2.1 We consider $(\mathbb{R}\backslash\{-1\}, \star)$, where

$$a \star b := ab + a + b, \qquad a, b \in \mathbb{R}\backslash\{-1\}$$

a. Show that $(\mathbb{R}\backslash\{-1\}, \star)$ is an abelian group

Solution.

We need to prove 5 properties:

Closure under $\star$, Associativity, Existence of Neutral Element, Existence of Inverse Element, and Commutativity.

Closure

We need to prove that $a \star b \in \mathbb{R}\backslash\{-1\}$

$$\begin{aligned}
&a \star b := ab + a + b \\
&= a(b+1) + b \\
&= a(b+1) + (b+1) - 1 \\
&= (a+1)(b+1) - 1
\end{aligned}$$

$(a+1)(b+1) - 1$ is a real number.

It cannot be equal to -1, since $(a+1)(b+1) - 1 = -1 \implies (a+1)(b+1) - 1 = 0 \implies a = -1$ or $b = -1$, which is not possible since $a, b \in \mathbb{R}\backslash\{-1\}$.

Therefore$(a+1)(b+1)-1 \in \mathbb{R}\backslash\{-1\}, \forall a, b \in \mathbb{R}\backslash\{-1\}$.

Closure proved.

Associativity

We need to prove that $a \star (b \star c) = (a \star b) \star c$

$$a \star (b \star c) = a(b \star c) + a + (b \star c)$$

$$= a(bc + b + c) + a + bc + b + c$$

$$= abc + ab + ac + a + bc + b + c$$

$$= (a + b + ab) + c + (a + b + ab)c$$

$$= (a \star b) + c + (a \star b)c$$

$$= (a \star b) \star c$$

Associativity proved

Existence of Neutral Element

We need to prove the following:

$$\exists e \in \mathbb{R}\backslash\{-1\} \forall x \in \mathbb{R}\backslash\{-1\} : x \star e = e \star x = x$$

$$
\begin{aligned}
&x \star e = x \\
&\implies x + e + ex = x \\
&\implies e + ex = 0 \\
&\implies e(1 + x) = 0 \\
&\implies e = 0 \text{ since } x \neq -1
\end{aligned}
$$

Also, if $e = 0, e \star x = e + x + ex = x = e \star x$.

Therefore, neutral element exists and it is 0.

Existence of Inverse Element

We need to prove that

$$\forall x \in \mathbb{R} \backslash \{-1\} \exists y \in \mathbb{R} \backslash \{-1\} : x \star y = e = y \star x$$

$x \star y = e$

$\implies x + y + xy = 0$

$\implies y = -x/(1+x)$

Also, $y \star x = e$

$\implies y + x + yx = 0$

$\implies y = -x/(1+x)$

Since $x \neq -1, -x/(1+x)$ exists.

Therefore, inverse element exists $\forall x \in \mathbb{R}\backslash\{-1\}$

Commutativity

We need to prove that

$$\forall x, y \in \mathbb{R}\backslash\{-1\}, a \star b = b \star a$$

$a \star b = a + b + ab$

$= b + a + ab$ scalar addition is commutative

$= b + a + ba$ scalar multiplication is commutative

$= b \star a$

Commutativity proved.

Hence proved that $(\mathbb{R}\backslash\{-1\}, \star)$ is an abelian group.

b. Solve

$$3 \star x \star x = 15$$

Solution.

$$\begin{aligned}
&3 \star x \star x = 15 \\
&\implies 3 \star (x \star x) = 15 \\
&\implies 3 \star (x + x + x^2) = 15 \\
&\implies 3 \star (2x + x^2) = 15 \\
&\implies 3 + (2x + x^2) + 3(2x + x^2) = 15 \\
&\implies 3 + 2x + x^2 + 6x + 3x^2 = 15 \\
&\implies 4x^2 + 8x - 12 = 0 \\
&\implies x^2 + 2x - 3 = 0 \\
&\implies x^2 - x + 3x - 3 = 0 \\
&\implies x(x-1) + 3(x-1) = 0 \\
&\implies (x+3)(x-1) = 0 \\
&\implies x = 1, -3
\end{aligned}$$