MTH301: Analysis I
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Assignment 4b

Question 12

Q12(a) Prove that if A is open or A is closed in M, then $(\partial A)^{\circ} = \phi$.
Solution:
Case 1: Let A be closed.
We will prove by contradiction. So, assume that $(\partial A)^{\circ} \neq \phi$.
Let $y \in (\partial A)^{\circ} \implies \exists r_y > 0$ s.t. $B(y, r_y) \subset (\partial A)^{\circ} \subset \partial A \subset A$
$\implies \exists r_y > 0$ s.t. $B(y, r_y) \cap A^c = \phi$
But $y \in (\partial A)^{\circ}$, so $\forall \epsilon > 0$, $B(y, \epsilon) \cap A^c \neq \phi$
Hence, by contradiction, $(\partial A)^{\circ} = \phi$

Case 2: Let A be open.
Then, A^c will be closed. We will again prove by contradiction. Assume that $(\partial A)^{\circ} \neq \phi$
Now, $\partial A = \partial A^c \implies (\partial A)^{\circ} = (\partial A^c)^{\circ}$
Let $y \in (\partial A)^{\circ} \implies y \in (\partial A^c)^{\circ}$
$\implies \exists r_y > 0$ s.t. $B(y, r_y) \subset (\partial A^c)^{\circ} \subset \partial A^c \subset A^c$
$\implies \exists r_y > 0$ s.t. $B(y, r_y) \cap A = \phi$
But $y \in \partial A$, so $\forall \epsilon > 0$, $B(y, \epsilon) \cap A \neq \phi$.
Hence, by contradiction, $(\partial A)^{\circ} = \phi$ ■

Q12(b) Give an example in which $(\partial A)^{\circ} = M$.
Solution:
Answer: $A = \mathbb{Q}$
Explanation For any $x \in \mathbb{R}$, $\exists (x_n) \subset \mathbb{Q}$ s.t. $x_n \xrightarrow{|\cdot|} x$
So, $x \in \bar{A} \implies \bar{A} = \mathbb{R}$
Claim: $A^{\circ} = \phi$
Proof: Let $x \in A^{\circ} \implies \exists \epsilon > 0$ s.t. $B(x, \epsilon) \subset A$.
Let $p \in B(x, \epsilon) \setminus \{x\}$
$\implies |x - p| < \epsilon$ i.e. $p - \epsilon < x < p + \epsilon$
But due to density of irrationals, $\exists r \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < r < p$
$\implies r \in B(x, \epsilon) \subset A$, but $r \notin \mathbb{Q}$
This is a contradiction. Hence, $A^{\circ} = \phi$ ■
$\partial A = \bar{A}/A^{\circ} = \mathbb{R}/\phi = \mathbb{R} = M$