5.10 Equation 5.39 gives us:

$$\mathbf{v}^T\mathbf{H}\mathbf{v} = \sum_i c_i^2\lambda_i$$

Substituting an eigenvector $\mathbf{u}_j$ for $\mathbf{v}$, we get:

$$L.H.S = \mathbf{u}_j^T\mathbf{H}\mathbf{u}_j$$

$$= \mathbf{u}_j^T\lambda_j\mathbf{u}_j$$

$$= \lambda_j\mathbf{u}_j^T\mathbf{u}_j$$

$$= \lambda_j$$

If $\mathbf{H}$ is positive definite, then $\mathbf{u}_j^T\mathbf{H}\mathbf{u}_j > 0 \quad \forall j, \implies \lambda_j > 0 \quad \forall j$, and all eigenvalues are positive.

Conversely, if all eigenvalues are positive, $\sum_i c_i^2\lambda_i > 0, \implies \mathbf{v}^T\mathbf{H}\mathbf{v} > 0$, and therefore $\mathbf{H}$ is positive definite.