

6.11 Iterated Expectations.

Consider two random variables x, y with joint distribution $p(x, y)$. Show that

$$\mathbb{E}_X[x] = \mathbb{E}_Y[\mathbb{E}_X[x|y]].$$

Here, $\mathbb{E}_Y[\mathbb{E}_X[x|y]]$ denotes the expected value of x under the conditional distribution $p(x|y)$.

Solution.

$$\begin{aligned}\mathbb{E}_X[x|y] &= \int_X p(x|y)x dx \\ \implies \mathbb{E}_Y[\mathbb{E}_X[x|y]] &= \int_Y \left(\int_X p(x|y)x dx \right) p(y) dy \\ &= \int_Y \int_X p(x|y)p(y)x dx dy \\ &= \int_Y \int_X p(x, y)x dx dy \\ &= \int_X x \int_Y (p(x, y) dy) dx \\ &= \int_X xp(x) dx \\ &= \mathbb{E}_X[x] = R.H.S\end{aligned}$$