MTH301: Analysis I
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Assignment 8

Question 11 and 16

Q11 Prove that the set $\{x \in \mathbb{R}^n : \|x\|_1 = 1\}$ is compact in $\mathbb{R}^n$ under the Euclidean norm.
Solution:
According to Heine-Borel theorem, a set is compact iff it is closed and bounded.
Hence, to prove that the set, say $X = \{x \in \mathbb{R}^n : \|x\|_1 = 1\}$ is compact, it suffices to show that X is (a) closed and (b) bounded.

(a) To prove- X is closed.
Let $(x_i)_{i=1}^{\infty} \in X$ s.t. $(x_j) \to x \in \mathbb{R}^n$.
$x_j = (x_j^{(1)}, x_j^{(2)}, \cdots, x_j^{(n)}) \in X$
$\implies |x_j^{(1)}| + |x_j^{(2)}| + \cdots + |x_j^{(n)}| = 1$
Convergence in $\mathbb{R}^n$ implies coordinate wise convergence. And $n \to \infty$
Hence, we have $|x^{(1)}| + |x^{(2)}| + \cdots + |x^{(n)}| = 1$
$\implies x \in X \implies$ X is closed. ■

(b) To prove- X is bounded.
Let $x := (x_1, x_2, \cdots, x_n) \in X$.
Then, $\|x\|_1 = 1 \implies \sum_{i=1}^{n} |x_i| = 1$
But since $|x_j| \le \sum_{i=1}^{n} |x_i| \; \forall j \in \{1, 2, \cdots, n\}$, we have $|x_j| \le 1 \; \forall j \in \{1, 2, \cdots, n\}$
$\implies |x_j|^2 \le 1 \; \forall j \in \{1, 2, \cdots, n\}$
$\implies \sum_{i=1}^{n} |x_i|^2 \le n$
$\implies \|x\|_2 \le \sqrt{n} < \infty$
Hence, X is bounded w.r.t to the Euclidean bound (as $\sqrt{n}$ is the upper bound and 0 is the lower bound). Hence, proved. ■

Q16 Given $f : [a, b] \to \mathbb{R}$. Define $G : [a, b] \to \mathbb{R}^2$ by $G(x) = (x, f(x))$. Prove that the following are equivalent-
(i) f is continuous.
(ii) G is continuous.
(iii) The graph of f is a compact subset of $\mathbb{R}^2$.
Solution:
To prove the equivalence of (i), (ii) and (iii), it suffices to proving the following-
(I) (i) $\iff$ (ii)
(II) (ii) $\implies$ (iii)
(III) (iii) $\implies$ (i)

Proof of (I)
f is continuous
$\implies$ for any $(x_n) \to x$ in $[a, b]$, $f(x_n) \to f(x)$ in $\mathbb{R}$

$\implies$ for any $(x_n) \to x$ in $[a, b]$, $(x_n, f(x_n)) \to (x, f(x))$ in $\mathbb{R}$
$\implies$ G is continuous.

G is continuous
$\implies$ as $|x - x_n| \to 0$, $|f(x_n) - f(x)| \leq \|(x_n, f(x_n)) - (x, f(x))\|_2 \to 0$
$\implies$ $f(x_n) \to f(x)$
$\implies$ f is continuous.

Hence, f is continuous $\iff$ G is continuous. ■

Proof of (II)
From the lecture on Compact Metric Spaces, we know that if $f : (M, d) \to (N, \rho)$ is a continuous map and K is compact in M, then $f(K)$ is compact in N.
Since G is continuous, $G([a, b])$ is compact in $\mathbb{R}^2$.
But $G([a, b]) = \{(x, f(x)) : x \in [a, b]\}$ that is the graph of f. Hence, the graph of f is a compact subset of $\mathbb{R}^2$. ■

Proof of (III)
Given that G is compact, we need to show that f is continuous.
We prove by contradiction. So, let us assume that f is not continuous.
Hence, $\exists x \in [a, b]$ and a sequence $(x_n) \in [a, b]$ such that $(x_n) \to x$ but $f(x_n) \not\to f(x)$.
Consider the sequence $((x_n, f(x_n)))$.
This sequence lies in $G([a, b])$ and G is compact, so it will have a convergent subsequence, which converges to (x, y) where $y \in \mathbb{R}$ and $y \neq f(x)$ (as $f(x_n) \not\to f(x)$)
$\implies$ $(x, y) \notin G \implies G$ is not sequentially compact $\implies$ G is not compact.
But this is a contradiction, as it is given that G is compact. Hence, our assumption is wrong. That is f must be continuous. Hence, proved. ■