Study Material

(Electromagnetism)

Chapter-30

Inductance

Topics covered:	• Self Inductance and Inductors • Magnetic-Field Energy • The R-L Circuit • The L-C Circuit • The L-R-C Series Circuits
Conceptual Problems:	TYU 30.2, 30.3, 30.4, 30.5, 30.6
Example Problems:	30.4, 30.9, 30.10
Exercise Problems:	30.25, 30.39

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Self-Inductance and Inductors

An inductor is a circuit element that stores magnetic field. If the current through the circuit is changing then the magnetic field will change accordingly. So, according to Faraday's law an emf will be induced across it.

The magnitude of the flux linked with the coil at any instant is directly proportional to the magnitude of the current. So,

$$\phi_B \, \alpha \, i$$

⇨ $\phi_B = L\,i$

If the coil has N turns, then

$$N\,\phi_B = L\,i$$

⇨ $N\dfrac{d\varphi_B}{dt} = L\dfrac{di}{dt}$ ------------------------(1)

The proportionality constant L is called the self inductance of the device. It is a property of the device (geometry, windings) and does not depend on the current.

Unit of L is "**henrys**",

Where 1 henry = 1 volt-second/ampere.

1 Henry = 1 Weber per 1 Ampere

1 Henry = 1 volt-second per 1 Ampere

1 Henry = 1 ohm-second per 1 Ampere

1 Henry = 1 joule per 1 Ampere

$1\ H = 1\ Wb/A = 1\ V\,.\,s/A = 1\ \Omega.\,s = 1\ J/A$

The Henry is a rather large unit of self inductance. Typical values of mutual inductance can be in the millihenry (mH) or microhenry 1mH.

From Faraday's law we have

$$\varepsilon = -N\frac{d\varphi_B}{dt}$$

⇨ $\varepsilon = -L\dfrac{di}{dt}$ -------------------------- (2)

The minus sign in Eq. (2) is from Lenz's law; it says that the self-induced emf in a circuit opposes any change in the current in that circuit.

Equation (2) also states that the self-inductance of a circuit is the magnitude of the self-induced emf per unit rate of change of current.

By measuring the rate of change of current in the circuit (di/dt) and the induced emf, we can determine L.

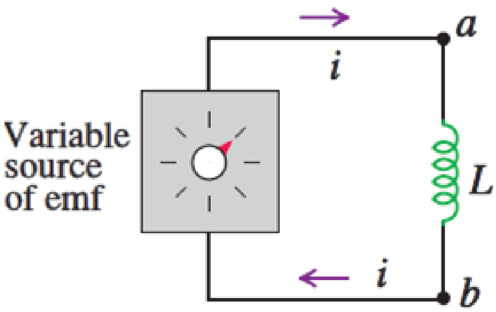
Inductors As Circuit Elements

A circuit device that is designed to have a particular inductance is called an inductor, or a choke.

The usual circuit symbol for an inductor is 

The purpose of an inductor is to oppose any variations in the current through the circuit.

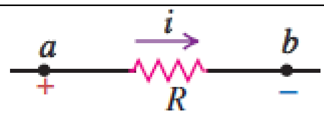
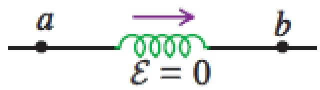
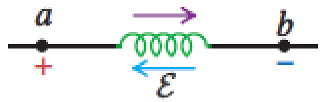
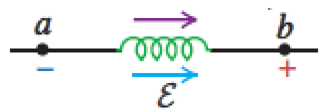
- An inductor in a direct-current circuit helps to maintain a steady current despite any fluctuations in the applied emf.
- In an alternating-current circuit, an inductor tends to suppress variations of the current that are more rapid than desired.



In the circuit the potential difference across the inductor (V_{ab}) is the magnitude of induced emf (ε).

Thus, $\varepsilon = V_a - V_b = L \dfrac{di}{dt}$

Comparison of the behaviors of a resistor and an inductor:

Resistor with current i flowing from a to b	potential drops from a to b is Vab = iR > 0	
Inductor with constant current i flowing from a to b (i constant: di/dt = 0)	Potential difference between a and b is $V_{ab} = L\dfrac{di}{dt} = 0$	
Inductor with increasing current i flowing from a to b (i increasing: di/dt > 0)	Potential drops from a to b is $V_{ab} = L\dfrac{di}{dt} > 0$	
Inductor with decreasing current i flowing from a to b (i decreasing: di/dt < 0)	Potential increases from a to b is $V_{ab} = L\dfrac{di}{dt} < 0$	

Applications of Inductors

Fluorescent light: As an inductor opposes changes in current, it plays an important role in fluorescent light fixtures. In such fixtures, current flows from the wiring into the gas that fills the tube, ionizing the gas and causing it to glow. For this purpose an inductor or magnetic ballast is put in series with the fluorescent tube so that the self-induced emf sustains the current and keeps the tube lit.

Streetlights: Magnetic ballasts are used to sustains the current and keeps the streetlights lit (which obtain their light from a glowing vapor of mercury or sodium atoms).

Automobiles: The self-inductance of a circuit depends on its size, shape, and number of turns. For N turns close together, it is always proportional to N^2. It also depends on the magnetic properties of the material enclosed by the circuit.

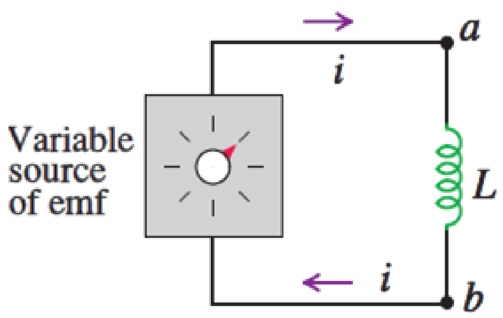
If the material is ferromagnetic, the difference is of crucial importance. A solenoid wound on a soft iron core having K_m = 5000 can have an inductance approximately 5000 times as great as that of the same solenoid with an air core. Ferromagnetic-core inductors are very widely used in a variety of electronic and electric-power applications.

An added complication is that with ferromagnetic materials the magnetization is in general not a linear function of magnetizing current, especially as saturation is approached. As a result, the inductance is not constant but can depend on current in a fairly complicated way. But we assume always that the inductance is constant. This is a reasonable assumption even for a ferromagnetic material if the magnetization remains well below the saturation level.

Because automobiles contain steel, a ferromagnetic material, driving an automobile over a coil causes an appreciable increase in the coil's inductance. This effect is used in traffic light sensors, which use a large, current carrying coil embedded under the road surface near an intersection. The circuitry connected to the coil detects the inductance change as a car drives over. When a pre-programmed number of cars have passed over the coil, the light changes to green to allow the cars through the intersection.

Magnetic-Field Energy:

Let us consider an inductor connected across a variable source of emf. It is shown in the figure. Let the current 'i' is increases in the inductor.



- Then an emf is induced between its terminals
- And a corresponding potential difference V_{ab} between the terminals of the source is developed.
- Here the point 'a' is at higher potential than point 'b'.
- Thus the source must be adding energy to the inductor, and the instantaneous power (rate of transfer of energy into the inductor) is given by

  $P = V_{ab}\, i.$
- Establishing a current in an inductor requires an input of energy, and an inductor carrying a current has energy stored in it.

Energy Stored in an Inductor

Let us consider an inductor with inductance L connected across a variable source of emf. We assume that the inductor has zero resistance, so no energy is dissipated within the inductor.Let the current 'i' is increases in the inductor.

The rate of change of current = $\dfrac{di}{dt}$

As i is increasing, we have, $\dfrac{di}{dt} > 0$

The voltage between the terminals a and b of the inductor at this instant is

$$V_{ab} = L\frac{di}{dt}$$

The instantaneous power (P) supplied by the external source (i.e the rate at which energy is being delivered to the inductor) is

$$P = V_{ab}\ i = \left(L\frac{di}{dt}\right)i$$

$$\Rightarrow\ P = L\,i\frac{di}{dt}$$

The energy dU supplied to the inductor during an infinitesimal time interval dt is

dU = Pdt, so

$$dU = P\,dt = \left(L\,i\frac{di}{dt}\right)dt$$

$$dU = L\,i\,di$$

The total energy U supplied while the current increases from zero to a final value I is

$$U = \int_0^I L\,i\,di = \frac{1}{2}L\,I^2$$

This is the energy stored in an inductor.

1. After the current has reached its final steady value I,
   - Current = constant
   - di/dt = 0
   - No more energy is input to the inductor.
2. When there is no current,
   - the stored energy is zero;
   - when the current is I, the energy is $\frac{1}{2}L\,I^2$
3. When the current decreases from I to zero
   - the inductor acts as a source that supplies a total amount of energy $\frac{1}{2}L\,I^2$ to the external circuit.
   - If the circuit is opened suddenly, the current decreases very rapidly,
   - the induced emf is very large,
   - and the energy may be dissipated in an arc across the switch contacts.
   - This large emf is the electrical analog of the large force exerted by a car running into a brick wall and stopping very suddenly.

Magnetic Energy Density(u)

The energy in an inductor is actually stored in the magnetic field within the coil. Let us consider an ideal toroidal solenoid.

A = cross-sectional area of toroidal solenoid

V = volume enclosed by the toroidal solenoid = 2 π r A

The magnetic energy density stored in an inductor is

$$u = \frac{\text{Magnetic Energy (U)}}{\text{Volume (V)}}$$

$$u = \frac{\frac{1}{2} L I^2}{(2\pi r) A} = \frac{1}{2}\left(\frac{L I^2}{2\pi r}\right) = \frac{1}{2}\frac{I^2}{(2\pi r) A}\left(\frac{\mu_0 N^2 A}{2\pi r}\right)$$

$$u = \frac{\mu_0}{2}\frac{I^2 N^2}{(2\pi r)^2}$$

$$u = \frac{1}{2\mu_0}\left(\frac{\mu_0^2 N^2 I^2}{(2\pi r)^2}\right) = \frac{B^2}{2\mu_0}$$

When the material inside the toroid is not vacuum but a material with magnetic permeability μ the magnenetic energy density is given by

$$u = \frac{B^2}{2\mu}$$

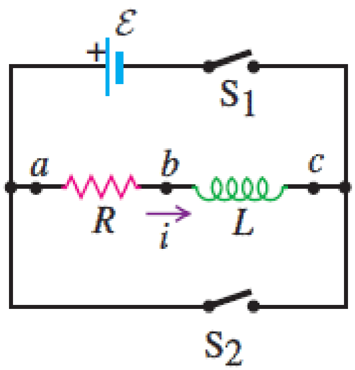
Some points regarding toroidal solenoid

- **Magnetic field (B) at a distance r from the toroid axis is**
  $$B = \frac{\mu_0 N i}{2\pi r}$$
- **Magnetic flux linked with the coil is**
  $$\varphi_B = B A = \left(\frac{\mu_0 N i}{2\pi r}\right) A$$
- **Self inductance of the coil is**
  $$L = \frac{N\varphi_B}{i} = \frac{N}{i}\left(\frac{\mu_0 N i}{2\pi r}\right) A = \frac{\mu_0 N^2 A}{2\pi r}$$
- **The magnetic energy stored in an inductor is**
  $$U = \frac{1}{2} L I^2$$

This expression for the energy density associated with any magnetic-field configuration in a material with constant permeability.

Magnetic-field energy plays an important role in the ignition systems of gasoline-powered automobiles.

The R-L Circuit



The circuit shown is a R-L combination in series with a source of emf ε. We assume that the source has zero internal resistance, so the terminal voltage equals the emf.

Let, i = instanteneous current in the circuit

v_R = instanteneous difference across the resistor

v_L = instanteneous difference across the inductor

So, $v_R = i R$ and

$$v_L = L\frac{di}{dt}$$

Applying Kirchhoff's loop rule, we get,

$$\varepsilon - v_R - v_L = 0 \qquad \Rightarrow \quad \varepsilon - i R - L\frac{di}{dt} = 0$$

$$\Rightarrow \quad L\frac{di}{dt} = \varepsilon - i R \qquad \Rightarrow \quad \frac{di}{dt} = \frac{\varepsilon}{L} - \frac{i R}{L} \qquad \text{----------------------(1)}$$

$$\Rightarrow \frac{di}{dt} = \frac{R}{L}\left(\frac{\varepsilon}{R} - i\right) \qquad \Rightarrow \frac{di}{\left(\varepsilon/R - i\right)} = \frac{R}{L}dt$$

$$\Rightarrow \frac{di}{\left(i - \varepsilon/R\right)} = -\frac{R}{L}dt \qquad \Rightarrow \int_0^i \frac{di}{\left(i - \varepsilon/R\right)} = \int_0^t -\frac{R}{L}dt$$

$$\Rightarrow \left[\ln\left(i - \varepsilon/R\right)\right]_0^i = \left[-\frac{R}{L}t\right]_0^t \qquad \Rightarrow \ln\left(i - \varepsilon/R\right) - \ln\left(-\varepsilon/R\right) = -\frac{R}{L}t$$

$$\Rightarrow \ln\frac{\left(i - \varepsilon/R\right)}{\left(-\varepsilon/R\right)} = -\frac{R}{L}t \qquad \Rightarrow \frac{\left(i - \varepsilon/R\right)}{\left(-\varepsilon/R\right)} = e^{-\left(\frac{R}{L}\right)t}$$

$$\Rightarrow \left(i - \frac{\varepsilon}{R}\right) = -\frac{\varepsilon}{R}e^{-\left(\frac{R}{L}\right)t} \qquad \Rightarrow i = \frac{\varepsilon}{R}\left[1 - e^{-\left(\frac{R}{L}\right)t}\right]$$

$$\Rightarrow i = I_0\left[1 - e^{-\left(\frac{R}{L}\right)t}\right] \qquad \text{---------------(2)}$$

Where, $I_0 = \frac{\varepsilon}{R}$ = maximum current ---------------(3)

This expression represent the current in an R-L circuit when current increses from its initial value $i = 0$

The rate of change of the current is given by

$$i = I_0\left[1 - e^{-\left(\frac{R}{L}\right)t}\right]$$

$$\frac{di}{dt} = \frac{\varepsilon}{R}\left(\frac{R}{L}\right)e^{-\left(\frac{R}{L}\right)t} = \frac{\varepsilon}{L}e^{-\left(\frac{R}{L}\right)t} \qquad \text{---------------(4)}$$

Analysis:

1. At $t = 0$ initial time
   - $i = 0$ [using eqn. (2)]
   - The rate of increase of current $\left(\frac{di}{dt}\right)_{initial}$ is

     $$\left(\frac{di}{dt}\right)_{initial} = \frac{\varepsilon}{L} \text{ [using eqn. (2) and (4)]}$$

2. At $t = \infty$ final time
   - $i = \frac{\varepsilon}{R} = I_o = \text{constant}$ [using eqn. (2)]
   - and $\frac{di}{dt} = 0$

3. As the current increases, the term $\left(\frac{R}{L}\right)i$ increases [using Eq. (1)]

So, the rate of increase of current $\left(\frac{di}{dt}\right)$ becomes smaller $\left[\text{using Eq. (1)}\right]$

As the current approaches a final, steady-state value I.

Then $\frac{di}{dt} = 0$, then $\frac{\varepsilon}{L} - \frac{I_o R}{L} = 0 \Rightarrow I_o = \frac{\varepsilon}{R}$ $\left[\text{using Eq. (1)}\right]$

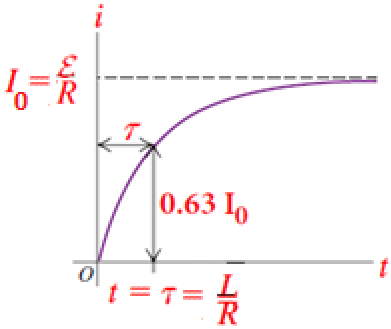
The final current I_0 does not depend on the inductance L; it is the same as it would be if the resistance R alone were connected to the source with emf ε.

4. At time $t = \frac{L}{R}$ = time constant (τ)

$$i = I_0\left[1 - e^{-\left(\frac{R}{L}\right)\frac{L}{R}}\right] = I_0\left[1 - e^{-1}\right] = I_0\left[1 - 0.37\right] = 0.63I_0$$

Thus time constant is the time in which the current will grow to 63% of its maximum value.

5. The variation of current(i) with time (t) is shown in the figure.



In the figure shows that

- The instantaneous current i first rises rapidly, then increases more slowly and approaches the final asymptotically.
- Maximum current = $I_0 = \frac{\varepsilon}{R}$
- At time $t = \frac{L}{R}$ = time constant (τ), the current grow upto about 63% of its maximum value.
- In t= 2τ, i = the current reaches 86% of I_0

  In t = 5τ, 99.3% of I_0

  In t = 10τ, 99.995%. of I_0

6. The graphs of i versus t have the same general shape for all values of L

For a given value of R, the time constant **τ** is greater for greater values of L.
When L is small, the current rises rapidly to its final value; when is large, it rises more slowly.

Energy considerations in R-L circuit

Energy considerations offer us additional insight into the behavior of an R-L circuit. The instantaneous rate at which the source delivers energy to the circuit is

$P = \varepsilon\, i$

The rate at which energy is dissipated in the resistor = i^2R

The rate at which energy is stored in the inductor = $i\, v_{bc} = Li\frac{di}{dt}$

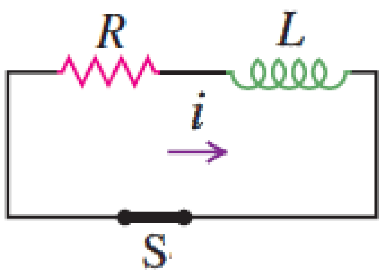
For a R-L circuit we have

$$\varepsilon = iR + L\frac{di}{dt}$$

$$\Rightarrow \quad \varepsilon i = i^2R + Li\frac{di}{dt}$$

$$\Rightarrow \quad \begin{bmatrix}\text{Energy supplied}\\ \text{by the source}\end{bmatrix} = \begin{bmatrix}\text{Energy dissipated}\\ \text{in the resistor}\end{bmatrix} + \begin{bmatrix}\text{Energy store energy}\\ \text{in the inductor}\end{bmatrix}$$

Current Decay in an R-L Circuit



When battery is removed from the circuit and the circuit is closed then the current will decrease from its maximum value I_0 to its minimum value zero.

Applying Kirchhoff's loop rule we get,

$$v_R + v_L = 0$$

$$\Rightarrow \quad iR + L\frac{di}{dt} = 0 \qquad \text{---------------(1)}$$

$$\Rightarrow \quad L\frac{di}{dt} = -iR$$

$$\Rightarrow \quad \frac{di}{i} = -\frac{R}{L}dt$$

$$\Rightarrow \quad \int_{I_0}^{i}\frac{di}{i} = -\int_0^t\left(\frac{R}{L}\right)dt$$

$$\Rightarrow \quad [\ln i]_{I_0}^{i} = -\left(\frac{R}{L}\right)t \qquad \Rightarrow \quad \ln i - \ln I_0 = -\left(\frac{R}{L}\right)t$$

$$\Rightarrow \quad \ln\left(\frac{i}{I_0}\right) = -\left(\frac{R}{L}\right)t$$

$$\Rightarrow \quad \frac{i}{I_0} = e^{-\left(\frac{R}{L}\right)t} \qquad \Rightarrow \quad i = I_0\, e^{-\left(\frac{R}{L}\right)t} \qquad \text{---------------(2)}$$

where I_0 is the initial current at time t = 0.

The time constant, $\tau = \frac{L}{R}$, is the time in which the current decreases to 1/e, or about 37%, of its initial value I_0.

- In, $t = 2\tau$, the current drop to 13.5% of I_0
  In, $t = 5\tau$, the current drop to 0.67% of I_0
  In, $t = 10\tau$, the current drop to 0.0045% of I_0

The energy that is needed to maintain the current during this decay is provided by the energy stored in the magnetic field of the inductor.

From eqn. (1) we have

$$iR + L\frac{di}{dt} = 0$$

$$\Rightarrow \quad i^2R + Li\frac{di}{dt} = 0$$

$$\Rightarrow \quad i^2R = -Li\frac{di}{dt} \qquad \text{---------------(3)}$$

In this case, $Li\frac{di}{dt}$ is negative.

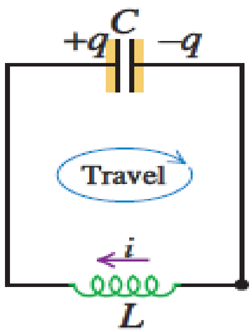
Eq. (3) shows that the energy stored in the inductor decreases at a rate equal to the rate of dissipation of energy i^2R in the resistor.

Electrical Oscillation In L-C Circuit

In an oscillating L-C circuit, the charge on the capacitor and the current through the inductor both vary sinusoidally with time.

From an energy viewpoint the oscillations of an electrical circuit, energy is transferred from electric energy in the capacitor (U_E) to the magnetic energy in the inductor (U_B) and back.The total energy associated with the circuit is constant. This is analogous to the transfer of energy in an oscillating mechanical system from potential energy to kinetic energy and back, with constant total energy.

A circuit containing an inductor and a capacitor is characterized by oscillating current and charge is shown. The steps involved in oscillation are given below.



i) In the L-C circuit we charge the capacitor to a potential difference V_m and the correspoding initial charge Q = C V_m on its left-hand plate and then close the switch.

ii) The capacitor begins to discharge through the inductor.

Because of the induced emf in the inductor, the current cannot change instantaneously; it starts at zero and eventually builds up to a maximum value I_m. During this buildup the capacitor is discharging.

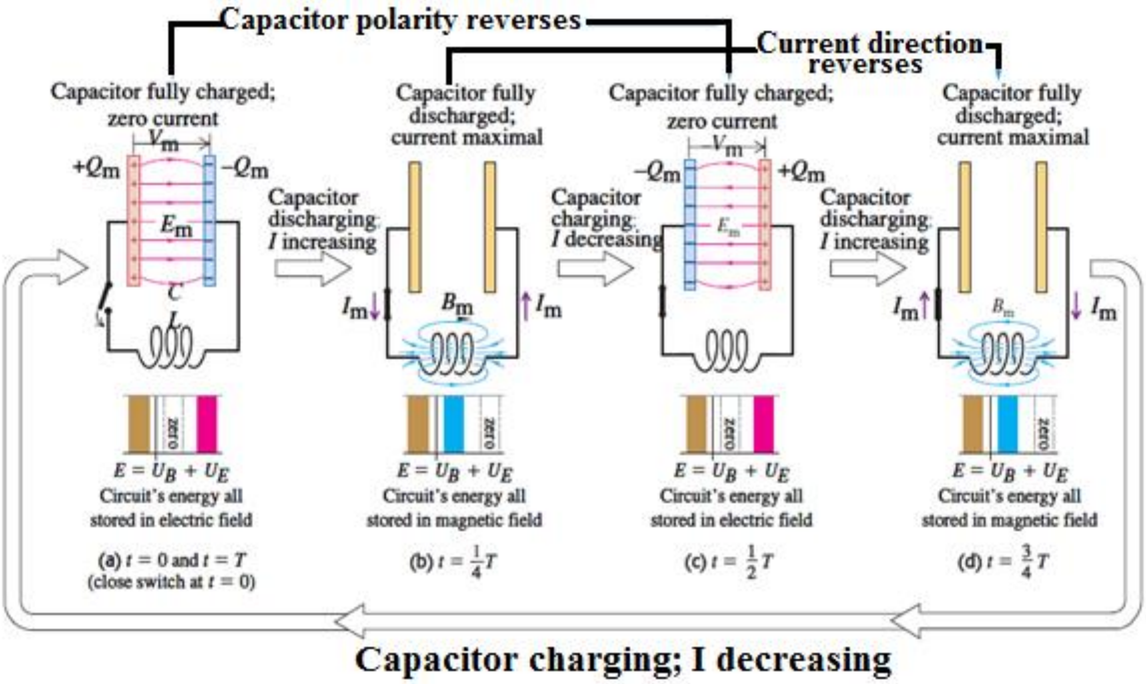
During the discharge of the capacitor, the increasing current in the inductor has established a magnetic field in the space around it, and the energy that was initially stored in the capacitor's electric field is now stored in the inductor's magnetic field.

At each instant the capacitor potential equals the induced emf, so as the capacitor discharges, the rate of change of current decreases. When the capacitor potential becomes zero, the induced emf is also zero, and the current has leveled off at its maximum value I_m. Figure b shows this situation; the capacitor has completely discharged. The potential dif-ference between its terminals (and those of the inductor) has decreased to zero, and the current has reached its maximum value I_m.

iii) Now the capacitor begins to charge with polarity opposite to that in the initial state. As the current decreases, the magnetic field also decreases, inducing an emf in the inductor in the same direction as the current; this slows down the decrease of the current.

Eventually, the current and the magnetic field reach zero, and the capacitor has been charged in the opposite to its initial polarity (Fig.c), with potential difference $-V_m$ and charge -Q on its left-hand plate.

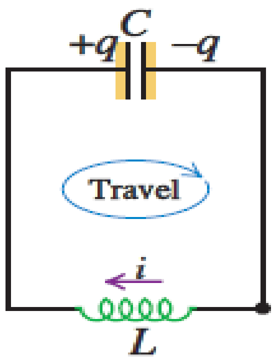
iv) The process now repeats in the reverse direction, the capacitor has again discharged, and there is a current in the inductor in the opposite direction (Fig.d).
Still later, the capacitor charge returns to its original value (Fig.a), and the whole process repeats.
If there are no energy losses, the charges on the capacitor continue to oscillate back and forth indefinitely. This process is called an **electrical oscillation**.



Mathematical treatment of Electrical Oscillation In L-C Circuit

Adjoining figure shows the chage q and and current i.

Applying Kirchhoff's loop rule, we get,



$$-L\frac{di}{dt} - \frac{q}{C} = 0 \quad \text{-----------------------(1)}$$

$$\Rightarrow \quad \frac{di}{dt} + \frac{q}{LC} = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{dq}{dt}\right) + \frac{q}{LC} = 0$$

$$\Rightarrow \quad \frac{d^2q}{dt^2} + \left(\frac{1}{LC}\right)q = 0 \quad \text{----------------------(2)}$$

This is a differntial equation of of order two. Equation (1) has exactly the same form as that of simple harmonic motion of a spring – mass system. The solution of eqn (2) is given by

$q = Q\cos(\omega t + \phi)$ ----------------------(3)

and the angular frequency ω of oscillation is given by

$$\omega = \sqrt{\frac{1}{LC}} \quad \text{----------------------(4)}$$

Instantaneous current i is given by

$$i = \frac{dq}{dt}$$

$$\Rightarrow \quad i = \frac{d}{dt}[Q\cos(\omega t + \varphi)]$$

$$\Rightarrow \quad i = -Q\omega\sin(\omega t + \varphi) \quad \text{----------------------(5)}$$

From eqns. (3) and (5) it is clear that the charge and current in an L-C circuit oscillate sinusoidally with time. The angular frequency determined by the values of L and C.

The constants Q and ϕ are determined by the initial conditions.

If at time t = 0, the left-hand capacitor plate in Fig. 30.15 has its maximum charge Q and the current i is zero, then $\phi = 0$. If q = 0, at time t = 0, then $\phi = \pi/2$ rad.

Energy in an L-C Circuit

The L-C circuit is a conservative system. Again let Q be the maximum capacitor charge. The magnetic-field energy (½) Li^2 in the inductor at any time and the electric-field energy $q^2/2C$

The sum of these energies equals the total energy $Q^2/2C$ of the system

$$\frac{1}{2}Li^2 + \frac{q^2}{2C} = \frac{Q^2}{2C} \quad \text{----------------------(6)}$$

The total energy in the L-C circuit is constant; it oscillates between the magnetic and the electric forms, just as the constant total mechanical energy in simple har-monic motion is constant and oscillates between the kinetic and potential forms.

Solving Eq. (7) for i, we get

$$\Rightarrow \quad Li^2 + \frac{q^2}{C} = \frac{Q^2}{C}$$

$$\Rightarrow \quad Li^2 = \frac{Q^2}{C} - \frac{q^2}{C}$$

$$\Rightarrow \quad i^2 = \frac{Q^2}{LC} - \frac{q^2}{LC}$$

$$\Rightarrow \quad i = \pm\sqrt{\frac{1}{LC}}\sqrt{(Q^2 - q^2)} \quad \text{----------------------(7)}$$

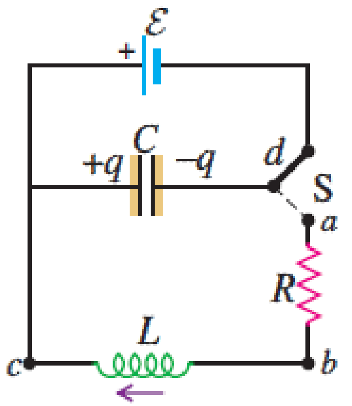
Equations (7) is another form of current in L-C circuit.

Following table shows the Oscillation of a Mass-Spring System Compared with Electrical Oscillation in an L-C Circuit.

Oscillation in L-C circuit is exactly the same form as the Simple Harmonic Motion.		
Sl. No	**Oscillation of a Mass-Spring System**	**Electrical Oscillation in an L-C Circuit**
1	Kinetic energy = $\frac{1}{2} m v^2$	Magnetic energy = $\frac{1}{2} Li^2$
2	Potential energy = $\frac{1}{2} k x^2$	Electric Energy = $\frac{1}{2}\frac{q^2}{C}$
3	Total Energy = $\frac{1}{2} k A^2$	Total Energy = $\frac{1}{2}\frac{Q^2}{C}$
4	Velocity = $v = \pm\sqrt{\frac{k}{m}}\sqrt{(A^2 - x^2)}$	Current = $i = \pm\sqrt{\frac{1}{LC}}\sqrt{(Q^2 - q^2)}$
5	Velocity = $v = \frac{dx}{dt}$	Current = $i = \frac{dq}{dt}$
6	Angular frequency = $\omega = \sqrt{\frac{k}{m}}$	Angular frequency = $\omega = \sqrt{\frac{1}{LC}}$
7	Differntial eqn. $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$	Differntial eqn. $\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0$
8	Displacement = x = A cos($\omega t + \phi$)	Charge = q = Q cos($\omega t + \phi$)
All the information for L-C circuit is obtained in the following way: Substitute: **q** for **x**, **L** for **m**, $\frac{1}{C}$ for **k** **i** for **v**		

The L-R-C Series Circuit

The figure shows the L-R-C series circuit with a constamt source of emf.



Let us connect the capacitor to a source of emf ε for a long enough time such that the capacitor acquires the final charge $Q = C\,\varepsilon$ and any initial oscillations have died out.

Then at time t = 0 we flip the switch to the downward position, removing the source from the circuit and placing the capacitor in series with the resistor and inductor.

Applying Kirchhoff's loop rule, we get,

$$L\frac{di}{dt} - iR - \frac{q}{C} = 0 \qquad \text{----------------------(1)}$$

$$\Rightarrow \quad L\frac{di}{dt} + iR + \frac{q}{C} = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{dq}{dt}\right) + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0$$

$$\Rightarrow \quad \frac{d^2q}{dt^2}+\frac{R}{L}\frac{dq}{dt}+\left(\frac{1}{LC}\right)q = 0 \text{ -----------------------(2)}$$

This is the differntial form for the damped oscillation in L-C-R circuit.

The damped oscillation in L-C-R circuit depends on the value of R.

- Under damped oscillation: The oscillation will under damped if the value of R will be such that, $\frac{1}{LC} > \frac{R^2}{4L^2}$
- Critical damped oscillation: The oscillation will critically damped if the value of R will be such that, $\frac{1}{LC} = \frac{R^2}{4L^2}$
- Over damped oscillation: The oscillation will over damped if the value of R will be such that, $\frac{1}{LC} < \frac{R^2}{4L^2}$

The solution for the underdamped (small R) is

$$q = Ae^{-(R/2L)t}\cos\left(\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\, t+\varphi\right) \text{-----------------------(3)}$$

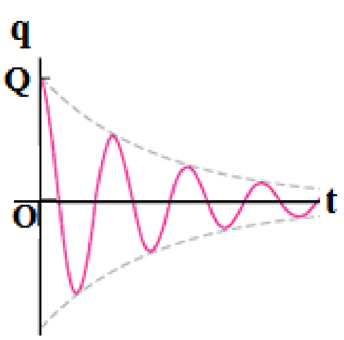
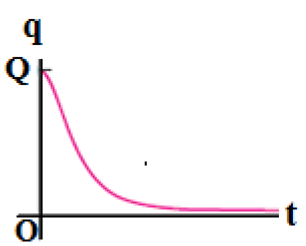
Where A and ϕ are constants and can be obtained from the initial conditions.

The angular frequency ω' of the under damped oscillations is given by

$$\omega' = \sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}} \qquad \text{-----------------------(4)}$$

In the underdamped case the phase constant ϕ in the cosine function of Eq. (3) provides for the possibility of both an initial charge and an initial current at time t = 0,

Graphical representation for damped vibration is shown below

Underdamped circuit (small resistance R)	Critically damped circuit (larger resistance R)	Overdamped circuit (very large resistance R)
		

Following table shows the damped Oscillation of a Mass-Spring System Compared with Electrical Oscillation in an R-L-C Circuit.

Oscillation in R-L-C circuit is exactly the same form as the damped oscillation		
Sl. No	**Damped Oscillation**	**Oscillation in an R-L-C Circuit**
1	Differntial eqn. $\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \frac{k}{m}x = 0$	Differntial eqn. $\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \left(\frac{1}{LC}\right)q = 0$
2	Displacement: $x = Ae^{-(b/2m)t}\cos\left(\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}\,t + \varphi\right)$	Charge: $q = Ae^{-(R/2L)t}\cos\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\,t + \varphi\right)$
3	Angular frequency for under damped oscillation: $\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$	Angular frequency for under damped oscillation: $\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$
	For under damping, $\frac{k}{m} > \frac{b^2}{4m^2}$ For critical damping, $\frac{k}{m} = \frac{b^2}{4m^2}$ For over damping, $\frac{k}{m} < \frac{b^2}{4m^2}$	For under damping, $\frac{1}{LC} > \frac{R^2}{4L^2}$ For critical damping, $\frac{1}{LC} = \frac{R^2}{4L^2}$ For over damping, $\frac{1}{LC} < \frac{R^2}{4L^2}$
All the information for R-L-C circuit is obtained in the following way: Substitute: **q** for **x**, **L** for **m**, $\frac{1}{C}$ for **k** , **R** for **b** (damping constant)		

Conceptual Problems:

Test Your Understanding of Section 30.2

Rank the following inductors in order of the potential difference v_{ab}, from most positive to most negative. In each case the inductor has zero resistance and the current flows from point ‘a’ through the inductor to point ‘b’.

i) The current through a 2.0-μH inductor increases from 1.0 A to 2.0 A in 0.50 s;
ii) The current through a 4.0-μH inductor decreases from 3.0 A to 0 in 2.0 s;
iii) The current through a 1.0-μH inductor remains constant at 4.0 A;
iv) The current through a 1.0-μH inductor increases from 0 to 4.0 A in 0.25 s.

Answer: (iv), (i), (iii), (ii)

The potential difference across the inductor is Vab = Ldi/dt. For the four cases we find

i) $V_{ab} = L\frac{di}{dt} = (2.0\ \mu H)\frac{(2.0\ A - 1.0\ A)}{0.50\ s} = 4.0\ \mu V$

ii) $V_{ab} = L\frac{di}{dt} = (4.0\ \mu H)\frac{(0.0\ A - 3.0\ A)}{2.0\ s} = -6.0\ \mu V$

iii) $V_{ab} = L\frac{di}{dt} = (1.0\ \mu H)\frac{(0.0\ A)}{2.0\ s} = 0$

iv) $V_{ab} = L\frac{di}{dt} = (1.0\ \mu H)\frac{(4.0\ A - 0.0\ A)}{0.25\ s} = 16.0\ \mu V$

Test Your Understanding of Section 30.3

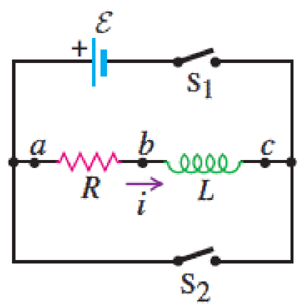
The current in a solenoid is reversed in direction while keeping the same magnitude.

a) Does this change the magnetic field within the solenoid?
b) Does this change the magnetic energy density in the solenoid?

Answers: (a) yes, (b) no

a) Reversing the direction of the current has no effect on the magnetic field magnitude, but it causes the direction of the magnetic field to reverse.
b) It has no effect on the magnetic-field energy density, which is proportional to the square of the magnitude of the magnetic field.

Test Your Understanding of Section 30.4



a) In figure what are the algebraic signs of the potential differences v_{ab} and v_{bc} when switch S_1 is closed and switch S_2 is open?

(i) $v_{ab} > 0, v_{bc} > 0$; (ii) $v_{ab} > 0, v_{bc} < 0$; (iii) $v_{ab} < 0, v_{bc} > 0$; (ii) $v_{ab} < 0, v_{bc} < 0$;

b) What are the signs of v_{ab} and v_{bc} when switch S_1 is open and switch S_2 is closed and current is flowing in the direction shown?

(i) $v_{ab} > 0, v_{bc} > 0$; (ii) $v_{ab} > 0, v_{bc} < 0$; (iii) $v_{ab} < 0, v_{bc} > 0$; (ii) $v_{ab} < 0, v_{bc} < 0$;

Answers: (a) (i), (b) (ii)

We know that $v_{ab} = v_a - v_b$ and $v_{bc} = v_b - v_c$

a) With S_1 closed and S_2 open, current flows through the resistor from a to b. The end a of the resistor is at the higher potential, so v_{ab} is positive.
The current through the inductor flows from b to c and is increasing. The self-induced emf opposes this increase and is therefore directed from c toward b, which means that b is at the higher potential. Hence v_{bc} is positive.

b) With S_1 open and S_2 closed, current flows through the resistor from a to b. The end a of the resistor is at the higher potential, so v_{ab} is positive.
The inductor current again flows from b to c but is now decreasing. The self-induced emf is directed from b to c in an effort to sustain the decaying current, so c is at the higher potential and v_{bc} is negative.

Test Your Understanding of Section 30.5

One way to think about the energy stored in an L-C circuit is to say that the circuit elements do positive or negative work on the charges that move back and forth through the circuit.

a) Between stages at t = 0 and t = T/4, does the capacitor do positive work or negative work on the charges?

b) What kind of force (electric or magnetic) does the capacitor exert on the charges to do this work?

c) During this process, does the inductor do positive or negative work on the charges?

d) What kind of force (electric or magnetic) does the inductor exert on the charges?

Answers: (a) positive, (b) electric, (c) negative, (d) electric

a) Between stages at t = 0 and t = T/4 the capacitor loses energy. So it does positive work on the charges.
b) The capacitor does positive work on the charges by exerting an electric force that pushes current away from the positively charged left-hand capacitor plate and toward the negatively charged right-hand plate.
c) As the capacitor loses energy, at the same time, the inductor gains energy and does negative work on the moving charges.
d) Although the inductor stores magnetic energy, the force that the inductor exerts is electric. This force comes about from the inductor's self-induced emf.

Test Your Understanding of Section 30.6

An L-R-C series circuit includes a 2.0 Ω resistor. At t = 0 the capacitor charge is 2.0 μC. For which of the following values of the inductance and capacitance will the charge on the capacitor not oscillate?

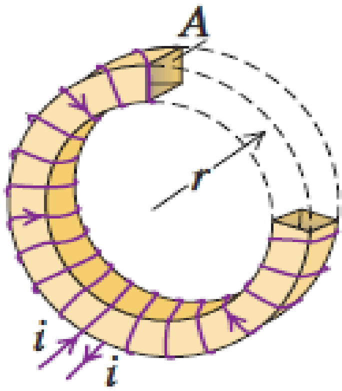
i) L = 3.0 μH, C = 6.0 μF
ii) L = 6.0 μH, C = 3.0 μF
iii) L = 3.0 μH, C = 3.0 μF

Answer: (i) and (iii)

<table>
<tr><td colspan="3">There are no oscillations if $\frac{R^2}{4L^2} \geq \frac{1}{LC} \quad \Rightarrow \quad R^2 \geq \frac{4L}{C}$</td></tr>
<tr><td>i) $R^2 = (2.0\ \Omega)^2 = 4.0\ \Omega^2$.
$\frac{4L}{C} = \frac{4(3.0\ \mu H)}{(6.0 \mu F)} = 2.0\ \Omega^2$
So there are no oscillations. The system is overdamped.</td><td>ii) $R^2 = (2.0\ \Omega)^2 = 4.0\ \Omega^2$.
$\frac{4L}{C} = \frac{4(6.0\ \mu H)}{(3.0 \mu F)} = 8.0\ \Omega^2$
So there are oscillations. The system is under damped.</td><td>iii) $R^2 = (2.0\ \Omega)^2 = 4.0\ \Omega^2$.
$\frac{4L}{C} = \frac{4(3.0\ \mu H)}{(3.0 \mu F)} = 4.0\ \Omega^2$
So there are no oscillations. The system is critically damped.</td></tr>
</table>

Example Problems:

Example 30.4 Calculating self-induced emf

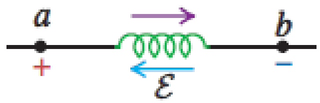


If the current in a toroidal solenoid, of self inductance 40×10^{-6} H,increases uniformly from 0 to 6.0 A in 3.0 μs, find the magnitude and direc-tion of the self-induced emf.

Solution: $L = 40 \times 10^{-6}$ H

Magnitude of the induced emf is

$$|\varepsilon| = L\frac{di}{dt} = \left(40 \times 10^{-6}\text{H}\right)\frac{(6\text{ A})}{3.0 \times 10^{-6}\text{ s}} = 80\text{ V}$$



The current is increasing, so according to Lenz's law the direction of the emf is **opposite** to that of the current. The emf is in the direction from b to a, like a battery with a as the + terminal and b the - terminal, tending to oppose the current increase from the external circuit.

This shows that even a small inductance can give rise to a substantial induced emf if the current changes rapidly.

Example 30.9: Energy in an oscillating circuit

For a L-C circuit a 300-V dc power supply is used to charge a 25-mF capacitor. After the capacitor is fully charged, it is disconnected from the power supply and connected across a 10-mH inductor. The resistance in the circuit is negligible. Fnd the magnetic and electric energies (a) at t = 0 and (b) at t = 1.2 ms.

Solution: $L = 10 \times 10^{-3}$ H, $C = 25 \times 10^{-6}$ F, $V = 300$ V.

a) At t = 0

- there is no current i = 0 and
- $q = Q = C\,\mathcal{E} = (25 \times 10^{-6}\text{ F})(300\text{ V}) = 7.5 \times 10^{-3}$ C.

$$\text{Magnetic energy} = U_E = \frac{1}{2}L\,I^2 = 0$$

$$\text{Electric energy} = U_E = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{\left(7.5 \times 10^{-3}\text{C}\right)^2}{\left(25 \times 10^{-6}\text{F}\right)} = 1.1\text{J}$$

Hence there is no magnetic energy, and all the energy in the form of electric energy in the capacitor.

b) At t = 1.2ms

$$\omega = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{\left(10 \times 10^{-3}\text{F}\right)\left(25 \times 10^{-6}\text{F}\right)}} = 2.0 \times 10^{36}\text{rad/s}$$

$$\text{Magnetic energy} = U_E = \frac{1}{2}L\,i^2 = \frac{1}{2}L\,(-\,\omega\,Q\sin\omega t)^2$$

$$U_E = \frac{1}{2}(10 \text{ x } 10^{-3}\text{H})\left[-(2.0 \text{ x } 10^3\text{rad/s})(7.5 \text{ x } 10^{-3}\text{C})\sin\left(2.0 \text{ x } 10^3\text{rad/s}\right)(1.2\text{x } 10^{-3}\text{s})\right]^2$$

$$U_E = 2.5\text{J}$$

$$\text{Electric energy} = U_E = \frac{q^2}{2C} = \frac{(Q\cos\omega t)^2}{2C}$$

$$U_E = \frac{\left[(7.5 \text{ x } 10^{-3}\text{C})\ \cos\left(2.0 \text{ x } 10^3\text{rad/s}\right)(1.2\text{x } 10^{-3}\text{s})\right]^2}{2\left(25 \text{ x } 10^{-6}\text{F}\right)} = 0.6 \text{ J}$$

Example 30.10: An underdamped L-R-C series circuit

What resistance R is required (in terms of L and C) to give an L-R-C series circuit a frequency that is one-half the undamped frequency?

Solution:

We want just enough resistance to reduce the oscillation frequency to one-half of the undamped value.

Angular frequency for under damped oscillation is given by

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

Angular frequency for undamped oscillation is given by $\omega = \sqrt{\frac{1}{LC}}$

Given that $\omega' = \frac{\omega}{2}$

$$\Rightarrow \quad \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \frac{\sqrt{\frac{1}{LC}}}{2}$$

$$\Rightarrow \quad \frac{1}{LC} - \frac{R^2}{4L^2} = \frac{1}{4\,LC} \qquad \Rightarrow \quad \frac{R^2}{4L^2} = \frac{1}{LC} - \frac{1}{4\,LC}$$

$$\Rightarrow \quad \frac{R^2}{4L^2} = \frac{3}{4LC} \qquad \Rightarrow \quad R^2 = \frac{3L}{C} \qquad \Rightarrow \quad R = \sqrt{\frac{3L}{C}}$$

Exercise Problems

30.25: A 35.0-V battery with negligible internal resistance, a resistor, 50.0Ω and a 1.25-mH inductor with negligible resistance are all connected in series with an open switch. The switch is suddenly closed. (a) How long after closing the switch will the current through the inductor reach one-half of its maximum value? (b) How long after closing the switch will the energy stored in the inductor reach one-half of its maximum value?

Ans: (a) $\varepsilon = 35$ V, $R = 50\ \Omega$, $L = 1.25$ mH

We know that

$$i = i_{max}\left(1 - e^{-\frac{R}{L}t}\right) \text{ where, } i_{max} = \frac{\varepsilon}{R}$$

$$\frac{i_{max}}{2} = i_{max}\left(1 - e^{-\frac{R}{L}t}\right) \Rightarrow \frac{1}{2} = 1 - e^{-\frac{R}{L}t} \Rightarrow e^{-\frac{R}{L}t} = \frac{1}{2}$$

$$\Rightarrow -\frac{R}{L}t = \ln\frac{1}{2} \quad \Rightarrow \quad t = -\left(\frac{L}{R}\right)\ln\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad t = \frac{L\ln 2}{R} = \frac{(\ln 2)(1.25 \text{ x } 10^3 \text{ H})}{50\ \Omega} = 17.3 \text{ s}$$

(b) Energy stored, $U = \frac{1}{2}L\,i^2$ so, $U_{max} = \frac{1}{2}L\,i_{max}^2$

Given that

$$U = \frac{U_{max}}{2} \Rightarrow \frac{1}{2}L\,i^2 = \frac{\frac{1}{2}L\,i_{max}^2}{2} \Rightarrow i^2 = \frac{i_{max}^2}{2} \Rightarrow i = \frac{i_{max}}{\sqrt{2}}$$

$$\Rightarrow i_{max}\left(1 - e^{-\frac{R}{L}t}\right) = \frac{i_{max}}{\sqrt{2}} \Rightarrow 1 - e^{-\frac{R}{L}t} = \frac{1}{\sqrt{2}} \Rightarrow e^{-\frac{R}{L}t} = 1 - \frac{1}{\sqrt{2}} = 0.2929$$

$$t = -\frac{L}{R}\ln(0.2929) = 30.7\ \mu\text{s}$$

$\tau = L/R = 2.5 \text{ x} .10^{-5}$ s = 25 μs. The time in part (a) is 0.692τ and the time in part (b) is 1.23τ

30.39: A 10 MFD capacitor is first charged and then discharged through $10^6\Omega$ resistor. Find the time the potential will take to fall to one-third of its initial value. Ans: t = 10.986s.

Ans: $C = 10 \text{ x } 10^{-6}\text{F}$, $R = 10^6\,\Omega$

$$v = v_0\,e^{-\frac{t}{RC}} \quad \Rightarrow \quad \frac{v_0}{3} = v_0\,e^{-\frac{t}{RC}}$$

$$\Rightarrow \frac{1}{3} = e^{-\frac{t}{RC}} \Rightarrow \ln\left(\frac{1}{3}\right) = -\frac{t}{RC}$$

$$\Rightarrow \quad t = -R\,C\ln\left(\frac{1}{3}\right) = -\left(10^6\Omega\right)\left(10 \text{ x } 10^{-6}\text{F}\right)\left(-1.0986\right) = 10.986 \text{ s}$$