2 The governing systems of equations

2.1 The basic equations

This chapter describes the governing systems of equations that can serve as the basis for atmospheric models used for both operational and research applications. Even though most models employ similar sets of equations, the exact formulation can affect the accuracy of model forecasts and simulations,[1] and can even preclude the existence in the model solution of certain types of atmospheric waves. Because these equations cannot be solved analytically, they must be converted to a form that can be. The numerical methods typically used to accomplish this are described in Chapter 3.

The equations that serve as the basis for most numerical weather and climate prediction models are described in all first-year atmospheric-dynamics courses. The momentum equations for a spherical Earth (Eqs. 2.1–2.3) represent Newton's second law of motion, which states that the rate of change of momentum of a body is proportional to the resultant force acting on the body, and is in the same direction as the force. The thermodynamic energy equation (Eq. 2.4) accounts for various effects, both adiabatic and diabatic, on temperature. The continuity equation for total mass (Eq. 2.5) states that mass is neither gained nor destroyed, and Eq. 2.6 is analogous, but applies only to water vapor. The ideal gas law (Eq. 2.7) relates temperature, pressure, and density. The variables have their standard meteorological meaning. The independent variables u, v, and w are the Cartesian velocity components, p is pressure, ρ is density, T is temperature, q_v is specific humidity, Ω is the rotational frequency of Earth, ϕ is latitude, a is the radius of Earth, γ is the lapse rate of temperature, γ_d is the dry adiabatic lapse rate, c_p is the specific heat of air at constant pressure, g is the acceleration of gravity, H represents a gain or loss of heat, Q_v is the gain or loss of water vapor through phase changes, and Fr is a generic friction term in each coordinate direction.

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} + \frac{uv\tan\phi}{a} - \frac{uw}{a} - \frac{1}{\rho}\frac{\partial p}{\partial x} - 2\Omega(w\cos\phi - v\sin\phi) + Fr_x \qquad (2.1)$$

$$\frac{\partial v}{\partial t} = -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - w\frac{\partial v}{\partial z} - \frac{u^2\tan\phi}{a} - \frac{uw}{a} - \frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi + Fr_y \qquad (2.2)$$

[1] In this text, the noun *simulation* refers to a model solution that is obtained for any purpose other than estimating the future state of the atmosphere (for example, for research). An estimate of the future state of the atmosphere is referred to as a *forecast*.

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$$\frac{\partial w}{\partial t} = -u\frac{\partial w}{\partial x} - v\frac{\partial w}{\partial y} - w\frac{\partial w}{\partial z} - \frac{u^2+v^2}{a} - \frac{1}{\rho}\frac{\partial p}{\partial z} + 2\Omega u\cos\phi - g + Fr_z \tag{2.3}$$

$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} + (\gamma - \gamma_d)w + \frac{1}{c_p}\frac{dH}{dt} \tag{2.4}$$

$$\frac{\partial \rho}{\partial t} = -u\frac{\partial \rho}{\partial x} - v\frac{\partial \rho}{\partial y} - w\frac{\partial \rho}{\partial z} - \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \tag{2.5}$$

$$\frac{\partial q_v}{\partial t} = -u\frac{\partial q_v}{\partial x} - v\frac{\partial q_v}{\partial y} - w\frac{\partial q_v}{\partial z} + Q_v \tag{2.6}$$

$$P = \rho R T \tag{2.7}$$

A complete model will also have continuity equations for cloud water, cloud ice, and the different types of precipitation (see Chapter 4). See Dutton (1976) and Holton (2004) for discussions of this set of prognostic,[2] coupled, nonlinear, nonhomogeneous partial differential equations. The equations are called the *primitive equations*, and models that are based on these equations are called *primitive-equation models*. This terminology is used to distinguish these models from ones that are based on differentiated versions of the equations, such as the vorticity equation. Virtually all contemporary research and operational models are based on some version of these primitive equations. Note that the terms in the equations related to diabatic effects (H), friction (Fr), and gains and losses of water through phase changes (Q_v) must be defined within the model. This particular example of the primitive equations has pressure as the vertical coordinate, but other options will be discussed in the next chapter.

2.2 Reynolds' equations: separating unresolved turbulence effects

The above equations apply to all scales of motion, even waves and turbulence that are too small to be represented by models designed for weather processes. Because this turbulence cannot be resolved explicitly in such models, the equations must be revised so that they apply only to larger nonturbulent motions. This can be accomplished by splitting all the dependent variables into mean and turbulent parts, or, analogously, spatially resolved and unresolved components, respectively. The mean is defined as an average over a grid cell, as described by Pielke (2002a). For example:

$$u = \bar{u} + u',$$
$$T = \bar{T} + T', \text{ and}$$
$$p = \bar{p} + p'.$$

[2] The word *prognostic* implies that an equation is predictive, in contrast to a *diagnostic* equation, which has no time derivative and simply relates the state of variables at the same time. For example, the ideal gas law is diagnostic.

These expressions are substituted into Eqs. 2.1–2.7, producing expansions such as the following one for the first term on the right side of Eq. 2.1:

$$u\frac{\partial u}{\partial x} = (\bar{u}+u')\frac{\partial}{\partial x}(\bar{u}+u') = \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{u}\frac{\partial u'}{\partial x} + u'\frac{\partial \bar{u}}{\partial x} + u'\frac{\partial u'}{\partial x}. \tag{2.8}$$

Because we want the equations to pertain to the mean motion, that is, the nonturbulent weather scales, we apply an averaging operator to all the terms. For the above term, we have

$$\overline{u\frac{\partial u}{\partial x}} = \overline{\bar{u}\frac{\partial \bar{u}}{\partial x}} + \overline{\bar{u}\frac{\partial u'}{\partial x}} + \overline{u'\frac{\partial \bar{u}}{\partial x}} + \overline{u'\frac{\partial u'}{\partial x}}. \tag{2.9}$$

Note that the last term on the right is a *covariance term*. Its value depends on whether the first quantity in the product covaries with the second. For example, if positive values of the first part tend to be paired with negative values of the second, the covariance, and the term, would be negative. If the two parts of the product are not physically correlated, the mean has a value of zero. We then simplify the equations using Reynolds' postulates (Reynolds 1895, Bernstein 1966). For variables a and b,

$$\overline{a'} = 0,$$

$$\bar{\bar{a}} = \bar{a} \text{ and } \overline{\bar{a}b} = \bar{\bar{a}}\bar{b} = \bar{a}\bar{b}\text{, and}$$

$$\overline{\bar{a}b'} = \bar{\bar{a}}\overline{b'} = \bar{a}\overline{b'} = 0\,.$$

Given these postulates, the terms in Eq. 2.9 become

$$\overline{u\frac{\partial u}{\partial x}} = \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{u}\cancelto{0}{\frac{\partial \overline{u'}}{\partial x}} + \cancelto{0}{\overline{u'}}\frac{\partial \bar{u}}{\partial x} + \overline{u'\frac{\partial u'}{\partial x}} = \bar{u}\frac{\partial \bar{u}}{\partial x} + \overline{u'\frac{\partial u'}{\partial x}}. \tag{2.10}$$

Before we show how to apply these methods to all the terms in Eqs. 2.1–2.7, let us rewrite Eq. 2.1 with a typical representation for the friction terms, Fr_x, without the Earth-curvature terms, and with only the dominant Coriolis term. In these equations, which explicitly represent turbulent motion, subgrid friction results only from viscous forces, which are a consequence of molecular motion.

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial x} + fv + \frac{1}{\rho}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right). \tag{2.11}$$

Here, τ_{zx} is the force per unit area, or the momentum or shearing stress, exerted in the x direction by the fluid on one side of a constant-z plane with the fluid on the other side of the z plane, and τ_{xx} and τ_{yx} are the forces in the x direction across the other two coordinate planes. In hypothetical, inviscid fluids, there would be no "communication" between the flow on either side of a plane. But, in real fluids, the molecular motion, or molecular

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diffusion, across each of the coordinate surfaces will allow for the exchange of properties. A typical representation for the stress is

$$\tau_{zx} = \mu \frac{\partial u}{\partial z},$$

where μ is dynamic viscosity coefficient. This is called Newtonian friction, or Newton's law for the stress. Referring to the two (infinitesimally shallow) layers of fluid on either side of the z plane, if there is no shear in the fluid, viscosity produces no stress, or force per unit area, of one layer on the other. Substituting these expressions for the Newtonian friction into the terms for Fr_x in Eq. 2.11, we have

$$\frac{\partial u}{\partial t} \propto \frac{1}{\rho}\left(\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial^2 u}{\partial z^2}\right) = \frac{\mu}{\rho}\nabla^2 u. \tag{2.12}$$

Now apply the averaging process to all the terms in Eq. 2.11. In particular, we represent each dependent variable by the sum of a resolved mean and an unresolved turbulent component, and then apply the averaging operator. Using Reynolds' postulates, and the assumption that $\rho' \ll \bar{\rho}$, we obtain

$$\frac{\partial \bar{u}}{\partial t} = -\bar{u}\frac{\partial \bar{u}}{\partial x} - \bar{v}\frac{\partial \bar{u}}{\partial y} - \bar{w}\frac{\partial \bar{u}}{\partial z} - \frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x} + f\bar{v} - \overline{u'\frac{\partial u'}{\partial x}} - \overline{v'\frac{\partial u'}{\partial y}} - \overline{w'\frac{\partial u'}{\partial z}} + \frac{1}{\bar{\rho}}\left(\frac{\partial \bar{\tau}_{xx}}{\partial x} + \frac{\partial \bar{\tau}_{yx}}{\partial y} + \frac{\partial \bar{\tau}_{zx}}{\partial z}\right). \tag{2.13}$$

Stull (1988) uses a scale analysis to show that, for turbulence scales of motion, the following continuity equation applies:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0. \tag{2.14}$$

Multiply this by u', average it, and add it to Eq. 2.13 to put the turbulent advection terms into *flux form*:

$$\frac{\partial \bar{u}}{\partial t} = -\bar{u}\frac{\partial \bar{u}}{\partial x} - \bar{v}\frac{\partial \bar{u}}{\partial y} - \bar{w}\frac{\partial \bar{u}}{\partial z} - \frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x} + f\bar{v} - \frac{\partial \overline{u'u'}}{\partial x} - \frac{\partial \overline{u'v'}}{\partial y} - \frac{\partial \overline{u'w'}}{\partial z} + \frac{1}{\bar{\rho}}\left(\frac{\partial \bar{\tau}_{xx}}{\partial x} + \frac{\partial \bar{\tau}_{yx}}{\partial y} + \frac{\partial \bar{\tau}_{zx}}{\partial z}\right). \tag{2.15}$$

By analogy with the molecular viscosity-related stresses, we define turbulent stresses (also, eddy stresses or Reynolds' stresses) as follows:

$$T_{xx} = -\bar{\rho}\overline{u'u'},$$
$$T_{yx} = -\bar{\rho}\overline{u'v'},$$
$$T_{zx} = -\bar{\rho}\ \overline{u'w'}.$$

Substituting these expressions into Eq. 2.15, and assuming that the spatial derivatives of the density are much smaller than those of the covariances, we have

$$\frac{\partial \bar{u}}{\partial t} = -\bar{u}\frac{\partial \bar{u}}{\partial x} - \bar{v}\frac{\partial \bar{u}}{\partial y} - \bar{w}\frac{\partial \bar{u}}{\partial z} - \frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x} + f\bar{v}$$
$$+ \frac{1}{\bar{\rho}}\left(\frac{\partial}{\partial x}(\tau_{xx} + T_{xx}) + \frac{\partial}{\partial y}(\tau_{yx} + T_{yx}) + \frac{\partial}{\partial z}(\tau_{zx} + T_{zx})\right). \tag{2.16}$$

This equation is the same as Eq. 2.11, except for the turbulent-stress terms and the mean-value symbols. The mean-value symbols are rarely used with the primitive equations, but it is still understood that the dependent variables represent only nonturbulent motions. And, the turbulent stresses are much larger than the viscous stresses, so the latter terms are usually not included. The turbulent-stress terms are sometimes represented symbolically as "*F*", referring to friction. The representation of the turbulent stresses in terms of variables predicted by the model is the subject of turbulence parameterizations for the boundary layer, or for above the boundary layer, described in Chapter 4.

2.3 Approximations to the equations

There are a few reasons why we might desire to use approximate sets of equations as the basis for a model.

- Some approximate sets are more efficient to solve numerically than the complete equations. For example, the hydrostatic, *Boussinesq*, and *anelastic approximations* described below do not permit sound waves in the solutions, which, for reasons that will be explained in the next chapter, means that less computing resources are required to produce a simulation or forecast of a given length.
- The complete equations describe a physical system that is so complex that it is challenging to use them in a model for research, to better understand cause and effect relationships in the atmosphere. Thus, sometimes specific terms and equations (and the associated processes) are removed from the set of equations. For example, removing equations for water in all its phases, and the thermodynamic effect of phase changes, allows the study of processes in a simpler setting.
- Very simple forms of the equations are more amenable for pedagogical applications and for initial testing of new numerical algorithms. For example, the shallow-fluid equations, described below, are used as the basis for "toy models" in NWP classes (and in this text). But, they contain enough of the dynamics of the full set of equations that they can be profitably used to test new differencing schemes, which can later be evaluated in complete models.

The approximations described in the following subsections are commonly used in research and operational models.

2.3.1 Hydrostatic approximation

The existence of relatively fast-propagating sound waves in a model solution means, as will be explained in the next chapter, that short time steps are required in order for the model's numerical solution to remain stable. The consequence of the short time step is that many more will be required in a model integration of a specific duration, and more computing resources will be required. Because sound waves are generally of no meteorological importance, it is desirable to use a form of the equations that does not admit them. One approach is to employ the hydrostatic approximation, wherein the complete third equation of motion (Eq. 2.3) is replaced by one containing only the gravity and vertical-pressure-gradient terms. That is

$$\frac{\partial p}{\partial z} = -\rho g .$$

This implies that the density is tied to the vertical pressure gradient. Because the propagation of sound waves requires that the density adjust to the longitudinal compression and expansion within the waves, sound waves are not possible in a hydrostatic atmosphere. For the hydrostatic assumption to be valid, the sum of all the terms eliminated in the complete equation must be, say, at least an order-of-magnitude smaller than the terms retained. Stated another way

$$\left|\frac{dw}{dt}\right| \ll g .$$

A scale analysis of the third equation of motion (e.g., Dutton 1976, Holton 2004) shows that the hydrostatic assumption is valid for synoptic-scale motions, but becomes less so for length scales of less than about 10 km on the mesoscale and convective-scale. Thus, coarser-resolution global models will tend to be based on the hydrostatic equations, while models of mesoscale processes will not. It will be shown in the next chapter that there are other approaches for dealing with the computational effects of fast waves on the model grid.

2.3.2 Boussinesq and anelastic approximations

As with the hydrostatic assumption, the Boussinesq and anelastic approximations are part of a family of approximations that directly filter sound waves from the equations by decoupling the pressure and density perturbations. However, their use is not limited to modeling larger horizontal length scales, as is the case with the hydrostatic approximation. Indeed, these approximations are widely used in models of mesoscale or cloud-scale processes. The Boussinesq approximation (Boussinesq 1903) is obtained by substituting the following for Eq. 2.5, the complete continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 .$$

This amounts to substituting volume conservation for mass conservation. For the anelastic approximation (Ogura and Phillips 1962, Lipps and Hemler 1982),

$$\frac{\partial}{\partial x}\bar{\rho}u + \frac{\partial}{\partial y}\bar{\rho}v + \frac{\partial}{\partial z}\bar{\rho}w = 0$$

is substituted for the complete continuity equation, where $\bar{\rho} = \bar{\rho}(z)$ is a steady reference-state density. In addition, both approximations involve simplifications in the momentum equations (see Durran 1999, pp. 20–26). Another type of approximation in this class is the pseudo-incompressible approximation described by Durran (1989).

2.3.3 Shallow-fluid equations

The *shallow-fluid equations*, sometimes called the shallow-water equations, can serve as the basis for a simple model that can be used to illustrate and evaluate the properties of numerical schemes. Inertia–gravity, advective, and Rossby waves can be represented. Not only is such a model useful for gaining experience with numerical methods, the fact that the equations represent much of the horizontal dynamics of full baroclinic models makes it a useful tool for testing numerical methods in a simple framework. For example, Williamson *et al.* (1992) used a shallow-fluid model applied to the sphere to test numerical methods that were proposed for climate modeling.

The name "shallow fluid" refers to the fact that the wavelengths simulated must be long relative to the depth of the fluid. There are various forms of this set of equations (Nadiga *et al.* 1996), but here the fluid is assumed to be autobarotropic (barotropic by definition, not by virtue of the prevailing atmospheric conditions), homogeneous, incompressible, hydrostatic, and inviscid. The homogeneity condition means that the density does not vary in space, and incompressibility means that density does not change in time following a parcel. The equations from which we begin the derivation are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0\,, \tag{2.17}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0\,, \tag{2.18}$$

$$\frac{\partial p}{\partial z} = -\rho g\,, \text{ and} \tag{2.19}$$

$$\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0\,. \tag{2.20}$$

Now, incompressibility and homogeneity imply

$$\frac{d\rho}{dt} = 0\,, \tag{2.21}$$

$$\rho = \rho_0\,,\ \text{for } \rho_0 \text{ a constant, and} \tag{2.22}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0\,. \tag{2.23}$$

The hydrostatic equation can thus be written

$$\frac{\partial p}{\partial z} = -\rho_0 g\,. \tag{2.24}$$

Differentiating Eq. 2.24 with respect to x, and using the fact that the right side is a constant, yields

$$\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial p}{\partial x}\right) = 0\,, \tag{2.25}$$

which means that there is no horizontal variation of the vertical pressure gradient or vertical variation of the horizontal pressure gradient (the definition of barotropy). Because the pressure-gradient force generates the wind, and the resulting Coriolis force, all forces are invariant with height. Integrating Eq. 2.24 over the depth of the fluid,

$$\int_{z(P_S)}^{z(P_T)} \frac{\partial p}{\partial z}dz = -\rho_0 g \int_{z(P_S)}^{z(P_T)} dz\,, \tag{2.26}$$

where P_T and P_S represent the pressure at the top and bottom boundaries of the fluid, respectively, yields

$$P_S - P_T = g\rho_0 h, \tag{2.27}$$

for h equal to the depth of the fluid. If $P_T = 0$, or $P_T \ll P_S$,

$$\frac{P_S}{\rho_0} = gh \ \text{ and} \tag{2.28}$$

$$\frac{1}{\rho_0}\frac{\partial P_S}{\partial x} = g\frac{\partial h}{\partial x}. \tag{2.29}$$

This statement that the horizontal pressure gradient at the bottom of the fluid is proportional to the gradient in the depth of the fluid provides a new form of the pressure-gradient term in Eqs. 2.17 and 2.18. The incompressible continuity equation (Eq. 2.23) can also be rewritten by integrating it with respect to z:

$$\int_0^z \frac{\partial w}{\partial z}dz = w_z - w_s = -\int_0^z\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dz\,. \tag{2.30}$$

If u and v are initially not a function of z, they will remain so because the pressure gradient is not a function of z. And, because u and v are not a function of z, neither are their derivatives, so that

$$w_h - w_S = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)h \tag{2.31}$$

for $z = h$. For a horizontal lower boundary, the kinematic boundary condition $w_S = 0$ prevails. Recognizing that

$$w_h = \frac{dh}{dt} \tag{2.32}$$

leads to a new continuity equation. There are now three equations in three variables, u, v, and h.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + g\frac{\partial h}{\partial x} = 0\,, \tag{2.33}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + g\frac{\partial h}{\partial y} = 0\,, \tag{2.34}$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0\,. \tag{2.35}$$

For simplicity, a one-dimensional version of this system of equations is frequently used. In order to permit a mean u component on which perturbations occur, a constant pressure gradient of the desired magnitude is specified in the y direction. The one-dimensional equations are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - fv + g\frac{\partial h}{\partial x} = 0\,, \tag{2.36}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + fu + g\frac{\partial H}{\partial y} = 0\,, \tag{2.37}$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial H}{\partial y} + h\frac{\partial u}{\partial x} = 0\,, \text{ where} \tag{2.38}$$

$$\frac{\partial H}{\partial y} = -\frac{f}{g}\overline{U} \tag{2.39}$$

and $\overline{U}$ is the specified, constant mean geostrophic speed on which the u perturbation is superimposed. Obviously there are limitations to the degree to which this system of equations can represent the real atmosphere, but one step toward more realism is to define the fluid depth to be consistent with the layer being represented, such as the boundary layer or the troposphere. The depth of the total atmosphere can be represented by the scale height

$$H = \frac{RT_0}{g}, \tag{2.40}$$

where T_0 is the surface temperature and H is about 8 km. If the model atmosphere is to represent the troposphere, it can be assumed that the active fluid layer of depth h is surmounted by an inert layer (Fig. 2.1) that represents the stratosphere. This exerts a buoyant

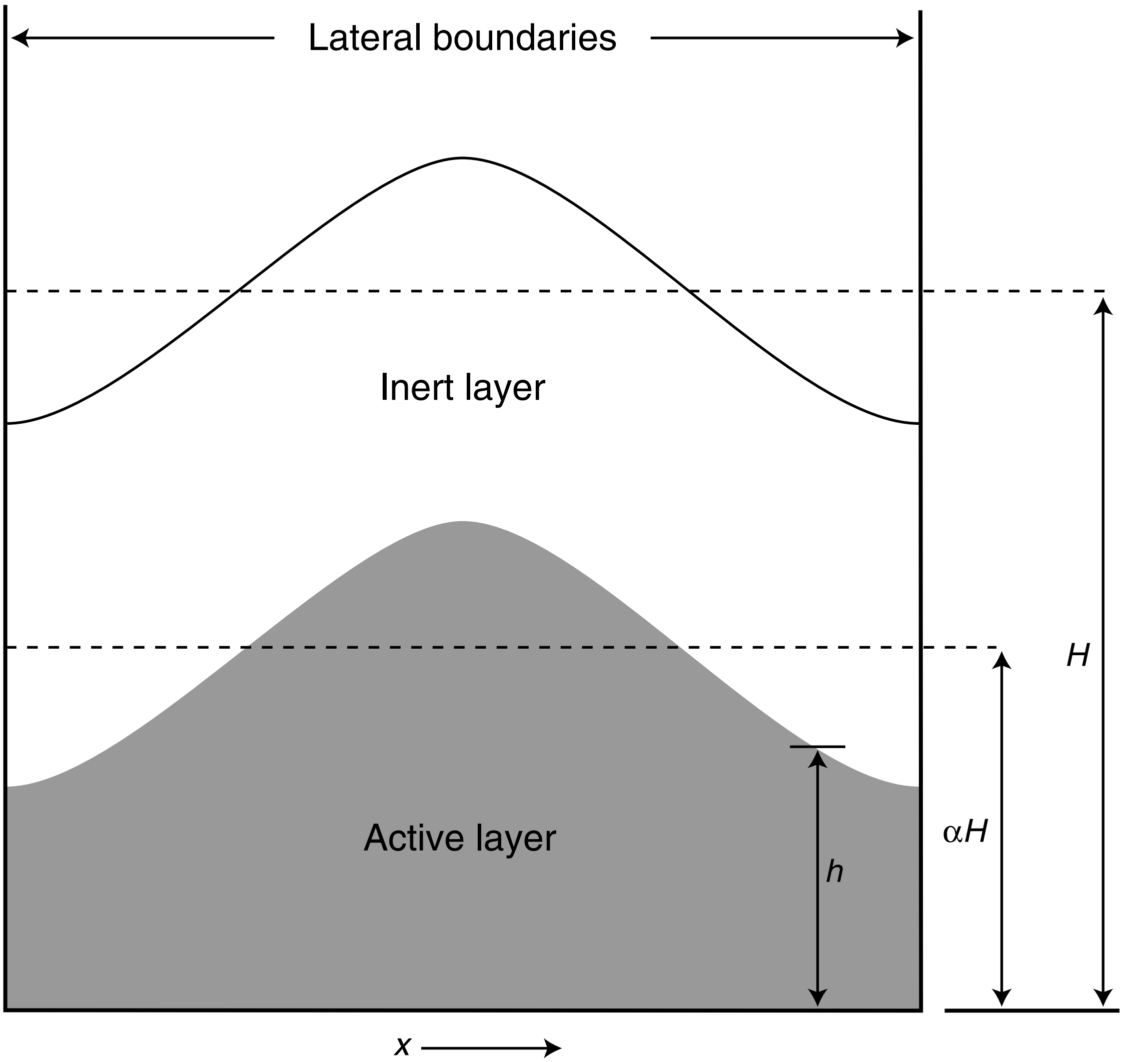

Fig. 2.1 Schematic showing the vertical structure of a shallow fluid model, for a situation where a wave ridge is centered in the computational domain. The lower shaded layer represents the active fluid for which the depth (h) and wind components are simulated. The depth, H, is the scale height of the atmosphere, and α is the factor by which the depth is reduced to account for the buoyancy of an inert layer above.

force on the lower layer that can be represented in the model by a reduced gravity. But this would impact the geostrophic relationship, so a better approach is to proportionately reduce the depth of the active layer. This is justified by the fact that, in the linear solution for the phase speed of external gravity waves, the acceleration of gravity is multiplied by the mean depth of the fluid. Application of either method would have the same effect of decreasing the phase speed of external gravity waves to one that is more characteristic of the internal waves at the layer interface. It can be shown that the gravity or layer depth should be reduced by a factor $\alpha = (\theta_T - \theta_B)/\theta_B$, which is based on the mean potential temperatures of the top and bottom layers. For the example where the lower layer represents the troposphere, this ratio is ~0.25 and the layer mean depth would be defined as 2 km.

When the above nonlinear shallow-fluid equations are used as the basis for a model, an explicit numerical diffusion term will need to be added to each equation to suppress the short wavelengths that will grow through the aliasing process, which will be described in the next chapter. Additional information on the shallow-fluid equations, and their numerical solution, may be found in Kinnmark (1985), Pedlosky (1987), Durran (1999), and McWilliams (2006).

PROBLEMS AND EXERCISES

1. Derive Reynolds' equations for Eqs. 2.2–2.7.
2. Reproduce the development of Reynolds' equations using tensor notation, and note the relative simplicity compared to the process in Section 2.2.