MTH301: Analysis I
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Assignment 5

Question 9

Q9 $I = (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1]$
$Q = \mathbb{Q} \cap [0, 1]$
To prove:
(i) There is continuous map from I onto Q
(ii) There does not exist a continuous map from $[0, 1]$ onto Q

Proof of (i):
Consider the partition of I into countably many nonempty pieces, given by-
$I_0 = (\frac{1}{2}, 1) \cap I$
$I_1 = (\frac{1}{4}, \frac{1}{2}) \cap I$
In general, $I_k = (\frac{1}{2^{k+1}}, \frac{1}{2^k}) \cap I$
Note that, each I_k is open in I and $I = \cup_{k=0}^{\infty} I_k$.
Let Q be enumerated as $\{q_0, q_1, q_2, \cdots\}$.
For any $x \in I$, $x \in I_k$ for exactly one k. Thereby, we define $f(x) = q_k$.
Since each I_k is non-empty, the surjectiveness of f follows.
We have seen in the lecture that proving f is continuous is equivalent to proving that for any open set O in Q, the set $f^{-1}(O)$ is open in I.
Now, consider any open set $O \subset Q$. So we can write, $O = \cup_{q_k \in O}\{q_k\}$
Therefore, $f^{-1}(O) = \cup_{q_k \in O} f^{-1}(q_k) = \cup_{q_k \in O} I_k$.
Since all I_k are open in I, union of open sets in I will also be open in I.
Therefore, by the theorem quoted above (from lecture), f is continuous.
Hence, proved. ■

Proof of (ii):
To prove (ii), we will use the following lemma:
Lemma: A non-empty connected subset of $\mathbb{Q}$ must be singleton.
The lemma's proof is provided in Appendix 1.
Now, let there be a function $f : [0, 1] \to Q$ s.t. f is continuous.
Since $[0, 1]$ is connected, $f([0, 1])$ will also be connected. (from lecture)
Since $f([0, 1])$ is the range, it must be a subset of the co-domain i.e. Q. Also, $Q \subseteq \mathbb{Q}$.
Therefore, $f([0, 1])$ is a connected subset of $\mathbb{Q}$.
Hence, by our lemma, $f([0, 1])$ is a singleton. But Q is not a singleton. So, range is not same as the co-domain. Hence, f is not onto.
Therefore, we have proved that any continuous function from $[0, 1]$ to Q cannot be surjective. In other words, there cannot be a continuous map from $[0, 1]$ onto Q.
Hence, proved. ■

Appendix 1: Proof of lemma

Let X be a nonempty connected subset of Q. We have to show that X is singleton. We prove by contradiction. Hence, assume that X is not singleton.
Therefore, $\exists a, b \in X$ s.t. $a < b$
Since irrationals are dense in rationals, $\exists c \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $a < c < b$.
The sets $(c, +\infty)$ and $(-\infty, c)$ are open (disjoint) subsets of X and $X = (c, +\infty) \cup (-\infty, c)$.
Therefore, X is disconnected, which is a contradiction.
Hence, our assumption was wrong. That is, X is singleton.
Hence, proved. ∎