3.20

$$\frac{\partial}{\partial \alpha} \ln p(\mathbf{t}|\alpha,\beta) = \frac{\partial}{\partial \alpha}\left(\frac{N}{2}\ln\beta - \frac{N}{2}\ln(2\pi) + \frac{M}{2}\ln\alpha - E(\mathbf{m}_N) - \frac{1}{2}\ln|\mathbf{A}|\right)$$

$$= 0 - 0 + \frac{M}{2}\frac{\partial \ln\alpha}{\partial\alpha} - \frac{\partial E(\mathbf{m}_N)}{\partial \alpha} - \frac{1}{2}\frac{\partial \ln|\mathbf{A}|}{\partial\alpha}$$

Applying 3.82, 3.87 amd 3.88, this becomes:

$$= \frac{M}{2}\frac{1}{\alpha} - \frac{\partial}{\partial\alpha}\left(\frac{\beta}{2}||\mathbf{t} - \mathbf{\Phi}\mathbf{m}_N||^2 + \frac{\alpha}{2}\mathbf{m}_N^T\mathbf{m}_N\right) - \frac{1}{2}\frac{\partial\left(\ln\prod_{i=1}^{M}(\alpha+\lambda_i)\right)}{\partial\alpha}$$

$$= \frac{M}{2\alpha} - \left(0 + \frac{1}{2}\mathbf{m}_N^T\mathbf{m}_N\right) - \frac{1}{2}\frac{\partial\left(\sum_{i=1}^{M}\ln(\alpha+\lambda_i)\right)}{\partial\alpha}$$

$$= \frac{M}{2\alpha} - \frac{1}{2}\mathbf{m}_N^T\mathbf{m}_N - \frac{1}{2}\left(\sum_{i=1}^{M}\frac{1}{(\alpha+\lambda_i)}\right)$$

Setting this derivative to 0, we get:

$$0 = \frac{M}{2\alpha} - \frac{1}{2}\mathbf{m}_N^T\mathbf{m}_N - \frac{1}{2}\left(\sum_{i=1}^{M}\frac{1}{(\alpha+\lambda_i)}\right)$$

$$\Longrightarrow 0 = M - \alpha\mathbf{m}_N^T\mathbf{m}_N - \alpha\left(\sum_{i=1}^{M}\frac{1}{(\alpha+\lambda_i)}\right)$$

$$\Longrightarrow \alpha\mathbf{m}_N^T\mathbf{m}_N = M - \alpha\left(\sum_{i=1}^{M}\frac{1}{(\alpha+\lambda_i)}\right)$$

$$\Longrightarrow \alpha\mathbf{m}_N^T\mathbf{m}_N = M - \left(\sum_{i=1}^{M}\frac{\alpha}{(\alpha+\lambda_i)}\right)$$

$$\Longrightarrow \alpha\mathbf{m}_N^T\mathbf{m}_N = \left(\sum_{i=1}^{M}\frac{(\alpha+\lambda_i)}{(\alpha+\lambda_i)}\right) - \left(\sum_{i=1}^{M}\frac{\alpha}{(\alpha+\lambda_i)}\right)$$

$$\Longrightarrow \alpha\mathbf{m}_N^T\mathbf{m}_N = \left(\sum_{i=1}^{M}\frac{(\alpha+\lambda_i)-\alpha}{(\alpha+\lambda_i)}\right)$$

$$\Longrightarrow \alpha \mathbf{m}_N^T \mathbf{m}_N = \left(\sum_{i=1}^{M} \frac{\lambda_i}{(\alpha + \lambda_i)} \right)$$

$$\Longrightarrow \alpha \mathbf{m}_N^T \mathbf{m}_N = \gamma$$

$$\Longrightarrow \alpha = \frac{\gamma}{\mathbf{m}_N^T \mathbf{m}_N}$$

which is the result in 3.92 that we wanted to verify.