2.17 Consider the linear mapping

$$\Phi : \mathbb{R}^3 \to \mathbb{R}^4$$

$$\Phi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 - 3x_2 \\ 2x_1 + 3x_2 + x_3 \end{bmatrix}$$

1. Find the transformation matrix $\mathbf{A}_\Phi$.

   Solution.

   Since the basis being used here are standard basis,

$$\begin{bmatrix} 3x_1 + 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 - 3x_2 \\ 2x_1 + 3x_2 + x_3 \end{bmatrix} = \mathbf{A}_\Phi \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\implies \mathbf{A}_\Phi = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

2. Determine $\mathrm{rk}(\mathbf{A}_\Phi)$.

Solution.

To obtain rank, we perform Gaussian elimination to find the number of linearly independent columns.

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{matrix} -2R_2 \\ \\ -R_2 \\ -2R_2 \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & -4 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{matrix} \\ -R_1 \\ \text{Swap with } R_4 \\ \text{Swap with } R_3 \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -4 & -1 \end{bmatrix} \begin{matrix} \\ \\ -R_2 \\ +4R_3 \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \\ 0 & 0 & -5 \end{bmatrix} \begin{matrix} \\ \\ \cdot -\frac{1}{3} \\ \cdot -\frac{1}{5} \end{matrix}$$

$$
\begin{array}{cc}
\rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} +R_3 \\ -2R_3 \\ \\ -R_3 \end{array}
\end{array}
$$

$$
\rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
$$

There are three pivot columns, so $\text{rk}(\mathbf{A}_\Phi) = 3$.

3. Compute the kernel and image of Φ. What are $\dim(\ker(\Phi))$ and $\dim(\text{Im}(\Phi))$?

Solution.

$\ker(\Phi) = \{\mathbf{v} \in V : \Phi(\mathbf{v}) = \mathbf{0}_4\}$

So to find the kernel, the following equation has to be solved for $\mathbf{v}$:

$$
\mathbf{A}_\Phi \mathbf{v} = \mathbf{0}_4
$$

Since we already have the reduced row echelon form of $\mathbf{A}_\Phi$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies v_1 = v_2 = v_3 = 0$$

$\ker(\Phi) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ and $\dim(\ker(\Phi)) = 0$.

The reduced row echelon form of $\mathbf{A}_\Phi$ has 3 linearly independent columns that form the standard basis for $\mathbb{R}_3$. Therefore, $\mathrm{Im}(\Phi) = \mathbb{R}_3$ and $\dim(\mathrm{Im}(\Phi)) = 3$.