6.13 Probability Integral Transformation

Given a continuous random variable X, with cdf $F_X(x)$, show that the random variable $Y := F_X(X)$ is uniformly distributed (Theorem 6.15).

Solution.

We need to show that the cumulative distribution function of Y defines a distribution of a uniform random variable. Recall that by the axioms of probability (Section 6.1) probabilities must be non-negative and sum/integrate to one. Therefore, the range of possible values of $Y = F_X(x)$ is the interval [0, 1]. For any $F_X(x)$, the inverse $F_X^{-1}(x)$ exists because we assumed that $F_X(x)$ is strictly monotonically increasing, which we will use in the following.

Given any continuous random variable X, the definition of a cdf gives

$$
\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(F_X(x) \leq y) \qquad \text{transformation of interest} \\
&= P(X \leq F_X^{-1}(y)) \qquad \text{inverse exists} \\
&= F_X(F_X^{-1}(y)) \qquad \text{definition of cdf} \\
&= y
\end{aligned}
$$

where the last line is due to the fact that $F_X(x)$ composed with its inverse results in an identity transformation. The statement $F_Y(y) = y$ along with the fact that y lies in the interval [0, 1] means that $F_Y(x)$ is the cdf of the uniform random variable on the unit interval.