2.5 We are given:

$$\Gamma(a)\Gamma(b) = \int_0^\infty exp(-x)x^{a-1}dx \int_0^\infty exp(-y)y^{b-1}dy$$

Bringing the integral over y inside the integrand of the integral over x:

$$\Gamma(a)\Gamma(b) = \int_0^\infty \int_0^\infty exp(-x)x^{a-1}\, exp(-y)y^{b-1}dy\, dx$$

$$\Gamma(a)\Gamma(b) = \int_0^\infty \int_0^\infty exp(-(x+y))x^{a-1}\, y^{b-1}dy\, dx$$

Making the change of variable t = y + x where x is fixed. For a fixed x, t ranges from x to ∞, giving us:

$$\Gamma(a)\Gamma(b) = \int_0^\infty \int_x^\infty exp(-t)\, x^{a-1}\, (t-x)^{b-1}dt\, dx$$

Interchanging the order of the x and t integrations, we need to find the limits for x and t. Source to understand this change in detail : Youtube video

The absolute range of t is 0 to ∞. For a fixed t, x ranges from 0 to t, giving us:

$$\Gamma(a)\Gamma(b) = \int_0^\infty \int_0^t exp(-t)\, x^{a-1}\, (t-x)^{b-1}dx\, dt$$

Making the change of variable x = tμ where t is fixed. The limits of μ become 0 to 1, giving us:

$$\Gamma(a)\Gamma(b) = \int_0^\infty \int_0^1 exp(-t)\, (t\mu)^{a-1}\, (t-t\mu)^{b-1}d(t\mu)\, dt$$

$$= \int_0^\infty \int_0^1 exp(-t)\, (t\mu)^{a-1}\, (t-t\mu)^{b-1}\, t\, d\mu\, dt$$

$$= \int_0^\infty \int_0^1 exp(-t)\, t^a \mu^{a-1}\, t^{b-1}(1-\mu)^{b-1}d\mu\, dt$$

$$= \int_0^\infty \int_0^1 exp(-t)\, t^{a+b-1}\mu^{a-1}\, (1-\mu)^{b-1}d\mu\, dt$$

$$= \int_0^\infty exp(-t)\, t^{a+b-1}\, dt \int_0^1 \mu^{a-1}\, (1-\mu)^{b-1}\, d\mu$$

$$\Longrightarrow \Gamma(a)\Gamma(b) = \Gamma(a+b) \int_0^1 \mu^{a-1} (1-\mu)^{b-1} \, d\mu$$

$$\Longrightarrow \int_0^1 \mu^{a-1} (1-\mu)^{b-1} \, d\mu = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$