5.4 Compute the Taylor Polynomials T_n, $n = 0, ..., 5$ of

$$f(x) = sin(x) + cos(x)$$

at $x_0 = 0$.

Solution.

Computing up till 5th derivative:

$f^{(0)}(x) = sin(x) + cos(x), \qquad f^{(0)}(0) = 0 + 1 = 1$

$f^{(1)}(x) = cos(x) - sin(x), \qquad f^{(1)}(0) = 1 - 0 = 1$

$f^{(2)}(x) = -sin(x) - cos(x), \qquad f^{(2)}(0) = -0 - 1 = -1$

$f^{(3)}(x) = -cos(x) + sin(x), \qquad f^{(3)}(0) = -1 + 0 = -1$

$f^{(4)}(x) = sin(x) + cos(x), \qquad f^{(4)}(0) = 0 + 1 = 1$

$f^{(5)}(x) = cos(x) - sin(x), \qquad f^{(5)}(0) = 1 - 0 = 1$

$$T_n \coloneqq \Sigma_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

$$T_5 \coloneqq \Sigma_{k=0}^{5} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

$T_5 = \left(\frac{f^{(0)}(0)}{0!}(x-0)^0\right) + \left(\frac{f^{(1)}(0)}{1!}(x)^1\right) + \left(\frac{f^{(2)}(0)}{2!}(x)^2\right) + \left(\frac{f^{(3)}(0)}{3!}(x)^3\right) + \left(\frac{f^{(4)}(0)}{4!}(x)^4\right) + \left(\frac{f^{(5)}(0)}{5!}(x)^5\right)$

$T_5 = (1) + (x) + \left(\frac{-1}{2}(x)^2\right) + \left(\frac{-1}{6}(x)^3\right) + \left(\frac{1}{24}(x)^4\right) + \left(\frac{1}{120}(x)^5\right)$

Therefore,

$$T_0 = 1$$

$$T_1 = 1 + x$$

$$T_2 = 1 + x - \frac{x^2}{2}$$

$$T_3 = 1 + x - \frac{x^2}{2} - \frac{x^3}{6}$$

$$T_4 = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}$$

$$T_5 = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$$