# Chapter-22: Gauss's Law

## Topics in the material

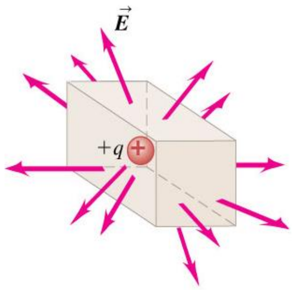
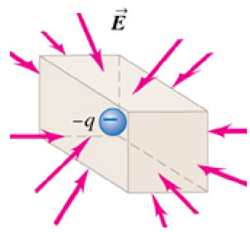
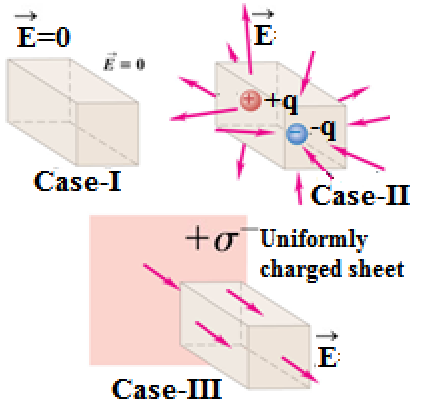
1. Learning Goals
2. Calculating Electric Flux
3. Gauss's Law
4. Applications of Gauss's Law
5. Test your understanding: TYU-22.2, TYU-22.3, TYU-22.4
6. In-class problems: 22.2, 22.3, 22.5, 22.9
7. Assignment problem: 22.14, 22.22

## LEARNING GOALS

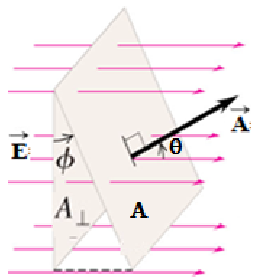
By studying this chapter, we will learn:

- To calculate the amount of charge within a closed surface from the type of the electric field on the surface.
- The definition of electric flux, and the method to calculate it.
- The Gauss's law in electrostatics.
- The use of Gauss's law to calculate the electric field due to a symmetric charge distribution.

## Charge and Electric flux

| **Positive charge inside a box** | **Negative charge inside a box** |
|---|---|
| Positive charge within the box produces **outward** electric flux through the surface of the box, i.e the flux is positive if the field line crosses a closed surface from **inside** to **outside.**  <br>• More charge inside the box means more flux exists<br>• Doubling charge within box doubles flux.<br>• Doubling size of box does NOT change flux. | Negative charge produces **inward** flux, i.e the flux is negative if the field line crosses a closed surface from **outside** to **inside.**  <br>• More negative charge more inward flux<br>• Doubling charge within box doubles flux.<br>• Doubling size of box does NOT change flux. |
| **Zero net charge inside a box**<br><br>Case-I: No net electric flux through surface of box<br><br>Case-II: Inward flux cancels outward flux<br><br>Case-III: Inward flux cancels outward flux |   |
| **The net electric flux through the surface of the box is directly proportional to the magnitude of the net charge enclosed by the box. This is independent of the size of the box.** | |

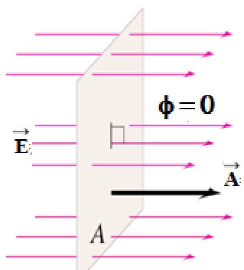
## Calculating electric flux in uniform fields




The product of an area and the component of E that is perpendicular to the area.
It can be written as:

$$\varphi_E = \vec{E}\cdot\vec{A} = E\,A\cos\theta = E_\perp\,A$$

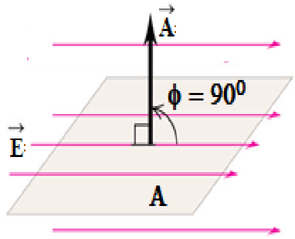
$$E_\perp = E\,A\cos\theta = \text{Component of } \vec{E} \text{ perpendicular to the area,}$$




### Case-I:

$\vec{E}$ and $\vec{A}$ are parallel (the angle between $\vec{E}$ and $\vec{A}$ is $\theta = 0$ ).
The flux $\varphi_E = \vec{E}\cdot\vec{A} = E\,A\cos\theta = E\,A$

**Case-II:**




$\vec{E}$ and $\vec{A}$ are perpendicular (the angle between $\vec{E}$ and $\vec{A}$ is $\theta = 90^0$ ).
The flux $\varphi_E = \vec{E}\cdot\vec{A} = E\,A\cos 90^0 = E\,A(0) = 0$

### Flux of a Non-uniform Electric Field

Let the electric field varies from point to point and is not uniform over the area A. Or a part of A is curved surface. Then the electric flux is given by

$$\varphi_E = \oint \vec{E}\cdot d\vec{A} = \oint E\cos\theta\, dA$$

## Gauss's Law:

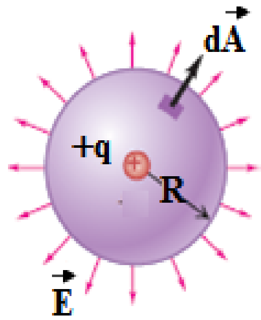
Gauss's law states that the total electric flux through any closed surface is equal $1/\varepsilon_0$ times the net electric charge inside the surface.

$$\varphi_E = \frac{q_{enclosed}}{\varepsilon_0}$$

In general it can be written as $\varphi_E = \oint \vec{E}\cdot d\vec{A} = \oint E\cos\theta\, dA = \frac{q_{enclosed}}{\varepsilon_0}$

Let us consider a single positive point charge +q. The field lines radiate out equally in all directions. We place this charge at the center of an imaginary spherical surface with radius R. The magnitude of the electric field at every point on the surface is given by

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}$$




At each point on the surface, is perpendicular to the surface, and its magnitude is the same at every point. The total electric flux is:

$$\varphi_E = E\,A\cos\theta = E\,A \quad (\theta = 0)$$

$$\varphi_E = \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}\right)\left(4\pi R^2\right) = \frac{q}{\varepsilon_0}$$

- The flux is independent of the radius R of the sphere.
- It depends only on the charge q enclosed by the sphere.

Again the Gauss's law also agrees with negative charge enclosed inside the surface

If $q_{enclosed} = -q$, then

$$\varphi_E = \frac{q_{enclosed}}{\varepsilon_0} = \frac{-q}{\varepsilon_0}$$

If $q_{enclosed} = 0$ , then the total flux through the Gaussian surface is zero $\left(\varphi_E = 0\right)$, even though some areas may have positive flux and others may have negative flux.

## Applications of Gauss's Law:

### 1. Field of a charged conducting sphere:

**Case-I: The Point is outside the sphere**

Let us consider a spherical charge conductor with of uniform charge density. 'q' amount of charge is enclosed in the sphere.

R = radius of the sphere and ρ = density of the material of the charged conductor

Let us consider a point outside the sphere. Let us construct a Gaussian surface passing through that point. Here, the Gaussian surface is a sphere of radius 'r'. Here, **r > R.**

The electric field has the same magnitude at every point of the surface and is directed outward.

$\vec{E}$ and $\vec{A}$ are in the same direction i.e $\theta = 0$.

So, the electric flux through the closed surface is:

$$\varphi_E = \vec{E}.\vec{A} = E\,A\cos 0 = E\,A \qquad \text{----------(1)}$$

From the Gauss' law we have, $\varphi_E = \dfrac{q}{\varepsilon_0}$ −−−−−−−−−−−(2)

Comparing eqn (1) and (2) we get

$$E\,A = \frac{q}{\varepsilon_0} \quad \Rightarrow \quad E = \frac{q}{\varepsilon_0 A}$$

$$\Rightarrow E = \frac{q}{\varepsilon_0\left(4\pi r^2\right)} \quad \Rightarrow E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \quad \Rightarrow \quad E_{out} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \qquad ---(3)$$

Thus, the electric field outside the sphere is $E_{out} \propto \dfrac{1}{r^2}$

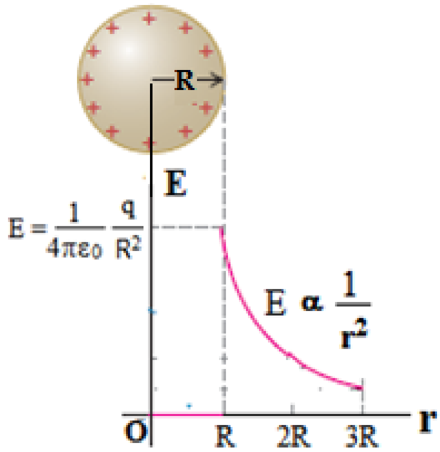
**Case-II: At a point on the surface of the sphere**

Here, the Gaussian surface is a sphere of radius r = **R.**

Putting this value in eqn (3) we get,

$$E_{surface} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2} \qquad ------(4)$$

Thus $E_{surface}$ = constant




**Case-III: The Point is inside the sphere**

Inside the sphere, $q_{enclosed} = 0$

So, from the Gauss' law we have,

$$\varphi_E = \frac{q}{\varepsilon_0} = \frac{0}{\varepsilon_0} = 0 \quad \Rightarrow \quad E\,A = 0 \quad \Rightarrow \quad E = 0$$

Thus, $E_{inside} = 0$

The figure shows 'E' as a function of the distance 'r' from the center of the sphere.

- Outside the sphere, the magnitude of the electric field decreases with the square of the radial distance from the center of the sphere:
- Inside the sphere, $q_{enclosed} = 0$, and $E_{inside} = 0$

## 2. Field of a uniformly insulating charged sphere

Positive electric charge 'q' is distributed uniformly throughout the volume of an insulating sphere.

R = radius of the sphere and $\rho$ = volume charge density ( charge per unit volume )

**Case-I: The Point is outside the sphere**

Let us consider a point outside the sphere. Let us construct a Gaussian surface passing through that point. Here, the Gaussian surface is a sphere of radius 'r'. and **r > R.**

This surface encloses the entire charged sphere, so, $q_{enclosed} = q$

The electric field has the same magnitude at every point of the surface and is directed outward.

$\vec{E}$ and $\vec{A}$ are in the same direction i.e $\theta = 0$.

So, the electric flux through the closed surface is:

$$\varphi_E = \vec{E}.\vec{A} = E\,A\cos 0 = E\,A$$

From the Gauss' law we have, $\varphi_E = \dfrac{q}{\varepsilon_0}$

Comparing the above two equations we get

$$E\,A = \frac{q}{\varepsilon_0} \quad \Rightarrow \quad E = \frac{q}{\varepsilon_0 A}$$

$$\Rightarrow E = \frac{q}{\varepsilon_0\left(4\pi r^2\right)} \quad \Rightarrow E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \quad \Rightarrow E_{out} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \quad ------(1)$$

Thus, the electric field outside the sphere is $E_{out}\ \alpha\ \dfrac{1}{r^2}$

**Case-II: At a point on the surface of the sphere**

Here, the Gaussian surface is a sphere of radius r = **R.**

Putting this value in either eqn (1) or eqn (2)we get,

$$E_{surface} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2} \quad ------(2)$$

**Case-III: The Point is inside the sphere**

The amount of charge enclosed within the Gaussian surface depends on 'r'. To find E inside the sphere, we choose

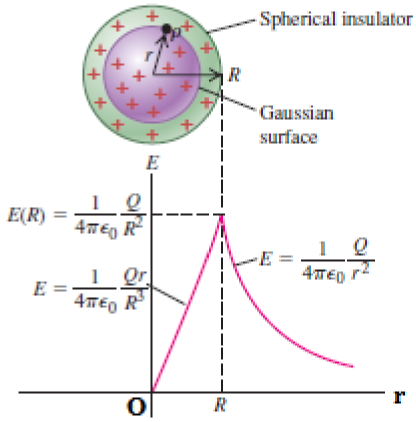
Let $V_R$ = volume of the sphere of radius R

$V_r$ = volume of the sphere of radius r

$q_r$ = charge enclosed in the Gaussian sphere of the sphere of radius r

Since the sphere has uniform charge density, we have,

$\rho$ = Uniform (i.e constant)




$$\Rightarrow \frac{\text{Charge}}{\text{Volume}} = \text{Constant} \quad \Rightarrow \frac{q}{V} = \text{Constant}$$

$$\Rightarrow \frac{q}{V} = \frac{q_r}{V_r} \qquad \Rightarrow q_r = \left(\frac{V_r}{V}\right)q$$

$$\Rightarrow q_r = \left(\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}\right)q \quad \Rightarrow q_r = \left(\frac{r^3}{R^3}\right)q$$

Gauss' law we have,

$$\varphi_E = \frac{q_{net}}{\varepsilon_0} \quad \Rightarrow E_{inside}\, A = \frac{q_r}{\varepsilon_0} \quad \Rightarrow E_{inside} = \frac{q_r}{\varepsilon_0 A}$$

$$\Rightarrow E_{inside} = \frac{\left(r^3/R^3\right)q}{\varepsilon_0\left(4\pi r^2\right)} \quad \Rightarrow E_{inside} = \left(\frac{q}{4\pi\varepsilon_0 R^3}\right)r \quad \text{--------(3)}$$

Thus, $E_{inside} \;\alpha\; r$

The figure shows 'E' as a function of the distance 'r' from the center of the sphere.

- Outside the sphere, the magnitude of the electric field decreases with the square of the radial distance from the center of the sphere $E_{out} \;\alpha\; \frac{1}{r^2}$ :
- Inside the sphere, E varies linearly with 'r' $E_{in} \;\alpha\; r$
- On the sphere, $E_{on}$ is constant and doesn't depends on 'r'

# Conceptual Problem

## Test Your Understanding: 22.2

Rank the following surfaces in order from most positive to most negative electric flux.

(i) a flat rectangular surface with vector area $\vec{A} = \left(6\,\text{m}^2\right)\hat{i}$ in a uniform electric field $\vec{E} = \left(4.0\ \text{N/C}\right)\hat{j}$

(ii) a flat circular surface with vector area $\vec{A} = \left(3\,\text{m}^2\right)\hat{j}$ in a uniform electric field $\vec{E} = \left(4.0\ \text{N/C}\right)\hat{i} + \left(2.0\ \text{N/C}\right)\hat{j}$

(iii) a flat square surface with vector area $\vec{A} = \left(3.0\ \text{m}^2\right)\hat{i} + \left(7.0\ \text{m}^2\right)\hat{j}$ in a uniform electric field $\vec{E} = \left(4.0\ \text{N/C}\right)\hat{i} - \left(2.0\ \text{N/C}\right)\hat{j}$

(iv) a flat oval surface with vector area $\vec{A} = \left(3.0\ \text{m}^2\right)\hat{i} - \left(7.0\ \text{m}^2\right)\hat{j}$ in a uniform electric field $\vec{E} = \left(4.0\ \text{N/C}\right)\hat{i} - \left(2.0\ \text{N/C}\right)\hat{j}$

**Answer: (iv), (ii), (i), (iii)**

In each case the electric field is uniform, so the flux is $\varphi = \vec{E}\,.\,\vec{A}$

$\varphi = E_x A_x + E_y A_y + E_z A_z$

i) $E_x = 0,\ E_y = 4,\ E_z = 0;\quad A_x = 6,\ A_y = 0,\ A_z = 0;$

$\varphi = E_x A_x + E_y A_y + E_z A_z = \mathbf{0}$

ii) $E_x = 4,\ E_y = 2,\ E_z = 0;\quad A_x = 0,\ A_y = 3,\ A_z = 0;$

$\varphi = E_x A_x + E_y A_y + E_z A_z = \mathbf{6}$

iii) $E_x = 4,\ E_y = -2,\ E_z = 0;\quad A_x = 3,\ A_y = 7,\ A_z = 0;$

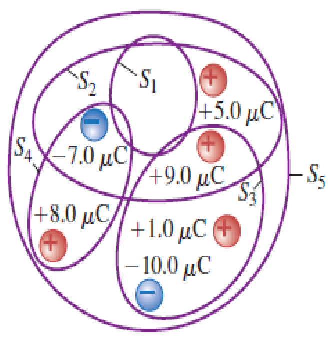
$\varphi = E_x A_x + E_y A_y + E_z A_z = \mathbf{-2}$

iv) $E_x = 4,\ E_y = -2,\ E_z = 0;\quad A_x = 3,\ A_y = -7,\ A_z = 0;$

$\varphi = E_x A_x + E_y A_y + E_z A_z = \mathbf{26}$

## Test Your Understanding: 22.3

Figure shows six point charges that all lie in the same plane. Five Gaussian surfaces— $S_1$, $S_2$, $S_3$, $S_4$, and $S_5$ —each enclose part of this plane and figure shows the intersection of each surface with the plane. Rank these five surfaces in order of the electric flux through them, from most positive to most negative.




**Answer: $S_2$, $S_5$ , $S_4$, $S_1$ and $S_3$**

Surface $S_1$ encloses no charge,

Surface $S_2$ encloses $9.0\ \mu C + 5.0\ \mu\ C + \ (-7.0\ \mu\ C) = 7.0\ \mu\ C$, Surface $S_3$ encloses 9.0 μC + 1.0 μ C + (-10.0 μ C) = 0 ,

Surface $S_4$ encloses 8.0 μC + (-7.0 μ C) = 1.0 μ C,

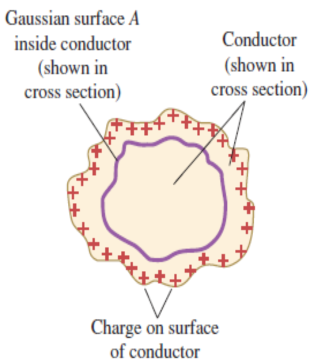
Surface $S_5$ encloses 8.0 μC + (-7.0 μ C) + (-10.0 μ C) + 1.0 μ C + 9.0 μC + 5.0 μC = 6.0 μ C

**Test Your Understanding: 22.4**

You place a known amount of Charge 'Q' on the irregularly shaped conductor shown in the figure. If you know the size and shape of the conductor, can you use Gauss's law to calculate the electric field at an arbitrary position outside the conductor?

**Answer: no**

You might be tempted to draw a Gaussian surface that is an enlarged version of the conductor, with the same shape and placed so that it completely encloses the conductor.




While you know the flux through this Gaussian surface (by Gauss's law, it's $\varphi_E = Q/\varepsilon_0$ ), the direction of the electric field need not be perpendicular to the surface and the magnitude of the field need not be the same at all points on the surface. It's not possible to do the flux integral and we can't calculate the electric field. Gauss's law is useful for calculating the electric field only when the charge distribution is highly symmetric.

## In Class Problems

Example:22.2: Electric flux through a cube

An imaginary cubical surface of side L is in a region of uniform electric field $\vec{E}$. Find the electric flux through each face of the cube and the total flux through the cube when

a) it is oriented with two of its faces perpendicular to $\vec{E}$ (Fig. 1) and

b) the cube is turned by an angle about a vertical axis (Fig. 2).

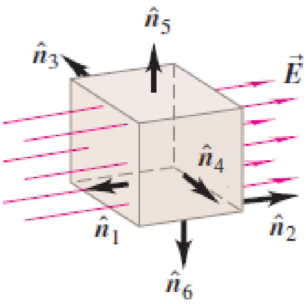



**Figure - 1**

Solution:

a) Figure-1 shows that each unit vector points outward from the cube's closed surface.

The angle between $\vec{E}$ and $\hat{n}_1$ is $180^0$,

The angle between $\vec{E}$ and $\hat{n}_2$ is $0^0$,

The angle between $\vec{E}$ and each of the other four unit vectors is $90^0$, Each face of the cube has area $L^2$, so the fluxes through the faces are

$$\varphi_{E1} = \vec{E}.\vec{A} = \vec{E}.\hat{n}_1\, A = E\,L^2 \cos 180^0 = -E\,L^2$$

$$\varphi_{E2} = \vec{E}.\vec{A} = \vec{E}.\hat{n}_2\, A = E\,L^2 \cos 0^0 = +E\,L^2$$

$$\varphi_{E3} = \varphi_{E4} = \varphi_{E5} = \varphi_{E6} = E\,L^2 \cos 90^0 = 0$$

The flux is negative on face 1, where $\vec{E}$ is directed into the cube,

The flux is positive on face 2, where $\vec{E}$ is directed out of the cube.

The total flux through the cube is

$$\varphi_E = \varphi_{E1} + \varphi_{E2} + \varphi_{E3} + \varphi_{E4} + \varphi_{E5} + \varphi_{E6}$$
$$\varphi_E = \left(-E\,L^2\right) + \left(+E\,L^2\right) + 0 + 0 + 0 + 0$$
$$\varphi_E = 0$$

b) Figure-2 shows that
$\vec{E}$ is directed into the faces 1 and 3. So the fluxes through them are negative.
$\vec{E}$ is directed outward from the faces 2 and 4. So the fluxes through them are positive.

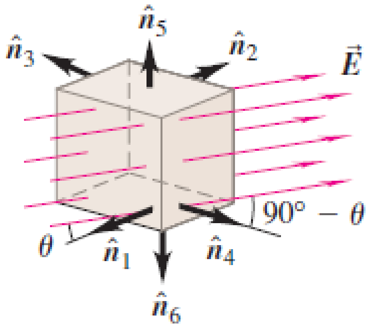


**Figure - 2**

So the fluxes through the faces are

$$\varphi_{E1} = \vec{E}.\vec{A} = \vec{E}.\hat{n}_1\, A = E\,L^2 \cos\left(180^0 - \theta\right) = -E\,L^2 \cos\theta$$

$$\varphi_{E2} = \vec{E}.\vec{A} = \vec{E}.\hat{n}_2\, A = E\,L^2 \cos\left(0^0 + \theta\right) = +E\,L^2 \cos\theta$$

$$\varphi_{E3} = \vec{E}.\hat{n}_3\, A = E\,L^2 \cos\left(90^0 + \theta\right) = -E\,L^2 \sin\theta$$

$$\varphi_{E4} = \vec{E}.\hat{n}_4\, A = E\,L^2 \cos\left(90^0 - \theta\right) = +E\,L^2 \sin\theta$$

$$\varphi_{E5} = \varphi_{E6} = E\,L^2 \cos 90^0 = 0$$

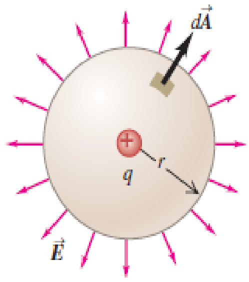
The total flux through the cube is

$$\varphi_E = \varphi_{E1} + \varphi_{E2} + \varphi_{E3} + \varphi_{E4} + \varphi_{E5} + \varphi_{E6}$$
$$\varphi_E = \left(-E\,L^2 \cos\theta\right) + \left(+E\,L^2 \cos\theta\right) + \left(-E\,L^2 \sin\theta\right) + \left(+E\,L^2 \sin\theta\right) + 0 + 0$$
$$\varphi_E = 0$$

From the results of (a) and (b) it is clear that if a closed surface does not contains any electric charge, then the net flux through it due to a uniform electric field is zero.

Example:22.3: Electric flux through a sphere

A point charge q = +3.0 μC is surrounded by an imaginary sphere of radius r = 0.2 m centered on the charge (shown in the adjoined figure). Find the resulting electric flux through the sphere.



Solution:

The sphere is centered on the point charge.
The magnitude of the electric field at every point on the surface is given by

$$E = \frac{1}{4\,\pi\,\varepsilon_0}\frac{q}{r^2}$$

At any point on the spherical surface $\vec{E}$ is directed out of the sphere perpendicular to the surface.
So, angle between $\vec{E}$ and $d\vec{A}$ is $\theta = 0$
So and the flux through a surface element $d\vec{A}$ is

$$\varphi_E = \oint \vec{E}\,.\,d\vec{A} = \oint E\cos\theta\,dA = E\oint \cos 0\,dA = \oint dA = E\left(4\pi r^2\right)$$

$$\varphi_E = \frac{1}{4\,\pi\,\varepsilon_0}\frac{q}{r^2}\left[A\right]_0^{2\pi r} = \frac{1}{4\,\pi\,\varepsilon_0}\frac{q}{r^2}\left(4\pi r^2\right)$$

$$\varphi_E = \frac{q}{\varepsilon_0} = \frac{3.0 \text{ x } 10^{-6}\text{ C}}{8.85 \text{ x } 10^{-12}C^2.N^{-1}.m^2} = 3.4 \text{ x } 10^5\text{ N.m}^2.C^{-1}$$

From the above it is clear that the flux of is independent of the size of the surface. This result also holds true for a cube. In fact, the flux through any surface enclosing a single point charge is independent of the shape or size of the surface.

Example: 22.5: Field of a charged conducting sphere

We place a total positive charge 'q' on a solid conducting sphere with radius R. Find electric field at any point inside and outside the sphere.

Solution: Application no. 1 of Gauss's law in the material

Example: 22.9: Field of a uniformly charged sphere

Positive electric charge Q is distributed uniformly throughout the volume of an insulating sphere with radius R. Find the magnitude of the electric field at a point a distance from the center of the sphere.

Solution: Application no. 2 of Gauss's law in the material

## Assignment Problems

22.14 A solid metal sphere with radius 0.45 m carries a net charge of 0.25 nC. Find the magnitude of the electric field

a) at a point 0.10 m outside the surface of the sphere and

b) at a point inside the sphere, 0.10 m below the surface.

Solution: Radius, R = 0.45 m, Net charge, Q = 0.25 nC = $0.25 \times 10^{-9}$C

a) Here, r = 0.45m + 0.10m = 0.55m

$$E_{out} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} = \left(9 \times 10^9 \text{ N.m}^2.\text{C}^{-2}\right)\frac{\left(0.25 \times 10^{-9}\text{C}\right)}{\left(0.55\text{m}\right)^2} = 7.44\text{N/C}$$

b) Here, r = 0.45m - 0.10m = 0.35m

Inside the conductor E = 0.

22.22 A point charge of -2.0 μC is located in the center of a spherical cavity of radius 6.5cm inside an insulating charged solid. The charge density in the solid is $\rho = 7.35 \times 10^{-4}$ C/m$^3$.Calculate the electric field inside the solid at a distance of 9.5 cm from the center of the cavity.

Solution: Radius, r = $6.5 \times 10^{-2}$ m, charge at the center of the cavity= = -0.2 μC = - $0.2 \times 10^{-6}$C

$\rho = 7.35 \times 10^{-4}$ C/m$^3$.

Radius of the Gaussian sphere = R = $9.5 \times 10^{-2}$ m

The charge within the solid is

$$q_{solid} = \rho V = \rho\left[\frac{4}{3}\pi\left(R^3 - r^3\right)\right]$$

$$\Rightarrow q_{solid} = \left(7.35 \times 10^{-4}\text{C/m}^3\right)\frac{4}{3}\pi\left[\left(9.5 \times 10^{-2}\text{m}\right)^3 - \left(9.5 \times 10^{-2}\text{m}\right)^3\right] = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

$$\Rightarrow q_{solid} = 1.794 \times 10^{-6}\text{C}$$

Total charge inside the Gaussian sphere = q = - $0.2 \times 10^{-6}$ C + $1.794 \times 10^{-6}$C = $0.2059 \times 10^{-6}$ C

The electric field = $E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} = \left(9 \times 10^9 \text{ N.m}^2.\text{C}^{-2}\right)\frac{\left(0.2059 \times 10^{-6}\text{C}\right)}{\left(9.5 \times 10^{-2}\text{m}\right)^2} = 2.05 \times 10^9\text{N/C}$