

3.11 Using 3.54, we can state that:

$$\mathbf{S}_N^{-1} = \alpha \mathbf{I} + \beta \Phi^T \Phi = \alpha \mathbf{I} + \beta \sum_{i=1}^N \phi(\mathbf{x}_i) \phi(\mathbf{x}_i)^T$$

and

$$\mathbf{S}_{N+1}^{-1} = \alpha \mathbf{I} + \beta \Phi^T \Phi = \alpha \mathbf{I} + \beta \sum_{i=1}^{N+1} \phi(\mathbf{x}_i) \phi(\mathbf{x}_i)^T$$

$$\implies \mathbf{S}_{N+1}^{-1} = \mathbf{S}_N^{-1} + \beta \phi(\mathbf{x}_{N+1}) \phi(\mathbf{x}_{N+1})^T$$

Applying 3.59,

$$\begin{aligned} \sigma_{N+1}^2(\mathbf{x}) &= \frac{1}{\beta} + \phi(\mathbf{x})^T \mathbf{S}_{N+1} \phi(\mathbf{x}) \\ &= \frac{1}{\beta} + \phi(\mathbf{x})^T (\mathbf{S}_N^{-1} + \beta \phi(\mathbf{x}_{N+1}) \phi(\mathbf{x}_{N+1})^T)^{-1} \phi(\mathbf{x}) \\ &= \frac{1}{\beta} + \phi(\mathbf{x})^T (\mathbf{S}_N^{-1} + (\sqrt{\beta} \phi(\mathbf{x}_{N+1})) (\sqrt{\beta} \phi(\mathbf{x}_{N+1}))^T)^{-1} \phi(\mathbf{x}) \end{aligned}$$

Applying the matrix identity from Appendix C, we get:

$$\begin{aligned} &= \frac{1}{\beta} + \phi(\mathbf{x})^T \left(\mathbf{S}_N - \frac{(\mathbf{S}_N (\sqrt{\beta} \phi(\mathbf{x}_{N+1}))) ((\sqrt{\beta} \phi(\mathbf{x}_{N+1}))^T \mathbf{S}_N)}{1 + (\sqrt{\beta} \phi(\mathbf{x}_{N+1}))^T \mathbf{S}_N (\sqrt{\beta} \phi(\mathbf{x}_{N+1}))} \right) \phi(\mathbf{x}) \\ &= \frac{1}{\beta} + \phi(\mathbf{x})^T \left(\mathbf{S}_N - \frac{\beta \mathbf{S}_N \phi(\mathbf{x}_{N+1}) \phi(\mathbf{x}_{N+1})^T \mathbf{S}_N}{1 + \beta \phi(\mathbf{x}_{N+1})^T \mathbf{S}_N \phi(\mathbf{x}_{N+1})} \right) \phi(\mathbf{x}) \\ &= \sigma_N^2(\mathbf{x}) - \left(\frac{\beta \phi(\mathbf{x})^T \mathbf{S}_N \phi(\mathbf{x}_{N+1}) \phi(\mathbf{x}_{N+1})^T \mathbf{S}_N \phi(\mathbf{x})}{1 + \beta \phi(\mathbf{x}_{N+1})^T \mathbf{S}_N \phi(\mathbf{x}_{N+1})} \right) \\ &= \sigma_N^2(\mathbf{x}) - \left(\frac{\beta (\phi(\mathbf{x})^T \mathbf{S}_N \phi(\mathbf{x}_{N+1})) (\phi(\mathbf{x}_{N+1}) \mathbf{S}_N^T \phi(\mathbf{x})^T)^T}{1 + \beta \phi(\mathbf{x}_{N+1})^T \mathbf{S}_N \phi(\mathbf{x}_{N+1})} \right) \end{aligned}$$

$\phi(\mathbf{x})^T \mathbf{S}_N \phi(\mathbf{x}_{N+1})$ and $\phi(\mathbf{x}_{N+1}) \mathbf{S}_N^T \phi(\mathbf{x})^T$ are scalars

Thus we can simplify the above expression to:

$$\sigma_{N+1}^2(\mathbf{x}) = \sigma_N^2(\mathbf{x}) - \left(\frac{\beta (\phi(\mathbf{x})^T \mathbf{S}_N \phi(\mathbf{x}_{N+1})) (\phi(\mathbf{x}_T) \mathbf{S}_N^T \phi(\mathbf{x}_{N+1}))^T}{1 + \beta \phi(\mathbf{x}_{N+1})^T \mathbf{S}_N \phi(\mathbf{x}_{N+1})} \right)$$

Covariance matrices are symmetric, so:

$$\begin{aligned} \sigma_{N+1}^2(\mathbf{x}) &= \sigma_N^2(\mathbf{x}) - \left(\frac{\beta (\phi(\mathbf{x})^T \mathbf{S}_N \phi(\mathbf{x}_{N+1})) (\phi(\mathbf{x}_T) \mathbf{S}_N \phi(\mathbf{x}_{N+1}))^T}{1 + \beta \phi(\mathbf{x}_{N+1})^T \mathbf{S}_N \phi(\mathbf{x}_{N+1})} \right) \\ \sigma_{N+1}^2(\mathbf{x}) &= \sigma_N^2(\mathbf{x}) - \left(\frac{\beta (\phi(\mathbf{x})^T \mathbf{S}_N \phi(\mathbf{x}_{N+1}))^2}{1 + \beta \phi(\mathbf{x}_{N+1})^T \mathbf{S}_N \phi(\mathbf{x}_{N+1})} \right) \end{aligned}$$

Here, $\beta \geq 0$, so the numerator is ≥ 0 .

\mathbf{S}_N is a covariance matrix, so it is positive semi-definite, implying that

$$\beta \phi(\mathbf{x}_{N+1})^T \mathbf{S}_N \phi(\mathbf{x}_{N+1}) \geq 0.$$

Therefore, term to the right is always ≥ 0 , and $\sigma_N^2(\mathbf{x}) \geq \sigma_{N+1}^2(\mathbf{x})$.