{R}$

$\mathbf{x}_1 + \mathbf{x}_2 = (\lambda_1 + \lambda_2, \lambda_1 + \mu_1^3 + \lambda_2 + \mu_2^3, \lambda_1 - \mu_1^3 + \lambda_2 - \mu_2^3)$

$= (\lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \mu_1^3 + \mu_2^3, \lambda_1 + \lambda_2 - (\mu_1^3 + \mu_2^3))$

Let $\lambda_3 = \lambda_1 + \lambda_2$ and $\mu_3 = (\mu_1^3 + \mu_2^3)^{\frac{1}{3}}$

$\implies \mathbf{x}_1 + \mathbf{x}_2 = (\lambda_3, \lambda_3 + \mu_3^3, \lambda_3 - \mu_3^3)$ where $\lambda_3, \mu_3 \in \mathbb{R}$

$\implies \mathbf{x}_1 + \mathbf{x}_2 \in A$

Therefore, A is a subspace of $\mathbb{R}^3$.

b. $B = \{(\lambda^2, -\lambda^2, 0) | \lambda \in \mathbb{R}\}$

Solution.

If $\lambda \in \mathbb{R}$, then it holds that $\lambda^2, -\lambda^2 \in \mathbb{R} \forall \lambda$.

So, $B \subseteq \mathbb{R}^3$.

Since $\mathbb{R}^3$ is a vector space, and $B \subseteq \mathbb{R}^3$, we need to show the following to prove that B is subspace of $\mathbb{R}^3$:

(a) $B \neq \emptyset$, in particular: $\mathbf{0} \in B$

Proof:

For $\lambda = 0, \mathbf{x} = (\lambda^2, -\lambda^2, 0) = (0, 0, 0)$.

$\implies \mathbf{0} \in B$ and $B \neq \emptyset$.

(b) Closure of B with respect to outer operation: $\forall \beta \in \mathbb{R}, \forall \mathbf{x} \in B :$ $\beta \mathbf{x} \in B$.

Proof:

Let $\beta \in \mathbb{R}$, and $\mathbf{x} = (\lambda^2, -\lambda^2, 0) | \lambda \in \mathbb{R}$.

$\implies \beta.\mathbf{x} = \beta.(\lambda^2, -\lambda^2, 0) = (\beta\lambda^2, -\beta\lambda^2, 0)$

Let $\gamma = \beta^{\frac{1}{2}}\lambda$.

$\implies \beta.\mathbf{x} = (\gamma^2, -\gamma^2, 0)$

Since $\gamma \in \mathbb{R}, \quad \beta.\mathbf{x} \in B$.

(c) Closure of B with respect to inner operation: $\forall \mathbf{x}, \mathbf{y} \in B : \mathbf{x} + \mathbf{y} \in B$.

Proof:

Let $\mathbf{x}_1 = (\lambda_1^2, -\lambda_1^2, 0), \mathbf{x}_2 = (\lambda_2^2, -\lambda_2^2, 0) | \lambda_1, \lambda_2 \in \mathbb{R}$

$\mathbf{x}_1 + \mathbf{x}_2 = (\lambda_1^2 + \lambda_2^2, -(\lambda_1^2 + \lambda_2^2), 0)$

Let $\lambda_3 = (\lambda_1^2 + \lambda_2^2)^{\frac{1}{2}}$

$\implies \mathbf{x}_1 + \mathbf{x}_2 = (\lambda_3^2, -\lambda_3^2, 0)$

Since $\lambda_3 \in \mathbb{R}, \mathbf{x}_1 + \mathbf{x}_2 \in B$.

Therefore, B is a subspace of $\mathbb{R}^3$.

c. Let γ be in $\mathbb{R}$. $C = \{(\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3 | \eta_1 - 2\eta_2 + 3\eta_3 = \gamma\}$

Solution.

Since $C \subseteq \mathbb{R}^3$, we need to show the following to prove that C is subspace of $\mathbb{R}^3$:

(a) $C \neq \emptyset$, in particular: $\mathbf{0} \in C$

Proof:

For $\eta_1 = \eta_2 = \eta_3 = 0, \mathbf{x} = (0, 0, 0)$ if $\gamma = 0$.

$\implies \mathbf{0} \in C$ and $C \neq \emptyset$ if $\gamma = 0$.

(b) Closure of C with respect to outer operation: $\forall \beta \in \mathbb{R}, \forall \mathbf{x} \in C$: $\beta\mathbf{x} \in C$.

Proof:

Let $\beta \in \mathbb{R}$, and $\mathbf{x} = (\eta_1, \eta_2, \eta_3) | \eta_1 - 2\eta_2 + 3\eta_3 = \gamma$.

$\implies \beta.\mathbf{x} = \beta.(\eta_1, \eta_2, \eta_3) = (\beta\eta_1, \beta\eta_2, \beta\eta_3)$

We can see that $\beta\eta_1, \beta\eta_2, \beta\eta_3 \in \mathbb{R} \implies (\beta\eta_1, \beta\eta_2, \beta\eta_3) \in \mathbb{R}^3$, and $\beta\eta_1 - 2\beta\eta_2 + 3\beta\eta_3 = \beta(\eta_1 - 2\eta_2 + 3\eta_3) = \beta\gamma$.

Therefore, closure w.r.t outer operation only holds if $\gamma = 0$.

(c) Closure of C with respect to inner operation: $\forall \mathbf{x}, \mathbf{y} \in C : \mathbf{x} + \mathbf{y} \in C$.

Proof:

Let $\mathbf{x}_1 = (\eta_1, \eta_2, \eta_3), \mathbf{x}_2 = (\eta_4, \eta_5, \eta_6) \quad | \quad \eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6 \in \mathbb{R}$, $\eta_1 - 2\eta_2 + 3\eta_3 = \gamma$, $\eta_4 - 2\eta_5 + 3\eta_6 = \gamma$

$\mathbf{x}_1 + \mathbf{x}_2 = (\eta_1 + \eta_4, \eta_2 + \eta_5, \eta_3 + \eta_6)$

Now, $(\eta_1 + \eta_4) - 2(\eta_2 + \eta_5) + 3(\eta_3 + \eta_6) = (\eta_1 - 2\eta_2 + 3\eta_3) + (\eta_4 - 2\eta_5 + 3\eta_6) = \gamma + \gamma = 2\gamma$

Therefore, closure w.r.t inner operation only holds if $\gamma = 0$.

Therefore, C is a subspace of $\mathbb{R}^3$ only if $\gamma = 0$.

d. $D = \{(\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3 | \eta_2 \in \mathbb{Z}\}$

Solution.

Since $\mathbb{Z} \subseteq \mathbb{R}$, $D \subseteq \mathbb{R}^3$. We need to show the following to prove that D is subspace of $\mathbb{R}^3$:

(a) $D \neq \emptyset$, in particular: $\mathbf{0} \in D$

Proof:

Since $0 \in \mathbb{Z}$, for $\eta_1 = \eta_2 = \eta_3 = 0, \mathbf{x} = (0, 0, 0)$ and $\mathbf{0} \in D$ and $D \neq \emptyset$.

(b) Closure of D with respect to outer operation: $\forall \beta \in \mathbb{R}, \forall \mathbf{x} \in D :$ $\beta\mathbf{x} \in D$.

Proof:

Let $\beta \in \mathbb{R}$, and $\mathbf{x} = (\eta_1, \eta_2, \eta_3) | \eta_1, \eta_3 \in \mathbb{R}, \eta_2 \in \mathbb{Z}$.

$\implies \beta.\mathbf{x} = \beta.(\eta_1, \eta_2, \eta_3) = (\beta\eta_1, \beta\eta_2, \beta\eta_3)$

We can see that $\beta\eta_1, \beta\eta_3 \in \mathbb{R}$, but $\beta\eta_2 \notin \mathbb{Z}$ as $\beta\eta_2$ can be a fraction.

Therefore, closure w.r.t outer operation does not hold.

Therefore, D is not a subspace of $\mathbb{R}^3$.