7.4 Consider whether the following statements are true or false:

a. The sum of any two convex functions is convex.

Solution.

Let f_1 and f_2 be two convex functions such that

$$f_1(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f_1(\mathbf{x}) + (1-\theta)f_1(\mathbf{y})$$

and

$$f_2(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f_2(\mathbf{x}) + (1-\theta)f_2(\mathbf{y})$$

where $0 \leq \theta \leq 1$.

Summing up the two equations, we get

$f_1(\theta\mathbf{x}+(1-\theta)\mathbf{y})+f_2(\theta\mathbf{x}+(1-\theta)\mathbf{y}) \leq \theta f_1(\mathbf{x})+(1-\theta)f_1(\mathbf{y})+\theta f_2(\mathbf{x})+(1-\theta)f_2(\mathbf{y})$

$\implies f_1(\theta\mathbf{x}+(1-\theta)\mathbf{y})+f_2(\theta\mathbf{x}+(1-\theta)\mathbf{y}) \leq \theta(f_1(\mathbf{x})+f_2(\mathbf{x}))+(1-\theta)(f_1(\mathbf{y})+f_2(\mathbf{y}))$

$\implies$ The sum of functions f_1 and f_2 is convex.

b. The difference of any two convex functions is convex.

Solution.

Let $f_1 = x^2$ and $f_2 = 2x^2$ be two convex functions.

Then, $f_2 - f_1 = -x^2$ which is not convex.

Therefore, the difference of any two convex functions is not convex.

c. The product of any two convex functions is convex.

Solution.

Let $f_1 = x^2$ and $f_2 = x$ be two convex functions.

Then, $f_2 * f_1 = x^3$ which is not convex.

Therefore, the product of any two convex functions is not convex.

d. The maximum of any two convex functions is convex.

Solution.

Let f_1 and f_2 be two convex functions such that

$$f_1(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \le \theta f_1(\mathbf{x}) + (1-\theta)f_1(\mathbf{y})$$

and

$$f_2(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \le \theta f_2(\mathbf{x}) + (1-\theta)f_2(\mathbf{y})$$

Let $h = max\{f_1, f_2\}$

To prove:

$$\theta h(\mathbf{x}) + (1-\theta)h(\mathbf{y}) \ge h(\theta\mathbf{x} + (1-\theta)\mathbf{y})$$

From the definitions of the convex function above, we can see that

$$max(\theta f_2(\mathbf{x}) + (1-\theta)f_2(\mathbf{y}), \theta f_1(\mathbf{x}) + (1-\theta)f_1(\mathbf{y}))$$

$$\ge$$

$$max(f_1(\theta\mathbf{x} + (1-\theta)\mathbf{y}), f_2(\theta\mathbf{x} + (1-\theta)\mathbf{y}))$$

$$\implies max(\theta f_2(\mathbf{x}) + \theta f_1(\mathbf{x})) + max((1-\theta)f_2(\mathbf{y}) + (1-\theta)f_1(\mathbf{y}))$$

$$\geq$$

$$max(\theta f_2(\mathbf{x}) + (1-\theta)f_2(\mathbf{y}), \theta f_1(\mathbf{x}) + (1-\theta)f_1(\mathbf{y}))$$

$$\geq$$

$$max(f_1(\theta\mathbf{x} + (1-\theta)\mathbf{y}), f_2(\theta\mathbf{x} + (1-\theta)\mathbf{y}))$$

$$\implies max(\theta f_2(\mathbf{x}) + \theta f_1(\mathbf{x})) + max((1-\theta)f_2(\mathbf{y}) + (1-\theta)f_1(\mathbf{y}))$$

$$\geq$$

$$max(f_1(\theta\mathbf{x} + (1-\theta)\mathbf{y}), f_2(\theta\mathbf{x} + (1-\theta)\mathbf{y}))$$

$$\implies \theta max(f_2(\mathbf{x}) + f_1(\mathbf{x})) + (1-\theta)max(f_2(\mathbf{y}) + f_1(\mathbf{y}))$$

$$\geq$$

$$max(f_1(\theta\mathbf{x} + (1-\theta)\mathbf{y}), f_2(\theta\mathbf{x} + (1-\theta)\mathbf{y}))$$

$$\implies \theta h(\mathbf{x}) + (1-\theta)h(\mathbf{y}) \geq [h(\theta\mathbf{x} + (1-\theta)\mathbf{y})$$

Thus, the maximum of any two convex functions is convex.