7.10 Consider the function

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x} + c,$$

where $\mathbf{A}$ is strictly positive definite, which means that it is invertible. Derive the convex conjugate of $f(\mathbf{x})$.

Hint: Take the gradient of an appropriate function and set the gradient to zero.

Solution.

From Definition 7.4,

$$f^*(\mathbf{s}) = \sup_{\mathbf{x}\in\mathbb{R}^D} (\langle \mathbf{s}, \mathbf{x}\rangle) - f(\mathbf{x})$$

$$\sup_{\mathbf{x}\in\mathbb{R}^D} (\mathbf{s}^T\mathbf{x}) - \left(\frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x} + c\right)$$

Taking the derivative w.r.t $\mathbf{x}$ and setting it to $\mathbf{0}$, we get:

$$\mathbf{s}^T - \left(\frac{1}{2}\mathbf{x}^T(\mathbf{A} + \mathbf{A}^T) + \mathbf{b}^T\right) = 0$$

$$\implies \mathbf{s}^T = \frac{1}{2}\mathbf{x}^T(2\mathbf{A}) + \mathbf{b}^T$$

(Strict positive definiteness implies symmetry of matrix, i.e.,$\mathbf{A} = \mathbf{A}^T$)

$$\implies \mathbf{s}^T - \mathbf{b}^T = \mathbf{x}^T\mathbf{A}$$

$$\implies \mathbf{x}^T = (\mathbf{s}^T - \mathbf{b}^T)\mathbf{A}^{-1}$$

$$\implies \mathbf{x} = (\mathbf{A}^{-1})^T(\mathbf{s} - \mathbf{b}) = (\mathbf{A}^T)^{-1}(\mathbf{s} - \mathbf{b}) = \mathbf{A}^{-1}(\mathbf{s} - \mathbf{b})$$

Substituting for $\mathbf{x}$ in $f^*(\mathbf{s})$, we get,

$$f^*(\mathbf{s}) = (\mathbf{s}^T\mathbf{A}^{-1}(\mathbf{s}-\mathbf{b})) - \left(\frac{1}{2}((\mathbf{s}^T - \mathbf{b}^T)\mathbf{A}^{-1})\mathbf{A}(\mathbf{A}^{-1}(\mathbf{s}-\mathbf{b})) + \mathbf{b}^T(\mathbf{A}^{-1}(\mathbf{s}-\mathbf{b})) + c\right)$$

$$= (\mathbf{s}^T\mathbf{A}^{-1}(\mathbf{s}-\mathbf{b})) - \left(\frac{1}{2}((\mathbf{s}^T - \mathbf{b}^T)\mathbf{A}^{-1}(\mathbf{s}-\mathbf{b})) + \mathbf{b}^T(\mathbf{A}^{-1}(\mathbf{s}-\mathbf{b})) + c\right)$$

$$= (\mathbf{s}^T\mathbf{A}^{-1}(\mathbf{s}-\mathbf{b})) - \frac{1}{2}((\mathbf{s}^T - \mathbf{b}^T)\mathbf{A}^{-1}(\mathbf{s}-\mathbf{b})) - \mathbf{b}^T(\mathbf{A}^{-1}(\mathbf{s}-\mathbf{b})) - c$$

$$= (\mathbf{s}^T\mathbf{A}^{-1}(\mathbf{s}-\mathbf{b})) - \frac{1}{2}(\mathbf{s}^T\mathbf{A}^{-1}(\mathbf{s}-\mathbf{b})) + \frac{1}{2}(\mathbf{b}^T\mathbf{A}^{-1}(\mathbf{s}-\mathbf{b})) - \mathbf{b}^T(\mathbf{A}^{-1}(\mathbf{s}-\mathbf{b})) - c$$

$$= \frac{1}{2}(\mathbf{s}^T\mathbf{A}^{-1}(\mathbf{s}-\mathbf{b})) - \frac{1}{2}(\mathbf{b}^T\mathbf{A}^{-1}(\mathbf{s}-\mathbf{b})) - c$$

$$= \frac{1}{2}((\mathbf{s}^T - \mathbf{b}^T)\mathbf{A}^{-1}(\mathbf{s}-\mathbf{b})) - c$$

$$\implies f^*(\mathbf{s}) = \frac{1}{2}((\mathbf{s}-\mathbf{b})^T\mathbf{A}^{-1}(\mathbf{s}-\mathbf{b})) - c$$