6.8 Express the Bernoulli distribution in the natural parameter form of the exponential family, see (6.107).

Solution. The Bernoulli distribution is given by

$$\begin{aligned} p(x|\mu) &= \mu^x (1-\mu)^{1-x} \\ &= \exp\left(\ln\left(\mu^x (1-\mu)^{1-x}\right)\right) \\ &= \exp\left(\ln\left(\mu^x\right) + \ln\left((1-\mu)^{1-x}\right)\right) \\ &= \exp\left(x \ln(\mu) + (1-x)\ln(1-\mu)\right) \\ &= \exp\left(x \ln\left(\frac{\mu}{1-\mu}\right) + \ln(1-\mu)\right) \end{aligned}$$

$$\text{Setting } \theta = \ln\left(\frac{\mu}{1-\mu}\right), \text{ we get } \mu = 1 - (\exp(\theta) + 1)^{-1}.$$

Additionally, setting

$$\begin{aligned} \phi(x) &= x, \\ h(x) &= 1 \text{ and} \\ A(\theta) &= \ln\left(1 - \left(1 - (\exp(\theta) + 1)^{-1}\right)\right) = (\exp(\theta) + 1)^{-1} \end{aligned}$$

We get

$$p(x|\theta) = h(x) \exp\left(\langle \theta, \phi(x) \rangle - A(\theta)\right).$$