

**1.16**  $n(D, M)$  is the number of independent parameters of order  $M$ . So it just follows logically that the total number  $N(D, M)$  of independent parameters in all of the terms up to and including the  $M^{th}$  order is given by:

$$N(D, M) = \sum_{m=0}^M n(D, m)$$

Now to prove 1.139.

First proving that the result holds for  $M = 0$  and arbitrary  $D \geq 1$ :

$$\begin{aligned} L.H.S &= N(D, 0) \\ &= \sum_{m=0}^0 n(D, m) \\ &= n(D, 0) \end{aligned}$$

Using 1.137, this becomes:

$$\begin{aligned} &= \frac{(D + 0 - 1)!}{(D - 1)! 0!} \\ &= \frac{(D - 1)!}{(D - 1)!} \\ &= 1 \end{aligned}$$

$$\begin{aligned} R.H.S &= \frac{(D + 0)!}{D! 0!} \\ &= \frac{D!}{D!} \\ &= 1 \end{aligned}$$

Thus proved for  $M = 0$ .

Now assuming that 1.139 holds at order  $M$ , we need to show that it holds at order  $M + 1$ .

$$\begin{aligned}
L.H.S &= N(D, (M + 1)) \\
&= \sum_{m=0}^{M+1} n(D, m) \\
&= \left( \sum_{m=0}^M n(D, m) \right) + n(D, (M + 1)) \\
&= N(D, M) + n(D, (M + 1))
\end{aligned}$$

Applying 1.137 and 1.139,

$$\begin{aligned}
&= \frac{(D + M)!}{D! M!} + \frac{(D + (M + 1) - 1)!}{(D - 1)! (M + 1)!} \\
&= \frac{(D + M)!}{D! M!} + \frac{(D + M)!}{(D - 1)! (M + 1)!} \\
&= \frac{(D + M)!}{(D - 1)! M!} \left( \frac{1}{D} + \frac{1}{(M + 1)} \right) \\
&= \frac{(D + M)!}{(D - 1)! M!} \left( \frac{(D + M + 1)}{D(M + 1)} \right) \\
&= \frac{(D + M + 1)!}{D! (M + 1)!}
\end{aligned}$$

Thus proved that 1.139 holds for  $M + 1$ .

Finally, applying Stirling's approximation for  $D \gg M$ ,

$$\begin{aligned}
N(D, M) &= \frac{(D + M)!}{D! M!} \\
&\simeq \frac{(D + M)^{(D+M)} e^{-(D+M)}}{D^D e^{-D} M!} \\
&= \frac{(D + M)^D (D + M)^M e^{-M}}{D^D M!}
\end{aligned}$$

$$= \frac{(1 + M/D)^D (D + M)^M e^{-M}}{M!}$$

Since  $D \gg M$ ,  $(1 + M/D)^D \simeq 1$ , and  $(D + M)^M \simeq D^M$ . So we can see that the quantity  $N(D, M)$  grows like  $D^M$ .

Similarly, considering the case of  $M \gg D$ ,

$$\begin{aligned} N(D, M) &= \frac{(D + M)!}{D! M!} \\ &\simeq \frac{(D + M)^{(D+M)} e^{-(D+M)}}{D! M^M e^{-M}} \\ &= \frac{(D + M)^D (D + M)^M e^{-D}}{D! M^M} \} \\ &= \frac{(D + M)^D (D/M + 1)^M e^{-D}}{D!} \end{aligned}$$

Since  $M \gg D$ ,  $(D/M + 1)^M \simeq 1$ , and  $(D + M)^D \simeq M^D$ . So we can see that the quantity  $N(D, M)$  grows like  $M^D$ .

Assuming  $M = 3$ , Total number of independent parameters for

$$(i) D = 10 = N(10, 3) = \frac{(10 + 3)!}{10! 3!} = \frac{13!}{10! 3!} = \frac{13 * 12 * 11}{3 * 2 * 1} = 286$$

$$(ii) D = 100 = N(100, 3) = \frac{(100 + 3)!}{100! 3!} = \frac{103!}{100! 3!} = \frac{103 * 102 * 101}{3 * 2 * 1} = 176851$$