2.5 Find the set S of all solutions in x of the following inhomogeneous linear system $Ax = b$, where A and b are defined as follows:

a.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 5 & -7 & -5 \\ 2 & -1 & 1 & 3 \\ 5 & 2 & -4 & 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 6 \end{bmatrix}$$

Solution.

$$\text{Augmented matrix}: \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 2 & 5 & -7 & -5 & -2 \\ 2 & -1 & 1 & 3 & 4 \\ 5 & 2 & -4 & 2 & 6 \end{array}\right]$$

Performing Gauss-Jordan elimination on the augmented matrix to obtain Reduced Row Echelon Form:

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 2 & 5 & -7 & -5 & -2 \\ 2 & -1 & 1 & 3 & 4 \\ 5 & 2 & -4 & 2 & 6 \end{array}\right] \begin{array}{c} \\ -2R_1 \\ -2R_1 \\ -5R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & -3 & 3 & 5 & 2 \\ 0 & -3 & 1 & 7 & 1 \end{array}\right] \begin{array}{c} -\frac{1}{3}R_2 \\ \cdot\frac{1}{3} \\ +R_2 \\ +R_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & 0 & \frac{7}{3} \\ 0 & 1 & -\frac{5}{3} & -1 & -\frac{4}{3} \\ 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & -4 & 4 & -3 \end{array}\right] \quad -2R_3$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & 0 & \frac{7}{3} \\ 0 & 1 & -\frac{5}{3} & -1 & -\frac{4}{3} \\ 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

The last row shows that the equation has no solution.

b.

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -3 & 0 \\ 2 & -1 & 0 & 1 & -1 \\ -1 & 2 & 0 & -2 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 6 \\ 5 \\ -1 \end{bmatrix}$$

Solution.

Augmented matrix :

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & -3 & 0 & 6 \\ 2 & -1 & 0 & 1 & -1 & 5 \\ -1 & 2 & 0 & -2 & -1 & -1 \end{array}\right]$$

Performing Gauss-Jordan elimination on the augmented matrix to obtain Reduced Row Echelon Form:

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & -3 & 0 & 6 \\ 2 & -1 & 0 & 1 & -1 & 5 \\ -1 & 2 & 0 & -2 & -1 & -1 \end{array}\right] \begin{array}{c} \\ -R_1 \\ -2R_1 \\ +R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 2 & 0 & -3 & -1 & 3 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 1 & 0 & -2 & 0 & 2 \end{array}\right] \begin{array}{c} \\ -R_3 \\ \\ -R_3 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -4 & 2 & 4 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & -3 & 3 & 3 \end{array}\right] \begin{array}{c} \\ \\ -R_2 \\ \cdot -\frac{1}{3} \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -4 & 2 & 4 \\ 0 & 0 & 0 & 5 & -5 & -5 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{array}\right] \begin{array}{c} \\ \\ \cdot\frac{1}{5} \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -4 & 2 & 4 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{array}\right] \begin{array}{c} \\ +4R_3 \\ \\ -R_3 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} +R_2 \\ \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} +R_2 \\ \\ \\ \\ \end{array}$$

$$\Longrightarrow x_1 - x_5 = 3, \quad x_2 - 2 * x_5 = 0, \quad x_4 - x_5 = -1$$

$$\Longrightarrow x_1 - x_5 = 3, \quad x_2 = 2 * x_5, \quad x_4 - x_5 = -1$$

Particular solution: $x_5 = x_3 = 0, \quad x_1 = 3, \quad x_2 = 0, \quad x_4 = -1$

$$= \mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

For General solution, we express each column as the sum of pivot columns on the left: $x_3 = 0 * x_1 + 0 * x_2, \quad x_5 = -1 * x_1 - 2 * x_2 - 1 * x_4$

Full solution:

$$S := \left\{ \mathbf{x} \in \mathbb{R}^6 : \mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} -1 \\ -2 \\ 0 \\ -1 \\ -1 \end{bmatrix} \middle| \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$