**6.9** Express the Binomial distribution as an exponential family distribution. Also express the Beta distribution as an exponential family distribution. Show that the product of the Beta and Binomial distribution is also a member of the exponential family.

**Solution.** N is not a parameter.

The Binomial distribution is given by:

$$p(m|N,\mu) = \binom{N}{m} \mu^m (1-\mu)^{N-m}$$

It needs to be expressed in the form : $p(m|\theta) = h(m)\exp(\langle\theta, \phi(m)\rangle - A(\theta))$ where $\theta$ is a function of $\mu$.

$$\binom{N}{m} \mu^m (1-\mu)^{N-m}$$

$$= \exp\left(\ln\left(\binom{N}{m} \mu^m (1-\mu)^{N-m}\right)\right)$$

$$= \exp\left(\ln\binom{N}{m} + m\ln(\mu) + (N-m)\ln(1-\mu)\right)$$

$$= \exp\left(\ln\binom{N}{m} + m\ln(\mu) + N\ln(1-\mu) - m\ln(1-\mu)\right)$$

$$= \exp\left(\ln\binom{N}{m} + m\ln(\frac{\mu}{1-\mu}) + N\ln(1-\mu)\right)$$

$$= \exp(\ln\binom{N}{m})\exp\left(m\ln(\frac{\mu}{1-\mu}) + N\ln(1-\mu)\right)$$

$$= \binom{N}{m}\exp\left(m\ln(\frac{\mu}{1-\mu}) + N\ln(1-\mu)\right)$$

$$\text{Let } \theta = \ln(\frac{\mu}{1-\mu})$$

$$\implies \exp(\theta) = \frac{\mu}{1-\mu}$$

$$\implies \exp(\theta) - \mu \exp(\theta) = \mu$$

$$\implies \exp(\theta) = \mu(1 + \exp(\theta))$$

$$\implies \mu = \frac{\exp(\theta)}{(1 + \exp(\theta))}$$

$$\implies N \ln(1-\mu) = N \ln\left(1 - \frac{\exp(\theta)}{(1 + \exp(\theta))}\right) = N \ln\left(\frac{1}{(1 + \exp(\theta))}\right) = -N \ln(1+\exp(\theta))$$

Now if we let $A(\theta) = N \ln(1+\exp(\theta)),$ and $h(m) = \binom{N}{m}$ and $\phi(m) = m,$ we get the form required.

The Beta distribution is given by:

$$p(\mu|\alpha, \beta) = \frac{\tau(\alpha + \beta)}{\tau(\alpha) + \tau(\beta)} \mu^{\alpha-1}(1-\mu)^{\beta-1}$$

It needs to be expressed in the form : $p(\mu|\boldsymbol{\theta}) = h(\mu) \exp(\langle \boldsymbol{\theta}, \boldsymbol{\phi}(\mu) \rangle - A(\boldsymbol{\theta}))$ where $\theta$ is a function of $\alpha$ and $\beta$.

$$p(\mu|\alpha, \beta) = \frac{\tau(\alpha + \beta)}{\tau(\alpha) + \tau(\beta)} \mu^{\alpha-1}(1-\mu)^{\beta-1}$$

$$= \exp\left(\ln\left(\frac{\tau(\alpha + \beta)}{\tau(\alpha) + \tau(\beta)} \mu^{\alpha-1}(1-\mu)^{\beta-1}\right)\right)$$

$$= \exp\left(\ln\left(\frac{\tau(\alpha+\beta)}{\tau(\alpha)+\tau(\beta)}\right) + \ln\left(\mu^{\alpha-1}(1-\mu)^{\beta-1}\right)\right)$$

$$= \exp\left(\ln\left(\frac{\tau(\alpha+\beta)}{\tau(\alpha)+\tau(\beta)}\right) + (\alpha-1)\ln(\mu) + (\beta-1)\ln(1-\mu)\right)$$

$$\text{Let } \boldsymbol{\theta} = \begin{bmatrix} \alpha-1 \\ \beta-1 \end{bmatrix} \text{ and let } \boldsymbol{\phi}(\mu) = \begin{bmatrix} ln(\mu) \\ ln(1-\mu) \end{bmatrix}$$

$$\text{Then, } \alpha = (\boldsymbol{\theta}^T\mathbf{k}) + 1, \text{ where } \mathbf{k} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\text{and } \beta = (\boldsymbol{\theta}^T\mathbf{l}) + 1, \text{ where } \mathbf{l} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\implies \ln\left(\frac{\tau(\alpha+\beta)}{\tau(\alpha)+\tau(\beta)}\right) = \ln\left(\frac{\tau((\boldsymbol{\theta}^T\mathbf{k})+1+(\boldsymbol{\theta}^T\mathbf{l})+1)}{\tau((\boldsymbol{\theta}^T\mathbf{k})+1)+\tau((\boldsymbol{\theta}^T\mathbf{l})+1)}\right) = \ln\left(\frac{\tau(\boldsymbol{\theta}^T(\mathbf{k}+\mathbf{l})+2)}{\tau((\boldsymbol{\theta}^T\mathbf{k})+1)+\tau((\boldsymbol{\theta}^T\mathbf{l})+1)}\right)$$

$$\text{Let } A(\boldsymbol{\theta}) = -\ln\left(\frac{\tau(\boldsymbol{\theta}^T(\mathbf{k}+\mathbf{l})+2)}{\tau((\boldsymbol{\theta}^T\mathbf{k})+1)+\tau((\boldsymbol{\theta}^T\mathbf{l})+1)}\right), \text{ and } h(\mu) = 1.$$

Thus, we get the form $h(\mu)\exp(\langle\boldsymbol{\theta}, \boldsymbol{\phi}(\mu)\rangle - A(\boldsymbol{\theta}))$.

Now we need to show that the product of the Beta and Binomial distribution is also a member of the exponential family.

$$p(m|N,\mu)p(\mu|\alpha,\beta) = \binom{N}{m}\mu^m(1-\mu)^{N-m}\frac{\tau(\alpha+\beta)}{\tau(\alpha)+\tau(\beta)}\mu^{\alpha-1}(1-\mu)^{\beta-1}$$

$$= \exp\left(\ln\left(\binom{N}{m}\mu^m(1-\mu)^{N-m}\frac{\tau(\alpha+\beta)}{\tau(\alpha)+\tau(\beta)}\mu^{\alpha-1}(1-\mu)^{\beta-1}\right)\right)$$

$$= \exp\left(\ln\left(\binom{N}{m}\mu^{m+\alpha-1}(1-\mu)^{N-m+\beta-1}\frac{\tau(\alpha+\beta)}{\tau(\alpha)+\tau(\beta)}\right)\right)$$

Assuming $\mu$ to be the random variable, and $\theta$ a parameter which is a function of $\alpha$ and $\beta$ ($m$ is treated as a constant), the above expression needs to be expressed in the form :

$$p(\mu|\boldsymbol{\theta}) = h(\mu)\exp(\langle\boldsymbol{\theta},\boldsymbol{\phi}(\mu)\rangle - A(\boldsymbol{\theta}))$$

$$= \exp\left(\ln\left(\binom{N}{m}\right) + \ln\left(\mu^{m+\alpha-1}\right) + \ln\left((1-\mu)^{N-m+\beta-1}\right) + \ln\left(\frac{\tau(\alpha+\beta)}{\tau(\alpha)+\tau(\beta)}\right)\right)$$

$$= \exp\left((m+\alpha-1)\ln(\mu) + (N-m+\beta-1)\ln(1-\mu) + \ln\left(\left(\binom{N}{m}\right)\frac{\tau(\alpha+\beta)}{\tau(\alpha)+\tau(\beta)}\right)\right)$$

We can see that the distribution belongs to exponential family by observing the following:

$$h(\mu) = 1$$

$$\boldsymbol{\theta} = \begin{bmatrix} m + \alpha - 1 \\ N - m + \beta - 1 \end{bmatrix}$$

where $\theta_0 = m + \alpha - 1$ and $\theta_1 = N - m + \beta - 1$

$$\implies \alpha + \beta = \theta_0 + \theta_1 - N + 2$$

$$\alpha = \theta_0 - m + 1$$

$$\beta = \theta_1 - N + m + 1$$

$$\boldsymbol{\phi}(\mu) = \begin{bmatrix} \ln(\mu) \\ \ln(1 - \mu) \end{bmatrix}$$

$$A(\boldsymbol{\theta}) = -\ln\left( \left(\binom{N}{m}\right) \frac{\tau(\theta_0 + \theta_1 - N + 2)}{\tau(\theta_0 - m + 1) + \tau(\theta_1 - N + m + 1)} \right).$$