2.46 From 2.158, we have:

$$p(x|\mu,a,b) = \int_0^\infty \frac{b^a e^{(-b\tau)}\tau^{a-1}}{\Gamma(a)} \left(\frac{\tau}{2\pi}\right)^{1/2} exp\left\{-\frac{\tau}{2}(x-\mu)^2\right\} d\tau$$

$$= \frac{b^a}{\Gamma(a)}\left(\frac{1}{2\pi}\right)^{1/2} \int_0^\infty e^{(-b\tau)}\tau^{a-1}\tau^{1/2} exp\left\{-\frac{\tau}{2}(x-\mu)^2\right\} d\tau$$

$$= \frac{b^a}{\Gamma(a)}\left(\frac{1}{2\pi}\right)^{1/2} \int_0^\infty \tau^{(a-1+1/2)} exp\left\{-\frac{\tau}{2}(x-\mu)^2 - b\tau\right\} d\tau$$

$$= \frac{b^a}{\Gamma(a)}\left(\frac{1}{2\pi}\right)^{1/2} \int_0^\infty \tau^{(a-1/2)} exp\left\{-\left(\frac{1}{2}(x-\mu)^2 + b\right)\tau\right\} d\tau$$

Let $z = \left(\frac{1}{2}(x-\mu)^2 + b\right)\tau$.

$\implies \dfrac{dz}{d\tau} = \left(\frac{1}{2}(x-\mu)^2 + b\right)$.

The integral becomes:

$$= \frac{b^a}{\Gamma(a)}\left(\frac{1}{2\pi}\right)^{1/2} \int_0^\infty \left(\frac{1}{2}(x-\mu)^2 + b\right)^{-(a-1/2)} z^{a-1/2}\, exp\left\{-z\right\}\, dz \left(\frac{d\tau}{dz}\right)$$

$$= \frac{b^a}{\Gamma(a)}\left(\frac{1}{2\pi}\right)^{1/2} \int_0^\infty \left(\frac{1}{2}(x-\mu)^2 + b\right)^{-(a-1/2)} z^{a-1/2}\, exp\left\{-z\right\}\, dz \left(\frac{1}{2}(x-\mu)^2 + b\right)^{-1}$$

$$= \frac{b^a}{\Gamma(a)}\left(\frac{1}{2\pi}\right)^{1/2} \int_0^\infty \left(\frac{1}{2}(x-\mu)^2 + b\right)^{-(a+1/2)} z^{a-1/2}\, exp\left\{-z\right\}\, dz$$

$$= \frac{b^a}{\Gamma(a)}\left(\frac{1}{2\pi}\right)^{1/2} \left(\frac{1}{2}(x-\mu)^2 + b\right)^{-(a+1/2)} \int_0^\infty z^{a-1/2}\, exp\left\{-z\right\}\, dz$$

$$= \frac{b^a}{\Gamma(a)}\left(\frac{1}{2\pi}\right)^{1/2} \left(\frac{1}{2}(x-\mu)^2 + b\right)^{-(a+1/2)} \int_0^\infty z^{(a-1/2+1)-1}\, exp\left\{-z\right\}\, dz$$

Applying equation 1.141, we get:

$$= \frac{b^a}{\Gamma(a)} \left(\frac{1}{2\pi}\right)^{1/2} \left(\frac{1}{2}(x-\mu)^2 + b\right)^{-(a+1/2)} \Gamma(a+1/2)$$

Doing the substitutions $\nu = 2a$ and $\lambda = a/b$, we get:

$$= \frac{\nu^{\nu/2}}{(2\lambda)^{\nu/2}\Gamma(\nu/2)} \left(\frac{1}{2\pi}\right)^{1/2} \left(\frac{1}{2}(x-\mu)^2 + \frac{\nu}{2\lambda}\right)^{-(\nu/2+1/2)} \Gamma(\nu/2+1/2)$$

$$= \frac{\nu^{\nu/2}}{(2\lambda)^{\nu/2}\Gamma(\nu/2)} \left(\frac{1}{2\pi}\right)^{1/2} \left(\frac{\lambda}{\nu}(x-\mu)^2 + 1\right)^{-(\nu/2+1/2)} \frac{\nu}{2\lambda}^{-(\nu/2+1/2)} \Gamma(\nu/2+1/2)$$

$$= \frac{\Gamma(\nu/2+1/2)}{\Gamma(\nu/2)} \left(\frac{1}{2\pi}\right)^{1/2} \left(\frac{\nu}{2\lambda}\right)^{\nu/2} \left(\frac{\nu}{2\lambda}\right)^{-(\nu/2+1/2)} \left(\frac{\lambda}{\nu}(x-\mu)^2 + 1\right)^{-(\nu/2+1/2)}$$

$$= \frac{\Gamma(\nu/2+1/2)}{\Gamma(\nu/2)} \left(\frac{1}{2\pi}\right)^{1/2} \left(\frac{\nu}{2\lambda}\right)^{-1/2} \left(\frac{\lambda}{\nu}(x-\mu)^2 + 1\right)^{-(\nu/2+1/2)}$$

$$= \frac{\Gamma(\nu/2+1/2)}{\Gamma(\nu/2)} \left(\frac{2\lambda}{2\pi\nu}\right)^{1/2} \left(\frac{\lambda}{\nu}(x-\mu)^2 + 1\right)^{-(\nu/2+1/2)}$$

$$= \frac{\Gamma(\nu/2+1/2)}{\Gamma(\nu/2)} \left(\frac{\lambda}{\pi\nu}\right)^{1/2} \left(\frac{\lambda}{\nu}(x-\mu)^2 + 1\right)^{-(\nu/2+1/2)}$$

which is the same as the result in 2.159.