3.24

$$p(\mathbf{t}) = \frac{p(\mathbf{t}|\mathbf{w}, \beta)\; p(\mathbf{w}, \beta)}{p(\mathbf{w}, \beta|\mathbf{t})}$$

where

$$p(\mathbf{t}|\mathbf{w}, \beta) = \left(\prod_{n=1}^{N} \mathcal{N}(t_n|\mathbf{w}^T \boldsymbol{\phi}(\mathbf{x}_n), \beta^{-1})^{N/2} \right)$$

$$= \frac{1}{(2\pi\beta^{-1})^{N/2}} \exp\left\{ -\frac{\beta}{2} (\mathbf{t} - \mathbf{\Phi}\mathbf{w})^T (\mathbf{t} - \mathbf{\Phi}\mathbf{w}) \right\}$$

$$p(\mathbf{w}, \beta) = \mathcal{N}(\mathbf{w}|\mathbf{m}_0, \beta^{-1}\mathbf{S}_0)\, Gam(\beta|a_0, b_0)$$

$$= \frac{1}{(2\pi)^{D/2} \quad |\beta^{-1}\mathbf{S}_0|^{1/2}} \exp\left\{ -\frac{1}{2} (\mathbf{w} - \mathbf{m}_0)^T (\beta^{-1}\mathbf{S}_0)^{-1} (\mathbf{w} - \mathbf{m}_0) \right\}$$

$$\frac{1}{\Gamma(a_0)} b_0^{a_0} \beta^{(a_0 - 1)} \exp\{-\beta b_0\}$$

$$p(\mathbf{w}, \beta|\mathbf{t}) = \mathcal{N}(\mathbf{w}|\mathbf{m}_N, \beta^{-1}\mathbf{S}_N)\; Gam(\beta|a_N, b_N)$$

$$= \frac{1}{(2\pi)^{D/2} \quad |\beta^{-1}\mathbf{S}_N|^{1/2}} \exp\left\{ -\frac{1}{2} (\mathbf{w} - \mathbf{m}_N)^T (\beta^{-1}\mathbf{S}_N)^{-1} (\mathbf{w} - \mathbf{m}_N) \right\}$$

$$\frac{1}{\Gamma(a_N)} b_N^{a_N} \beta^{(a_N - 1)} \exp\{-\beta b_N\}$$

Therefore, $p(\mathbf{t})$ is given by:

$$\frac{\frac{1}{(2\pi\beta^{-1})^{N/2}}\exp\left\{-\frac{\beta}{2}(\mathbf{t}-\mathbf{\Phi w})^T(\mathbf{t}-\mathbf{\Phi w})\right\}\frac{1}{(2\pi)^{D/2}|\beta^{-1}\mathbf{S}_0|^{1/2}}\exp\left\{-\frac{1}{2}(\mathbf{w}-\mathbf{m}_0)^T(\beta^{-1}\mathbf{S}_0)^{-1}(\mathbf{w}-\mathbf{m}_0)\right\}}{\frac{1}{(2\pi)^{D/2}\quad|\beta^{-1}\mathbf{S}_N|^{1/2}}\exp\left\{-\frac{1}{2}(\mathbf{w}-\mathbf{m}_N)^T(\beta^{-1}\mathbf{S}_N)^{-1}(\mathbf{w}-\mathbf{m}_N)\right\}}$$

$$*\frac{\frac{1}{\Gamma(a_0)}b_0^{a_0}\beta^{(a_0-1)}\exp-\beta b_0}{\frac{1}{\Gamma(a_N)}b_N^{a_N}\beta^{(a_N-1)}\exp-\beta b_N}$$

$$=\frac{|\mathbf{S}_N|^{1/2}}{(2\pi\beta^{-1})^{N/2}|\mathbf{S}_0|^{1/2}}$$

$$\exp\left\{-\frac{\beta}{2}((\mathbf{t}-\mathbf{\Phi w})^T(\mathbf{t}-\mathbf{\Phi w})+(\mathbf{w}-\mathbf{m}_0)^T\mathbf{S}_0^{-1}(\mathbf{w}-\mathbf{m}_0)-(\mathbf{w}-\mathbf{m}_N)^T\mathbf{S}_N^{-1}(\mathbf{w}-\mathbf{m}_N))\right\}$$

$$\frac{\Gamma(a_N)}{\Gamma(a_0)}\frac{b_0^{a_0}\beta^{(a_0-1)}\exp\{-\beta b_0\}}{b_N^{a_N}\beta^{(a_N-1)}\exp\{-\beta b_N\}}$$

$$=\left(\frac{|\mathbf{S}_N|^{1/2}}{(2\pi)^{N/2}|\mathbf{S}_0|^{1/2}}\frac{\Gamma(a_N)}{\Gamma(a_0)}\frac{b_0^{a_0}}{b_N^{a_N}}\right)\beta^{N/2}$$

$$\exp\left\{-\frac{\beta}{2}((\mathbf{t}-\mathbf{\Phi w})^T(\mathbf{t}-\mathbf{\Phi w})+(\mathbf{w}-\mathbf{m}_0)^T\mathbf{S}_0^{-1}(\mathbf{w}-\mathbf{m}_0)-(\mathbf{w}-\mathbf{m}_N)^T\mathbf{S}_N^{-1}(\mathbf{w}-\mathbf{m}_N))\right\}$$

$$\frac{\beta^{(a_0-1)}\exp\{-\beta b_0\}}{\beta^{(a_N-1)}\exp\{-\beta b_N\}}$$

The exponent of the middle the middle term can be simplified as:

$$(\mathbf{t}-\mathbf{\Phi w})^T(\mathbf{t}-\mathbf{\Phi w})+(\mathbf{w}-\mathbf{m}_0)^T\mathbf{S}_0^{-1}(\mathbf{w}-\mathbf{m}_0)-(\mathbf{w}-\mathbf{m}_N)^T\mathbf{S}_N^{-1}(\mathbf{w}-\mathbf{m}_N)$$

$$\begin{aligned}&=\mathbf{t}^T\mathbf{t}-2\mathbf{t}^T\mathbf{\Phi w}+\mathbf{w}^T\mathbf{\Phi}^T\mathbf{\Phi w}\\&+\mathbf{w}^T\mathbf{S}_0^{-1}\mathbf{w}-2\mathbf{m}_0^T\mathbf{S}_0^{-1}\mathbf{w}+\mathbf{m}_0^T\mathbf{S}_0^{-1}\mathbf{m}_0\\&-\mathbf{w}^T\mathbf{S}_N^{-1}\mathbf{w}+2\mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{w}-\mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{m}_N\end{aligned}$$

$$=\mathbf{w}^T(\mathbf{\Phi}^T\mathbf{\Phi}+\mathbf{S}_0^{-1}-\mathbf{S}_N^{-1})\mathbf{w}$$

$$-2(\mathbf{t}^T\mathbf{\Phi} + \boldsymbol{m_0^T S_0^{-1}} - \mathbf{m}_N^T\mathbf{S}_N^{-1})\mathbf{w}$$
$$+\mathbf{m}_0^T\mathbf{S}_0^{-1}\mathbf{m}_0 + \mathbf{t}^T\mathbf{t} - \mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{m}_N$$

From Exercise 3.12, we know that $\mathbf{S}_N^{-1} = \mathbf{\Phi}^T\mathbf{\Phi} + \mathbf{S}_0^{-1}$ and $\mathbf{m}_N^T\mathbf{S}_N^{-1} = \mathbf{t}^T\mathbf{\Phi} + \boldsymbol{m_0^T S_0^{-1}}$.

Therefore, the term becomes:

$$= \mathbf{m}_0^T\mathbf{S}_0^{-1}\mathbf{m}_0 + \mathbf{t}^T\mathbf{t} - \mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{m}_N$$

Substituting this result back into the expression for the posterior, we get:

$$= \left(\frac{|\mathbf{S}_N|^{1/2}}{(2\pi)^{N/2}|\mathbf{S}_0|^{1/2}} \frac{\Gamma(a_N)}{\Gamma(a_0)} \frac{b_0^{a_0}}{b_N^{a_N}} \right) \beta^{N/2}$$
$$\exp\left\{ -\frac{\beta}{2}(\mathbf{m}_0^T\mathbf{S}_0^{-1}\mathbf{m}_0 + \mathbf{t}^T\mathbf{t} - \mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{m}_N) \right\}$$
$$\frac{\beta^{(a_0-1)}\exp\{-\beta b_0\}}{\beta^{(a_N-1)}\exp\{-\beta b_N\}}$$

From Exercise 3.12, we also know that $a_N = a_0 + N/2$, and $b_N = b_0 + \frac{1}{2}(\mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{m}_N + \mathbf{m}_0^T\mathbf{S}_0^{-1}\mathbf{m}_0 + \mathbf{t}^T\mathbf{t})$.

Substituting these into the posterior, we get:

$$= \left(\frac{|\mathbf{S}_N|^{1/2}}{(2\pi)^{N/2}|\mathbf{S}_0|^{1/2}} \frac{\Gamma(a_N)}{\Gamma(a_0)} \frac{b_0^{a_0}}{b_N^{a_N}} \right) \beta^{N/2}$$
$$\exp\left\{ -\frac{\beta}{2}(\mathbf{m}_0^T\mathbf{S}_0^{-1}\mathbf{m}_0 + \mathbf{t}^T\mathbf{t} - \mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{m}_N) \right\}$$
$$\frac{\beta^{(a_0-1)}\exp\{-\beta b_0\}}{\beta^{(a_0+N/2-1)}\exp\{-\beta \left(b_0 + \frac{1}{2}(\mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{m}_N + \mathbf{m}_0^T\mathbf{S}_0^{-1}\mathbf{m}_0 + \mathbf{t}^T\mathbf{t})\}\right)}$$

$$= \left(\frac{|\mathbf{S}_N|^{1/2}}{(2\pi)^{N/2}|\mathbf{S}_0|^{1/2}} \frac{\Gamma(a_N)}{\Gamma(a_0)} \frac{b_0^{a_0}}{b_N^{a_N}} \right) \beta^{N/2}$$
$$\exp\left\{ -\frac{\beta}{2}(\mathbf{m}_0^T\mathbf{S}_0^{-1}\mathbf{m}_0 + \mathbf{t}^T\mathbf{t} - \mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{m}_N) \right\}$$

$$\frac{1}{\beta^{N/2}\exp\{-\beta\left(\frac{1}{2}(\mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{m}_N+\mathbf{m}_0^T\mathbf{S}_0^{-1}\mathbf{m}_0+\mathbf{t}^T\mathbf{t})\}\right)}$$

$$= \left(\frac{|\mathbf{S}_N|^{1/2}}{(2\pi)^{N/2}|\mathbf{S}_0|^{1/2}}\frac{\Gamma(a_N)}{\Gamma(a_0)}\frac{b_0^{a_0}}{b_N^{a_N}}\right)\beta^{N/2}\frac{1}{\beta^{N/2}}$$

$$= \frac{|\mathbf{S}_N|^{1/2}}{(2\pi)^{N/2}|\mathbf{S}_0|^{1/2}}\frac{\Gamma(a_N)}{\Gamma(a_0)}\frac{b_0^{a_0}}{b_N^{a_N}}$$

which is the same as the result in 3.118.