1.7 We are given:

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} exp\left(-\frac{1}{2\sigma^2}x^2 - \frac{1}{2\sigma^2}y^2\right) dx\, dy$$

Now we make the transformation from Cartesian coordinates (x, y) to polar coordinates (r, θ) using this,

where :

$$\int\int_R f(x, y)\, dx\, dy = \int\int_D f(rcos\,\theta, rsin\,\theta)\, r\, dr\, d\,\theta$$

which gives us:

$$I^2 = \int_0^{\infty} \int_0^{2\pi} exp\left(-\frac{1}{2\sigma^2}(r\,cos\,\theta)^2 - \frac{1}{2\sigma^2}(r\,sin\,\theta)^2\right) r\, dr\, d\,\theta$$

since r is a distance, it goes from 0 to ∞.

$$= \int_0^{\infty} \int_0^{2\pi} exp\left(-\frac{r^2}{2\sigma^2}\right) r\, dr\, d\,\theta$$

Substituting $u = r^2$, the limits for u become 0 to ∞ as it cannot be a negative number:

$$I^2 = \int_0^{\infty} \int_0^{2\pi} exp\left(-\frac{u}{2\sigma^2}\right) \sqrt{u} \left(\frac{d\,r}{d\,u}\right) d\,u\, d\,\theta$$

$$= \int_0^{\infty} \int_0^{2\pi} exp\left(-\frac{u}{2\sigma^2}\right) \sqrt{u} \left(\frac{1}{2\sqrt{u}}\right) d\,u\, d\,\theta$$

$$= \int_0^{\infty} \int_0^{2\pi} exp\left(-\frac{u}{2\sigma^2}\right) \left(\frac{1}{2}\right) d\,u\, d\,\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left(\int_0^{\infty} exp\left(-\frac{u}{2\sigma^2}\right) d\,u\right) d\,\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left(\int_0^{\infty} exp\left(-\frac{u}{2\sigma^2}\right) d\,u\right) d\,\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left(\left[(-2\sigma^2)\, exp\left(-\frac{u}{2\sigma^2}\right)\right]_0^{\infty}\right) d\,\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \big((0-1)\,(-2\sigma^2)\big)\,d\,\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 2\sigma^2\,d\,\theta$$

$$= \frac{1}{2}(2\pi\,2\sigma^2)$$

$$= 2\pi\sigma^2$$

$$\Longrightarrow I = \sqrt{2\pi\sigma^2}$$

The Gaussian distribution is given by:

$$f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

The integral of the Gaussian distribution is given by:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\,dx$$

Let $y = x - \mu$,

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{y^2}{2\sigma^2}\right)\,dy\,\left(\frac{dx}{dy}\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} exp\left(-\frac{y^2}{2\sigma^2}\right)\,dy\,(1)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\,\sqrt{2\pi\sigma^2}$$

$$= 1.$$