4.5 From 4.25, we have:

$$J(\mathbf{w}) = \frac{(m_2 - m_1)^2}{s_1^2 + s_2^2}$$

First we consider the numerator:

$$(m_2 - m_1)^2 = (\mathbf{w}^T(\mathbf{m}_2 - \mathbf{m}_1))^2$$

$$= (\mathbf{w}^T(\mathbf{m}_2 - \mathbf{m}_1))(\mathbf{w}^T(\mathbf{m}_2 - \mathbf{m}_1))$$

$$= (\mathbf{w}^T(\mathbf{m}_2 - \mathbf{m}_1))((\mathbf{m}_2 - \mathbf{m}_1)^T\mathbf{w})$$

$$= (\mathbf{w}^T(\mathbf{m}_2 - \mathbf{m}_1)(\mathbf{m}_2 - \mathbf{m}_1)^T\mathbf{w})$$

$$= \mathbf{w}^T\mathbf{S}_B\mathbf{w} \qquad \text{using 4.27}$$

Now, we consider equation 4.24:

$$s_k^2 = \sum_{n \in \mathcal{C}_k} (y_n - m_k)^2$$

$$= \sum_{n \in \mathcal{C}_k} (\mathbf{w}^T\mathbf{x}_n - \mathbf{w}^T\mathbf{m}_k)^2$$

$$= \sum_{n \in \mathcal{C}_k} (\mathbf{w}^T\mathbf{x}_n - \mathbf{w}^T\mathbf{m}_k)(\mathbf{w}^T\mathbf{x}_n - \mathbf{w}^T\mathbf{m}_k)$$

$$= \sum_{n \in \mathcal{C}_k} (\mathbf{w}^T(\mathbf{x}_n - \mathbf{m}_k))((\mathbf{x}_n^T - \mathbf{m}_k^T)\mathbf{w})$$

$$= \sum_{n \in \mathcal{C}_k} \mathbf{w}^T(\mathbf{x}_n - \mathbf{m}_k)(\mathbf{x}_n - \mathbf{m}_k)^T\mathbf{w}$$

$$= \mathbf{w}^T \left(\sum_{n \in \mathcal{C}_k} (\mathbf{x}_n - \mathbf{m}_k)(\mathbf{x}_n - \mathbf{m}_k)^T \right) \mathbf{w}$$

Applying this result to the denominator of $J(\mathbf{w})$,

$$s_1^2 + s_2^2 = \mathbf{w}^T \left(\sum_{n \in \mathcal{C}_1} (\mathbf{x}_n - \mathbf{m}_1)(\mathbf{x}_n - \mathbf{m}_1)^T \right) \mathbf{w} + \mathbf{w}^T \left(\sum_{n \in \mathcal{C}_2} (\mathbf{x}_n - \mathbf{m}_2)(\mathbf{x}_n - \mathbf{m}_2)^T \right) \mathbf{w}$$

$$= \mathbf{w}^T \left(\sum_{n \in \mathcal{C}_1} (\mathbf{x}_n - \mathbf{m}_1)(\mathbf{x}_n - \mathbf{m}_1)^T + \sum_{n \in \mathcal{C}_2} (\mathbf{x}_n - \mathbf{m}_2)(\mathbf{x}_n - \mathbf{m}_2)^T \right) \mathbf{w}$$

$$= \mathbf{w}^T \mathbf{S}_W \mathbf{w} \qquad \text{using 4.28}$$

Substituting these results into the expression for $J(\mathbf{w})$,we get:

$$J(\mathbf{w}) = \frac{\mathbf{w}^T \mathbf{S}_B \mathbf{w}}{\mathbf{w}^T \mathbf{S}_W \mathbf{w}}$$

which is the same as 4.26.