3.8 The posterior distribution is given by:

$$p(\mathbf{w}|\mathbf{t}_{N+1}, \mathbf{t}_N) \propto p(t_{N+1}|\mathbf{w}, \beta) * p(\mathbf{w}|\mathbf{t}_N)$$

$$R.H.S = \frac{1}{\sqrt{2\pi\beta^{-1}}} \exp\left\{-\frac{\beta}{2}(t_{N+1} - \mathbf{w}^T\phi(\mathbf{x}_{N+1}))^2\right\}$$
$$\frac{1}{(2\pi)^{D/2}|\mathbf{S}_N|^{(1/2)}} \exp\left\{-\frac{1}{2}\left((\mathbf{w}-\mathbf{m}_N)^T\mathbf{S}_N^{-1}(\mathbf{w}-\mathbf{m}_N)\right)\right\}$$

$$= \frac{\beta^{(1/2)}}{(2\pi)^{(D+1)/2}|\mathbf{S}_N|^{(1/2)}} \exp\left\{-\frac{\beta}{2}(t_{N+1} - \mathbf{w}^T\phi(\mathbf{x}_{N+1}))^2\right\}$$
$$\exp\left\{-\frac{1}{2}(\mathbf{w}-\mathbf{m}_N)^T\mathbf{S}_N^{-1}(\mathbf{w}-\mathbf{m}_N)\right\}$$

$$= \frac{\beta^{(1/2)}}{(2\pi)^{(D+1)/2}|\mathbf{S}_N|^{(1/2)}} \exp\left\{-\frac{\beta}{2}\left(t_{N+1}^2 + \mathbf{w}^T\phi(\mathbf{x}_{N+1})\phi(\mathbf{x}_{N+1})^T\mathbf{w} - 2t_{N+1}\mathbf{w}^T\phi(\mathbf{x}_{N+1})\right)\right\}$$
$$\exp\left\{-\frac{1}{2}\left(\mathbf{w}^T\mathbf{S}_N^{-1}\mathbf{w} - 2\mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{w} + \mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{m}_N\right)\right\}$$

Rearranging the terms in the exponent, we get:

$$= \frac{\beta^{(1/2)}}{(2\pi)^{(D+1)/2}|\mathbf{S}_N|^{(1/2)}}$$

$$\exp\left\{-\frac{1}{2}\left(\mathbf{w}^T(\mathbf{S}_N^{-1} + \beta\phi(\mathbf{x}_{N+1})\phi(\mathbf{x}_{N+1})^T)\mathbf{w} - 2\mathbf{w}^T(\beta t_{N+1}\phi(\mathbf{x}_{N+1}) + \mathbf{S}_N^{-1}\mathbf{m}_N)\right)\right\}$$

$$\exp\left\{-\frac{1}{2}\left(\beta t_{N+1}^2 + \mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{m}_N\right)\right\}$$

Simplifying the exponent of the second term:

$$\mathbf{w}^T(\mathbf{S}_N^{-1} + \beta\phi(\mathbf{x}_{N+1})\phi(\mathbf{x}_{N+1})^T)\mathbf{w} - 2\mathbf{w}^T(\beta t_{N+1}\phi(\mathbf{x}_{N+1}) + \mathbf{S}_N^{-1}\mathbf{m}_N)$$

$$= \mathbf{w}^T\left(\mathbf{S}_0^{-1} + \beta\boldsymbol{\Phi_N}^T\boldsymbol{\Phi_N} + \beta\phi(\mathbf{x}_{N+1})\phi(\mathbf{x}_{N+1})^T)\mathbf{w} - 2\mathbf{w}^T(\beta t_{N+1}\phi(\mathbf{x}_{N+1}) + (\mathbf{S}_0^{-1}\mathbf{m}_0 + \beta\boldsymbol{\Phi_N}^T\mathbf{t}_N)\right)$$

$$= \mathbf{w}^T\left(\mathbf{S}_0^{-1} + \beta(\boldsymbol{\Phi_N}^T\boldsymbol{\Phi_N} + \phi(\mathbf{x}_{N+1})\phi(\mathbf{x}_{N+1})^T)\right)\mathbf{w} - 2\mathbf{w}^T\left(\mathbf{S}_0^{-1}\mathbf{m}_0 + \beta(t_{N+1}\phi(\mathbf{x}_{N+1}) + \boldsymbol{\Phi_N}^T\mathbf{t}_N)\right)$$

$$= \mathbf{w}^T\left(\mathbf{S}_0^{-1} + \beta\boldsymbol{\Phi_{N+1}}^T\boldsymbol{\Phi_{N+1}}\right)\mathbf{w} - 2\mathbf{w}^T\left(\mathbf{S}_0^{-1}\mathbf{m}_0 + \beta\boldsymbol{\Phi_{N+1}}^T\mathbf{t}_{N+1}\right)$$

$$= \mathbf{w}^T\mathbf{S}_{N+1}^{-1}\mathbf{w} - 2\mathbf{w}^T\mathbf{S}_{N+1}^{-1}\mathbf{m}_{N+1}$$

Adding and subtracting $\mathbf{m}_{N+1}^T\mathbf{S}_{N+1}^{-1}\mathbf{m}_{N+1}$, the term becomes:

$$= \mathbf{w}^T\mathbf{S}_{N+1}^{-1}\mathbf{w} - 2\mathbf{w}^T\mathbf{S}_{N+1}^{-1}\mathbf{m}_{N+1} + \mathbf{m}_{N+1}^T\mathbf{S}_{N+1}^{-1}\mathbf{m}_{N+1} - \mathbf{m}_{N+1}^T\mathbf{S}_{N+1}^{-1}\mathbf{m}_{N+1}$$

$$= (\mathbf{w} - \mathbf{m}_{N+1})\mathbf{S}_{N+1}^{-1}(\mathbf{w} - \mathbf{m}_{N+1}) - \mathbf{m}_{N+1}^T\mathbf{S}_{N+1}^{-1}\mathbf{m}_{N+1}$$

Substituting this back into the expression for posterior distribution, we get:

$$= \frac{\beta^{(1/2)}}{(2\pi)^{(D+1)/2}|\mathbf{S}_N|^{(1/2)}}$$

$$\exp\left\{-\frac{1}{2}\left((\mathbf{w} - \mathbf{m}_{N+1})\mathbf{S}_{N+1}^{-1}(\mathbf{w} - \mathbf{m}_{N+1}) - \mathbf{m}_{N+1}^T\mathbf{S}_{N+1}^{-1}\mathbf{m}_{N+1}\right)\right\}$$

$$\exp\left\{-\frac{1}{2}\left(\beta t_{N+1}^2 + \mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{m}_N\right)\right\}$$

$$= \frac{\beta^{(1/2)}}{(2\pi)^{(D+1)/2}|\mathbf{S}_N|^{(1/2)}}$$

$$\exp\left\{-\frac{1}{2}\left((\mathbf{w}-\mathbf{m}_{N+1})\mathbf{S}_{N+1}^{-1}(\mathbf{w}-\mathbf{m}_{N+1})\right)\right\}$$

$$\exp\left\{-\frac{1}{2}\left(\beta t_{N+1}^2+\mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{m}_N-\mathbf{m}_{N+1}^T\mathbf{S}_{N+1}^{-1}\mathbf{m}_{N+1}\right)\right\}$$

Multiplying and dividing the fraction by $|\mathbf{S}_{N+1}|^{(1/2)}$, we get:

$$= \frac{\beta^{(1/2)}|\mathbf{S}_{N+1}|^{(1/2)}}{(2\pi)^{(D+1)/2}|\mathbf{S}_N|^{(1/2)}|\mathbf{S}_{N+1}|^{(1/2)}}$$

$$\exp\left\{-\frac{1}{2}\left((\mathbf{w}-\mathbf{m}_{N+1})\mathbf{S}_{N+1}^{-1}(\mathbf{w}-\mathbf{m}_{N+1})\right)\right\}$$

$$\exp\left\{-\frac{1}{2}\left(\beta t_{N+1}^2+\mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{m}_N-\mathbf{m}_{N+1}^T\mathbf{S}_{N+1}^{-1}\mathbf{m}_{N+1}\right)\right\}$$

$$= \frac{1}{(2\pi)^{D/2}|\mathbf{S}_{N+1}|^{(1/2)}}\exp\left\{-\frac{1}{2}\left((\mathbf{w}-\mathbf{m}_{N+1})\mathbf{S}_{N+1}^{-1}(\mathbf{w}-\mathbf{m}_{N+1})\right)\right\}$$

$$\frac{\beta^{(1/2)}|\mathbf{S}_{N+1}|^{(1/2)}}{(2\pi)^{(D+1)/2}|\mathbf{S}_N|^{(1/2)}}\exp\left\{-\frac{1}{2}\left(\beta t_{N+1}^2+\mathbf{m}_N^T\mathbf{S}_N^{-1}\mathbf{m}_N-\mathbf{m}_{N+1}^T\mathbf{S}_{N+1}^{-1}\mathbf{m}_{N+1}\right)\right\}$$

$$= c\mathcal{N}\left(\mathbf{w}|\mathbf{m}_{N+1},\mathbf{S}_{N+1}\right)$$

where c is the constant second term.