5.5 Consider the following functions:

$$f_1(\mathbf{x}) = sin(x_1)cos(x_2), \qquad \mathbf{x} \in \mathbb{R}^2$$

$$f_2(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T\mathbf{y}, \qquad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

$$f_3(\mathbf{x}) = \mathbf{x}\mathbf{x}^T, \qquad \mathbf{x} \in \mathbb{R}^n$$

a. What are the dimensions of $\frac{\partial f_i}{\partial \mathbf{x}}$?

Solution.

For $f_1(\mathbf{x}) = sin(x_1)cos(x_2)$,

It has 2 independent variables and 1 dependent variable. So the dimensions of $\frac{\partial f_1}{\partial \mathbf{x}}$ are 1x2.

$$\text{Jacobian } J = \begin{bmatrix} \frac{\partial sin(x_1)cos(x_2)}{\partial x_1} & \frac{\partial sin(x_1)cos(x_2)}{\partial x_2} \end{bmatrix}$$

$$\implies J = \begin{bmatrix} cos(x_1)cos(x_2) & -sin(x_1)sin(x_2) \end{bmatrix}$$

For $f_2(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T\mathbf{y}, \qquad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$\implies f_2(\mathbf{x}, \mathbf{y}) = \Sigma_{i=1}^n x_i y_i$

It has n independent variables (since derivative is w.r.t x and y is being treated as a constant) and 1 dependent variable. So the dimensions of $\frac{\partial f_1}{\partial \mathbf{x}}$ are 1xn.

$$\text{Jacobian } J = \begin{bmatrix} \frac{\partial \Sigma_{i=1}^{n} x_i y_i}{\partial x_1} & \frac{\partial \Sigma_{i=1}^{n} x_i y_i}{\partial x_2} & \dots & \frac{\partial \Sigma_{i=1}^{n} x_i y_i}{\partial x_n} & \frac{\partial \Sigma_{i=1}^{n} x_i y_i}{\partial y_1} & \frac{\partial \Sigma_{i=1}^{n} x_i y_i}{\partial y_2} & \dots & \frac{\partial \Sigma_{i=1}^{n} x_i y_i}{\partial y_n} \end{bmatrix}$$

$$\implies \text{Jacobian } J = \begin{bmatrix} y_1 & y_2 & \dots & y_n & x_1 & x_2 & \dots & x_n \end{bmatrix}$$

For $f_3(\mathbf{x}) = \mathbf{x}\mathbf{x}^T, \qquad \mathbf{x} \in \mathbb{R}^n,$

$$f_3(\mathbf{x}) = \begin{bmatrix} x_1x_1 & x_1x_2 & \dots & x_1x_n \\ x_2x_1 & x_2x_2 & \dots & x_2x_n \\ \vdots & \vdots & \dots & \vdots \\ x_nx_1 & x_nx_2 & \dots & x_nx_n \end{bmatrix}$$

It is an nxn matrix and the derivative is w.r.t a n size vector. So the dimensions of $\frac{\partial f_3}{\partial \mathbf{x}}$ are (nxn)xn.

$$J = \begin{bmatrix} \dfrac{\partial \begin{bmatrix} x_1x_1 & x_1x_2 & \dots & x_1x_n \\ x_2x_1 & x_2x_2 & \dots & x_2x_n \\ \vdots & \vdots & \dots & \vdots \\ x_nx_1 & x_nx_2 & \dots & x_nx_n \end{bmatrix}}{\partial x_1} & \dots & \dfrac{\partial \begin{bmatrix} x_1x_1 & x_1x_2 & \dots & x_1x_n \\ x_2x_1 & x_2x_2 & \dots & x_2x_n \\ \vdots & \vdots & \dots & \vdots \\ x_nx_1 & x_nx_2 & \dots & x_nx_n \end{bmatrix}}{\partial x_n} \end{bmatrix}$$

$$\implies \text{Jacobian } J = \begin{bmatrix} \begin{bmatrix} 2x_1 & x_2 & \dots & x_n \\ x_2 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ x_n & 0 & \dots & 0 \end{bmatrix} & \begin{bmatrix} 0 & x_1 & \dots & 0 \\ x_2 & 2x_2 & \dots & x_n \\ \vdots & \vdots & \dots & \vdots \\ 0 & x_n & \dots & 0 \end{bmatrix} & \dots & \begin{bmatrix} 0 & 0 & \dots & x_1 \\ 0 & 0 & \dots & x_2 \\ \vdots & \vdots & \dots & \vdots \\ x_1 & x_2 & \dots & 2x_n \end{bmatrix} \end{bmatrix}$$