4.1 First we prove that if the two sets of points are linearly separable, their convex hulls do not intersect.

Let $\mathbf{x}$ be a point belonging to the convex hull of $\{\mathbf{x}_n\}$.

Then,

$$\hat{\mathbf{w}}^T\mathbf{x} + w_0 = \hat{\mathbf{w}}^T\left(\sum_n \alpha_n \mathbf{x}_n\right) + w_0$$

$$= \sum_n (\hat{\mathbf{w}}^T \alpha_n \mathbf{x}_n) + w_0$$

where $0 \leq \alpha_n \leq 1$ and $\sum_n \alpha_n = 1$.

$$= \sum_n (\hat{\mathbf{w}}^T \alpha_n \mathbf{x}_n) + w_0.1$$

$$= \sum_n (\hat{\mathbf{w}}^T \alpha_n \mathbf{x}_n) + w_0 \sum_n \alpha_n \qquad \text{since } \sum_n \alpha_n = 1$$

$$= \sum_n \alpha_n (\hat{\mathbf{w}}^T \mathbf{x}_n + w_0)$$

Since $\alpha_n \geq 0$ and $(\hat{\mathbf{w}}^T\mathbf{x}_n + w_0) > 0,\ \forall n$, we can say that $\hat{\mathbf{w}}^T\mathbf{x} + w_0 > 0$, $\forall \mathbf{x}$.

Similarly, we can prove that $\hat{\mathbf{w}}^T\mathbf{y} + w_0 < 0$, $\forall \mathbf{y}$.

This proves that if the two sets of points are linearly separable, their convex hulls do not intersect.

Now, let's assume that their convex hulls intersect. We have to prove that the two sets of points cannot be linearly separable.

Let $\mathbf{z}$ be a point $\in$ Convex Hull($\{\mathbf{x}_n\}$) $\cap$ Convex Hull($\{\mathbf{y}_n\}$)

Therefore, $\mathbf{z}$ can be represented as the combination of all the points $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$, such that:

$$\mathbf{z} = \sum_n \alpha_n \mathbf{x}_n = \sum_n \beta_n \mathbf{y}_n$$

$$\text{where } \alpha_n, \beta_n \geq 0 \text{ for all } n, \text{ and } \sum_n \alpha_n = 1, \sum_n \beta_n = 1$$

Now let's assume that the 2 sets of points $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ are linearly separable.

So, there exists a vector $\hat{\mathbf{w}}$ and a scalar w_0 such that $\hat{\mathbf{w}}^T\mathbf{x}_n + w_0 > 0 \; \forall \; \mathbf{x}_n$ and $\hat{\mathbf{w}}^T\mathbf{y}_n + w_0 < 0 \; \forall \; \mathbf{y}_n$.

$$\implies w_0 + \hat{\mathbf{w}}^T\mathbf{z} = w_0 + \hat{\mathbf{w}}^T \sum_n (\alpha_n \mathbf{x}_n) = w_0 + \hat{\mathbf{w}}^T \sum_n (\beta_n \mathbf{y}_n)$$

$$\implies w_0 + \hat{\mathbf{w}}^T\mathbf{z} = w_0 + \sum_n \alpha_n \hat{\mathbf{w}}^T\mathbf{x}_n = w_0 + \sum_n \beta_n \hat{\mathbf{w}}^T\mathbf{y}_n$$

$$\implies w_0 + \hat{\mathbf{w}}^T\mathbf{z} = w_0.1 + \sum_n \alpha_n \hat{\mathbf{w}}^T\mathbf{x}_n = w_0.1 + \sum_n \beta_n \hat{\mathbf{w}}^T\mathbf{y}_n$$

$$\implies w_0 + \hat{\mathbf{w}}^T\mathbf{z} = w_0 \sum_n \alpha_n + \sum_n \alpha_n \hat{\mathbf{w}}^T\mathbf{x}_n = w_0 \sum_n \alpha_n + \sum_n \beta_n \hat{\mathbf{w}}^T\mathbf{y}_n$$

$$\implies w_0 + \hat{\mathbf{w}}^T\mathbf{z} = \sum_n (\alpha_n \hat{\mathbf{w}}^T\mathbf{x}_n + \alpha_n w_0) = \sum_n (\beta_n \hat{\mathbf{w}}^T\mathbf{y}_n + \beta_n w_0)$$

$$\implies w_0 + \hat{\mathbf{w}}^T\mathbf{z} = \sum_n \alpha_n (\hat{\mathbf{w}}^T\mathbf{x}_n + w_0) = \sum_n \beta_n (\hat{\mathbf{w}}^T\mathbf{y}_n + w_0)$$

Again, we can see that $\sum_n \alpha_n(\hat{\mathbf{w}}^T\mathbf{x}_n + w_0) > 0$ and $\sum_n \beta_n(\hat{\mathbf{w}}^T\mathbf{y}_n + w_0) < 0$.

So the equality cannot hold, and we have a contradiction here. This proves that if the convex hulls of the two sets of point intersect, the two sets of points cannot be linearly separable,