

**3.20** Taking the derivative of 3.86:

$$\begin{aligned}\frac{\partial}{\partial \alpha} \ln p(\mathbf{t}|\alpha, \beta) &= \frac{\partial}{\partial \alpha} \left( \frac{N}{2} \ln \beta - \frac{N}{2} \ln(2\pi) + \frac{M}{2} \ln \alpha - E(\mathbf{m}_N) - \frac{1}{2} \ln |\mathbf{A}| \right) \\ &= 0 - 0 + \frac{M}{2} \frac{\partial \ln \alpha}{\partial \alpha} - \frac{\partial E(\mathbf{m}_N)}{\partial \alpha} - \frac{1}{2} \frac{\partial \ln |\mathbf{A}|}{\partial \alpha}\end{aligned}$$

Applying 3.82, 3.87 and 3.88, this becomes:

$$\begin{aligned}&= \frac{M}{2} \frac{1}{\alpha} - \frac{\partial}{\partial \alpha} \left( \frac{\beta}{2} \|\mathbf{t} - \Phi \mathbf{m}_N\|^2 + \frac{\alpha}{2} \mathbf{m}_N^T \mathbf{m}_N \right) - \frac{1}{2} \frac{\partial \left( \ln \prod_{i=1}^M (\alpha + \lambda_i) \right)}{\partial \alpha} \\ &= \frac{M}{2\alpha} - \left( 0 + \frac{1}{2} \mathbf{m}_N^T \mathbf{m}_N \right) - \frac{1}{2} \frac{\partial \left( \sum_{i=1}^M \ln(\alpha + \lambda_i) \right)}{\partial \alpha} \\ &= \frac{M}{2\alpha} - \frac{1}{2} \mathbf{m}_N^T \mathbf{m}_N - \frac{1}{2} \left( \sum_{i=1}^M \frac{1}{(\alpha + \lambda_i)} \right)\end{aligned}$$

Setting this derivative to 0, we get:

$$\begin{aligned}0 &= \frac{M}{2\alpha} - \frac{1}{2} \mathbf{m}_N^T \mathbf{m}_N - \frac{1}{2} \left( \sum_{i=1}^M \frac{1}{(\alpha + \lambda_i)} \right) \\ \implies 0 &= M - \alpha \mathbf{m}_N^T \mathbf{m}_N - \alpha \left( \sum_{i=1}^M \frac{1}{(\alpha + \lambda_i)} \right) \\ \implies \alpha \mathbf{m}_N^T \mathbf{m}_N &= M - \alpha \left( \sum_{i=1}^M \frac{1}{(\alpha + \lambda_i)} \right) \\ \implies \alpha \mathbf{m}_N^T \mathbf{m}_N &= M - \left( \sum_{i=1}^M \frac{\alpha}{(\alpha + \lambda_i)} \right) \\ \implies \alpha \mathbf{m}_N^T \mathbf{m}_N &= \left( \sum_{i=1}^M \frac{(\alpha + \lambda_i)}{(\alpha + \lambda_i)} \right) - \left( \sum_{i=1}^M \frac{\alpha}{(\alpha + \lambda_i)} \right) \\ \implies \alpha \mathbf{m}_N^T \mathbf{m}_N &= \left( \sum_{i=1}^M \frac{(\alpha + \lambda_i) - \alpha}{(\alpha + \lambda_i)} \right)\end{aligned}$$

$$\begin{aligned}
\Rightarrow \alpha \mathbf{m}_N^T \mathbf{m}_N &= \left( \sum_{i=1}^M \frac{\lambda_i}{(\alpha + \lambda_i)} \right) \\
\Rightarrow \alpha \mathbf{m}_N^T \mathbf{m}_N &= \gamma \\
\Rightarrow \alpha &= \frac{\gamma}{\mathbf{m}_N^T \mathbf{m}_N}
\end{aligned}$$

which is the result in 3.92 that we wanted to verify.