2.14 Consider two subspaces U_1 and U_2, where U_1 is spanned by the columns of A_1 and U_2 is spanned by the columns of A_2 with

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \qquad \mathbf{A}_2 = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix}.$$

Solution.

a. Determine the dimension of U_1, U_2.

Solution.

In 2.13, Gauss-Jordan reduction of showed that the reduced row echelon forms of $\mathbf{A}_1$ and $\mathbf{A}_2$ are the same, and have 2 pivot columns. Therefore, dim(U_1)=dim(U_2)=2.

b. Determine bases of U_1 and U_2.

Solution.

The first 2 columns of both U_1 and U_2 are linearly independent.

$$\text{So, basis of } U_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}, \qquad \text{Basis of } U_2 = \left\{ \begin{bmatrix} 3 \\ 1 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -5 \\ -1 \end{bmatrix} \right\}$$

c. Determine a basis of $U_1 \cap U_2$.

Solution.

Every $\mathbf{v} \in U_1 \cap U_2$ can be represented a linear combination of the basis vectors of U_1 and U_2 such that

$\forall \mathbf{v} \in U_1 \cap U_2,$

$$\mathbf{v} = a \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} = c \begin{bmatrix} 3 \\ 1 \\ 7 \\ 3 \end{bmatrix} + d \begin{bmatrix} -3 \\ 2 \\ -5 \\ -1 \end{bmatrix} | a, b, c, d \in \mathbb{R}$$

$$\implies a \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} - c \begin{bmatrix} 3 \\ 1 \\ 7 \\ 3 \end{bmatrix} - d \begin{bmatrix} -3 \\ 2 \\ -5 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 0 & 3 & -3 \\ 1 & -2 & 1 & 2 \\ 2 & 1 & 7 & -5 \\ 1 & 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ -c \\ -d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

a, b, c, d can be solved for by reducing the following augmented matrix to reduced row echelon form:

$$\begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & 3 & -3 & 0 \\ 1 & -2 & 1 & 2 & 0 \\ 2 & 1 & 7 & -5 & 0 \\ 1 & 0 & 3 & -1 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ -R_1 \\ -2R_2 \\ -R_2 \end{array}$$

$$\rightsquigarrow \begin{array}{cl} \left[\begin{array}{cccc|c} 1 & 0 & 3 & -3 & 0 \\ 0 & -2 & -2 & 5 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right] & \begin{array}{l} \\ \cdot -\frac{1}{2} \\ \\ \cdot\frac{1}{2} \end{array} \end{array}$$

$$\rightsquigarrow \begin{array}{cl} \left[\begin{array}{cccc|c} 1 & 0 & 3 & -3 & 0 \\ 0 & 1 & 1 & -\frac{5}{2} & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] & \begin{array}{l} \\ \\ -R_2 \\ \\ \end{array} \end{array}$$

$$\rightsquigarrow \begin{array}{cl} \left[\begin{array}{cccc|c} 1 & 0 & 3 & -3 & 0 \\ 0 & 1 & 1 & -\frac{5}{2} & 0 \\ 0 & 0 & 0 & \frac{7}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] & \begin{array}{l} +3R_4 \\ +\frac{5}{2}R_4 \\ \cdot\frac{2}{7} \\ \\ \end{array} \end{array}$$

$$\rightsquigarrow \begin{array}{cl} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] & \begin{array}{l} \\ \\ \\ -R_3 \end{array} \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\implies -d = 0, \quad b - c = 0, \quad a - 3c = 0$$

$$\implies a = 3c, \quad b = c, \quad c = c, \quad d = 0$$

$$\mathbf{v} = a \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\implies \mathbf{v} = 3c \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \quad = c \begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix} + c \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \quad = c \begin{bmatrix} 3 \\ 1 \\ 7 \\ 3 \end{bmatrix} \quad | c \in \mathbb{R}$$

$$\implies \text{Basis of } U_1 \cap U_2 = \left\{ \begin{bmatrix} 3 \\ 1 \\ 7 \\ 3 \end{bmatrix} \right\}$$