5.31 From 5.139, we have,

$$\tilde{E}(\mathbf{w}) = E(\mathbf{w}) + \lambda\mathbf{\Omega}(\mathbf{w})$$

$$\implies \frac{\partial \tilde{E}}{\partial \sigma_j} = \frac{\partial E}{\partial \sigma_j} + \frac{\partial \lambda \mathbf{\Omega}(\mathbf{w})}{\partial \sigma_j}$$

$$= 0 + \lambda \frac{\partial \mathbf{\Omega}(\mathbf{w})}{\partial \sigma_j}$$

$$\frac{\partial \mathbf{\Omega}(\mathbf{w})}{\partial \sigma_j} = \frac{\partial}{\partial \sigma_j}\left(-\sum_i \ln\left(\sum_{j=1}^{M} \pi_j \mathcal{N}(w_i|\mu_j, \sigma_j^2)\right)\right)$$

$$= -\sum_i \frac{\partial}{\partial \sigma_j} \ln\left(\sum_{j=1}^{M} \pi_j \mathcal{N}(w_i|\mu_j, \sigma_j^2)\right)$$

$$= -\sum_i \frac{1}{\left(\sum_{j=1}^{M} \pi_j \mathcal{N}(w_i|\mu_j, \sigma_j^2)\right)} \frac{\partial}{\partial \sigma_j}\left(\sum_{j=1}^{M} \pi_j \mathcal{N}(w_i|\mu_j, \sigma_j^2)\right)$$

Since this partial derivative is w.r.t a specific j, it becomes:

$$= -\sum_i \frac{1}{\left(\sum_{j=1}^{M} \pi_j \mathcal{N}(w_i|\mu_j, \sigma_j^2)\right)} \left(\pi_j \frac{\partial \mathcal{N}(w_i|\mu_j, \sigma_j^2)}{\partial \sigma_j}\right)$$

Simplifying the derivative inside the braces:

$$\frac{\partial \mathcal{N}(w_i|\mu_j, \sigma_j^2)}{\partial \sigma_j}$$

$$= \frac{\partial}{\partial \sigma_j}\left(\frac{1}{(2\pi\sigma_j^2)^{(1/2)}} \exp\left(-\frac{1}{2}\left(\frac{w_i - \mu_j}{\sigma_j}\right)^2\right)\right)$$

$$= -\frac{1}{(2\pi)^{(1/2)}\sigma_j^2} \exp\left(-\frac{1}{2}\left(\frac{w_i - \mu_j}{\sigma_j}\right)^2\right)$$

$$+\frac{1}{(2\pi\sigma_j^2)^{(1/2)}} \left(\frac{\partial}{\partial \sigma_j} \exp\left(-\frac{1}{2}\left(\frac{w_i - \mu_j}{\sigma_j}\right)^2\right)\right)$$

$$= -\frac{1}{(2\pi)^{(1/2)}\sigma_j^2}\ \exp\left(-\frac{1}{2}\left(\frac{w_i-\mu_j}{\sigma_j}\right)^2\right)$$

$$+\frac{1}{(2\pi\sigma_j^2)^{(1/2)}}\ \exp\left(-\frac{1}{2}\left(\frac{w_i-\mu_j}{\sigma_j}\right)^2\right)\frac{\partial}{\partial\sigma_j}\left(-\frac{1}{2}\left(\frac{w_i-\mu_j}{\sigma_j}\right)^2\right)$$

$$= -\frac{1}{(2\pi)^{(1/2)}\sigma_j^2}\ \exp\left(-\frac{1}{2}\left(\frac{w_i-\mu_j}{\sigma_j}\right)^2\right)$$

$$+\frac{1}{(2\pi\sigma_j^2)^{(1/2)}}\ \exp\left(-\frac{1}{2}\left(\frac{w_i-\mu_j}{\sigma_j}\right)^2\right)\left(\frac{(w_i-\mu_j)^2}{\sigma_j^3}\right)$$

$$= -\frac{1}{\sigma_j}\mathcal{N}(w_i|\mu_j.\sigma_j^2) + \mathcal{N}(w_i|\mu_j.\sigma_j^2)\left(\frac{(w_i-\mu_j)^2}{\sigma_j^3}\right)$$

$$= \mathcal{N}(w_i|\mu_j.\sigma_j^2)\left(\frac{(w_i-\mu_j)^2}{\sigma_j^3}-\frac{1}{\sigma_j}\right)$$

Substituting this result in the partial derivative above, we get:

$$\frac{\partial\tilde{E}}{\partial\sigma_j} = -\lambda\sum_i \frac{1}{\left(\sum_{j=1}^{M}\pi_j\,\mathcal{N}(w_i|\mu_j,\sigma_j^2)\right)}\left(\pi_j\,\mathcal{N}(w_i|\mu_j.\sigma_j^2)\left(\frac{(w_i-\mu_j)^2}{\sigma_j^3}-\frac{1}{\sigma_j}\right)\right)$$

$$= -\lambda\sum_i\left(\frac{\pi_j\,\mathcal{N}(w_i|\mu_j,\sigma_j^2)}{\left(\sum_{j=1}^{M}\pi_j\,\mathcal{N}(w_i|\mu_j,\sigma_j^2)\right)}\left(\frac{(w_i-\mu_j)^2}{\sigma_j^3}-\frac{1}{\sigma_j}\right)\right)$$

$$= \lambda\sum_i\gamma_j(w_i)\left(\frac{1}{\sigma_j}-\frac{(w_i-\mu_j)^2}{\sigma_j^3}\right)$$

which is the same as the result in 5.143.