3.8 Using the Gram-Schmidt method, turn the basis $B = (\mathbf{b}_1, \mathbf{b}_2)$ of a two-dimensional subspace $U \subseteq \mathbb{R}^3$ into an ONB $C = (\mathbf{c}_1, \mathbf{c}_2)$ of U, where

$$\mathbf{b}_1 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{b}_2 := \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

Solution. Applying Gram-Schmidt Orthogonalization,

$$\mathbf{c}_1 := \frac{\mathbf{b}_1}{\|\mathbf{b}_1\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

To find a vector orthogonal to $\mathbf{c}_1$, we calculate $\mathbf{b}_2 - \pi_{span[\mathbf{c}_1]}(\mathbf{b}_2)$

$$\pi_{span[\mathbf{c}_1]}(\mathbf{b}_2) = \frac{\mathbf{c}_1\mathbf{c}_1^T}{\|\mathbf{c}_1\|^2}\mathbf{b}_2$$

$$\mathbf{c}_1\mathbf{c}_1^T = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\|\mathbf{c}_1\| = \frac{1}{\sqrt{3}}$$

Therefore, $\pi_{span[\mathbf{c}_1]}(\mathbf{b}_2) = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Now we subtract the projection of $\mathbf{b}_2$ onto $\mathbf{c}_1$ from $\mathbf{b}_2$ to find the vector orthogonal to $\mathbf{c}_1$:

$$\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1-\frac{1}{3} \\ 2-\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} \\ \frac{5}{3} \\ -\frac{1}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -4 \\ 5 \\ -1 \end{bmatrix}$$

Normalizing it, we get $\mathbf{c}_2 = \frac{1}{3\sqrt{16+25+1}}\begin{bmatrix} -4 \\ 5 \\ -1 \end{bmatrix} = \frac{1}{3\sqrt{42}}\begin{bmatrix} -4 \\ 5 \\ -1 \end{bmatrix}$

So,

$$\mathbf{c}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{c}_2 = \frac{1}{3\sqrt{42}}\begin{bmatrix} -4 \\ 5 \\ -1 \end{bmatrix}.$$