

Problem Sheet For Analysis

MATHEMATICS SUMMER PROGRAM

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For this tutorial sheet, we let (X, d) be a metric space.

§1 Basics of Metric-Topology

Problem 1.1

Let $A \subseteq X$. Then, prove that

$$\overline{(A^c)} = \text{int}(A)^c.$$

Problem 1.2

Is $\text{int}(E) = \text{int}(\overline{E})$ for all subsets $E \subseteq \mathbb{R}$? In the other direction, does $\overline{E} = \overline{\text{int}(E)}$ hold for all subsets $E \subseteq \mathbb{R}$?

Problem 1.3

Let A_1, A_2, \dots, A_n be subsets of X . Then, prove that

$$\bigcup_{i=1}^n \overline{A_i} = \overline{\bigcup_{i=1}^n A_i}.$$

Show that

$$\bigcup_{i=1}^{\infty} \overline{A_i} \subseteq \overline{\bigcup_{i=1}^{\infty} A_i}.$$

Also, show via an example that inclusion can be proper.

Problem 1.4

Let $A \subseteq X$ be nonempty and distance of $x \in X$ from A is defined as

$$\text{dist}(x, A) = \inf\{d(x, a) : a \in A\}.$$

Then, prove that $x \in \overline{A}$ if and only if $\text{dist}(x, A) = 0$.

§2 Completeness of Metric Spaces

Problem 2.1

Let $X = \mathbb{N}$ and $d(m, n) = \left| \frac{1}{m} - \frac{1}{n} \right|$. Show that (X, d) is incomplete.

Problem 2.2

Let $X = \mathbb{Z}$ and $d(m, n) = |m - n|$. Show that (X, d) is complete.

Problem 2.3

Let $X = \mathbb{R}$ and $d(x, y) = |\arctan(x) - \arctan(y)|$. Show that (X, d) is incomplete.

§3 Baire Category Theorem

Problem 3.1

Let (X, d) be a complete metric space, a countable collection $\{E_i\}$ of subsets of X and

$$X = \bigcup_{i=1}^{\infty} E_i.$$

Then, prove that there exists an m such that $\text{int}(\overline{E_m}) \neq \emptyset$.

Definition 3.2

We say X to be a *Baire* if it satisfies the following condition: Given any countable collection $\{A_n\}$ of closed sets in X , each of which has an empty interior in X , their union $\bigcup A_n$ also has an empty interior in X .

Problem 3.3

Prove that \mathbb{Q} is not a Baire space while \mathbb{Z}_+ is.

Problem 3.4

Any open subspace of a Baire space is also a Baire space.