6.26 From 6.78, we have :

$$p(a_{N+1}|\mathbf{a}_N) = \mathcal{N}\left(a_{N+1}|\mathbf{k}^T\mathbf{C}_N^{-1}\mathbf{a}_N, c - \mathbf{k}^T\mathbf{C}_N^{-1}\mathbf{k}\right)$$

and from 6.86, we have :

$$p(\mathbf{a}_N|\mathbf{t}_N) = \mathcal{N}\left(\mathbf{a}_N|\mathbf{a}_N^{\star}, \mathbf{H}^{-1}\right)$$

Substituting these into 6.77, and applying 2.115, we get:

$$p(a_{N+1}|\mathbf{t}_N) = \int p(a_{N+1}|\mathbf{a}_N)\, p(\mathbf{a}_N|\mathbf{t}_N)\, d\mathbf{a}_N$$

$$= \int \mathcal{N}\left(a_{N+1}|\mathbf{k}^T\mathbf{C}_N^{-1}\mathbf{a}_N, c - \mathbf{k}^T\mathbf{C}_N^{-1}\mathbf{k}\right) \mathcal{N}\left(\mathbf{a}_N|\mathbf{a}_N^{\star}, \mathbf{H}^{-1}\right)\, d\mathbf{a}_N$$

Equating the parameters to the ones in 2.115,

$$\mathbf{x} = \mathbf{a}_N$$
$$\boldsymbol{\mu} = \mathbf{a}_N^{\star}$$
$$\boldsymbol{\Lambda} = \mathbf{H}$$
$$\mathbf{L}^{-1} = c - \mathbf{k}^T\mathbf{C}_N^{-1}\mathbf{k}$$
$$\mathbf{A} = \mathbf{k}^T\mathbf{C}_N^{-1}$$
$$\mathbf{b} = \mathbf{0}$$
$$\mathbf{y} = a_{N+1}$$

$$\implies p(a_{N+1}|\mathbf{t}_N) = \mathcal{N}\left(a_{N+1}\middle|\mathbf{k}^T\mathbf{C}_N^{-1}\mathbf{a}_N^{\star}, \left(c - \mathbf{k}^T\mathbf{C}_N^{-1}\mathbf{k}\right) + \left(\mathbf{k}^T\mathbf{C}_N^{-1}\right)\mathbf{H}^{-1}\left(\mathbf{k}^T\mathbf{C}_N^{-1}\right)^T\right)$$

$$\mathbb{E}[a_{N+1}|\mathbf{t}_N] = \mathbf{k}^T\mathbf{C}_N^{-1}\mathbf{a}_N^{\star}$$

$$= \mathbf{k}^T\mathbf{C}_N^{-1}\left(\mathbf{C}_N(\mathbf{t}_N - \boldsymbol{\sigma}_N)\right)$$

$$= \mathbf{k}^T(\mathbf{t}_N - \boldsymbol{\sigma}_N)$$

which is the same as the result in 6.87.

$$\begin{aligned}
\text{var}[a_{N+1}|\mathbf{t}_N] &= \left(c - \mathbf{k}^T\mathbf{C}_N^{-1}\mathbf{k}\right) + \left(\mathbf{k}^T\mathbf{C}_N^{-1}\right)\mathbf{H}^{-1}\left(\mathbf{k}^T\mathbf{C}_N^{-1}\right)^T \\
&= c - \mathbf{k}^T\mathbf{C}_N^{-1}\mathbf{k} + \mathbf{k}^T\mathbf{C}_N^{-1}\mathbf{H}^{-1}\mathbf{C}_N^{-T}\mathbf{k} \\
&= c - \mathbf{k}^T\mathbf{C}_N^{-1}\mathbf{k} + \mathbf{k}^T\mathbf{C}_N^{-1}\left(\mathbf{W}_N + \mathbf{C}_N^{-1}\right)^{-1}\mathbf{C}_N^{-T}\mathbf{k}
\end{aligned}$$

$\mathbf{C}_N$ is symmetric.

$$\begin{aligned}
&\implies c - \mathbf{k}^T\mathbf{C}_N^{-1}\mathbf{k} + \mathbf{k}^T\mathbf{C}_N^{-1}\left(\mathbf{W}_N + \mathbf{C}_N^{-1}\right)^{-1}\mathbf{C}_N^{-1}\mathbf{k} \\
&= c - \mathbf{k}^T\mathbf{C}_N^{-1}\mathbf{k} + \mathbf{k}^T\mathbf{C}_N^{-1}\left(\left(\mathbf{W}_N\mathbf{C}_N + \mathbf{I}\right)\mathbf{C}_N^{-1}\right)^{-1}\mathbf{C}_N^{-1}\mathbf{k} \\
&= c - \mathbf{k}^T\mathbf{C}_N^{-1}\mathbf{k} + \mathbf{k}^T\mathbf{C}_N^{-1}\left(\mathbf{W}_N\left(\mathbf{C}_N + \mathbf{W}_N^{-1}\right)\mathbf{C}_N^{-1}\right)^{-1}\mathbf{C}_N^{-1}\mathbf{k} \\
&= c - \mathbf{k}^T\mathbf{C}_N^{-1}\mathbf{k} + \mathbf{k}^T\mathbf{C}_N^{-1}\left(\mathbf{C}_N\left(\mathbf{C}_N + \mathbf{W}_N^{-1}\right)^{-1}\mathbf{W}_N^{-1}\right)\mathbf{C}_N^{-1}\mathbf{k} \\
&= c - \mathbf{k}^T\left(\mathbf{C}_N^{-1} - \mathbf{C}_N^{-1}\mathbf{C}_N\left(\mathbf{C}_N + \mathbf{W}_N^{-1}\right)^{-1}\mathbf{W}_N^{-1}\mathbf{C}_N^{-1}\right)\mathbf{k}
\end{aligned}$$

The Woodbury Matrix identity is:

$$(\mathbf{A} + \mathbf{UCV})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}(\mathbf{C}^{-1} + \mathbf{VA}^{-1}\mathbf{U})^{-1}\mathbf{VA}^{-1}$$

Comparing the R.H.S to:

$$\mathbf{C}_N^{-1} - \mathbf{C}_N^{-1}\mathbf{C}_N\left(\mathbf{C}_N + \mathbf{W}_N^{-1}\right)^{-1}\mathbf{W}_N^{-1}\mathbf{C}_N^{-1}$$

such that:

$$\begin{aligned}
\mathbf{A} &= \mathbf{C}_N \\
\mathbf{U} &= \mathbf{C}_N \\
\mathbf{C} &= \mathbf{C}_N^{-1} \\
\mathbf{V} &= \mathbf{W}_N^{-1}
\end{aligned}$$

we get:

$$= \left(\mathbf{C}_N + \mathbf{C}_N \mathbf{C}_N^{-1} \mathbf{W}_N^{-1}\right)^{-1}$$

$$= \left(\mathbf{C}_N + \mathbf{W}_N^{-1}\right)^{-1}$$

Substituting this back, we get:

$$\mathrm{var}[a_{N+1}|\mathbf{t}_N] = c - \mathbf{k}^T \left(\mathbf{C}_N + \mathbf{W}_N^{-1}\right)^{-1} \mathbf{k}$$

$$= c - \mathbf{k}^T \left(\mathbf{W}_N^{-1} + \mathbf{C}_N\right)^{-1} \mathbf{k}$$

which is the same as the result in 6.88.