MTH204 : Assignment

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[Q1] Show that every group of order < 60 is solvable.

Solution :

We present the following Lemmas

~~In~~ In all the Lemmas, it is assumed that $|G| < 60$

- Lemma 1 : if $|G| = 2^m \cdot 3^n$, then G is solvable.

  [Proof in Appendix A1]

- Lemma 2 : ~~If~~ if largest prime factor of $|G| > 7$

  then ~~[illegible]~~ G is solvable

  [Proof in Appendix A2]

- Lemma 3 : if Largest prime factor of $|G| = 7$

  then G is solvable.

  [Proof in Appendix A3]

- Lemma 4 : if Largest prime factor of $|G| = 5$

  then G is solvable.

  [Proof in Appendix A4]

We will use one or more of the following Lemmas to show that all groups of order < 60 are solvable.

The following table shows that the above 4 Lemmas are sufficient to prove that every group of order < 60 is solvable.

Order	Factorisation ~~Lemma app~~	Lemma applicable
1	1	only one group of order 1 exists
2	$1\cdot 2$	1
3	$1\cdot 3$	1
4	2^2	1
5	5	4
6	$2\cdot 3$	1
7	7	3
8	2^3	1
9	3^2	1
10	$2\cdot 5$	4
11	11	2
12	$2^2\cdot 3$	1
13	13	2
14	$2\cdot 7$	3
15	$3\cdot 5$	4
16	2^4	1
17	17	2
18	$3^2\cdot 2$	1
19	19	2
20	$2^2\cdot 5$	4
21	$3\cdot 7$	3
22	$2\cdot 11$	2
23	23	2
24	$2^2\cdot 3$	1
25	5^2	4
26	$13\cdot 2$	2
27	3^3	1
28	$7\cdot 2^2$	3
29	29	2
30	$2\cdot 3\cdot 5$	4

Order	Factorisation	Lemma applicable
31	31	2
32	2^5	1
33	$3\cdot 11$	2
34	$2\cdot 17$	2
35	$5\cdot 7$	3
36	$3^2\cdot 2^2$	1
37	37	2
38	$19\cdot 2$	2
39	39	2
40	$2^3\cdot 5$	4
41	41	2
42	$7\cdot 2\cdot 3$	3
43	43	2
44	$11\cdot 4$	2
45	$3^2\cdot 5$	4
46	$23\cdot 2$	2
47	47	2
48	$2^4\cdot 3$	1
49	7^2	3
50	$5^2\cdot 2$	4
51	$17\cdot 3$	2
52	$2^2\cdot 13$	2
53	53	2
54	$3^3\cdot 2$	1
55	$5\cdot 11$	2
56	$7\cdot 2^3$	3
57	$3\cdot 19$	2
58	$29\cdot 2$	2
59	59	2

Appendix A1: Proof of Lemma 1

Lemma 1 : If $|G| < 60$ and $|G| = 2^m \cdot 3^n$ then G is solvable

Proof

We prove by induction

- Base case: ~~[illegible]~~
  - for $n = 0$, $|G| = 2$ which is cyclic, hence solvable.
  - for $m = 0$, $|G| = 3$ which is cyclic, hence solvable
  - for $m = n = 1$, $|G| = 6$ ~~which~~ only 2 such groups exist and are both solvable, as discussed in the lectures

- For $m \leq 3$

$n_3 \in \{1, 4\}$

- if $n_3 = 1$

then the Sylow-3 subgroup is normal and cyclic and thereby solvable

$\Rightarrow$ By induction G/P is solvable

$\Rightarrow$ G is solvable

- if $n_3 = 4$

then we have a non-trivial homomorphism $\phi : G \to S_4$ ~~[illegible]~~ G acts on the 4-element set of Sylow-3 subgroup by conjugation.

$\Rightarrow$ $Im(\phi)$ is solvable (as a subgroup of a solvable group).

$Ker\,\phi$ is solvable (by induction hypothesis)

Since $Im\,\phi \cong G/Ker(\phi)$

$\Rightarrow$ G is solvable.

- For $m \geq 4$

- if $n = 0$

$\Rightarrow G$ is a p-group $\Rightarrow G$ is solvable

~~N/G~~

- if $n = 1$

then $|G| = 48$, $m = 4$ (as $|G| < 60$)

The number of Sylow 2-subgroup $n_2 \in \{1, 3\}$

if $n_2 = 1$, then G is solvable by normality & induction (as shown in the previous case)

(i.e. $m \leq 3$, $n_3 = 1$)

if $n_2 = 3$

then we have a non-trivial homomorphism $\phi : G \to S_3$

(G acts on 3-element set of Sylow 2-subgroups)

Since S_3 is solvable, $\text{Im}\,\phi$ is solvable

$\ker \phi$ is solvable (by Induction hypothesis)

But $G / \ker\phi = \text{Im}\,\phi$

Hence, G is solvable.

Hence, proved $\square$ $\blacksquare$

Appendix A2: Proof of Lemma 2

Lemma 2: ~~if larges~~ given $|G| < 60$
if largest prime factor of $|G|$, $p \geq 7$
then G is solvable.

Proof: Since $p \geq 7$
~~If~~ so, $|G| = pk$ with $k < p$ (as $|G| < 60$)

As $n_p \cong 1 \bmod p$, the only possible $n_p = 1$

So, the Sylow p-subgroups are normal

By induction, it is solvable.

$\Rightarrow$ G/P is solvable

$\Rightarrow$ G is solvable.

$\square$

Appendix A 3: Proof of Lemma 3

Lemma 3: Given $|G| < 60$
if the largest prime factor of $|G|$, $p = 7$
then G is solvable.

Proof: Since $p = 7$, ~~where~~ $|G| = 7k$
~~the only~~ When $k < 8$, the Sylow-7-subgroup is normal
hence, solvable.

Now, for $k \geq 8$, there is only one possible value of
$|G| < 60$ i.e. $|G| = 56$ i.e. $k = 8$

$\Rightarrow n_7 \in \{1, 8\}$

if $n_7 = 1$,
then G is solvable by normality and induction hypothesis
(analogous to proof of similar case in Appendix A1)

if $n_7 = 8$.
we have $48 \ (6 \cdot 8)$ ~~of~~ elements of order 6
$\Rightarrow$ we can only have 1 Sylow 2-subgroup.
But Sylow 2-subgroup is normal
$\Rightarrow G$ is solvable by induction

Hence, proved. $\square$

Appendix A4 : Proof of Lemma 4

Lemma 4 : Given $|G| < 60$
If the largest prime factor of $|G|$, $p = 5$
then G is solvable.

Proof : Since $p = 5$, $|G| = 5k$

- if $k \leq 5$
then $n_5 = 1$.
$\Rightarrow$ Sylow 5-subgroup is normal.
By induction, G is solvable.

- if $k > 5$
then,
Case 1 : $k \not\equiv 1 \bmod 5$
then G is solvable (as shown above).
Case 2 : $k \equiv 1 \bmod 5$.
then the only such possibility for $|G| < 60$ is $|G| = 30$.
i.e. $n_5 \in \{1, 6\}$.

if $n_5 = 1$,
then G is solvable by normality & induction hypothesis
(analogous to proof of similar case in Appendix A1 & A2)

if $n_5 = 6$
Then ~~there~~ we have 24 $(4 \cdot 6)$ elements of order 4.
$\Rightarrow$ we can have only 1 Sylow 3-subgroup.
$Q \in Syl_3(G)$ is normal and G/Q and Q are solvable by induction
Hence, G is also solvable. Hence, proved. $\square$

Q2 List all groups with proof of order < 60 which are nilpotent

Solution:

Order of group	# G nilpotent
1	1
2	1
3	1
4	2
5	1
6	1
7	1
8	5
9	2
10	1
11	1
12	2
13	1
14	1
15	1
16	14
17	1
18	2
19	1
20	2
21	1
22	1
23	1
24	5
25	2
26	1
27	5
28	2
29	1
30	1
31	1
32	51
33	1
34	1
35	1
36	4
37	1
38	1
39	1
40	5
41	1
42	1
43	1
44	2
45	2
46	1
47	1
48	14
49	2
50	2
51	1
52	2
53	1
54	5
55	1
56	5
57	1
58	1
59	1

Proof :

Let $|G| = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$
and let G be nilpotent
Then $G \cong S_1 \times S_2 \times \cdots \times S_k$
where S_i = no. of unique p_i-Sylow subgroups

So, no. of nilpotent groups $= |S_1 \times S_2 \times \cdots \times S_k|$
(# G) $= n(S_1) \cdot n(S_2) \cdots n(S_k)$

where $n(S_i)$ = total no. of possible ~~p_i-subgps~~ p_i-Sylow subgroups upto isomorphism

Now, we present the following 3 Lemmas in this content.

- Lemma 1 : if $|G| = p_1 \cdot p_2 \cdot p_3 \cdots p_k$
  where p_i's are distinct primes
  then no. of nilpotent G (# G) $= 1 \cdot 1 \cdot 1 \cdots 1 = 1$
  [Proof in Appendix B1]
- Lemma 2 : if $|G| = p^2$
  then no. of nilpotent G (# G) $= 2$
  [Proof in Appendix B2]
- Lemma 3 : if $|G| = p^3$ where $p \in \{2, 3\}$
  then no. of nilpotent G (# G) $= 5$
  [Proof in Appendix B3]

Now, we will factorise the orders and apply appropriate lemma to find nilpotent groups of order < 60

Order	Factorisation	lemma(s) applied	No. of nilpotent groups
1	1	only 1 group of order 1 exists	$\boxed{1}$
2	2	1	$\boxed{1}$
3	3	1	$\boxed{1}$
4	2^2	2	$\boxed{2}$
5	5	1	$\boxed{1}$
6	$2 \cdot 3$	1	$\boxed{1}$
7	7	1	$\boxed{1}$
8	2^3	3	$\boxed{5}$
9	3^2	2	$\boxed{2}$
10	$2 \cdot 5$	1	$\boxed{1}$
11	11	1	$\boxed{1}$
12	$2^2 \cdot 3$	1 & 2	$2 \cdot 1 = \boxed{2}$
13	13	1	$\boxed{1}$
14	$2 \cdot 7$	1	$\boxed{1}$
15	$3 \cdot 5$	1	$\boxed{1}$
16	2^4	Sir asked to take this as given	$\boxed{14}$
17	17	1	$\boxed{1}$
18	$2 \cdot 3^2$	1 & 2	$1 \cdot 2 = \boxed{2}$
19	19	1	$\boxed{1}$
20	$2^2 \cdot 5$	1 & 2	$2 \cdot 1 = \cancel{\boxed{2}}\ \boxed{2}$

Order	Factorisation	Lemmas used	No. of nilpotent groups
21	$=3\cdot 7$	1	$\boxed{1}$
22	$=2\cdot 11$	1	$\boxed{1}$
23	$=23$	1	$\boxed{1}$
24	$=2^3\cdot 3$	1 & 3	$5\cdot 1=\boxed{5}$
25	$=5^2$	2	$\boxed{2}$
26	$2\cdot 13$	1	$\boxed{1}$
27	3^3	3	$\boxed{5}$
28	$2^2\cdot 7$	1 & 2	$2\cdot 1=\boxed{2}$
29	29	1	$\boxed{1}$
30	$2\cdot 3\cdot 5$	1	$\boxed{1}$
31	31	1	$\boxed{1}$
32	2^5	Sir asked to take as given	$\boxed{51}$
33	$3\cdot 11$	1	$\boxed{1}$
34	$2\cdot 17$	1	$\boxed{1}$
35	$5\cdot 7$	1	$\boxed{1}$
36	$2^2\cdot 3^2$	2	$2\cdot 2=\boxed{4}$
37	37	1	$\boxed{1}$
38	$2\cdot 19$	1	$\boxed{1}$
39	$3\cdot 13$	1	$\boxed{1}$
40	$2^3\cdot 5$	1 & 3	$1\cdot 5=\boxed{5}$

Order	Factorisation	Lemmas used	No. of nilpotent groups
41	41	1	$\boxed{1}$
42	$2 \cdot 3 \cdot 7$	1	$\boxed{1}$
43	43	1	$\boxed{1}$
44	$2^2 \cdot 11$	1 & 2	$2 \cdot 1 = \boxed{2}$
45	$3^2 \cdot 5$	1 & 2	$2 \cdot 1 = \boxed{2}$
46	$23 \cdot 2$	1	$\boxed{1}$
47	47.	1	$\boxed{1}$
48	$16 \cdot 3$	1 and 8 is asked to assume 14 for 16.	$14 \cdot 1 = \boxed{14}$
49	7^2	2	$\boxed{2}$
50	$2 \cdot 5^2$	1&2	$1 \cdot 2 = \boxed{2}$
51	$3 \cdot 17$	1	$\boxed{1}$
52	$2^2 \cdot 13$	1&2	$2 \cdot 1 = \boxed{2}$
53	53	1	$\boxed{1}$
54	$3^3 \cdot 2$	1& 3	$5 \cdot 1 = \boxed{5}$
55	$5 \cdot 11$	1	$\boxed{1}$
56	$2^3 \cdot 7$	1&3	$5 \cdot 1 = \boxed{5}$
57	$3 \cdot 19$	1	$\boxed{1}$
58	$2 \cdot 29$	1	$\boxed{1}$
59	59	1	$\boxed{1}$
~~60~~	~~$2^2 \cdot 3 \cdot 5$~~	[illegible]	[illegible] $\boxed{2}$

Appendix B1: Proof of Lemma 1

Lemma 1 : If $|G| = p_1 \cdot p_2 \cdots p_k$ where p_i's are distinct primes then number of nilpotent groups $G = 1 \cdot 1 \cdot 1 \ldots 1 = 1$.

Proof

If say $|G| = p$ (where p is a prime)
then G is cyclic and thereby abelian. (as taught in lectures)

$\Rightarrow G \cong \mathbb{Z}/p\mathbb{Z}$

~~We already know that~~ every finite p-group is nilpotent.
~~And since all groups of p…~~
So, the only group of order p is $\mathbb{Z}/p\mathbb{Z}$ which is nilpotent.

Hence, if $|G| = p$, $n(G) = 1$.

So, if $|G| = p_1 \cdot p_2 \cdot p_3 \cdots p_k$
then $n(G) = n(p_1) \cdot n(p_2) \cdots n(p_k) = 1 \cdot 1 \cdots 1 = 1$

Hence, proved. $\square$

Appendix B2: Proof of lemma 2

Lemma 2: If $|G| = p^2$ where p is a prime then number of nilpotent groups $G = 2$

Proof:

Since all finite p-groups are nilpotent $\Rightarrow$ all groups of order p^2 are nilpotent

So the problem reduces to finding total number of groups of order p^2

Let G be the group. We know that any group of order p^2 is abelian

- Case 1 – there is an element of order p^2
  $\Rightarrow$ G is cyclic $\Rightarrow$ $G \cong \mathbb{Z}_{p^2}$

- Case 2 – $\forall g \in G \setminus \{e\}$, $o(g) = p$.
  Let $x \in G \setminus \{e\}$
  So, $\langle x \rangle$ is cyclic
  and $G/\langle x \rangle$ is also cyclic (since $|G/\langle x \rangle| = p$)

  So, consider the homomorphism $\phi : G \to G/\langle x \rangle$
  which maps $g \mapsto g\langle x \rangle$
  (It is a homomorphism because $\langle x \rangle$ is normal)

  So, by Isomorphism Theorem, $G \cong \mathrm{Ker}\,\phi \times \mathrm{Im}\,\phi$
  $\mathrm{Im}\,\phi = G/\langle x \rangle$
  $\mathrm{Ker}\,\phi = \langle x \rangle$., So, $G \cong G/\langle x \rangle \times \langle x \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$

So, number of groups of order $p^2 = 2$. Hence, proved. $\square$

Appendix B3 : Proof of Lemma 3

Lemma 3 : if $|G| = p^3$ where $p \in \{2,3\}$
then $\underset{\text{no. of nilpotent groups}}{\#G} = 5$

• Proof

For $p=2$ ie. $|G| = 8$

• Lemma - If each element $1 \neq g \in G$ is of order 2 then G is abelian and isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots \mathbb{Z}/2\mathbb{Z}$ and $|G|$ is a power of 2

Proof : Assume $1 \neq a \neq b \in G$

So, we have. $a^2 = b^2 = 1 \Rightarrow a = a^{-1}, \; b = b^{-1}$.

$\Rightarrow ab \neq 1$ (as if $ab = 1 \Rightarrow a = b^{-1} = b$)

$\Rightarrow 1 = (ab)^2 = a(ba)b. \Rightarrow ba = a^{-1}b^{-1} = ab.$

Thus G is abelian

Also, since G is finite, it has a finite set of independent generators $a_1, \ldots, a_n$.

Since G is abelian, we can write any $g \in G$ as $g = a_1^{e_1} \cdots a_n^{e_n}$ for $e_i \in \{0,1\}$

$\Rightarrow G = \langle a_1 \rangle \times \cdots \times \langle a_n \rangle$ and $|G| = 2 \times 2 \ldots 2 = 2^n$.

$\square$

- For $|G| = 8$, there are 3 abelian and 2 non-abelian groups.

The 3 abelian groups are $\mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $(\mathbb{Z}/2\mathbb{Z})^3$

We have already seen that ~~a~~ finite - p group is ~~cyclic~~ nilpotent.

So, the 3 abelian groups are ~~cu~~ clearly nilpotent.

~~Since every p finite p-group will be nilpotent~~
~~and $|G| = 8$ has only 2 Sgroups~~

- The other groups must have max order of any element > 2 but < 8
$\Rightarrow \exists$ an element a of order 4
all others (besides 1) ~~b~~ have order 2 or 4
Let b be an element not generated by a.

So, we have elements $1, a, a^2, a^3, b, ab, a^2b, a^3b$

- Now, b^2 can only be one of the 4
But $b^2 = a, a^3 \Rightarrow b$ is not of order 2 or 4
So, $b^2 = 1$ or $b^2 = a^2$.

- Case 1: $b^2 = 1$
So, ba should be equal to the last 3 elements.
If $ba = ab$, then the group is abelian & we get $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
If $ba = a^2b$, then $b^{-1}a^2b = a \Rightarrow a^2 = 1$ which is contradiction
So, we must have $ba = a^3b \Rightarrow (ab)^2 = 1$.
$\Rightarrow a^4 = b^2 = (ab)^2 = 1 \Rightarrow$ dihedral group of order 8.

- Case 2 : $b^2 = a^2$

$\Rightarrow$ b has order 4

If $ba = ab$, then the group is abelian so we get $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

If $ba = a^2 b$, $\Rightarrow ba = b^3$ which is a contradiction
(because then $a = b^2 = a^2$)

Hence, $ba = a^3 b$.

So, we get a group with $a^4 = 1$, $a^2 = b^2$, $ba = a^3 b$.

$\Rightarrow$ quaternion group

- for $|G| = 27$ i.e. $p = 3$

We get 3 Abelian groups; $\mathbb{Z}/27\mathbb{Z}$, $\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$
and $(\mathbb{Z}/3\mathbb{Z})^3$.

So, now we have a nonabelian group of order 27.
Since it is a p-group, the order of the centre is either 3 or 9.
~~But for any group in which~~ We'll have 2 non-Abelian groups. □