

**3.8** The posterior distribution is given by:

$$p(\mathbf{w}|t_{N+1}, \mathbf{t}_N) \propto p(t_{N+1}|\mathbf{w}, \beta) * p(\mathbf{w}|\mathbf{t}_N)$$

$$\begin{aligned} R.H.S &= \frac{1}{\sqrt{2\pi\beta^{-1}}} \exp \left\{ -\frac{\beta}{2} (t_{N+1} - \mathbf{w}^T \phi(\mathbf{x}_{N+1}))^2 \right\} \\ &\frac{1}{(2\pi)^{D/2} |\mathbf{S}_N|^{(1/2)}} \exp \left\{ -\frac{1}{2} ((\mathbf{w} - \mathbf{m}_N)^T \mathbf{S}_N^{-1} (\mathbf{w} - \mathbf{m}_N)) \right\} \\ &= \frac{\beta^{(1/2)}}{(2\pi)^{(D+1)/2} |\mathbf{S}_N|^{(1/2)}} \exp \left\{ -\frac{\beta}{2} (t_{N+1} - \mathbf{w}^T \phi(\mathbf{x}_{N+1}))^2 \right\} \\ &\quad \exp \left\{ -\frac{1}{2} (\mathbf{w} - \mathbf{m}_N)^T \mathbf{S}_N^{-1} (\mathbf{w} - \mathbf{m}_N) \right\} \\ &= \frac{\beta^{(1/2)}}{(2\pi)^{(D+1)/2} |\mathbf{S}_N|^{(1/2)}} \exp \left\{ -\frac{\beta}{2} (t_{N+1}^2 + \mathbf{w}^T \phi(\mathbf{x}_{N+1}) \phi(\mathbf{x}_{N+1})^T \mathbf{w} - 2t_{N+1} \mathbf{w}^T \phi(\mathbf{x}_{N+1})) \right\} \\ &\quad \exp \left\{ -\frac{1}{2} (\mathbf{w}^T \mathbf{S}_N^{-1} \mathbf{w} - 2\mathbf{m}_N^T \mathbf{S}_N^{-1} \mathbf{w} + \mathbf{m}_N^T \mathbf{S}_N^{-1} \mathbf{m}_N) \right\} \end{aligned}$$

Rearranging the terms in the exponent, we get:

$$\begin{aligned} &= \frac{\beta^{(1/2)}}{(2\pi)^{(D+1)/2} |\mathbf{S}_N|^{(1/2)}} \\ &\exp \left\{ -\frac{1}{2} (\mathbf{w}^T (\mathbf{S}_N^{-1} + \beta \phi(\mathbf{x}_{N+1}) \phi(\mathbf{x}_{N+1})^T) \mathbf{w} - 2\mathbf{w}^T (\beta t_{N+1} \phi(\mathbf{x}_{N+1}) + \mathbf{S}_N^{-1} \mathbf{m}_N)) \right\} \\ &\quad \exp \left\{ -\frac{1}{2} (\beta t_{N+1}^2 + \mathbf{m}_N^T \mathbf{S}_N^{-1} \mathbf{m}_N) \right\} \end{aligned}$$

Simplifying the exponent of the second term:

$$\begin{aligned}
& \mathbf{w}^T (\mathbf{S}_N^{-1} + \beta \phi(\mathbf{x}_{N+1}) \phi(\mathbf{x}_{N+1})^T) \mathbf{w} - 2\mathbf{w}^T (\beta t_{N+1} \phi(\mathbf{x}_{N+1}) + \mathbf{S}_N^{-1} \mathbf{m}_N) \\
&= \mathbf{w}^T \left( \mathbf{S}_0^{-1} + \beta \Phi_N^T \Phi_N + \beta \phi(\mathbf{x}_{N+1}) \phi(\mathbf{x}_{N+1})^T \right) \mathbf{w} - 2\mathbf{w}^T (\beta t_{N+1} \phi(\mathbf{x}_{N+1}) + (\mathbf{S}_0^{-1} \mathbf{m}_0 + \beta \Phi_N^T \mathbf{t}_N)) \\
&= \mathbf{w}^T \left( \mathbf{S}_0^{-1} + \beta (\Phi_N^T \Phi_N + \phi(\mathbf{x}_{N+1}) \phi(\mathbf{x}_{N+1})^T) \right) \mathbf{w} - 2\mathbf{w}^T \left( \mathbf{S}_0^{-1} \mathbf{m}_0 + \beta (t_{N+1} \phi(\mathbf{x}_{N+1}) + \Phi_N^T \mathbf{t}_N) \right) \\
&= \mathbf{w}^T \left( \mathbf{S}_0^{-1} + \beta \Phi_{N+1}^T \Phi_{N+1} \right) \mathbf{w} - 2\mathbf{w}^T \left( \mathbf{S}_0^{-1} \mathbf{m}_0 + \beta \Phi_{N+1}^T \mathbf{t}_{N+1} \right) \\
&= \mathbf{w}^T \mathbf{S}_{N+1}^{-1} \mathbf{w} - 2\mathbf{w}^T \mathbf{S}_{N+1}^{-1} \mathbf{m}_{N+1}
\end{aligned}$$

Adding and subtracting  $\mathbf{m}_{N+1}^T \mathbf{S}_{N+1}^{-1} \mathbf{m}_{N+1}$ , the term becomes:

$$\begin{aligned}
&= \mathbf{w}^T \mathbf{S}_{N+1}^{-1} \mathbf{w} - 2\mathbf{w}^T \mathbf{S}_{N+1}^{-1} \mathbf{m}_{N+1} + \mathbf{m}_{N+1}^T \mathbf{S}_{N+1}^{-1} \mathbf{m}_{N+1} - \mathbf{m}_{N+1}^T \mathbf{S}_{N+1}^{-1} \mathbf{m}_{N+1} \\
&= (\mathbf{w} - \mathbf{m}_{N+1}) \mathbf{S}_{N+1}^{-1} (\mathbf{w} - \mathbf{m}_{N+1}) - \mathbf{m}_{N+1}^T \mathbf{S}_{N+1}^{-1} \mathbf{m}_{N+1}
\end{aligned}$$

Substituting this back into the expression for posterior distribution, we get:

$$\begin{aligned}
&= \frac{\beta^{(1/2)}}{(2\pi)^{(D+1)/2} |\mathbf{S}_N|^{(1/2)}} \\
&\exp \left\{ -\frac{1}{2} ((\mathbf{w} - \mathbf{m}_{N+1}) \mathbf{S}_{N+1}^{-1} (\mathbf{w} - \mathbf{m}_{N+1}) - \mathbf{m}_{N+1}^T \mathbf{S}_{N+1}^{-1} \mathbf{m}_{N+1}) \right\} \\
&\exp \left\{ -\frac{1}{2} (\beta t_{N+1}^2 + \mathbf{m}_N^T \mathbf{S}_N^{-1} \mathbf{m}_N) \right\}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\beta^{(1/2)}}{(2\pi)^{(D+1)/2} |\mathbf{S}_N|^{(1/2)}} \\
&\exp \left\{ -\frac{1}{2} ((\mathbf{w} - \mathbf{m}_{N+1}) \mathbf{S}_{N+1}^{-1} (\mathbf{w} - \mathbf{m}_{N+1})) \right\} \\
&\exp \left\{ -\frac{1}{2} (\beta t_{N+1}^2 + \mathbf{m}_N^T \mathbf{S}_N^{-1} \mathbf{m}_N - \mathbf{m}_{N+1}^T \mathbf{S}_{N+1}^{-1} \mathbf{m}_{N+1}) \right\}
\end{aligned}$$

Multiplying and dividing the fraction by  $|\mathbf{S}_{N+1}|^{(1/2)}$ , we get:

$$\begin{aligned}
&= \frac{\beta^{(1/2)} |\mathbf{S}_{N+1}|^{(1/2)}}{(2\pi)^{(D+1)/2} |\mathbf{S}_N|^{(1/2)} |\mathbf{S}_{N+1}|^{(1/2)}} \\
&\exp \left\{ -\frac{1}{2} ((\mathbf{w} - \mathbf{m}_{N+1}) \mathbf{S}_{N+1}^{-1} (\mathbf{w} - \mathbf{m}_{N+1})) \right\} \\
&\exp \left\{ -\frac{1}{2} (\beta t_{N+1}^2 + \mathbf{m}_N^T \mathbf{S}_N^{-1} \mathbf{m}_N - \mathbf{m}_{N+1}^T \mathbf{S}_{N+1}^{-1} \mathbf{m}_{N+1}) \right\} \\
&= \frac{1}{(2\pi)^{D/2} |\mathbf{S}_{N+1}|^{(1/2)}} \exp \left\{ -\frac{1}{2} ((\mathbf{w} - \mathbf{m}_{N+1}) \mathbf{S}_{N+1}^{-1} (\mathbf{w} - \mathbf{m}_{N+1})) \right\} \\
&\frac{\beta^{(1/2)} |\mathbf{S}_{N+1}|^{(1/2)}}{(2\pi)^{(D+1)/2} |\mathbf{S}_N|^{(1/2)}} \exp \left\{ -\frac{1}{2} (\beta t_{N+1}^2 + \mathbf{m}_N^T \mathbf{S}_N^{-1} \mathbf{m}_N - \mathbf{m}_{N+1}^T \mathbf{S}_{N+1}^{-1} \mathbf{m}_{N+1}) \right\} \\
&= c \mathcal{N}(\mathbf{w} | \mathbf{m}_{N+1}, \mathbf{S}_{N+1})
\end{aligned}$$

where c is the constant second term.