4.26 4.152 gives us:

$$\int \mathbf{\Phi}(\lambda a)\,\mathcal{N}(a|\mu,\sigma^2)\,da = \mathbf{\Phi}\left(\frac{\mu}{(\lambda^{-2}+\sigma^2)^{1/2}}\right)$$

$$\int \mathbf{\Phi}(\lambda a)\,\mathcal{N}(a|\mu,\sigma^2)\,da = \int_{-\infty}^{\left(\frac{\mu}{(\lambda^{-2}+\sigma^2)^{1/2}}\right)} \mathcal{N}(\theta|0,1)\,d\theta$$

$$\int \mathbf{\Phi}(\lambda a)\,\frac{1}{(2\pi\sigma^2)^{1/2}}\exp\left\{-\frac{1}{2}\left(\frac{a-\mu}{\sigma}\right)^2\right\}\,da = \int_{-\infty}^{\left(\frac{\mu}{(\lambda^{-2}+\sigma^2)^{1/2}}\right)} \mathcal{N}(\theta|0,1)\,d\theta$$

Introducing a change of variable given by $\frac{a-\mu}{\sigma} = z$ on the left hand side :

$$\int \mathbf{\Phi}(\lambda(z\sigma+\mu))\,\frac{1}{(2\pi\sigma^2)^{1/2}}\exp\left\{-\frac{z^2}{2}\right\}\,dz\,\left(\frac{da}{dz}\right)$$

$$= \int \mathbf{\Phi}(\lambda(z\sigma+\mu))\,\frac{1}{(2\pi\sigma^2)^{1/2}}\exp\left\{-\frac{z^2}{2}\right\}\,dz\,(\sigma)$$

$$= \int \mathbf{\Phi}(\lambda(z\sigma+\mu))\,\frac{1}{(2\pi)^{1/2}}\exp\left\{-\frac{z^2}{2}\right\}\,dz$$

Taking derivative w.r.t μ,

$$\frac{\partial}{\partial\mu}\int \mathbf{\Phi}(\lambda(z\sigma+\mu))\,\frac{1}{(2\pi)^{1/2}}\exp\left\{-\frac{z^2}{2}\right\}\,dz = \frac{\partial}{\partial\mu}\int_{-\infty}^{\left(\frac{\mu}{(\lambda^{-2}+\sigma^2)^{1/2}}\right)} \mathcal{N}(\theta|0,1)\,d\theta$$

NOTE : μ is a constant w.r.t z.

$$\int \frac{1}{(2\pi)^{1/2}}\exp\left\{-\frac{z^2}{2}\right\}\frac{\partial}{\partial\mu}\left(\mathbf{\Phi}(\lambda(z\sigma+\mu))\right)\,dz$$

$$= \left(\frac{\partial\left(\frac{\mu}{(\lambda^{-2}+\sigma^2)^{1/2}}\right)}{\partial\mu}\right)\left(\frac{\partial}{\partial\left(\frac{\mu}{(\lambda^{-2}+\sigma^2)^{1/2}}\right)}\int_{-\infty}^{\left(\frac{\mu}{(\lambda^{-2}+\sigma^2)^{1/2}}\right)} \mathcal{N}(\theta|0,1)\,d\theta\right)$$

$$\int \frac{1}{(2\pi)^{1/2}} \exp\left\{-\frac{z^2}{2}\right\} \frac{\partial}{\partial \mu} \left(\int_{-\infty}^{\lambda(z\sigma+\mu)} \mathcal{N}(\theta|0,1)\, d\theta \right) dz$$

$$= \left(\frac{1}{(\lambda^{-2}+\sigma^2)^{1/2}} \right) \mathcal{N}\left(\left(\frac{\mu}{(\lambda^{-2}+\sigma^2)^{1/2}} \right) |0,1 \right)$$

Simplifying the L.H.S:

$$\int \frac{1}{(2\pi)^{1/2}} \exp\left\{-\frac{z^2}{2}\right\} \left(\frac{\partial(\lambda(z\sigma+\mu))}{\partial \mu} \right) \frac{\partial}{\partial(\lambda(z\sigma+\mu))} \left(\int_{-\infty}^{\lambda(z\sigma+\mu)} \mathcal{N}(\theta|0,1)\, d\theta \right) dz$$

$$= \int \frac{1}{(2\pi)^{1/2}} \exp\left\{-\frac{z^2}{2}\right\} (\lambda) \left(\mathcal{N}(\lambda(z\sigma+\mu)|0,1) \right) dz$$

$$= \int \lambda \frac{1}{(2\pi)^{1/2}} \exp\left\{-\frac{z^2}{2}\right\} \frac{1}{(2\pi)^{1/2}} \exp\left\{-\frac{1}{2}\lambda^2(z\sigma+\mu)^2\right\} dz$$

$$= \int \frac{\lambda}{(2\pi)^{1/2}(2\pi)^{1/2}} \exp\left\{-\frac{z^2}{2} - \frac{1}{2}\lambda^2(z\sigma+\mu)^2\right\} dz$$

$$= \int \frac{\lambda}{(2\pi)^{1/2}(2\pi)^{1/2}} \exp\left\{-\frac{z^2}{2} - \frac{\lambda^2 z^2\sigma^2 + \lambda^2\mu^2 + 2\lambda^2 z\sigma\mu}{2}\right\} dz$$

$$= \int \frac{\lambda}{(2\pi)^{1/2}(2\pi)^{1/2}} \exp\left\{-\frac{1}{2}(z^2 + \lambda^2 z^2\sigma^2 + \lambda^2\mu^2 + 2\lambda^2 z\sigma\mu)\right\} dz$$

$$= \int \frac{\lambda}{(2\pi)^{1/2}(2\pi)^{1/2}} \exp\left\{-\frac{1}{2}(z^2(1+\lambda^2\sigma^2) + \lambda^2\mu^2 + 2\lambda^2 z\sigma\mu)\right\} dz$$

$$= \int \frac{\lambda}{(2\pi)^{1/2}(2\pi)^{1/2}} \exp\left\{-\frac{(1+\lambda^2\sigma^2)}{2}\left(z^2 + 2z\frac{\lambda^2\sigma\mu}{(1+\lambda^2\sigma^2)} + \frac{\lambda^2\mu^2}{1+\lambda^2\sigma^2}\right)\right\} dz$$

$$= \int \frac{\lambda}{(2\pi)^{1/2}(2\pi)^{1/2}} \exp\left\{-\frac{(1+\lambda^2\sigma^2)}{2}\left(z^2 + 2z\frac{\lambda^2\sigma\mu}{(1+\lambda^2\sigma^2)} + \left(\frac{\lambda^2\sigma\mu}{(1+\lambda^2\sigma^2)}\right)^2\right)\right\}$$
$$\exp\left\{-\frac{(1+\lambda^2\sigma^2)}{2}\left(-\left(\frac{\lambda^2\sigma\mu}{(1+\lambda^2\sigma^2)}\right)^2 + \frac{\lambda^2\mu^2}{1+\lambda^2\sigma^2}\right)\right\} dz$$

$$= \int \frac{\lambda(1+\lambda^2\sigma^2)^{1/2}}{(2\pi)^{1/2}(2\pi(1+\lambda^2\sigma^2))^{1/2}} \exp\left\{-\frac{(1+\lambda^2\sigma^2)}{2}\left(z + \frac{\lambda^2\sigma\mu}{(1+\lambda^2\sigma^2)}\right)^2\right\}$$
$$\exp\left\{-\frac{(1+\lambda^2\sigma^2)}{2}\left(-\left(\frac{\lambda^2\sigma\mu}{(1+\lambda^2\sigma^2)}\right)^2 + \frac{\lambda^2\mu^2}{1+\lambda^2\sigma^2}\right)\right\} dz$$

$$= \frac{\lambda}{(2\pi)^{1/2}(1+\lambda^2\sigma^2)^{1/2}} \exp\left\{-\frac{(1+\lambda^2\sigma^2)}{2}\left(-\left(\frac{\lambda^2\sigma\mu}{(1+\lambda^2\sigma^2)}\right)^2 + \frac{\lambda^2\mu^2}{1+\lambda^2\sigma^2}\right)\right\}$$
$$\int \frac{1}{(2\pi(1+\lambda^2\sigma^2)^{-1})^{1/2}} \exp\left\{-\frac{(1+\lambda^2\sigma^2)}{2}\left(z + \frac{\lambda^2\sigma\mu}{(1+\lambda^2\sigma^2)}\right)^2\right\} dz$$

$$= \frac{\lambda}{(2\pi)^{1/2}(1+\lambda^2\sigma^2)^{1/2}} \exp\left\{-\frac{(1+\lambda^2\sigma^2)}{2}\left(-\left(\frac{\lambda^2\sigma\mu}{(1+\lambda^2\sigma^2)}\right)^2 + \frac{\lambda^2\mu^2}{1+\lambda^2\sigma^2}\right)\right\}$$
$$\int \mathcal{N}\left(z|\left(-\frac{\lambda^2\sigma\mu}{(1+\lambda^2\sigma^2)}\right), (1+\lambda^2\sigma^2)^{-1}\right) dz$$

$$= \frac{\lambda}{(2\pi)^{1/2}(1+\lambda^2\sigma^2)^{1/2}} \exp\left\{-\frac{(1+\lambda^2\sigma^2)}{2}\left(-\left(\frac{\lambda^2\sigma\mu}{(1+\lambda^2\sigma^2)}\right)^2 + \frac{\lambda^2\mu^2}{1+\lambda^2\sigma^2}\right)\right\}$$

$$= \frac{\lambda}{(2\pi)^{1/2}(1+\lambda^2\sigma^2)^{1/2}} \exp\left\{\frac{\lambda^4\sigma^2\mu^2}{2(1+\lambda^2\sigma^2)} - \frac{\lambda^2\mu^2}{2}\right\}$$

$$= \frac{\lambda}{(2\pi)^{1/2}(1+\lambda^2\sigma^2)^{1/2}} \exp\left\{\frac{\lambda^4\sigma^2\mu^2 - \lambda^2\mu^2 - \lambda^4\mu^2\sigma^2}{2(1+\lambda^2\sigma^2)}\right\}$$

$$= \frac{\lambda}{(2\pi)^{1/2}(1+\lambda^2\sigma^2)^{1/2}} \exp\left\{\frac{-\lambda^2\mu^2}{2(1+\lambda^2\sigma^2)}\right\}$$

$$= \frac{1}{(2\pi)^{1/2}(\lambda^{-2}+\sigma^2)^{1/2}} \exp\left\{\frac{-\mu^2}{2(\lambda^{-2}+\sigma^2)}\right\}$$

$$= \frac{1}{(\lambda^{-2}+\sigma^2)^{1/2}} \left(\frac{1}{(2\pi)^{1/2}} \exp\left\{-\frac{1}{2}\left(\frac{\mu}{(\lambda^{-2}+\sigma^2)^{1/2}}\right)^2\right\}\right)$$

$$= \left(\frac{1}{(\lambda^{-2}+\sigma^2)^{1/2}}\right) \mathcal{N}\left(\left(\frac{\mu}{(\lambda^{-2}+\sigma^2)^{1/2}}\right)|0,1\right)$$

$$= R.H.S$$

Thus, the derivative of the left-hand side with respect to μ is equal to the derivative of the right-hand side.

The integral of this, w.r.t μ is:

$$\int \left(\frac{1}{(\lambda^{-2}+\sigma^2)^{1/2}}\right) \mathcal{N}\left(\left(\frac{\mu}{(\lambda^{-2}+\sigma^2)^{1/2}}\right)|0,1\right) d\mu$$

$$= \int \mathcal{N}\left(\left(\frac{\mu}{(\lambda^{-2}+\sigma^2)^{1/2}}\right)|0,1\right) d\left(\frac{\mu}{(\lambda^{-2}+\sigma^2)^{1/2}}\right)$$

$$= \mathbf{\Phi}\left(\frac{\mu}{(\lambda^{-2}+\sigma^2)^{1/2}}\right)$$

The derivatives of both sides being the same, if at some arbitrary point the functions have the same value, then they are equal. At $-\infty$, both the L.H.S and R.H.S go to 0, so there is no constant of integration.

Explanation and Formal Statement:

Uniqueness Theorem for First-Order ODEs:

If $f(x)$ and $g(x)$ are two functions such that $f'(x) = g'(x)$ for all x in some interval and $f(a) = g(a)$ for some point a in that interval, then $f(x) = g(x)$ for all x in that interval.

Proof Outline:

1. **Given**: $f'(x) = g'(x)$ for all x.
2. Define $h(x) = f(x) - g(x)$.
3. Then $h'(x) = f'(x) - g'(x) = 0$ for all x.
4. Since $h'(x) = 0$, $h(x)$ must be a constant function.
5. Given $f(a) = g(a)$, we have $h(a) = f(a) - g(a) = 0$.
6. Therefore, $h(x) = 0$ for all x, meaning $f(x) = g(x)$ for all x.

Figure 1: Reasoning using fundamental theorem of calculus