7.7 Consider the quadratic program illustrated in Figure 7.4,

$$\min_{\mathbf{x} \in \mathbb{R}^2} \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
subject to
$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Derive the dual quadratic program using Lagrange duality.

Solution.

Using 7.48a, the Lagrangian is given by

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \boldsymbol{\lambda}^T \begin{pmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{pmatrix}$$

$$= \left(\frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \boldsymbol{\lambda}^T \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) - \boldsymbol{\lambda}^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Taking the derivative with respect to \mathbf{x} and setting it to zero gives (Using 5.107)

$$\frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \left(\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}^T \right) + \begin{bmatrix} 5 \\ 3 \end{bmatrix}^T + \boldsymbol{\lambda}^T \begin{vmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{vmatrix} = \mathbf{0}$$

$$\Rightarrow \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \left(\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \right) + \begin{bmatrix} 5 \\ 3 \end{bmatrix}^T + \boldsymbol{\lambda}^T \begin{vmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{vmatrix} = \mathbf{0}$$

$$\Rightarrow \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \left(\begin{bmatrix} 4 & 2 \\ 2 & 8 \end{bmatrix} \right) + \begin{bmatrix} 5 \\ 3 \end{bmatrix}^T + \boldsymbol{\lambda}^T \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix}^T + \boldsymbol{\lambda}^T \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} = \mathbf{0}$$

Matrix
$$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$
 is invertible since determinant = $4 * 2 - 1 * 1 = 7$.

Therefore:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T = -\left(\begin{bmatrix} 5 \\ 3 \end{bmatrix}^T + \boldsymbol{\lambda}^T \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \right) \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}^{-1}$$

$$\Longrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T = -\left(\begin{bmatrix} 5 \\ 3 \end{bmatrix}^T + \boldsymbol{\lambda}^T \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \right) \begin{bmatrix} \frac{4}{7} & -\frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

$$\implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T = \begin{bmatrix} -\frac{17}{7} \\ -\frac{1}{7} \end{bmatrix}^T + \lambda^T \begin{bmatrix} -\frac{4}{7} & \frac{1}{7} \\ \frac{4}{7} & -\frac{1}{7} \\ \frac{1}{7} & -\frac{2}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

Substituting in the expression for $\mathcal{L}(\mathbf{x}, \lambda)$, we get

$$\mathcal{D}(\boldsymbol{\lambda}) = \frac{1}{2} \left(\begin{bmatrix} -\frac{17}{7} \\ -\frac{17}{7} \\ -\frac{1}{7} \end{bmatrix}^{T} + \boldsymbol{\lambda}^{T} \begin{bmatrix} -\frac{4}{7} & \frac{1}{7} \\ \frac{4}{7} & -\frac{1}{7} \\ \frac{1}{7} & -\frac{2}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix} \right) \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \left(\begin{bmatrix} -\frac{17}{7} \\ -\frac{1}{7} \end{bmatrix}^{T} + \boldsymbol{\lambda}^{T} \begin{bmatrix} -\frac{4}{7} & \frac{1}{7} \\ \frac{4}{7} & -\frac{1}{7} \\ \frac{1}{7} & -\frac{2}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix} \right)^{T} + \left[\begin{bmatrix} 5 \\ 3 \end{bmatrix}^{T} \left(\begin{bmatrix} -\frac{17}{7} \\ -\frac{1}{7} \end{bmatrix}^{T} + \boldsymbol{\lambda}^{T} \begin{bmatrix} -\frac{4}{7} & \frac{1}{7} \\ \frac{4}{7} & -\frac{1}{7} \\ \frac{1}{7} & -\frac{2}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix} \right)^{T}$$

$$+\lambda^{T} \begin{pmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \begin{bmatrix} -\frac{17}{7} \\ -\frac{1}{7} \\ -\frac{1}{7} \end{bmatrix}^{T} + \lambda^{T} \begin{pmatrix} \frac{4}{7} & -\frac{1}{7} \\ \frac{1}{7} & -\frac{2}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix} \end{pmatrix}^{T} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} \begin{bmatrix} -5 \\ -3 \end{bmatrix}^{T} + \lambda^{T} \begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{bmatrix} -\frac{17}{7} \\ -\frac{1}{7} \end{bmatrix}^{T} + \lambda^{T} \begin{pmatrix} \frac{4}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{2}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix} \end{pmatrix}^{T}$$

$$+ \begin{pmatrix} -\frac{88}{7} + \begin{pmatrix} -\frac{17}{7} \\ \frac{17}{7} \\ -\frac{1}{7} \\ -\frac{1}{7} \\ -\frac{1}{7} \\ -\frac{1}{7} \end{pmatrix} + \begin{pmatrix} -\frac{4}{7} & \frac{4}{7} & \frac{1}{7} & -\frac{1}{7} \\ \frac{1}{7} & -\frac{1}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{1}{7} & -\frac{1}{7} & \frac{2}{7} & -\frac{2}{7} \\ -\frac{1}{7} & \frac{1}{7} & -\frac{1}{7} & \frac{2}{7} & -\frac{2}{7} \\ -\frac{1}{7} & \frac{1}{7} & -\frac{1}{7} & \frac{1}{7} & \frac{2}{7} & -\frac{2}{7} \\ -\frac{1}{7} & \frac{1}{7} & -\frac{1}{7} & \frac{2}{7} & -\frac{2}{7} \end{pmatrix} \lambda - \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$=\frac{1}{2}\left(\begin{bmatrix} -5 \\ -3 \end{bmatrix}^T + \boldsymbol{\lambda}^T \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}\right) \left(\begin{bmatrix} -\frac{17}{7} \\ -\frac{1}{7} \end{bmatrix} + \begin{bmatrix} -\frac{4}{7} & \frac{1}{7} \\ \frac{4}{7} & -\frac{1}{7} \\ \frac{1}{7} & -\frac{2}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}^T \boldsymbol{\lambda}\right)$$

$$+ \begin{pmatrix} -\frac{88}{7} + \begin{bmatrix} -\frac{17}{7} \\ \frac{17}{7} \\ -\frac{1}{7} \\ \frac{1}{7} \end{bmatrix}^{T} \\ \boldsymbol{\lambda} \\ + \boldsymbol{\lambda}^{T} \begin{pmatrix} \begin{bmatrix} -\frac{24}{7} \\ \frac{10}{7} \\ -\frac{8}{7} \\ -\frac{6}{7} \end{bmatrix} + \begin{bmatrix} -\frac{4}{7} & \frac{4}{7} & \frac{1}{7} & -\frac{1}{7} \\ \frac{4}{7} & -\frac{4}{7} & -\frac{1}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{1}{7} & -\frac{2}{7} & \frac{2}{7} \\ -\frac{1}{7} & \frac{1}{7} & \frac{2}{7} & -\frac{2}{7} \end{bmatrix} \boldsymbol{\lambda} \end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix} \frac{17}{7} \\ \frac{17}{7} \\ -\frac{17}{7} \\ \frac{1}{7} \\ -\frac{1}{7} \end{pmatrix}^{T} \boldsymbol{\lambda} + \boldsymbol{\lambda}^{T} \begin{bmatrix} \frac{17}{7} \\ -\frac{17}{7} \\ \frac{1}{7} \\ -\frac{1}{7} \end{bmatrix} - \boldsymbol{\lambda}^{T} \begin{bmatrix} -\frac{4}{7} & \frac{4}{7} & \frac{1}{7} & -\frac{1}{7} \\ \frac{4}{7} & -\frac{4}{7} & -\frac{1}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{1}{7} & -\frac{2}{7} & \frac{2}{7} \\ -\frac{1}{7} & \frac{1}{7} & \frac{2}{7} & -\frac{2}{7} \end{bmatrix} \boldsymbol{\lambda}$$

$$+\begin{pmatrix} -\frac{88}{7} + \begin{bmatrix} -\frac{17}{7} \\ \frac{17}{7} \\ -\frac{1}{7} \\ \frac{1}{7} \end{bmatrix} \boldsymbol{\lambda} \\ -\frac{1}{7} \end{bmatrix} \boldsymbol{\lambda}$$

$$+\boldsymbol{\lambda}^{T} \begin{pmatrix} \begin{bmatrix} -\frac{24}{7} \\ \frac{10}{7} \\ -\frac{8}{7} \\ -\frac{6}{7} \end{bmatrix} + \begin{bmatrix} -\frac{4}{7} & \frac{4}{7} & \frac{1}{7} & -\frac{1}{7} \\ \frac{4}{7} & -\frac{4}{7} & -\frac{1}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{1}{7} & -\frac{2}{7} & \frac{2}{7} \\ -\frac{1}{7} & \frac{1}{7} & \frac{2}{7} & -\frac{2}{7} \end{bmatrix} \boldsymbol{\lambda}$$

$$=\frac{1}{2}\left(-\frac{88}{7}-\begin{bmatrix} \frac{17}{7} \\ -\frac{17}{7} \\ \frac{1}{7} \\ -\frac{1}{7} \end{bmatrix}^{T} \boldsymbol{\lambda}+\boldsymbol{\lambda}^{T}\begin{bmatrix} -\frac{31}{7} \\ \frac{3}{7} \\ -\frac{15}{7} \\ -\frac{13}{7} \end{bmatrix}+\boldsymbol{\lambda}^{T}\begin{bmatrix} -\frac{4}{7} & \frac{4}{7} & \frac{1}{7} & -\frac{1}{7} \\ \frac{4}{7} & -\frac{4}{7} & -\frac{1}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{1}{7} & -\frac{2}{7} & \frac{2}{7} \\ -\frac{1}{7} & \frac{1}{7} & \frac{2}{7} & -\frac{2}{7} \end{bmatrix}\boldsymbol{\lambda}\right)$$

$$= \frac{1}{2} \begin{pmatrix} -\frac{88}{7} - \boldsymbol{\lambda}^T \begin{bmatrix} \frac{17}{7} \\ -\frac{17}{7} \\ \frac{1}{7} \\ -\frac{1}{7} \end{bmatrix} + \boldsymbol{\lambda}^T \begin{bmatrix} -\frac{31}{7} \\ \frac{3}{7} \\ -\frac{15}{7} \\ -\frac{13}{7} \end{bmatrix} + \boldsymbol{\lambda}^T \begin{bmatrix} -\frac{4}{7} & \frac{4}{7} & \frac{1}{7} & -\frac{1}{7} \\ \frac{4}{7} & -\frac{4}{7} & -\frac{1}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{1}{7} & -\frac{2}{7} & \frac{2}{7} \\ -\frac{1}{7} & \frac{1}{7} & \frac{2}{7} & -\frac{2}{7} \end{bmatrix} \boldsymbol{\lambda} \end{pmatrix}$$

$$=rac{1}{2}egin{pmatrix} -rac{88}{7}+oldsymbol{\lambda}^T egin{bmatrix} -rac{48}{7} \ rac{20}{7} \ -rac{16}{7} \ -rac{12}{7} \end{bmatrix}+oldsymbol{\lambda}^T egin{bmatrix} -rac{4}{7} & rac{4}{7} & rac{1}{7} & -rac{1}{7} \ rac{4}{7} & -rac{4}{7} & -rac{1}{7} & rac{1}{7} \ rac{1}{7} & -rac{1}{7} & -rac{2}{7} & rac{2}{7} \ -rac{1}{7} & rac{1}{7} & rac{2}{7} & -rac{2}{7} \end{bmatrix}oldsymbol{\lambda}$$

Therefore, the dual optimization problem is given by

$$\max_{\boldsymbol{\lambda} \in \mathbb{R}^4} \frac{1}{2} \begin{pmatrix} -\frac{88}{7} + \boldsymbol{\lambda}^T & -\frac{18}{7} \\ -\frac{88}{7} + \boldsymbol{\lambda}^T & \frac{20}{7} \\ -\frac{16}{7} & -\frac{12}{7} \end{pmatrix} + \boldsymbol{\lambda}^T \begin{pmatrix} -\frac{4}{7} & \frac{4}{7} & \frac{1}{7} & -\frac{1}{7} \\ \frac{4}{7} & -\frac{4}{7} & -\frac{1}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{1}{7} & -\frac{2}{7} & \frac{2}{7} \\ -\frac{1}{7} & \frac{1}{7} & \frac{2}{7} & -\frac{2}{7} \end{pmatrix} \boldsymbol{\lambda}$$

subject to $\lambda \geq 0$.