2.48 From 2.161, we have:

$$St(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Lambda},\nu) = \int_0^\infty \mathcal{N}(\mathbf{x}|\boldsymbol{\mu},(\eta\boldsymbol{\Lambda})^{-1})\, Gam(\eta|\nu/2,\nu/2)\,\mathrm{d}\eta$$

$$= \int_0^\infty \frac{1}{(2\pi)^{D/2}|(\eta\boldsymbol{\Lambda})^{-1}|^{1/2}} exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T(\eta\boldsymbol{\Lambda})(\mathbf{x}-\boldsymbol{\mu})\right\} \frac{1}{\Gamma(\nu/2)}\left(\frac{\nu}{2}\right)^{\nu/2}\eta^{\nu/2-1} exp\left(-\frac{\nu\eta}{2}\right)\mathrm{d}\eta$$

Using equation 19 from the Matrix Cookbook, we know that $det(c\mathbf{A}) = c^n det(\mathbf{A})$. Which gives us:

$$= \int_0^\infty \frac{1}{(2\pi)^{D/2}\,\eta^{(-D/2)}|\boldsymbol{\Lambda}^{-1}|^{1/2}} exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T(\eta\boldsymbol{\Lambda})(\mathbf{x}-\boldsymbol{\mu})\right\} \frac{1}{\Gamma(\nu/2)}\left(\frac{\nu}{2}\right)^{\nu/2}\eta^{\nu/2-1} exp\left(-\frac{\nu\eta}{2}\right)\mathrm{d}\eta$$

$$= \int_0^\infty \frac{\eta^{(D/2+\nu/2-1)}}{(2\pi)^{D/2}\,|\boldsymbol{\Lambda}^{-1}|^{1/2}} exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T(\eta\boldsymbol{\Lambda})(\mathbf{x}-\boldsymbol{\mu})\right\} \frac{1}{\Gamma(\nu/2)}\left(\frac{\nu}{2}\right)^{\nu/2} exp\left(-\frac{\nu\eta}{2}\right)\mathrm{d}\eta$$

$$= \frac{\nu^{\nu/2}}{(2\pi)^{D/2}\,2^{\nu/2}\,|\boldsymbol{\Lambda}^{-1}|^{1/2}\,\Gamma(\nu/2)}\int_0^\infty \eta^{(D/2+\nu/2-1)} exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T(\eta\boldsymbol{\Lambda})(\mathbf{x}-\boldsymbol{\mu}) - \frac{\nu\eta}{2}\right)\mathrm{d}\eta$$

$$= \frac{\nu^{\nu/2}}{(2\pi)^{D/2}\,2^{\nu/2}\,|\boldsymbol{\Lambda}^{-1}|^{1/2}\,\Gamma(\nu/2)}\int_0^\infty \eta^{(D/2+\nu/2-1)} exp\left\{-\frac{1}{2}\eta\left((\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Lambda}(\mathbf{x}-\boldsymbol{\mu})+\nu\right)\right\}\mathrm{d}\eta$$

Let $z = \frac{1}{2}\eta\left((\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Lambda}(\mathbf{x}-\boldsymbol{\mu})+\nu\right)$.

Then, $\eta = \frac{2z}{((\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Lambda}(\mathbf{x}-\boldsymbol{\mu})+\nu)}$, and $\frac{\mathrm{d}\eta}{\mathrm{d}z} = \frac{2}{((\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Lambda}(\mathbf{x}-\boldsymbol{\mu})+\nu)}$.

Substituting into the integral, we get:

$$= \frac{\nu^{\nu/2}}{(2\pi)^{D/2}\,2^{\nu/2}\,|\boldsymbol{\Lambda}^{-1}|^{1/2}\,\Gamma(\nu/2)}\int_0^\infty \left(\frac{2z}{((\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Lambda}(\mathbf{x}-\boldsymbol{\mu})+\nu)}\right)^{(D/2+\nu/2-1)} exp\{-z\}\,\mathrm{d}z\left(\frac{\mathrm{d}\eta}{\mathrm{d}z}\right)$$

$$= \frac{\nu^{\nu/2}}{(2\pi)^{D/2}\,2^{\nu/2}\,|\boldsymbol{\Lambda}^{-1}|^{1/2}\,\Gamma(\nu/2)}\int_0^\infty \left(\frac{2z}{(\Delta^2+\nu)}\right)^{(D/2+\nu/2-1)} exp\{-z\}\,\mathrm{d}z\left(\frac{\mathrm{d}\eta}{\mathrm{d}z}\right)$$

$$= \frac{\nu^{\nu/2}}{(2\pi)^{D/2}\, 2^{\nu/2}\, |\mathbf{\Lambda}^{-1}|^{1/2}\, \Gamma(\nu/2)} \int_0^\infty \left(\frac{2z}{(\Delta^2 + \nu)} \right)^{(D/2+\nu/2-1)} exp\{-z\}\, \mathrm{d}z \, \left(\frac{2}{(\Delta^2+\nu)} \right)$$

$$= \frac{\nu^{\nu/2}}{(2\pi)^{D/2}\, 2^{\nu/2}\, |\mathbf{\Lambda}^{-1}|^{1/2}\, \Gamma(\nu/2)} \left(\frac{2}{(\Delta^2 + \nu)} \right)^{(D/2+\nu/2-1)+1} \int_0^\infty z^{(D/2+\nu/2-1)} exp\{-z\}\, \mathrm{d}z$$

Using equation 1.141,

$$= \frac{\nu^{\nu/2}}{(2\pi)^{D/2}\, 2^{\nu/2}\, |\mathbf{\Lambda}^{-1}|^{1/2}\, \Gamma(\nu/2)} \left(\frac{2}{(\Delta^2 + \nu)} \right)^{(D/2+\nu/2)} \Gamma(D/2 + \nu/2)$$

$$= \frac{\nu^{\nu/2}}{(2\pi)^{D/2}\, 2^{\nu/2}\, |\mathbf{\Lambda}^{-1}|^{1/2}\, \Gamma(\nu/2)} \left(\frac{2}{(\Delta^2 + \nu)} \right)^{(\nu/2+D/2)} \Gamma(\nu/2 + D/2)$$

$$= \frac{\nu^{\nu/2}}{(2\pi)^{D/2}\, 2^{\nu/2}\, |\mathbf{\Lambda}^{-1}|^{1/2}\, \Gamma(\nu/2)} \left(\frac{(\Delta^2 + \nu)}{2} \right)^{-(\nu/2+D/2)} \Gamma(\nu/2 + D/2)$$

$$= \frac{\nu^{\nu/2}}{(2\pi)^{D/2}\, 2^{\nu/2}\, |\mathbf{\Lambda}^{-1}|^{1/2}\, \Gamma(\nu/2)} \left(\frac{\nu}{2} \frac{(\Delta^2 + \nu)}{\nu} \right)^{-(\nu/2+D/2)} \Gamma(\nu/2 + D/2)$$

$$= \frac{\nu^{\nu/2}}{(2\pi)^{D/2}\, 2^{\nu/2}\, |\mathbf{\Lambda}^{-1}|^{1/2}\, \Gamma(\nu/2)} \left(\frac{\nu}{2} \right)^{-(\nu/2+D/2)} \left(\frac{\Delta^2 + \nu}{\nu} \right)^{-(\nu/2+D/2)} \Gamma(\nu/2{+}D/2)$$

$$= \frac{\nu^{-D/2}\, |\mathbf{\Lambda}|^{1/2}}{(2\pi)^{D/2}\, 2^{-D/2}\, \Gamma(\nu/2)} \left(1 + \frac{\Delta^2}{\nu} \right)^{-(\nu/2+D/2)} \Gamma(\nu/2 + D/2)$$

$$= \frac{\Gamma(\nu/2 + D/2)}{\Gamma(\nu/2)} \frac{\nu^{-D/2}\,|\mathbf{\Lambda}|^{1/2}}{(2\pi)^{D/2}\,2^{-D/2}} \left(1 + \frac{\Delta^2}{\nu}\right)^{-(\nu/2+D/2)}$$

$$= \frac{\Gamma(\nu/2 + D/2)}{\Gamma(\nu/2)} \frac{|\mathbf{\Lambda}|^{1/2}}{(\pi\nu)^{D/2}} \left(1 + \frac{\Delta^2}{\nu}\right)^{-D/2-\nu/2}$$

which is the same as the result in 2.162.

Now to prove that the multivariate t-distribution is correctly normalized.

$$\int_{\mathbf{x}} St(\mathbf{x}|\boldsymbol{\mu}, \mathbf{\Lambda}, \nu)\, d\mathbf{x} = \int_{\mathbf{x}} \int_0^\infty \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, (\eta\mathbf{\Lambda})^{-1})\, Gam(\eta|\nu/2, \nu/2)\, \mathrm{d}\eta\, d\mathbf{x}$$

$$= \int_0^\infty \left(\int_{\mathbf{x}} \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, (\eta\mathbf{\Lambda})^{-1})\, d\mathbf{x} \right) Gam(\eta|\nu/2, \nu/2)\, \mathrm{d}\eta$$

$$= \int_0^\infty (1)\; Gam(\eta|\nu/2, \nu/2)\, \mathrm{d}\eta$$

$$= \int_0^\infty Gam(\eta|\nu/2, \nu/2)\, \mathrm{d}\eta$$

$$= 1$$