2.19 Consider an endomorphism $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ whose transformation matrix (with respect to the standard basis in $\mathbb{R}^3$) is

$$\mathbf{A}_\Phi = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

a. Determine $ker(\Phi)$ and $Im(\Phi)$.

Solution.

$ker(\Phi)$ is the solution space of $\mathbf{A}_\Phi \mathbf{x} = \mathbf{0}$ and $Im(\Phi)$ is the column space of $\mathbf{A}_\Phi$.

$$Im(\Phi) = \text{span}\,[\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}]$$

To find the kernel, we get reduced row echelon form of $\mathbf{A}_\Phi$:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{matrix} \\ -R_1 \\ -R_1 \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \cdot -\frac{1}{2} \\ \\ \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} -R_2 \\ \\ \\ \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\implies ker(\mathbf{A}_\Phi) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

b. Determine the transformation matrix $\tilde{\mathbf{A}}_\Phi$ with respect to the basis

$$B = (\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}),$$

i.e., perform a basis change toward the new basis B.

Solution.

From 2.113 in the book, we know that

$$\tilde{\mathbf{A}}_\Phi = \mathbf{T}^{-1}\mathbf{A}_\Phi\mathbf{S}$$

Here, $\mathbf{T}$ is a transformation matrix that maps coordinates w.r.t basis

B onto coordinates w.r.t standard basis, and $\mathbf{S}$ is a transformation matrix that maps coordinates w.r.t basis B onto coordinates w.r.t standard basis. So, $\mathbf{S} = \mathbf{T}$.

Let $\tilde{\mathbf{B}}$ be the matrix formed by the basis vectors of B such that

$$\tilde{\mathbf{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Let $\hat{\mathbf{x}}$ represent the coordinates w.r.t standard basis and $\hat{\mathbf{y}}$ represent the coordinates w.r.t B.

Then,

$$\mathbf{I}_3\hat{\mathbf{x}} = \tilde{\mathbf{B}}\hat{\mathbf{y}}$$

$$\implies \hat{\mathbf{x}} = \tilde{\mathbf{B}}\hat{\mathbf{y}}$$

This implies that $\tilde{\mathbf{B}}$ is the transformation matrix that maps coordinates w.r.t basis B (represented by $\hat{\mathbf{y}}$) onto coordinates w.r.t standard basis (represented by $\hat{\mathbf{x}}$).

$$\implies \mathbf{T} = \tilde{\mathbf{B}} = \mathbf{S}$$

To find $\mathbf{T}^{-1}$, we can perform Gauss-Jordan reduction to convert the left side of $[\mathbf{T}|\mathbf{I}_3]$ to $\mathbf{I}_3$.

$$\begin{array}{cc} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] & \begin{array}{c} \\ -R_1 \\ -R_1 \end{array} \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} -R_2 \\ \\ \cdot -1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & -1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \begin{array}{l} -2R_3 \\ +R_3 \\ \cdot -1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$\Longrightarrow \mathbf{T}^{-1} = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

Therefore,

$$\tilde{\mathbf{A}}_{\Phi} = \mathbf{T}^{-1}\mathbf{A}_{\Phi}\mathbf{S} = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 9 & 1 \\ -3 & -5 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{A}}_{\Phi} = \begin{bmatrix} 6 & 9 & 1 \\ -3 & -5 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$