1.31 We need to prove two things:

$$\mathbf{H}[\mathbf{x},\mathbf{y}] = \mathbf{H}[\mathbf{x}] + \mathbf{H}[\mathbf{y}] \Longrightarrow p(\mathbf{x},\mathbf{y}) = p(\mathbf{x})\,p(\mathbf{y})$$

and

$$p(\mathbf{x},\mathbf{y}) = p(\mathbf{x})\,p(\mathbf{y}) \Longrightarrow \mathbf{H}[\mathbf{x},\mathbf{y}] = \mathbf{H}[\mathbf{x}] + \mathbf{H}[\mathbf{y}]$$

Proving the first statement:

$$\mathbf{H}[\mathbf{x},\mathbf{y}] = \mathbf{H}[\mathbf{x}] + \mathbf{H}[\mathbf{y}]$$

$$\Longrightarrow \mathbf{H}[\mathbf{x}|\mathbf{y}] + \mathbf{H}[\mathbf{y}] = \mathbf{H}[\mathbf{x}] + \mathbf{H}[\mathbf{y}]$$

$$\Longrightarrow \mathbf{H}[\mathbf{x}|\mathbf{y}] = \mathbf{H}[\mathbf{x}]$$

Applying 1.121, we get:

$$\Longrightarrow \mathbf{I}[\mathbf{x},\mathbf{y}] = 0$$

$$\Longrightarrow \mathbf{x} \text{ and } \mathbf{y} \text{ are independent.}$$

Proving the second statement:

$$\mathbf{H}[\mathbf{x},\mathbf{y}] = -\int_{\mathbf{x}}\int_{\mathbf{y}} p(\mathbf{x},\mathbf{y})\;\ln p(\mathbf{x},\mathbf{y})\,d\mathbf{x}\,d\mathbf{y}$$

$$= -\int_{\mathbf{x}}\int_{\mathbf{y}} p(\mathbf{x})\,p(\mathbf{y})\;\ln(p(\mathbf{x})\,p(\mathbf{y}))\,d\mathbf{x}\,d\mathbf{y}$$

$$= -\int_{\mathbf{x}}\int_{\mathbf{y}} p(\mathbf{x})\,p(\mathbf{y})\;\ln p(\mathbf{x})\,d\mathbf{x}\,d\mathbf{y} - \int_{\mathbf{x}}\int_{\mathbf{y}} p(\mathbf{x})\,p(\mathbf{y})\;\ln p(\mathbf{y})\,d\mathbf{x}\,d\mathbf{y}$$

$$= -\int_{\mathbf{x}}\left(\int_{\mathbf{y}} p(\mathbf{y})\,d\mathbf{y}\right) p(\mathbf{x})\;\ln p(\mathbf{x})\,d\mathbf{x} - \int_{\mathbf{y}}\left(\int_{\mathbf{x}} p(\mathbf{x})\,d\mathbf{x}\right)\,p(\mathbf{y})\;\ln p(\mathbf{y})\,d\mathbf{y}$$

$$
\begin{aligned}
&= -\int_{\mathbf{x}} (1)\, p(\mathbf{x})\ \ln p(\mathbf{x})\, d\mathbf{x} - \int_{\mathbf{y}} (1)\ p(\mathbf{y})\ \ln p(\mathbf{y})\, d\mathbf{y} \\
&= -\int_{\mathbf{x}} p(\mathbf{x})\ \ln p(\mathbf{x})\, d\mathbf{x} - \int_{\mathbf{y}} p(\mathbf{y})\ \ln p(\mathbf{y})\, d\mathbf{y} \\
&= \mathbf{H}[\mathbf{x}] + \mathbf{H}[\mathbf{y}]
\end{aligned}
$$