7.6 Consider the linear program illustrated in Figure 7.9,

$$\min_{\mathbf{x}\in\mathbb{R}^2} -\begin{bmatrix} 5 \\ 3 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{subject to} \begin{bmatrix} 2 & 2 \\ 2 & -4 \\ -2 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 33 \\ 8 \\ 5 \\ -1 \\ 8 \end{bmatrix}$$

Derive the dual linear program using Lagrange duality.

Solution.

Using 7.40,

$$\mathcal{L}(\mathbf{x},\boldsymbol{\lambda}) = -\begin{bmatrix} 5 \\ 3 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \boldsymbol{\lambda}^T \left(\begin{bmatrix} 2 & 2 \\ 2 & -4 \\ -2 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 33 \\ 8 \\ 5 \\ -1 \\ 8 \end{bmatrix} \right)$$

Rearranging the terms corresponding to $\mathbf{x}$ yields

$$\mathcal{L}(\mathbf{x},\boldsymbol{\lambda}) = \left(\boldsymbol{\lambda}^T \begin{bmatrix} 2 & 2 \\ 2 & -4 \\ -2 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 \\ 3 \end{bmatrix}^T\right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \boldsymbol{\lambda}^T \begin{bmatrix} 33 \\ 8 \\ 5 \\ -1 \\ 8 \end{bmatrix}$$

$$= \left(\left(\begin{bmatrix} 2 & 2 \\ 2 & -4 \\ -2 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}^T \boldsymbol{\lambda}\right)^T - \begin{bmatrix} 5 \\ 3 \end{bmatrix}^T\right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \boldsymbol{\lambda}^T \begin{bmatrix} 33 \\ 8 \\ 5 \\ -1 \\ 8 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 2 & 2 \\ 2 & -4 \\ -2 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}^T \boldsymbol{\lambda} - \begin{bmatrix} 5 \\ 3 \end{bmatrix}\right)^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \boldsymbol{\lambda}^T \begin{bmatrix} 33 \\ 8 \\ 5 \\ -1 \\ 8 \end{bmatrix}$$

Taking the derivative with respect to $\mathbf{x}$ and setting it to zero gives us (Using 5.105)

$$\begin{bmatrix} 2 & 2 \\ 2 & -4 \\ -2 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}^T \boldsymbol{\lambda} - \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \mathbf{0}$$

Since this term becomes $\mathbf{0}$,

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = -\boldsymbol{\lambda}^T \begin{bmatrix} 33 \\ 8 \\ 5 \\ -1 \\ 8 \end{bmatrix}$$

Therefore, the dual optimization problem becomes:

$$\max_{\boldsymbol{\lambda} \in \mathbb{R}^5} -\boldsymbol{\lambda}^T \begin{bmatrix} 33 \\ 8 \\ 5 \\ -1 \\ 8 \end{bmatrix}$$

$$\text{subject to} \begin{bmatrix} 2 & 2 \\ 2 & -4 \\ -2 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}^T \boldsymbol{\lambda} - \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \mathbf{0}$$

$$\boldsymbol{\lambda} \geq \mathbf{0}.$$