3.7 Let V be a vector space, and π an endomorphism of V.

a. Prove that π is a projection if and only if id_V - π is a projection, where id_V is the identity endomorphism on V.

Solution.

We need to prove that $\pi \circ \pi = \pi \iff (id_V - \pi) \circ (id_V - \pi) = (id_V - \pi)$

So, we have to prove 2 statements:

$\pi \circ \pi = \pi \implies (id_V - \pi) \circ (id_V - \pi) = (id_V - \pi)$ - (1)

and

$(id_V - \pi) \circ (id_V - \pi) = (id_V - \pi) \implies \pi \circ \pi = \pi$ -(2)

First we simplify $(id_V - \pi) \circ (id_V - \pi)$

$(id_V - \pi) \circ (id_V - \pi)$

$= id_V(id_V(x) - \pi(x)) - \pi(id_V(x) - \pi(x)) \; \forall x \in V$

$= id_V(x - \pi(x)) - \pi(x - \pi(x)) \qquad \because id_V(x) = x \; \forall x \in V$

$= id_V(x) - id_V(\pi(x)) - \pi(x) + \pi(\pi(x)) \qquad \because id_V$ and π are both endomorphisms and the property of linearity holds (eqn 2.15 of book)

$= x - id_V(\pi(x)) - \pi(x) + \pi(\pi(x))$

$= x - \pi(x) - \pi(x) + \pi(\pi(x)) \qquad \because \pi : V \to V$

- (3)

Now, $\pi(\pi(x)) = \pi(x) \forall x \in V \implies x - \pi(x) - \pi(x) + \pi(\pi(x)) = x - \pi(x) - \pi(x) + \pi(x)$

$= x - \pi(x)$

(1) is proved.

Now let's assume $(id_V - \pi) \circ (id_V - \pi) = (id_V - \pi)$

Applying (3), it becomes

$x - \pi(x) - \pi(x) + \pi(\pi(x)) = x - \pi(x)$

$\implies -\pi(x) + \pi(\pi(x)) = 0$

$\implies \pi(\pi(x)) = \pi(x)$

(2) is proved.

b. Assume now that π is a projection. Calculate $Im(id_V - \pi)$ and $ker(id_V - \pi)$ as a function of $Im(\pi)$ and $ker(\pi)$.

Solution.

Since $\pi(id_V - \pi) = \pi - \pi \circ \pi = \pi - \pi = 0_V$, $\mathbf{x} - \pi(\mathbf{x}) \in ker(\pi) \forall \mathbf{x} \in V$

$\implies Im(id_V - \pi) \subseteq ker(\pi)$

Now, $\forall \mathbf{x} \in ker(\pi)$, $\pi(\mathbf{x}) = 0$, and $(id_V - \pi)(\mathbf{x}) = \mathbf{x}$

This means that $\forall \mathbf{x} \in ker(\pi), \mathbf{x} \in Im(id_V - \pi)$

$\implies ker(\pi) \subseteq Im(id_V - \pi)$

Therefore, $Im(id_V - \pi) = ker(\pi)$

Now, $ker(id_V - \pi)$

$= \{\mathbf{x} \in V | \mathbf{x} - \pi(\mathbf{x}) = 0_V\}$

$= \{\mathbf{x} \in V | \mathbf{x} = \pi(\mathbf{x})\}$

Therefore, $\forall \mathbf{x} \in ker(id_V - \pi), \mathbf{x} = \pi(\mathbf{x})$, and $ker(id_V - \pi) \subseteq Im(\pi)$

Now, $\forall \mathbf{x} \in Im(\pi)$, $\mathbf{x} = \pi(\mathbf{y}) | \mathbf{y} \in V$

$(id_V - \pi)(\mathbf{x}) = (id_V - \pi)(\pi(\mathbf{y})) = \pi(\mathbf{y}) - \pi(\pi(\mathbf{y})) = \pi(\mathbf{y}) - \pi(\mathbf{y}) =$

0_V

Therefore, $\forall \mathbf{x} \in Im(\pi)$, $\mathbf{x} \in ker(id_V - \pi)$

and $Im(\pi) \subseteq ker(id_V - \pi)$

So, $Im(\pi) = ker(id_V - \pi)$