MTH301: Analysis I
Abhimanyu Sethia
190023
sethia@iitk.ac.in

Assignment 7

Question 7

Q7 If A is not totally bounded, show that A has an infinite subset B that is homomorphic to a discrete space.
Solution: A set $A \subset M$ is said to be totally bounded,
iff for all $\epsilon > 0$, $\exists x_1, x_2, \cdots, x_n \in M$ s.t. $A \subset \bigcup_{i=1}^{n} B(x_i, \epsilon)$.

Since A is **not** totally bounded, $\exists \epsilon > 0$ s.t. for all $x_1, x_2, \cdots, x_n$, $A \not\subset \bigcup_{i=1}^{n} B(x_i, \epsilon)$.
Since this is true for all n, take $x_1 \in A$.
Since $A \not\subset B(x_1, \epsilon)$, we can take $x_2 \in A \setminus B(x_1, \epsilon)$.
But since $A \not\subset B(x_1, \epsilon) \cup B(x_2, \epsilon)$ we can take $x_3 \in A \setminus B(x_1, \epsilon) \cup B(x_2, \epsilon)$
We can continue this process to construct a sequence of points $\{x_n\}_{n=1}^{\infty} \in A$ s.t. $d(x_n, x_m) \geq \epsilon \forall n \neq m$.

Now, consider the set $B := \{x_n | n \in \mathbb{N}\}$.
B is clearly an infinite subset of A. To prove that B is homeomorphic to a discrete subspace of A, it suffices to prove that B is discrete subspace of A(as every set is homeomorphic to itself through the identity function).

To show that B is discrete, it suffices to show that every singleton subset of B is open in B.
Let $x_n \in B$.
Since $d(x_n, m) > \epsilon \ \forall m (\neq x_n) \in B$, $\{x_n\} = B(x_n, \epsilon) \cap B$.
$\implies \{x_n\}$ is open in B.
Hence, B is a discrete subspace of A.
Hence, proved. ■