1.8 We are given:

$$\mathcal{N}(x|\mu,\sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}}\, exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

We need to prove that:

$$\int_{-\infty}^{\infty} \frac{1}{(2\pi\sigma^2)^{1/2}}\, exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} x\, dx = \mu$$

$$L.H.S = \frac{1}{(2\pi\sigma^2)^{1/2}} \int_{-\infty}^{\infty} exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} x\, dx$$

Let $y = x - \mu$. That gives us:

$$L.H.S = \frac{1}{(2\pi\sigma^2)^{1/2}} \int_{-\infty}^{\infty} exp\left\{-\frac{y^2}{2\sigma^2}\right\} (y+\mu)\, dy$$

$$= \left(\frac{1}{(2\pi\sigma^2)^{1/2}} \int_{-\infty}^{\infty} exp\left\{-\frac{y^2}{2\sigma^2}\right\} y\, dy\right) + \left(\frac{1}{(2\pi\sigma^2)^{1/2}} \int_{-\infty}^{\infty} exp\left\{-\frac{y^2}{2\sigma^2}\right\} \mu\, dy\right)$$

Lets simplify the integral inside first term, by substituting $z = y^2$.
We get this term:

$$\int exp\left\{-\frac{z}{2\sigma^2}\right\} \sqrt{z}\, dz \left(\frac{dy}{dz}\right)$$

$$= \int exp\left\{-\frac{z}{2\sigma^2}\right\} \sqrt{z}\, dz \left(\frac{1}{2\sqrt{z}}\right)$$

$$= \frac{1}{2} \int exp\left\{-\frac{z}{2\sigma^2}\right\} dz$$

$$= \frac{1}{2} (-2\sigma^2)\, exp\left\{-\frac{z}{2\sigma^2}\right\}$$

$$= -\sigma^2\, exp\left\{-\frac{z}{2\sigma^2}\right\}$$

Putting back $z = y^2$, we get

$$\left[-\sigma^2\, exp\left\{-\frac{y^2}{2\sigma^2}\right\}\right]_{-\infty}^{\infty}$$

$$= 0$$

since y^2 gives us the same value for every +a and -a from $-\infty$ to $+\infty$.

The second term becomes:

$$\frac{1}{(2\pi\sigma^2)^{1/2}}\,\mu \int_{-\infty}^{\infty} exp\left\{-\frac{y^2}{2\sigma^2}\right\} dy$$

This integral was evaluated in previous exercise, and gives us:

$$\frac{1}{(2\pi\sigma^2)^{1/2}}\,\mu\,(2\pi\sigma^2)^{1/2}$$

$$= \mu$$

Therefore, we proved that 1.46 satisfies 1.49.

Now we differentiate both sides of the normalization condition:

$$\int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2)\, dx = 1$$

$$\implies \int_{-\infty}^{\infty} exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx = (2\pi\sigma^2)^{1/2}$$

Taking derivative w.r.t σ^2 on both sides, we get:

$$\frac{d}{d\sigma^2}\int_{-\infty}^{\infty} exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx = \frac{d}{d\sigma^2}(2\pi\sigma^2)^{1/2}$$

$$\implies \int_{-\infty}^{\infty} \frac{d}{d\sigma^2}\, exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx = (2\pi)^{1/2}\frac{d(\sigma^2)^{1/2}}{d\sigma^2}$$

$$\implies \int_{-\infty}^{\infty} exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \left(-\frac{(x-\mu)^2}{2}\right) \left(-\frac{1}{(\sigma^2)^2}\right) dx = (2\pi)^{1/2} \frac{1}{2(\sigma^2)^{1/2}}$$

$$\implies \int_{-\infty}^{\infty} exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} (x-\mu)^2 \left(\frac{1}{\sigma^3}\right) dx = (2\pi)^{1/2}$$

$$\implies \int_{-\infty}^{\infty} exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} (x-\mu)^2 \, dx = (2\pi)^{1/2}\sigma^3$$

$$\implies \int_{-\infty}^{\infty} exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} (x-\mu)^2 \, dx = (2\pi\sigma^2)^{1/2}\sigma^2$$

$$\begin{gathered}
\implies \frac{1}{(2\pi\sigma^2)^{1/2}} \int_{-\infty}^{\infty} exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} x^2 \, dx \\
+\frac{1}{(2\pi\sigma^2)^{1/2}} \int_{-\infty}^{\infty} exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} (-2\mu x) \, dx \\
+\frac{1}{(2\pi\sigma^2)^{1/2}} \int_{-\infty}^{\infty} exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \mu^2 \, dx \\
= \sigma^2
\end{gathered}$$

$$\begin{gathered}
\implies \frac{1}{(2\pi\sigma^2)^{1/2}} \int_{-\infty}^{\infty} exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} x^2 \, dx \\
-2\mu \, \frac{1}{(2\pi\sigma^2)^{1/2}} \int_{-\infty}^{\infty} exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} x \, dx \\
+\mu^2 \, \frac{1}{(2\pi\sigma^2)^{1/2}} \int_{-\infty}^{\infty} exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx \\
= \sigma^2
\end{gathered}$$

$$\begin{gathered}
\implies \int_{-\infty}^{\infty} \mathcal{N}(x|\mu,\sigma^2) \, x^2 \, dx \\
-2\mu \int_{-\infty}^{\infty} \mathcal{N}(x|\mu,\sigma^2) \, x \, dx \\
+\mu^2 \int_{-\infty}^{\infty} \mathcal{N}(x|\mu,\sigma^2) \, dx \\
= \sigma^2
\end{gathered}$$

$$\implies \int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2)\, x^2 \, dx$$
$$-2\mu \int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2)\, x \, dx$$
$$+\mu^2 \int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2)\, dx$$
$$= \sigma^2$$

$$\implies \int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2)\, x^2 \, dx$$
$$-2\mu\,(\mu)$$
$$+\mu^2\,(1)$$
$$= \sigma^2$$

$$\implies \int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2)\, x^2 \, dx = 2\mu^2 - \mu^2 + \sigma^2$$
$$\implies \int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2)\, x^2 \, dx = \mu^2 + \sigma^2$$

1.50 is proved.

Now to prove 1.51:

$$E[x^2] - E[x]^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2.$$