2.15 Let

$$F = \{(x, y, z) \in \mathbb{R}^3 | x + y - z = 0\}$$

and

$$G = \{(a - b, a + b, a - 3b) | a, b \in \mathbb{R}\}.$$

a. Show that F and G are subspaces of $\mathbb{R}^3$.

Solution.

$F \subseteq \mathbf{R}^3, G \subseteq \mathbf{R}^3$

So, to prove that F and G are subspaces of R^3, we need to prove 3 properties:

1. $F \neq \emptyset$ and $G \neq \emptyset$, in particular: $\mathbf{0} \in F$ and $\mathbf{0} \in G$.

Proof.

If $x = y = z = 0, x + y - z = 0 \implies (0, 0, 0) \in F$ and $F \neq \emptyset$.

If $a = b = c = 0, (a - b, a + b, a - 3b) = (0, 0, 0) \implies (0, 0, 0) \in G$ and $G \neq \emptyset$.

2. Closure of F and G with respect to outer operation: $\forall \beta \in \mathbb{R}, \forall \mathbf{t} \in F : \beta \mathbf{t} \in F$

and

$\forall \beta \in \mathbb{R}, \forall \mathbf{t} \in G : \beta \mathbf{t} \in G$

Proof.

$\forall \beta \in \mathbb{R}, \forall \mathbf{t} \in F : \beta \mathbf{t} = \beta(x, y, z) = (\beta x, \beta y, \beta z)$

$\beta x + \beta y - \beta z = \beta(x + y - z) = \beta 0 = 0$

$\implies \beta \mathbf{t} \in F$

Now let $\mathbf{t} \in G | \mathbf{t} = (a - b, a + b, a - 3b), a, b \in \mathbb{R}$

$\forall \beta \in \mathbb{R} : \beta \mathbf{t} = \beta(a - b, a + b, a - 3b) = (\beta a - \beta b, \beta a + \beta b, \beta a - 3\beta b)$

Let $\beta a = c$ and $\beta b = d$

We get $\beta \mathbf{t} = (c - d, c + d, c - 3d) | c, d \in \mathbb{R}$

$\implies \beta \mathbf{t} \in G.$

3. Closure of F and G with respect to inner operation:

$\forall \mathbf{t}, \mathbf{u} \in F : \mathbf{t} + \mathbf{u} \in F$

and

$\forall \mathbf{t}, \mathbf{u} \in G : \mathbf{t} + \mathbf{u} \in G.$

Proof.

Let $\mathbf{t}, \mathbf{u} \in F$ such that $\mathbf{t} = (x_1, y_1, z_1) | x_1 + y_1 - z_1 = 0$ and $\mathbf{u} = (x_2, y_2, z_2) | x_2 + y_2 - z_2 = 0$.

Let $\mathbf{t} + \mathbf{u} = (x_3, y_3, z_3)$.

$(x_3, y_3, z_3) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$x_3 + y_3 - z_3 = (x_1 + x_2) + (y_1 + y_2) - (z_1 + z_2) = (x_1 + y_1 - z_1) + (x_2 + y_2 - z_2) = 0 + 0 = 0$

$\implies \mathbf{t} + \mathbf{u} \in F.$

Let $\mathbf{t} = (a_1 - b_1, a_1 + b_1, a_1 - 3b_1) | a_1, b_1 \in \mathbb{R}$

and

$\mathbf{u} = (a_2 - b_2, a_2 + b_2, a_2 - 3b_2) | a_2, b_2 \in \mathbb{R}$

$\mathbf{t} + \mathbf{u} = (a_1 - b_1 + a_2 - b_2, a_1 + b_1 + a_2 + b_2, a_1 - 3b_1 + a_2 - 3b_2)$

$= ((a_1 + a_2) - (b_1 + b_2), (a_1 + a_2) + (b_1 + b_2), (a_1 + a_2) - 3(b_1 + b_2))$

Let $a_1 + a_2 = a_3$ and $b_1 + b_2 = b_3$

$\Longrightarrow \mathbf{t} + \mathbf{u} = (a_3 - b_3, a_3 + b_3, a_3 - 3b_3) | a_3, b_3 \in \mathbb{R}$

$\Longrightarrow \mathbf{t} + \mathbf{u} \in G.$

b. Calculate $F \cap G$ without resorting to any basis vector.

Solution.

$\forall \mathbf{t} = (x, y, z) \in F \cap G, x + y - z = 0$, and $\exists a, b \in \mathbb{R} : x = a - b, y = a + b, z = a - 3b$

$\Longrightarrow a - b + a + b - (a - 3b) = 0$

$\Longrightarrow a + 3b = 0 \Longrightarrow a = -3b$

$\Longrightarrow \mathbf{t} = (-4b, -2b, -6b) = -2b(2, 1, 3)$

$\Longrightarrow \mathbf{t} = \lambda(2, 1, 3) \forall \lambda \in \mathbb{R}.$

$$\Longrightarrow F \cap G = \text{span}\left[\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}\right]$$

c. Find one basis for F and one basis for G, calculate $F \cap G$ using the basis vectors previously found and check your result with the previous question.

Solution.

$F = \{(x, y, x + y)|x, y, \in \mathbb{R}\} = \{x(1, 0, 1) + y(0, 1, 1)|x, y, \in \mathbb{R}\}$

$$\implies \text{Basis of } F = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

It is a basis because the two vectors are linearly independent, as neither of the vectors is a multiple of the other.

$G = \{(a-b, a+b, a-3b)|a, b \in \mathbb{R}\} = \{(a, a, a)+(-b, b, -3b)|a, b \in \mathbb{R}\}$
$= \{a(1, 1, 1) + b(-1, 1, -3)|a, b \in \mathbb{R}\}$

$$\implies \text{Basis of } G = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} \right\}$$

It is a basis because the two vectors are linearly independent, as neither of the vectors is a multiple of the other.

Any vector $\mathbf{v} \in F \cap G$ can be represented as a linear combination of basis vectors of F and G respectively, such that

$$\mathbf{v} = m\begin{bmatrix}1\\0\\1\end{bmatrix} + n\begin{bmatrix}0\\1\\1\end{bmatrix} = o\begin{bmatrix}1\\1\\1\end{bmatrix} + p\begin{bmatrix}-1\\1\\-3\end{bmatrix} | m,n,o,p \in \mathbb{R}$$

$$\implies m\begin{bmatrix}1\\0\\1\end{bmatrix} + n\begin{bmatrix}0\\1\\1\end{bmatrix} - o\begin{bmatrix}1\\1\\1\end{bmatrix} - p\begin{bmatrix}-1\\1\\-3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\implies \begin{bmatrix}1 & 0 & 1 & -1\\0 & 1 & 1 & 1\\1 & 1 & 1 & -3\end{bmatrix}\begin{bmatrix}m\\n\\-o\\-p\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Solving for m, n, o, p by applying Gauss-Jordan reduction to get reduced row echelon form of the following augmented matrix:

$$\left[\begin{array}{cccc|c}1 & 0 & 1 & -1 & 0\\0 & 1 & 1 & 1 & 0\\1 & 1 & 1 & -3 & 0\end{array}\right]\begin{array}{l}\\ \\ -R_1\end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c}1 & 0 & 1 & -1 & 0\\0 & 1 & 1 & 1 & 0\\0 & 1 & 0 & -2 & 0\end{array}\right]\begin{array}{l}\\ \\ -R_2\end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c}1 & 0 & 1 & -1 & 0\\0 & 1 & 1 & 1 & 0\\0 & 0 & -1 & -3 & 0\end{array}\right]\begin{array}{l}+R_3\\ +R_3\\ \cdot -1\end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -4 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right]$$

$$\implies m + 4p = 0, \quad n + 2p = 0, \quad o - 3p = 0$$

$$\implies m = -4p, \quad n = -2p, \quad o = 3p$$

Substituting in $\mathbf{v}$,

$$\mathbf{v} = m \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + n \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = -4p \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 2p \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = -2p \left(\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$$

$$= -2p \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} | p \in \mathbb{R}$$

$$\implies F \cap G = \text{span}\ [\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}]$$