6.12 Manipulation of Gaussian Random Variables.

Consider a Gaussian random variable $\mathbf{x} \sim \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_x, \boldsymbol{\Sigma}_x)$, where $\mathbf{x} \in \mathbb{R}^D$. Furthermore, we have

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b} + \mathbf{w}$$

,

where $\mathbf{y} \in \mathbb{R}^E, \mathbf{A} \in \mathbb{R}^{E \times D}, \mathbf{b} \in \mathbb{R}^E$, and $\mathbf{w} \sim \mathcal{N}(\mathbf{w}|\mathbf{0}, \mathbf{Q})$ is independent Gaussian noise. "Independent" implies that $\mathbf{x}$ and $\mathbf{w}$ are independent random variables and that $\mathbf{Q}$ is diagonal.

a. Write down the likelihood $p(\mathbf{y}|\mathbf{x})$.

Solution.

$$p(\mathbf{y}) = p(\mathbf{A}\mathbf{x} + \mathbf{b} + \mathbf{w})$$

Linearity properties of Gaussian mixtures tell us that $\mathbf{A}\mathbf{x} + \mathbf{b} + \mathbf{w}$ is a Gaussian distribution.

$$\begin{aligned}
\mathbb{E}[\mathbf{y}] &= \mathbb{E}[\mathbf{A}\mathbf{x} + \mathbf{b} + \mathbf{w}] \\
&= \mathbb{E}[\mathbf{A}\mathbf{x}] + \mathbb{E}[\mathbf{b}] + \mathbb{E}[\mathbf{w}] \\
&= \mathbf{A}\mathbb{E}[\mathbf{x}] + \mathbf{b} + \mathbf{0} \\
&= \mathbf{A}\boldsymbol{\mu}_{\mathbf{x}} + \mathbf{b}
\end{aligned}$$

$$\mathbb{V}[\mathbf{y}] = \mathbb{V}[\mathbf{A}\mathbf{x} + \mathbf{b} + \mathbf{w}]$$

$$= \mathbb{V}[\mathbf{Ax} + \mathbf{w}] \qquad \text{Since } \mathbf{b} \text{ is a constant}$$

$$= \mathbb{V}[\mathbf{Ax}] + \mathbb{V}[\mathbf{w}] + Cov[\mathbf{Ax}, \mathbf{w}] + Cov[\mathbf{w}, \mathbf{Ax}]$$

$$= \mathbf{A}\mathbb{V}[\mathbf{x}]\mathbf{A}^T + \mathbf{Q} + \mathbf{0} + \mathbf{0} \qquad \text{since } \mathbf{x} \text{ and } \mathbf{w} \text{ are independent}$$

$$= \mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{A}^T + \mathbf{Q}$$

$$\implies \boldsymbol{\mu}_{\mathbf{y}} = \mathbf{A}\boldsymbol{\mu}_{\mathbf{x}} + \mathbf{b}, \qquad \boldsymbol{\Sigma}_{\mathbf{yy}} = \mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{A}^T + \mathbf{Q}$$

From 6.65, 6.66 and 6.67, we know that

$$p(\mathbf{y}|\mathbf{x}) = \mathcal{N}(\boldsymbol{\mu}_{\mathbf{y}|\mathbf{x}}, \boldsymbol{\Sigma}_{\mathbf{y}|\mathbf{x}})$$

$$\text{where, } \boldsymbol{\mu}_{\mathbf{y}|\mathbf{x}} = \boldsymbol{\mu}_{\mathbf{y}} + \boldsymbol{\Sigma}_{\mathbf{yx}}\boldsymbol{\Sigma}_{\mathbf{xx}}^{-1}(\mathbf{x} - \boldsymbol{\mu}_{\mathbf{x}})$$

$$\text{and } \boldsymbol{\Sigma}_{\mathbf{y}|\mathbf{x}} = \boldsymbol{\Sigma}_{\mathbf{yy}} - \boldsymbol{\Sigma}_{\mathbf{yx}}\boldsymbol{\Sigma}_{\mathbf{xx}}^{-1}\boldsymbol{\Sigma}_{\mathbf{xy}}.$$

From 6.37, we know that

$$\boldsymbol{\Sigma}_{\mathbf{yx}} = \mathbb{E}[\mathbf{y}\mathbf{x}^T] - \mathbb{E}[\mathbf{y}]\mathbb{E}[\mathbf{x}]^T = \boldsymbol{\Sigma}_{\mathbf{xy}}^T$$

$$\mathbb{E}[\mathbf{y}\mathbf{x}^T] = \mathbb{E}[(\mathbf{Ax} + \mathbf{b} + \mathbf{w})\mathbf{x}^T]$$

$$= \mathbb{E}[\mathbf{Axx}^T] + \mathbb{E}[\mathbf{bx}^T] + \mathbb{E}[\mathbf{wx}^T]$$

$$= \mathbf{A}\mathbb{E}[\mathbf{xx}^T] + \mathbf{b}\mathbb{E}[\mathbf{x}^T] + \mathbb{E}[\mathbf{wx}^T]$$

$$= \mathbf{A}(\boldsymbol{\Sigma}_{\mathbf{xx}} + \boldsymbol{\mu}_{\mathbf{x}}\boldsymbol{\mu}_{\mathbf{x}}^T) + \mathbf{b}\boldsymbol{\mu}_{\mathbf{x}}^T + \mathbf{0}$$

$$= \mathbf{A}(\boldsymbol{\Sigma}_{\mathbf{xx}} + \boldsymbol{\mu}_{\mathbf{x}}\boldsymbol{\mu}_{\mathbf{x}}^T) + \mathbf{b}\boldsymbol{\mu}_{\mathbf{x}}^T$$

$$\implies \boldsymbol{\Sigma}_{\mathbf{yx}} = \mathbf{A}(\boldsymbol{\Sigma}_{\mathbf{xx}} + \boldsymbol{\mu}_{\mathbf{x}}\boldsymbol{\mu}_{\mathbf{x}}^T) + \mathbf{b}\boldsymbol{\mu}_{\mathbf{x}}^T - (\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}} + \mathbf{b})(\boldsymbol{\mu}_{\mathbf{x}})^T$$

$$= \mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}} + \mathbf{A}\boldsymbol{\mu}_{\mathbf{x}}\boldsymbol{\mu}_{\mathbf{x}}^T + \mathbf{b}\boldsymbol{\mu}_{\mathbf{x}}^T - \mathbf{A}\boldsymbol{\mu}_{\mathbf{x}}\boldsymbol{\mu}_{\mathbf{x}}^T - \mathbf{b}\boldsymbol{\mu}_{\mathbf{x}}^T$$

$$= \mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}}$$

$$\implies \boldsymbol{\mu}_{\mathbf{y}|\mathbf{x}} = (\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}} + \mathbf{b}) + (\mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}})\boldsymbol{\Sigma}_{\mathbf{xx}}^{-1}(\mathbf{x} - \boldsymbol{\mu}_{\mathbf{x}})$$

$$= \mathbf{A}\boldsymbol{\mu}_{\mathbf{x}} + \mathbf{b} + \mathbf{A}\mathbf{x} - \mathbf{A}\boldsymbol{\mu}_{\mathbf{x}}$$

$$= \mathbf{A}\mathbf{x} + \mathbf{b}$$

and

$$\boldsymbol{\Sigma}_{\mathbf{y}|\mathbf{x}} = (\mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{A}^T + \mathbf{Q}) - (\mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}})\boldsymbol{\Sigma}_{\mathbf{xx}}^{-1}(\mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}})^T$$

$$= (\mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{A}^T + \mathbf{Q}) - \mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}}^T\mathbf{A}^T$$

$$= \mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{A}^T + \mathbf{Q} - \mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{A}^T$$

$$= \mathbf{Q}$$

$$\implies p(\mathbf{y}|\mathbf{x}) = \mathcal{N}(\mathbf{A}\mathbf{x} + \mathbf{b}, \mathbf{Q})$$

b. The distribution $p(\mathbf{y}) = \int p(\mathbf{y}|\mathbf{x})p(\mathbf{x})d\mathbf{x}$ is Gaussian. Compute the mean $\boldsymbol{\mu}_{\mathbf{y}}$ and the covariance $\boldsymbol{\Sigma}_{\mathbf{y}}$. Derive your result in detail.

Solution.

$\boldsymbol{\mu}_{\mathbf{y}} = \mathbf{A}\boldsymbol{\mu}_{\mathbf{x}} + \mathbf{b}, \qquad \boldsymbol{\Sigma}_{\mathbf{yy}} = \mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{A}^T + \mathbf{Q}$

Already calculated above.

c. The random variable $\mathbf{y}$ is being transformed according to the measurement mapping

$$\mathbf{z} = \mathbf{C}\mathbf{y} + \mathbf{v}$$

where $\mathbf{z} \in \mathbb{R}^F, \mathbf{C} \in \mathbb{R}^{F\times E}$, and $\mathbf{v} \sim (\mathbf{v}|\mathbf{0}, \mathbf{R})$ is independent Gaussian (measurement) noise.

- Write down $p(\mathbf{z}|\mathbf{y})$.

  Solution.

  Linear transformation of a Gaussian variable gives us another Gaussian variable.

$$\mathbf{z} \sim \mathcal{N}(\mathbf{z}|\boldsymbol{\mu}_z, \boldsymbol{\Sigma}_z)$$

  From 6.65, 6.66 and 6.67, we know that

$$p(\mathbf{z}|\mathbf{y}) = \mathcal{N}(\boldsymbol{\mu}_{\mathbf{z}|\mathbf{y}}, \boldsymbol{\Sigma}_{\mathbf{z}|\mathbf{y}})$$

$$\text{where, } \boldsymbol{\mu}_{\mathbf{z}|\mathbf{y}} = \boldsymbol{\mu}_{\mathbf{z}} + \boldsymbol{\Sigma}_{\mathbf{zy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{y}})$$

$$\text{and } \boldsymbol{\Sigma}_{\mathbf{z}|\mathbf{y}} = \boldsymbol{\Sigma}_{\mathbf{zz}} - \boldsymbol{\Sigma}_{\mathbf{zy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\boldsymbol{\Sigma}_{\mathbf{yz}}.$$

$$\boldsymbol{\mu}_{\mathbf{z}} = \mathbb{E}[\mathbf{Cy} + \mathbf{v}] = \mathbb{E}[\mathbf{Cy}] + \mathbb{E}[\mathbf{v}] = \mathbf{C}\mathbb{E}[\mathbf{y}] + \mathbf{0} = \mathbf{C}(\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}} + \mathbf{b})$$

$$\boldsymbol{\Sigma}_{\mathbf{z}} = \mathbb{V}[\mathbf{Cy} + \mathbf{v}] = \mathbb{V}[\mathbf{Cy}] + \mathbb{V}[\mathbf{v}] + Cov[\mathbf{Cy}, \mathbf{v}] + Cov[\mathbf{v}, \mathbf{Cy}]$$

$$= \mathbf{C}\boldsymbol{\Sigma}_{\mathbf{y}}\mathbf{C}^T + \mathbf{R} + \mathbf{0} + \mathbf{0} = \mathbf{C}(\mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{A}^T + \mathbf{Q})\mathbf{C}^T + \mathbf{R}$$

$$\boldsymbol{\Sigma}_{\mathbf{zy}} = \boldsymbol{\Sigma}_{\mathbf{yz}}^T = \mathbb{E}[\mathbf{z}\mathbf{y}^T] - \mathbb{E}[\mathbf{z}]\mathbb{E}[\mathbf{y}]^T$$

$$= \mathbb{E}[(\mathbf{Cy}+\mathbf{v})\mathbf{y}^T] - (\mathbf{C}(\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}} + \mathbf{b}))(\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}} + \mathbf{b})^T$$

$$= \mathbb{E}[\mathbf{Cy}\mathbf{y}^T] + \mathbb{E}[\mathbf{v}\mathbf{y}^T] - (\mathbf{C}(\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}} + \mathbf{b}))(\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}} + \mathbf{b})^T$$

$$= \mathbf{C}\mathbb{E}[\mathbf{y}\mathbf{y}^T] + \mathbf{0} - (\mathbf{C}(\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}} + \mathbf{b}))(\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}} + \mathbf{b})^T$$

$$= \mathbf{C}(\boldsymbol{\Sigma}_{\mathbf{yy}} + \boldsymbol{\mu}_{\mathbf{y}}\boldsymbol{\mu}_{\mathbf{y}}^T) - (\mathbf{C}(\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}} + \mathbf{b}))(\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}} + \mathbf{b})^T$$

$$= \mathbf{C}((\mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{A}^T+\mathbf{Q})+(\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}}+\mathbf{b})(\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}}+\mathbf{b})^T)-(\mathbf{C}(\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}}+\mathbf{b}))(\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}}+\mathbf{b})^T$$

$$= \mathbf{C}(\mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{A}^T + \mathbf{Q})$$

$$\boldsymbol{\mu}_{\mathbf{z}|\mathbf{y}} = \mathbf{C}(\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}}+\mathbf{b})+\mathbf{C}(\mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{A}^T+\mathbf{Q})(\mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{A}^T+\mathbf{Q})^{-1}(\mathbf{y}-(\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}}+\mathbf{b}))$$

$$= \mathbf{C}(\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}} + \mathbf{b}) + \mathbf{C}(\mathbf{y} - (\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}} + \mathbf{b}))$$

$$= \mathbf{Cy}$$

$$\boldsymbol{\Sigma}_{\mathbf{z}|\mathbf{y}} = \boldsymbol{\Sigma}_{\mathbf{zz}} - \boldsymbol{\Sigma}_{\mathbf{zy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\boldsymbol{\Sigma}_{\mathbf{yz}}$$

$$= (\mathbf{C}(\mathbf{A}\mathbf{\Sigma}_{\mathbf{xx}}\mathbf{A}^T+\mathbf{Q})\mathbf{C}^T+\mathbf{R})-\mathbf{C}(\mathbf{A}\mathbf{\Sigma}_{\mathbf{xx}}\mathbf{A}^T+\mathbf{Q})(\mathbf{A}\mathbf{\Sigma}_{\mathbf{xx}}\mathbf{A}^T+\mathbf{Q})^{-1}(\mathbf{C}(\mathbf{A}\mathbf{\Sigma}_{\mathbf{xx}}\mathbf{A}^T+\mathbf{Q}))^T$$

$$= \mathbf{C}(\mathbf{A}\mathbf{\Sigma}_{\mathbf{xx}}\mathbf{A}^T + \mathbf{Q})\mathbf{C}^T + \mathbf{R} - \mathbf{C}(\mathbf{C}(\mathbf{A}\mathbf{\Sigma}_{\mathbf{xx}}\mathbf{A}^T + \mathbf{Q}))^T$$

$$= \mathbf{C}(\mathbf{A}\mathbf{\Sigma}_{\mathbf{xx}}\mathbf{A}^T + \mathbf{Q})\mathbf{C}^T + \mathbf{R} - \mathbf{C}(\mathbf{A}\mathbf{\Sigma}_{\mathbf{xx}}\mathbf{A}^T + \mathbf{Q})^T\mathbf{C}^T$$

$$= \mathbf{C}(\mathbf{A}\mathbf{\Sigma}_{\mathbf{xx}}\mathbf{A}^T + \mathbf{Q})\mathbf{C}^T + \mathbf{R} - \mathbf{C}(\mathbf{A}{\mathbf{\Sigma}_{\mathbf{xx}}}^T\mathbf{A}^T + \mathbf{Q}^T)\mathbf{C}^T$$

$$= \mathbf{C}(\mathbf{A}\mathbf{\Sigma}_{\mathbf{xx}}\mathbf{A}^T + \mathbf{Q})\mathbf{C}^T + \mathbf{R} - \mathbf{C}(\mathbf{A}\mathbf{\Sigma}_{\mathbf{xx}}\mathbf{A}^T + \mathbf{Q})\mathbf{C}^T$$

$$= \mathbf{R}$$

$$\implies p(\mathbf{z}|\mathbf{y}) = \mathcal{N}(\mathbf{C}\mathbf{y}, \mathbf{R})$$

- Compute $p(\mathbf{z})$, i.e., the mean $\boldsymbol{\mu}_{\mathbf{z}}$, and the covariance $\mathbf{\Sigma}_{\mathbf{z}}$. Derive your result in detail.

  Solution.

  As calculated above,

$$\boldsymbol{\mu}_{\mathbf{z}} = \mathbf{C}(\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}} + \mathbf{b})$$

$$\mathbf{\Sigma}_{\mathbf{z}} = \mathbf{C}(\mathbf{A}\mathbf{\Sigma}_{\mathbf{xx}}\mathbf{A}^T + \mathbf{Q})\mathbf{C}^T + \mathbf{R}$$

$$\implies p(\mathbf{z}) = \mathcal{N}(\mathbf{C}(\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}} + \mathbf{b}), \mathbf{C}(\mathbf{A}\mathbf{\Sigma}_{\mathbf{xx}}\mathbf{A}^T + \mathbf{Q})\mathbf{C}^T + \mathbf{R})$$

d. Now, a value $\hat{\mathbf{y}}$ is measured. Compute the posterior distribution $p(\mathbf{x}|\hat{\mathbf{y}})$. Hint for solution: This posterior is also Gaussian, i.e., we need to determine only its mean and covariance matrix. Start by explicitly computing the joint Gaussian $p(\mathbf{x}, \mathbf{y})$. This also requires us to compute the cross-covariances $Cov_{\mathbf{x},\mathbf{y}}[\mathbf{x}, \mathbf{y}]$ and $Cov_{\mathbf{y},\mathbf{x}}[\mathbf{y}, \mathbf{x}]$. Then apply the rules for Gaussian conditioning.

Solution.

Since conditionals of Gaussians are Gaussians, $p(\mathbf{x}|\mathbf{y})$ is also a Gaussian.

From 6.65, 6.66 and 6.67, we know that

$$p(\mathbf{x}|\mathbf{y}) = \mathcal{N}(\boldsymbol{\mu}_{\mathbf{x}|\mathbf{y}}, \boldsymbol{\Sigma}_{\mathbf{x}|\mathbf{y}})$$

$$\text{where, } \boldsymbol{\mu}_{\mathbf{x}|\mathbf{y}} = \boldsymbol{\mu}_{\mathbf{x}} + \boldsymbol{\Sigma}_{\mathbf{xy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{y}})$$

$$\text{and } \boldsymbol{\Sigma}_{\mathbf{x}|\mathbf{y}} = \boldsymbol{\Sigma}_{\mathbf{xx}} - \boldsymbol{\Sigma}_{\mathbf{xy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\boldsymbol{\Sigma}_{\mathbf{yx}}.$$

$$\boldsymbol{\mu}_{\mathbf{y}} = \mathbf{A}\boldsymbol{\mu}_{\mathbf{x}} + \mathbf{b}$$

$$\boldsymbol{\Sigma}_{\mathbf{yy}} = \mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{A}^T + \mathbf{Q}$$

$$\boldsymbol{\Sigma}_{\mathbf{yx}} = \mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}}$$

$$\boldsymbol{\Sigma}_{\mathbf{xy}} = (\mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}})^T = \boldsymbol{\Sigma}_{\mathbf{xx}}^T\mathbf{A}^T = \boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{A}^T$$

$$\implies \boldsymbol{\mu}_{\mathbf{x}|\mathbf{y}} = \boldsymbol{\mu}_{\mathbf{x}} + (\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{A}^T)(\mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{A}^T + \mathbf{Q})^{-1}(\mathbf{y} - (\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}} + \mathbf{b}))$$

and

$$\implies \boldsymbol{\Sigma}_{\mathbf{x}|\mathbf{y}} = \boldsymbol{\Sigma}_{\mathbf{xx}} - (\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{A}^T)(\mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{A}^T + \mathbf{Q})^{-1}\mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}}$$

$$\implies p(\mathbf{x}|\hat{\mathbf{y}})$$

$$=$$

$$\mathcal{N}\left(\boldsymbol{\mu}_{\mathbf{x}} + (\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{A}^T)(\mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{A}^T + \mathbf{Q})^{-1}(\hat{\mathbf{y}} - (\mathbf{A}\boldsymbol{\mu}_{\mathbf{x}} + \mathbf{b})), \boldsymbol{\Sigma}_{\mathbf{xx}} - (\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{A}^T)(\mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}}\mathbf{A}^T + \mathbf{Q})^{-1}\mathbf{A}\boldsymbol{\Sigma}_{\mathbf{xx}}\right)$$