2.3 Consider the set G of 3×3 matrices defined as follows:

$$G = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times3} \,\middle|\, x, y, z \in \mathbb{R} \right\}$$

We define . as the standard matrix multiplication.

Is $(G, \cdot)$ a group? If yes, is it Abelian? Justify your answer.

Solution.

Checking for closure

$$\text{Let } A = \left\{ \begin{bmatrix} 1 & x_1 & z_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times3} \,\middle|\, x_1, y_1, z_1 \in \mathbb{R} \right\}$$

$$\text{and let } B = \left\{ \begin{bmatrix} 1 & x_2 & z_2 \\ 0 & 1 & y_2 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times3} \,\middle|\, x_2, y_2, z_2 \in \mathbb{R} \right\}$$

$$\text{A} \cdot \text{B} = \left\{ \begin{bmatrix} 1 & x_1 + x_2 & z_1 + x_1 y_2 + z_2 \\ 0 & 1 & y_1 + y_2 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times3} \right\}$$

Since $(x_1 + x_2), (z_1 + x_1 y_2 + z_2), (y_1 + y_2) \in R$, $A \cdot B \in G$

Associativity

Associativity holds in normal matrix multiplication.

Existence of neutral element

The identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the neutral element for matrices since $\mathrm{A} \cdot \mathrm{I} = \mathrm{I} \cdot \mathrm{A} = \mathrm{A}$

Existence of inverse element

Let use the definitions of A and B we used previously in this question.

$$\mathrm{A} \cdot \mathrm{B} = \left\{ \begin{bmatrix} 1 & x_1 + x_2 & z_1 + x_1 y_2 + z_2 \\ 0 & 1 & y_1 + y_2 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3} \;\middle|\; x_2, y_2, z_2 \in \mathbb{R} \right\}$$

$$\mathrm{B} \cdot \mathrm{A} = \left\{ \begin{bmatrix} 1 & x_2 + x_1 & z_2 + x_2 y_1 + z_1 \\ 0 & 1 & y_2 + y_1 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3} \;\middle|\; x_2, y_2, z_2 \in \mathbb{R} \right\}$$

If B is the inverse of A, then $A \cdot B = B \cdot A = I$

$\implies x_1 + x_2 = x_2 + x_1 = 0$ and $y_1 + y_2 = y_2 + y_1 = 0$

$\implies x_2 = -x_1$ and $y_2 = -y_1$

If we substitute for x_2 and y_2 in the last remaining element,

$z_1 + x_1 y_2 + z_2 = z_2 + x_2 y_1 + z_1 \implies x_1 y_2 = x_2 y_1 \implies x_1(-y_1) = (-x_1) y_1$ $\implies -x_1 y_1 = -x_1 y_1$

We can see this equality holds.

Now to find the inverse,

$z_1 + x_1 y_2 + z_2 = 0$

$\implies z_2 = -z_1 - x_1 y_2$

$\implies z_2 = -z_1 + x_1 y_1$

$$\text{So, if } A = \left\{ \begin{bmatrix} 1 & x_1 & z_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3} \;\middle|\; x_1, y_1, z_1 \in \mathbb{R} \right\}$$

$$\text{the inverse of } A \text{ is } \begin{bmatrix} 1 & -x_1 & x_1 y_1 - z_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{bmatrix}$$

Commutativity

From the above conditions, we can see that if $A \cdot B = B \cdot A$, $x_1 y_2 = x_2 y_1$ needs to be true. This is not necessarily the case, so $(G, \cdot)$ is a group, but is not Abelian.