4.5 Diagonalizability of a matrix is unrelated to its invertibility. Determine for the following four matrices whether they are diagonalizable and/or invertible

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Solution.

A Symmetric, square matrix can always be diagonalized. So, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ can be diagonalized.

For $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, we can check if it can be diagonalized by obtaining its eigenvalues and eigenvectors.

We can see that the determinant is 1 and trace is 2, so it has only one eigenvalue : 1, with algebraic multiplicity of 2.

To obtain eigenvectors, we solve for $(\mathbf{A} - \lambda\mathbf{I}_2)\mathbf{x} = 0$

$$\implies \begin{bmatrix} 1-1 & 1 \\ 0 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\implies \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\implies x_2 = 0, \quad x_1 = x_1$$

$$\text{Eigenvector is } \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since it doesn't have 2 linearly independent eigenvectors, matrix is not diagonalizable.

For $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, we can check if it can be diagonalized by obtaining its eigenvalues and eigenvectors.

We can see that the determinant is 0 and trace is also 0. So it has only one eigenvalue : 0, with algebraic multiplicity of 2.

To obtain eigenvectors, we solve for $(\mathbf{A} - \lambda \mathbf{I}_2)\mathbf{x} = 0$

$$\implies \begin{bmatrix} 0-0 & 1 \\ 0 & 0-0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\implies \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\implies x_2 = 0, x_1 = x_1$$

$$\text{Eigenvector is } \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since it doesn't have 2 linearly independent eigenvectors, matrix is not diagonalizable.

Checking invertibility,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{is invertible as determinant} = 1$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{is not invertible as determinant} = 0$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{is invertible as determinant} = 1$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{is not invertible as determinant} = 0$$