7.7 Consider the quadratic program illustrated in Figure 7.4,

$$\min_{\mathbf{x}\in\mathbb{R}^2} \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{subject to} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Derive the dual quadratic program using Lagrange duality.

Solution.

Using 7.48a, the Lagrangian is given by

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \boldsymbol{\lambda}^T \left(\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

$$= \left(\frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \boldsymbol{\lambda}^T \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) - \boldsymbol{\lambda}^T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Taking the derivative with respect to $\mathbf{x}$ and setting it to zero gives (Using 5.107)

$$\frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \left(\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}^T \right) + \begin{bmatrix} 5 \\ 3 \end{bmatrix}^T + \boldsymbol{\lambda}^T \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} = \mathbf{0}$$

$$\implies \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \left(\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \right) + \begin{bmatrix} 5 \\ 3 \end{bmatrix}^T + \boldsymbol{\lambda}^T \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} = \mathbf{0}$$

$$\implies \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \left(\begin{bmatrix} 4 & 2 \\ 2 & 8 \end{bmatrix} \right) + \begin{bmatrix} 5 \\ 3 \end{bmatrix}^T + \boldsymbol{\lambda}^T \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} = \mathbf{0}$$

$$\implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix}^T + \boldsymbol{\lambda}^T \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} = \mathbf{0}$$

Matrix $\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ is invertible since determinant $= 4 * 2 - 1 * 1 = 7$.

Therefore :

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T = -\left(\begin{bmatrix} 5 \\ 3 \end{bmatrix}^T + \boldsymbol{\lambda}^T \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}\right) \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}^{-1}$$

$$\implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T = -\left(\begin{bmatrix} 5 \\ 3 \end{bmatrix}^T + \boldsymbol{\lambda}^T \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}\right) \begin{bmatrix} \frac{4}{7} & -\frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

$$\implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T = \begin{bmatrix} -\frac{17}{7} \\ -\frac{1}{7} \end{bmatrix}^T + \boldsymbol{\lambda}^T \begin{bmatrix} -\frac{4}{7} & \frac{1}{7} \\ \frac{4}{7} & -\frac{1}{7} \\ \frac{1}{7} & -\frac{2}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

Substituting in the expression for $\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda})$, we get

$$\mathcal{D}(\boldsymbol{\lambda}) = \tfrac{1}{2}\left(\begin{bmatrix} -\frac{17}{7} \\ -\frac{1}{7} \end{bmatrix}^T + \boldsymbol{\lambda}^T \begin{bmatrix} -\frac{4}{7} & \frac{1}{7} \\ \frac{4}{7} & -\frac{1}{7} \\ \frac{1}{7} & -\frac{2}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}\right) \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \left(\begin{bmatrix} -\frac{17}{7} \\ -\frac{1}{7} \end{bmatrix}^T + \boldsymbol{\lambda}^T \begin{bmatrix} -\frac{4}{7} & \frac{1}{7} \\ \frac{4}{7} & -\frac{1}{7} \\ \frac{1}{7} & -\frac{2}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}\right)^T$$

$$+ \begin{bmatrix} 5 \\ 3 \end{bmatrix}^T \left(\begin{bmatrix} -\frac{17}{7} \\ -\frac{1}{7} \end{bmatrix}^T + \boldsymbol{\lambda}^T \begin{bmatrix} -\frac{4}{7} & \frac{1}{7} \\ \frac{4}{7} & -\frac{1}{7} \\ \frac{1}{7} & -\frac{2}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}\right)^T$$

$$+\boldsymbol{\lambda}^T \left(\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \left(\begin{bmatrix} -\frac{17}{7} \\ -\frac{1}{7} \end{bmatrix}^T + \boldsymbol{\lambda}^T \begin{bmatrix} -\frac{4}{7} & \frac{1}{7} \\ \frac{4}{7} & -\frac{1}{7} \\ \frac{1}{7} & -\frac{2}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix} \right)^T - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

$$= \tfrac{1}{2} \left(\begin{bmatrix} -5 \\ -3 \end{bmatrix}^T + \boldsymbol{\lambda}^T \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \right) \left(\begin{bmatrix} -\frac{17}{7} \\ -\frac{1}{7} \end{bmatrix}^T + \boldsymbol{\lambda}^T \begin{bmatrix} -\frac{4}{7} & \frac{1}{7} \\ \frac{4}{7} & -\frac{1}{7} \\ \frac{1}{7} & -\frac{2}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix} \right)^T$$

$$+ \left(-\tfrac{88}{7} + \begin{bmatrix} -\frac{17}{7} \\ \frac{17}{7} \\ -\frac{1}{7} \\ \frac{1}{7} \end{bmatrix}^T \boldsymbol{\lambda} \right)$$

$$+\boldsymbol{\lambda}^T \left(\begin{bmatrix} -\frac{17}{7} \\ \frac{17}{7} \\ -\frac{1}{7} \\ \frac{1}{7} \end{bmatrix} + \begin{bmatrix} -\frac{4}{7} & \frac{4}{7} & \frac{1}{7} & -\frac{1}{7} \\ \frac{4}{7} & -\frac{4}{7} & -\frac{1}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{1}{7} & -\frac{2}{7} & \frac{2}{7} \\ -\frac{1}{7} & \frac{1}{7} & \frac{2}{7} & -\frac{2}{7} \end{bmatrix} \boldsymbol{\lambda} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

$$= \tfrac{1}{2} \left(\begin{bmatrix} -5 \\ -3 \end{bmatrix}^T + \boldsymbol{\lambda}^T \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \right) \left(\begin{bmatrix} -\frac{17}{7} \\ -\frac{1}{7} \end{bmatrix} + \begin{bmatrix} -\frac{4}{7} & \frac{1}{7} \\ \frac{4}{7} & -\frac{1}{7} \\ \frac{1}{7} & -\frac{2}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}^T \boldsymbol{\lambda} \right)$$

$$
+\left(-\frac{88}{7}+\begin{bmatrix}-\frac{17}{7}\\ \frac{17}{7}\\ -\frac{1}{7}\\ \frac{1}{7}\end{bmatrix}^T\boldsymbol{\lambda}\right)
$$

$$
+\boldsymbol{\lambda}^T\left(\begin{bmatrix}-\frac{24}{7}\\ \frac{10}{7}\\ -\frac{8}{7}\\ -\frac{6}{7}\end{bmatrix}+\begin{bmatrix}-\frac{4}{7} & \frac{4}{7} & \frac{1}{7} & -\frac{1}{7}\\ \frac{4}{7} & -\frac{4}{7} & -\frac{1}{7} & \frac{1}{7}\\ \frac{1}{7} & -\frac{1}{7} & -\frac{2}{7} & \frac{2}{7}\\ -\frac{1}{7} & \frac{1}{7} & \frac{2}{7} & -\frac{2}{7}\end{bmatrix}\boldsymbol{\lambda}\right)
$$

$$
=\frac{1}{2}\left(\frac{88}{7}+\begin{bmatrix}\frac{17}{7}\\ -\frac{17}{7}\\ \frac{1}{7}\\ -\frac{1}{7}\end{bmatrix}^T\boldsymbol{\lambda}+\boldsymbol{\lambda}^T\begin{bmatrix}\frac{17}{7}\\ -\frac{17}{7}\\ \frac{1}{7}\\ -\frac{1}{7}\end{bmatrix}-\boldsymbol{\lambda}^T\begin{bmatrix}-\frac{4}{7} & \frac{4}{7} & \frac{1}{7} & -\frac{1}{7}\\ \frac{4}{7} & -\frac{4}{7} & -\frac{1}{7} & \frac{1}{7}\\ \frac{1}{7} & -\frac{1}{7} & -\frac{2}{7} & \frac{2}{7}\\ -\frac{1}{7} & \frac{1}{7} & \frac{2}{7} & -\frac{2}{7}\end{bmatrix}\boldsymbol{\lambda}\right)
$$

$$
+\left(-\frac{88}{7}+\begin{bmatrix}-\frac{17}{7}\\ \frac{17}{7}\\ -\frac{1}{7}\\ \frac{1}{7}\end{bmatrix}^T\boldsymbol{\lambda}\right)
$$

$$
+\boldsymbol{\lambda}^T\left(\begin{bmatrix}-\frac{24}{7}\\ \frac{10}{7}\\ -\frac{8}{7}\\ -\frac{6}{7}\end{bmatrix}+\begin{bmatrix}-\frac{4}{7} & \frac{4}{7} & \frac{1}{7} & -\frac{1}{7}\\ \frac{4}{7} & -\frac{4}{7} & -\frac{1}{7} & \frac{1}{7}\\ \frac{1}{7} & -\frac{1}{7} & -\frac{2}{7} & \frac{2}{7}\\ -\frac{1}{7} & \frac{1}{7} & \frac{2}{7} & -\frac{2}{7}\end{bmatrix}\boldsymbol{\lambda}\right)
$$

$$= \frac{1}{2}\left(-\frac{88}{7} - \begin{bmatrix} \frac{17}{7} \\ -\frac{17}{7} \\ \frac{1}{7} \\ -\frac{1}{7} \end{bmatrix}^T \boldsymbol{\lambda} + \boldsymbol{\lambda}^T \begin{bmatrix} -\frac{31}{7} \\ \frac{3}{7} \\ -\frac{15}{7} \\ -\frac{13}{7} \end{bmatrix} + \boldsymbol{\lambda}^T \begin{bmatrix} -\frac{4}{7} & \frac{4}{7} & \frac{1}{7} & -\frac{1}{7} \\ \frac{4}{7} & -\frac{4}{7} & -\frac{1}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{1}{7} & -\frac{2}{7} & \frac{2}{7} \\ -\frac{1}{7} & \frac{1}{7} & \frac{2}{7} & -\frac{2}{7} \end{bmatrix} \boldsymbol{\lambda}\right)$$

$$= \frac{1}{2}\left(-\frac{88}{7} - \boldsymbol{\lambda}^T \begin{bmatrix} \frac{17}{7} \\ -\frac{17}{7} \\ \frac{1}{7} \\ -\frac{1}{7} \end{bmatrix} + \boldsymbol{\lambda}^T \begin{bmatrix} -\frac{31}{7} \\ \frac{3}{7} \\ -\frac{15}{7} \\ -\frac{13}{7} \end{bmatrix} + \boldsymbol{\lambda}^T \begin{bmatrix} -\frac{4}{7} & \frac{4}{7} & \frac{1}{7} & -\frac{1}{7} \\ \frac{4}{7} & -\frac{4}{7} & -\frac{1}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{1}{7} & -\frac{2}{7} & \frac{2}{7} \\ -\frac{1}{7} & \frac{1}{7} & \frac{2}{7} & -\frac{2}{7} \end{bmatrix} \boldsymbol{\lambda}\right)$$

$$= \frac{1}{2}\left(-\frac{88}{7} + \boldsymbol{\lambda}^T \begin{bmatrix} -\frac{48}{7} \\ \frac{20}{7} \\ -\frac{16}{7} \\ -\frac{12}{7} \end{bmatrix} + \boldsymbol{\lambda}^T \begin{bmatrix} -\frac{4}{7} & \frac{4}{7} & \frac{1}{7} & -\frac{1}{7} \\ \frac{4}{7} & -\frac{4}{7} & -\frac{1}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{1}{7} & -\frac{2}{7} & \frac{2}{7} \\ -\frac{1}{7} & \frac{1}{7} & \frac{2}{7} & -\frac{2}{7} \end{bmatrix} \boldsymbol{\lambda}\right)$$

Therefore, the dual optimization problem is given by

$$\max_{\boldsymbol{\lambda} \in \mathbb{R}^4} \frac{1}{2}\left(-\frac{88}{7} + \boldsymbol{\lambda}^T \begin{bmatrix} -\frac{48}{7} \\ \frac{20}{7} \\ -\frac{16}{7} \\ -\frac{12}{7} \end{bmatrix} + \boldsymbol{\lambda}^T \begin{bmatrix} -\frac{4}{7} & \frac{4}{7} & \frac{1}{7} & -\frac{1}{7} \\ \frac{4}{7} & -\frac{4}{7} & -\frac{1}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{1}{7} & -\frac{2}{7} & \frac{2}{7} \\ -\frac{1}{7} & \frac{1}{7} & \frac{2}{7} & -\frac{2}{7} \end{bmatrix} \boldsymbol{\lambda}\right)$$

$$\text{subject to } \boldsymbol{\lambda} \geq \mathbf{0}.$$