2.42 Finding mean:

$$\mathbb{E}[\lambda] = \int_0^\infty \lambda \, Gam(\lambda|a,b)\, d\lambda$$

$$= \int_0^\infty \lambda \, \frac{1}{\Gamma(a)} b^a \lambda^{a-1} exp(-b\lambda)\, d\lambda$$

$$= \frac{1}{\Gamma(a)} \int_0^\infty b^a \lambda^a exp(-b\lambda)\, d\lambda$$

$$= \frac{1}{\Gamma(a)} \int_0^\infty (b\lambda)^a exp(-(b\lambda))\, \frac{1}{b}\, d(b\lambda)$$

$$= \frac{1}{b\Gamma(a)} \int_0^\infty (b\lambda)^a exp(-(b\lambda))\, d(b\lambda)$$

Let $u = b\lambda$. Then we get:

$$= \frac{1}{b\Gamma(a)} \int_0^\infty u^a exp(-u)\, du$$

$$= \frac{1}{b\Gamma(a)} \Gamma(a+1)$$

$$= \frac{1}{b\Gamma(a)} a\Gamma(a)$$

$$= \frac{a}{b}$$

Finding variance:

$$var[\lambda] = \mathbb{E}[\lambda^2] - \mathbb{E}[\lambda]^2$$

$$\mathbb{E}[\lambda^2] = \int_0^\infty \lambda^2 \, Gam(\lambda|a,b)\, d\lambda$$

$$= \int_0^\infty \lambda^2 \, \frac{1}{\Gamma(a)} b^a \lambda^{a+1} exp(-b\lambda)\, d\lambda$$

$$= \frac{1}{\Gamma(a)} \int_0^\infty b^a \lambda^{a+1} exp(-b\lambda)\, d\lambda$$

$$= \frac{1}{\Gamma(a)} \frac{1}{b^2} \int_0^\infty (b\lambda)^{a+1} exp(-(b\lambda)) \, d(b\lambda)$$

$$= \frac{1}{b\Gamma(a)} \frac{1}{b^2} \int_0^\infty (b\lambda)^{a+1} exp(-(b\lambda)) \, d(b\lambda)$$

Let $u = b\lambda$. Then we get:

$$= \frac{1}{b^2\Gamma(a)} \int_0^\infty u^{a+1} exp(-u) \, du$$

$$= \frac{1}{b^2\Gamma(a)} \Gamma(a+2)$$

$$= \frac{1}{b^2\Gamma(a)} (a+1)(a)\Gamma(a)$$

$$= \frac{(a+1)a}{b^2}$$

$$\implies \frac{(a+1)a}{b^2} - \left(\frac{a}{b}\right)^2$$

$$= \frac{a^2 + a - a^2}{b^2}$$

$$= \frac{a}{b^2}$$

To find mode, we take derivative w.r.t λ and set it to 0:

$$\frac{\partial}{\partial\lambda} Gam(\lambda|a, b)$$

$$= \frac{\partial}{\partial\lambda} \frac{1}{\Gamma(a)} b^a \lambda^{a-1} exp(-b\lambda)$$

$$= \frac{1}{\Gamma(a)} b^a \left(\left(\frac{\partial\lambda^{a-1}}{\partial\lambda} \right) exp(-b\lambda) + \lambda^{a-1} \left(\frac{\partial exp(-b\lambda)}{\partial\lambda} \right) \right)$$

$$= \frac{1}{\Gamma(a)} b^a \left((a-1)\lambda^{a-2} exp(-b\lambda) + \lambda^{a-1}(-b) exp(-b\lambda) \right)$$

$$= \frac{1}{\Gamma(a)} b^a \lambda^{a-2} exp(-b\lambda) \left((a-1) + \lambda(-b)\right)$$

Setting the derivative to 0, we get the maxima which is the mode:

$$0 = \frac{1}{\Gamma(a)} b^a \lambda^{a-2} exp(-b\lambda) \left((a-1) + \lambda(-b)\right)$$

$$\implies 0 = ((a-1) + \lambda(-b))$$

$$\implies \lambda = \frac{a-1}{b}$$

Mode doesn't exist for a < 1 since probability should be ≥ 0.