2.20 To prove necessity and sufficiency, we need to prove that $\mathbf{a}^T\boldsymbol{\Sigma}\mathbf{a} > 0 \iff \lambda_i > 0\,\forall i$.

$$\mathbf{a}^T\boldsymbol{\Sigma}\mathbf{a}$$

$$= \mathbf{a}^T\left(\sum_{i=1}^{D}\lambda_i\mathbf{u}_i\mathbf{u}_i^T\right)\mathbf{a}$$

$$= \sum_{i=1}^{D}\mathbf{a}^T\lambda_i\mathbf{u}_i\mathbf{u}_i^T\mathbf{a}$$

$$= \sum_{i=1}^{D}\lambda_i\mathbf{a}^T\mathbf{u}_i\mathbf{u}_i^T\mathbf{a}$$

$$= \sum_{i=1}^{D}\lambda_i(\mathbf{u}_i^T\mathbf{a})^T(\mathbf{u}_i^T\mathbf{a})$$

$\mathbf{u}_i^T\mathbf{a}$ is a scalar, so we get:

$$= \sum_{i=1}^{D}\lambda_i(\mathbf{u}_i^T\mathbf{a})^2$$

$(\mathbf{u}_i^T\mathbf{a})^2 \geq 0\,\forall\mathbf{a}$, so all the eigenvalues being positive is a sufficient condition for $\boldsymbol{\Sigma}$ being positive definite, and $\lambda_i > 0\,\forall i \implies \mathbf{a}^T\boldsymbol{\Sigma}\mathbf{a} > 0$.

Now to prove that eigenvalues being positive is a necessary condition for $\boldsymbol{\Sigma}$ being positive definite, we need to prove that $\mathbf{a}^T\boldsymbol{\Sigma}\mathbf{a} > 0$ implies that all eigenvalues are positive.

Let $\mathbf{a} = \mathbf{u}_j$.

$$\sum_{i=1}^{D}\lambda_i(\mathbf{u}_i^T\mathbf{a})^2 = \sum_{i=1}^{D}\lambda_i(\mathbf{u}_i^T\mathbf{u}_j)^2$$

Orthogonality of eigenvectors implies that or $i = j$, $\mathbf{u}_i^T\mathbf{u}_j = 1$ and for $i \neq j$, $\mathbf{u}_i^T\mathbf{u}_j = 0$.

$$\implies \sum_{i=1}^{D}\lambda_i(\mathbf{u}_i^T\mathbf{u}_j)^2 = \lambda_j$$

Since this is true for all j, we can say that $\mathbf{a}^T\mathbf{\Sigma}\mathbf{a} > 0 \implies \lambda_j > 0 \, \forall j$.