5.39 We have to apply 2.115 to marginalize out $\mathbf{w}$ in 5.174.

Using 5.162,

$$p(\mathbf{w}|\alpha) = \mathcal{N}(\mathbf{w}|\mathbf{0}, \alpha^{-1}\mathbf{I})$$

And using 5.163,

$$p(\mathcal{D}|\mathbf{w}, \beta) = \prod_{n=1}^{N} \mathcal{N}(t_n|y(\mathbf{x}, \mathbf{w}), \beta^{-1})$$

Assuming that the mode is at $\mathbf{w}_{MAP}$, and identifying that

$$f(\mathbf{w}) = \left(\prod_{n=1}^{N} \mathcal{N}(t_n|y(\mathbf{x}, \mathbf{w}), \beta^{-1})\right) \left(\mathcal{N}(\mathbf{w}|\mathbf{0}, \alpha^{-1}\mathbf{I})\right)$$

and

$$Z = p(\mathcal{D}|\alpha, \beta)$$

Applying 4.135, we get:

$$p(\mathcal{D}|\alpha, \beta) \simeq \left(\prod_{n=1}^{N} \mathcal{N}(t_n|y(\mathbf{x}, \mathbf{w}_{MAP}), \beta^{-1})\right) \left(\mathcal{N}(\mathbf{w}_{MAP}|\mathbf{0}, \alpha^{-1}\mathbf{I})\right) \frac{(2\pi)^{W/2}}{|\mathbf{A}|^{1/2}}$$

$$\implies \ln p(\mathcal{D}|\alpha, \beta) \simeq \ln \left(\left(\prod_{n=1}^{N} \mathcal{N}(t_n|y(\mathbf{x}, \mathbf{w}_{MAP}), \beta^{-1})\right) \right.$$
$$\left. \left(\mathcal{N}(\mathbf{w}_{MAP}|\mathbf{0}, \alpha^{-1}\mathbf{I})\right) \frac{(2\pi)^{W/2}}{|\mathbf{A}|^{1/2}} \right)$$

$$= \left(\sum_{n=1}^{N} \ln \mathcal{N}(t_n|y(\mathbf{x}, \mathbf{w}_{MAP}), \beta^{-1})\right)$$
$$+ \left(\ln \mathcal{N}(\mathbf{w}_{MAP}|\mathbf{0}, \alpha^{-1}\mathbf{I})\right) + \frac{W}{2}\ln(2\pi) - \frac{1}{2}\ln|\mathbf{A}|$$

$$
\begin{aligned}
&= \sum_{n=1}^{N} \ln\left(\frac{1}{(2\pi)^{1/2}(\beta^{-1})^{1/2}} \exp\left\{-\frac{\beta\,(t_n - y(\mathbf{x}, \mathbf{w}_{MAP}))^2}{2}\right\}\right) \\
&+ \ln\left(\frac{1}{(2\pi)^{W/2}|\alpha^{-1}\mathbf{I}|^{1/2}} \exp\left\{-\frac{\alpha \mathbf{w}_{MAP}^T \mathbf{w}_{MAP}}{2}\right\}\right) + \frac{W}{2}\ln(2\pi) - \frac{1}{2}\ln|\mathbf{A}|
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=1}^{N} \left(-\frac{N}{2}\ln(2\pi) + \frac{1}{2}\ln\beta - \frac{\beta\,(t_n - y(\mathbf{x}, \mathbf{w}_{MAP}))^2}{2}\right) \\
&-\frac{W}{2}\ln(2\pi) + \frac{W}{2}\ln\alpha - \frac{\alpha \mathbf{w}_{MAP}^T \mathbf{w}_{MAP}}{2} + \frac{W}{2}\ln(2\pi) - \frac{1}{2}\ln|\mathbf{A}|
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{N}{2}\ln(2\pi) + \frac{N}{2}\ln\beta - \sum_{n=1}^{N} \frac{\beta\,(t_n - y(\mathbf{x}, \mathbf{w}_{MAP}))^2}{2} \\
&+\frac{W}{2}\ln\alpha - \frac{\alpha \mathbf{w}_{MAP}^T \mathbf{w}_{MAP}}{2} - \frac{1}{2}\ln|\mathbf{A}|
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\frac{\beta}{2}\sum_{n=1}^{N}(t_n - y(\mathbf{x}, \mathbf{w}_{MAP}))^2 + \frac{\alpha}{2}\mathbf{w}_{MAP}^T \mathbf{w}_{MAP}\right) \\
&-\frac{N}{2}\ln(2\pi) + \frac{N}{2}\ln\beta + \frac{W}{2}\ln\alpha - \frac{1}{2}\ln|\mathbf{A}|
\end{aligned}
$$

which is the same as the result in 5.175.