2.8 Determine the inverses of the following matrices if possible:

a.

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

Solution.

$$[\mathbf{A}|\mathbf{I}_3] = \left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 3 & 4 & 5 & 0 & 1 & 0 \\ 4 & 5 & 6 & 0 & 0 & 1 \end{array}\right]$$

Applying Gaussian elimination to $[\mathbf{A}|\mathbf{I}_3]$,

and converting the left side to $\mathbf{I}_3$ will give us $[\mathbf{I}_3|\mathbf{A}^{-1}]$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 3 & 4 & 5 & 0 & 1 & 0 \\ 4 & 5 & 6 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -\frac{3}{2}R_1 \\ -2R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -1 & -\frac{3}{2} & 1 & 0 \\ 0 & -1 & -2 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \cdot -2 \\ -2R_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & -2 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

Here, we can see that inverse does not exist.

b.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Solution.

$$[\mathbf{A}|\mathbf{I}_4] = \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Applying Gaussian elimination to $[\mathbf{A}|\mathbf{I}_4]$,

and converting the left side to $\mathbf{I}_4$ will give us $[\mathbf{I}_4|\mathbf{A}^{-1}]$

$$\begin{array}{cc} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] & \begin{array}{c} \\ \\ -R_1 \\ -R_1 \end{array} \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ -R_2 \\ -R_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} +R_4 \\ +R_4 \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \text{Swap with } R_4 \\ \text{Swap with } R_3 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & -2 & 1 & -1 & -1 & 1 & 0 \end{array}\right] \begin{array}{l} \\ \\ \cdot -1 \\ -2R_3 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & -2 \end{array}\right]$$

So, $\mathbf{A}^{-1} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & -2 \end{bmatrix}$