4.4 Compute all the eigenspaces of

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & 1 & 1 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$$

Solution.

To find eigenvalues, we set $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$

Let $\mathbf{B} = \mathbf{A} - \lambda \mathbf{I}$

$$\mathbf{B} = \begin{bmatrix} -\lambda & -1 & 1 & 1 \\ -1 & 1-\lambda & -2 & 3 \\ 2 & -1 & -\lambda & 0 \\ 1 & -1 & 1 & -\lambda \end{bmatrix}$$

We can apply Gauss Jordan reduction to simplify the matrix.

$$\begin{bmatrix} -\lambda & -1 & 1 & 1 \\ -1 & 1-\lambda & -2 & 3 \\ 2 & -1 & -\lambda & 0 \\ 1 & -1 & 1 & -\lambda \end{bmatrix} \begin{array}{l} \\ \\ -2R_4 \\ +R_2 \end{array}$$

$$\rightsquigarrow \begin{bmatrix} -\lambda & -1 & 1 & 1 \\ -1 & 1-\lambda & -2 & 3 \\ 0 & 1 & -\lambda-2 & 2\lambda \\ 0 & -\lambda & -1 & 3-\lambda \end{bmatrix} \begin{array}{l} \\ -R_3 - R_4 \\ \\ \\ \end{array}$$

$$\rightsquigarrow \begin{bmatrix} -\lambda & -1 & 1 & 1 \\ -1 & 0 & 1+\lambda & -\lambda \\ 0 & 1 & -\lambda-2 & 2\lambda \\ 0 & -\lambda & -1 & 3-\lambda \end{bmatrix}$$

Now using Laplace expansion along first column:

$$det(\mathbf{B}) = \Sigma_{k=1}^{n}(-1)^{k+j}a_{kj}det(\mathbf{B}_{k,j})$$

$$= \Sigma_{k=1}^{n}(-1)^{k+1}a_{k1}det(\mathbf{B}_{k,1})$$

$$= (-1)^{1+1}(-\lambda) \cdot \begin{vmatrix} 0 & 1+\lambda & -\lambda \\ 1 & -\lambda-2 & 2\lambda \\ -\lambda & -1 & 3-\lambda \end{vmatrix} + (-1)^{2+1}(-1) \cdot \begin{vmatrix} -1 & 1 & 1 \\ 1 & -\lambda-2 & 2\lambda \\ -\lambda & -1 & 3-\lambda \end{vmatrix}$$

$$= (-\lambda) \cdot \begin{vmatrix} 0 & 1+\lambda & -\lambda \\ 1 & -\lambda-2 & 2\lambda \\ -\lambda & -1 & 3-\lambda \end{vmatrix} + \begin{vmatrix} -1 & 1 & 1 \\ 1 & -\lambda-2 & 2\lambda \\ -\lambda & -1 & 3-\lambda \end{vmatrix}$$

Applying Sarrus rule to obtain both determinants,

$$\begin{vmatrix} 0 & 1+\lambda & -\lambda \\ 1 & -\lambda-2 & 2\lambda \\ -\lambda & -1 & 3-\lambda \end{vmatrix} = \begin{matrix} (0)(-\lambda-2)(3-\lambda) + (1)(-1)(-\lambda) + (-\lambda)(1+\lambda)(2\lambda) \\ -(-\lambda)(-\lambda-2)(-\lambda) - (0)(-1)(2\lambda) - (1)(1+\lambda)(3-\lambda) \end{matrix}$$

$$= (0) + (\lambda) + (-2\lambda^2 - 2\lambda^3) - (-\lambda^3 - 2\lambda^2) - (0) - (-\lambda^2 + 2\lambda + 3)$$

$$= -2\lambda^3 + \lambda^3 - 2\lambda^2 + 2\lambda^2 + \lambda^2 - 2\lambda + \lambda - 3$$

$$= -\lambda^3 + \lambda^2 - \lambda - 3$$

$$\begin{vmatrix} -1 & 1 & 1 \\ 1 & -\lambda - 2 & 2\lambda \\ -\lambda & -1 & 3 - \lambda \end{vmatrix} = \begin{array}{c} (-1)(-\lambda - 2)(3 - \lambda) + (1)(-1)(1) + (-\lambda)(1)(2\lambda) \\ -(-\lambda)(-\lambda - 2)(1) - (-1)(-1)(2\lambda) - (1)(1)(3 - \lambda) \end{array}$$

$$= (-\lambda^2 + \lambda + 6) + (-1) + (-2\lambda^2) - (\lambda^2 + 2\lambda) - (2\lambda) - (3 - \lambda)$$

$$= -\lambda^2 - 2\lambda^2 - \lambda^2 + \lambda - 2\lambda - 2\lambda + \lambda + 6 - 1 - 3$$

$$= -4\lambda^2 - 2\lambda + 2$$

$$= -2(2\lambda^2 + \lambda - 1)$$

Therefore, $\det(\mathbf{A}) = (-\lambda)(-\lambda^3 + \lambda^2 - \lambda - 3) - 2(2\lambda^2 + \lambda - 1)$

$= \lambda^4 - \lambda^3 + \lambda^2 + 3\lambda - 4\lambda^2 - 2\lambda + 2$

$= \lambda^4 - \lambda^3 - 3\lambda^2 + \lambda + 2$

$= \lambda^3(\lambda - 1) - (3\lambda^2 - \lambda - 2)$

$= \lambda^3(\lambda - 1) - (3\lambda^2 - 3\lambda + 2\lambda - 2)$

$= \lambda^3(\lambda - 1) - (3\lambda(\lambda - 1) + 2(\lambda - 1))$

$= \lambda^3(\lambda - 1) - (3\lambda + 2)(\lambda - 1)$

$= (\lambda^3 - 3\lambda - 2)(\lambda - 1)$

$= (\lambda^3 + \lambda^2 - \lambda^2 - 3\lambda - 2)(\lambda - 1)$

$= (\lambda^2(\lambda + 1) - (\lambda^2 + 3\lambda + 2))(\lambda - 1)$

$= (\lambda^2(\lambda + 1) - (\lambda^2 + \lambda + 2\lambda + 2))(\lambda - 1)$

$= (\lambda^2(\lambda + 1) - (\lambda + 2)(\lambda + 1))(\lambda - 1)$

$= (\lambda^2 - \lambda - 2)(\lambda + 1)(\lambda - 1)$

$= (\lambda^2 - 2\lambda + \lambda - 2)(\lambda + 1)(\lambda - 1)$

$= (\lambda(\lambda - 2) + 1(\lambda - 2))(\lambda + 1)(\lambda - 1)$

$= (\lambda - 2)(\lambda + 1)^2(\lambda - 1)$

Eigenvalues are : $2, -1, 1$

Now we obtain the eigenvectors, we solve the following for $\mathbf{x}$:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$$

For eigenvalue $= 2$,

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0 \qquad \Longrightarrow \qquad (\mathbf{A} - 2 \cdot \mathbf{I})\mathbf{x} = 0$$

$$\Longrightarrow \begin{bmatrix} 0-2 & -1 & 1 & 1 \\ -1 & 1-2 & -2 & 3 \\ 2 & -1 & 0-2 & 0 \\ 1 & -1 & 1 & 0-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} -2 & -1 & 1 & 1 \\ -1 & -1 & -2 & 3 \\ 2 & -1 & -2 & 0 \\ 1 & -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix :

$$\left[\begin{array}{cccc|c} -2 & -1 & 1 & 1 & 0 \\ -1 & -1 & -2 & 3 & 0 \\ 2 & -1 & -2 & 0 & 0 \\ 1 & -1 & 1 & -2 & 0 \end{array}\right]$$

To solve this homogeneous system of equations, we need to get reduced row echelon form using Gauss-Jordan reduction.

$$\left[\begin{array}{cccc|c} -2 & -1 & 1 & 1 & 0 \\ -1 & -1 & -2 & 3 & 0 \\ 2 & -1 & -2 & 0 & 0 \\ 1 & -1 & 1 & -2 & 0 \end{array}\right] \begin{array}{l} \\ \\ +R_1 \\ +R_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} -2 & -1 & 1 & 1 & 0 \\ -1 & -1 & -2 & 3 & 0 \\ 0 & -2 & -1 & 1 & 0 \\ 0 & -2 & -1 & 1 & 0 \end{array}\right] \begin{array}{l} \cdot -1 \\ \cdot -1 \\ \\ -R_3 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 2 & 1 & -1 & -1 & 0 \\ 1 & 1 & 2 & -3 & 0 \\ 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \text{Swap with } R_2 \\ \text{Swap with } R_1 \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & -3 & 0 \\ 2 & 1 & -1 & -1 & 0 \\ 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ -2R_1 \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & -3 & 0 \\ 0 & -1 & -5 & 5 & 0 \\ 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \cdot -1 \\ -2R_2 \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & -3 & 0 \\ 0 & 1 & 5 & -5 & 0 \\ 0 & 0 & 9 & -9 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \cdot \frac{1}{9} \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & -3 & 0 \\ 0 & 1 & 5 & -5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} -2R_3 \\ -5R_3 \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} -R_2 \\ \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

After reducing it to row echelon form, equation becomes:

$$\implies \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies x_1 - x_4 = 0, \quad x_2 = 0, \quad x_3 - x_4 = 0$

$\implies x_1 = x_4, \quad x_2 = 0, \quad x_3 = x_4$

$$\implies E_2 = \text{Span}\left[\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right]$$

For eigenvalue = 1,

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0 \qquad \implies \qquad (\mathbf{A} - 1 \cdot \mathbf{I})\mathbf{x} = 0$$

$$\Longrightarrow \begin{bmatrix} 0-1 & -1 & 1 & 1 \\ -1 & 1-1 & -2 & 3 \\ 2 & -1 & 0-1 & 0 \\ 1 & -1 & 1 & 0-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & 0 & -2 & 3 \\ 2 & -1 & -1 & 0 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix :

$$\left[\begin{array}{cccc|c} -1 & -1 & 1 & 1 & 0 \\ -1 & 0 & -2 & 3 & 0 \\ 2 & -1 & -1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 \end{array}\right]$$

To solve this homogeneous system of equations, we need to get reduced row echelon form using Gauss-Jordan reduction.

$$\begin{array}{c} \left[\begin{array}{cccc|c} -1 & -1 & 1 & 1 & 0 \\ -1 & 0 & -2 & 3 & 0 \\ 2 & -1 & -1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 \end{array}\right] \end{array} \begin{array}{c} \cdot -1 \\ \cdot -1 \\ +2R_1 \\ +R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ 1 & 0 & 2 & -3 & 0 \\ 0 & -3 & 1 & 2 & 0 \\ 0 & -2 & 2 & 0 & 0 \end{array}\right] \begin{array}{l} \\ -R_1 \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ 0 & -1 & 3 & -2 & 0 \\ 0 & -3 & 1 & 2 & 0 \\ 0 & -2 & 2 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ -3R_2 \\ -2R_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ 0 & -1 & 3 & -2 & 0 \\ 0 & 0 & -8 & 8 & 0 \\ 0 & 0 & -4 & 4 & 0 \end{array}\right] \begin{array}{l} +R_2 \\ \\ \cdot -\frac{1}{8} \\ \cdot -\frac{1}{4} \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & -3 & 0 \\ 0 & -1 & 3 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ -R_3 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & -3 & 0 \\ 0 & -1 & 3 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \cdot -1 \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & -3 & 0 \\ 0 & 1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ +3R_3 \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & -3 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} -2R_3 \\ \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

After reducing it to row echelon form, equation becomes:

$$\implies \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies x_1 - x_4 = 0, \quad x_2 - x_4 = 0, \quad x_3 - x_4 = 0$

$\implies x_1 = x_4, \quad x_2 = x_4 = x_1, \quad x_3 = x_4 = x_1$

$\implies E_1 = \text{Span}\ [\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}]$

For eigenvalue = -1,

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \qquad \implies \qquad (\mathbf{A} + 1 \cdot \mathbf{I})\mathbf{x} = 0$$

$$\Longrightarrow \begin{bmatrix} 0+1 & -1 & 1 & 1 \\ -1 & 1+1 & -2 & 3 \\ 2 & -1 & 0+1 & 0 \\ 1 & -1 & 1 & 0+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 2 & -2 & 3 \\ 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix :

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ -1 & 2 & -2 & 3 & 0 \\ 2 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 1 & 0 \end{array}\right]$$

To solve this homogeneous system of equations, we need to get reduced row echelon form using Gauss-Jordan reduction.

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ -1 & 2 & -2 & 3 & 0 \\ 2 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 1 & 0 \end{array}\right] \begin{array}{c} \\ +R_1 \\ -2R_1 \\ -R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 4 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} +R_2 \\ \\ -R_2 \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & -1 & 4 & 0 \\ 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \cdot -\frac{1}{6} \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & -1 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} -5R_3 \\ -4R_3 \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

After reducing it to row echelon form, equation becomes:

$$\Longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow x_1 = 0, \quad x_2 - x_3 = 0, \quad x_4 = 0$

$$\Longrightarrow x_1 = 0, \quad x_2 = x_3, \quad x_4 = 0$$

$$\Longrightarrow E_{-1} = \text{Span}\,[\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}]$$