7.8 Consider the following convex optimization problem

$$\min_{\mathbf{w}\in\mathbb{R}^D} \quad \frac{1}{2}\mathbf{w}^T\mathbf{w}$$

$$\text{subject to} \quad \mathbf{w}^T\mathbf{x} \geq 1.$$

Derive the Lagrangian dual by introducing the Lagrange multiplier λ.

Solution.

The Lagrangian is given by

$$\mathcal{L}(\mathbf{w},\lambda) = \frac{1}{2}\mathbf{w}^T\mathbf{w} + \lambda(1 - \mathbf{w}^T\mathbf{x})$$

Taking the derivative of $\mathcal{L}(\mathbf{w},\lambda)$ with respect to $\mathbf{w}$ and setting it to zero gives (applying 5.104 and 5.106):

$$\mathbf{w}^T - \lambda\mathbf{x}^T = 0$$

$$\implies \mathbf{w}^T = \lambda\mathbf{x}^T$$

$$\implies \mathbf{w} = \lambda\mathbf{x}$$

Substituting into the primal Lagrangian $\mathcal{L}(\mathbf{w},\lambda)$, we get the dual Lagrangian $\mathcal{D}(\lambda)$:

$$\mathcal{D}(\lambda) = \frac{1}{2}\lambda\mathbf{x}^T\lambda\mathbf{x} + \lambda(1 - \lambda\mathbf{x}^T\mathbf{x})$$

$$= \frac{1}{2}\lambda^2\mathbf{x}^T\mathbf{x} + \lambda - \lambda^2\mathbf{x}^T\mathbf{x}$$

$$= -\frac{1}{2}\lambda^2 \mathbf{x}^T \mathbf{x} + \lambda$$

Therefore, the dual optimization problem is given by

$$\max_{\lambda \in \mathbb{R}^D} \quad -\frac{1}{2}\lambda^2 \mathbf{x}^T \mathbf{x} + \lambda$$

$$\text{subject to} \quad \lambda \geq 0.$$