CS648 : Randomized Algorithms
Semester I, 2011-12, CSE, IIT Kanpur

Practice problem sheet - 1

Note : With a positive attitude and fresh mind, try to solve the following problems. If need arises, we may discuss these problems in some optional extra class next week. Each student should be able to solve at least 7 problems out of the first 10. Each student is also strongly recommended to solve the last two problems (11 and 12th).

1. A fair coin is thrown repeatedly. What is the probability that on the nth throw :
   (a) a head appears for the first time ?
   (b) the number of heads and tails to date are equal ?
   (c) exactly two heads have appeared altogether to date ?
   (d) at least two heads have appeared to date ?

2. Let A and B be two events defined over a probability space $(\Omega, \mathbf{P})$ with probabilities $\mathbf{P}(A) = 3/4$ and $\mathbf{P}(B) = 1/3$. Show that $1/12 \leq \mathbf{P}(A \cap B) \leq 1/3$, and give examples to show that both extremes are possible. Find corresponding bounds for $\mathbf{P}(A \cup B)$.

3. You are given that at least one of the events $A_r, 1 \leq r \leq n$ is certain to occur, but certainly no more than two occur. If $\mathbf{P}(A_r) = p$ and $\mathbf{P}(A_r \cap A_s) = q$ for all $r \neq s$, show that $p \geq 1/n$ and $q \leq 2/n$.

4. You are given a coin which gives heads with probability p and tail otherwise. What is the average number of times you need to toss the coin until you get a head ? (Answer : $1/p$)

5. There are 3 red balls and 2 black balls in a bag. We take out the balls uniformly randomly without replacement. What is the average number of red balls preceding the first black ball ?

6. There are 15 red balls and 7 black balls in a bag. We conduct the following experiment until there is only one ball left in the bag :

   Select 2 balls uniformly at random from the bag.
   (a) if the two balls are of same color, throw them away and place a new red ball back into the bag.
   (b) if the two balls are of different colors, throw them away and place a new black ball into the bag.

   What is the probability that the last ball remaining in the bag is red ? (Answer : 0)

7. An urn contains b blue balls and r red balls. They are removed at random and not replaced. Show that the probability that the first red ball drawn appears is the $(k+1)$th ball drawn equals $\binom{r+b-k-1}{r-1} / \binom{r+b}{b}$. Find the probability that the last ball drawn is red. (Answer : $r/(r+b)$)

8. There are n balls to be thrown and m bins placed in a row. Each ball selects its destination bin randomly uniformly and independently. What is the probability that the first bin remains empty.

9. A, B, C, and D were friends at school. Subsequently each of the $\binom{4}{2} = 6$ subpairs meet up; at each of the six meetings the pair gets involved in a quarrel with some fixed probability p, or becomes firm friends with probability $1 - p$. Quarrels take place independently of each other. In future, if any of the four hears rumour, then she tells it to her firm friends only. If A hears a rumour, what is the probability that :

(a) D hears it ? (Answer : $(1-(1-(1-p)^2)^2)p+(1-p^2)^2(1-p)$)

(b) D hears it if A and B have quarrelled ? (Answer : $(1-(1-(1-p)^2)^2)p+(1-p)$)

(c) D hears it if B and C have quarrelled ? (Answer : $(1-p)(1-p(1-(1-p)^2))p+(1-p)$)

(d) D hears if if she has quarrelled with A ?
(Answer : $(1-(1-(1-p)^2)^2)p^2+(1-p^2)^2p(1-p)+(1-p)$)

10. A bowl contains twenty cherries, exactly fifteen of which have had their stones removed. A greedy pig eats five whole cherries, picked at random, without remarking on the presence or absence of stones. Subsequently, a cherry is picked randomly from the remaining fifteen.

    (a) What is the probability that this cherry contains a stone ? (Answer : $1/4$)

    (b) Given that this cherry contains a stone, what is the probability that the pig consumed at least one stone ? (Answer : $1-\frac{15}{19}\frac{14}{18}\frac{13}{17}\frac{12}{16}\frac{11}{15}$)

11. ** You are lost in the National Park of Bandrika. Tourists comprise two-thirds of the visitors to the park, and give a correct answer to requests for directions with probability 3/4. (Answers to repeated questions are independent, even if the questions and the person are the same). If you ask a Bandrikan for directions, the answer is always false.

    (a) You ask a passer-by whether the exit from the Park is East or West. The answer is East. What is the probability this is correct ?
    (Answer : $1/2$)

    (b) You ask the same person again, and receive the same reply. Show the probability that it is correct is $1/2$.

    (c) You ask the same person again, and receive the same reply. What is the probability that it is correct ?

    (d) You ask for the fourth time, and receive the answer East. Show that the probability that it is correct is $27/70$.

    (e) Show that, had the fourth answer been West instead, the probability that East is nevertheless correct is $9/10$.

12. ** A man possesses five coins, two of which are double-headed, one is double-tailed, and two are normal. He shuts his eyes, picks a coin at random, and tosses it. What is the probability that the lower face of the coin is a head ?

    He opens his eyes and sees that the coin is showing heads; what is the probability that the lower face is a head ? He shuts his eyes again, and tosses the coin again. What is the probability that the lower face is a head ? He opens his eyes again and sees that the coin is showing heads; what is the probability that the lower face is a head ?

    He discards this coin, picks another at random, and tosses it. What is the probability that it shows heads ? (Answer : $3/5,\ 2/3,\ 5/6,\ 4/5$.)