Wigner Description in Quantum Optics and its Application in Two Mode Entanglement

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Contents

1 Introduction

In the context of Quantum Information Theory, different quantum optical states have been used successfully to realise practical implication of quantum information protocols. The Continuous Variable Quantum Information Theory has been found to be of profound importance in recent years.
In this work, the Wigner description of quantum states has been discussed in detail which is one of the most important description of CV systems. We started with definition of different quantum states and introduction to continuous variable systems. Then we calculated Wigner function of different quantum states and their Photon added forms. Finally the Wigner description of two mode squeezed state has been described which is important to describe Entanglement and correlations.

2 Quantum States in Optics

In the quantum theory of light, the electric and magnetic fields are Hermitian operators. The simplest method of constructing these operators is to decompose the fields into modes and associate each mode with a quantum mechanical harmonic oscillator. The most efficient way is to describe the observable associated with such oscillator in terms of creation and annihilation operators a and $a^\dagger$. These non-Hermitian operators do not commute, having the commutator

$$[a, a^\dagger] = 1.$$

The Hamiltonian for the oscillator, or the field mode is

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}) = \hbar\omega(n + \frac{1}{2})$$

where the operator n is defined as $n = a^\dagger a$. This operator is a Hermitian operator known as the Number operator.
Let us now discuss the important classes of quantum states. These usually describe various states of electromagnetic field.

2.1 Fock State

The single mode states which are eigenstates of the number operator $n = a^\dagger a$ as well as Free field hamiltonians are known as Number states or Fock states. The corresponding eigenvalues are the energy of corresponding mode. Fock states are denoted by $|n\rangle$ with n = 0,1,2 .... The state $|0\rangle$ is known as the Ground state or Vaccum state.
Fock states are orthonormal and complete.

$$\langle m|n\rangle = \delta_{mn}$$

and

$$\sum_n |n\rangle\langle n| = I$$

The action of $a^\dagger$ and a on Fock states result as follows:

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$
$$\text{and } a|n\rangle = \sqrt{n}|n-1\rangle$$

2.2 Coherent State

The Fock states are eigenstates of the Hamiltonian but they can not represent fields with well-defined amplitudes and phase at a classical level. This is because for the Fock states the expectation values of the quadrature operators vanish. Therefore the state that represents a certain electromagnetic state with certain phase is known as coherent state. By definition, Coherent states are eigenstates of the annihilation operator a, defined by the equation

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

where α is a complex number. Using the above relation and then normalizing, the state $a|\alpha\rangle$ can be expressed in terms of number states as

$$|\alpha\rangle = e^{-\frac{1}{2}-|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

. Thus the probability of finding n photons in a coherent state is given by the Poisson distribution. The coherent states can also be generated by acting the Weyl displacement operator on single mode vaccum state $|0\rangle$.

$$|\alpha\rangle = D(\alpha)|0\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle$$

Out of all states of radiation field, the coherent staets are the most important and arise frequently. Not only they be an accurate representation of the field produced by a stabilized laser, but many study of fields rely on the mathematics of coherent state.

2.3 Squeezed State

Squeezed states arise in simple quantum models of a number of nonlinear optical processes including optical parametric oscillation and four-wave mixing.
The squeezed vacuum state $|r\rangle$ is obtained by applying the squeezing operator $S(r)$ to the vacuum state $|0\rangle$:

$$|r\rangle = S(r)|0\rangle$$

The single-mode squeezing operator $S(r)$ is defined as:

$$S(r) = e^{\frac{r}{2}(a^2 - a^{\dagger 2})}$$

where $a^\dagger$ and a are the creation and annihilation operators, respectively, and r is the squeezing parameter. The squeezed vacuum state in terms of Fock states is given by

$$|r\rangle = \frac{1}{\sqrt{\cosh(r)}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} \tanh^n(r)|2n\rangle$$

3 Continuous Variable Systems

The quantum states described above are states of a single particle or mode. However, generalization of the quantum states to arbitarily large number of particles or modes leads to continuous variable systems.

3.1 Definition

Continuous variable systems are usually the systems that has infinite dimensional Hilbert spaces associated with them. For N canonical modes the hilbert space is $\mathcal{H} = \bigotimes_{k=1}^{N} \mathcal{H}_k$ resulting from the tensor product structure of infinite dimensional Fock spaces $\mathcal{H}_k$'s, each associated with a single mode.
The state of a N bosonic mode system with different frequencies having Hamiltonian

$$H = \sum_{k=1}^{N} \hbar\omega_k \left(a_k^\dagger a_k + \frac{1}{2} \right)$$

is an example of continuous variable system(CV system).

3.2 Description of CV system

The operators $a_k^\dagger$ and a_k can be expressed in term of canonical operators q_k and p_k as

$$a_k = \frac{1}{2}(q_k + ip_k) \text{ and } a_k^\dagger = \frac{1}{2}(q_k - ip_k)$$

which satisfy the relations

$$[q_k, p_k] = \delta_{kk'},$$

$$[a_k, a_k^\dagger] = \delta_{kk'} \text{ and } [a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0$$

For convenience we used natural units having $\hbar = 2$. The canonical operators can be grouped together in the vector

$$R = (q_1, p_1, q_2, p_2, \ldots, q_N, p_N)^{\mathrm{T}}$$

which can be used to write the bosonic commutation relation is compact form as

$$[R_k, R_l] = 2i\Omega_{kl},$$

where Ω is the symplectic form

$$\Omega = \bigoplus_{k=1}^{N} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

4 Phase Space Description

In the phase-space formulation, we treat position and momentum variables on equal footing in phase space. Unlike the Schrödinger picture, which uses position or momentum representations, the phase-space approach employs a quasiprobability distribution to describe the quantum state. Instead of a wave function, state vector, or density matrix, we use a quasiprobability distribution $f(x,p)$ that describes how the state is distributed in phase space.

4.1 Characteristic function

The complete description of a quantum state ρ is provided by one of its s-ordered characteristic functions defined by

$$\chi_s(\xi) = \mathrm{Tr}[\rho D(\xi)] e^{s||\xi||^2/2},$$

with $\xi \in \mathbb{R}^{2N}$, $||\xi||$ denoting the Eucledian norm of ξ in $\mathbb{R}^{2N}$. The vector ξ belongs to the 2N dimensional Phase space. This vector groups all the canonical variables and hence its complex fourier transform gives us a quasi-probability distribution of the state over phase space.

4.2 Quasi-probability distribution

The Fourier transform of the characteristic function is given by

$$W_s(\xi) = \frac{1}{\pi^2} \int_{\mathbb{R}^{2N}} \chi_s(\alpha) e^{i\alpha^{\mathrm{T}}\Omega\xi} d^{2N}\alpha$$

This constitutes another set of complete description of the quantum state. There exists cases where W_s is not regular probability distribution because it can be singular or assume negative values. There are different representations depending on the value of s.

- The case $s = -1$ represents regular distribution known as the *Husimi Q function*.
- The case $s = 0$ is known as the *Wigner function* or *Wigner distribution* which is more widely used.
- Finally for $s = 1$, the function is known as the *Glauber-Sudarshan P representation.*

5 Wigner Distribution

In this section Wigner function is defined and its various properties are discussed.

5.1 Definition of the Wigner Function

Wigner function is a generating function for all spatial auto correlation functions of a given quantum-mechanical wave function $\psi(x)$. Thus, it maps on the quantum density matrix in the map between real phase-space functions and Hermitian operators.

For a pure state $\psi(x)$, the Wigner distribution is given by,

$$W(x,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} \psi^*(x+y)\psi(x-y)e^{2ipy/\hbar}dy$$

The Wigner function for a mixed state denoted by density operator ρ is given by:

$$W(\alpha) = \frac{1}{\pi^2}\int_{\mathbb{R}^{2N}} \mathrm{Tr}[\rho D(\xi)]e^{i\alpha\Omega\xi}d^2\xi$$

where Ω is the symplectic matrix and α is any complex number expressing canonical position and momentum.

5.2 Properties of Wigner function

Some important properties and features of Wigner function are as follows.

- For a Physical state ρ and its Wigner presentation W(x,p),

  $$\int_{-\infty}^{\infty} W(x,p)dxdp = \mathrm{Tr}(\rho) = 1$$

  For a n-mode state,

  $$\int_{\mathbb{R}^{2N}} W(\kappa)d^{2N}\kappa = 1$$

- The purity(μ) of a state in terms of Wigner distribution $W(\kappa)$ is given by

  $$\mu = \mathrm{Tr}\rho^2 = \int_{\mathbb{R}^{2N}} W^2(\kappa)d^{2N}\kappa$$

- $W(x,p)$ obeys the reflection symmetries. Which means,

  $$\psi(x) \to \psi^*(x) \implies W(x,p) \to W(x,-p),$$

  $$\psi(x) \to \psi(-x) \implies W(x,p) \to W(-x,-p)$$

- The negativity of Wigner function signifies that it certainly describes any quantum mechanical state opposed to classical state where probabilities can not be negative. The nonclassical features of electromagnetic field and their Wigner descriptions in phase space implies that Wigner negativity is a indicator of non-classicallity.

6 Wigner Function of various states

In this section Wigner function of some important quantum optical states are derived.

6.1 Coherent State

Let's start with the wave function of a coherent state, denoted by $|\alpha\rangle$, where α is a complex number. The coherent state is defined as the eigenstate of the annihilation operator $\hat{a}$:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

Let us now try to calculate Wigner Function of a single mode coherent state $|\beta\rangle$. To evaluate this we find the expectation value of the displacement operator $D(\alpha)$ over a state $|\beta\rangle$.

The displacement operator $D(\alpha)$ is defined as:

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$$

where $a^\dagger$ and a are the creation and annihilation operators, respectively.

We want to find the expectation value of $D(\alpha)$ in the state $|\beta\rangle$:

$$\langle D(\alpha)\rangle = \langle\beta|D(\alpha)|\beta\rangle$$

The BCH formula allows us to simplify the exponential of operators:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$$

Let's apply the BCH formula to our displacement operator:

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} = e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-\frac{1}{2}|\alpha|^2}$$

Now we can compute the expectation value:

$$\langle D(\alpha)\rangle = \langle\beta|e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-\frac{1}{2}|\alpha|^2}|\beta\rangle$$

We know that $a|\beta\rangle = \beta|\beta\rangle$ and $a^\dagger|\beta\rangle = \beta^*|\beta\rangle$. Using these relations, we can simplify the expression.

After simplification, we get:

$$\langle D(\alpha)\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\beta^* - \alpha^*\beta}$$

Wigner function of state ρ is given by

$$W(\xi) = \frac{1}{\pi^2}\int_C \mathrm{Tr}[\rho D(\alpha)] e^{i\alpha^T\Omega\xi} d^2\alpha$$

For a single mode state

$$\begin{aligned} W(\xi) &= \frac{1}{\pi^2}\int_C \langle D(\alpha)\rangle e^{\xi\alpha^*-\alpha\xi^*} d^2\alpha \\ &= \frac{1}{\pi^2}\int_C e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\beta^*-\alpha^*\beta} e^{\xi\alpha^*-\alpha\xi^*} d^2\alpha \\ &= \frac{1}{\pi^2}\int_C e^{-\frac{1}{2}|\alpha|^2} e^{\alpha(\beta-\xi)^*-\alpha^*(\beta-\xi)} d^2\alpha \end{aligned}$$

Taking $\gamma = \beta - \xi$,

$$W(\gamma) = \frac{1}{\pi^2}\int_C e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\gamma^*-\alpha^*\gamma} d^2\alpha \tag{1}$$

From appendix A, the integral can be found to be

$$\begin{aligned} W(\gamma) &= \frac{2}{\pi} e^{-2|\gamma|^2} \\ &\Rightarrow \boxed{W_{|\beta\rangle}(\xi) = \frac{2}{\pi} e^{-2|\beta-\xi|^2}} \end{aligned}$$

This is the expression of Wigner function of state $|\beta\rangle$.

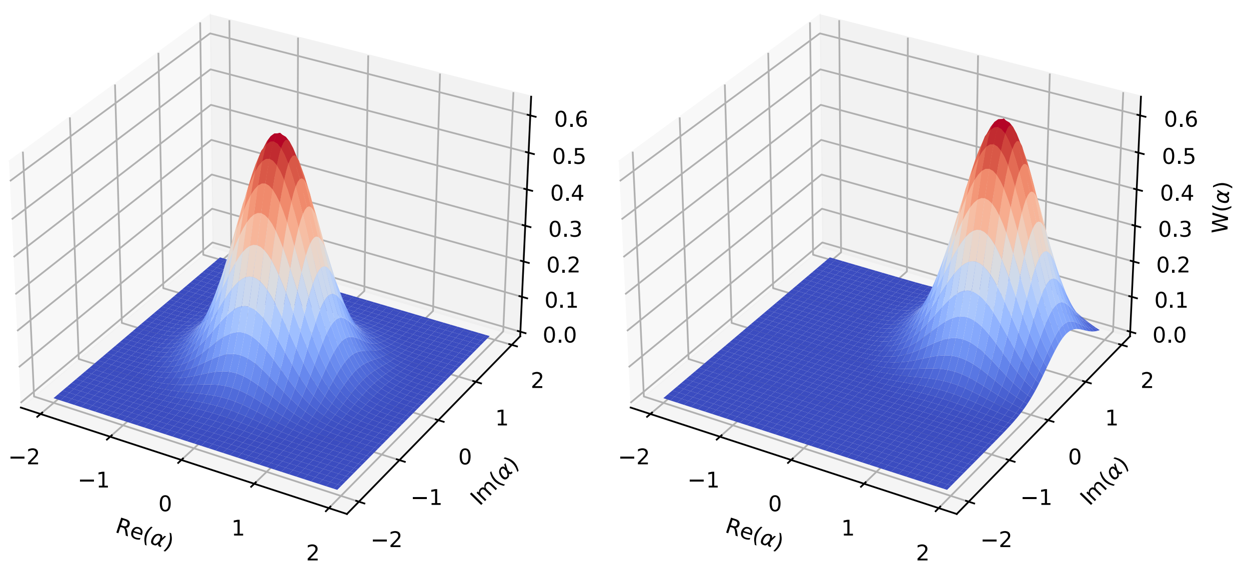

Figure 1 : Wigner Function of two coherent states : $|0\rangle$ on left and $|1+i\rangle$ on right.

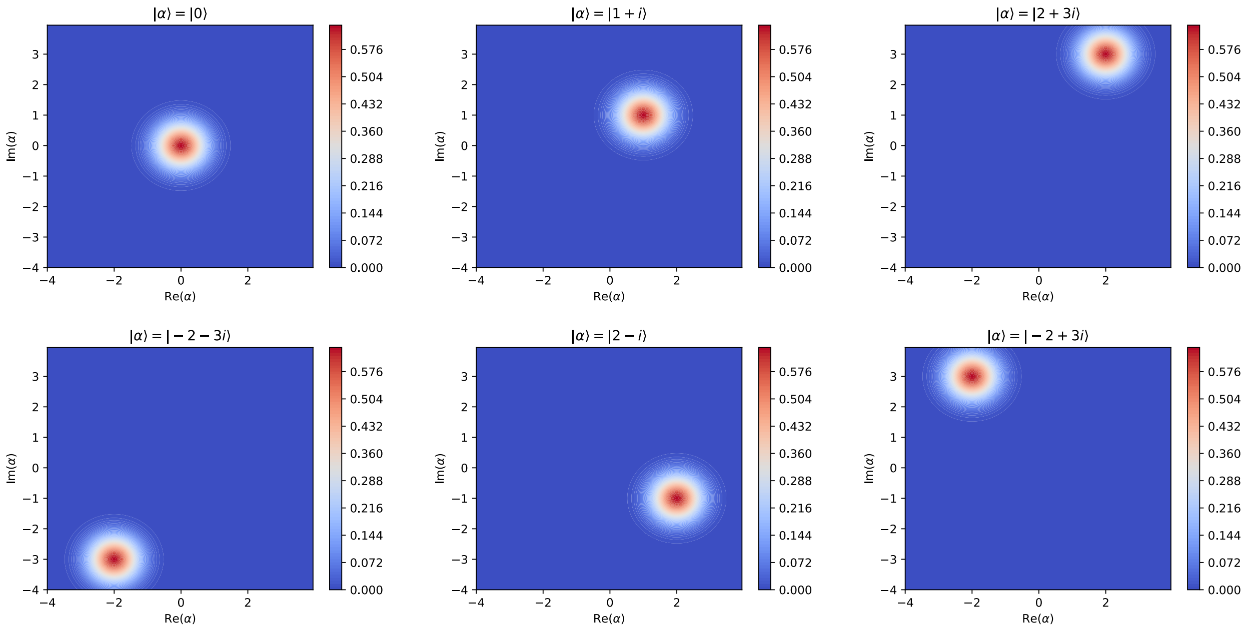

Figure 2 : Density plot of Wigner Function of a few coherent states.

6.2 Squeezed State

The Weyl displacement operator is defined as:

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$$

Here, $a^\dagger$ and a are the creation and annihilation operators for the harmonic oscillator (single-mode field), respectively. The parameter α is a complex number.

Let's find the hermitian conjugate of the squeezing operator:

$$S^\dagger(r) = e^{\frac{r}{2}(a^{\dagger 2} - a^2)} = e^{-\frac{r}{2}(a^2 - a^{\dagger 2})}$$

Therefore,

$$S^\dagger(r)S(r) = e^{\frac{r}{2}(a^2 - a^{\dagger 2})} e^{-\frac{r}{2}(a^2 - a^{\dagger 2})} = \mathrm{I}$$

also

$$S(r)S^\dagger(r) = \mathrm{I}$$

Now let's apply $S^\dagger(r)$ to the annihilation operator a:

$$S^\dagger(r)aS(r) = e^{\frac{r}{2}(a^{\dagger 2} - a^2)} a e^{-\frac{r}{2}(a^{\dagger 2} - a^2)}$$

We can use the BCH formula to simplify the exponential:

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \ldots$$

The commutation relations for the annihilation and creation operators are:

$$[a, a^\dagger] = 1 \quad (1)$$

$$[a, a] = [a^\dagger, a^\dagger] = 0 \quad (2)$$

Let's apply the BCH formula to our expression:

$$S(r)^\dagger a S(r) = a + [\frac{r}{2}(a^{\dagger 2} - a^2), a] + \frac{1}{2}[\frac{r}{2}(a^{\dagger 2} - a^2), [\frac{r}{2}(a^{\dagger 2} - a^2), a]] + \dots$$

Using the commutation relations (1) and (2), we get:

$$[\frac{r}{2}(a^{\dagger 2} - a^2), a] = \frac{r}{2}[a^{\dagger 2}, a] = -ra^\dagger$$

Similarly,

$$[\frac{r}{2}(a^{\dagger 2} - a^2), [\frac{r}{2}(a^{\dagger 2} - a^2), a]] = r^2 a$$

Thus,

$$\begin{aligned} S(r)^\dagger a S(r) &= a - ra^\dagger + \frac{r^2}{2!}a - \frac{r^3}{3!}a^\dagger + \frac{r^4}{4!}a + \dots \\ &= a(1 + \frac{r^2}{2!} + \frac{r^4}{4!} + \dots) - a^\dagger(r + \frac{r^3}{3!} + \frac{r^5}{5!} + \dots) \\ &= a \cosh r - a^\dagger \sinh r \end{aligned}$$

Therefore, we have proved that:

$$S(r)^\dagger a S(r) = a \cosh r - a^\dagger \sinh r$$

Similarly

$$S(r)^\dagger a^\dagger S(r) = a^\dagger \cosh r - a \sinh r$$

Again

$$S(r)^\dagger a^2 S(r) = S(r)^\dagger a S(r) S(r)^\dagger a S(r) = (a \cosh r - a^\dagger \sinh r)^2$$

and

$$S(r)^\dagger a^{\dagger 2} S(r) = S(r)^\dagger a^\dagger S(r) S(r)^\dagger a^\dagger S(r) = (a^\dagger \cosh r - a \sinh r)^2$$

Thus for any arbitrary operator $G(a, a^\dagger)$,

$$S(r)^\dagger G(a, a^\dagger) S(r) = G(a \cosh r - a^\dagger \sinh r, a^\dagger \cosh r - a \sinh r)$$

Then,

$$\begin{aligned} \langle r|D(\alpha)|r\rangle &= \langle 0|S^\dagger(r) \exp(\alpha a^\dagger - \alpha^* a) S(r)|0\rangle \\ &= \langle 0| \exp(\alpha(a^\dagger \cosh r - a \sinh r) - \alpha^*(a \cosh r - a^\dagger \sinh r))|0\rangle \\ &= \langle 0| \exp(a^\dagger(\alpha \cosh r + a^* \sinh r) - a(\alpha^* \cosh r + \alpha \sinh r))|0\rangle \end{aligned}$$

Let, $\gamma = \alpha\cosh r + \alpha^* \sinh r$. Then,

$$\langle r|D(\alpha)|r\rangle = \langle 0|\exp(a^\dagger\gamma - a\gamma^*)|0\rangle$$

We know,

$$e^{-\gamma^* a}|0\rangle = |0\rangle \text{ and } \langle 0|e^{\gamma a^\dagger} = \langle 0|$$

Using BCH equality,

$$\begin{aligned}\langle D(\alpha)\rangle = \langle r|D(\alpha)|r\rangle \quad &= \langle 0|e^{\gamma a^\dagger} e^{-\gamma^* a} e^{-\frac{1}{2}|\gamma|^2}|0\rangle \\ &= e^{-\frac{1}{2}|\gamma|^2}\langle 0|0\rangle \\ &= e^{-\frac{1}{2}|\gamma|^2}\end{aligned}$$

For a single mode state,

$$\begin{aligned}W(\xi) &= \frac{1}{\pi^2}\int_C \langle D(\alpha)\rangle e^{\xi\alpha^* - \alpha\xi^*} d^2\alpha \\ &= \frac{1}{\pi^2}\int_C e^{-\frac{1}{2}|\gamma|^2} e^{\xi\alpha^* - \alpha\xi^*} d^2\alpha\end{aligned}$$

where, $\gamma = \alpha\cosh r + \alpha^* \sinh r$.
To carry out the integration we change the variable from α to γ. Therefore expressing α in term of γ, we obtain ,

$$\alpha = \gamma\cosh r - \gamma^* \sinh r$$

$$\text{and, } \alpha^* = \gamma^* \cosh r - \gamma \sinh r$$

$$\begin{aligned}W(\xi) &= \frac{1}{\pi^2}\int_C e^{-\frac{1}{2}|\gamma|^2} e^{\xi(\gamma^*\cosh r - \gamma\sinh r) - (\gamma\cosh r - \gamma^*\sinh r)\xi^*} d^2\gamma \\ &= \frac{1}{\pi^2}\int_C e^{-\frac{1}{2}|\gamma|^2} e^{\gamma(-\xi^*\cosh r - \xi\sinh r)} e^{-\gamma^*(-\xi\cosh r - \xi^*\sinh r)\xi^*} d^2\gamma \\ &= \frac{1}{\pi^2}\int_C e^{-\frac{1}{2}|\gamma|^2} e^{\gamma\chi^*} e^{-\gamma^*\chi} d^2\gamma\end{aligned}$$

where, $\chi = -\xi\cosh r - \xi^* \sinh r$

Which can be evaluated as same form as coherent state from Appendix A. So,

$$\begin{aligned}W(\xi) &= \frac{1}{\pi^2}\int_C e^{-\frac{1}{2}|\gamma|^2} e^{\gamma\chi^*} e^{-\gamma^*\chi} d^2\gamma \\ &= \frac{2}{\pi} e^{-2|\chi|^2} \\ &= \frac{2}{\pi} e^{-2|-\xi\cosh r - \xi^*\sinh r|^2} \\ &= \frac{2}{\pi} e^{-2|\xi\cosh r + \xi^*\sinh r|^2} \\ &\Rightarrow \boxed{W_r(\xi) = \frac{2}{\pi} e^{-2|\xi\cosh r + \xi^*\sinh r|^2}}\end{aligned}$$

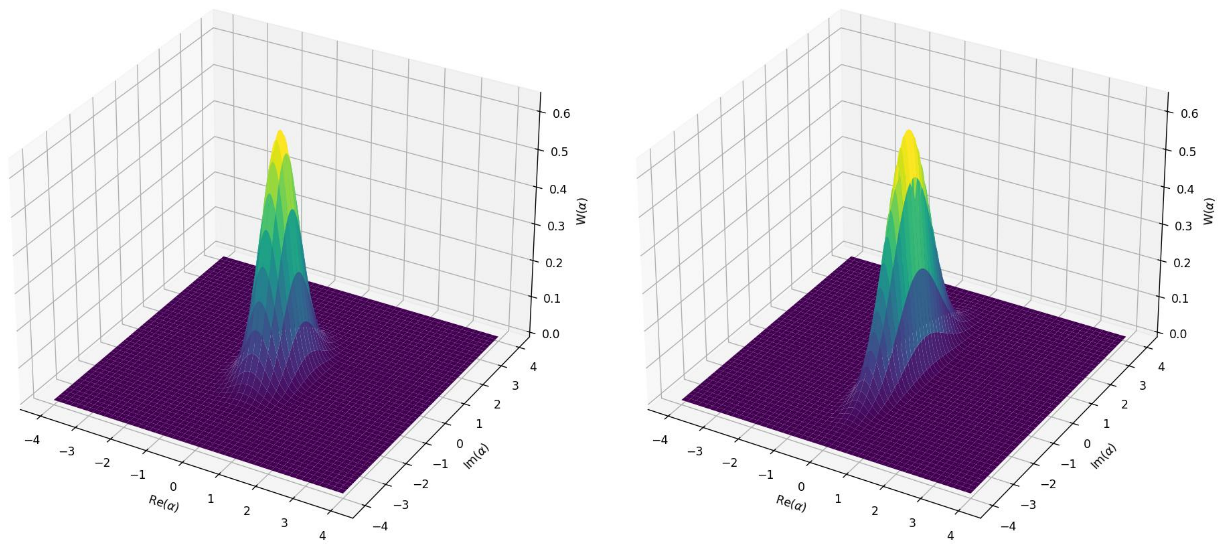

Figure 3 : Wigner distribution of Two single mode squeezed state : r= 0.3 (on left) and r=0.7 (on right).

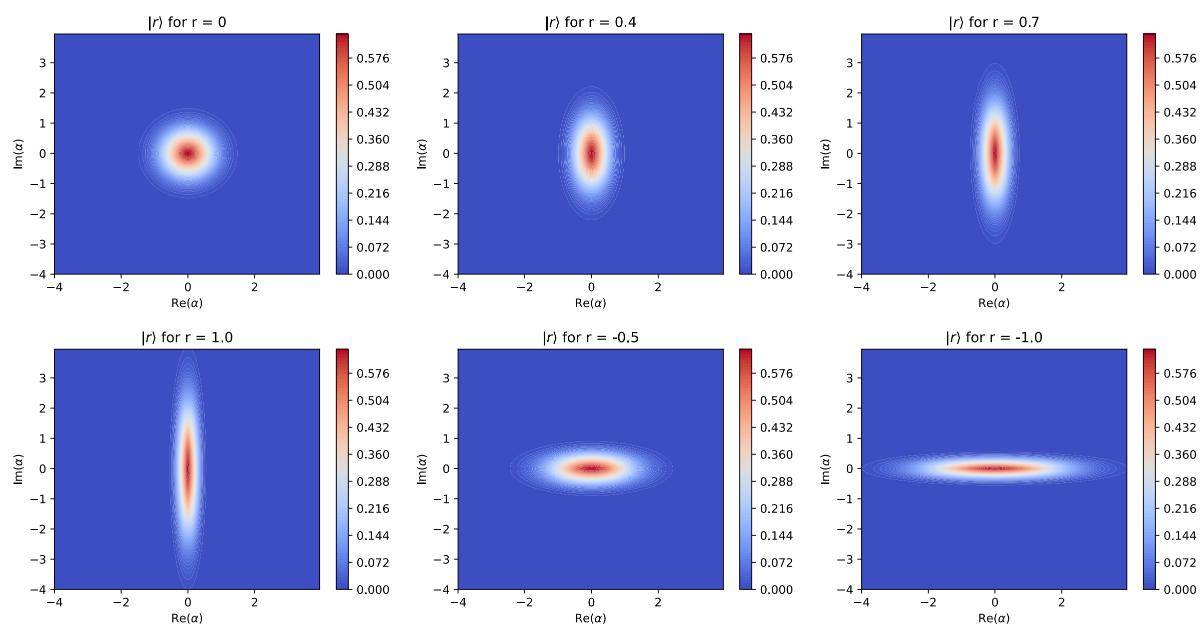

Figure 4 : Wigner distribution of a few squeezed states centered at origin with different squeezing factor.

6.3 Photon added coherent State

The Weyl displacement operator is defined as:

$$D(\beta) = \exp(\beta a^\dagger - \beta^* a)$$

Here, $a^\dagger$ and a are the creation and annihilation operators for the harmonic oscillator (single-mode field), respectively. The parameter β is a complex number.

The displaced vacuum state $|\beta\rangle$ is obtained by applying the Displacement operator $D(\beta)$ to the vacuum state $|0\rangle$:

$$|\beta\rangle = D(\beta)|0\rangle$$

Let's find the hermitian conjugate of the displacement operator:

$$D^\dagger(\beta) = \exp(\beta^* a - \beta a^\dagger)$$

Therefore,

$$D^\dagger(\beta)D(\beta) = \mathrm{I}$$

also

$$D(\beta)D^\dagger(\beta) = \mathrm{I}$$

Now let's apply $D(\beta)$ and $D^\dagger(\beta)$ on annihilation operator:

$$D^\dagger(\beta)aD(\beta) = e^{\beta^* a - \beta a^\dagger} a e^{\beta a^\dagger - \beta^* a}$$

We can use the BCH formula and bosonic commutation relations, we get

$$\begin{aligned}
D^\dagger(\beta)aD(\beta) &= e^{\beta^* a - \beta a^\dagger} a e^{\beta a^\dagger - \beta^* a} \\
&= a + [\beta^* a - \beta a^\dagger, a] + \frac{1}{2}[\beta^* a - \beta a^\dagger, [\beta^* a - \beta a^\dagger, a]] + \ldots \\
&= a - \beta[a^\dagger, a] \\
&= a + \beta
\end{aligned}$$

Similarly it can be shown that

$$D^\dagger(\beta)a^\dagger D(\beta) = a^\dagger + \beta^*$$

And for any general operator in the form $G(a, a^\dagger)$,

$$D^\dagger(\beta)G(a, a^\dagger)D(\beta) = G(a + \beta, a^\dagger + \beta^*)$$

Single Photon added coherent state is given by $|\beta\rangle = a^\dagger D(\beta)|0\rangle$ where $D(\beta)$ is single mode Weyl displacement operator. The expectation value of $D(\alpha)$ over the state $|\beta\rangle$ is given by

$$
\begin{aligned}
\langle D(\alpha)\rangle &= \langle\beta|D(\alpha)|\beta\rangle \\
&= \langle 0|D^\dagger(\beta)aD(\alpha)a^\dagger D(\beta)|0\rangle \\
&= \langle 0|(a+\beta)D(\alpha)(a^\dagger+\beta^*)|0\rangle \\
&= \langle 0|(a+\beta)\exp(\alpha(a^\dagger+\beta^*)-\alpha^*(a+\beta))(a^\dagger+\beta^*)|0\rangle \\
&= e^{\alpha\beta^*-\alpha^*\beta}\langle 0|(a+\beta)e^{\alpha a^\dagger-\alpha^* a}(a^\dagger+\beta^*)|0\rangle \\
&= e^{\alpha\beta^*-\alpha^*\beta}\langle 0|(a+\beta)e^{\alpha a^\dagger-\alpha^* a}(|1\rangle+\beta^*|0\rangle) \\
&= e^{\alpha\beta^*-\alpha^*\beta}\langle 0|(a+\beta)e^{-\frac{|\alpha|^2}{2}}e^{\alpha a^\dagger}e^{-\alpha^* a}(|1\rangle+\beta^*|0\rangle) \\
&= e^{\alpha\beta^*-\alpha^*\beta}e^{-\frac{|\alpha|^2}{2}}\langle 0|(a+\beta)e^{\alpha a^\dagger}((\mathrm{I}-\alpha^* a)|1\rangle+\beta^*|0\rangle) \\
&= e^{\alpha\beta^*-\alpha^*\beta}e^{-\frac{|\alpha|^2}{2}}\langle 0|(a+\beta)e^{\alpha a^\dagger}(|1\rangle-\alpha^*|0\rangle+\beta^*|0\rangle) \\
&= e^{\alpha\beta^*-\alpha^*\beta}e^{-\frac{|\alpha|^2}{2}}(\langle 1|-\alpha\langle 0|+\beta\langle 0|)(|1\rangle-\alpha^*|0\rangle+\beta^*|0\rangle) \\
&= e^{\alpha\beta^*-\alpha^*\beta}e^{-\frac{|\alpha|^2}{2}}(\langle 1|+\alpha\langle 0|+\beta\langle 0|)(|1\rangle-\alpha^*|0\rangle+\beta^*|0\rangle) \\
&= e^{\alpha\beta^*-\alpha^*\beta}e^{-\frac{|\alpha|^2}{2}}(1+(\alpha+\beta)(\beta^*-\alpha^*)) \\
&= e^{\alpha\beta^*-\alpha^*\beta}e^{-\frac{|\alpha|^2}{2}}(1+\alpha\beta^*-\alpha^*\beta-|\alpha|^2+|\beta|^2)
\end{aligned}
$$

For a single mode state,

$$
\begin{aligned}
W(\xi) &= \frac{1}{\pi^2}\int_C \langle D(\alpha)\rangle e^{\xi\alpha^*-\alpha\xi^*}d^2\alpha \\
&= \frac{1}{\pi^2}\int_C e^{\alpha\beta^*-\alpha^*\beta}e^{-\frac{|\alpha|^2}{2}}(1+\alpha\beta^*-\alpha^*\beta-|\alpha|^2+|\beta|^2)e^{\xi\alpha^*-\alpha\xi^*}d^2\alpha
\end{aligned}
$$

The integral from the previous expression :

$$
\int_C e^{\alpha\beta^*-\alpha^*\beta}e^{-\frac{|\alpha|^2}{2}}(1+\alpha\beta^*-\alpha^*\beta-|\alpha|^2+|\beta|^2)e^{\xi\alpha^*-\alpha\xi^*}d^2\alpha \tag{2}
$$

From appendix B, The value of the integral is found to be

$$
\begin{aligned}
W(\xi) &= \frac{1}{\pi^2}\int_C \langle D(\alpha)\rangle e^{\xi\alpha^*-\alpha\xi^*}d^2\alpha \\
&= \frac{1}{\pi^2}\int_C e^{\alpha\beta^*-\alpha^*\beta}e^{-\frac{|\alpha|^2}{2}}(1+\alpha\beta^*-\alpha^*\beta-|\alpha|^2+|\beta|^2)e^{\xi\alpha^*-\alpha\xi^*}d^2\alpha \\
&= \frac{2}{\pi}(|2\xi-\beta|^2-1)e^{-2|\xi-\beta|^2} \\
&\Rightarrow \boxed{W_\beta(\xi) = \frac{2}{\pi}(|2\xi-\beta|^2-1)e^{-2|\xi-\beta|^2}}
\end{aligned}
$$

This is the Wigner function of SPACS (Single Photon added Coherent state) $|\beta\rangle$.

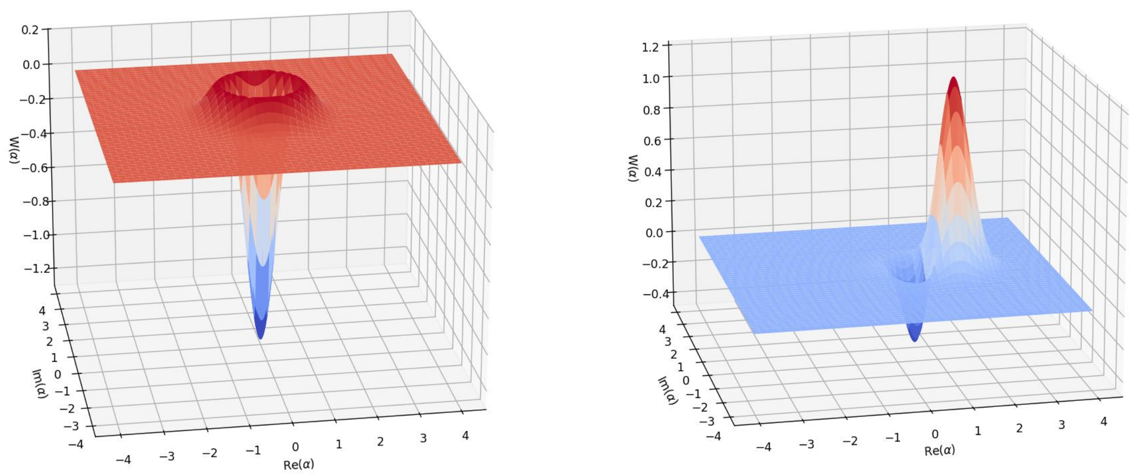

Figure 5 : Single Photon added Coherent State for $|\beta\rangle = |0\rangle$ (0n left) and $|\beta\rangle = |1 + i0\rangle$ (On right).

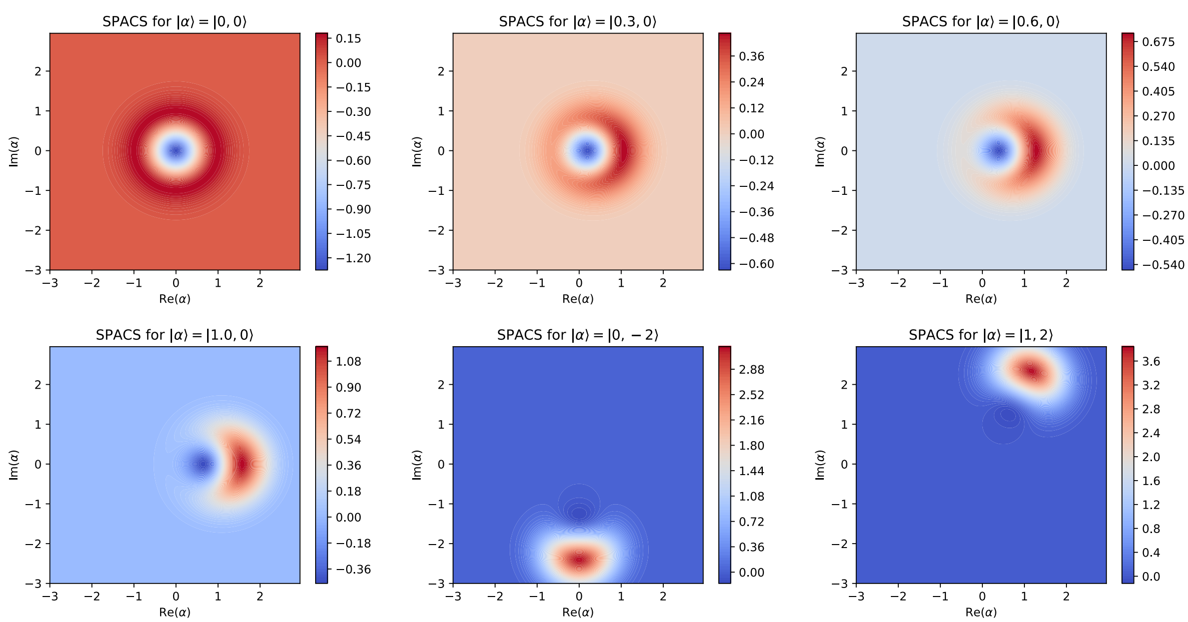

Figure 6 : Density plot of Single Photon added Coherent States for different $|\alpha\rangle$.

6.4 Photon added squeezed State

From section 6.2 we know that, For any arbitrary operator $G(a, a^\dagger)$,

$$S(r)^\dagger G(a, a^\dagger) S(r) = G(a \cosh r - a^\dagger \sinh r, a^\dagger \cosh r - a \sinh r)$$

Single Photon added squeezed state is given by

$$|r\rangle = a^\dagger S(r)|0\rangle$$

where S(r) is single mode squeezing operator. The expectation value of $D(\alpha)$ over the state $|r\rangle$ is given by

$$\begin{aligned}
\langle D(\alpha)\rangle &= \langle r|D(\alpha)|r\rangle \\
&= \langle 0|S^\dagger(r) a D(\alpha) a^\dagger S(r)|0\rangle \\
&= \langle 0|S^\dagger(r) a e^{\alpha a^\dagger - \alpha^* a} a^\dagger S(r)|0\rangle \\
&= \langle 0|(a\cosh r - a^\dagger \sinh r) e^{\alpha(a^\dagger \cosh r - a \sinh r) - \alpha^*(a\cosh r - a^\dagger \sinh r)}(a^\dagger \cosh r - a\sinh r)|0\rangle \\
&= \langle 0|(a\cosh r - a^\dagger \sinh r) e^{\gamma a^\dagger - \gamma^* a}(a^\dagger \cosh r - a\sinh r)|0\rangle \\
&\text{where } \gamma = \alpha\cosh r + \alpha^* \sinh r
\end{aligned}$$

Now,

$$(a^\dagger \cosh r - a\sinh r)|0\rangle = \cosh r|1\rangle$$

and

$$\langle 0|(a\cosh r - a^\dagger \sinh r) = \cosh r\langle 1|$$

So,

$$\begin{aligned}
\langle D(\alpha)\rangle &= \cosh^2 r\langle 1|e^{\gamma a^\dagger - \gamma^* a}|1\rangle \\
&= \cosh^2 r\langle 1|e^{-\frac{|\gamma|^2}{2}} e^{\gamma a^\dagger} e^{-\gamma^* a}|1\rangle \\
&= \cosh^2 r e^{-\frac{|\gamma|^2}{2}}\langle 1|e^{\gamma a^\dagger}(\mathrm{I} - \gamma^* a + \cdots)|1\rangle \\
&= \cosh^2 r e^{-\frac{|\gamma|^2}{2}}\langle 1|e^{\gamma a^\dagger}(|1\rangle - \gamma^*|0\rangle) \\
&= \cosh^2 r e^{-\frac{|\gamma|^2}{2}}(\langle 1| + \gamma\langle 0|)(|1\rangle - \gamma^*|0\rangle) \\
&= \cosh^2 r e^{-\frac{|\gamma|^2}{2}}(1 - \gamma\gamma^*) \\
&= \cosh^2 r e^{-\frac{|\gamma|^2}{2}}(1 - |\gamma|^2)
\end{aligned}$$

For a single mode state,

$$\begin{aligned}
W(\xi) &= \frac{1}{\pi^2}\int_C \langle D(\alpha)\rangle e^{\xi\alpha^* - \alpha\xi^*} d^2\alpha \\
&= \frac{1}{\pi^2}\int_C \cosh^2 r e^{-\frac{|\gamma|^2}{2}}(1 - |\gamma|^2) e^{\xi\alpha^* - \alpha\xi^*} d^2\alpha
\end{aligned}$$

From appendix C, The value of the integral is found to be

$$W(\xi) = \frac{1}{\pi^2}\int_C \cosh^2 r e^{-\frac{|\gamma|^2}{2}}(1 - |\gamma|^2) e^{\xi\alpha^* - \alpha\xi^*} d^2\alpha$$

where, $\gamma = \alpha \cosh r + \alpha^* \sinh r$.
To carry out the integration we change the variable from α to γ. Therefore expressing α in term of γ, we obtain ,

$$\alpha = \gamma \cosh r - \gamma^* \sinh r$$
$$\text{and, } \alpha^* = \gamma^* \cosh r - \gamma \sinh r$$

Thus changing from α to γ,

$$W(\xi) = \frac{\cosh^2 r}{\pi^2} \int_C e^{-\frac{|\gamma|^2}{2}} (1 - |\gamma|^2) \exp[\xi(\gamma^* \cosh r - \gamma \sinh r) - \xi^*(\gamma \cosh r - \gamma^* \sinh r)] d^2\gamma$$
$$\Rightarrow W(\chi) = \frac{\cosh^2 r}{\pi^2} \int_C e^{-\frac{|\gamma|^2}{2}} (1 - |\gamma|^2) e^{\chi\gamma^* - \chi^*\gamma} d^2\gamma$$

The integral from the previous expression:

$$I = \int_C e^{-\frac{|\gamma|^2}{2}} (1 - |\gamma|^2) e^{\chi\gamma^* - \chi^*\gamma} d^2\gamma \tag{3}$$

where $\chi = \xi \cosh r + \xi^* \sinh r$.
The integration is carried out in **Appendix C** and found out to be

$$W(\chi) = \frac{2\cosh^2 r}{\pi} (4|\chi|^2 - 1) e^{-2|\chi|^2}$$
$$\Rightarrow \boxed{W(\xi) = \frac{2\cosh^2 r}{\pi} (4|\xi \cosh r + \xi^* \sinh r|^2 - 1) e^{-2|\xi \cosh r + \xi^* \sinh r|^2}}$$

This is the Wigner function of SPASS (Single Photon added Squeezed state) $|r\rangle$.

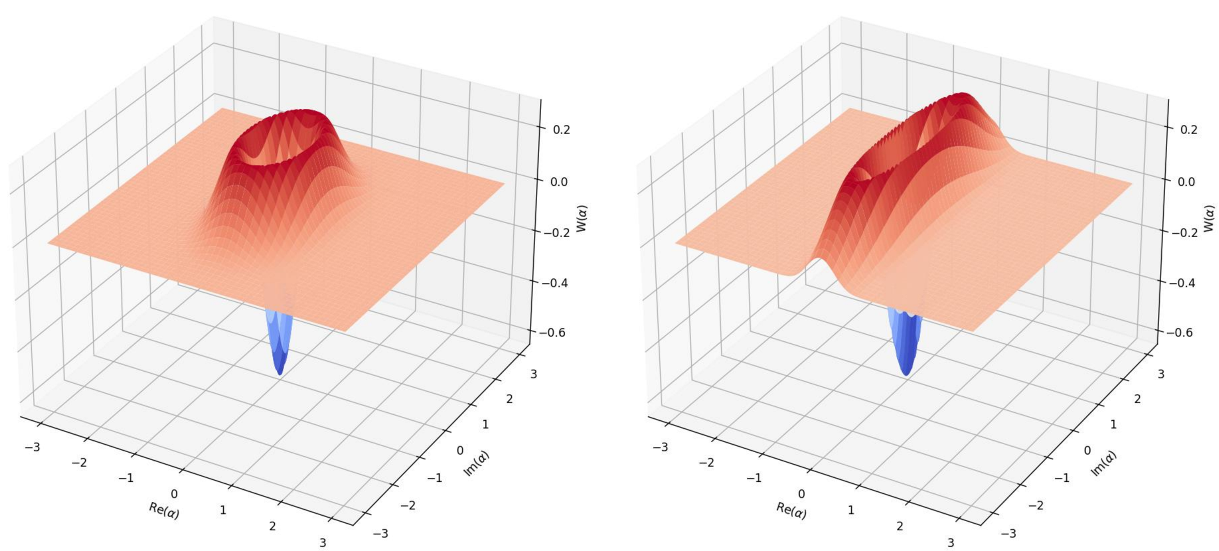

Figure 7 : Single Photon added Squeezed State for r = 0.3 (0n left) and r = 0.8 (On right).

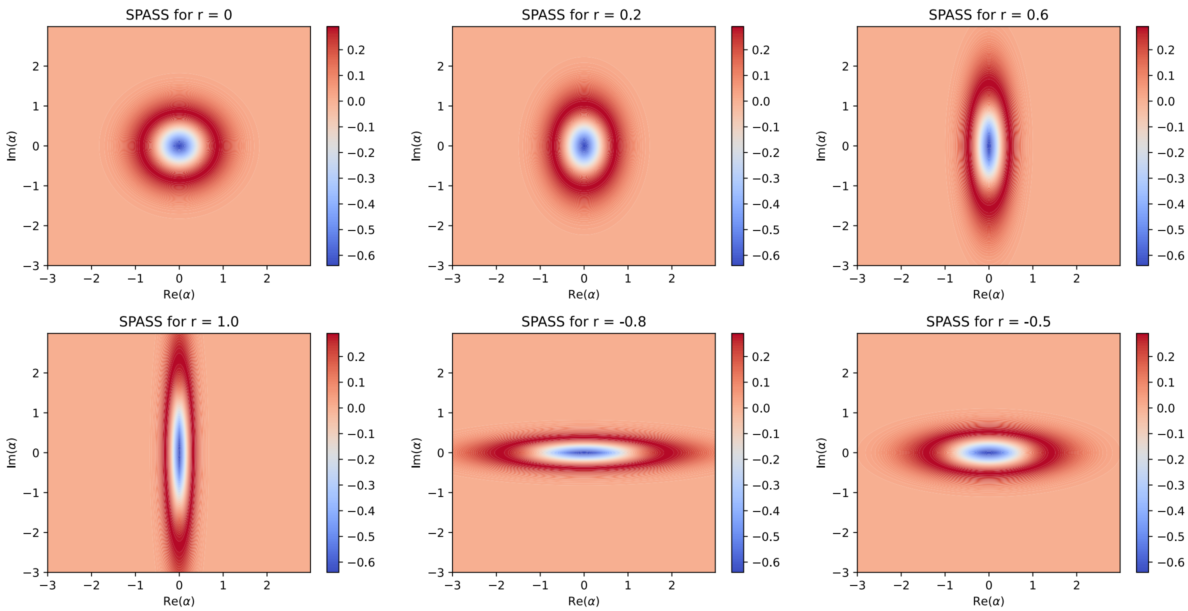

Figure 8 : Density plot of Single Photon added Squeezed States for different **r**.

7 Two mode squeezed state and entanglement

In this section we will discuss entanglement in two mode squeezed state using Wigner function representation.

7.1 Definition

Let a and $a^\dagger$ be respectively annihilation and creation operators of mode 1 and b and $b^\dagger$ be respectively annihilation and creation operators of mode 2. Let $|\eta\rangle$ be the two mode squeezed state. The state $|\eta\rangle$ is obtained by applying the Two mode squeezing operator $S_{12}(r)$ to the two mode vacuum state $|00\rangle$:

$$|\eta\rangle = S_{12}(r)|00\rangle$$

The two-mode squeezing operator $S_{12}(r)$ is defined as:

$$S(r) = \exp\left[-\frac{r}{2}(a^\dagger b^\dagger - ab)\right]$$

where r is the squeezing parameter.
Then using the BCH equality of section 6.2, it can be shown that,

$$\langle\eta|G(a,b)|\eta\rangle = \langle 00|G(a\cosh r + b^\dagger e^{i\phi}\sinh r, b\cosh r + a^\dagger e^{i\phi}\sinh r)|00\rangle.$$

Using this it can be proved that,

$$\langle a^\dagger a\rangle = \langle b^\dagger b\rangle = \sinh^2 r,$$

$$\langle ab \rangle = \sinh r \cosh r e^{i\phi}, \langle a^\dagger b \rangle = 0$$

To evaluate the Wigner function in conventional way using integration is difficult. Therefore here we introduce the idea of covariance matrix.

7.2 Covariance Matrix Formalism

The elements of a covariance matrix σ is given by

$$\sigma_{ij} = \frac{1}{2}\langle R_i R_j + R_j R_i \rangle - \langle R_i \rangle \langle R_j \rangle$$

where R_i denotes the i-th element of the vector containing the canonical variables. However, the expectation value of the first moments can be adjusted to zero using local unitary transformations. Thus the expression for σ_{ij} reduces to

$$\sigma_{ij} = \frac{1}{2}\langle R_i R_j + R_j R_i \rangle$$

Therefore $\sigma_{ij} = \sigma_{ji}$.
The Wigner function in terms of the covariance matrix is written as

$$W(R) = \frac{1}{\pi\sqrt{\text{Det } \sigma}} e^{-\frac{1}{2} R \sigma^{-1} R^{\mathrm{T}}}$$

Now for a two mode state,

$$R = (q_1, p_1, q_2, p_2)^{\mathrm{T}}$$

$$\sigma_{ii} = \frac{1}{2}\langle R_i R_i + R_i R_i \rangle = \langle {R_i}^2 \rangle$$

7.3 Calculation of Covariance Matrix

The relations from Section 7.1 are used to evaluate the expectation value of the aforementioned operators to evaluate the Covariance matrix. So,

$$\begin{aligned}
\sigma_{11} &= \langle {q_1}^2 \rangle \\
&= \langle {(a + a^\dagger)}^2 \rangle \\
&= \langle a^2 + {a^\dagger}^2 + a a^\dagger + a^\dagger a \rangle \\
&= \langle a^2 \rangle + \langle {a^\dagger}^2 \rangle + \langle a a^\dagger \rangle + \langle a^\dagger a \rangle \\
&= \langle a a^\dagger \rangle + \langle a^\dagger a \rangle \\
&= \langle I + a^\dagger a \rangle + \langle a^\dagger a \rangle \\
&= \langle I + 2 a^\dagger a \rangle \\
&= I + 2\langle a^\dagger a \rangle \\
&= 1 + 2\sinh^2 r \\
&= \cosh 2r.
\end{aligned}$$

Similarly,

$$\sigma_{33} = \langle {q_2}^2 \rangle = 1 + 2\langle b^\dagger b \rangle = \cosh 2r.$$

Now,

$$\begin{aligned}
\sigma_{22} &= \langle {p_1}^2 \rangle \\
&= \langle (\frac{a - a^\dagger}{i})^2 \rangle \\
&= -\langle (a - a^\dagger)^2 \rangle \\
&= -\langle a^2 + {a^\dagger}^2 - aa^\dagger - a^\dagger a \rangle \\
&= -\langle a^2 \rangle - \langle {a^\dagger}^2 \rangle + \langle aa^\dagger \rangle + \langle a^\dagger a \rangle \\
&= \langle aa^\dagger \rangle + \langle a^\dagger a \rangle \\
&= \cosh 2r.
\end{aligned}$$

Similarly,

$$\sigma_{44} = \langle {p_2}^2 \rangle = \langle bb^\dagger \rangle + \langle b^\dagger b \rangle = \cosh 2r.$$

Now,

$$\begin{aligned}
\sigma_{12} &= \frac{1}{2}\langle q_1 p_1 + p_1 q_1 \rangle \\
&= \frac{1}{2}\langle q_1 p_1 + p_1 q_1 \rangle \\
&= \frac{1}{2}\langle 2i + 2p_1 q_1 \rangle \text{ (Since } [q_1, p_1] = 2i \text{ from section 3.2)} \\
&= \langle p_1 q_1 \rangle + i \\
&= \langle (\frac{a - a^\dagger}{i})(a + a^\dagger) \rangle + i \\
&= \frac{\langle aa^\dagger - a^\dagger a \rangle}{i} + i \\
&= \frac{1}{i} + i = 0
\end{aligned}$$

Now,

$$\begin{aligned}
\sigma_{13} &= \frac{1}{2}\langle q_1 q_2 + q_2 q_1 \rangle \\
&= \langle q_1 q_1 \rangle \\
&= \langle (a + a^\dagger)(b + b^\dagger) \rangle \\
&= \langle ab + a^\dagger b^\dagger + ab^\dagger + a^\dagger b \rangle \\
&= \langle ab + a^\dagger b^\dagger + ab^\dagger + a^\dagger b \rangle \\
&= \langle ab \rangle + \langle a^\dagger b^\dagger \rangle + \langle ab^\dagger \rangle + \langle a^\dagger b \rangle \\
&= e^{i\phi} \sinh r \cosh r + e^{-i\phi} \sinh r \cosh r \\
&= 2\cos\phi \sinh r \cosh r \\
&= \cos\phi \sinh 2r.
\end{aligned}$$

Now,

$$\begin{aligned}
\sigma_{14} &= \frac{1}{2}\langle q_1p_2 + p_2q_1\rangle \\
&= \langle q_1p_2\rangle \\
&= \langle (a+a^\dagger)\frac{(b-b^\dagger)}{i}\rangle \\
&= -i\langle ab - a^\dagger b^\dagger - ab^\dagger + a^\dagger b\rangle \\
&= -i\langle ab\rangle + i\langle a^\dagger b^\dagger\rangle + i\langle ab^\dagger\rangle - \langle a^\dagger b\rangle \\
&= -i(e^{i\phi}\sinh r\cosh r - e^{-i\phi}\sinh r\cosh r) \\
&= -2i^2\sin\phi\sinh r\cosh r \\
&= \sin\phi\sinh 2r.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sigma_{23} &= \frac{1}{2}\langle p_1p_2 + p_2p_1\rangle \\
&= \langle q_2p_1\rangle \\
&= \langle (b+b^\dagger)\frac{(a-a^\dagger)}{i}\rangle \\
&= -i\langle ba - b^\dagger a^\dagger\rangle \\
&= -i(e^{i\phi}\sinh r\cosh r - e^{-i\phi}\sinh r\cosh r) \\
&= -2i^2\sin\phi\sinh r\cosh r \\
&= \sin\phi\sinh 2r.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sigma_{24} &= \frac{1}{2}\langle p_1q_2 + q_2p_1\rangle \\
&= \langle p_1p_2\rangle \\
&= \langle \frac{(b+b^\dagger)(b-b^\dagger)}{i^2}\rangle \\
&= -\langle ab + a^\dagger b^\dagger\rangle \\
&= -i(e^{i\phi}\sinh r\cosh r - e^{-i\phi}\sinh r\cosh r) \\
&= -2i^2\sin\phi\sinh r\cosh r \\
&= \sin\phi\sinh 2r.
\end{aligned}$$

Again,

$$\sigma_{34} = \frac{1}{2}\langle q_2p_2 + p_2q_2\rangle$$

So just as σ_{12},

$$\sigma_{34} = 0.$$

$$\sigma = \begin{bmatrix} \cosh 2r & 0 & \sinh 2r\cos\phi & \sinh 2r\sin\phi \\ 0 & \cosh 2r & \sinh 2r\sin\phi & -\sinh 2r\cos\phi \\ \sinh 2r\cos\phi & \sinh 2r\sin\phi & \cosh 2r & 0 \\ \sinh 2r\sin\phi & -\sinh 2r\cos\phi & 0 & \cosh 2r \end{bmatrix}$$

If r is real, $\phi = 0$

$$\sigma = \begin{bmatrix} \cosh 2r & 0 & \sinh 2r & 0 \\ 0 & \cosh 2r & 0 & -\sinh 2r \\ \sinh 2r & 0 & \cosh 2r & 0 \\ 0 & -\sinh 2r & 0 & \cosh 2r \end{bmatrix}$$

7.4 The Wigner Function

The inverse and determinant of σ can be calculated by standard procedure and obtained to be

$$\text{Det } \sigma = 1$$

and

$$\sigma^{-1} = \begin{bmatrix} \cosh 2r & 0 & -\sinh 2r & 0 \\ 0 & \cosh 2r & 0 & \sinh 2r \\ -\sinh 2r & 0 & \cosh 2r & 0 \\ 0 & \sinh 2r & 0 & \cosh 2r \end{bmatrix}$$

So,

$$R\sigma^{-1}R^T = \begin{bmatrix} q_1 & p_1 & q_2 & p_2 \end{bmatrix} \begin{bmatrix} \cosh 2r & 0 & -\sinh 2r & 0 \\ 0 & \cosh 2r & 0 & \sinh 2r \\ -\sinh 2r & 0 & \cosh 2r & 0 \\ 0 & \sinh 2r & 0 & \cosh 2r \end{bmatrix} \begin{bmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{bmatrix}$$

$$\begin{aligned} &= q_1(q_1 \cosh 2r - q_2 \sinh 2r) + p_1(p_1 \cosh 2r + p_2 \sinh 2r) \\ &+ q_2(-q_1 \sinh 2r + q_2 \cosh 2r) + p_2(p_1 \sinh 2r + p_2 \cosh 2r) \end{aligned}$$

So,

$$\begin{aligned} R\sigma^{-1}R^T &= \cosh 2r(q_1^2 + p_1^2 + q_2^2 + p_2^2) + \sinh 2r(2p_1p_2 - 2q_1q_2) \\ &= \frac{e^{2r} + e^{-2r}}{2}(q_1^2 + p_1^2 + q_2^2 + p_2^2) + \frac{e^{2r} - e^{-2r}}{2}(2p_1p_2 - 2q_1q_2) \\ &= \frac{e^{2r}}{2}(q_1^2 + p_1^2 + q_2^2 + p_2^2 + 2p_1p_2 - 2q_1q_2) + \frac{e^{-2r}}{2}(q_1^2 + p_1^2 + q_2^2 + p_2^2 - 2p_1p_2 + 2q_1q_2) \\ &= \frac{e^{2r}}{2}((q_1 - q_2)^2 + (p_1 + p_2)^2) + \frac{e^{-2r}}{2}((q_1 + q_2)^2 + (p_1 - p_2)^2) \end{aligned}$$

So,

$$W(q_1, q_2, p_1, p_2) = \frac{1}{\pi^2} \exp\left[-\frac{e^{2r}}{4}((q_1 - q_2)^2 + (p_1 + p_2)^2) - \frac{e^{-2r}}{4}((q_1 + q_2)^2 + (p_1 - p_2)^2)\right]$$

7.5 EPR state and Entanglement

Let us now discuss the case when squeezing operator is very large.

$$\text{As } r \longrightarrow \infty,$$

$$e^{2r} \longrightarrow \infty \text{ and } e^{-2r} \longrightarrow 0.$$

So in limit of infinite squeezing, the Wigner function becomes

$$\begin{aligned} W(q_1, q_2, p_1, p_2) &= \frac{1}{\pi^2} \exp\left[-\frac{e^{2r}}{4}((q_1 - q_2)^2 + (p_1 + p_2)^2)\right] \\ &= \frac{1}{\pi} \exp\left[-\frac{e^{2r}}{4}(q_1 - q_2)^2\right] \frac{1}{\pi} \exp\left[-\frac{e^{2r}}{4}(p_1 + p_2)^2)\right] \end{aligned}$$

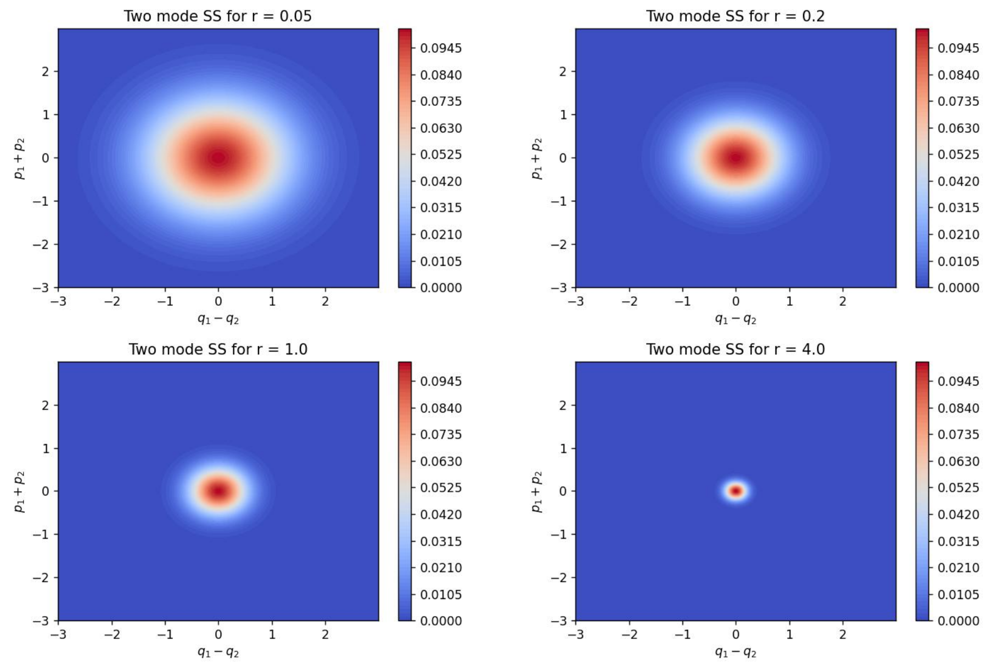

Figure 9 : Density plot of Two mode squeezed state in $q_1 - q_2$, $p_1 - p_2$ space with increasing **r**.

As seen from the expression the correlations contained in the state are between $(q_A - q_B)$ and $(p_A + p_B)$. Clearly the state of the two modes can not be written as a factorized product of the states for the individual modes. They are correlated but not separable. Such a state is called an entangled state. The entanglement in the state depends on the parameter r. According to the properties of the Dirac Delta function, As r tends to ∞ this state can be approximated to the form :

$$W(q_1, q_2, p_1, p_2) \longrightarrow \delta(q_1 - q_2)\delta(p_1 + p_2)$$

Corresponding state will be

$$\Psi_{AB} \longrightarrow \delta(q_A - q_B)\delta(p_A + p_B)$$

This state is known as the ideal Einstein-Podolsky- Rosen (EPR) state which is simultaneous eigenstate of total momentum and relative position of the two subsystems, which thus share infinite entanglement.

The EPR state is unnormalizable and unphysical. However as we have shown an EPR state can be approximated with an arbitrarily high degree of accuracy by two-mode squeezed states with sufficiently large squeezing. Therefore, two-mode squeezed states are of key importance as entangled resources for practical implementations of CV quantum information protocols.

8 Conclusion

The Wigner function of quantum states are a key tool for their description in Phase Space. The other Phase Space distributions i.e. P-representation and Q-functions can also be shown equivalent to Wigner description, however their applications are limited to only few special cases.
Our derivation of Wigner representation of coherent and squeezed states and their photon added versions can be really insightful for practical implications of these states in laboratory while preparing these for CV protocols.
Also these states are Gaussian states and hence minimum uncertainty states which distinguishes them in their applications. Their analysis using covariance matrix formalism can be extended to entanglement measures and finding negativity.
Finally the two mode entanglement of two mode squeezed states and their photon added states are subject of active research in recent years in Quantum Information theory. As we showed that Wigner description is very much insightful in this analysis, the description may be extended to the entanglement in more number of modes.

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A Appendix : Derivation of integral (1)

Given integral :

$$W(\gamma) = \frac{1}{\pi^2} \int_C e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\gamma^* - \alpha^*\gamma} d^2\alpha$$

Now let, $\alpha = x + iy$ and $\gamma = a + ib$,

$$\alpha\gamma^* - \alpha^*\gamma = (x+iy)(a-ib) - (x-iy)(a+ib) = 2i(ay - bx)$$

Substituting, we get

$$W(a,b) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} e^{2i(ay-bx)} dxdy$$

which can be written by changing the order of variables and dropping the pre-term as

$$I = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} e^{2i(ay-bx)} \, dy \right) dx$$

The inner integral is

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} e^{2i(ay-bx)} \, dy$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2-4iay+4ibx))} \, dy$$

To simplify, let's complete the square in the exponent

$$x^2 + y^2 - 4iay + 4ibx = (x - 2ib)^2 + (y - 2ia)^2 + 4a^2 + 4b^2$$

Now we have

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}[(x-2ib)^2+(y-2ia)^2+4a^2+4b^2]} \, dy$$

$$= e^{-2(a^2+b^2)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[(x-2ib)^2+(y-2ia)^2]} \, dy$$

$$= e^{-2(a^2+b^2)} e^{(x-2ib)^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-2ia)^2} \, dy$$

Using the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} \, dz = \sqrt{2\pi}$$

Applying this to our integral

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} e^{2i(ay-bx)} \, dy = \sqrt{2\pi} e^{-2(a^2+b^2)} e^{-\frac{1}{2}(x-2ib)^2}$$

Now integrating with respect to x,

$$I = e^{-2(a^2+b^2)} \int_{-\infty}^{\infty} \sqrt{2\pi} e^{-\frac{1}{2}(x-2ib)^2} \, dx$$

$$I = \sqrt{2\pi} e^{-2(a^2+b^2)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-2ib)^2} \, dx$$

$$I = \sqrt{2\pi} \cdot \sqrt{2\pi} \cdot e^{-2(a^2+b^2)} = 2\pi e^{-2(a^2+b^2)} = 2\pi e^{-2|\gamma|^2}$$

Therefore,

$$\boxed{W(\gamma) = \frac{1}{\pi^2} \int_C e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\gamma^* - \alpha^*\gamma} d^2\alpha = \frac{2\pi}{\pi^2} e^{-2|\gamma|^2} = \frac{2}{\pi} e^{-2|\gamma|^2}}$$

B Appendix : Derivation of integral (2)

Given integral:

$$W(\xi) = \frac{1}{\pi^2} \int_C e^{\alpha\beta^* - \alpha^*\beta} e^{-\frac{|\alpha|^2}{2}} (1 + \alpha\beta^* - \alpha^*\beta - |\alpha|^2 + |\beta|^2) e^{\xi\alpha^* - \alpha\xi^*} d^2\alpha$$

Let's consider

$$\alpha = x + iy,$$
$$\beta = a + ib,$$
$$\xi = c + id$$

Then,

$$\alpha\beta^* - \alpha^*\beta = 2i(ay - bx),$$
$$\alpha^*\xi - \alpha\xi^* = 2i(xd - yc).$$

And, the integral becomes

$$W(c,d) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2i(ay-bx)} e^{2i(xd-yc)} e^{-\frac{1}{2}(x^2+y^2)} (1 + 2iay - 2ibx - x^2 - y^2 + a^2 + b^2) dxdy$$

$$1 + 2iay - 2ibx - x^2 - y^2 + a^2 + b^2 = (a + iy)^2 + (b - ix)^2 + 1,$$

$$\begin{aligned} e^{2i(ay-bx)} e^{2i(xd-yc)} e^{-\frac{1}{2}(x^2+y^2)} &= exp[-\frac{1}{2}(x^2 + 4ibx - 4idx + y^2 - 4iay + 4iyc)] \\ &= e^{-\frac{1}{2}[x+2i(b-d)]^2} e^{-\frac{1}{2}[y+2i(c-a)]^2} e^{-2[(b-d)^2+(c-a)^2]} \end{aligned}$$

Now let, $p = x + 2i(b-d)$ and $q = y + 2i(c-a)$. So,

$$x^2 + 4ibx - 4idx + y^2 - 4iay + 4iyc = (2c - a + qi)^2 + (2d - b - ip)^2 + 1$$

So,

$$W(p,q) = \frac{1}{\pi^2} e^{-|\beta-\xi|^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(p^2+q^2)} [(2c - a + qi)^2 + (2d - b - ip)^2 + 1] dpdq$$

Only the even terms will survive and

$$W(p,q) = \frac{1}{\pi^2} e^{-|\beta-\xi|^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(p^2+q^2)}[(2c-a)^2 - q^2 + (2d-b)^2 - p^2 + 1]dpdq$$

Using standard Gaussian integrals this is found out to be

$$\begin{aligned} W(\xi) &= \frac{2\pi}{\pi^2} e^{-|\beta-\xi|^2}[(2c-a)^2 + (2d-b)^2 - 1] \\ &= \frac{2}{\pi} e^{-|\beta-\xi|^2}(4|\xi|^2 + |\beta|^2 - 2\beta\xi^* - 2\xi\beta^* - 1) \\ &= \frac{2}{\pi} e^{-|\beta-\xi|^2}(|2\xi-\beta|^2 - 1) \end{aligned}$$

$$\boxed{W(\xi) = \frac{2}{\pi}(|2\xi-\beta|^2 - 1)e^{-2|\xi-\beta|^2}}$$

C Appendix : Derivation of integral (3)

Given integral:

$$\begin{aligned} I &= \int_C e^{-\frac{|\gamma|^2}{2}}(1-|\gamma|^2)e^{\chi\gamma^* - \chi^*\gamma} d^2\gamma \\ &= \int_C e^{-\frac{|\gamma|^2}{2}} e^{\chi\gamma^* - \chi^*\gamma} d^2\gamma - \int_C |\gamma|^2 e^{-\frac{|\gamma|^2}{2}} e^{\chi\gamma^* - \chi^*\gamma} d^2\gamma \end{aligned}$$

The value of first integral will be $2\pi e^{-2|\chi|^2}$. For the second part let us consider,

$$\gamma = x + iy,$$

$$\chi = a + ib.$$

So,

$$\begin{aligned} \int_C |\gamma|^2 e^{-\frac{|\gamma|^2}{2}} e^{\chi\gamma^* - \chi^*\gamma} d^2\gamma &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x^2+y^2)e^{-\frac{1}{2}(x^2+y^2)} e^{2i(ay-bx)} dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x^2+y^2)e^{-\frac{1}{2}(x^2+2x\cdot 2ib+y^2-2y\cdot 2ia)} dxdy \end{aligned}$$

Making full squares in the exponentials and then using the properties of Gaussian integrals, we get

$$\begin{aligned} \int_C |\gamma|^2 e^{-\frac{|\gamma|^2}{2}} e^{\chi\gamma^* - \chi^*\gamma} d^2\gamma &= 2\pi e^{-2(a^2+b^2)}(2 - 4a^2 - 4b^2) \\ &= 2\pi e^{-2|\chi|^2}(2 - 4|\chi|^2) \end{aligned}$$

Then,

$$\begin{aligned} I &= 2\pi e^{-2|\chi|^2} - 2\pi e^{-2|\chi|^2}(2 - 4|\chi|^2) \\ &= 2\pi(4|\chi|^2 - 1)e^{-2|\chi|^2} \end{aligned}$$

$$\boxed{I = 2\pi(4|\chi|^2 - 1)e^{-2|\chi|^2}}$$