5.41 Using 5.162,

$$p(\mathbf{w}|\alpha) = \mathcal{N}(\mathbf{w}|\mathbf{0}, \alpha^{-1}\mathbf{I})$$

And using 5.181,

$$p(\mathcal{D}|\mathbf{w}) = \prod_{n=1}^{N} y(\mathbf{x}_n, \mathbf{w})^{t_n}(1 - y(\mathbf{x}_n, \mathbf{w}))^{(1-t_n)}$$

Assuming that the mode is at $\mathbf{w}_{MAP}$, and identifying that

$$f(\mathbf{w}) = \left(\prod_{n=1}^{N} y(\mathbf{x}_n, \mathbf{w})^{t_n}(1 - y(\mathbf{x}_n, \mathbf{w}))^{(1-t_n)}\right)\left(\mathcal{N}(\mathbf{w}|\mathbf{0}, \alpha^{-1}\mathbf{I})\right)$$

and

$$Z = p(\mathcal{D}|\alpha)$$

Applying 4.135, we get:

$$p(\mathcal{D}|\alpha) \simeq \left(\prod_{n=1}^{N} y(\mathbf{x}_n, \mathbf{w})^{t_n}(1 - y(\mathbf{x}_n, \mathbf{w}))^{(1-t_n)}\right)\left(\mathcal{N}(\mathbf{w}_{MAP}|\mathbf{0}, \alpha^{-1}\mathbf{I})\right)\frac{(2\pi)^{W/2}}{|\mathbf{A}|^{1/2}}$$

$$\implies \ln p(\mathcal{D}|\alpha) \simeq \ln\left(\left(\prod_{n=1}^{N} y_n^{t_n}(1 - y_n)^{(1-t_n)}\right)\left(\mathcal{N}(\mathbf{w}_{MAP}|\mathbf{0}, \alpha^{-1}\mathbf{I})\right)\frac{(2\pi)^{W/2}}{|\mathbf{A}|^{1/2}}\right)$$

$$= \left(\sum_{n=1}^{N}(t_n \ln y_n + (1 - t_n)\ln(1 - y_n))\right) + \left(\ln \mathcal{N}(\mathbf{w}_{MAP}|\mathbf{0}, \alpha^{-1}\mathbf{I})\right) + \frac{W}{2}\ln(2\pi) - \frac{1}{2}\ln|\mathbf{A}|$$

$$= \sum_{n=1}^{N}(t_n \ln y_n + (1 - t_n)\ln(1 - y_n))$$

$$+\ln\left(\frac{1}{(2\pi)^{W/2}|\alpha^{-1}\mathbf{I}|^{1/2}}\exp\left\{-\frac{\alpha\mathbf{w}_{MAP}^T\mathbf{w}_{MAP}}{2}\right\}\right)+\frac{W}{2}\ln(2\pi)-\frac{1}{2}\ln|\mathbf{A}|$$

$$=\sum_{n=1}^{N}(t_n\ln y_n+(1-t_n)\ln(1-y_n))$$
$$-\frac{W}{2}\ln(2\pi)+\frac{W}{2}\ln\alpha-\frac{\alpha\mathbf{w}_{MAP}^T\mathbf{w}_{MAP}}{2}+\frac{W}{2}\ln(2\pi)-\frac{1}{2}\ln|\mathbf{A}|$$

$$=-\left(-\sum_{n=1}^{N}(t_n\ln y_n+(1-t_n)\ln(1-y_n))+\frac{\alpha}{2}\mathbf{w}_{MAP}^T\mathbf{w}_{MAP}\right)$$
$$+\frac{W}{2}\ln\alpha-\frac{1}{2}\ln|\mathbf{A}|$$

$$=-E(\mathbf{w}_{MAP})+\frac{W}{2}\ln\alpha-\frac{1}{2}\ln|\mathbf{A}|$$

which is the same as the result in 5.183.

NOTE: In PRML, the final "const" term in Equation (5.183) should be ommitted.