1.17 The gamma function is defined by:

$$\Gamma(x) \equiv \int_0^\infty u^{x-1} e^{-u} du.$$

$$\Longrightarrow \Gamma(x+1) \equiv \int_0^\infty u^x e^{-u} du = \int_0^\infty u^{x-1} u\, e^{-u} du$$

The formula for integration by parts is:

$$\int_a^b f(x)\, g'(x)\, dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\, dx$$

Applying this to the expression for $\Gamma(x+1)$, by setting $f(u) = u^{x-1}\, e^{-u}$ and $g'(u) = u$. This gives us $g(u) = \dfrac{u^2}{2}$, and we get:

$$\left[u^{x-1}\, e^{-u}\, \frac{u^2}{2}\right]_0^\infty - \int_0^\infty ((x-1)u^{x-2}e^{-u} + (u^{x-1}e^{-u}(-1)))\frac{u^2}{2}\, du$$

$$= 0 - \int_0^\infty ((x-1)u^{x-2}e^{-u} + (u^{x-1}e^{-u}(-1)))\frac{u^2}{2}\, du$$

$$= -\frac{1}{2}\int_0^\infty ((x-1)u^x e^{-u} - u^{x+1}e^{-u})\, du$$

$$= -\frac{1}{2}\int_0^\infty (x-1)u^x e^{-u} + \frac{1}{2}\int_0^\infty u^{x+1}e^{-u}\, du$$

$$\Longrightarrow \Gamma(x+1) = -\frac{1}{2}(x-1)\Gamma(x+1) + \frac{1}{2}\Gamma(x+2)$$

$$\Longrightarrow \left(1 + \frac{x}{2} - \frac{1}{2}\right)\Gamma(x+1) = \frac{1}{2}\Gamma(x+2)$$

$$\Longrightarrow \left(\frac{x}{2} + \frac{1}{2}\right)\Gamma(x+1) = \frac{1}{2}\Gamma(x+2)$$

$$\Longrightarrow (x+1)\Gamma(x+1) = \Gamma(x+2)$$

$$\Longrightarrow x\Gamma(x) = \Gamma(x+1)$$

$$\Gamma(1) = \int_0^\infty u^{1-1}e^{-u}du = \int_0^\infty e^{-u}du = [-e^{-u}]_0^\infty = -(0-1) = 1.$$

$$
\begin{aligned}
\Gamma(x+1) &= x\Gamma(x) \\
&= x.(x-1).\Gamma(x-1) \\
&= x.(x-1).\ldots.1.\Gamma(1) \\
&= x(x-1)\ldots 1 \\
&= x!
\end{aligned}
$$