2.43

$$\int_{-\infty}^{\infty} p(x|\sigma^2, q)\, dx = \int_{-\infty}^{\infty} \frac{q}{2(2\sigma^2)^{1/q}\Gamma(1/q)} exp\left(-\frac{|x|^q}{2\sigma^2}\right)\, dx$$
$$= \frac{q}{2(2\sigma^2)^{1/q}\Gamma(1/q)} \int_{-\infty}^{\infty} exp\left(-\frac{|x|^q}{2\sigma^2}\right)\, dx$$

Focusing on the integral only, the function inside is symmetric. So, we get:

$$\int_{-\infty}^{\infty} exp\left(-\frac{|x|^q}{2\sigma^2}\right)\, dx = 2\int_{0}^{\infty} exp\left(-\frac{x^q}{2\sigma^2}\right)\, dx$$

Let $u = \dfrac{x^q}{2\sigma^2}$. The limits of u become 0 to ∞.

$\dfrac{du}{dx} = \dfrac{qx^{q-1}}{2\sigma^2} = \dfrac{qu}{x} = \dfrac{qu}{(2u\sigma^2)^{1/q}} = q(2\sigma^2)^{-1/q}u^{1-1/q}$.

The integral becomes:

$$2\int_{0}^{\infty} exp\left(-u\right)\, du\, \frac{dx}{du}$$
$$= 2\int_{0}^{\infty} exp(-u)\, (q(2\sigma^2)^{-1/q}u^{1-1/q})^{-1}\, du$$
$$= 2\frac{1}{q}(2\sigma^2)^{1/q}\int_{0}^{\infty} exp(-u)\, u^{1/q-1}\, du$$
$$= 2\frac{1}{q}(2\sigma^2)^{1/q}\Gamma(1/q)$$

Substituting into the original expression, we get:

$$= \frac{q}{2(2\sigma^2)^{1/q}\Gamma(1/q)}\, 2\frac{1}{q}(2\sigma^2)^{1/q}\Gamma(1/q) = 1.$$

Therefore, this distribution is normalized.

For $q = 2$,

$$\frac{q}{2(2\sigma^2)^{1/q}\Gamma(1/q)} exp\left(-\frac{|x|^q}{2\sigma^2}\right) = \frac{2}{2(2\sigma^2)^{1/2}\Gamma(1/2)} exp\left(-\frac{|x|^2}{2\sigma^2}\right)$$

$$= \frac{1}{(2\sigma^2)^{1/2}\Gamma(1/2)} exp\left(-\frac{x^2}{2\sigma^2}\right)$$

$\Gamma(1/2) = \sqrt{\pi}$ as per this source. This gives us:

$$= \frac{1}{(2\sigma^2)^{1/2}\pi^{1/2}} exp\left(-\frac{x^2}{2\sigma^2}\right)$$

$$= \frac{1}{(2\pi\sigma^2)^{1/2}} exp\left(-\frac{x^2}{2\sigma^2}\right)$$

which is a 0 mean Gaussian distribution.

$$p(\mathbf{t}|\mathbf{X}, \mathbf{w}, \sigma^2) = \prod_{n=1}^{N} p(t_n|\mathbf{x}_n, \mathbf{w}, \sigma^2) = \prod_{n=1}^{N} p(\epsilon_n|\mathbf{x}_n, \mathbf{w}, \sigma^2)$$

(since ϵ is the only random variable in the expression for y)

$$= \prod_{n=1}^{N} \frac{q}{2(2\sigma^2)^{1/q}\Gamma(1/q)} exp\left(-\frac{|(t_n - y(\mathbf{w}, \mathbf{x}_n))|^q}{2\sigma^2}\right)$$

$$\implies \ln p(\mathbf{t}|\mathbf{X}, \mathbf{w}, \sigma^2)$$

$$= \ln\left(\prod_{n=1}^{N} \frac{q}{2(2\sigma^2)^{1/q}\Gamma(1/q)} exp\left(-\frac{|(t_n - y(\mathbf{w}, \mathbf{x}_n))|^q}{2\sigma^2}\right)\right)$$

$$= \sum_{n=1}^{N} \ln\left(\frac{q}{2(2\sigma^2)^{1/q}\Gamma(1/q)} exp\left(-\frac{|(t_n - y(\mathbf{w}, \mathbf{x}_n))|^q}{2\sigma^2}\right)\right)$$

$$= N\ln\left(\frac{q}{2(2\sigma^2)^{1/q}\Gamma(1/q)}\right) + \sum_{n=1}^{N}\left(-\frac{|(t_n - y(\mathbf{w}, \mathbf{x}_n))|^q}{2\sigma^2}\right)$$

$$= N\ln\left(\frac{q}{2(2\sigma^2)^{1/q}\Gamma(1/q)}\right) - \frac{1}{2\sigma^2}\sum_{n=1}^{N}|(t_n - y(\mathbf{w}, \mathbf{x}_n))|^q$$

$$= N \ln\left(\frac{q}{2\Gamma(1/q)}\right) + N \ln\left(\frac{1}{(2\sigma^2)^{1/q}}\right) - \frac{1}{2\sigma^2}\sum_{n=1}^{N} |(t_n - y(\mathbf{w}, \mathbf{x}_n))|^q$$

$$= N \ln\left(\frac{q}{2\Gamma(1/q)}\right) - \frac{N}{q}\ln(2\sigma^2) - \frac{1}{2\sigma^2}\sum_{n=1}^{N} |(t_n - y(\mathbf{w}, \mathbf{x}_n))|^q$$

$$= -\frac{1}{2\sigma^2}\sum_{n=1}^{N} |(t_n - y(\mathbf{w}, \mathbf{x}_n))|^q - \frac{N}{q}\ln(2\sigma^2) + const$$

where const $= N \ln\left(\frac{q}{2\Gamma(1/q)}\right)$