MTH301 : Analysis I

Assignment 2 – Question 7

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Q7 (a) Show that every Cauchy sequence in $(\mathbb{R}^n, \|\cdot\|_2)$ cvg. in $\mathbb{R}^n$ for $n \geq 2$

(b) Does every Cauchy sequence in $(\mathbb{R}^n, \|\cdot\|_\infty)$ and in $(\mathbb{R}^n, \|\cdot\|_1)$ also cvg. in $\mathbb{R}^n$ w.r.t. these norms?

(c) Does every Cauchy seq. in $(\ell^1, \|\cdot\|_\infty)$ cvg in ℓ^1?

(d) Is the $\|\cdot\|_1$-norm equivalent to $\|\cdot\|_\infty$-norm?

Solution

- Some Results (proof in apendix – (as already proved in lectures))

① — $|x_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$ $\forall\, 1 \leq i \leq n,\ p \geq 1,\ n \geq 2$ $(i, p, n \in \mathbb{N})$

② — If $x^{(m)}$ denotes a sequence in $\mathbb{R}^n$

and $x^{(m)} := (x_1^{(m)}, x_2^{(m)}, \dots x_n^{(m)})$

and $x_i^{(m)} \to x_i$ $\forall\ 1 \leq i \leq n$

then $x^{(m)} \xrightarrow{\|\cdot\|_p} x$ $\forall p \geq 1$

where $x := (x_1, x_2 \dots x_n)$

- Proof of (a)

Given: $\forall\, \varepsilon > 0, \exists\, N_\varepsilon$ s.t. $\forall m_1, m_2 \geq N_\varepsilon$, $\|x^{(m_1)} - x^{(m_2)}\|_2 < \varepsilon$

$\Rightarrow \left(\sum_{i=1}^{n} |x_i^{(m_1)} - x_i^{(m_2)}|^2 \right)^{1/2} < \varepsilon$

$\Rightarrow |x_i^{(m_1)} - x_i^{(m_2)}| < \varepsilon$ $\forall\ 1 \leq i \leq n$ $\forall m_1, m_2 \geq N_\varepsilon$ (from ①)

$\Rightarrow x_i^{(m)}$ is a ~~[illegible]~~ Cauchy sequence in $\mathbb{R}$ for all $1 \leq i \leq n$.

Since every Cauchy seq. in $\mathbb{R}$ cvg, ~~[illegible]~~ $x_i^{(m)} \to x_i$. $\forall i$ s.t. $1 \leq i \leq n$

$\Rightarrow (x^{(m)}) \xrightarrow{\|\cdot\|_2} x$ in $\mathbb{R}^n$ (from ②) $\square$

~~Proof of (b)~~

Yes

- <u>Answer of (b)</u> : Yes, every Cauchy sequence in $(\mathbb{R}^n, \|\cdot\|_1)$ and $(\mathbb{R}^n, \|\cdot\|_\infty)$ converges with respect to their respective norms

<u>Proof</u> :

- for $\|\cdot\|_1$

$\forall \varepsilon > 0, \exists N_\varepsilon$ s.t. $\forall m_1, m_2 \geq N_\varepsilon, \|x^{(m_1)} - x^{(m_2)}\|_1 < \varepsilon$

$\Rightarrow \left(\sum_{i=1}^{n} |x_i^{(m_1)} - x_i^{(m_2)}\right) < \varepsilon$

$\Rightarrow |x_i^{(m_1)} - x_i^{(m_2)}| < \varepsilon \quad \forall m_1, m_2 \geq N_\varepsilon, \forall 1 \leq i \leq n$

$\Rightarrow x_i^{(m)}$ is a Cauchy seq. in $\mathbb{R}$ $\forall i$ s.t. $1 \leq i \leq n$.

$\Rightarrow x_i^{(m)}$ converges in $\mathbb{R}$ $\forall i$ s.t $1 \leq i \leq n$

i.e. $x_i^{(m)} \longrightarrow x_i \quad x_i \in \mathbb{R}$

$\Rightarrow x^{(m)} \xrightarrow{\|\cdot\|_1} x$ in $\mathbb{R}^n$

- for $\|\cdot\|_\infty$

$\forall \varepsilon > 0, \exists N_\varepsilon$ s.t. $\forall m_1, m_2 \geq N_\varepsilon, \|x^{(m_1)} - x^{(m_2)}\|_\infty < \varepsilon$

$\Rightarrow \max_{1 \leq i \leq n} |x_i^{(m_1)} - x_i^{(m_2)}| < \varepsilon$

But $|x_i^{(m_1)} - x_i^{(m_2)}| \leq \max_{1 \leq i \leq n} |x_i^{(m_1)} - x_i^{(m_2)}| \quad \forall 1 \leq i \leq n$

$\Rightarrow |x_i^{(m_1)} - x_i^{(m_2)}| < \varepsilon \quad \forall 1 \leq i \leq n.$

$\Rightarrow x_i^{(m)}$ is a Cauchy seq. in $\mathbb{R}$ $\forall 1 \leq i \leq n$

$\Rightarrow$ $x_i^{(m)}$ converges in $\mathbb{R}$ $\quad \forall\, 1 \leq i \leq n$

i.e. $x_i^{(m)} \longrightarrow x_i \quad \forall\, x_i \in \mathbb{R}$

$$\Rightarrow \quad x^{(m)} \xrightarrow{\|\cdot\|_\infty} x \quad \text{in } \mathbb{R}^n$$

$\square$

- <u>Answer of (c)</u>

No, every Cauchy seq. in $(\ell^1, \|\cdot\|_\infty)$ does not cvg in ℓ^1.

One counterexample is enough to ~~g~~ disprove the assertion

Consider $x^{(n)}$ in ℓ^1 ~~i.e.~~

s.t. $x_i^{(n)} = (1, 1/2, 1/3, \ldots, 1/n, 0, 0, 0, 0 \ldots)$

$\left[x^{(n)} \text{ is in } \ell \text{ because } \sum_{i=1}^{n} \frac{1}{i} \leq n < \infty \quad \forall\, n \in \mathbb{N} \right]$

- $x^{(n)}$ is a Cauchy sequence in $(\ell^1, \|\cdot\|_\infty)$

~~Sup~~

How?

Assume $m > n$ (no loss in generality as true otherwise as well)

$$\sup_{i \geq 1} |x_i^{(m)} - x_i^{(n)}| = \frac{1}{n+1} \longrightarrow 0$$

• Answer of (d): No, $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are not equivalent on l_1

Let the metrics induced by $\|\cdot\|_1$ and $\|\cdot\|_\infty$ be d_1 and d_∞.

To show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are not equivalent on l_1, it suffices

to ~~st~~ give a sequence $x^{(n)}$ in l^1 and $x \in l^1$

s.t. $d_\infty(x^{(n)}, x) \to 0$

but $d_1(x^{(n)}, x) \not\to 0$.

Consider the same sequence ~~as~~ used in part (c)

$$x_i^{(n)} = (1, 1/2, 1/3 \dots 1/n, 0, 0, 0 \dots)$$

Let $x_i := 1/i \quad \forall i \in \mathbb{N}$

- $d_\infty(x^{(n)}, x) = \|x^{(n)} - x\|_\infty$

$$= \sup_{i \geq 1} |x_i^{(n)} - x_i|$$

$$= \frac{1}{n+1}.$$

We know that $\frac{1}{n+1} \xrightarrow{n \to \infty} 0$

So, $d_\infty(x^{(n)}, x) \to 0$ as $n \to \infty$.

- $d_1(x^{(n)}, x) = \|x^{(n)} - x\|_1$

$$= \lim_{m \to \infty} \sum_{i=1}^{m} |x_i^{(n)} - x_i|$$

$$= \lim_{m \to \infty} \sum_{i=n+1}^{M} \frac{1}{i}$$

This limit does not converge (harmonic series diverge)

So, $d_1(x^{(n)}, x) \not\to 0$ as $n \to \infty$.

$\square$

Appendix A : Proof of Result ①

Result ① : $|x_i| \leqslant \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \quad \forall\, 1 \leqslant i \leqslant n,\ p \geqslant 1,\ n \geqslant 2$
$(i, p, n \in \mathbb{N})$

Proof : $|x_i|^p \leqslant |x_i|^p \quad \forall\, 1 \leqslant i \leqslant n,\ p \geqslant 1$

$\Rightarrow |x_i|^p \leqslant \sum_{i=1}^{n} |x_i|^p$ [Since $|x_i|^p$ s are positive or 0 hence their sum $\geqslant$ each term]

$\Rightarrow |x_i| \leqslant \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \quad \forall\, 1 \leqslant i \leqslant n$ $\square$

Appendix B : Proof of Result ②

Result ② : If $x^{(m)}$ denotes a sequence of terms in $\mathbb{R}^n$
and $x^{(m)} := (x_1^{(m)}, x_2^{(m)} \ldots x_n^{(m)})$
and $x_i^{(m)} \longrightarrow x_i \quad \forall\, 1 \leqslant i \leqslant n$
then $x^{(m)} \xrightarrow{\|\cdot\|_p} x \quad \forall\, p \geqslant 1$
where $x := (x_1, x_2 \ldots x_n)$

Proof : Given - $x_i^{(m)} \longrightarrow x_i \quad \forall\, 1 \leqslant i \leqslant n$
To show - $\forall\, \varepsilon > 0, \exists\, N_\varepsilon$ s.t. $\forall\, m \geqslant N_\varepsilon$ $\|x^{(m)} - x\|_p < \varepsilon$
where $x := \{x_1, x_2 \ldots x_n)$

Since $\frac{\varepsilon}{n^{1/p}} > 0$, $\forall\, i$, $\exists\, N'_\varepsilon$ s.t $|x_i^{(m)} - x_i| < \frac{\varepsilon}{n^{1/p}} \quad \forall\, m \geqslant N'_\varepsilon$

$\Rightarrow |x_i^{(m)} - x_i|^p < \frac{\varepsilon^p}{n} \quad \forall\, m \geqslant N'_\varepsilon$

Take $N_\varepsilon = \max_{1 \leqslant i \leqslant n} \{N'_{\varepsilon_i}\}$

So, $\forall\, m \geqslant N_\varepsilon$, $\sum_{i=1}^{n} |x_i^{(m)} - x_i|^p < \frac{\varepsilon^p}{n} \cdot n$

(Since each term is less than $\frac{\varepsilon^p}{n}$)

$$\Rightarrow \left(\sum_{i=1}^{n} |x_i^{(m)} - x_i|^p \right)^{1/p} < \varepsilon \quad \forall\, m \geq N_\varepsilon$$

$$\Rightarrow \|x^{(m)} - x\|_p < \varepsilon \quad \forall\, m \geq N_\varepsilon$$

$$\Rightarrow x^{(m)} \xrightarrow{\|\cdot\|_p} x \qquad \square$$