2.7 Find all solutions in $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ of the equation system $\mathbf{Ax} = 12\mathbf{x}$, where

$$\mathbf{A} = \begin{bmatrix} 6 & 4 & 3 \\ 6 & 0 & 9 \\ 0 & 8 & 0 \end{bmatrix}$$

and $\Sigma_{i=1}^{3} x_i = 1$

Solution.

$\mathbf{Ax} = 12\mathbf{x}$ can be written as $\mathbf{Ax} = 12\mathbf{Ix}$

$$\implies \mathbf{Ax} - 12\mathbf{Ix} = 0$$

$$\implies \begin{bmatrix} 6-12 & 4 & 3 \\ 6 & 0-12 & 9 \\ 0 & 8 & 0-12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} -6 & 4 & 3 \\ 6 & -12 & 9 \\ 0 & 8 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Also $\Sigma_{i=1}^{3} x_i = 1 \implies \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

This can be added to the the above expression, giving us

$$\implies \begin{bmatrix} -6 & 4 & 3 \\ 6 & -12 & 9 \\ 0 & 8 & -12 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Augmented matrix } \tilde{A} = \left[\begin{array}{ccc|c} -6 & 4 & 3 & 0 \\ 6 & -12 & 9 & 0 \\ 0 & 8 & -12 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

Applying Gauss Jordan reduction to convert it into Row Echelon form:

$$\left[\begin{array}{ccc|c} -6 & 4 & 3 & 0 \\ 6 & -12 & 9 & 0 \\ 0 & 8 & -12 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} +7R_4 \\ -6R_4 \\ \\ -R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 11 & 10 & 7 \\ 0 & -18 & 3 & -6 \\ 0 & 8 & -12 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} \\ \cdot -\frac{1}{3} \\ \cdot \frac{1}{4} \\ -R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 11 & 10 & 7 \\ 0 & 6 & -1 & 2 \\ 0 & 2 & -3 & 0 \\ 0 & -10 & -9 & -6 \end{array}\right] \begin{array}{l} \\ \text{Swap with } R_3 \\ \text{Swap with } R_2 \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 11 & 10 & 7 \\ 0 & 2 & -3 & 0 \\ 0 & 6 & -1 & 2 \\ 0 & -10 & -9 & -6 \end{array}\right] \begin{array}{c} \\ \\ -3R_2 \\ +5R_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 11 & 10 & 7 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 8 & 2 \\ 0 & 0 & -24 & -6 \end{array}\right] \begin{array}{c} \\ \\ \cdot\frac{1}{2} \\ \cdot-\frac{1}{6} \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 11 & 10 & 7 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 4 & 1 \end{array}\right] \begin{array}{c} \\ \cdot\frac{1}{2} \\ \\ -R_3 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 11 & 10 & 7 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} -11R_2 \\ \\ \cdot\frac{1}{4} \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{53}{2} & 7 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} -\frac{53}{2}R_3 \\ +\frac{3}{2}R_3 \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{8} \\ 0 & 1 & 0 & \frac{3}{8} \\ 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So, we have a unique solution,

$$\mathbf{x} = \begin{bmatrix} \frac{3}{8} \\ \frac{3}{8} \\ \frac{1}{4} \end{bmatrix}$$