2.6 Calculating mean:

$$\mathbb{E}[\mu] = \int_0^1 \mu\, Beta(\mu|a,b)\, d\mu$$

$$= \int_0^1 \mu\, \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \mu^{a-1}(1-\mu)^{b-1}\, d\mu$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 \mu^{a}(1-\mu)^{b-1}\, d\mu$$

Using 2.265,

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\, \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+1+b)}$$

$$\Gamma(x+1) = x\Gamma(x), \text{ which can applied to give:}$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\, \frac{a\Gamma(a)\Gamma(b)}{(a+b)\Gamma(a+b)}$$

$$= \frac{a}{a+b}$$

Calculating variance:

$$var[\mu] = E[\mu^2] - E[\mu]^2$$

$$E[\mu^2] = \int_0^1 \mu^2\, Beta(\mu|a,b)\, d\mu$$

$$= \int_0^1 \mu^2\, \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \mu^{a-1}(1-\mu)^{b-1}\, d\mu$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 \mu^{a+1}(1-\mu)^{b-1}\, d\mu$$

Using 2.265,

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\, \frac{\Gamma(a+2)\Gamma(b)}{\Gamma(a+2+b)}$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{(a+1)a\Gamma(a)\Gamma(b)}{(a+b+1)(a+b)\Gamma(a+b)}$$

$$= \frac{(a+1)a}{(a+b+1)(a+b)}$$

Substituting to get variance:

$$var[\mu] = \frac{(a+1)a}{(a+b+1)(a+b)} - \left(\frac{a}{(a+b)}\right)^2$$

$$= \frac{a}{a+b}\left(\frac{a+1}{a+b+1} - \frac{a}{a+b}\right)$$

$$= \frac{a}{a+b}\left(\frac{(a+1)(a+b) - a(a+b+1)}{(a+b+1)(a+b)}\right)$$

$$= \frac{a}{a+b}\left(\frac{(a^2+ab+a+b) - (a^2+ab+a)}{(a+b+1)(a+b)}\right)$$

$$= \frac{a}{a+b}\left(\frac{b}{(a+b+1)(a+b)}\right)$$

$$= \frac{ab}{(a+b)^2(a+b+1)}$$

Calculating mode by maximizing Beta distribution w.r.t μ:

$$\frac{\partial}{\partial\,\mu}\left(\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\mu^{a-1}(1-\mu)^{b-1}\right)$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\partial}{\partial\,\mu}\left(\mu^{a-1}(1-\mu)^{b-1}\right)$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\left((a-1)\mu^{a-2}(1-\mu)^{b-1} + \mu^{a-1}(-1)(b-1)(1-\mu)^{b-2}\right)$$

Setting it to 0, we get:

$$0 = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\left((a-1)\mu^{a-2}(1-\mu)^{b-1} + \mu^{a-1}(-1)(b-1)(1-\mu)^{b-2}\right)$$

$$\implies 0 = \big((a-1)\mu^{a-2}(1-\mu)^{b-1} + \mu^{a-1}(-1)(b-1)(1-\mu)^{b-2}\big)$$

$$\implies (a-1)\mu^{a-2}(1-\mu)^{b-1} = \mu^{a-1}(b-1)(1-\mu)^{b-2}$$

$$\implies (a-1)(1-\mu)^{b-1} = \mu(b-1)(1-\mu)^{b-2}$$

$$\implies (a-1)(1-\mu) = \mu(b-1)$$

$$\implies a - a\mu - 1 + \mu = b\mu - \mu$$

$$\implies -a\mu + \mu - b\mu + \mu = 1 - a$$

$$\implies (2-a-b)\mu = (1-a)$$

$$\implies Mode[\mu] = \frac{1-a}{2-a-b} = \frac{a-1}{a+b-2}$$