3.4 Compute the angle between

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

using

a. $\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^T \mathbf{y}$

Solution.

If the angle between $\mathbf{x}$ and $\mathbf{y}$ is θ,

$$cos\theta = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|\|\mathbf{y}\|}$$

$$\langle \mathbf{x}, \mathbf{y} \rangle = -1 - 2 = -3$$

$$\|\mathbf{x}\| = \sqrt{1+4} = \sqrt{5}, \qquad \|\mathbf{y}\| = \sqrt{1+1} = \sqrt{2}$$

$$\implies \theta = cos^{-1}\left(\frac{-3}{\sqrt{10}}\right)$$

b. $\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^T \mathbf{B} \mathbf{y}, \qquad \mathbf{B} := \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

Solution.

Similarly,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -11$$

$$\|\mathbf{x}\| = \sqrt{\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}} = \sqrt{\begin{bmatrix} 4 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}} = \sqrt{18}$$

$$\|\mathbf{y}\| = \sqrt{\begin{bmatrix} -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix}} = \sqrt{\begin{bmatrix} -3 & -4 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix}} = \sqrt{7}$$

$$\implies \theta = cos^{-1}\left(\frac{-11}{\sqrt{18*7}}\right) = cos^{-1}\left(\frac{-11}{\sqrt{126}}\right)$$

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