MTH301: Analysis I

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Assignment 9

Question 12

Q12 (a) If $f, g : \mathbb{R} \to \mathbb{R}$ s.t. f and g are uniformly continuous and bounded, then show that $f \cdot g$ is also uniformly continuous.

Solution:

To prove that $f \cdot g$ is uniformly continuous, it suffices to show that, $\forall \varepsilon$, there exists a δ such that $|x - y| < \delta \implies |f(x)g(x) - f(y)g(y)| < \varepsilon \ \forall x, y \in \mathbb{R}$.

Consider $\alpha > 0$.
f and g are uniformly continuous
$\implies \exists \delta_1 > 0$ s.t. $|x - y| < \delta_1 \implies |f(x) - f(y)| < \alpha \forall x, y \in \mathbb{R}$
$\implies \exists \delta_2 > 0$ s.t. $|x - y| < \delta_2 \implies |g(x) - g(y)| < \alpha \forall x, y \in \mathbb{R}$
Take $\delta = \min\{\delta_1, \delta_2\}$ and assume $|x - y| < \delta$.
Then, $|f(x) - f(y)| < \alpha$ and $|g(x) - g(y)| < \alpha$

Now, $|f(x)g(x) - f(y)g(y)| = |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)|$
But by triangle inequality,
$|f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \leq |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)|$
$\implies |f(x)g(x) - f(y)g(y)| \leq |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)|$
$\implies |f(x)g(x) - f(y)g(y)| \leq |f(x)| \cdot |g(x) - g(y)| + |g(y)| \cdot |f(x) - f(y)|$
But $|f(x) - f(y)| < \alpha$ and $|g(x) - g(y)| < \alpha$ (from above)
Also, f and g are bounded (given). Let their upper bounds be M_1 and M_2
i.e. $|f(x)| < M_1 \forall x \in \mathbb{R}$
and $|g(x)| < M_2 \forall x \in \mathbb{R}$
So we have, $|f(x)g(x) - f(y)g(y)| < M_1 \cdot \alpha + M_2 \cdot \alpha$

Let $\varepsilon = M_1 \cdot \alpha + M_2 \cdot \alpha$.
So, for this ϵ, we have a δ (=min$\{\delta_1, \delta_2\}$) s.t.
$|x - y| < \delta \implies |f(x)g(x) - f(y)g(y)| < \varepsilon \ \forall x, y \in \mathbb{R}$
Since this holds for all $\alpha > 0$, it holds for all $\varepsilon > 0$ as well.
Hence, $f \cdot g$ is uniformly continuous. ■

Q12 (b) Suppose $f : (M, d) \to (N, \rho)$ is continuous.
Prove or disprove: f is uniformly continuous $\iff$ f maps Cauchy sequence to Cauchy sequence.

Solution:

We will disprove the statement by giving a counterexample.
Consider the function $f : \mathbb{R} \to \mathbb{R} : f(x) = x^2$
If (x_n) is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$
$\implies (x_n)$ is a convergent sequence in $(\mathbb{R}, |\cdot|)$

$\implies (x_n^2)$ is a convergent sequence in $(\mathbb{R}, |\cdot|)$
$\implies (x_n^2)$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$
Hence, f maps a Cauchy sequence to a Cauchy sequence.
Now, we will prove that f is not uniformly continuous. We prove this by contradiction. So assume that f is uniformly continuous.
$\implies \forall \varepsilon > 0$, $\exists \delta$ s.t. $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \ \forall x, y \in \mathbb{R}$
Take $\varepsilon = 1$ and $y = x + \frac{\delta}{2}$.
Clearly, $|x - y| = \frac{\delta}{2} < \delta \forall x \in \mathbb{R}$
$\implies |f(x) - f(y)| = |x^2 - y^2| < 1$
$\implies |x^2 - (x + \frac{\delta}{2})^2| < 1 \forall x \in \mathbb{R}$
$\implies |x \cdot \delta + \frac{\delta^4}{4}| < 1 \forall x \in \mathbb{R}$
$\implies |x \cdot \delta| < 1 \forall x \in \mathbb{R}$.
This is a contradiction. Hence, our assumption was false i.e. f is not uniformly continuous.
Therefore, f is a function which maps a Cauchy sequence to Cauchy sequence but is not uniformly continuous.
One counterexample, suffices to disprove the statement given in the question. $\blacksquare$