3.2 Consider $\mathbb{R}^2$ with $\langle \cdot, \cdot \rangle$ defined for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$ as

$$\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^T \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \mathbf{y}.$$

Is $\langle \cdot, \cdot \rangle$ an inner product?

Solution. To prove that $\langle \cdot, \cdot \rangle$ is an inner product, we need to prove three properties:

(i) Bilinearity

Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$.

We need to prove that

$$\langle \lambda \mathbf{x} + \psi \mathbf{y}, \mathbf{z} \rangle = \lambda \langle \mathbf{x}, \mathbf{z} \rangle + \psi \langle \mathbf{y}, \mathbf{z} \rangle \forall \lambda, \psi \in \mathbb{R}.$$

$$\text{L.H.S} = (\lambda \mathbf{x} + \psi \mathbf{y})^T \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \mathbf{z}$$

$$= (\lambda \mathbf{x})^T \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \mathbf{z} + (\psi \mathbf{y})^T \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \mathbf{z}$$

$$= \lambda \mathbf{x}^T \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \mathbf{z} + \psi \mathbf{y}^T \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \mathbf{z}$$

$$= \lambda \langle \mathbf{x}, \mathbf{z} \rangle + \psi \langle \mathbf{y}, \mathbf{z} \rangle$$

$$= \text{R.H.S}$$

We also need to prove that

$$\langle \mathbf{x}, \lambda\mathbf{y} + \psi\mathbf{z} \rangle = \lambda\langle \mathbf{x}, \mathbf{y} \rangle + \psi\langle \mathbf{x}, \mathbf{z} \rangle \forall \lambda, \psi \in \mathbb{R}.$$

$$\text{L.H.S} = \mathbf{x}^T \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} (\lambda\mathbf{y} + \psi\mathbf{z})$$

$$= \mathbf{x}^T \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} (\lambda\mathbf{y}) + \mathbf{x}^T \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} (\psi\mathbf{z})$$

$$= \lambda\mathbf{x}^T \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \mathbf{y} + \psi\mathbf{x}^T \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \mathbf{z}$$

$$= \lambda\langle \mathbf{x}, \mathbf{y} \rangle + \psi\langle \mathbf{x}, \mathbf{z} \rangle$$

$$=\text{R.H.S}$$

Bilinearity proved.

(ii) Symmetry

We need to prove that $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$

Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and let $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

$$\langle \mathbf{x}, \mathbf{y} \rangle = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 + x_2 & 2x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 2x_1y_1 + x_2y_1 + 2x_2y_2$$

$$\langle \mathbf{y}, \mathbf{x} \rangle = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^T \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2y_1 + y_2 & 2y_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2y_1x_1 + y_2x_1 + 2y_2x_2 = 2x_1y_1 + y_2x_1 + 2x_2y_2$$

$$2x_1y_1 + x_2y_1 + 2x_2y_2 \neq 2x_1y_1 + y_2x_1 + 2x_2y_2 \qquad \text{as } x_2y_1 \neq y_2x_1 \forall \mathbf{x}, \mathbf{y}.$$

Therefore $\langle \cdot, \cdot \rangle$ is not symmetric.

(iii) Positive definiteness

We need to prove that $\forall \mathbf{x} \in \mathbb{R}^2 \backslash \{\mathbf{0}\} : \langle \mathbf{x}, \mathbf{x} \rangle > 0, \langle \mathbf{0}, \mathbf{0} \rangle = 0$.

No need to prove.

$\langle \cdot, \cdot \rangle$ is not an inner product.