Approximation and error in computing.

Error:

Defference between true value and expected value or estimated value/approximated value.

eg: difference between population mean and sample value. ~ estimated mean

→ **Significant digits.** or imp digits

Significant digits of a given no are those imp digits which convey their meaning according to their accuracy.

→ **Rules for determining significant digits:**

Let $x \geq 0$

1. If x represented as $d_1 d_2 d_3 \ldots d_m . e_1 e_2 \ldots e_n$, then where $d_1 \neq 0$, then no of significant digit $= m+n$
2. If $x = 0.00 e_1 e_2 e_3 \ldots e_n$ then where $e_1 \neq 0$, no of significant digits $= n$
3. If $x = a \times 10^n$ where a is a non negative real number. then no of significant digits of $x =$ no of digits of a.

Provide no of significant digits for each representation.

1. 2340000

   $= 234 \times 10^4 = 3$

2. 0.02965

   $= 4$

3. 1.011

   $= 4$

4. 2.23×10^3

   $= 3$

5. 9.569×10^2

   $= 4$

6. 2314×10

   $= 4$

7. 200.000

   $= 6$

8. 30.001

   $= 5$

9. 45

   $= 2$

10. 0.046.

    $= 2$

11. 7.4220

    $= 4$

12. 5002 $= 4$

13. 3800 $= 2$

14. 1070m $= 4$

15. 0.0079800 $= 5$

16. 0.00798 $= 3$

17. 108.0097 $= 7$.

URBAN EDGE ___/___/___

Significant digits

$2.56 \times 10^4 = 3$, $2.560 \times 10^4 \rightarrow 4$, $2.5600 \times 10^4 - 5$

Exact and approximate number:

Exact number: $1, 2, 3, \sqrt{2}, \sqrt{3}, 1/2, 3/4, \pi, e$.

Approximate number: Approx numbers are those numbers that represent to a certain degree of accuracy

for eg:-

An approx value of π is 3.1416, better approx for π is 3.14159265.

- Round off error and truncation error:

Rules: It is usual to round off number according to foll rule:-

To round off a number to 'n' significant digit, discard all digits to the right of nth digit and if this discarded number is

1) less than half a unit in the n'th place leave the n'th place unaltered

2) if '>' half a unit in the place, increase n'th place by a unit.

3.

URBAN EDGE ___/___/___

Errors.

1. Absolute Error 3. Percentage Error.
2. Relative Error

~~Error or~~ Absolute error

E_A = True value − approximate value.
$= x - x' = \delta x$

Relative error:

$$E_R = \frac{E_A}{x} \Rightarrow \frac{\text{Absolute error}}{\text{True value}} = \frac{\delta x}{x}$$

Percentage error.

$$E_p = 100 \times E_R = 100 * \frac{(x - x')}{x} = \underline{100\,E_R\,\%}$$

(b) Let Δx be a number such that $|x - x_1| \leq \Delta x$

→ Absolute accuracy and Relative Accuracy

• Absolute accuracy ⇒ ~~Δx~~

Since Δx is an upper limit, on the magnitude of absolute error this said to measure absolute accuracy.

Absolute ~~error~~ accuracy $=$ ~~δx~~ Δx

The quantity $\frac{\Delta x}{|x|}$ approximately equal ($\simeq$) $\frac{\Delta x}{x'}$

example

Q) If number x is rounded to N decimal places then $\Delta X = \frac{1}{2} \times 10^{-N}$

→ If $x = 0.51$ and is corrected to 2 decimal places find ΔX and percentage accuracy.

$x = 0.51$ ~~Δx~~ $N = 2$

$$\Delta X = \frac{1}{2} \times 10^{-2} = 0.5 \times 10^{-2} = 0.005$$

Relative accuracy $= \frac{0.5 \times 10^{-2}}{0.51} = \frac{50}{51} \times 10^{-2}$

$= 0.98 \cancel{\times 10^{-2} = 0.098}\ 0.98 \times 10^{-2}$

Percentage accuracy $= 0.98$.

$$\begin{array}{r} 0.98 \\ 51 \enclose{longdiv}{500} \\ 459 \\ \hline 410 \end{array}$$

$51 \times 9 = 459$

$51 \times 8 = 40.8$

Q) An approximate value of π is given by $X_1 = \frac{22}{7} = 3.1428571$ and its true value is $X = 3.1415926$. Find E_A and E_R.

Absolute error $= E_A = \overbrace{3.1415926 - 3.1428571}^{X - X'}$

$= -0.0012645$

$$\begin{array}{r} 3.1415926 \\ -3.1428571 \\ \hline 355 \end{array}$$

$$\begin{array}{r} 3.1428571 \\ -3.1415926 \\ \hline 0.0012645 \end{array}$$

Relative error $= \frac{-0.0012645}{3.1415926}$

$= -0.00040250255$

Q) Find the relative error of the number 8.6 if both of its digits are correct.

significant digit = 2 decimal points = 1

Q) calculate EA $= \frac{1}{2} \times 10^{-N} = \frac{1}{2} \times 10^{-1} = 0.05$

ΔX

$X = 8.6$

$EA = 8.6 - 0.05 = 8.55$

$ER = \frac{X - X'}{X} = \frac{8.55}{8.6} = \frac{855}{860}$

$8.60 - 0.05 = 8.55$

$860 \overline{)8550}$ 0.

-860

9

Q) Find the difference $\sqrt{6.37} - \sqrt{6.36}$ upto 3 significant figures.

$\sqrt{6.37} = 2.52388589$

$\sqrt{6.36} = 2.52190404$

diff $= 2.52388589 - 2.52190404$

$= 0.00198185$

$= 0.00198$

$2.52388589 - 2.52190404 = 0.00198185$

Method - 2

$$\sqrt{6.37} - \sqrt{6.36} = \frac{(\sqrt{6.37})^2 - (\sqrt{6.36})^2}{\sqrt{6.37} + \sqrt{6.36}}$$

$$= \frac{6.37 - 6.36}{2.524 + 2.522} = 0.01$$

We want to find the ΔX for each number and find min digit decimal numbers

Sum the foll numbers. 0.1532, 15.45, 0.000354, 305.1, 8.012, 143.3, 0.0212, 0.643, 0.1734 where in each of which all the digits are correct. Find absolute error.

~~$X \cdot \frac{472.853154}{9} = 52.5512343$~~

~~$\Delta X = \frac{1}{2} \times 10^{-N} = \frac{1}{2} \times 10^{-6} = 0.5 \times 10^{-6}$~~

~~Absolute error = $52.5512343 - 0.5 \times 10^{-6}$~~

$0.1532 \longrightarrow 0.15$

$15.45 \longrightarrow 15.45$

$0.000354 \longrightarrow 0.00$

~~Find absolute error.~~ $305.1 \longrightarrow 305.1$

$8.012 \longrightarrow 8.12$

$143.3 \longrightarrow 143.3$

$0.0212 \longrightarrow 0.02$

$0.643 \longrightarrow 0.64$

$0.1734 \longrightarrow 0.17$

472.95

$EA = 0.05(2) + 0.005(7)$
$= 0.135$
≈ 0.14 } There is a small error of 0.05 q approx

so we should add $0.005 \approx 0.01$

Two numbers are given as 2.5 and 48.289 both q which being correct to the significant figures given find the product.

$\Delta X = \frac{1}{2} \times 10^{-1} = 0.05$ //

$2.5 \longrightarrow 2.5$

$48.289 \longrightarrow 48.29 \longrightarrow 48.3$.

$\Longrightarrow 2.5 \times 48.3 \Longrightarrow 120.75$ //.

$120.8 \approx \underline{1.2 \times 10^{2}}$

In the product, we retain only 2 significant ~~digits since~~ one of the given no.s i.e ~~2.5 contained~~ only 2 sig figures.

Find the absolute error in the sum of the numbers 105.6, 27.28, 5.63, 0.1467, 0.000523, 208.5, 0.0235, 0.432, 0.0467. Thus all the numbers where each number is correct to the digit given.

$\Delta X = \frac{1}{2} \times 10^{-1} = 0.05$ 2 ~~significe~~ 1

$105.6 \longrightarrow 105.6$
$27.28 \longrightarrow 27.28$
$5.63 \longrightarrow 5.63$
$0.1467 \longrightarrow 0.15$
$0.000523 \longrightarrow 0.00$
$208.5 \longrightarrow 208.5$
$0.0235 \longrightarrow 0.02$
$0.432 \longrightarrow 0.43$
$0.0467 \longrightarrow 0.05$

347.66

≈ 347.7

± 0.15 //

Calculate the value of $\sqrt{102} - \sqrt{101}$ correct to 4 significant figures.

$\sqrt{102} = 10.0995$

$\sqrt{101} = 10.0499$

difference of $\sqrt{102} - \sqrt{101}$

$= 0.0496$

Round of the foll nos to 4 ~~38~~ 38.46235, 0.70029, 0.0022218, 19.235101, 2.36425

Explain the term round off error and roundoff the foll number to 2 decimal places

48.21416, 2.3742, 52.275, 2.375, 2.385, 81.255.

Error Propagation

The purpose of error propagation is to study how errors in numbers can propagate ~~to~~ through mathematical functions. for eg: If we multiply two numbers having errors. we would like to estimate the errors in product.

A General error formula.

Let $u = f(x, y, z) \longrightarrow ①$

Let errors in x, y, z be $\Delta x, \Delta y, \Delta z$ respectively.

Then the error ~~in~~ Δu in u is given by

$u + \Delta u = f(x + \Delta x, y + \Delta y, z + \Delta z) \longrightarrow ②$

expanding RHS of eq ② by Taylor series we obtain

$$u + \Delta u = f(x, y, z) + \frac{\partial u}{\partial x} \times \Delta x + \frac{\partial u}{\partial y} \times \Delta y + \frac{\partial u}{\partial z} \times \Delta z$$

+ terms including higher power of Δx, Δy and Δz.

Assuming that the errors Δx, Δy and Δz are small the higher powers can be neglected.

Now eqn ③ becomes.

$$\Delta u = \frac{\partial u}{\partial x} \times \Delta x + \frac{\partial u}{\partial y} \times \Delta y + \frac{\partial u}{\partial z} \times \Delta z$$

Relative error in u is given by $\frac{\Delta u}{u}$

$$E_R = \frac{\Delta u}{u} = \frac{\partial u}{\partial x}\frac{\Delta x}{u} + \frac{\partial u}{\partial y} \times \frac{\Delta y}{u} + \frac{\partial u}{\partial z} \times \frac{\Delta z}{u}$$

Q) Find the value of $s = \frac{a^2\sqrt{b}}{c^3}$ where $a = 6.54 \pm 0.01$, $b = 48.64 \pm 0.012$, $c = 13.5 \pm 0.03$ also find relative error in the result. c^{-3}

$$S = \frac{6.4646 \times 6.9622}{405.7317}$$

$$\Delta S = \cancel{\frac{2a\sqrt{b}}{c^3}} \frac{\partial s}{\partial a}\Delta a + \frac{\partial s}{\partial b}\Delta b + \frac{\partial s}{\partial c}\Delta c$$

$$\frac{\sqrt{b} \times 2a}{c^3} \times \Delta a + \frac{a^2}{c^3}\frac{1}{2\sqrt{b}}\Delta b + -3c^{-3-1}a^2\sqrt{b}$$

URBAN EDGE

$$= \frac{\sqrt{48.64} \times 2 \times 6.54 \times 0.01}{(13.5)^3} + \frac{1}{2\sqrt{48.64}} \frac{\times (6.54)^2}{(13.5)^3} \times 0.02$$

$$\cancel{=} \frac{3 \times (6.54)^2 \times \sqrt{48.64} \times 0.03}{(13.5)^4}$$

$$\frac{6.97 \times 0.1308}{2460.375} +$$

OR

$$s = \frac{a^2\sqrt{b}}{c^3}$$

Applying log on both side

$$\log s = \log(a^2) + \log\sqrt{b} - \log c^3$$

$$\log s = 2\log a + \frac{1}{2}\log b - 3\log c$$

$$\frac{\Delta s}{s} \leq 2 \times \frac{\Delta a}{a} + \frac{\Delta b}{2b} + 3 \times \frac{\Delta c}{c}$$

$$= \frac{2 \times 0.01}{6.54} + \frac{1}{2} \times \frac{0.012}{48.64} + \frac{3 \times 0.03}{13.5}$$

$$= \frac{0.02}{6.54} + \frac{1}{2} \times 2.467 \times 10^{-4} + 3 \times 2.22 \times 10^{-3}$$

$3.0581 \times 10^{-3} + 1.2335 \times 10^{-4} + 6.66 \times 10^{-3}$

$= 3.0581 \times 10^{-3} + 12.335 \times 10^{-3} + 6.66$

$= 22.0531 \times 10^{-3}$

$=$

Q Given that $u = \frac{5xy^2}{z^3}$. Find the relative error at $x = y = z = 1$. When the errors in each of x, y, z is 0.001

E_R max

Ans $= 0.006$.

taking log on both sides

$\log u = 5 \cdot x(2\log y - 3\log z)$

$\frac{\Delta u}{u} \leq 5 (2 \frac{\Delta y}{y} + 3 \frac{\Delta z}{z})$

$= 5 (2 \times \frac{0.001}{1} + 3 \times \frac{0.001}{1})$

$= 5(0.002 + 0.003)$

$= (0.005) 5$

$0.005 \times 5 = 0.025$

$= 5(0.002 - 0.003)$

$= 5(0.001)$

$\frac{\Delta u}{u} \leq 0.005$

Root of

system

Trans

eg: $f(x$

$\phi(c$

$f(x$

$\rightarrow$ fn need

General

method 1) fn(x)

Bisection

Find

$f(1) =$

$f(2)$

$f(-1) =$

$f(x_0)$

3.375

=

MODULE-2

URBAN EDGE

Roots of non linear equations and solution of system of linear equation. (non algebraic eqns)

Transcendental equations

dependent

$y = f(x)$

independent variable

eg: $f(x) = \ln x^3 - 0.7$

$\phi(x) = e^{-0.5x} - 5x$

$f(x) = \sin^2 x - x^2 - 2$

→ fn needs ~~atleast~~ ~~atmost~~ atleast 2 variables

General form of an algebraic equation

$fn(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$.

method 1) Bisection method

Find the real root of eqn $f(x) = x^3 - x - 1 = 0$

$f(1) = 1 - 1 - 1 = -1$ //

$f(2) = 8 - 2 - 1 = 5$ //

$f(-1) = -1 + 1 - 1 = -1$ //

$f(3) = 9 - 3 - 1 = 6 - 1 = 5$ //

root lies b/w [1, 2]

$x_0 = \frac{1+2}{2} = 1.5$ //

$f(x_0) = f(1.5)$

$(1.5)^3 - (1.5) - 1$

~~$2.25 - 1.5 - 1$~~

~~$= -0.25$~~

$3.375 - 1.5 - 1$

$= 0.875$ //

15
15
175
150
225

12
2.25
1.50
~~1.00~~
0.75
1.00
0.

[1, 1.5]

$1 + 3/2 = 5/2 = 1.25 = +ve.$

$x_0 = 1.5$

$x_1 = 1.25$

$x_2 = \frac{1.5 + 1.25}{2} = 1.375$

$f(x_2) = f(1.375) = (1.375)^3 - 1.375 - 1$

$= 2.59960 - 1.375$

$1.2246 - 1$

$= 0.2246$

$x_3 = \frac{1.25 + 1.375}{2} = 1.3125$

$x_4 = \frac{1.375 + 1.3125}{2} = 1.3437$

$x_5 = \frac{1.3125 + 1.3437}{2} = 1.328125$

Q) Find a real root of the eqn $x^3 - 2x - 5 = 0$.

$f(1) = 1 - 2(1) - 5 = 1 - 2 - 5 = 1 - 7 = -6$

$f(2) = 8 - 4 - 5 = 8 - 9 = -1$

$f(3) = 27 - 6 - 5 = 21 - 5 = 16$ //

$[2, 3]$

$x_1 = \frac{2+3}{2} = \frac{5}{2} = 2.5$

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$f(2.5) = (2.5)^3 - 2(2.5) - 5$

$15.625 - 5 - 5 = 5.625$

$[2, 2.5]$

$x_2 = \frac{2+2.5}{2} = 2.25$ //

$f(2.25) = (2.25)^3 - 2(2.25) - 5 = 11.390625 - 4.5 - 5$

$= 1.890625$

$x_3 = \frac{2+2.25}{2} = 2.125$ //

$f(2.125) = 0.34$ +ve

$x_3 = \frac{2+2.125}{2} = 2.0625$ //

$F(2.0625) = -0.3513$ −ve

$x_4 = \frac{2.0625 + 2.125}{2} = 2.04375$

$f(2.09375) = -0.008941650$

$x_5 = \frac{2.09375 + 2.125}{2}$

$= 2.109375$

Find the real root of $f(x) = x^3 + x^2 + x + 7 = 0$.

correct to 3 decimal places.

error $\Delta x = \frac{1}{2} \times 10^{-3} = 0.5 \times 10^{-3} = 0.0005$

$f(-1) = (-1)^3 + (-1)^2 + -1 + 7$

$= \cancel{-1+1} - 1 + 7 = 6$ //

$f(0) = 0 + 0 + 0 + 7 = 7$

$f(-2) = -8 + 4 - 2 + 7 = -4 - 2 + 7 = 1$

$f(-3) = -27 + 9 + -3 + 7 = -18 - 3 + 7$

$= -21 + 7 = -12$

$[-3, -2]$

$x_0 = \frac{-3 + -2}{2} = \frac{-5}{2} = -2.5$

$f(-2.5) = -15.625 + 6.25 + -2.5 + 7$

$= -4.875$ //

$[-2.5, -2]$

$x_1 = -2.25$

$f(-2.25) = (-2.25)^3 + (-2.25)^2 + (-2.25) + 7$

$= -11.390625 + 5.0625 - 2.25 + 7$

$= -1.578125$ //

$x_2 = \frac{-2.25 - 2}{2} = -2.125$ //

$f(x_2) =$

$x_3 =$

$F(2.$

$x_4 =$

$F(x_4$

$x_5 =$

$F(x_5$

x_6

$F(x$

x_7

F

$f(x_2) = \bar{9}.59570313$
$+ 4.515625$
$- 2.25 + 7 = -0.330007813$

$x_3 = 2.0625$

$F(2.0625) = (-8.77368164) + 4.25390625$
$- 2.0625 + 7 = 0.4177246$ //

$x_4 = \dfrac{-2.125 - 2.0625}{2} = \bar{2}.09375$

$F(x_4) = \bar{9}.17855835 + 4.38878906 - 2.09375$
$+ 7 = 0.11148071$

$x_5 = \dfrac{\bar{2}.125 + \bar{2}.0935}{2} = 2.109218$

$F(x_5) = -9.3859195 + 4.44956836$
$- 2.1094 + 7$
$= -0.04575114$ //

$x_6 = -2.10165$ $\quad F(x_7) = -ve.$

$F(x_6) = +ve.$

$x_7 = \bar{2}.1094 + 2.1016$ $\quad$ 2.1055
$= -2.1055$ $\quad$ 2.10165

$F(x_8) =$

$x_9 = -2.1635 - 2.1094/2 =$

$x_{10} = \frac{2.1094 + 2.1045}{2}$

$= 2.10695$

URBAN EDGE

1. Find ~~real~~ the +ve root b/w 0 and 1 of the eqn $x = e^{-x}$ to a tolerance of 0.05%.

Find the root correct to 3 decimal places and lying between 0 and 0.5 of the eqn $e^{-x} + \sin x - 1 = 0$

1. $x = e^{-x}$

$xe^x = e^{-x} \times e^x \Rightarrow xe^x = 1$ / $f(x) = xe^x - 1 = 0$

$f(0) = 0 \times e^0 - 1 = 0$

-1 //

$f(1) = 1e^1 - 1 = e - 1 = 1.71$ //

$x_1 = \frac{0+1}{2} = 1/2 = 0.5 = +ve$

$f(x_1) = (0.5)e^{(0.5)} - 1 = 0$

$(0.5)(1.64872127) - 1 = -0.175639$

$-ve$.

$x_2 = \frac{0.5+1}{2} = 0.75$ //

$\varepsilon_1 = \left|\frac{x_2 - x_1}{x_2}\right| \times 100 = \frac{0.25}{0.75} \times 100 = 33.3\%$ //

$f(x_2) = 0.58$

$x_3 = 0.5$

$\varepsilon_2 =$

$x_4 = 0.5625$

$x_6 = 0.5781$

~~$x_8 = 0.560$~~

$x_8 = 0.5664$

$x_{10} = 0.5674$

$x_{12} = 0.5671$

$\varepsilon_{11} = 0.0$

$f(x_2) = 0.5878$ root lies 0.5 and 0.75

$x_3 = \frac{0.5+0.75}{2} = 0.625$

$\varepsilon_2 = \frac{0.625-0.75}{0.625} \times 100 = 20\%.$

$x_4 = 0.5625$ $\varepsilon_3 = 11.1\%$ $x_5 = 0.5938$ $\varepsilon_4 = 5.26\%$

$x_6 = 0.5781$ $\varepsilon_5 = 2.71\%$ $x_7 = 0.5703$ $\varepsilon_6 = 1.37\%$

~~$x_8 = 0.568$~~

$x_8 = 0.5664$ $\varepsilon_7 = 0.69\%$ $x_9 = 0.5684$ $\varepsilon_8 = 0.35\%.$

$x_{10} = 0.5674$ $\varepsilon_9 = 0.18\%$ $x_{11} = 0.5669$ $\varepsilon_{10} = 0.09\%.$

$x_{12} = 0.5671$ $\varepsilon_{11} = 0.035\%.$

$\varepsilon_{11} = 0.035\% < 0.05\%.$ root is 0.567

2) P $4e^{-x}\sin x - 1 = 0$

$f(x) = 4e^{-x}(\sin x) - 1 = 0.$

$f(0) = 4e^{-0}\sin 0 - 1 = -1$

$\cancel{f(0.5)} = f(1) = 4e^{-1}\cancel{\sin 1}\ 0.163185$

$x_1 = 0.25$

$f(0.25) = -0.22929$

$x_2 = \frac{0.75}{2} = 0.375$ //

$x_3 = 0.3125$ $\quad x_4 = 0.3438$ $\quad x_5 = 0.3594$