4.7 Are the following matrices diagonalizable? If yes, determine their diagonal form and a basis with respect to which the transformation matrices are diagonal. If no, give reasons why they are not diagonalizable.

a.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}$$

Solution.

To find eigenvalues, we set $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

Let $\mathbf{B} = \mathbf{A} - \lambda\mathbf{I}$

$$\mathbf{B} = \begin{bmatrix} 0 - \lambda & 1 \\ -8 & 4 - \lambda \end{bmatrix}$$

$$det(\mathbf{B}) = \begin{vmatrix} 0 - \lambda & 1 \\ -8 & 4 - \lambda \end{vmatrix}$$

$$= (0 - \lambda)(4 - \lambda) - (1 * -8)$$

$$= -4\lambda + \lambda^2 + 8$$

$$= \lambda^2 - 4\lambda + 8$$

$$\implies \lambda = \frac{4 \pm \sqrt{(-4)^2 - 4 * 1 * 8}}{2 * 1}$$

$$\implies \lambda = \frac{4 \pm \sqrt{16 - 32}}{2}$$

$$\implies \lambda = \frac{4 \pm \sqrt{-16}}{2}$$

$$\implies \lambda = 2 \pm 2i$$

To find eigenvectors, we solve the following for $\mathbf{x}$:

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$$

For $\lambda = 2 + 2i$,

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$$

$$\implies (\mathbf{A} - (2+2i) \cdot \mathbf{I})\mathbf{x} = 0$$

$$\implies \begin{bmatrix} 0-(2+2i) & 1 \\ -8 & 4-(2+2i) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} -2-2i & 1 \\ -8 & 2-2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We need to apply Gauss Jordan reduction to the following augmented matrix to solve this:

$$\left[\begin{array}{cc|c} -2-2i & 1 & 0 \\ -8 & 2-2i & 0 \end{array}\right] \begin{array}{l} \cdot -\frac{1}{2} \\ \cdot -\frac{1}{8} \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1+i & -\frac{1}{2} & 0 \\ 1 & \frac{-1+i}{4} & 0 \end{array}\right] \begin{array}{l} \\ -\frac{(1-i)}{2}R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1+i & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\implies (1+i)x_1 - \frac{1}{2}x_2 = 0$$

$$\implies x_2 = 2(1+i)x_1$$

$$\implies E_{2+2i} = \text{Span}\ [\begin{bmatrix} 1 \\ 2(1+i) \end{bmatrix}]$$

For $\lambda = 2 - 2i$,

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$$

$$\implies (\mathbf{A} - (2+2i) \cdot \mathbf{I})\mathbf{x} = 0$$

$$\implies \begin{bmatrix} 0-(2-2i) & 1 \\ -8 & 4-(2-2i) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} -2+2i & 1 \\ -8 & 2+2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We need to apply Gauus Jordan reduction to the following augmented matrix to solve this:

$$\left[\begin{array}{cc|c} -2+2i & 1 & 0 \\ -8 & 2+2i & 0 \end{array}\right] \begin{array}{l} \cdot -\frac{1}{2} \\ \cdot -\frac{1}{8} \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1-i & -\frac{1}{2} & 0 \\ 1 & \frac{-1-i}{4} & 0 \end{array}\right] \begin{array}{l} \\ -\frac{(1+i)}{2}R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1-i & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\implies (1-i)x_1 - \frac{1}{2}x_2 = 0$$

$$\implies x_2 = 2(1-i)x_1$$

$$\implies E_{2-2i} = \text{Span } [\begin{bmatrix} 1 \\ 2(1-i) \end{bmatrix}]$$

$$\text{Therefore, Basis } \mathbb{B} = \left\{ \begin{bmatrix} 1 \\ 2(1+i) \end{bmatrix}, \begin{bmatrix} 1 \\ 2(1-i) \end{bmatrix} \right\}$$

Now, we get the diagonal form $\mathbf{A} = \mathbf{PDP}^{-1}$

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 2(1+i) & 2(1-i) \end{bmatrix}$$

$$\mathbf{P}^{-1} = \frac{1}{1 * 2(1-i) - 1 * 2(1+i)} \begin{bmatrix} 2(1-i) & -1 \\ -2(1+i) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2(1-i)}{-4i} & \frac{-1}{-4i} \\ \frac{-2(1+i)}{-4i} & \frac{1}{-4i} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{i+1}{2} & \frac{-i}{4} \\ \frac{1-i}{2} & \frac{i}{4} \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 2(1+i) & 0 \\ 0 & 2(1-i) \end{bmatrix}$$

$$\mathbf{PDP}^{-1} = \begin{bmatrix} 1 & 1 \\ 2(1+i) & 2(1-i) \end{bmatrix} \begin{bmatrix} 2(1+i) & 0 \\ 0 & 2(1-i) \end{bmatrix} \begin{bmatrix} \frac{i+1}{2} & \frac{-i}{4} \\ \frac{1-i}{2} & \frac{i}{4} \end{bmatrix}$$

b.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution.

This is a square, symmetric matrix. So it is diagonalizable.

First we have to find eigenvalues, for which we set $det(\mathbf{A} - \lambda \mathbf{I}_3 = 0)$

$$\Longrightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

Using Sarrus rule,

$$\Longrightarrow ((1-\lambda)*(1-\lambda)*(1-\lambda))+(1*1*1)+(1*1*1)-(1*(1-\lambda)*1)-(1*1*(1-\lambda))-((1-\lambda)*1*1) = 0$$

$$\Longrightarrow (-\lambda^3 + 3\lambda^2 - 3\lambda + 1) + (1) + (1) - (1-\lambda) - (1-\lambda) - (1-\lambda) = 0$$

$$\implies -\lambda^3 + 3\lambda^2 - 3\lambda + 1 + 1 + 1 - 1 + \lambda - 1 + \lambda - 1 + \lambda = 0$$

$$\implies -\lambda^3 + 3\lambda^2 = 0$$

$$\implies -\lambda^2(\lambda - 3) = 0$$

$$\implies \lambda = 0, 0, 3$$

Now, we need to find eigenvectors by solving the following for $\mathbf{x}$:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$$

For $\lambda = 0$,

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$$

$$\implies \begin{bmatrix} 1-0 & 1 & 1 \\ 1 & 1-0 & 1 \\ 1 & 1 & 1-0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies x_1 + x_2 + x_3 = 0$$

$$\implies x_1 = -x_2 - x_3$$

$$\Longrightarrow E_0 = \text{Span}\left[\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right]$$

For $\lambda = 3$,

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$$

$$\Longrightarrow \begin{bmatrix} 1-3 & 1 & 1 \\ 1 & 1-3 & 1 \\ 1 & 1 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To solve this, we apply Gauss Jordan reduction to the following augmented matrix:

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \begin{array}{l} \cdot -\frac{1}{2} \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \begin{array}{l} \\ -R_1 \\ -R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 0 \end{array}\right] \begin{array}{l} \\ \cdot -\frac{2}{3} \\ +R_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} +\frac{1}{2}R_2 \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\implies x_1 = x_3, x_2 = x_3$$

$$\implies E_3 = \text{Span}\ [\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}]$$

$$\text{Therefore, Basis } \mathbb{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Now we can construct $\mathbf{P}$ and $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$

$$\mathbf{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

To get $\mathbf{P}^{-1}$, we can apply Gauss Jordan reduction to $\mathbf{P}|\mathbf{I}_3$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -R_1 \\ -R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} +R_3 \\ \cdot\frac{1}{2} \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ -R_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right] \begin{array}{l} \\ \\ \cdot\frac{2}{3} \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{array}\right] \begin{array}{l} -R_3 \\ \cdot(-\frac{1}{2}R_3) \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{array}\right]$$

$$\Longrightarrow \mathbf{P}^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$\Longrightarrow \mathbf{A} = \mathbf{PDP}^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

c.

$$\mathbf{A} = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$

Solution.

First we have to find the eigenvalues, for which we set $det(\mathbf{A}-\lambda\mathbf{I}_4) = 0$

$$\implies \begin{vmatrix} 5-\lambda & 4 & 2 & 1 \\ 0 & 1-\lambda & -1 & -1 \\ -1 & -1 & 3-\lambda & 0 \\ 1 & 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

Applying Laplace expansion along first row,

$$\begin{vmatrix} 5-\lambda & 4 & 2 & 1 \\ 0 & 1-\lambda & -1 & -1 \\ -1 & -1 & 3-\lambda & 0 \\ 1 & 1 & -1 & 2-\lambda \end{vmatrix} = \Sigma_{k=1}^{n}(-1)^{k+1}a_{1k}det(\mathbf{A}_{1,k})$$

$$= (-1)^{1+1}a_{11}det(\mathbf{A}_{1,1}) + (-1)^{2+1}a_{12}det(\mathbf{A}_{1,2}) + (-1)^{3+1}a_{13}det(\mathbf{A}_{1,3}) + (-1)^{4+1}a_{14}det(\mathbf{A}_{1,4})$$

$$= (-1)^2(5-\lambda) \cdot \begin{vmatrix} 1-\lambda & -1 & -1 \\ -1 & 3-\lambda & 0 \\ 1 & -1 & 2-\lambda \end{vmatrix}$$

$$+(-1)^3(4) \cdot \begin{vmatrix} 0 & -1 & -1 \\ -1 & 3-\lambda & 0 \\ 1 & -1 & 2-\lambda \end{vmatrix}$$

$$+(-1)^4(2) \cdot \begin{vmatrix} 0 & 1-\lambda & -1 \\ -1 & -1 & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix}$$

$$+(-1)^5(1)\cdot\begin{vmatrix} 0 & 1-\lambda & -1 \\ -1 & -1 & 3-\lambda \\ 1 & 1 & -1 \end{vmatrix}$$

$$=(5-\lambda)*(((1-\lambda)*(3-\lambda)*(2-\lambda))+((-1)*0*1)+((-1)*(-1)*(-1))$$

$$-(1*(3-\lambda)*(-1))-((-1)*0*(1-\lambda))-((2-\lambda)*(-1)*(-1)))$$

$$-4*(((0)*(3-\lambda)*(2-\lambda))+((-1)*(0)*(1))+((-1)*(-1)*(-1))$$

$$-((1)*(3-\lambda)*(-1))-((-1)*(0)*(0))-((2-\lambda)*(-1)*(-1)))$$

$$+2*(((0)*(-1)*(2-\lambda))+((1-\lambda)*(0)*(1))+((-1)*(-1)*(1)$$

$$-((1)*(-1)*(-1))-((1)*(0)*(0))-((2-\lambda)*(-1)*(1-\lambda)))$$

$$-(((0)*(-1)*(-1))+((1-\lambda)*(3-\lambda)*(1))+((-1)*(-1)*(1))$$

$$-((1)*(-1)*(-1))-((1)*(3-\lambda)*(0))-((-1)*(-1)*(1-\lambda)))$$

$$=(5-\lambda)((-\lambda^3+6\lambda^2-11\lambda+6)+(0)+(-1)$$

$$-(\lambda-3)-(0)-(2-\lambda))$$

$$-4*((0)+(0)+(-1)$$

$$-(\lambda-3)-(0)-(2-\lambda))$$

$$+2*((0)+(0)+(1)$$

$$-(1)-(0)-(-\lambda^2+3\lambda-2))$$

$$-((0)+(\lambda^2-4\lambda+3)+(1)$$

$$-(1)-(0)-(1-\lambda))$$

$$=(5-\lambda)(-\lambda^3+6\lambda^2-11\lambda+6-1-\lambda+3-2+\lambda)$$

$$-4*(-1-\lambda+3-2+\lambda)$$

$$+2*(1-1+\lambda^2-3\lambda+2)$$

$$-(\lambda^2-4\lambda+3+1-1-1+\lambda)$$

$$=(5-\lambda)(-\lambda^3+6\lambda^2-11\lambda+6)$$

$$-4*(0)$$

$$+2*(\lambda^2-3\lambda+2)$$

$$-(\lambda^2-3\lambda+2)$$

$$=(5-\lambda)*(\lambda-1)*(3-\lambda)*(\lambda-2)$$

$$+2*(\lambda-1)(\lambda-2)$$

$$-(\lambda-1)(\lambda-2)$$

$$=((5-\lambda)*(3-\lambda)+2-1)(\lambda-1)(\lambda-2)$$

$$= (\lambda^2 - 8\lambda + 15 + 1)(\lambda - 1)(\lambda - 2)$$

$$= (\lambda^2 - 8\lambda + 16)(\lambda - 1)(\lambda - 2)$$

$$= (\lambda - 4)^2(\lambda - 1)(\lambda - 2)$$

$$\implies \lambda = 1, 2, 4, 4$$

Now, we need to calculate eigenvectors by solving for $(\mathbf{A} - \lambda\mathbf{I}_4)\mathbf{x} = 0$

For $\lambda = 1$,

$$(\mathbf{A} - \lambda\mathbf{I}_4)\mathbf{x} = 0$$

$$\implies \begin{bmatrix} 5-1 & 4 & 2 & 1 \\ 0 & 1-1 & -1 & -1 \\ -1 & -1 & 3-1 & 0 \\ 1 & 1 & -1 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 4 & 4 & 2 & 1 \\ 0 & 0 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

To solve this, we need to convert the following augmented matrix to row-echelon form using Gauss Jordan Reduction

$$\begin{array}{c} \left[\begin{array}{cccc|c} 4 & 4 & 2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 \end{array}\right] \end{array} \begin{array}{l} -3R_4 \\ \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & 5 & -2 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 \end{array}\right] \begin{array}{l} \\ \\ +R_1 \\ -R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & 5 & -2 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 7 & -2 & 0 \\ 0 & 0 & -6 & 3 & 0 \end{array}\right] \begin{array}{l} \\ \cdot -1 \\ +6R_2 \\ -7R_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & 5 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -8 & 0 \\ 0 & 0 & 1 & 10 & 0 \end{array}\right] \begin{array}{l} \\ \\ -R_2 \\ -R_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & 5 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -9 & 0 \\ 0 & 0 & 0 & 9 & 0 \end{array}\right] \begin{array}{l} -5R_2 \\ \\ \cdot -\frac{1}{9} \\ +R_3 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & -7 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} +7R_3 \\ -R_3 \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\implies x_1 + x_2 = 0, x_3 = 0, x_4 = 0$$

$$E_1 = \text{Span}\ [\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}]$$

For $\lambda = 2$,

$$(\mathbf{A} - \lambda \mathbf{I}_4)\mathbf{x} = 0$$

$$\implies \begin{bmatrix} 5-2 & 4 & 2 & 1 \\ 0 & 1-2 & -1 & -1 \\ -1 & -1 & 3-2 & 0 \\ 1 & 1 & -1 & 2-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 3 & 4 & 2 & 1 \\ 0 & -1 & -1 & -1 \\ -1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

To solve this, we need to convert the following augmented matrix to row-echelon form using Gauss Jordan Reduction

$$\left[\begin{array}{cccc|c} 3 & 4 & 2 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \end{array}\right] \begin{array}{l} -3R_4 \\ \\ +R_4 \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 0 & 1 & 5 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \end{array}\right] \begin{array}{l} \text{Swap with } R_4 \\ \\ \\ \text{Swap with } R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 1 & 0 \end{array}\right] \begin{array}{l} \\ \text{Swap with } R_4 \\ \\ \text{Swap with } R_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 1 & 0 \end{array}\right] \begin{array}{l} \\ \\ \text{Swap with } R_4 \\ \text{Swap with } R_3 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 1 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \cdot -1 \\ +R_2 \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} -R_2 \\ \\ \cdot\frac{1}{4} \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} +2R_3 \\ -R_3 \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\implies x_1 - x_4 = 0, x_2 + x_4 = 0, x_3 = 0$$

$$\implies x_1 = x_4, x_2 = -x_4, x_3 = 0$$

$$\implies E_2 = \text{Span}\ [\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda = 4$,

$$(\mathbf{A} - \lambda \mathbf{I}_4)\mathbf{x} = 0$$

$$\implies \begin{bmatrix} 5-4 & 4 & 2 & 1 \\ 0 & 1-4 & -1 & -1 \\ -1 & -1 & 3-4 & 0 \\ 1 & 1 & -1 & 2-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & -3 & -1 & -1 \\ -1 & -1 & -1 & 0 \\ 1 & 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

To solve this, we need to convert the following augmented matrix to row-echelon form using Gauss Jordan Reduction

$$\left[\begin{array}{cccc|c} 1 & 4 & 2 & 1 & 0 \\ 0 & -3 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -2 & 0 \end{array}\right] \begin{array}{l} \\ \\ +R_1 \\ -R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 4 & 2 & 1 & 0 \\ 0 & -3 & -1 & -1 & 0 \\ 0 & 3 & 1 & 1 & 0 \\ 0 & -3 & -3 & -3 & 0 \end{array}\right] \begin{array}{l} \\ \\ +R_2 \\ -R_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 4 & 2 & 1 & 0 \\ 0 & -3 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & 0 \end{array}\right] \begin{array}{c} \\ +R_4 \\ \\ \cdot -\frac{1}{2} \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 4 & 2 & 1 & 0 \\ 0 & -3 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{c} \\ \cdot -\frac{1}{3} \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 4 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{c} -4R_2 \\ -R_4 \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{c} +2R_4 \\ \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{c} +2R_4 \\ \\ \text{Swap with } R_4 \\ \text{Swap with } R_3 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\implies x_1 - x_4 = 0, x_2 = 0, x_3 + x_4 = 0$$

$$\implies x_1 = x_4, x_2 = 0, x_3 = -x_4$$

$$\implies E_4 = \text{Span}\ [\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}]$$

Total number of eigenvectors = 3, but matrix is 4x4. Therefore, it is a defective matrix and is not diagonalizable.

d.

$$\mathbf{A} = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

Solution.

First we have to find eigenvalues, for which we set $det(\mathbf{A} - \lambda\mathbf{I}_3 = 0)$

$$\implies \begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} = 0$$

Using Sarrus rule,

$$\implies ((5-\lambda)*(4-\lambda)*(-4-\lambda))+((-6)*(2)*(3))+((-6)*(-1)*(-6))$$

$$-((3)*(4-\lambda)*(-6))-((-6)*(2)*(5-\lambda))-((-4-\lambda)*(-1)*(-6)) = 0$$

$$\implies (\lambda^3+5\lambda^2+16\lambda-80)+(-36)+(-36)-(18\lambda-72)-(12\lambda-60)-(-6\lambda-24) = 0$$

$$\implies -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

$$\implies (\lambda - 1)(\lambda - 2)(-\lambda + 2) = 0$$

$$\implies \lambda = 1, 2$$

Now, we need to calculate eigenvectors by solving for $(\mathbf{A} - \lambda\mathbf{I}_3)\mathbf{x} = 0$

For $\lambda = 1$,

$$(\mathbf{A} - \lambda\mathbf{I}_3)\mathbf{x} = 0$$

$$\implies \begin{bmatrix} 5-1 & -6 & -6 \\ -1 & 4-1 & 2 \\ 3 & -6 & -4-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To solve this, we apply Gauss Jordan reduction to convert the following augmented matrix to reduced row echelon form:

$$\left[\begin{array}{ccc|c} 4 & -6 & -6 & 0 \\ -1 & 3 & 2 & 0 \\ 3 & -6 & -5 & 0 \end{array}\right] \begin{array}{l} \cdot\frac{1}{4} \\ \cdot -1 \\ +3R_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & -\frac{3}{2} & 0 \\ 1 & -3 & -2 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right] \begin{array}{l} \\ -R_1 \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & -\frac{3}{2} & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 3 & 1 & 0 \end{array}\right] \begin{array}{l} +R_2 \\ \cdot\frac{2}{3} \\ \cdot\frac{1}{3} \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \end{array}\right] \begin{array}{l} \\ \\ -R_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\implies x_1 - x_3 = 0, x_2 + \frac{x_3}{3} = 0$$

$$\implies x_1 = x_3, x_2 = -\frac{x_3}{3}$$

$$\implies E_1 = \text{Span}\ [\begin{bmatrix} 1 \\ -\frac{1}{3} \\ 1 \end{bmatrix}]$$

For $\lambda = 2$,

$$(\mathbf{A} - \lambda\mathbf{I}_3)\mathbf{x} = 0$$

$$\implies \begin{bmatrix} 5-2 & -6 & -6 \\ -1 & 4-2 & 2 \\ 3 & -6 & -4-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To solve this, we apply Gauss Jordan reduction to convert the following augmented matrix to reduced row echelon form:

$$\left[\begin{array}{ccc|c} 3 & -6 & -6 & 0 \\ -1 & 2 & 2 & 0 \\ 3 & -6 & -6 & 0 \end{array}\right] \begin{array}{c} \cdot\frac{1}{3} \\ \cdot -1 \\ -3R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 1 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ -R_1 \\ -3R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\implies x_1 - 2x_2 - 2x_3 = 0$$

$$\implies x_1 = 2x_2 + 2x_3 = 0$$

$$\implies E_2 = \text{Span }[\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}]$$

Since we have 3 distinct eigenvectors and matrix is 3x3, matrix is diagonalizable.

$$\text{Therefore, Basis } \mathbb{B} = \left\{ \begin{bmatrix} 1 \\ -\frac{1}{3} \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Now we can construct $\mathbf{P}$ and $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 1 & 2 & 2 \\ -\frac{1}{3} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

To obtain $\mathbf{P}^{-1}$, we apply Gauss Jordan reduction to $\mathbf{P}|\mathbf{I}_3$, to convert the left side to reduced row echelon form

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \\ +\frac{1}{3}R_1 \\ -R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & \frac{5}{3} & \frac{2}{3} & \frac{1}{3} & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] \begin{array}{c} \\ \cdot\frac{3}{5} \\ \cdot -\frac{1}{2} \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{5} & \frac{1}{5} & \frac{3}{5} & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right] \begin{array}{c} -2R_2 \\ \\ -R_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{6}{5} & \frac{3}{5} & -\frac{6}{5} & 0 \\ 0 & 1 & \frac{2}{5} & \frac{1}{5} & \frac{3}{5} & 0 \\ 0 & 0 & \frac{1}{10} & \frac{3}{10} & -\frac{3}{5} & -\frac{1}{2} \end{array}\right] \begin{array}{c} \\ \\ \cdot 10 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{6}{5} & \frac{3}{5} & -\frac{6}{5} & 0 \\ 0 & 1 & \frac{2}{5} & \frac{1}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 & 3 & -6 & -5 \end{array}\right] \begin{array}{c} -\frac{6}{5}R_3 \\ -\frac{2}{5}R_3 \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 6 & 6 \\ 0 & 1 & 0 & -1 & 3 & 2 \\ 0 & 0 & 1 & 3 & -6 & -5 \end{array}\right]$$

$$\implies \mathbf{P}^{-1} = \begin{bmatrix} -3 & 6 & 6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix}$$

$$\text{Therefore, } \mathbf{PDP}^{-1} = \begin{bmatrix} 1 & 2 & 2 \\ -\frac{1}{3} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -3 & 6 & 6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix}$$