1.19 We know that volume of a sphere of radius a in D dimensions is given by:

$$V_D = \int_0^a S_D r^{D-1} dr$$

where S_D is area of unit sphere.

This gives us :

$$V_D = \left[\frac{2(\pi)^{D/2}}{\Gamma(D/2)} \frac{1}{D} r^D \right]_0^a$$

$$= \frac{2a^D(\pi)^{D/2}}{D\Gamma(D/2)}$$

Volume of hypercube of side $2a$ in D dimensions is given by $(2a)^D$.

Therefore,

$$\frac{\text{volume of sphere}}{\text{volume of cube}} = \frac{2a^D(\pi)^{D/2}}{D\Gamma(D/2)(2a)^D}$$

$$= \frac{(\pi)^{D/2}}{D\Gamma(D/2)2^{D-1}}$$

Using Stirling's formula, for $D >> 1$,

$$\Gamma(D/2) \simeq (2\pi)^{1/2} e^{-((D/2)-1)} (D/2-1)^{((D/2-1)+1/2)}$$

$$= (2\pi)^{1/2} e^{-D/2} e (D/2-1)^{(D-1)/2}$$

$$\simeq (2\pi)^{1/2} e^{-D/2} e (D/2)^{(D-1)/2}$$

Substituting in the expression for the ratio of volume of sphere to volume of cube, we get:

$$\frac{(\pi)^{D/2}}{D\Gamma(D/2)2^{D-1}}$$

$$= \left(\frac{(\pi)^{D/2}}{D2^{D-1}} \right) \left(\frac{1}{\Gamma(D/2)} \right)$$

$$= \left(\frac{\pi^{D/2}}{D2^{D-1}}\right)\left(\frac{1}{(2\pi)^{1/2}e^{-D/2}e(D/2)^{(D-1)/2}}\right)$$

$$= \left(\frac{\pi^{D/2}}{D2^{D-1}}\right)\left(\frac{2^{(D-1)/2}}{(2\pi)^{1/2}e^{-D/2}eD^{(D-1)/2}}\right)$$

$$= \left(\frac{\pi^{D/2}}{D2^{(D-1)/2}}\right)\left(\frac{1}{(2\pi)^{1/2}e^{-D/2}eD^{(D-1)/2}}\right)$$

$$= \left(\frac{\pi^{D/2}}{2^{(D-1)/2}}\right)\left(\frac{1}{D}\right)\left(\frac{1}{D^{(D-1)/2}}\right)\left(\frac{1}{(2\pi)^{1/2}e}\right)\left(\frac{1}{e^{-D/2}}\right)$$

$$= \left(\frac{\pi^{D/2}}{2^{(D-1)/2}}\right)\left(\frac{1}{D^{(D+1)/2}}\right)\left(\frac{1}{(2\pi)^{1/2}e}\right)\left(\frac{1}{e^{-D/2}}\right)$$

$$= \left(\frac{2^{1/2}\pi^{D/2}}{2^{D/2}}\right)\left(\frac{1}{D^{1/2}D^{D/2}}\right)\left(\frac{1}{(2\pi)^{1/2}e}\right)\left(\frac{1}{e^{-D/2}}\right)$$

$$= \left(\frac{2^{1/2}}{(2\pi)^{1/2}e}\right)\left(\frac{1}{D^{1/2}}\right)\left(\frac{\pi^{D/2}e^{D/2}}{D^{D/2}2^{D/2}}\right)$$

$$= \left(\frac{1}{(\pi)^{1/2}e}\right)\left(\frac{1}{D^{1/2}}\right)\left(\frac{\pi e}{2D}\right)^{D/2}$$

The first term is a constant, the second term goes to 0 as $D \to \infty$, and the fraction in the third term also goes to 0 as $D \to \infty$.

Therefore, the ratio goes to 0 as $D \to \infty$.

Distance from the centre of the hypercube to a corners in D dimensions is given by : $\sqrt{a^2 + a^2 + \ldots + a^2} = \sqrt{Da^2}$.

Perpendicular distance to one of the sides $= a$.

Ratio of the distance from the centre of the hypercube to one of the corners, divided by the perpendicular distance to one of the sides $= \dfrac{\sqrt{Da^2}}{a} = \sqrt{D}$.