MTH301: Analysis I
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Assignment 12

Question 5

Q5 For $a, b \in \mathbb{R}$ with $a < b$, let (f_n) be a sequence of differential functions on $[a, b]$. Suppose (f_n) and (f'_n) are uniformly bounded. Show that (f_n) is equicontinuous and has a uniformly convergent convergent subsequence.
Solution:
Since (f'_n) is bounded, there exists $M \in \mathbb{R}$ s.t. $|f'_n(x)| \leq M \ \forall x \in [a, b] \ \forall n \in \mathbb{N}$.
So, by Mean Value Theorem, $|f_n(x) - f_n(y)| \leq M|x - y|$, $x, y \in [a, b] \forall n \in \mathbb{N}$.
Now, for any $\varepsilon > 0$, take $\delta_\varepsilon = \frac{\varepsilon}{M} > 0$. So, $d(x, y) < \delta_\varepsilon \implies |f(x) - f(y)| < \varepsilon \ \forall f \in (f_n)$.
Since this is true for all $\varepsilon > 0$, (f_n) is equicontinuous, by the definition of equicontinuity.

Moreover, $[a, b]$ is compact. We have already seen that (f_n) is uniformly bounded and equicontinuous. Hence, from the proof of Arzela-Ascoli theorem as done in the lecture, we can obtain a subsequence of (f_n) which is uniformly Cauchy, and thereby, uniformly convergent.
Hence, proved. ■