4.9 Find the SVD of

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

Solution.

To obtain SVD, we need to compute the right singular vectors $\mathbf{v}_j$, the singular values σ_k, and the left-singular vectors $\mathbf{u}_i$.

Step 1: Right Singular vectors as the eigenbasis of $\mathbf{A}^T\mathbf{A}$

We start by computing

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

We compute the singular values and right-singular vectors $\mathbf{v}_j$ through the eigenvalue decomposition of $\mathbf{A}^T\mathbf{A}$.

For a 2x2 matrix, we can only have a maximum of 2 eigenvalues.

The determinant is $5 * 5 - 3 * 3 = 16$, and the trace is $5 + 15 = 10$.

It's trivial to see that the eigenvalues are $\lambda = 2, 8$

Now, to obtain eigenvectors, we solve for $(\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I}_3)\mathbf{x} = 0$

For $\lambda = 2$,

$$(\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I}_2)\mathbf{x} = 0$$

$$\implies (\mathbf{A}^T\mathbf{A} - 2\mathbf{I}_2)\mathbf{x} = 0$$

$$\implies \begin{bmatrix} 5-2 & 3 \\ 3 & 5-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies x_1 + x_2 = 0$$

$$\implies x_1 = -x_2$$

$$\text{Eigenvector is : } \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\text{Normalized eigenvector } \mathbf{v}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

For $\lambda = 8$,

$$(\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I}_2)\mathbf{x} = 0$$

$$\implies (\mathbf{A}^T\mathbf{A} - 8\mathbf{I}_2)\mathbf{x} = 0$$

$$\implies \begin{bmatrix} 5-8 & 3 \\ 3 & 5-8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies -x_1 + x_2 = 0$$

$$\Longrightarrow x_1 = x_2$$

$$\text{Eigenvector is : } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{Normalized eigenvector } \mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\text{Therefore, } \mathbf{V} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Step 2: Singular-value matrix

Singular values are square roots of the eigenvalues of $\mathbf{A}^T\mathbf{A}$.

$$\text{Therefore, } \mathbf{\Sigma} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{8} \end{bmatrix}$$

Step 3: Left-singular vectors as the normalized image of the right-singular vectors

$$\mathbf{u}_1 = \frac{1}{\sigma_1}\mathbf{A}\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\sigma_2}\mathbf{A}\mathbf{v}_2 = \frac{1}{\sqrt{8}} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\implies \mathbf{U} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\implies \mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{8} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$$