1.12 When $n = m$, $I_{nm} = 1$, and $E[x_n x_m] = E[x_n^2] = \mu^2 + \sigma^2 = \mu^2 + I_{nm}\sigma^2$.

When $n \neq m$, $I_{nm} = 0$, and $E[x_n x_m] = E[x_n]E[x_m]$ since x_n and x_m are independent variables.

$E[x_n]E[x_m] = \mu^2 = \mu^2 + I_{nm}\sigma^2$, since $I_{nm} = 0$.

Now to prove 1.57:

We know that $\mu_{ML} = \frac{1}{N}\sum_{n=1}^{N} x_n$.

$$\implies E[\mu_{ML}] = E\left[\frac{1}{N}\sum_{n=1}^{N} x_n\right]$$

$$= \frac{1}{N}\sum_{n=1}^{N} E[x_n]$$

$$= \frac{1}{N}\sum_{n=1}^{N} \mu$$

$$= \frac{1}{N} N\mu$$

$$= \mu$$

Now to prove 1.58:

We know that $\sigma_{ML}^2 = \frac{1}{N}\sum_{n=1}^{N}(x_n - \mu)^2$

$$\implies E[\sigma_{ML}^2] = E\left[\frac{1}{N}\sum_{n=1}^{N}(x_n - \mu_{ML})^2\right]$$

$$= \frac{1}{N}\sum_{n=1}^{N} E[(x_n - \mu_{ML})^2]$$

$$= \frac{1}{N}\sum_{n=1}^{N} E\left[\left(x_n - \left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)\right)^2\right]$$

$$= \frac{1}{N} \sum_{n=1}^{N} E \left[x_n^2 + \left(\frac{1}{N} \sum_{i=1}^{N} x_i \right)^2 - 2x_n \left(\frac{1}{N} \sum_{i=1}^{N} x_i \right) \right]$$

$$= \frac{1}{N} \sum_{n=1}^{N} \left(E[x_n^2] + E \left[\left(\frac{1}{N} \sum_{i=1}^{N} x_i \right)^2 \right] - E \left[2x_n \left(\frac{1}{N} \sum_{i=1}^{N} x_i \right) \right] \right)$$

Evaluating these three terms separately:

First term:

$$E[x_n^2] = \mu^2 + \sigma^2$$

Second term:

$$E \left[\left(\frac{1}{N} \sum_{i=1}^{N} x_i \right)^2 \right] = E \left[\frac{1}{N^2} \left(\sum_{i=1}^{N} x_i \right) \left(\sum_{j=1}^{N} x_j \right) \right]$$

$$= E \left[\left(\frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \right) \right]$$

$$= \frac{1}{N} \left(\sum_{i=1}^{N^2} \sum_{j=1}^{N} E[x_i x_j] \right)$$

There are N^2 terms, out of which for N, $i = j$ and for the rest, $i \neq j$.

When $i = j$, $E[x_i x_j] = E[x_i^2] = \mu^2 + \sigma^2$.

When $i \neq j$, $E[x_i x_j] = E[x_i]E[x_j] = \mu^2$ (since x_i and x_j are independent).

Substituting these results, we get:

$$= \frac{1}{N^2} \left(N(\mu^2 + \sigma^2) + (N^2 - N)(\mu^2) \right)$$

$$= \frac{1}{N} (\mu^2 + \sigma^2 + N\mu^2 - \mu^2)$$

$$= \frac{1}{N}\sigma^2 + \mu^2$$

Third term:

$$E\left[2x_n\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)\right] = \frac{2}{N}\sum_{i=1}^{N} E[x_i x_n]$$

$$= \frac{2}{N}\left(\sum_{i=1, i\neq n}^{N} E[x_i x_n] + E[x_n^2]\right)$$

$$= \frac{2}{N}((N-1)\mu^2 + (\mu^2 + \sigma^2))$$

$$= \frac{2}{N}(N\mu^2 + \sigma^2)$$

Substituting back the values for the three terms back into the expression for $E[\sigma_{ML}^2]$, we get:

$$E[\sigma_{ML}^2] = \frac{1}{N}\sum_{n=1}^{N}\left((\mu^2 + \sigma^2) + \left(\frac{1}{N}\sigma^2 + \mu^2\right) - \left(\frac{2}{N}(N\mu^2 + \sigma^2)\right)\right)$$

$$= \frac{1}{N}\sum_{n=1}^{N}\frac{1}{N}\left(N\mu^2 + N\sigma^2 + \sigma^2 + N\mu^2 - 2N\mu^2 - 2\sigma^2\right)$$

$$= \frac{1}{N^2}\sum_{n=1}^{N}(N-1)\sigma^2$$

$$= \frac{1}{N^2}N(N-1)\sigma^2$$

$$= \frac{(N-1)}{N}\sigma^2$$