5.32 From 5.139, we have,

$$\tilde{E}(\mathbf{w}) = E(\mathbf{w}) + \lambda\boldsymbol{\Omega}(\mathbf{w})$$

$$\begin{aligned}\implies \frac{\partial \tilde{E}}{\partial \eta_j} &= \frac{\partial E}{\partial \eta_j} + \frac{\partial \lambda\boldsymbol{\Omega}(\mathbf{w})}{\partial \eta_j} \\ &= 0 + \lambda\frac{\partial \boldsymbol{\Omega}(\mathbf{w})}{\partial \eta_j} \\ &= \lambda\frac{\partial \boldsymbol{\Omega}(\mathbf{w})}{\partial \eta_j}\end{aligned}$$

Solving the partial derivative:

$$\begin{aligned}\frac{\partial \boldsymbol{\Omega}(\mathbf{w})}{\partial \eta_j} &= \frac{\partial}{\partial \eta_j}\left(-\sum_i \ln\left(\sum_{k=1}^{M} \pi_k\,\mathcal{N}(w_i|\mu_k, \sigma_k^2)\right)\right) \\ &= -\sum_i \frac{\partial}{\partial \eta_j} \ln\left(\sum_{k=1}^{M} \pi_k\,\mathcal{N}(w_i|\mu_k, \sigma_k^2)\right) \\ &= -\sum_i \frac{1}{\left(\sum_{j=1}^{M} \pi_j\,\mathcal{N}(w_i|\mu_j, \sigma_j^2)\right)} \frac{\partial}{\partial \eta_j}\left(\sum_{k=1}^{M} \pi_k\,\mathcal{N}(w_i|\mu_k, \sigma_k^2)\right)\end{aligned}$$

Note: Here, η_j affects all the $\pi_k s$. It does not affect the σ_j as can be seen in the errata of the book, that for equation 5.144, η should be replaced with ξ.

$$\begin{aligned}&\frac{\partial}{\partial \eta_j}\left(\sum_{k=1}^{M} \pi_k\,\mathcal{N}(w_i|\mu_k, \sigma_k^2)\right) \\ &= \sum_{k=1}^{M}\left(\mathcal{N}(w_i|\mu_k, \sigma_k^2)\,\frac{\partial \pi_k}{\partial \eta_j}\right) \\ &= \sum_{k=1}^{M}\mathcal{N}(w_i|\mu_k, \sigma_k^2)\,(\pi_k(I_{jk} - \pi_j)) \\ &= \sum_{k=1}^{M}\mathcal{N}(w_i|\mu_k, \sigma_k^2)\,\pi_k I_{jk} - \sum_{k=1}^{M}\mathcal{N}(w_i|\mu_k, \sigma_k^2)\,\pi_k\pi_j \\ &= \pi_j\,\mathcal{N}(w_i|\mu_j, \sigma_j^2) - \pi_j\sum_{k=1}^{M}\mathcal{N}(w_i|\mu_k, \sigma_k^2)\,\pi_k\end{aligned}$$

$$= \pi_j \, \mathcal{N}(w_i|\mu_j, \sigma_j^2) - \pi_j \sum_{k=1}^{M} \mathcal{N}(w_i|\mu_k, \sigma_k^2) \, \pi_k$$

Substituting this result back into the expression for the partial derivative of $\mathbf{\Omega}$,

$$\frac{\partial \mathbf{\Omega}(\mathbf{w})}{\partial \eta_j} = - \sum_i \frac{1}{\left(\sum_{j=1}^{M} \pi_j \, \mathcal{N}(w_i|\mu_j, \sigma_j^2)\right)}$$

$$\left(\pi_j \, \mathcal{N}(w_i|\mu_j, \sigma_j^2) - \pi_j \sum_{k=1}^{M} \mathcal{N}(w_i|\mu_k, \sigma_k^2) \, \pi_k\right)$$

$$= - \sum_i \left(\gamma_j(w_i) - \pi_j\right)$$

$$= \sum_i (\pi_j - \gamma_j(w_i))$$

With one final substitution into the expression for the partial derivative of the error function, we get:

$$\implies \frac{\partial \tilde{E}}{\partial \eta_j} = \lambda \sum_i (\pi_j - \gamma_j(w_i))$$