2.16 Are the following mappings linear?

a. Let $a, b \in \mathbb{R}$.

$$\phi : L^1([a,b]) \to \mathbb{R}$$

$$f \mapsto \phi(f) = \int_a^b f(x)dx,$$

where $L^1([a,b])$ denotes the set of integrable functions on $[a,b]$.

Solution.

Let $f,\ g \in L^1([a,b])$.

To prove that ϕ is a linear mapping, we need to show that:

$$\phi(\lambda f + \psi g) = \lambda\phi(f) + \psi\phi(g) | f, g \in L^1([a,b])$$

$$\begin{aligned}
\text{L.H.S} &= \int_a^b (\lambda f(x) + \psi g(x))dx \\
&= \int_a^b \lambda f(x)dx + \int_a^b \psi g(x)dx \\
&= \lambda \int_a^b f(x)dx + \psi \int_a^b g(x)dx \\
&= \lambda\phi(f) + \psi\phi(g) \\
&= \text{R.H.S}
\end{aligned}$$

Therefore, it is a linear mapping.

b.

$$\phi : C^1 \to C^0$$

$$f \mapsto \phi(f) = f^{'},$$

where for $k \geq 1, C^k$ denotes the set of k times continuously differentiable functions, and C^0 denotes the set of continuous functions.

Solution.

Let $f,\ g \in C^1$.

To prove that ϕ is a linear mapping, we need to show that:

$$\phi(\lambda f + \psi g) = \lambda\phi(f) + \psi\phi(g) | f, g \in C^1$$

$$\text{L.H.S} = \frac{d}{dx}(\lambda f(x) + \psi g(x))$$

$$= \frac{d(\lambda f(x))}{dx} + \frac{d(\psi g(x))}{dx}$$

$$= \lambda\frac{df(x)}{dx} + \psi\frac{dg(x)}{dx}$$

$$= \lambda\phi(f) + \psi\phi(g)$$

$$= \text{R.H.S}$$

Therefore, it is a linear mapping.

c.

$$\phi : \mathbb{R} \to \mathbb{R}$$

$$x \mapsto \phi(x) = cos(x)$$

Solution.

Let $x, y \in \mathbb{R}$.

To prove that ϕ is a linear mapping, we need to show that:

$$\phi(\lambda x + \psi y) = \lambda\phi(y) + \psi\phi(y)|x, r \in \mathbb{R}$$

$$\text{L.H.S} = cos(\lambda x + \psi y)$$

$$= cos(\lambda x)cos(\psi y) - sin(\lambda x)sin(\psi y)$$

$$\neq \lambda cos(x) + \psi cos(y)$$

Therefore, it is not a linear mapping.

d.

$$\phi : \mathbb{R}^3 \to \mathbb{R}^2$$

$$\mathbf{x} \mapsto \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} \mathbf{x}$$

Solution.

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$.

To prove that ϕ is a linear mapping, we need to show that:

$$\phi(\lambda\mathbf{x} + \psi\mathbf{y}) = \lambda\phi(\mathbf{y}) + \psi\phi(\mathbf{y})|\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$$

$$\text{L.H.S} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} (\lambda\mathbf{x} + \psi\mathbf{y})$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} \lambda\mathbf{x} + \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} \psi\mathbf{y}$$

$$= \lambda \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} \mathbf{x} + \psi \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} \mathbf{y}$$

$$= \lambda\phi(\mathbf{y}) + \psi\phi(\mathbf{y})$$

$$= \text{R.H.S}$$

Therefore, it is a linear mapping.

e. Let θ be in $[0, 2\pi[$ and

$$\phi : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\mathbf{x} \mapsto \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix} \mathbf{x}$$

Solution.

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$.

To prove that ϕ is a linear mapping, we need to show that:

$$\phi(\lambda \mathbf{x} + \psi \mathbf{y}) = \lambda \phi(\mathbf{y}) + \psi \phi(\mathbf{y}) | \mathbf{x}, \mathbf{y} \in \mathbb{R}^2$$

$$\text{L.H.S} = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix} (\lambda \mathbf{x} + \psi \mathbf{y})$$

$$= \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix} \lambda \mathbf{x} + \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix} \psi \mathbf{y}$$

$$= \lambda \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix} \mathbf{x} + \psi \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix} \mathbf{y}$$

$$= \lambda \phi(\mathbf{y}) + \psi \phi(\mathbf{y})$$

$$= \text{R.H.S}$$

Therefore, it is a linear mapping.