3.19

$$\int \exp\left\{-E(\mathbf{w})\right\}\, d\mathbf{w}$$

Applying the result from 3.80, this becomes:

$$= \int \exp\left\{-\left(E(\mathbf{m}_N) + \frac{1}{2}(\mathbf{w}-\mathbf{m}_N)^T\mathbf{A}(\mathbf{w}-\mathbf{m}_N)\right)\right\}\, d\mathbf{w}$$

$$= \int \exp\left\{-E(\mathbf{m}_N)\right\} \exp\left\{-\frac{1}{2}(\mathbf{w}-\mathbf{m}_N)^T\mathbf{A}(\mathbf{w}-\mathbf{m}_N)\right\}\, d\mathbf{w}$$

$$= \exp\left\{-E(\mathbf{m}_N)\right\} \int \exp\left\{-\frac{1}{2}(\mathbf{w}-\mathbf{m}_N)^T\mathbf{A}(\mathbf{w}-\mathbf{m}_N)\right\}\, d\mathbf{w}$$

$$= \exp\left\{-E(\mathbf{m}_N)\right\} \int \frac{(2\pi)^{M/2}\,|\mathbf{A}^{-1}|^{1/2}}{(2\pi)^{M/2}\,|\mathbf{A}^{-1}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{w}-\mathbf{m}_N)^T\mathbf{A}(\mathbf{w}-\mathbf{m}_N)\right\}\, d\mathbf{w}$$

$$= \exp\left\{-E(\mathbf{m}_N)\right\} (2\pi)^{M/2}\,|\mathbf{A}^{-1}|^{1/2} \int \frac{1}{(2\pi)^{M/2}\,|\mathbf{A}^{-1}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{w}-\mathbf{m}_N)^T\mathbf{A}(\mathbf{w}-\mathbf{m}_N)\right\}\, d\mathbf{w}$$

$$= \exp\left\{-E(\mathbf{m}_N)\right\} (2\pi)^{M/2}\,|\mathbf{A}^{-1}|^{1/2} \int \mathcal{N}(\mathbf{w}|\mathbf{m}_N, \mathbf{A}^{-1})\, d\mathbf{w}$$

$$= \exp\left\{-E(\mathbf{m}_N)\right\} (2\pi)^{M/2}\,|\mathbf{A}^{-1}|^{1/2}(1)$$

$$= \exp\left\{-E(\mathbf{m}_N)\right\} (2\pi)^{M/2}\,|\mathbf{A}|^{-1/2}$$

which is the same 3.85.

Now, using 3.78 and this result, the marginal likelihood becomes:

$$p(\mathbf{t}|\alpha,\beta) = \left(\frac{\beta}{2\pi}\right)^{N/2} \left(\frac{\alpha}{2\pi}\right)^{M/2} \exp\left\{-E(\mathbf{m}_N)\right\} (2\pi)^{M/2}\,|\mathbf{A}|^{-1/2}$$

$$= \left(\frac{\beta}{2\pi}\right)^{N/2} \alpha^{M/2} \exp\left\{-E(\mathbf{m}_N)\right\}\,|\mathbf{A}|^{-1/2}$$

The log-likelihood becomes:

$$\ln p(\mathbf{t}|\alpha, \beta) = \frac{N}{2} \ln \beta - \frac{N}{2} \ln(2\pi) + \frac{M}{2} \ln \alpha - E(\mathbf{m}_N) - \frac{1}{2} \ln |\mathbf{A}|$$

which is the same as 3.86.