3.22 Taking the derivative of 3.86:

$$\frac{\partial}{\partial\beta}\ln p(\mathbf{t}|\alpha,\beta) = \frac{\partial}{\partial\beta}\left(\frac{N}{2}\ln\beta - \frac{N}{2}\ln(2\pi) + \frac{M}{2}\ln\alpha - E(\mathbf{m}_N) - \frac{1}{2}\ln|\mathbf{A}|\right)$$

$$= \frac{N}{2}\frac{1}{\beta} - 0 + 0 - \frac{\partial E(\mathbf{m}_N)}{\partial\beta} - \frac{1}{2}\frac{\partial\ln|\mathbf{A}|}{\partial\alpha}$$

Applying 3.82, 3.87 amd 3.88, this becomes:

$$= \frac{N}{2}\frac{1}{\beta} - \frac{\partial}{\partial\beta}\left(\frac{\beta}{2}||\mathbf{t} - \mathbf{\Phi}\mathbf{m}_N||^2 + \frac{\alpha}{2}\mathbf{m}_N^T\mathbf{m}_N\right) - \frac{1}{2}\frac{\partial\left(\ln\prod_{i=1}^{M}(\alpha+\lambda_i)\right)}{\partial\beta}$$

$$= \frac{N}{2}\frac{1}{\beta} - \left(\frac{1}{2}||\mathbf{t} - \mathbf{\Phi}\mathbf{m}_N||^2 + 0\right) - \frac{1}{2}\frac{\partial\left(\sum_{i=1}^{M}\ln(\alpha+\lambda_i)\right)}{\partial\beta}$$

$$= \frac{N}{2}\frac{1}{\beta} - \frac{1}{2}||\mathbf{t} - \mathbf{\Phi}\mathbf{m}_N||^2 - \frac{1}{2}\sum_{i=1}^{M}\left(\frac{\partial\ln(\alpha+\lambda_i)}{\partial\beta}\right)$$

$$= \frac{N}{2}\frac{1}{\beta} - \frac{1}{2}||\mathbf{t} - \mathbf{\Phi}\mathbf{m}_N||^2 - \frac{1}{2}\sum_{i=1}^{M}\left(\frac{1}{(\alpha+\lambda_i)}\frac{\partial(\alpha+\lambda_i)}{\partial\beta}\right)$$

$$= \frac{N}{2}\frac{1}{\beta} - \frac{1}{2}||\mathbf{t} - \mathbf{\Phi}\mathbf{m}_N||^2 - \frac{1}{2}\sum_{i=1}^{M}\left(\frac{1}{(\alpha+\lambda_i)}\left(\frac{\partial\alpha}{\partial\beta} + \frac{\partial\lambda_i}{\partial\beta}\right)\right)$$

To calculate $\frac{\partial\lambda_i}{\partial\beta}$, we consider 3.87:

$$(\beta\mathbf{\Phi}^T\mathbf{\Phi})\mathbf{u}_i = \lambda_i\mathbf{u}_i$$

$$\Longrightarrow \frac{\partial((\beta\mathbf{\Phi}^T\mathbf{\Phi})\mathbf{u}_i)}{\partial\beta} = \frac{\partial(\lambda_i\mathbf{u}_i)}{\partial\beta}$$

$$\Longrightarrow (\mathbf{\Phi}^T\mathbf{\Phi})\mathbf{u}_i = \left(\frac{\partial\lambda_i}{\partial\beta}\right)\mathbf{u}_i$$

Multiplying and dividing the R.H.S by β:

$$\Longrightarrow \frac{\beta}{\beta}(\mathbf{\Phi}^T\mathbf{\Phi})\mathbf{u}_i = \left(\frac{\partial\lambda_i}{\partial\beta}\right)\mathbf{u}_i$$

$$\implies \frac{\lambda_i \mathbf{u}_i}{\beta} = \left(\frac{\partial \lambda_i}{\partial \beta}\right) \mathbf{u}_i$$

$$\implies \frac{\lambda_i}{\beta} = \frac{\partial \lambda_i}{\partial \beta}$$

Substituting this into the derivative, we get:

$$= \frac{N}{2}\frac{1}{\beta} - \frac{1}{2}||\mathbf{t} - \mathbf{\Phi}\mathbf{m}_N||^2 - \frac{1}{2}\sum_{i=1}^{M}\left(\frac{1}{(\alpha+\lambda_i)}\left(0 + \frac{\lambda_i}{\beta}\right)\right)$$

$$= \frac{N}{2}\frac{1}{\beta} - \frac{1}{2}||\mathbf{t} - \mathbf{\Phi}\mathbf{m}_N||^2 - \frac{1}{2\beta}\sum_{i=1}^{M}\left(\frac{\lambda_i}{(\alpha+\lambda_i)}\right)$$

Setting this derivative to 0, we get:

$$0 = \frac{N}{2}\frac{1}{\beta} - \frac{1}{2}||\mathbf{t} - \mathbf{\Phi}\mathbf{m}_N||^2 - \frac{1}{2\beta}\sum_{i=1}^{M}\left(\frac{\lambda_i}{(\alpha+\lambda_i)}\right)$$

$$\implies \frac{1}{2}||\mathbf{t} - \mathbf{\Phi}\mathbf{m}_N||^2 = \frac{N}{2}\frac{1}{\beta} - \frac{1}{2\beta}\sum_{i=1}^{M}\left(\frac{\lambda_i}{(\alpha+\lambda_i)}\right)$$

$$\implies ||\mathbf{t} - \mathbf{\Phi}\mathbf{m}_N||^2 = \frac{N}{\beta} - \frac{1}{\beta}\sum_{i=1}^{M}\left(\frac{\lambda_i}{(\alpha+\lambda_i)}\right)$$

$$\implies ||\mathbf{t} - \mathbf{\Phi}\mathbf{m}_N||^2 = \frac{N}{\beta} - \frac{1}{\beta}\gamma$$

$$\implies \beta||\mathbf{t} - \mathbf{\Phi}\mathbf{m}_N||^2 = (N - \gamma)$$

$$\implies \beta = \frac{(N-\gamma)}{||\mathbf{t} - \mathbf{\Phi}\mathbf{m}_N||^2}$$

$$\implies \frac{1}{\beta} = \frac{||\mathbf{t} - \mathbf{\Phi}\mathbf{m}_N||^2}{(N-\gamma)}$$

which is the result in 3.95 that we wanted to verify.