6.5 Verifying 6.13:

Let us assume that the Gram matrix $\mathbf{K}_1$'s elements are given by the valid kernel $k_1(\mathbf{x}, \mathbf{x'})$, making it positive semidefinite.

Then, if we multiply each element of the matrix by a constant $c > 0$, then the matrix just becomes $c\mathbf{K}_1$, which is also positive semidefinite. Therefore, $ck_1(\mathbf{x}, \mathbf{x'})$ is also a valid kernel.

Verifying 6.14:

We know the following properties for scaling the columns and rows of a matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} a_{11}d_1 & a_{12}d_2 & a_{13}d_3 \\ a_{21}d_1 & a_{22}d_2 & a_{23}d_3 \\ a_{31}d_1 & a_{32}d_2 & a_{33}d_3 \end{bmatrix}$$

and

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_1a_{12} & d_1a_{13} \\ d_2a_{21} & d_2a_{22} & d_2a_{23} \\ d_3a_{31} & d_3a_{32} & d_3a_{33} \end{bmatrix}$$

If each element of the Gram matrix $\mathbf{K}$ corresponding to the kernel $k(\mathbf{x}, \mathbf{x'})$ is given by $f(\mathbf{x})k_1(\mathbf{x}, \mathbf{x'})f(\mathbf{x'})$, then in the resulting Gram matrix, the columns and rows are simply being scaled as follows:

$$\mathbf{K} = \mathbf{F}^T\mathbf{K}_1\mathbf{F}$$

$$= \begin{bmatrix} f(\mathbf{x}_1) & 0 & \dots & 0 \\ 0 & f(\mathbf{x}_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f(\mathbf{x}_N) \end{bmatrix} \begin{bmatrix} k_1(\mathbf{x}_1, \mathbf{x}_1) & k_1(\mathbf{x}_1, \mathbf{x}_2) & \dots & k_1(\mathbf{x}_1, \mathbf{x}_N) \\ k_1(\mathbf{x}_2, \mathbf{x}_1) & k_1(\mathbf{x}_2, \mathbf{x}_2) & \dots & k_1(\mathbf{x}_2, \mathbf{x}_N) \\ \vdots & \vdots & \ddots & \vdots \\ k_1(\mathbf{x}_N, \mathbf{x}_1) & k_1(\mathbf{x}_N, \mathbf{x}_2) & \dots & k_1(\mathbf{x}_N, \mathbf{x}_N) \end{bmatrix} \begin{bmatrix} f(\mathbf{x}_1) & 0 & \dots & 0 \\ 0 & f(\mathbf{x}_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f(\mathbf{x}_N) \end{bmatrix}$$

For any vector $\mathbf{y}$ of size N,

$$\mathbf{y}^T\mathbf{K}\mathbf{y}$$

$$= \mathbf{y}^T \begin{bmatrix} f(\mathbf{x}_1) & 0 & \dots & 0 \\ 0 & f(\mathbf{x}_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f(\mathbf{x}_N) \end{bmatrix} \begin{bmatrix} k_1(\mathbf{x}_1,\mathbf{x}_1) & k_1(\mathbf{x}_1,\mathbf{x}_2) & \dots & k_1(\mathbf{x}_1,\mathbf{x}_N) \\ k_1(\mathbf{x}_2,\mathbf{x}_1) & k_1(\mathbf{x}_2,\mathbf{x}_2) & \dots & k_1(\mathbf{x}_2,\mathbf{x}_N) \\ \vdots & \vdots & \ddots & \vdots \\ k_1(\mathbf{x}_N,\mathbf{x}_1) & k_1(\mathbf{x}_N,\mathbf{x}_2) & \dots & k_1(\mathbf{x}_N,\mathbf{x}_N) \end{bmatrix} \begin{bmatrix} f(\mathbf{x}_1) & 0 & \dots & 0 \\ 0 & f(\mathbf{x}_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f(\mathbf{x}_N) \end{bmatrix} \mathbf{y}$$

$$= (\mathbf{F}\mathbf{y})^T\mathbf{K}_1(\mathbf{F}\mathbf{y})$$

Since $\mathbf{K}_1$ is positive semidefinite, the above product will be ≥ 0, implying that $\mathbf{y}^T\mathbf{K}\mathbf{y} \geq 0$ and therefore, $\mathbf{K}$ is also positive semidefinite, making $k(\mathbf{x}, \mathbf{x'})$ a valid kernel.