3.7 The posterior distribution is given by:

$$p(\mathbf{w}|t) \propto p(t|\mathbf{X}, \mathbf{w}, \beta) * p(\mathbf{w})$$

From 3.10, and 3.48, this becomes:

$$= \left(\prod_{n=1}^{N} \mathcal{N}\left(t_n | \mathbf{w}^T \phi(\mathbf{x}_n), \beta^{-1}\right) \right) \mathcal{N}(\mathbf{w}|\mathbf{m}_0, \mathbf{S}_0)$$

$$= \left(\prod_{n=1}^{N} \frac{1}{\sqrt{2\pi\beta^{-1}}} \exp\left\{ -\frac{\beta}{2}(t_n - \mathbf{w}^T \phi(\mathbf{x}_n))^2 \right\} \right) \mathcal{N}(\mathbf{w}|\mathbf{m}_0, \mathbf{S}_0)$$

$$= \left(\frac{1}{(2\pi\beta^{-1})^{(N/2)}} \exp\left\{ \sum_{n=1}^{N} -\frac{\beta}{2}(t_n - \mathbf{w}^T \phi(\mathbf{x}_n))^2 \right\} \right) \mathcal{N}(\mathbf{w}|\mathbf{m}_0, \mathbf{S}_0)$$

$$= \left(\frac{1}{(2\pi\beta^{-1})^{(N/2)}} \exp\left\{ -\frac{\beta}{2} \sum_{n=1}^{N} (t_n - \mathbf{w}^T \phi(\mathbf{x}_n))^2 \right\} \right) \mathcal{N}(\mathbf{w}|\mathbf{m}_0, \mathbf{S}_0)$$

$$= \left(\frac{1}{(2\pi\beta^{-1})^{(N/2)}} \exp\left\{ -\frac{\beta}{2} \sum_{n=1}^{N} (t_n - \mathbf{w}^T \phi(\mathbf{x}_n))^2 \right\} \right)$$
$$\frac{1}{(2\pi)^{(D/2)} |\mathbf{S}_0|^{(1/2)}} \exp\left\{ -\frac{1}{2}(\mathbf{w} - \mathbf{m}_0)^T \mathbf{S}_0^{-1} (\mathbf{w} - \mathbf{m}_0) \right\}$$

$$= \frac{\beta^{(N/2)}}{(2\pi)^{(D+N)/2} |\mathbf{S}_0|^{(1/2)}} \exp\left\{ -\frac{\beta}{2} \sum_{n=1}^{N} (t_n - \mathbf{w}^T \phi(\mathbf{x}_n))^2 \right\}$$
$$\exp\left\{ -\frac{1}{2}(\mathbf{w} - \mathbf{m}_0)^T \mathbf{S}_0^{-1} (\mathbf{w} - \mathbf{m}_0) \right\}$$

$$= \frac{\beta^{(N/2)}}{(2\pi)^{(D+N)/2} |\mathbf{S}_0|^{(1/2)}} \exp\left\{ -\frac{\beta}{2} \sum_{n=1}^{N} (t_n^2 - 2t_n \mathbf{w}^T \phi(\mathbf{x}_n) + (\mathbf{w}^T \phi(\mathbf{x}_n))^2) \right\}$$
$$\exp\left\{ -\frac{1}{2}(\mathbf{w} - \mathbf{m}_0)^T \mathbf{S}_0^{-1} (\mathbf{w} - \mathbf{m}_0) \right\}$$

$$= \frac{\beta^{(N/2)}}{(2\pi)^{(D+N)/2}|\mathbf{S}_0|^{(1/2)}} \exp\left\{-\frac{\beta}{2}\sum_{n=1}^{N}(t_n^2 - 2t_n\mathbf{w}^T\phi(\mathbf{x}_n) + \mathbf{w}^T\phi(\mathbf{x}_n)\phi(\mathbf{x}_n)^T\mathbf{w})\right\}$$

$$\exp\left\{-\frac{1}{2}(\mathbf{w}-\mathbf{m}_0)^T\mathbf{S}_0^{-1}(\mathbf{w}-\mathbf{m}_0)\right\}$$

$$= \frac{\beta^{(N/2)}}{(2\pi)^{(D+N)/2}|\mathbf{S}_0|^{(1/2)}} \exp\left\{-\frac{\beta}{2}\left(\sum_{n=1}^{N} t_n^2 - 2\mathbf{w}^T\mathbf{\Phi}^T\mathbf{t} + \mathbf{w}^T\mathbf{\Phi}^T\mathbf{\Phi}\mathbf{w}\right)\right\}$$

$$\exp\left\{-\frac{1}{2}(\mathbf{w}-\mathbf{m}_0)^T\mathbf{S}_0^{-1}(\mathbf{w}-\mathbf{m}_0)\right\}$$

$$= \frac{\beta^{(N/2)}}{(2\pi)^{(D+N)/2}|\mathbf{S}_0|^{(1/2)}} \exp\left\{-\frac{\beta}{2}\left(\sum_{n=1}^{N} t_n^2 - 2\mathbf{w}^T\mathbf{\Phi}^T\mathbf{t} + \mathbf{w}^T\mathbf{\Phi}^T\mathbf{\Phi}\mathbf{w}\right)\right\}$$

$$\exp\left\{-\frac{1}{2}\left(\mathbf{w}^T\mathbf{S}_0^{-1}\mathbf{w} - 2\mathbf{w}^T\mathbf{S}_0^{-1}\mathbf{m}_0 + \mathbf{m}_0^T\mathbf{S}_0^{-1}\mathbf{m}_0\right)\right\}$$

Rearranging the terms in the exponents, we get:

$$= \frac{\beta^{(N/2)}}{(2\pi)^{(D+N)/2}|\mathbf{S}_0|^{(1/2)}} \exp\left\{-\frac{1}{2}\left(\mathbf{w}^T(\beta\mathbf{\Phi}^T\mathbf{\Phi} + \mathbf{S}_0^{-1})\mathbf{w} - 2\mathbf{w}^T(\beta\mathbf{\Phi}^T\mathbf{t} + \mathbf{S}_0^{-1}\mathbf{m}_0)\right)\right\}$$

$$\exp\left\{-\frac{1}{2}\left(\mathbf{m}_0^T\mathbf{S}_0^{-1}\mathbf{m}_0 + \beta\sum_{n=1}^{N} t_n^2\right)\right\}$$

$$= \frac{\beta^{(N/2)}}{(2\pi)^{(D+N)/2}|\mathbf{S}_0|^{(1/2)}} \exp\left\{-\frac{1}{2}\left(\mathbf{w}^T\mathbf{S}_N^{-1}\mathbf{w} - 2\mathbf{w}^T\mathbf{S}_N^{-1}\mathbf{m}_N\right)\right\}$$

$$\exp\left\{-\frac{1}{2}\left(\mathbf{m}_0^T\mathbf{S}_0^{-1}\mathbf{m}_0 + \beta\sum_{n=1}^{N} t_n^2\right)\right\}$$

Adding and subtracting $\mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{m}_N$ in the exponents,

$$= \frac{\beta^{(N/2)}}{(2\pi)^{(D+N)/2}|\mathbf{S}_0|^{(1/2)}} \exp\left\{-\frac{1}{2}\left(\mathbf{w}^T\mathbf{S}_N^{-1}\mathbf{w} - 2\mathbf{w}^T\mathbf{S}_N^{-1}\mathbf{m}_N + \mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{m}_N\right)\right\}$$

$$\exp\left\{-\frac{1}{2}\left(\mathbf{m}_0^T\mathbf{S}_0^{-1}\mathbf{m}_0 - \mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{m}_N + \beta\sum_{n=1}^{N} t_n^2\right)\right\}$$

$$= \frac{\beta^{(N/2)}}{(2\pi)^{(D+N)/2}|\mathbf{S}_0|^{(1/2)}} \exp\left\{-\frac{1}{2}\left((\mathbf{w} - \mathbf{m}_N)^T\mathbf{S}_N^{-1}(\mathbf{w} - \mathbf{m}_N)\right)\right\}$$

$$\exp\left\{-\frac{1}{2}\left(\mathbf{m}_0^T\mathbf{S}_0^{-1}\mathbf{m}_0 - \mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{m}_N + \beta\sum_{n=1}^{N} t_n^2\right)\right\}$$

Multiplying and dividing the fraction by $|\mathbf{S}_N|^{(1/2)}$, we get:

$$= \frac{\beta^{(N/2)}|\mathbf{S}_N|^{(1/2)}}{(2\pi)^{(D+N)/2}|\mathbf{S}_0|^{(1/2)}|\mathbf{S}_N|^{(1/2)}} \exp\left\{-\frac{1}{2}\left((\mathbf{w} - \mathbf{m}_N)^T\mathbf{S}_N^{-1}(\mathbf{w} - \mathbf{m}_N)\right)\right\}$$

$$\exp\left\{-\frac{1}{2}\left(\mathbf{m}_0^T\mathbf{S}_0^{-1}\mathbf{m}_0 - \mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{m}_N + \beta\sum_{n=1}^{N} t_n^2\right)\right\}$$

$$= \frac{1}{(2\pi)^{D/2}|\mathbf{S}_N|^{(1/2)}} \exp\left\{-\frac{1}{2}\left((\mathbf{w} - \mathbf{m}_N)^T\mathbf{S}_N^{-1}(\mathbf{w} - \mathbf{m}_N)\right)\right\}$$

$$\frac{\beta^{(N/2)}|\mathbf{S}_N|^{(1/2)}}{(2\pi)^{N/2}|\mathbf{S}_0|^{(1/2)}} \exp\left\{-\frac{1}{2}\left(\mathbf{m}_0^T\mathbf{S}_0^{-1}\mathbf{m}_0 - \mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{m}_N + \beta\sum_{n=1}^{N} t_n^2\right)\right\}$$

$$= c\mathcal{N}\left(\mathbf{w}|\mathbf{m}_N, \mathbf{S}_N\right)$$

where c is the constant second term.