5.36

$$E_n = -\ln\left\{\sum_{k=1}^{K}\pi_k(\mathbf{x}_n,\mathbf{w})\,\mathcal{N}\left(\mathbf{t}|\boldsymbol{\mu}_k(\mathbf{x}_n,\mathbf{w}),\left(\sigma_k(\mathbf{x}_n,\mathbf{w})^2\mathbf{I}\right)\right)\right\}$$

Given that $\sigma_k = \exp(a_k^\sigma)$,

$$\frac{\partial\sigma_k}{\partial a_k^\sigma} = \exp(a_k^\sigma) = \sigma_k$$

Applying this, we get:

$$\frac{\partial E_n}{\partial a_k^\sigma} = \left(\frac{\partial E_n}{\partial\sigma_k}\right)\left(\frac{\partial\sigma_k}{\partial a_k^\sigma}\right) = -\frac{1}{\left\{\sum_{k=1}^{K}\pi_k\,\mathcal{N}_{nk}\right\}}\left(\pi_k\frac{\partial\mathcal{N}_{nk}}{\partial\sigma_k}\right)(\sigma_k)$$

$$\frac{\partial\mathcal{N}_{nk}}{\partial\sigma_k} = \frac{\partial}{\partial\sigma_k}\left(\frac{1}{(2\pi)^{L/2}|\left(\sigma_k^2\mathbf{I}\right)|^{(1/2)}}\exp\left\{-\frac{1}{2}(\mathbf{t}-\boldsymbol{\mu}_k)^T\left(\sigma_k^2\mathbf{I}\right)^{-1}(\mathbf{t}-\boldsymbol{\mu}_k)\right\}\right)$$

$$= \frac{\partial}{\partial\sigma_k}\left(\frac{1}{(2\pi)^{L/2}\sigma_k^L}\exp\left\{-\frac{1}{2\sigma_k^2}(\mathbf{t}-\boldsymbol{\mu}_k)^T(\mathbf{t}-\boldsymbol{\mu}_k)\right\}\right)$$

$$= \left(\frac{\partial\sigma_k^{-L}}{\partial\sigma_k}\right)\left(\frac{1}{(2\pi)^{L/2}}\exp\left\{-\frac{1}{2\sigma_k^2}(\mathbf{t}-\boldsymbol{\mu}_k)^T(\mathbf{t}-\boldsymbol{\mu}_k)\right\}\right)$$
$$+\frac{1}{(2\pi)^{L/2}\sigma_k^L}\left(\frac{\partial}{\partial\sigma_k}\left(\exp\left\{-\frac{1}{2\sigma_k^2}(\mathbf{t}-\boldsymbol{\mu}_k)^T(\mathbf{t}-\boldsymbol{\mu}_k)\right\}\right)\right)$$

$$= \left(-L\sigma_k^{-L-1}\right)\left(\frac{1}{(2\pi)^{L/2}}\exp\left\{-\frac{1}{2\sigma_k^2}(\mathbf{t}-\boldsymbol{\mu}_k)^T(\mathbf{t}-\boldsymbol{\mu}_k)\right\}\right)$$
$$+\frac{1}{(2\pi)^{L/2}\sigma_k^L}\left(\exp\left\{-\frac{1}{2\sigma_k^2}(\mathbf{t}-\boldsymbol{\mu}_k)^T(\mathbf{t}-\boldsymbol{\mu}_k)\right\}\right)\left(\frac{\partial}{\partial\sigma_k}\left\{-\frac{1}{2\sigma_k^2}(\mathbf{t}-\boldsymbol{\mu}_k)^T(\mathbf{t}-\boldsymbol{\mu}_k)\right\}\right)$$

$$= \left(-L\sigma_k^{-1}\right)\left(\frac{1}{(2\pi)^{L/2}\sigma_k^L}\exp\left\{-\frac{1}{2\sigma_k^2}(\mathbf{t}-\boldsymbol{\mu}_k)^T(\mathbf{t}-\boldsymbol{\mu}_k)\right\}\right)$$

$$+\frac{1}{(2\pi)^{L/2}\sigma_k^L}\left(\exp\left\{-\frac{1}{2\sigma_k^2}(\mathbf{t}-\boldsymbol{\mu}_k)^T(\mathbf{t}-\boldsymbol{\mu}_k)\right\}\right)\left(\frac{\partial\sigma_k^{-2}}{\partial\sigma_k}\right)\left\{-\frac{1}{2}(\mathbf{t}-\boldsymbol{\mu}_k)^T(\mathbf{t}-\boldsymbol{\mu}_k)\right\}$$

$$= \left(-L\sigma_k^{-1}\right)\mathcal{N}_{nk} + \mathcal{N}_{nk}\left(-2\sigma_k^{-3}\right)\left\{-\frac{1}{2}(\mathbf{t}-\boldsymbol{\mu}_k)^T(\mathbf{t}-\boldsymbol{\mu}_k)\right\}$$

$$= \left(-L\sigma_k^{-1}\right)\mathcal{N}_{nk} + \mathcal{N}_{nk}\left(\sigma_k^{-3}\right)\left\{(\mathbf{t}-\boldsymbol{\mu}_k)^T(\mathbf{t}-\boldsymbol{\mu}_k)\right\}$$

$$= \left(-L\sigma_k^{-1}\right)\mathcal{N}_{nk} + \mathcal{N}_{nk}\left(\sigma_k^{-3}\right)||\mathbf{t}-\boldsymbol{\mu}_k||^2$$

$$= \mathcal{N}_{nk}\left(\frac{||\mathbf{t}-\boldsymbol{\mu}_k||^2}{\sigma_k^3} - \frac{L}{\sigma_k}\right)$$

Substituting this result back, we get:

$$\frac{\partial E_n}{\partial a_k^\sigma} = -\frac{1}{\left\{\sum_{k=1}^K \pi_k \mathcal{N}_{nk}\right\}}\left(\pi_k\mathcal{N}_{nk}\left(\frac{||\mathbf{t}-\boldsymbol{\mu}_k||^2}{\sigma_k^3} - \frac{L}{\sigma_k}\right)\right)(\sigma_k)$$

$$= \frac{\pi_k\mathcal{N}_{nk}}{\left\{\sum_{k=1}^K \pi_k \mathcal{N}_{nk}\right\}}\left(\frac{L}{\sigma_k} - \frac{||\mathbf{t}-\boldsymbol{\mu}_k||^2}{\sigma_k^3}\right)(\sigma_k)$$

$$= \gamma_{nk}\left(L - \frac{||\mathbf{t}-\boldsymbol{\mu}_k||^2}{\sigma_k^2}\right)$$

which is the result we want. Note: 5.157 is incorrect, check solution manual and errata for correct result.