**2.49** From 2.161, we have:

$$St(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Lambda},\nu) = \int_0^\infty \mathcal{N}(\mathbf{x}|\boldsymbol{\mu},(\eta\boldsymbol{\Lambda})^{-1})\,Gam(\eta|\nu/2,\nu/2)\,\mathrm{d}\eta$$

$$= \int_0^\infty \frac{1}{(2\pi)^{D/2}|(\eta\boldsymbol{\Lambda})^{-1}|^{1/2}} exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T(\eta\boldsymbol{\Lambda})(\mathbf{x}-\boldsymbol{\mu})\right\}\frac{1}{\Gamma(\nu/2)}\left(\frac{\nu}{2}\right)^{\nu/2}\eta^{\nu/2-1}exp\left(-\frac{\nu\eta}{2}\right)\mathrm{d}\eta$$

And from 2.162, we have:

$$= \frac{\Gamma(\nu/2+D/2)}{\Gamma(\nu/2)}\frac{|\boldsymbol{\Lambda}|^{1/2}}{(\pi\nu)^{D/2}}\left(1+\frac{\Delta^2}{\nu}\right)^{-D/2-\nu/2}$$

Finding Expectation:

$$\mathbb{E}[\mathbf{x}] = \int_{\mathbf{x}} St(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Lambda},\nu)\,\mathbf{x}\,d\mathbf{x}$$

$$= \int_{\mathbf{x}}\int_0^\infty \mathcal{N}(\mathbf{x}|\boldsymbol{\mu},(\eta\boldsymbol{\Lambda})^{-1})\,Gam(\eta|\nu/2,\nu/2)\,\mathbf{x}\,\mathrm{d}\eta\,d\mathbf{x}$$

$$= \int_0^\infty \left(\int_{\mathbf{x}} \mathcal{N}(\mathbf{x}|\boldsymbol{\mu},(\eta\boldsymbol{\Lambda})^{-1})\,\mathbf{x}\,d\mathbf{x}\right) Gam(\eta|\nu/2,\nu/2)\,\mathrm{d}\eta$$

$$= \int_0^\infty (\boldsymbol{\mu})\;Gam(\eta|\nu/2,\nu/2)\,\mathrm{d}\eta$$

$$= \boldsymbol{\mu}\int_0^\infty Gam(\eta|\nu/2,\nu/2)\,\mathrm{d}\eta$$

$$= \boldsymbol{\mu}$$

Unexplained: why does 2.164 require $\nu > 1$? As per page 100, for a finite integral it is required that $\nu/2 > 0 \Longrightarrow \nu > 0$.

Finding covariance:

$$cov[\mathbf{x}] = \mathbb{E}[\mathbf{x}\mathbf{x}^T] - \mathbb{E}[\mathbf{x}]\mathbb{E}[\mathbf{x}]^T$$

$$\mathbb{E}[\mathbf{x}\mathbf{x}^T] = \int_{\mathbf{x}} St(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Lambda}, \nu)\, \mathbf{x}\mathbf{x}^T\, d\mathbf{x}$$

$$= \int_{\mathbf{x}} \int_0^\infty \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, (\eta\boldsymbol{\Lambda})^{-1})\, Gam(\eta|\nu/2, \nu/2)\, \mathbf{x}\mathbf{x}^T\, \mathrm{d}\eta\, d\mathbf{x}$$

$$= \int_0^\infty \left( \int_{\mathbf{x}} \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, (\eta\boldsymbol{\Lambda})^{-1})\, \mathbf{x}\mathbf{x}^T\, d\mathbf{x} \right) Gam(\eta|\nu/2, \nu/2)\, \mathrm{d}\eta$$

$$= \int_0^\infty (\eta\boldsymbol{\Lambda})^{-1} + \boldsymbol{\mu}\boldsymbol{\mu}^T)\, Gam(\eta|\nu/2, \nu/2)\, \mathrm{d}\eta$$

$$= \int_0^\infty (\eta\boldsymbol{\Lambda})^{-1}\, Gam(\eta|\nu/2, \nu/2)\, \mathrm{d}\eta + \int_0^\infty \boldsymbol{\mu}\boldsymbol{\mu}^T\, Gam(\eta|\nu/2, \nu/2)\, \mathrm{d}\eta$$

$$= \boldsymbol{\Lambda}^{-1} \int_0^\infty \eta^{-1}\, Gam(\eta|\nu/2, \nu/2)\, \mathrm{d}\eta + \boldsymbol{\mu}\boldsymbol{\mu}^T \int_0^\infty Gam(\eta|\nu/2, \nu/2)\, \mathrm{d}\eta$$

$$= \boldsymbol{\Lambda}^{-1} \int_0^\infty \eta^{-1}\, Gam(\eta|\nu/2, \nu/2)\, \mathrm{d}\eta + \boldsymbol{\mu}\boldsymbol{\mu}^T (1)$$

Focusing on the integral,

$$\int_0^\infty \eta^{-1}\, Gam(\eta|\nu/2, \nu/2)\, \mathrm{d}\eta$$

$$= \int_0^\infty \eta^{-1} \frac{1}{\Gamma(\nu/2)} (\nu/2)^{\nu/2} \eta^{(\nu/2-1)} exp(-(\nu/2)\eta)\, \mathrm{d}\eta$$

$$= \frac{1}{\Gamma(\nu/2)} (\nu/2)^{\nu/2} \int_0^\infty \eta^{-1}\, \eta^{(\nu/2-1)} exp(-(\nu/2)\eta)\, \mathrm{d}\eta$$

$$= \frac{1}{\Gamma(\nu/2)} (\nu/2)^{\nu/2} \int_0^\infty \eta^{(\nu/2-2)} exp(-(\nu/2)\eta)\, \mathrm{d}\eta$$

Let $z = (\nu/2)\eta$. $\dfrac{\mathrm{d}\eta}{\mathrm{d}z} = \dfrac{2}{\nu}$. Limits of z stay the same.

$$= \frac{1}{\Gamma(\nu/2)} (\nu/2)^{\nu/2} \int_0^\infty \left(\frac{2z}{\nu}\right)^{(\nu/2-2)} exp(-z)\, \mathrm{d}z \left(\frac{2}{\nu}\right)$$

$$= \frac{1}{\Gamma(\nu/2)} \left(\frac{\nu}{2}\right)^{\nu/2} \left(\frac{2}{\nu}\right)^{(\nu/2-2)} \left(\frac{2}{\nu}\right) \int_0^\infty z^{(\nu/2-2)} exp(-z)\, \mathrm{d}z$$

$$= \frac{\nu}{2\Gamma(\nu/2)} \int_0^\infty z^{(\nu/2-2)} exp(-z)\, \mathrm{d}z$$

Using 1.141, this becomes:

$$= \frac{\nu}{2\Gamma(\nu/2)} \Gamma(\nu/2-1)$$

$$= \frac{\nu}{2(\nu/2-1)\Gamma(\nu/2-1)} \Gamma(\nu/2-1)$$

$$= \frac{\nu}{2(\nu/2-1)}$$

$$= \frac{\nu}{\nu-2}$$

Substituting all the results into the expression for covariance, we get,

$$cov[\mathbf{x}] = \mathbb{E}[\mathbf{x}\mathbf{x}^T] - \mathbb{E}[\mathbf{x}]\mathbb{E}[\mathbf{x}]^T$$

$$= \mathbf{\Lambda}^{-1} \frac{\nu}{(\nu-2)} + \boldsymbol{\mu}\boldsymbol{\mu}^T - \boldsymbol{\mu}\boldsymbol{\mu}^T$$

$$= \mathbf{\Lambda}^{-1} \frac{\nu}{(\nu-2)}$$

Note: $\nu > 2$ as covariance matrix has to be positive definite.

Finding mode by taking derivative of Student's t-distribution w.r.t $\mathbf{x}$:

$$\frac{\partial}{\partial \mathbf{x}} St(\mathbf{x}|\boldsymbol{\mu}, \mathbf{\Lambda}, \nu) = \frac{\partial}{\partial \mathbf{x}} \int_0^\infty \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, (\eta\mathbf{\Lambda})^{-1})\, Gam(\eta|\nu/2, \nu/2)\, \mathrm{d}\eta$$

$$= \int_0^\infty \frac{\partial}{\partial \mathbf{x}} \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, (\eta\mathbf{\Lambda})^{-1})\, Gam(\eta|\nu/2, \nu/2)\, \mathrm{d}\eta$$

$$= \int_0^\infty Gam(\eta|\nu/2, \nu/2) \left( \frac{\partial \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, (\eta\mathbf{\Lambda})^{-1})}{\partial \mathbf{x}} \right) \mathrm{d}\eta$$

Setting it to $\mathbf{0}$, we get the same result as the mode of the Gaussian distribution, which is at $\mathbf{x} = \boldsymbol{\mu}$.