4.12 Show that for $\mathbf{x} \neq \mathbf{0}$ Theorem 4.24 holds, i.e., show that

$$\max_{x} \frac{\|\mathbf{A}\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \sigma_1$$

where σ_1 is the largest singular value of $\mathbf{A} \in \mathbb{R}^{m \times n}$.

Solution.

The SVD of $\mathbf{A}$ gives us:

$$\mathbf{A}_{m\times n} = \mathbf{U}_{m\times m} \mathbf{\Sigma}_{m\times n} \mathbf{V}^T_{n\times n}$$

Now,

$$\|\mathbf{A}\mathbf{x}\|_2^2 = ((\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x}))$$

$$\implies \|\mathbf{A}\mathbf{x}\|_2^2 = (\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T\mathbf{x})^T(\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T\mathbf{x})$$

$$\implies \|\mathbf{A}\mathbf{x}\|_2^2 = \mathbf{x}^T\mathbf{V}\mathbf{\Sigma}^T\mathbf{U}^T\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T\mathbf{x}$$

$$\implies \|\mathbf{A}\mathbf{x}\|_2^2 = \mathbf{x}^T\mathbf{V}\mathbf{\Sigma}^T\mathbf{\Sigma}\mathbf{V}^T\mathbf{x}$$

$$\implies \|\mathbf{A}\mathbf{x}\|_2^2 = \mathbf{x}^T\mathbf{V}\mathbf{\Sigma}^2\mathbf{V}^T\mathbf{x}$$

$$\implies \|\mathbf{A}\mathbf{x}\|_2^2 = (\mathbf{V}^T\mathbf{x})^T\mathbf{\Sigma}^2(\mathbf{V}^T\mathbf{x})$$

Let $\mathbf{y} = \mathbf{V}^T\mathbf{x}$. We get

$$\|\mathbf{A}\mathbf{x}\|_2^2 = \mathbf{y}^T\mathbf{\Sigma}^2\mathbf{y}$$

Since $\mathbf{\Sigma}$ is a diagonal matrix containing the singular values, the expression becomes

$$\|\mathbf{Ax}\|_2^2 = \begin{bmatrix} y_0 & y_1 & \dots & y_n \end{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & \sigma_n^2 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$\implies \|\mathbf{Ax}\|_2^2 = \Sigma_{k=1}^n (y_i \sigma_i)^2 \qquad Eqn.(1)$$

Let us assume that the singular values are ordered, and σ_1 is the largest singular value. Then we can say that

$$\forall \mathbf{y} \in \mathbb{R}^n, \sigma_1^2 \Sigma_{k=1}^n (y_i)^2 \geq \Sigma_{k=1}^n (y_i \sigma_i)^2 \qquad Eqn.(2)$$

Also,

$$\|\mathbf{y}\|_2^2 = \Sigma_{k=1}^n (y_i)^2 = (\mathbf{V}^T\mathbf{x})^T(\mathbf{V}^T\mathbf{x}) = (\mathbf{x}^T\mathbf{V}\mathbf{V}^T\mathbf{x}) = \mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|_2^2$$

$$\implies \Sigma_{k=1}^n (y_i)^2 = \|\mathbf{x}\|_2^2 \qquad Eqn.(3)$$

Substituting the results of $Eqn.(1)$ and $Eqn.(3)$ into $Eqn.(2)$, we get

$$\sigma_1^2 \|\mathbf{x}\|_2^2 \geq \|\mathbf{Ax}\|_2^2$$

$$\implies \sigma_1^2 \geq \frac{\|\mathbf{Ax}\|_2^2}{\|\mathbf{x}\|_2^2}$$

$$\implies \sigma_1 \geq \frac{\|\mathbf{Ax}\|}{\|\mathbf{x}\|}$$

Thus, σ_1 is the upper bound on $\frac{\|\mathbf{Ax}\|}{\|\mathbf{x}\|}$, and therefore $\max\limits_x \frac{\|\mathbf{Ax}\|}{\|\mathbf{x}\|} = \sigma_1, \forall \mathbf{x} \neq \mathbf{0}$.