6.18 Given:

$$f(x - x_n, t - t_n) = \mathcal{N}\left(\left[\begin{array}{c} x - x_n \\ t - t_n \end{array}\right]\middle|\mathbf{0}, \sigma^2\mathbf{I}\right)$$

Using equation 6.42:

$$p(x,t) = \frac{1}{N}\sum_{n=1}^{N}\mathcal{N}\left(\left[\begin{array}{c} x - x_n \\ t - t_n \end{array}\right]\middle|\mathbf{0}, \sigma^2\mathbf{I}\right)$$

$$= \frac{1}{N}\sum_{n=1}^{N}\frac{1}{(2\pi)^{2/2}|\sigma^2\mathbf{I}|^{1/2}}\exp\left\{-\frac{1}{2\sigma^2}\left((x-x_n)^2 + (t-t_n)^2\right)\right\}$$

$$= \frac{1}{N}\sum_{n=1}^{N}\frac{1}{(2\pi\sigma^2)}\exp\left\{-\frac{1}{2\sigma^2}\left((x-x_n)^2 + (t-t_n)^2\right)\right\}$$

$$= \frac{1}{N}\sum_{n=1}^{N}\left(\frac{1}{\sqrt{(2\pi\sigma^2)}}\exp\left\{-\frac{1}{2\sigma^2}(x-x_n)^2\right\}\right)\left(\frac{1}{\sqrt{(2\pi\sigma^2)}}\exp\left\{-\frac{1}{2\sigma^2}(t-t_n)^2\right\}\right)$$

$$\implies p(x,t) = \frac{1}{N}\sum_{n}\mathcal{N}(x - x_n|0,\sigma^2)\,\mathcal{N}(t - t_n|0,\sigma^2)$$

$$\implies \int p(x,t)\,dt = \frac{1}{N}\sum_{n}\mathcal{N}(x - x_n|0,\sigma^2)$$

Using equation 6.48, the conditional density is given by:

$$p(t|x) = \frac{p(t,x)}{\int p(t,x)\,dt} = \frac{\frac{1}{N}\sum_n \mathcal{N}(x - x_n|0,\sigma^2)\,\mathcal{N}(t - t_n|0,\sigma^2)}{\frac{1}{N}\sum_m \mathcal{N}(x - x_m|0,\sigma^2)}$$

$$= \frac{\sum_n \mathcal{N}(x - x_n|0,\sigma^2)\,\mathcal{N}(t - t_n|0,\sigma^2)}{\sum_m \mathcal{N}(x - x_m|0,\sigma^2)}$$

Using equation 6.43, the conditional mean is given by:

$$\mathbb{E}[t|x] = \frac{\int t\,p(x,t)\,dt}{\int p(x,t)\,dt}$$

$$= \frac{\int t\,\frac{1}{N}\sum_n \mathcal{N}(x - x_n|0,\sigma^2)\,\mathcal{N}(t - t_n|0,\sigma^2)\,dt}{\frac{1}{N}\sum_m \mathcal{N}(x - x_m|0,\sigma^2)}$$

$$= \frac{\int t \sum_n \mathcal{N}(x - x_n|0, \sigma^2) \mathcal{N}(t - t_n|0, \sigma^2) \, dt}{\sum_m \mathcal{N}(x - x_m|0, \sigma^2)}$$

$$= \frac{\sum_n \mathcal{N}(x - x_n|0, \sigma^2) \int t \mathcal{N}(t - t_n|0, \sigma^2) \, dt}{\sum_m \mathcal{N}(x - x_m|0, \sigma^2)}$$

Solving the integral:

$$\int t \mathcal{N}(t - t_n|0, \sigma^2) \, dt$$

$$= \left(\int (t - t_n) \mathcal{N}(t - t_n|0, \sigma^2) \, dt \right) + \left(\int t_n \mathcal{N}(t - t_n|0, \sigma^2) \, dt \right)$$

$$= 0 + t_n \left(\int \mathcal{N}(t - t_n|0, \sigma^2) \, dt \right)$$

$$= t_n$$

$$\implies \mathbb{E}[t|x] = \frac{\sum_n \mathcal{N}(x - x_n|0, \sigma^2) \, t_n}{\sum_m \mathcal{N}(x - x_m|0, \sigma^2)}$$

Comparing the result to equations 6.45 and 6.46:

$$\implies g(x - x_n) = \mathcal{N}(x - x_n|0, \sigma^2)$$

$$\implies k(x, x_n) = \frac{\mathcal{N}(x - x_n|0, \sigma^2)}{\sum_m \mathcal{N}(x - x_m|0, \sigma^2)}$$

$$\implies \mathbb{E}[t|x] = \sum_n k(x, x_n) t_n$$

The conditional variance is given by:

$$var[t|x] = \mathbb{E}[(t - \mathbb{E}[t|x])^2 | x]$$

$$= \int (t - \mathbb{E}[t|x])^2 \, p(t|x) \, dt$$

$$= \int (t^2 + \mathbb{E}[t|x]^2 - 2t\mathbb{E}[t|x]) \, p(t|x) \, dt$$

Solving the three integrals separately:

First term:

$$\int t^2\, p(t|x)\, dt$$

$$= \int t^2\, \frac{\sum_n \mathcal{N}(x - x_n|0, \sigma^2)\, \mathcal{N}(t - t_n|0, \sigma^2)}{\sum_m \mathcal{N}(x - x_m|0, \sigma^2)}\, dt$$

$$= \frac{\sum_n \mathcal{N}(x - x_n|0, \sigma^2)\, \int t^2\, \mathcal{N}(t - t_n|0, \sigma^2)\, dt}{\sum_m \mathcal{N}(x - x_m|0, \sigma^2)}$$

Evaluating the integral:

$$\int t^2\, \mathcal{N}(t - t_n|0, \sigma^2)\, dt$$

$$= \int (t - t_n + t_n)^2\, \mathcal{N}(t - t_n|0, \sigma^2)\, dt$$

$$= \int \left((t - t_n)^2 + t_n^2 + 2(t - t_n)t_n\right)\, \mathcal{N}(t - t_n|0, \sigma^2)\, dt$$

$$= \int (t - t_n)^2\, \mathcal{N}(t - t_n|0, \sigma^2)\, dt$$
$$+ \int t_n^2\, \mathcal{N}(t - t_n|0, \sigma^2)\, dt$$
$$+ \int 2(t - t_n)t_n\, \mathcal{N}(t - t_n|0, \sigma^2)\, dt$$

$$= \sigma^2 + t_n^2 - 0$$

$$= \sigma^2 + t_n^2$$

$$\implies \int t^2\, p(t|x)\, dt = \frac{\sum_n \mathcal{N}(x - x_n|0, \sigma^2)\, (\sigma^2 + t_n^2)}{\sum_m \mathcal{N}(x - x_m|0, \sigma^2)}$$
$$= \sum_n k(x, x_n)\, (\sigma^2 + t_n^2)$$

Second term:

$$\int \mathbb{E}[t|x]^2 \, p(t|x) \, dt$$

$$= \mathbb{E}[t|x]^2 \int p(t|x) \, dt$$

$$= \mathbb{E}[t|x]^2$$

Third term:

$$\int 2t\mathbb{E}[t|x] \, p(t|x) \, dt$$

$$= 2\mathbb{E}[t|x] \int t \, p(t|x) \, dt$$

$$= 2\mathbb{E}[t|x]^2$$

Substituting the results back into the expression for conditional variance:

$$\begin{aligned} var[t|x] &= \sum_n k(x, x_n) \, (\sigma^2 + t_n^2) + \mathbb{E}[t|x]^2 - 2\mathbb{E}[t|x]^2 \\ &= \sum_n k(x, x_n) \, (\sigma^2 + t_n^2) - \mathbb{E}[t|x]^2 \\ &= \sum_n k(x, x_n) \, \sigma^2 + \sum_n k(x, x_n) \, t_n^2 - \mathbb{E}[t|x]^2 \\ &= \sigma^2 - \mathbb{E}[t|x]^2 + \sum_n k(x, x_n) \, t_n^2 \end{aligned}$$