4.6 4.33 gives us:

$$\sum_{n=1}^{N}(\mathbf{w}^T\mathbf{x}_n + w_0 - t_n)\mathbf{x}_n = 0$$

$$\implies \sum_{n=1}^{N}(\mathbf{w}^T\mathbf{x}_n)\mathbf{x}_n + \sum_{n=1}^{N} w_0\mathbf{x}_n - \sum_{n=1}^{N} t_n\mathbf{x}_n = 0$$

Simplifying these terms separately:

First term:

$$\sum_{n=1}^{N}(\mathbf{w}^T\mathbf{x}_n)\mathbf{x}_n$$

$$= \sum_{n=1}^{N}\mathbf{x}_n(\mathbf{w}^T\mathbf{x}_n)$$

$$= \sum_{n=1}^{N}\mathbf{x}_n(\mathbf{x}_n^T\mathbf{w})$$

$$= \sum_{n=1}^{N}(\mathbf{x}_n\mathbf{x}_n^T)\mathbf{w}$$

Using 4.28, we can expressing $\sum_{n=1}^{N}\mathbf{x}_n\mathbf{x}_n^T$ in terms of $\mathbf{S}_W$:

$$\mathbf{S}_W = \sum_{n\in\mathcal{C}_1}(\mathbf{x}_n - \mathbf{m}_1)(\mathbf{x}_n - \mathbf{m}_1)^T + \sum_{n\in\mathcal{C}_2}(\mathbf{x}_n - \mathbf{m}_2)(\mathbf{x}_n - \mathbf{m}_2)^T$$

$$= \sum_{n\in\mathcal{C}_1}\left(\mathbf{x}_n\mathbf{x}_n^T - \mathbf{x}_n\mathbf{m}_1^T - \mathbf{m}_1\mathbf{x}_n^T + \mathbf{m}_1\mathbf{m}_1^T\right)$$
$$+ \sum_{n\in\mathcal{C}_2}\left(\mathbf{x}_n\mathbf{x}_n^T - \mathbf{x}_n\mathbf{m}_2^T - \mathbf{m}_2\mathbf{x}_n^T + \mathbf{m}_2\mathbf{m}_2^T\right)$$

$$= \left(\sum_{n\in\mathcal{C}_1}\mathbf{x}_n\mathbf{x}_n^T\right) + \left(-N_1\mathbf{m}_1\mathbf{m}_1^T - N_1\mathbf{m}_1\mathbf{m}_1^T + N_1\mathbf{m}_1\mathbf{m}_1^T\right)$$

$$+\left(\sum_{n\in\mathcal{C}_2}\mathbf{x}_n\mathbf{x}_n^T\right)+\left(-N_2\mathbf{m}_2\mathbf{m}_2^T-N_2\mathbf{m}_2\mathbf{m}_2^T+N_2\mathbf{m}_2\mathbf{m}_2^T\right)$$

$$=\left(\sum_{n=1}^{N}\mathbf{x}_n\mathbf{x}_n^T\right)-N_1\mathbf{m}_1\mathbf{m}_1^T-N_2\mathbf{m}_2\mathbf{m}_2^T$$

$$\implies\sum_{n=1}^{N}\mathbf{x}_n\mathbf{x}_n^T=\mathbf{S}_W+N_1\mathbf{m}_1\mathbf{m}_1^T+N_2\mathbf{m}_2\mathbf{m}_2^T$$

Therefore, the first term becomes:

$$=(\mathbf{S}_W+N_1\mathbf{m}_1\mathbf{m}_1^T+N_2\mathbf{m}_2\mathbf{m}_2^T)\mathbf{w}$$

Second term:

$$\sum_{n=1}^{N}w_0\mathbf{x}_n$$

$$=w_0\sum_{n=1}^{N}\mathbf{x}_n$$

$$=w_0N\mathbf{m}$$

$$=-N(\mathbf{w}^T\mathbf{m})\mathbf{m}\qquad\text{using 4.34}$$

$$=-N\mathbf{m}(\mathbf{m}^T\mathbf{w})$$

$$=-N\left(\frac{N_1}{N}\mathbf{m}_1+\frac{N_2}{N}\mathbf{m}_2\right)\left(\frac{N_1}{N}\mathbf{m}_1+\frac{N_2}{N}\mathbf{m}_2\right)^T\mathbf{w}\qquad\text{using 4.36}$$

$$=-\frac{1}{N}(N_1\mathbf{m}_1+N_2\mathbf{m}_2)(N_1\mathbf{m}_1+N_2\mathbf{m}_2)^T\mathbf{w}$$

$$=-\frac{1}{N}\left(N_1^2\mathbf{m}_1\mathbf{m}_1^T+N_1N_2(\mathbf{m}_1\mathbf{m}_2^T+\mathbf{m}_2\mathbf{m}_1^T)+N_2^2\mathbf{m}_2\mathbf{m}_2^T\right)\mathbf{w}$$

Here, we can apply 4.27 to utilize $\mathbf{S}_B$, as

$$\mathbf{S}_B = (\mathbf{m}_2 - \mathbf{m}_1)(\mathbf{m}_2 - \mathbf{m}_1)^T$$

$$= \mathbf{m}_2\mathbf{m}_2^T - \mathbf{m}_2\mathbf{m}_1^T - \mathbf{m}_1\mathbf{m}_2^T + \mathbf{m}_1\mathbf{m}_1^T$$

$$\implies \mathbf{m}_1\mathbf{m}_2^T + \mathbf{m}_2\mathbf{m}_1^T = \mathbf{m}_1\mathbf{m}_1^T + \mathbf{m}_2\mathbf{m}_2^T - \mathbf{S}_B$$

Substituting into the expression for the second term, we get:

$$= -\frac{1}{N}((N_1^2 + N_1N_2)\mathbf{m}_1\mathbf{m}_1^T + (N_2^2 + N_1N_2)\mathbf{m}_2\mathbf{m}_2^T - N_1N_2\mathbf{S}_B)\mathbf{w}$$

$$= -\frac{1}{N}((N_1N)\mathbf{m}_1\mathbf{m}_1^T + (N_2N)\mathbf{m}_2\mathbf{m}_2^T - N_1N_2\mathbf{S}_B)\mathbf{w}$$

$$= \left(-N_1\mathbf{m}_1\mathbf{m}_1^T - N_2\mathbf{m}_2\mathbf{m}_2^T + \frac{N_1N_2}{N}\mathbf{S}_B\right)\mathbf{w}$$

Third term:

$$\sum_{n=1}^{N} t_n\mathbf{x}_n$$

$$= \sum_{n\in\mathcal{C}_1} t_n\mathbf{x}_n + \sum_{n\in\mathcal{C}_2} t_n\mathbf{x}_n$$

$$= \sum_{n\in\mathcal{C}_1} \frac{N}{N_1}\mathbf{x}_n - \sum_{n\in\mathcal{C}_2} \frac{N}{N_2}\mathbf{x}_n$$

$$= N(\mathbf{m}_1 - \mathbf{m}_2)$$

Substituting the simplifications of the three terms into 4.33:

$$\Longrightarrow \left(\mathbf{S}_W + N_1\mathbf{m}_1\mathbf{m}_1^T + N_2\mathbf{m}_2\mathbf{m}_2^T\right)\mathbf{w}$$
$$+\left(-N_1\mathbf{m}_1\mathbf{m}_1^T - N_2\mathbf{m}_2\mathbf{m}_2^T + \frac{N_1N_2}{N}\mathbf{S}_B\right)\mathbf{w} = N(\mathbf{m}_1 - \mathbf{m}_2)$$

$$\Longrightarrow \left(\mathbf{S}_W + N_1\mathbf{m}_1\mathbf{m}_1^T + N_2\mathbf{m}_2\mathbf{m}_2^T - N_1\mathbf{m}_1\mathbf{m}_1^T - N_2\mathbf{m}_2\mathbf{m}_2^T + \frac{N_1N_2}{N}\mathbf{S}_B\right)\mathbf{w} = N(\mathbf{m}_1 - \mathbf{m}_2)$$

$$\Longrightarrow \left(\mathbf{S}_W + \frac{N_1N_2}{N}\mathbf{S}_B\right)\mathbf{w} = N(\mathbf{m}_1 - \mathbf{m}_2)$$

which is the same as 4.37.