4.11 Show that for any $\mathbf{A} \in \mathbb{R}^{m \times n}$ the matrices $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$ possess the same nonzero eigenvalues.

Solution.

Let λ be an eigenvalue of $\mathbf{A}^T\mathbf{A}$, and $\mathbf{x}$ the corresponding eigenvector.

$\Longrightarrow (\mathbf{A}^T\mathbf{A})\mathbf{x} = \lambda\mathbf{x}$

$\Longrightarrow \mathbf{A}(\mathbf{A}^T\mathbf{A})\mathbf{x} = \mathbf{A}\lambda\mathbf{x} = \lambda\mathbf{A}\mathbf{x}$

$\Longrightarrow (\mathbf{A}\mathbf{A}^T)(\mathbf{A}\mathbf{x}) = \lambda(\mathbf{A}\mathbf{x}) \qquad \text{Associativity of matrix multiplication}$

$\Longrightarrow \lambda$ is an eigenvalue of $\mathbf{A}\mathbf{A}^T,$ and $\mathbf{A}\mathbf{x}$ the corresponding eigenvector.

Therefore, the matrices $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$ possess the same nonzero eigenvalues.