MTH301: Analysis I
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Assignment 11

Question 4

Q4 Suppose $(f_n) \in C[0,1]$ and that $f_n \to f$ uniformly on $[0,1]$. Is it true that $\int_0^{1-\frac{1}{n}} f_n \to \int_0^1 f$? Explain.
Solution:
Yes, the statement is true. The proof is as follows.
We need to show that $|\int_0^{1-\frac{1}{n}} f_n - \int_0^1 f| \to 0$ as $n \to \infty$.
Now, $|\int_0^{1-\frac{1}{n}} f_n - \int_0^1 f| = |\int_0^1 f - \int_0^{1-\frac{1}{n}} f_n| = |\int_0^1 f - \int_0^{1-\frac{1}{n}} f + \int_0^{1-\frac{1}{n}} f - \int_0^{1-\frac{1}{n}} f_n|$
$\implies |\int_0^{1-\frac{1}{n}} f_n - \int_0^1 f| \leq |\int_0^1 f - \int_0^{1-\frac{1}{n}} f| + |\int_0^{1-\frac{1}{n}} f - \int_0^{1-\frac{1}{n}} f_n|$

Now, since $f_n \to f$ uniformly in $[0,1]$ and f_n are continuous functions, $f \in C[0,1]$.
By Intermediate Value theorem (since f is continuous and $[0,1]$ is closed and connected in $\mathbb{R}$),$f([0,1])$ is closed, connected and thereby bounded.
So let $X = \|f\|_\infty$.
$\implies |\int_0^1 f - \int_0^{1-\frac{1}{n}} f| \leq (1-(1-\frac{1}{n}))\|f\|_\infty = \frac{X}{n}$
$\implies |\int_0^{1-\frac{1}{n}} f - \int_0^{1-\frac{1}{n}} f_n| \leq (1-\frac{1}{n})\|f - f_n\|_\infty$

Hence, we have from above, $|\int_0^{1-\frac{1}{n}} f_n - \int_0^1 f| \leq \frac{X}{n} + (1-\frac{1}{n})\|f - f_n\|_\infty$
As $n \to \infty$, $\frac{X}{n} \to 0$ and $\|f - f_n\| \to 0$ (as $f_n \to f$ uniformly on $[0,1]$).
$\implies \frac{X}{n} + (1-\frac{1}{n})\|f - f_n\|_\infty \to 0$ as $n \to \infty$
$\implies |\int_0^{1-\frac{1}{n}} f_n - \int_0^1 f| \to 0$ as $n \to \infty$
Hence, proved. ■