2.20 Let us consider $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}'_1, \mathbf{b}'_2$, 4 vectors of $\mathbb{R}^2$ expressed in the standard basis of $\mathbb{R}^2$ as

$$\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \mathbf{b}'_1 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \quad \mathbf{b}'_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and let us define two ordered bases $B = (\mathbf{b}_1, \mathbf{b}_2)$ and $B' = (\mathbf{b}'_1, \mathbf{b}'_2)$ of $\mathbb{R}^2$.

a. Show that B and B' are two bases of $\mathbb{R}^2$ and draw those basis vectors.

Solution.

To prove that B and B' are two bases of $\mathbb{R}^2$, we need to show that their constituent vectors are linearly independent.

This can be done by performing Gaussian reduction to convert them to row echelon form and counting the number of pivot columns.

For B:

$$\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{matrix} \cdot \frac{1}{2} \\ -\frac{1}{2}R_1 \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{matrix} \\ \cdot -2 \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 2 \end{bmatrix}$$

$\implies B$ has 2 independent vectors so it is a basis for $\mathbb{R}^2$.

For B':

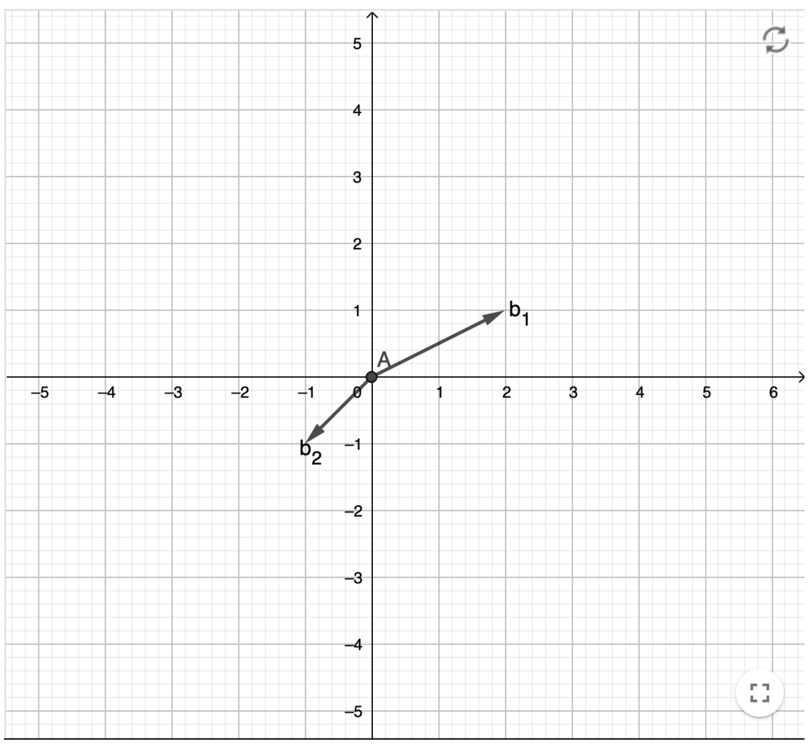
$$\begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix} \begin{matrix} \cdot\frac{1}{2} \\ +R_1 \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 2 \end{bmatrix} \begin{matrix} \\ \cdot\frac{1}{2} \end{matrix}$$

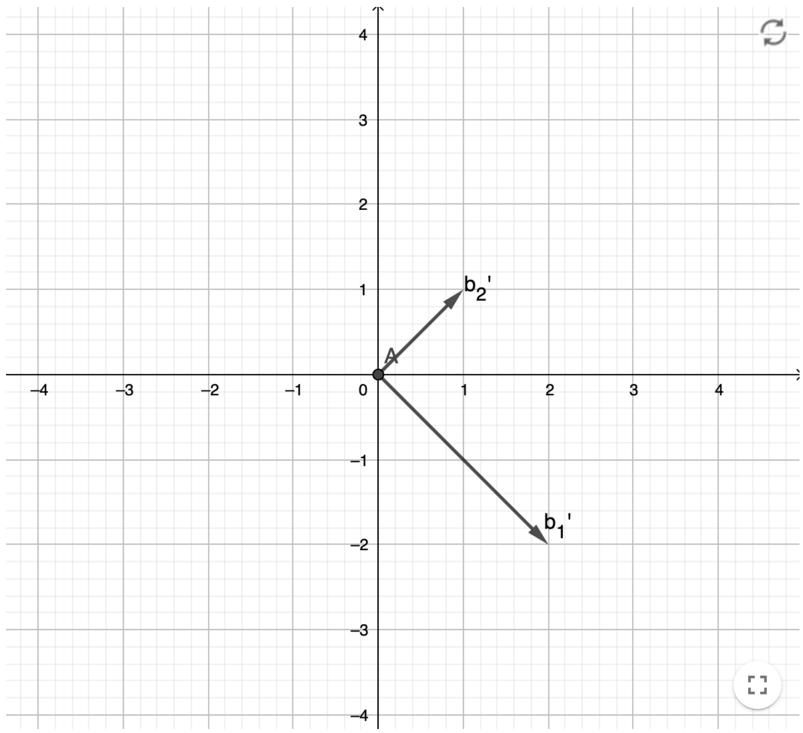
$$\rightsquigarrow \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

$\implies B'$ has 2 independent vectors so it is also a basis for $\mathbb{R}^2$.

Drawings of basis vectors of B:



Drawings of basis vectors of B':



b. Compute the matrix $\mathbf{P}_1$ that performs a basis change from B' to B.

Solution.

Let $\mathbf{x}$ be the coordinates w.r.t basis B and $\mathbf{y}$ be the coordinates w.r.t basis B'.

Then, $B\mathbf{x} = B'\mathbf{y} \qquad \Longrightarrow \mathbf{x} = B^{-1}B'\mathbf{y} \qquad \Longrightarrow \mathbf{P}_1 = B^{-1}B'$

$$B^{-1} = \frac{1}{2*(-1)-(-1)*1}\begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix} = -1\begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$$

$$\mathbf{P}_1 = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix}$$

c. We consider $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$, three vectors of $\mathbb{R}^3$ defined in the standard basis of $\mathbb{R}^3$ as

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \qquad \mathbf{c}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \qquad \mathbf{c}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and we define $C = (\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$.

(i) Show that C is a basis of $\mathbb{R}^3$, e.g. by using determinants (see Section 4.1).

Solution.

Using Sarrus rule,

$$\det(\mathrm{C}) = \begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{vmatrix}$$

= (1*(-1)*(-1))+(0*0*1)+(1*2*2)-(1*(-1)*1)-(2*0*1)-((-1)*2*0)

$$= 1 + 0 + 4 + 1 - 0 - 0$$

$$= 6$$

A non-zero determinant implies that the three columns are linearly independent, and therefore, form a basis of $\mathbb{R}^3$.

(ii) Let us call $C' = (\mathbf{c}'_1, \mathbf{c}'_2, \mathbf{c}'_3)$ the standard basis of $\mathbb{R}^3$. Determine the matrix $\mathbf{P}_2$ that performs the basis change from C to C'.

Solution.

Let $\mathbf{x}$ be the coordinates w.r.t basis C and $\mathbf{y}$ be the coordinates w.r.t basis C'.

Then, $C\mathbf{x} = C'\mathbf{y} \qquad \Longrightarrow \mathbf{y} = C'^{-1}C\mathbf{x} \qquad \Longrightarrow \mathbf{P}_2 = C'^{-1}C$

Since C' is $\mathbf{I}_3$, $C'^{-1} = \mathbf{I}_3$.

$\Longrightarrow \mathbf{P}_2 = C'^{-1}C = \mathbf{I}_3C = C$.

d. We consider a homomorphism $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, such that

$$\Phi(\mathbf{b}_1 + \mathbf{b}_2) = \mathbf{c}_2 + \mathbf{c}_3$$

$$\Phi(\mathbf{b}_1 - \mathbf{b}_2) = 2\mathbf{c}_1 - \mathbf{c}_2 + 3\mathbf{c}_3$$

where $B = (\mathbf{b}_1, \mathbf{b}_2)$ and $C = (\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)$ are ordered bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

Determine the transformation matrix $\mathbf{A}_\Phi$ of Φ with respect to the ordered bases B and C.

Solution.

Homomorphisms are linear, so,

$$\Phi(\mathbf{b}_1 + \mathbf{b}_2) = \Phi(\mathbf{b}_1) + \Phi(\mathbf{b}_2)$$

and

$$\Phi(\mathbf{b}_1 - \mathbf{b}_2) = \Phi(\mathbf{b}_1) - \Phi(\mathbf{b}_2)$$

Therefore,

$$2\Phi(\mathbf{b}_1) = \mathbf{c}_2 + \mathbf{c}_3 + 2\mathbf{c}_1 - \mathbf{c}_2 + 3\mathbf{c}_3 = 2\mathbf{c}_1 + 4\mathbf{c}_3$$

$$\implies \Phi(\mathbf{b}_1) = \mathbf{c}_1 + 2\mathbf{c}_3$$

and

$$2\Phi(\mathbf{b}_2) = \mathbf{c}_2 + \mathbf{c}_3 - 2\mathbf{c}_1 + \mathbf{c}_2 - 3\mathbf{c}_3 = -2\mathbf{c}_1 + 2\mathbf{c}_2 - 2\mathbf{c}_3$$

$$\implies \Phi(\mathbf{b}_2) = -\mathbf{c}_1 + \mathbf{c}_2 - \mathbf{c}_3$$

Therefore, if α_{ij} represents the elements of $\mathbf{A}_\Phi$,

$$\alpha_{11} = 1, \qquad \alpha_{21} = 0, \qquad \alpha_{31} = 2$$

$$\alpha_{12} = -1, \qquad \alpha_{22} = 1, \qquad \alpha_{32} = -1$$

$$\implies \mathbf{A}_\Phi = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}$$

e. Determine $\mathbf{A}'$, the transformation matrix of Φ with respect to the bases B' and C'.

Solution.

$$\mathbf{A}' = \mathbf{T}^{-1}\mathbf{A}_{\Phi}\mathbf{S}$$

where $\mathbf{T}$ is a transformation matrix that maps coordinates w.r.t basis C' onto coordinates w.r.t C, and $\mathbf{S}$ is a transformation matrix that maps coordinates w.r.t basis B' onto coordinates w.r.t B.

From previous results, we can see that $\mathbf{S} = \mathbf{P}_1$, and $\mathbf{T} = \mathbf{P}_2^{-1}$.

$$\implies \mathbf{A}' = (\mathbf{P}_2^{-1})^{-1}\mathbf{A}_{\Phi}\mathbf{P}_1$$

$$= \mathbf{P}_2\mathbf{A}_{\Phi}\mathbf{P}_1$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 \\ 2 & -3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 \\ -10 & 3 \\ 12 & -4 \end{bmatrix}$$

f. Let us consider the vector $\mathbf{x} \in \mathbb{R}^2$ whose coordinates in B' are $[2, 3]^T$. In other words, $\mathbf{x} = 2\mathbf{b}'_1 + 3\mathbf{b}'_2$.

(i) Calculate the coordinates of $\mathbf{x}$ in B.

Solution.

Let $\mathbf{x}_B$ be the coordinate vector of $\mathbf{x}$ in B.

$\mathbf{P}_1$ can perform basis change from B' to B.

$$\implies \mathbf{x}_B = \mathbf{P}_1\mathbf{x}$$

$$\implies \mathbf{x}_B = \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$$

(ii) Based on that, compute the coordinates of $\Phi(\mathbf{x})$ expressed in C.

Solution.

The matrix $\mathbf{A}_\Phi$ performs transformation with respect to the ordered bases B and C.

Let $\mathbf{x}_C$ be the coordinate vector of $\mathbf{x}$ in C.

$$\implies \mathbf{x}_C = \mathbf{A}_\Phi \mathbf{x}_B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 7 \end{bmatrix}$$

(iii) Then, write $\Phi(\mathbf{x})$ in terms of $\mathbf{c}'_1, \mathbf{c}'_2, \mathbf{c}'_3$.

Solution.

The matrix $\mathbf{P}_2$ performs the basis change from C to C'.

Let $\mathbf{x}_{C'}$ be the coordinate vector of $\mathbf{x}$ in C'.

$$\implies \mathbf{x}_{C'} = \mathbf{P}_2\mathbf{x}_C = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 9 \\ 7 \end{bmatrix} = \begin{bmatrix} 6 \\ -11 \\ 12 \end{bmatrix}$$

$$\implies \quad \Phi(\mathbf{x}) \text{ in terms of } \mathbf{c}_1', \mathbf{c}_2', \mathbf{c}_3' \quad = \quad 6\mathbf{c}_1' - 11\mathbf{c}_2' + 12\mathbf{c}_3'$$

(iv) Use the representation of $\mathbf{x}$ in B' and the matrix $\mathbf{A}'$ to find this result directly.

Solution.

To find the result directly, we multiply $\mathbf{A}'$ with $\mathbf{x}$,

$$\implies \begin{bmatrix} 0 & 2 \\ -10 & 3 \\ 12 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -11 \\ 12 \end{bmatrix}$$