MTH301: Analysis I
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Assignment 2

Question 3

Given: function $f : [0, \infty) \to [0, \infty)$ is increasing and $f(0) = 0$ and $f(x) > 0 \forall x > 0$
To show: Each of the following conditions
(i) f has a second derivative satisfying $f'' \leq 0$ i.e.

$$f''(x) \leq 0 \forall x \tag{1}$$

(ii) f has a decreasing first derivative i.e.

$$f'(x) \leq f'(y) \forall x > y \geq 0 \tag{2}$$

(iii) $f(x)/x$ is decreasing for $x > 0$ i.e.

$$\frac{f(x)}{x} \leq \frac{f(y)}{y} \forall x > y > 0 \tag{3}$$

is sufficient to ensure that

$$f(x+y) \leq f(x) + f(y) \forall x, y \geq 0 \tag{4}$$

Proof:
It is sufficient to prove that (1) $\implies$ (2) $\implies$ (3) $\implies$ (4)

(i) To Prove: (1) $\implies$ (2)
According to the Mean Value Theorem,
$\forall x, y$ s.t. $y > x \geq 0$, $\exists c \in (x, y)$ s.t. $f''(c) = \frac{f'(y) - f'(x)}{y - x}$
But $f''(c) \leq 0$ (from (1))
$\implies \frac{f'(y) - f'(x)}{y - x} \leq 0$
$\implies f'(y) \leq f'(x)$ (since $y - x > 0$) ■

(ii) To Prove: (2) $\implies$ (3)
According to Mean Value Theorem,
$\forall x > 0$, $\exists c \in (0, x)$ s.t. $f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}$
To prove that $\frac{f(x)}{x}$ is decreasing, it suffices to show that $\frac{d(f(x)/x)}{dx} \leq 0 \ \forall x > 0$
Now, $\frac{d(f(x)/x)}{dx} = \frac{xf'(x) - f(x)}{x^2} = \frac{f'(x) - \frac{f(x)}{x}}{x} = \frac{f'(x) - f'(c)}{x}$
Since $c \in (0, x)$ and f' is decreasing, $f'(c) \geq f'(x)$
$\implies \frac{d(f(x)/x)}{dx} \leq 0$ ■

(iii) To Prove: (3) $\implies$ (4)
Since $\frac{f(x)}{x}$ is decreasing,
we have $\frac{f(x)}{x} \geq \frac{f(x+y)}{x+y}$ and $\frac{f(y)}{y} \geq \frac{f(x+y)}{x+y}$ $\forall x, y \geq 0$
$\implies f(x) \geq \frac{xf(x+y)}{x+y}$ and $f(y) \geq \frac{yf(x+y)}{x+y}$ $\forall x, y \geq 0$
By adding the two we get, $f(x) + f(y) \geq f(x+y)$ $\forall x, y \geq 0$ ■

Hence, proved.