Robotics and Automation

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Chapter Three

Spatial Description and Transformation

Notations

1. Usually, variables written in uppercase represent vectors or matrices. Lowercase variables are scalars.
2. Leading subscripts and superscripts identify which coordinate system a quantity is written in. For example, ${}^{A}P$ represents a position vector written in coordinate system $\{A\}$, and ${}^{A}_{B}R$ is a rotation matrix[3] that specifies the relationship between coordinate systems $\{A\}$ and $\{B\}$.
3. Trailing superscripts are used (as widely accepted) for indicating the inverse or transpose of a matrix (e.g., R^{-1}, R^{T}).
4. Trailing subscripts are not subject to any strict convention but may indicate a vector component (e.g., x, y, or z) or may be used as a description—as in P_{bolt}, the position of a bolt.
5. We will use many trigonometric functions. Our notation for the cosine of an angle θ_1 may take any of the following forms: $\cos\theta_1 = c\theta_1 = c_1$.

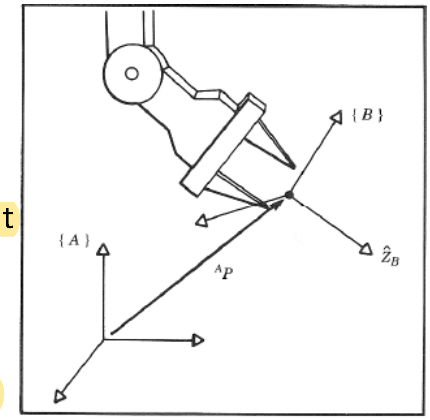
Vectors are taken to be column vectors; hence, row vectors will have the transpose indicated explicitly.

Basic Principles and Terminologies

- Orthogonal and Orthonormal vectors.
- Dot product and projection.
- Cross product.
- Free vector.
- Matrix operations are associative but not commutative.
- Identity matrix.
- Transpose and inverse of a matrix.
- Orthogonal and Orthonormal matrix.
- Trace of a matrix.
- The determinant of a matrix.
- Singular matrix.
- Trigonometric functions.

Spatial Description and Transformation

- **Problem:** Robotic manipulation, by definition, implies that parts and tools will be moving around in space by the manipulator mechanism. This naturally leads to the need of representing positions and orientations of the parts, tools, and the mechanism it self.
- **Solution:** Mathematical tools for representing position and orientation of objects / frames in a 3Dspace.



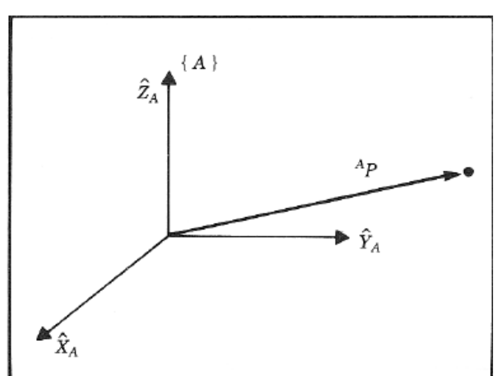
Description of a Position

- The location of any point can be described as a 3x1 ***position vector in a*** reference coordinate system.

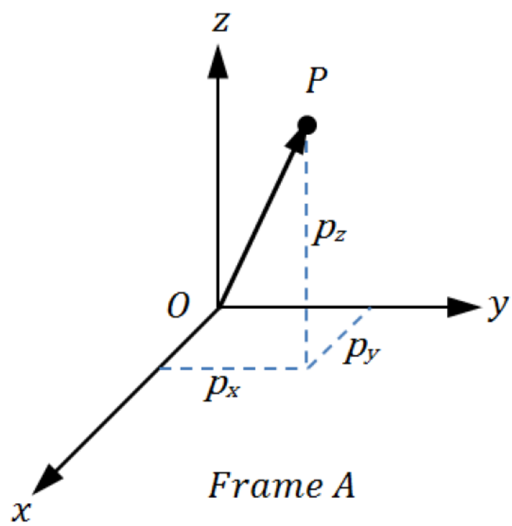
Coordinate System

$$^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

Position vector



Description of a Position



Description of an Orientation

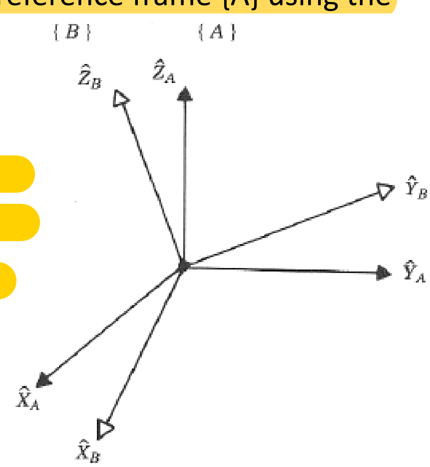
- The orientation of a body is described by attaching a coordinate system to the body {B} and then defining the relationship between the body frame and the reference frame {A} using the rotation matrix.

Reference Frame

$$ {}^{A}_{B}R = [{}^{A}\hat{X}_{B}, {}^{A}\hat{Y}_{B}, {}^{A}\hat{Z}_{B}] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

Body Frame

The rotation matrix describing frame {B} relative to frame {A}



Description of an Orientation

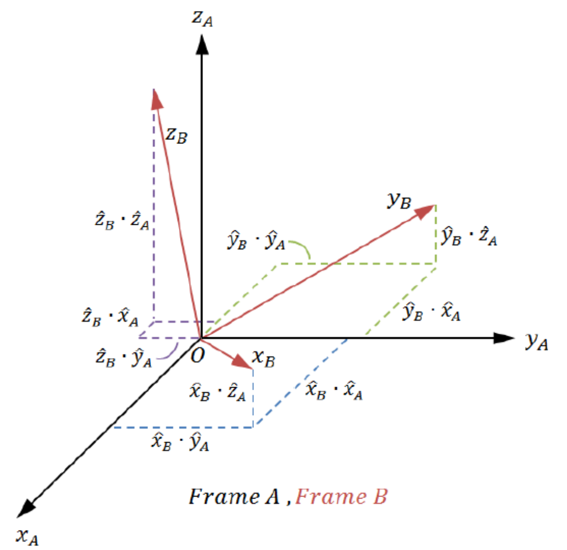
- In summary, a set of three vectors may be used to specify an orientation. For convenience, we will construct a 3 x 3 matrix that has these three vectors as its columns. Hence, whereas the position of a point is represented with a vector, the orientation of a body is represented with a matrix.
- The rotation matrix is built by projecting the axes of the coordinate system {B} onto the axes of coordinate system {A}. **Recalling that the dot product of two unit vectors gives the projection of one onto the other, we obtain:**

Description of an Orientation

$$ {}^A_B R = \left[{}^A\hat{X}_B \quad {}^A\hat{Y}_B \quad {}^A\hat{Z}_B \right] = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}. $$

- The dot product of two unit vectors yields the cosine of the angle between them, so it is clear why the components of rotation matrices are often referred to as **direction cosines**.
- The columns of ${}^A_B R$ specify the direction cosines of the {B} coordinate axes relative to the {A} coordinate axes.

Description of an Orientation



Description of an Orientation

- Further inspection on the rotation matrix shows that the rows of the matrix are the unit vectors of {A} expressed in {B}; that is,

$$ {}^{A}_{B}R = \left[\ {}^{A}\hat{X}_B \quad {}^{A}\hat{Y}_B \quad {}^{A}\hat{Z}_B\ \right] = \begin{bmatrix} {}^{B}\hat{X}^T_A \\ {}^{B}\hat{Y}^T_A \\ {}^{B}\hat{Z}^T_A \end{bmatrix} $$

$$ {}^{B}_{A}R = {}^{A}_{B}R^T $$

${}^{B}_{A}R$, the description of frame $\{A\}$ relative to $\{B\}$

Note: The dot product is commutative operation.

Description of an Orientation

- Additionally, it can be noticed that,

$$ {}^{A}_{B}R^{T}\,{}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}^{T}_{B} \\ {}^{A}\hat{Y}^{T}_{B} \\ {}^{A}\hat{Z}^{T}_{B} \end{bmatrix} \left[{}^{A}\hat{X}_{B} \;\; {}^{A}\hat{Y}_{B} \;\; {}^{A}\hat{Z}_{B} \right] = I_3, $$

- Hence,

$$ {}^{A}_{B}R = {}^{B}_{A}R^{-1} = {}^{B}_{A}R^{T} $$

Description of an Orientation

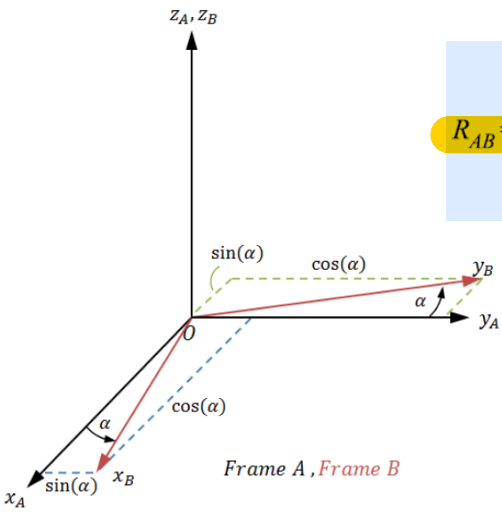
Rotation Matrix Properties

1. All the columns of a rotation matrix are orthogonal to each other.
2. The determinant of a rotation matrix is 1.
3. The inverse of a rotation matrix is equal to its transpose.

Description of an Orientation

Rotation about **Z axis**

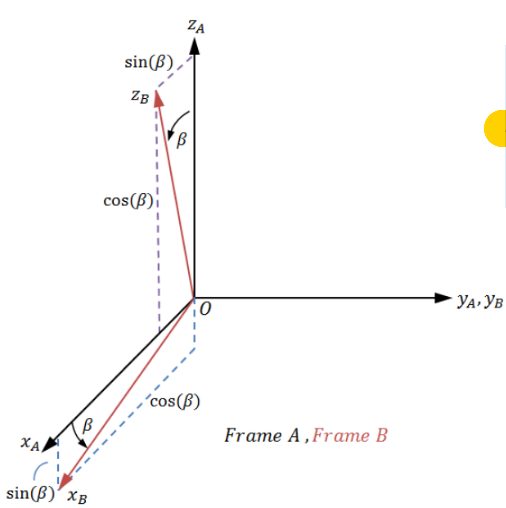
$$R_{AB} = R_z(\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Description of an Orientation

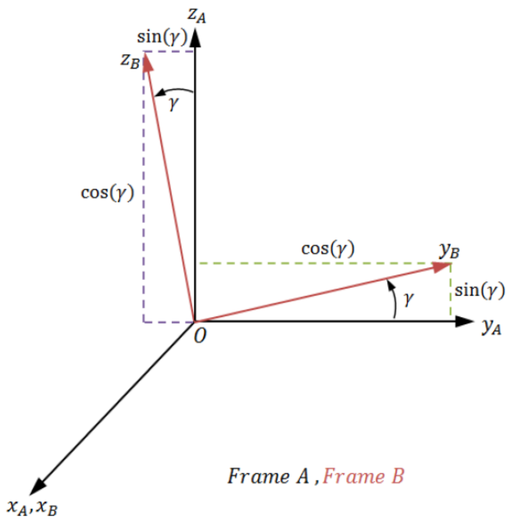
Rotation about **Y axis**

$$R_{AB} = R_y(\beta) = \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}$$



Description of an Orientation

Rotation about X axis



$$R_{AB} = R_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{bmatrix}$$

Description of a Frame

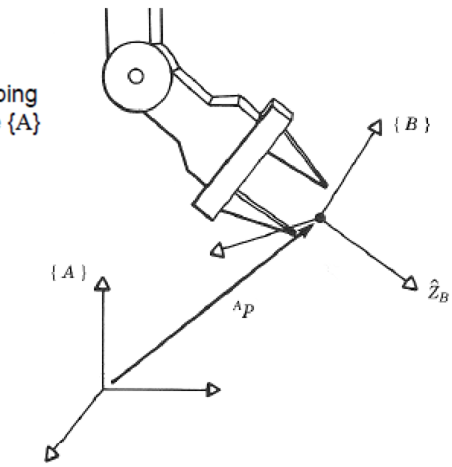
- The information needed to completely specify where is the manipulator hand is a position and an orientation.
- The point on the body whose position we describe could be chosen arbitrarily. However, for convenience, the point whose position we will describe is chosen as the origin of the body-attached frame.
- The situation of a position and an orientation pair arises so often in robotics that we define an entity called a **frame**, which is a set of four vectors giving position and orientation information.
- Note that a frame is a coordinate system where, in addition to the orientation, we give a position vector which locates its origin relative to some other embedding frame.

Description of a Frame

The rotation matrix describing frame {B} relative to frame {A}

$$\{B\} = \{{}^{A}_{B}R, {}^{A}P_{BORG}\}$$

The origin of frame {B} relative to frame {A}



Mappings: Changing Description from Frame to Frame

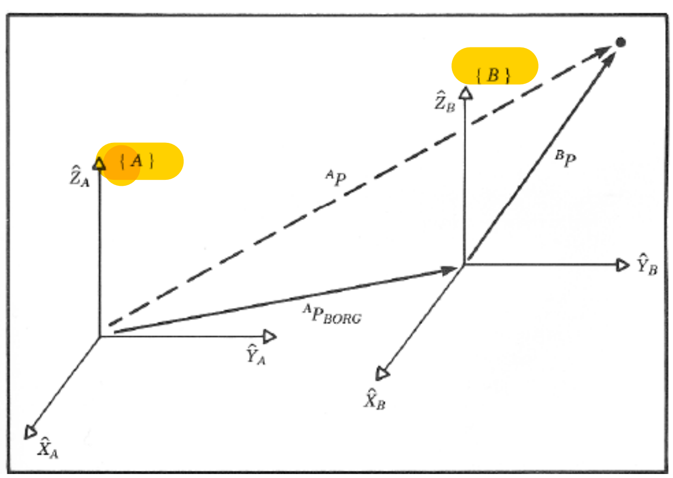
- In robotics, it is required to express the same quantity in terms of various reference coordinate systems.
- The previous section introduced descriptions of positions, orientations, and frames. Now, the mathematics of mapping is considered in order to change descriptions from frame to frame.

Mapping: Translated Frames

- In the following Figure, we have a position defined by the vector ^{B}P. We wish to express this point in space in terms of frame {A}, when {A} has the same orientation as {B}. In this case, {B} differs from {A} only by a translation, which is given by $^{A}P_{BORG}$, a vector that locates the origin of {B} relative to {A}.
- Because both vectors are defined relative to frames of the same orientation, we calculate the description of point *P* relative to {A}, ^{A}P, by vector addition.

Mapping: Translated Frames

$$^{A}P = {}^{B}P + {}^{A}P_{BORG}$$

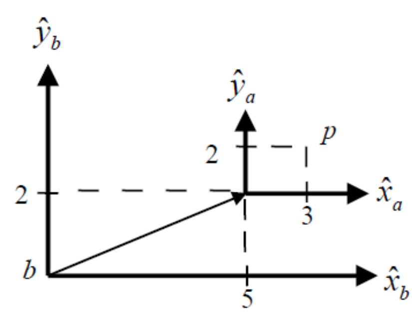


Mapping: Translated Frames

- The quantity itself (here, a point in space) is not changed; only its description is changed. i.e. The point described by ${}^{B}P$ is not translated, but remains the same, and instead we have computed a new description of the same point, but now with respect to system {A}.
- The vector ${}^{A}P_{BORG}$ defines this mapping because all the information needed to perform the change in description is contained in in this vector.
- Note: in this case ${}^{A}_{B}R = I$.

Mapping: Translated Frames

- Example:



$${}^{b}p = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \qquad {}^{a}p = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Same point, two different reference frames

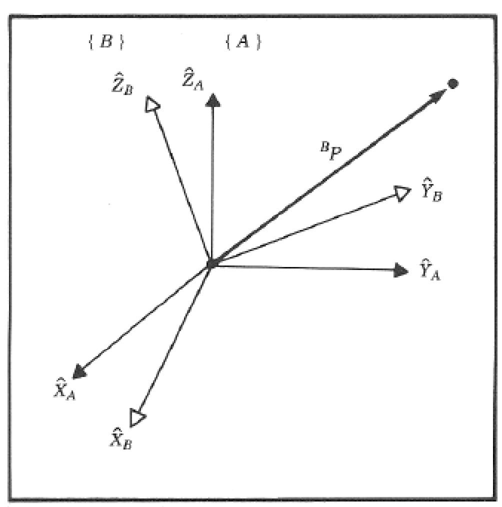
Mapping: Rotated Frames

- Assuming that **frame {B} is only rotated (not translated) with respect to frame {A}** (the origins of the two frames are located at the same point) the position of the point in frame {B} can be expressed in frame {A} using the rotation matrix.
- The components of any vector are simply the projections of that vector onto the unit directions of its frame. The projection is calculated as the vector dot product. Thus, we see that the components of ^{A}P may be calculated as:

Mapping: Rotated Frames

$$^{A}px = {}^{B}\hat{X}_A \cdot {}^{B}P,$$
$$^{A}p_y = {}^{B}\hat{Y}_A \cdot {}^{B}P,$$
$$^{A}p_z = {}^{B}\hat{Z}_A \cdot {}^{B}P.$$

$$^{A}P = {}^{A}_{B}R \; {}^{B}P$$



Mapping: Rotated Frames

- The equation implements a mapping (i.e. it changes the description of a vector) from ${}^{B}P$ which describes a point in space relative to {B}, into ${}^{A}P$, which is a description of the same point, but expressed relative to {A}.
- Note: in this case ${}^{A}P_{BROG} = 0$.
- A helpful way of viewing the notation we have introduced is to imagine that leading subscripts cancel the leading superscripts of the following entity, for example the Bs in the previous equation.

Mapping: Rotated Frames (example)

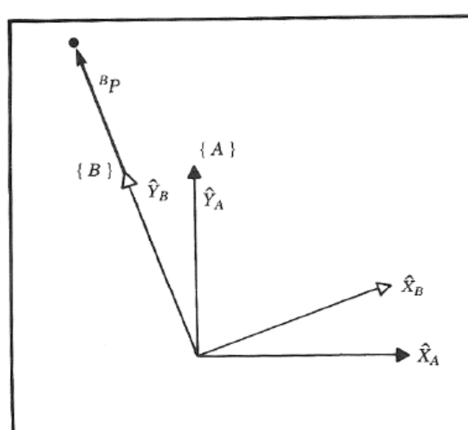
Given :

$$ {}^{B}P = \begin{bmatrix} 0 \\ {}^{B}p_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} $$

$$ \theta = 30^o $$

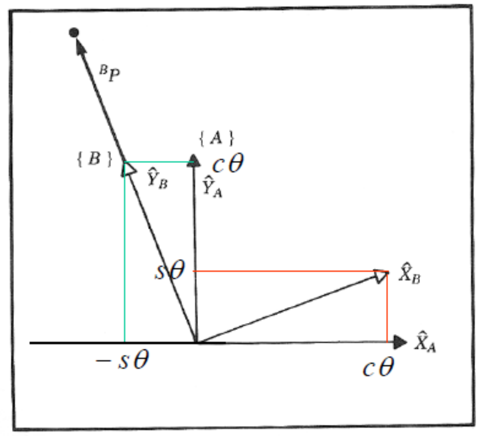
Compute: ${}^{A}P$

Solution: ${}^{A}P = {}^{A}_{B}R \; {}^{B}P$



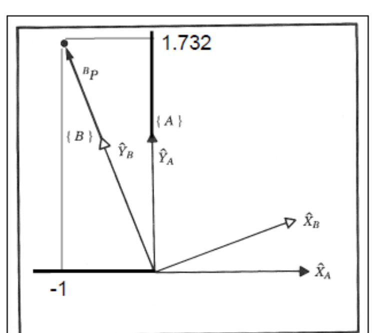
Mapping: Rotated Frames (example)

$$ {}^{A}_{B}R = [{}^{A}\hat{X}_B, {}^{A}\hat{Y}_B, {}^{A}\hat{Z}_B] = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} $$



Mapping: Rotated Frames (example)

$$ {}^{A}P = {}^{A}_{B}R\ {}^{B}P = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ {}^{B}p_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \begin{bmatrix} 0.000 \\ 2.000 \\ 0.000 \end{bmatrix} = \begin{bmatrix} -1.000 \\ 1.732 \\ 0.000 \end{bmatrix} $$



Mapping: General Frames

- Assuming that frame {B} is both ***translated and rotated with respect to frame {A}.*** The description of a vector with respect to frame {B} is known (^{B}P), and it is required to describe the vector with respect to frame {A} (^{A}P).
- **It is the general case of mapping.**
- The origin of frame {B} is not coincident with that of frame {A} but has a general vector offset.

Mapping: General Frames

- We can first change ^{B}P to its description relative to an intermediate frame that has the same orientation as {A}, but whose origin is coincident with the origin of {B}. **This is done by premultiplying by** $^{A}_{B}R$
- We then account for the translation between origins by simple vector addition, as before, and obtain:

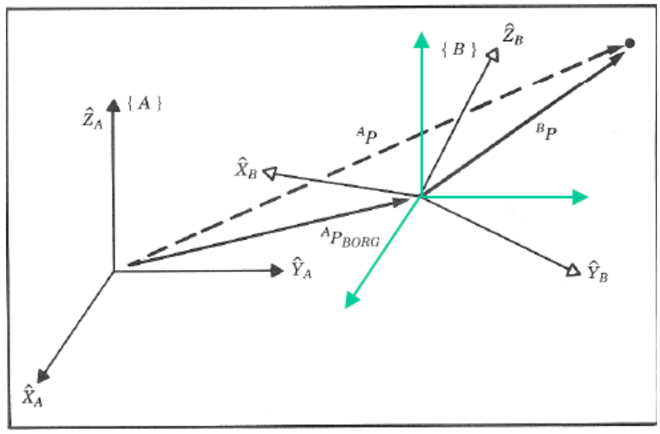
$$^{A}P = {}^{A}_{B}R \; {}^{B}P + {}^{A}P_{BORG}$$

Mapping: General Frames

$$\{B\} = \{{}^{A}_{B}R, {}^{A}P_{BORG}\}$$

$${}^{A}P = {}^{A}_{B}R\ {}^{B}P + {}^{A}P_{BORG}$$

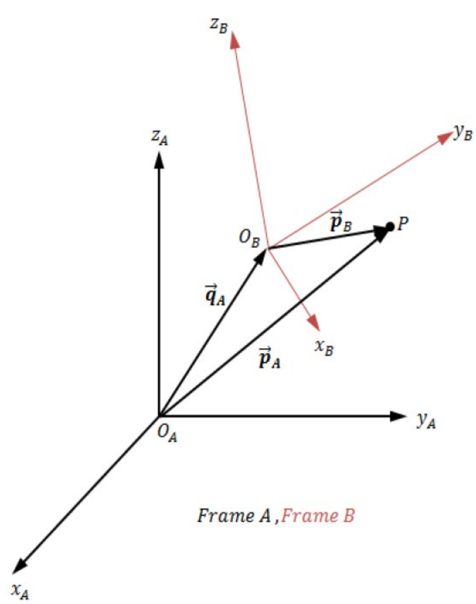
$${}^{A}P = {}^{A}_{B}T\, {}^{B}P$$



Mapping: General Frames

$$\overrightarrow{O_A P} = \overrightarrow{O_A O_B} + \overrightarrow{O_B P}$$

$$\vec{p}_A = \vec{q}_A + R_{AB} \cdot \vec{p}_B$$



Mapping: General Frames

- It is required to think of a mapping from one frame to another as an **operator in matrix form**. This aids in writing compact equations and is conceptually clearer than the previous equation.

$$ {}^{A}P = {}^{A}_{B}T \; {}^{B}P. $$

$$ \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array} \right] \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}. $$

1. a “1” is added as the last element of the 4×1 vectors;
2. a row “[0 0 0 1]” is added as the last row of the 4×4 matrix

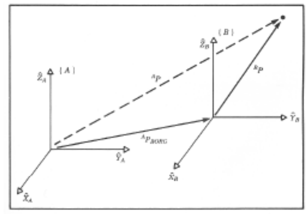
Mapping: General Frames

- The 4 x 4 matrix in the previous equation is called a **homogeneous transform**.
- The homogeneous transform is a 4x4 matrix casting the ***rotation and translation of*** a general transform into a single matrix.
- The description of frame {B} relative to {A} is ${}^{A}_{B}T$.

Mapping: General Frames

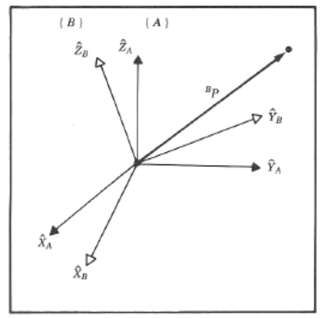
Translation

$$ {}^{A}_{B}T = \begin{bmatrix} 1 & 0 & 0 & {}^{A}P_{BORGx} \\ 0 & 1 & 0 & {}^{A}P_{BORGy} \\ 0 & 0 & 1 & {}^{A}P_{BORGz} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



Rotation

$$ {}^{A}_{B}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



Mapping: General Frames (Example)

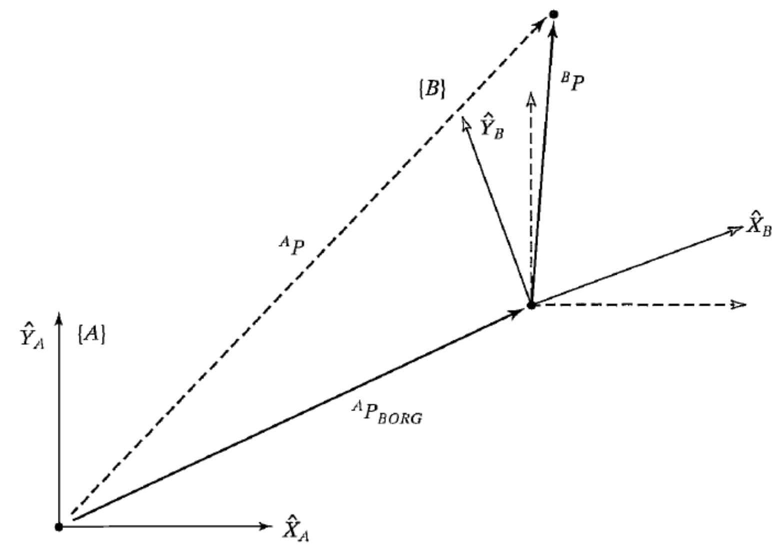
Given:

$$ {}^{B}P = \begin{bmatrix} 3.0 \\ 7.0 \\ 0.0 \end{bmatrix}, $$

Frame {B} is rotated relative to frame {A} about $\hat{Z}$ by 30 degrees, and translated 10 units in $\hat{X}_A$ and 5 units in $\hat{Y}_A$

Calculate: The vector ${}^{A}P$ expressed in frame {A}.

Mapping: General Frames (Example)



Mapping: General Frames (Example)

$$^AP = {}^A_BT\,{}^BP = \begin{bmatrix} {}^AP \\ \\ 1 \end{bmatrix} = \begin{bmatrix} & {}^A_BR & & {}^AP_{BORG} \\ & & & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^BP \\ \\ 1 \end{bmatrix}$$

$$^AP = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 10.0 \\ 0.500 & 0.866 & 0.000 & 5.0 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3.0 \\ 7.0 \\ 0.0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9.098 \\ 12.562 \\ 0.0 \\ 1 \end{bmatrix}$$

Operators: Translation, Rotation, and Transformation

- Transformation Operator - Operates on a a vector ${}^{A}P_1$ and changes that vector to a new vector ${}^{A}P_2$, by means of a rotation by R and translation by Q

$$ {}^{A}P_2 = T\ {}^{A}P_1 $$

- Note: The matrix of the transform operator T which rotates vectors by R and translation by Q , is the same as the transformation matrix which describes a frame rotated by R and translated by Q relative to the reference frame

$$ {}^{A}P_2 = T\ {}^{A}P_1 \quad \Leftrightarrow \quad {}^{A}P = {}^{A}_{B}T\ {}^{B}P $$

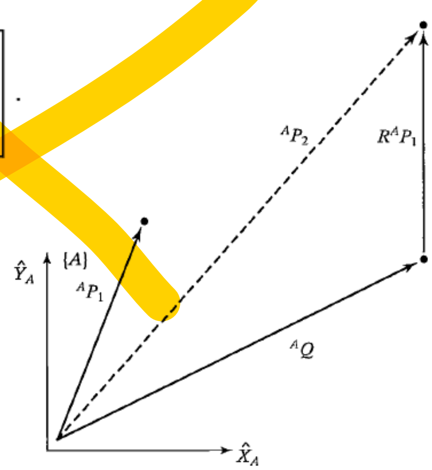
Operator **Mapping**

Operators: Translation, Rotation, and Transformation (Example)

Figure 2.11 shows a vector ${}^{A}P_1$. We wish to rotate it about $\hat{Z}$ by 30 degrees and translate it 10 units in $\hat{X}_A$ and 5 units in $\hat{Y}_A$. Find ${}^{A}P_2$, where ${}^{A}P_1 = [3.0\ 7.0\ 0.0]^T$.

$$ T = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 10.0 \\ 0.500 & 0.866 & 0.000 & 5.0 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

$$ {}^{A}P_2 = T\ {}^{A}P_1 = \begin{bmatrix} 9.098 \\ 12.562 \\ 0.000 \end{bmatrix}. $$



Homogeneous Transform - Summary of Interpretation

- As a general tool to represent frames, we have introduced the homogeneous transform, a 4 x 4 matrix containing orientation and position information. We have introduced three interpretations of this homogeneous transform:

1. It is a *description of a frame*. ${}^{A}_{B}T$ describes the frame $\{B\}$ relative to the frame $\{A\}$. Specifically, the columns of ${}^{A}_{B}R$ are unit vectors defining the directions of the principal axes of $\{B\}$, and ${}^{A}P_{BORG}$ locates the position of the origin of $\{B\}$.
2. It is a *transform mapping*. ${}^{A}_{B}T$ maps ${}^{B}P \rightarrow {}^{A}P$.
3. It is a *transform operator*. T operates on ${}^{A}P_1$ to create ${}^{A}P_2$.

Transformation Arithmetic

- In this section, we look at the multiplication of transforms and the inversion of transforms. These two elementary operations form a functionally complete set of transform operators.
  - **Compound transformation.**
  - **Inverted transformation.**

Transformation Arithmetic: Compound Transformation

Given: Vector ${}^{C}P$

Frame {C} is known relative to frame {B} - ${}^{B}_{C}T$

Frame {B} is known relative to frame {A} - ${}^{A}_{B}T$

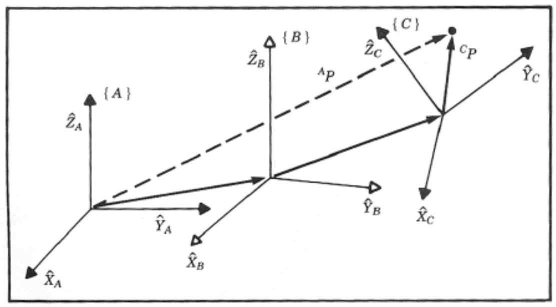
Calculate: Vector ${}^{A}P$

$${}^{B}P={}^{B}_{C}T\,{}^{C}P$$

$${}^{A}P={}^{A}_{B}T\,{}^{B}P$$

$${}^{A}P={}^{A}_{B}T\,{}^{B}_{C}T\,{}^{C}P$$

$${}^{A}_{C}T = {}^{A}_{B}T\,{}^{B}_{C}T.$$



$${}^{A}_{C}T = \left[\begin{array}{ccc|c} \multicolumn{3}{c|}{{}^{A}_{B}R\,{}^{B}_{C}R} & {}^{A}_{B}R\,{}^{B}P_{CORG} + {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

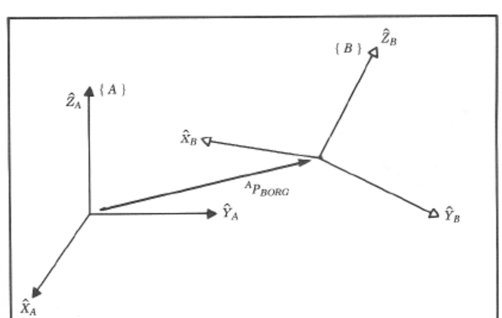
Transformation Arithmetic: Inverted Transformation

Given: Description of frame {B} relative to frame {A} - ${}^{A}_{B}T$ $({}^{A}_{B}R, {}^{A}P_{BORG})$

Calculate: Description of frame {A} relative to frame {B} - Homogeneous Transform ${}^{B}_{A}T$ $({}^{B}_{A}R, {}^{B}P_{AORG})$

$${}^{B}_{A}R={}^{A}_{B}R^{T}$$

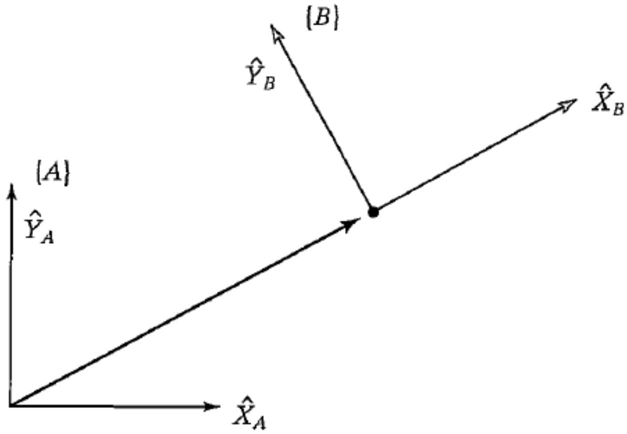
$${}^{B}_{A}T = \left[\begin{array}{ccc|c} \multicolumn{3}{c|}{{}^{A}_{B}R^{T}} & -{}^{A}_{B}R^{T}\,{}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$



Note: ${}^{B}_{A}T={}^{A}_{B}T^{-1}$

Transformation Arithmetic (Example)

Figure 2.13 shows a frame $\{B\}$ that is rotated relative to frame $\{A\}$ about $\hat{Z}$ by 30 degrees and translated four units in $\hat{X}_A$ and three units in $\hat{Y}_A$. Thus, we have a description of ${}^A_BT$. Find ${}^B_AT$.



Transformation Arithmetic (Example)

$$ {}^A_BT = \left[\begin{array}{ccc|c} c\theta & -s\theta & 0 & {}^AP_{BORGx} \\ s\theta & c\theta & 0 & {}^AP_{BORGy} \\ 0 & 0 & 1 & {}^AP_{BORGz} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 0.866 & -0.500 & 0.000 & 4.000 \\ 0.500 & 0.866 & 0.000 & 3.000 \\ 0.000 & 0.000 & 1.000 & 0.000 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

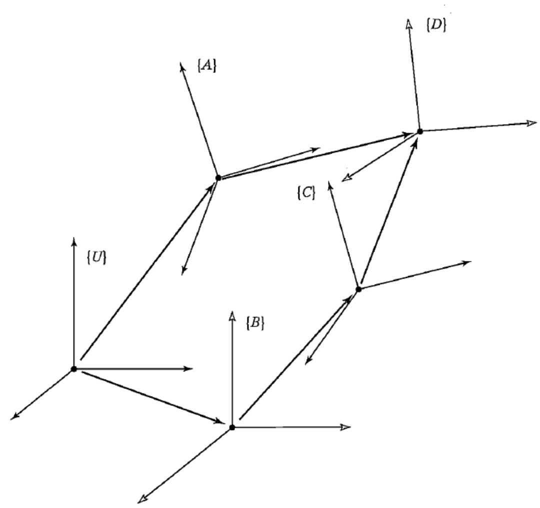
$$ {}^B_AT = \left[\begin{array}{c|c} {}^A_BR^T & -{}^A_BR^T\,{}^AP_{BORG} \\ \hline 0 \quad 0 \quad 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 0.866 & 0.500 & 0.000 & -4.964 \\ -0.500 & 0.866 & 0.000 & -0.598 \\ 0.000 & 0.000 & 1.000 & 0.000 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

Transform Equation

- The following Figure indicates a situation in which a frame {D} can be expressed relative to frame {U} as products of transformations in two different ways:

$$ {}^{U}_{D}T = {}^{U}_{A}T \, {}^{A}_{D}T; $$

$$ {}^{U}_{D}T = {}^{U}_{B}T \, {}^{B}_{C}T \, {}^{C}_{D}T. $$



Transform Equation

- We can set these two descriptions of ${}^{U}_{D}T$ equal to construct **transform equation:**

$$ {}^{U}_{A}T \, {}^{A}_{D}T = {}^{U}_{B}T \, {}^{B}_{C}T \, {}^{C}_{D}T. $$

- Transform equations can be used to solve for transforms in the case of *n* unknown transforms and *n* transform equations.
- For example, in the previous situation, if all the transformations are known except ${}^{B}_{C}T$. Here, we have one transform equation and one unknown transform; hence, we easily find its solution to be:

$$ {}^{B}_{C}T = {}^{U}_{B}T^{-1} \, {}^{U}_{A}T \, {}^{A}_{D}T \, {}^{C}_{D}T^{-1}. $$

Transform Equation

- A graphical representation of frames as an arrow pointing from one origin to another origin is usually used. The arrow's direction indicates which way the frames are defined.
- In order to compound frames when the arrows line up, we simply compute the product of the transforms. If an arrow points the opposite way in a chain of transforms, we simply compute its inverse first.
- For example, in the following Figure, there are two possible descriptions of frame {C}:

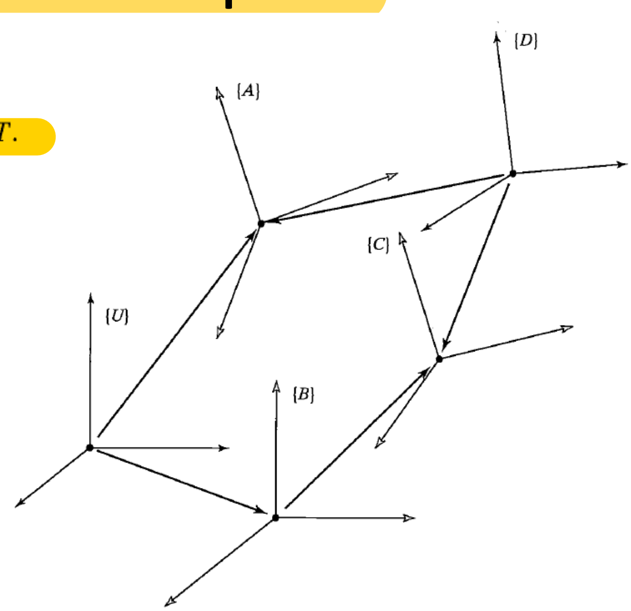
$$ {}^{U}_{C}T = {}^{U}_{A}T \, {}^{D}_{A}T^{-1} \, {}^{D}_{C}T $$

$$ {}^{U}_{C}T = {}^{U}_{B}T \, {}^{B}_{C}T. $$

Transform Equation

- To find ${}^{U}_{A}T$

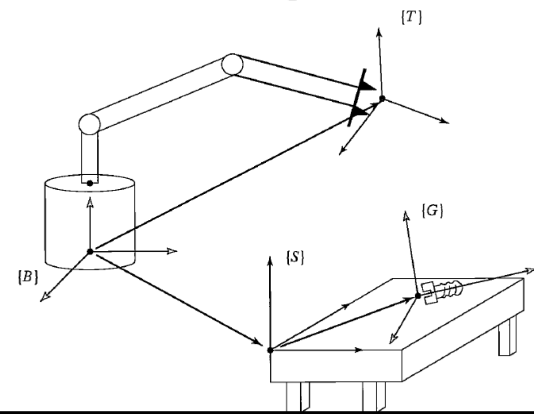
$$ {}^{U}_{A}T = {}^{U}_{B}T \, {}^{B}_{C}T \, {}^{D}_{C}T^{-1} \, {}^{D}_{A}T. $$



Transform Equation (Example)

Assume that we know the transform ${}^{B}_{T}T$ in Fig. 2.16, which describes the frame at the manipulator's fingertips $\{T\}$ relative to the base of the manipulator, $\{B\}$, that we know where the tabletop is located in space relative to the manipulator's base (because we have a description of the frame $\{S\}$ that is attached to the table as shown, ${}^{B}_{S}T$), and that we know the location of the frame attached to the bolt lying on the table relative to the table frame—that is, ${}^{S}_{G}T$. Calculate the position and orientation of the bolt relative to the manipulator's hand, ${}^{T}_{G}T$.

$$ {}^{T}_{G}T = {}^{B}_{T}T^{-1}\, {}^{B}_{S}T\, {}^{S}_{G}T. $$



Different Methods in Representing Orientation

The rotation matrices with respect to the reference frame are defined as follows:

$$ R_X(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

$$ R_Y(\beta) = \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} $$

$$ R_Z(\alpha) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

Different Methods in Representing Orientation

- The nine elements of a rotation matrix are dependent.
- The orientation can be represented using only three independent variables.
- A human operator at a computer terminal who wishes to type in the specification of the desired orientation of a robot's hand would have a hard time inputting a nine-element matrix with orthonormal columns. A representation that requires only three numbers would be simpler.

Different Methods in Representing Orientation

- **X-Y-Z Fixed Angles (Roll, Pitch, Yaw angles):**
  The rotations perform about an axis of a fixed reference frame
- **Z-Y-X Euler Angles**
  The rotations perform about an axis of a moving reference frame.
- Note: The final orientation of three successive rotations made about moving axes is the same as the final orientation of the three same rotations taken in the opposite order about fixed axes.

$$R_{AB,\ moving}(\alpha, \beta, \gamma) = R_{AB,\ fixed}(\gamma, \beta, \alpha) = R_z(\alpha) \cdot R_y(\beta) \cdot R_x(\gamma)$$

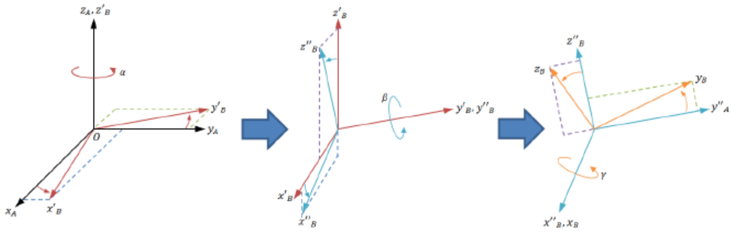


Fig. 8: Three rotations about moving axes

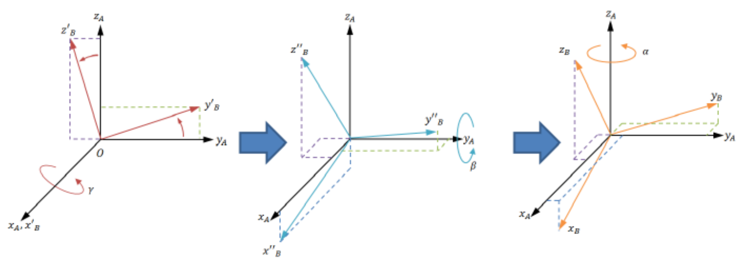


Fig. 9: Three rotations about fixed axes

X-Y-Z Fixed Angles (Roll, Pitch, Yaw angles)

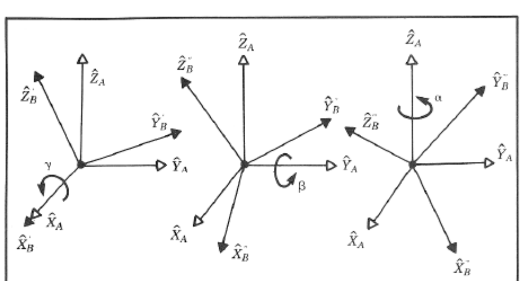
- One method of describing the orientation of a frame {B} is as follows:

Start with frame {B} coincident with a known reference frame {A}.

- Rotate frame {B} about $\hat{X}_A$ by an angle γ
- Rotate frame {B} about $\hat{Y}_A$ by an angle β
- Rotate frame {B} about $\hat{Z}_A$ by an angle α

Fixed Angles

Note *- Each of the three rotations takes place about an axis in the fixed reference frame {A}*



X-Y-Z Fixed Angles (Roll, Pitch, Yaw angles)

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} $$

X-Y-Z Fixed Angles (Roll, Pitch, Yaw angles)

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} $$

$$ \beta = \text{Atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}), $$

$$ \alpha = \text{Atan2}(r_{21}/c\beta, r_{11}/c\beta), $$

$$ \gamma = \text{Atan2}(r_{32}/c\beta, r_{33}/c\beta), $$

X-Y-Z Fixed Angles (Roll, Pitch, Yaw angles)

Four-quadrant inverse tangent (arctangent) in the range of

$$\text{Atan}\,2(y,x) = \tan^{-1}(y/x)$$

For example

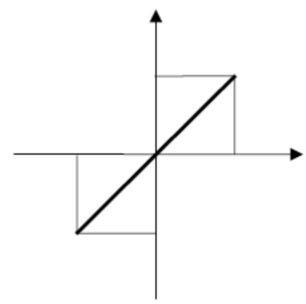
$$[-\pi \quad \pi]$$

$$\text{Atan}(+1,+1) = 45^o$$

$$\text{Atan}\,2(+1,+1) = 45^o$$

$$\text{Atan}(-1,-1) = 45^o$$

$$\text{Atan}\,2(-1,-1) = -135^o$$



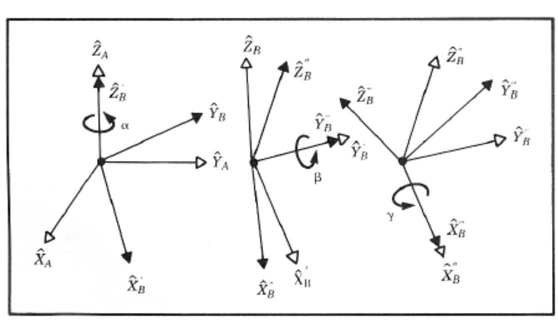
Z-Y-X Euler Angles

Start with frame {B} coincident with a known reference frame {A}.

- Rotate frame {B} about $\hat{Z}_B$ by an angle α
- Rotate frame {B} about $\hat{Y}_B$ by an angle β
- Rotate frame {B} about $\hat{X}_B$ by an angle γ

$\Big\}$ **Euler Angles**

Note *- Each rotation is preformed about an axis of the the moving reference frame {B}, rather then a fixed reference frame {A}.*



Z-Y-X Euler Angles

$$ {}^{A}_{B}R = {}^{A}_{B'}R \; {}^{B'}_{B''}R \; {}^{B''}_{B}R, $$

$$ \begin{aligned} {}^{A}_{B}R_{Z'Y'X'} &= R_Z(\alpha)R_Y(\beta)R_X(\gamma) \\ &= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}, \end{aligned} $$

$$ {}^{A}_{B}R_{Z'Y'X'}(\alpha, \beta, \gamma) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}. $$

X-Y-Z Fixed Angles Versus Z-Y-X Euler Angles

$$ {}^{A}_{B}R_{XYZ}(\gamma, \beta, \alpha) = {}^{A}_{B}R_{Z'Y'X'} $$

Three rotations taken about fixed axes (Fixed Angles) yield the same final orientation as the same three rotation taken in an opposite order about the axes of the moving frame (Euler Angles)

Transformation of Free Vectors

- A **free vector** refers to a vector that may be positioned anywhere in space without loss or change of meaning, provided that magnitude and direction are preserved.
- Velocity, moment, force and acceleration are free vectors.
- In other words, all that counts is the magnitude and direction (in the case of a free vector), so only the rotation matrix relating the two systems is used in transforming. The relative locations of the origins do not enter into the calculation.

Likewise, a velocity vector written in $\{B\}$, ${}^{B}V$, is written in $\{A\}$ as

$$ {}^{A}V = {}^{A}_{B}R \, {}^{B}V. $$

Transformation of Free Vectors

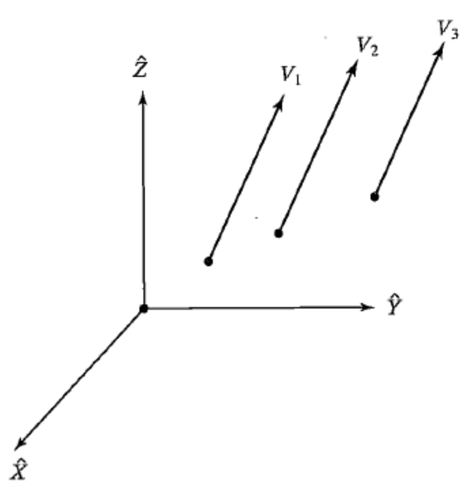


FIGURE 2.21: Equal velocity vectors.