2.15

$$H[\mathbf{x}] = -\int p(\mathbf{x})\ \ln(p(\mathbf{x}))\,d\mathbf{x}$$

$$= -\int \mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma})\ \ln(\mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma}))\,d\mathbf{x}$$

$$= -\int \mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma})\ \ln\left(\frac{1}{(2\pi)^{D/2}}\frac{1}{|\boldsymbol{\Sigma}|^{1/2}}exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}\right)\ d\mathbf{x}$$

$$= -\int \mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma})\ \left(\ln\left(\frac{1}{(2\pi)^{D/2}}\right)+\ln\left(\frac{1}{|\boldsymbol{\Sigma}|^{1/2}}\right)-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)\ d\mathbf{x}$$

$$= -\int \mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma})\ \left(-\frac{D}{2}\ln 2\pi-\frac{1}{2}\ln|\boldsymbol{\Sigma}|-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)\ d\mathbf{x}$$

$$= \int \mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma})\ \left(\frac{D}{2}\ln 2\pi+\frac{1}{2}\ln|\boldsymbol{\Sigma}|+\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)\ d\mathbf{x}$$

$$\begin{aligned}
&= \int \frac{D}{2}\ln 2\pi\,\mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma})\,d\mathbf{x}\\
&+\int \frac{1}{2}\ln|\boldsymbol{\Sigma}|\,\mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma})\,d\mathbf{x}\\
&+\int \frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\,\mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma})\,d\mathbf{x}
\end{aligned}$$

$$\begin{aligned}
&= \frac{D}{2}\ln 2\pi\ \int \mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma})\,d\mathbf{x}\\
&+\frac{1}{2}\ln|\boldsymbol{\Sigma}|\ \int \mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma})\,d\mathbf{x}\\
&+\frac{1}{2}\int(\mathbf{x}^T\boldsymbol{\Sigma}^{-1}\mathbf{x}-\mathbf{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}-\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\mathbf{x}+\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})\,\mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma})\,d\mathbf{x}
\end{aligned}$$

$$\begin{aligned}
&= \frac{D}{2}\ln 2\pi\,(1)\\
&+\frac{1}{2}\ln|\boldsymbol{\Sigma}|\,(1)
\end{aligned}$$

$$+\frac{1}{2}\int (\mathbf{x}^T\mathbf{\Sigma}^{-1}\mathbf{x} - \mathbf{x}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu} - \mathbf{x}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu} + \boldsymbol{\mu}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu})\,\mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\mathbf{\Sigma})\,d\mathbf{x}$$

$$= \frac{D}{2}\ln 2\pi$$
$$+\frac{1}{2}\ln|\mathbf{\Sigma}|$$
$$+\frac{1}{2}\int (\mathbf{x}^T\mathbf{\Sigma}^{-1}\mathbf{x} - 2\mathbf{x}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu} + \boldsymbol{\mu}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu})\,\mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\mathbf{\Sigma})\,d\mathbf{x}$$

$$= \frac{D}{2}\ln 2\pi + \frac{1}{2}\ln|\mathbf{\Sigma}|$$
$$+\frac{1}{2}\int \mathbf{x}^T\mathbf{\Sigma}^{-1}\mathbf{x}\,\mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\mathbf{\Sigma})\,d\mathbf{x}$$
$$-\frac{1}{2}\int 2\mathbf{x}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\,\mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\mathbf{\Sigma})\,d\mathbf{x}$$
$$+\frac{1}{2}\int \boldsymbol{\mu}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\,\mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\mathbf{\Sigma})\,d\mathbf{x}$$

Using equation 318 from Matrix Cookbook, we get

$$\int \mathbf{x}^T\mathbf{\Sigma}^{-1}\mathbf{x}\,\mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\mathbf{\Sigma})\,d\mathbf{x} = tr(\mathbf{\Sigma}^{-1}\mathbf{\Sigma}) + \boldsymbol{\mu}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}$$

This gives us:

$$= \frac{D}{2}\ln 2\pi + \frac{1}{2}\ln|\mathbf{\Sigma}|$$
$$+\frac{1}{2}(tr(\mathbf{\Sigma}^{-1}\mathbf{\Sigma}) + \boldsymbol{\mu}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu})$$
$$-\boldsymbol{\mu}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}$$
$$+\frac{1}{2}\boldsymbol{\mu}^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\int \mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\mathbf{\Sigma})\,d\mathbf{x}$$

$$= \frac{D}{2}\ln 2\pi + \frac{1}{2}\ln|\mathbf{\Sigma}|$$

$$+\frac{1}{2}(tr(\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}) + \boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})$$
$$-\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$$
$$+\frac{1}{2}\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\,(1)$$

$$= \frac{D}{2}\ln 2\pi + \frac{1}{2}\ln|\boldsymbol{\Sigma}| + \frac{1}{2}tr(\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}) + \frac{1}{2}\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \frac{1}{2}\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$$

$$= \frac{D}{2}\ln 2\pi + \frac{1}{2}\ln|\boldsymbol{\Sigma}| + \frac{1}{2}tr(\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma})$$

$$= \frac{D}{2}\ln 2\pi + \frac{1}{2}\ln|\boldsymbol{\Sigma}| + \frac{1}{2}D$$

$$= \frac{1}{2}\ln|\boldsymbol{\Sigma}| + \frac{D}{2}(1 + \ln(2\pi))$$