2.38

$$p(\mu|\mathbf{X}) \propto p(\mathbf{X}|\mu)\, p(\mu)$$

From 2.137 and 2.138, we get:

$$R.H.S = \frac{1}{(2\pi\sigma^2)^{N/2}} exp\left\{-\frac{1}{2\sigma^2}\sum_{n=1}^{N}(x_n-\mu)^2\right\}\ \frac{1}{2\pi\sigma_0^2} exp\left\{-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right\}$$

$$= \frac{1}{(2\pi\sigma^2)^{N/2}} exp\left\{-\frac{1}{2\sigma^2}\sum_{n=1}^{N}(x_n-\mu)^2\right\}\ \frac{1}{2\pi\sigma_0^2} exp\left\{-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right\}$$

$$= \frac{1}{(2\pi\sigma^2)^{N/2}}\ \frac{1}{2\pi\sigma_0^2}\ exp\left\{-\frac{1}{2\sigma^2}\sum_{n=1}^{N}(x_n-\mu)^2 - \frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right\}$$

$$= \frac{1}{(2\pi\sigma^2)^{N/2}}\ \frac{1}{2\pi\sigma_0^2}\ exp\left\{-\frac{1}{2}\sum_{n=1}^{N}\left(\frac{(x_n-\mu)^2}{\sigma^2} + \frac{(\mu-\mu_0)^2}{N\sigma_0^2}\right)\right\}$$

$$= \frac{1}{(2\pi\sigma^2)^{N/2}}\ \frac{1}{2\pi\sigma_0^2}\ exp\left\{-\frac{1}{2}\sum_{n=1}^{N}\left(\frac{(x_n^2+\mu^2-2x_n\mu)}{\sigma^2} + \frac{(\mu^2+\mu_0^2-2\mu\mu_0)}{N\sigma_0^2}\right)\right\}$$

$$= \frac{1}{(2\pi\sigma^2)^{N/2}}\ \frac{1}{2\pi\sigma_0^2}\ exp\left\{-\frac{1}{2}\sum_{n=1}^{N}\left(\frac{N\sigma_0^2x_n^2 + N\sigma_0^2\mu^2 - 2N\sigma_0^2x_n\mu + \sigma^2\mu^2 + \sigma^2\mu_0^2 - 2\sigma^2\mu\mu_0}{N\sigma_0^2\sigma^2}\right)\right\}$$

$$= \frac{1}{(2\pi\sigma^2)^{N/2}}\ \frac{1}{2\pi\sigma_0^2}\ exp\left\{-\frac{1}{2}\sum_{n=1}^{N}\frac{1}{N}\left(\frac{(\sigma^2+N\sigma_0^2)\mu^2 - 2\mu(N\sigma_0^2x_n+\sigma^2\mu_0) + N\sigma_0^2x_n^2 + \sigma^2\mu_0^2}{\sigma_0^2\sigma^2}\right)\right\}$$

$$= \frac{1}{(2\pi\sigma^2)^{N/2}}\ \frac{1}{2\pi\sigma_0^2}\ exp\left\{-\frac{1}{2}\sum_{n=1}^{N}\frac{1}{N}\left(\frac{\mu^2 - 2\mu\frac{(N\sigma_0^2x_n+\sigma^2\mu_0)}{(\sigma^2+N\sigma_0^2)} + \frac{(N\sigma_0^2x_n^2+\sigma^2\mu_0^2)}{(\sigma^2+N\sigma_0^2)}}{\frac{\sigma_0^2\sigma^2}{(\sigma^2+N\sigma_0^2)}}\right)\right\}$$

$$= \frac{1}{(2\pi\sigma^2)^{N/2}}\ \frac{1}{2\pi\sigma_0^2}\ exp\left\{-\frac{1}{2\left(\frac{\sigma_0^2\sigma^2}{(\sigma^2+N\sigma_0^2)}\right)}\sum_{n=1}^{N}\frac{1}{N}\left(\mu^2 - 2\mu\frac{(N\sigma_0^2x_n+\sigma^2\mu_0)}{(\sigma^2+N\sigma_0^2)} + \frac{(N\sigma_0^2x_n^2+\sigma^2\mu_0^2)}{(\sigma^2+N\sigma_0^2)}\right)\right\}$$

$$= \frac{1}{(2\pi\sigma^2)^{N/2}} \frac{1}{2\pi\sigma_0^2} exp \left\{ - \frac{1}{2\left(\frac{\sigma_0^2\sigma^2}{(\sigma^2+N\sigma_0^2)}\right)} \left(\mu^2 - 2\mu \left(\frac{(N\sigma_0^2\mu_{ML} + \sigma^2\mu_0)}{(\sigma^2 + N\sigma_0^2)} \right) + \sum_{n=1}^{N} \frac{1}{N} \frac{(N\sigma_0^2 x_n^2 + \sigma^2\mu_0^2)}{(\sigma^2 + N\sigma_0^2)} \right) \right\}$$

From the term inside the exponent, we can see that:

$$\mu_n = \left(\frac{(N\sigma_0^2\mu_{ML} + \sigma^2\mu_0)}{(\sigma^2 + N\sigma_0^2)} \right)$$
$$= \frac{\sigma^2}{N\sigma_0^2 + \sigma^2}\mu_0 + \frac{N\sigma_0^2}{N\sigma_0^2 + \sigma^2}\mu_{ML}$$

The denominator in the exponent gives us the variance such that:

$$\sigma_N^2 = \frac{\sigma_0^2\sigma^2}{(\sigma^2 + N\sigma_0^2)}$$
$$\implies \frac{1}{\sigma_N^2} = \frac{\sigma^2 + N\sigma_0^2}{\sigma_0^2\sigma^2} = \frac{1}{\sigma_0^2} + \frac{N}{\sigma^2}$$