3.11 Using 3.54, we can state that:

$$\mathbf{S}_N^{-1} \quad = \quad \alpha\mathbf{I} + \beta\mathbf{\Phi}_N^T\mathbf{\Phi}_N \quad = \quad \alpha\mathbf{I} + \beta\sum_{i=1}^{N}\boldsymbol{\phi}(\mathbf{x}_i)\boldsymbol{\phi}(\mathbf{x}_i)^T$$

and

$$\mathbf{S}_{N+1}^{-1} \quad = \quad \alpha\mathbf{I} + \beta\mathbf{\Phi}_{N+1}^T\mathbf{\Phi}_{N+1} \quad = \quad \alpha\mathbf{I} + \beta\sum_{i=1}^{N+1}\boldsymbol{\phi}(\mathbf{x}_i)\boldsymbol{\phi}(\mathbf{x}_i)^T$$

where $\mathbf{\Phi}_N^T\mathbf{\Phi}_N = \sum_{i=1}^{N}\boldsymbol{\phi}(\mathbf{x}_i)\boldsymbol{\phi}(\mathbf{x}_i)^T$ and $\mathbf{\Phi}_{N+1}^T\mathbf{\Phi}_{N+1} = \sum_{i=1}^{N+1}\boldsymbol{\phi}(\mathbf{x}_i)\boldsymbol{\phi}(\mathbf{x}_i)^T$.

$$\implies \mathbf{S}_{N+1}^{-1} = \mathbf{S}_N^{-1} + \beta\boldsymbol{\phi}(\mathbf{x}_{N+1})\boldsymbol{\phi}(\mathbf{x}_{N+1})^T$$

Applying 3.59,

$$\sigma_{N+1}^2(\mathbf{x}) = \frac{1}{\beta} + \boldsymbol{\phi}(\mathbf{x})^T\mathbf{S}_{N+1}\boldsymbol{\phi}(\mathbf{x})$$

$$= \frac{1}{\beta} + \boldsymbol{\phi}(\mathbf{x})^T(\mathbf{S}_N^{-1} + \beta\boldsymbol{\phi}(\mathbf{x}_{N+1})\boldsymbol{\phi}(\mathbf{x}_{N+1})^T)^{-1}\boldsymbol{\phi}(\mathbf{x})$$

$$= \frac{1}{\beta} + \boldsymbol{\phi}(\mathbf{x})^T(\mathbf{S}_N^{-1} + \left(\sqrt{\beta}\boldsymbol{\phi}(\mathbf{x}_{N+1})\right)\left(\sqrt{\beta}\boldsymbol{\phi}(\mathbf{x}_{N+1})\right)^T)^{-1}\boldsymbol{\phi}(\mathbf{x})$$

Applying the matrix identity from Appendix C, we get:

$$= \frac{1}{\beta} + \boldsymbol{\phi}(\mathbf{x})^T\left(\mathbf{S}_N - \frac{\left(\mathbf{S}_N\left(\sqrt{\beta}\boldsymbol{\phi}(\mathbf{x}_{N+1})\right)\right)\left(\left(\sqrt{\beta}\boldsymbol{\phi}(\mathbf{x}_{N+1})\right)^T\mathbf{S}_N\right)}{1 + \left(\sqrt{\beta}\boldsymbol{\phi}(\mathbf{x}_{N+1})\right)^T\mathbf{S}_N\left(\sqrt{\beta}\boldsymbol{\phi}(\mathbf{x}_{N+1})\right)}\right)\boldsymbol{\phi}(\mathbf{x})$$

$$= \frac{1}{\beta} + \boldsymbol{\phi}(\mathbf{x})^T\left(\mathbf{S}_N - \frac{\beta\mathbf{S}_N\boldsymbol{\phi}(\mathbf{x}_{N+1})\boldsymbol{\phi}(\mathbf{x}_{N+1})^T\mathbf{S}_N}{1 + \beta\boldsymbol{\phi}(\mathbf{x}_{N+1})^T\mathbf{S}_N\boldsymbol{\phi}(\mathbf{x}_{N+1})}\right)\boldsymbol{\phi}(\mathbf{x})$$

$$= \sigma_N^2(\mathbf{x}) - \left(\frac{\beta\boldsymbol{\phi}(\mathbf{x})^T\mathbf{S}_N\boldsymbol{\phi}(\mathbf{x}_{N+1})\boldsymbol{\phi}(\mathbf{x}_{N+1})^T\mathbf{S}_N\boldsymbol{\phi}(\mathbf{x})}{1 + \beta\boldsymbol{\phi}(\mathbf{x}_{N+1})^T\mathbf{S}_N\boldsymbol{\phi}(\mathbf{x}_{N+1})}\right)$$

$$= \sigma_N^2(\mathbf{x}) - \left(\frac{\beta\left(\boldsymbol{\phi}(\mathbf{x})^T\mathbf{S}_N\boldsymbol{\phi}(\mathbf{x}_{N+1})\right)\left(\boldsymbol{\phi}(\mathbf{x}_{N+1})\mathbf{S}_N^T\boldsymbol{\phi}(\mathbf{x})^T\right)^T}{1 + \beta\boldsymbol{\phi}(\mathbf{x}_{N+1})^T\mathbf{S}_N\boldsymbol{\phi}(\mathbf{x}_{N+1})}\right)$$

$$\boldsymbol{\phi}(\mathbf{x})^T \mathbf{S}_N \boldsymbol{\phi}(\mathbf{x}_{N+1}) \text{ and } \boldsymbol{\phi}(\mathbf{x}_{N+1}) \mathbf{S}_N^T \boldsymbol{\phi}(\mathbf{x})^T \text{ are scalars}$$

Thus we can simplify the above expression to:

$$\sigma_{N+1}^2(\mathbf{x}) = \sigma_N^2(\mathbf{x}) - \left(\frac{\beta \left(\boldsymbol{\phi}(\mathbf{x})^T \mathbf{S}_N \boldsymbol{\phi}(\mathbf{x}_{N+1}) \right) \left(\boldsymbol{\phi}(\mathbf{x}_T) \mathbf{S}_N^T \boldsymbol{\phi}(\mathbf{x}_{N+1}) \right)^T}{1 + \beta \boldsymbol{\phi}(\mathbf{x}_{N+1})^T \mathbf{S}_N \boldsymbol{\phi}(\mathbf{x}_{N+1})} \right)$$

Covariance matrices are symmetric, so:

$$\sigma_{N+1}^2(\mathbf{x}) = \sigma_N^2(\mathbf{x}) - \left(\frac{\beta \left(\boldsymbol{\phi}(\mathbf{x})^T \mathbf{S}_N \boldsymbol{\phi}(\mathbf{x}_{N+1}) \right) \left(\boldsymbol{\phi}(\mathbf{x}_T) \mathbf{S}_N \boldsymbol{\phi}(\mathbf{x}_{N+1}) \right)^T}{1 + \beta \boldsymbol{\phi}(\mathbf{x}_{N+1})^T \mathbf{S}_N \boldsymbol{\phi}(\mathbf{x}_{N+1})} \right)$$

$$\sigma_{N+1}^2(\mathbf{x}) = \sigma_N^2(\mathbf{x}) - \left(\frac{\beta \left(\boldsymbol{\phi}(\mathbf{x})^T \mathbf{S}_N \boldsymbol{\phi}(\mathbf{x}_{N+1}) \right)^2}{1 + \beta \boldsymbol{\phi}(\mathbf{x}_{N+1})^T \mathbf{S}_N \boldsymbol{\phi}(\mathbf{x}_{N+1})} \right)$$

Here, $\beta \geq 0$, so the numerator is ≥ 0.

$\mathbf{S}_N$ is a covariance matrix, so it is positive semi-definite, implying that

$$\beta \boldsymbol{\phi}(\mathbf{x}_{N+1})^T \mathbf{S}_N \boldsymbol{\phi}(\mathbf{x}_{N+1}) \geq 0.$$

Therefore, term to the right is always ≥ 0, and $\sigma_N^2(\mathbf{x}) \geq \sigma_{N+1}^2(\mathbf{x})$.