3.23 The model evidence is given by:

$$p(\mathbf{t}) = \int \int p\left(\mathbf{t}|\mathbf{X}, \mathbf{w}, \beta\right) \, p(\mathbf{w}, \beta) \, d\mathbf{w} \, d\beta$$

From 3.10 and 3.112, this becomes:

$$= \int \int \left(\prod_{n=1}^{N} \mathcal{N}(t_n|\mathbf{w}^T \boldsymbol{\phi}(\mathbf{x}_n), \beta^{-1}) \right) \mathcal{N}(\mathbf{w}|\mathbf{m}_0, \beta^{-1}\mathbf{S}_0) \, Gam(\beta|a_0, b_0) \, d\mathbf{w} \, d\beta$$

Directly applying the result from exercise 3.12, we get:

$$= \int \int \mathcal{N}(\mathbf{w}|\mathbf{m}_N, \beta^{-1}\mathbf{S}_N) \left(\frac{|\mathbf{S}_N|^{1/2}}{(2\pi)^{N/2}|\mathbf{S}_0|^{1/2}} \frac{\Gamma(a_N)}{\Gamma(a_0)} \frac{b_0^{a_0}}{b_N^{a_N}} \right)$$
$$\frac{1}{\Gamma(a_N)} b_N^{a_N} \beta^{(a_N-1)} \exp\left\{-\beta b_N\right\} \quad d\mathbf{w} \, d\beta$$

$$= \int \left(\int \mathcal{N}(\mathbf{w}|\mathbf{m}_N, \beta^{-1}\mathbf{S}_N) \quad d\mathbf{w} \right) \left(\frac{|\mathbf{S}_N|^{1/2}}{(2\pi)^{N/2}|\mathbf{S}_0|^{1/2}} \frac{\Gamma(a_N)}{\Gamma(a_0)} \frac{b_0^{a_0}}{b_N^{a_N}} \right)$$
$$\frac{1}{\Gamma(a_N)} b_N^{a_N} \beta^{(a_N-1)} \exp\left\{-\beta b_N\right\} \quad d\beta$$

$$= \int (1) \left(\frac{|\mathbf{S}_N|^{1/2}}{(2\pi)^{N/2}|\mathbf{S}_0|^{1/2}} \frac{\Gamma(a_N)}{\Gamma(a_0)} \frac{b_0^{a_0}}{b_N^{a_N}} \right) \frac{1}{\Gamma(a_N)} b_N^{a_N} \beta^{(a_N-1)} \exp\left\{-\beta b_N\right\} \quad d\beta$$

$$= \left(\frac{|\mathbf{S}_N|^{1/2}}{(2\pi)^{N/2}|\mathbf{S}_0|^{1/2}} \frac{\Gamma(a_N)}{\Gamma(a_0)} \frac{b_0^{a_0}}{b_N^{a_N}} \right) \int \frac{1}{\Gamma(a_N)} b_N^{a_N} \beta^{(a_N-1)} \exp\left\{-\beta b_N\right\} \quad d\beta$$

$$= \left(\frac{|\mathbf{S}_N|^{1/2}}{(2\pi)^{N/2}|\mathbf{S}_0|^{1/2}} \frac{\Gamma(a_N)}{\Gamma(a_0)} \frac{b_0^{a_0}}{b_N^{a_N}} \right) \int Gamma(\beta|a_N, b_N) \quad d\beta$$

$$= \left(\frac{|\mathbf{S}_N|^{1/2}}{(2\pi)^{N/2}|\mathbf{S}_0|^{1/2}} \frac{\Gamma(a_N)}{\Gamma(a_0)} \frac{b_0^{a_0}}{b_N^{a_N}} \right) (1)$$

$$\implies p(\mathbf{t}) = \frac{|\mathbf{S}_N|^{1/2}}{(2\pi)^{N/2}|\mathbf{S}_0|^{1/2}} \frac{\Gamma(a_N)}{\Gamma(a_0)} \frac{b_0^{a_0}}{b_N^{a_N}}$$

which is the same as 3.118.