1.30

Given:

$$p(x) = \mathcal{N}(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

$$q(x) = \mathcal{N}(x|m, s^2) = \frac{1}{\sqrt{2\pi s^2}} exp\left(-\frac{1}{2}\left(\frac{x-m}{s}\right)^2\right)$$

Using 1.113,

$$KL(p||q) = -\int p(x) ln\left\{\frac{q(x)}{p(x)}\right\} dx$$

$$= -\int \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) ln\left\{\frac{\frac{1}{\sqrt{2\pi s^2}} exp\left(-\frac{1}{2}\left(\frac{x-m}{s}\right)^2\right)}{\frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)}\right\} dx$$

$$= -\int \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) ln\left\{\frac{\sigma\, exp\left(-\frac{1}{2}\left(\frac{x-m}{s}\right)^2\right)}{s\, exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)}\right\} dx$$

$$= -\int \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \left(ln\left(\frac{\sigma}{s}\right) - \frac{1}{2}\left(\frac{x-m}{s}\right)^2 + \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx$$

$$= -\int \mathcal{N}(x|\mu, \sigma^2)\left(ln\left(\frac{\sigma}{s}\right) - \frac{1}{2}\left(\frac{x-m}{s}\right)^2 + \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx$$

This gives us three integrals, which can be solved using 1.48, 1.49 and 1.50:

Part 1:

$$\int \mathcal{N}(x|\mu,\sigma^2) ln\left(\frac{\sigma}{s}\right) dx$$

$$= ln\left(\frac{\sigma}{s}\right) \int \mathcal{N}(x|\mu,\sigma^2)\, dx$$

$$= ln\left(\frac{\sigma}{s}\right)(1)$$

$$= ln\left(\frac{\sigma}{s}\right)$$

Part 2:

$$\int \mathcal{N}(x|\mu,\sigma^2)\left(-\frac{1}{2}\left(\frac{x-m}{s}\right)^2\right) dx$$

$$= -\frac{1}{2s^2}\int \mathcal{N}(x|\mu,\sigma^2)(x^2-2xm+m^2)\, dx$$

$$= -\frac{1}{2s^2}\left(\int \mathcal{N}(x|\mu,\sigma^2)x^2\, dx - 2m\int \mathcal{N}(x|\mu,\sigma^2)x\, dx + m^2\int \mathcal{N}(x|\mu,\sigma^2)\, dx\right)$$

$$= -\frac{1}{2s^2}\left(\mu^2+\sigma^2-2m\mu+m^2 1\right)$$

$$= -\frac{1}{2s^2}\left(\mu^2+\sigma^2-2m\mu+m^2\right)$$

Part 3:

$$= \int \mathcal{N}(x|\mu,\sigma^2)\left(\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx$$

$$= \frac{1}{2\sigma^2}\int \mathcal{N}(x|\mu,\sigma^2)(x^2+\mu^2-2\mu x)\, dx$$

$$= \frac{1}{2\sigma^2}\left(\int \mathcal{N}(x|\mu,\sigma^2)x^2\, dx + \mu^2\int \mathcal{N}(x|\mu,\sigma^2)\, dx - 2\mu\int \mathcal{N}(x|\mu,\sigma^2)x\, dx\right)$$

$$= \frac{1}{2\sigma^2}\left(\mu^2 + \sigma^2 + \mu^2(1) - 2\mu\mu\right)$$

$$= \frac{1}{2\sigma^2}\left(\mu^2 + \sigma^2 + \mu^2 - 2\mu^2\right)$$

$$= \frac{1}{2\sigma^2}\left(\sigma^2\right)$$

$$= \frac{1}{2}$$

Putting all the parts together, we get:

$$= -\left(ln\left(\frac{\sigma}{s}\right) - \frac{1}{2s^2}\left(\mu^2 + \sigma^2 - 2m\mu + m^2\right) + \frac{1}{2}\right)$$

$$= ln\left(\frac{s}{\sigma}\right) + \frac{1}{2s^2}\left(\mu^2 + \sigma^2 - 2m\mu + m^2\right) - \frac{1}{2}$$