6

Forecast value: prototype decision-making models

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1. Introduction

In this chapter, theoretical relationships between the scientific quality and economic value of imperfect weather forecasts are considered. Prototype decision-making models are treated that, while relatively simple in structure, still capture some of the essential features (e.g., the "dynamics") of real-world situations. The emphasis is on analytical results that can be obtained concerning the structure of the optimal policy and the shape of the "quality/value curve." It is anticipated that knowledge of such theoretical properties for prototype models will provide insight into analogous properties for real-world decision-making problems that are inherently much more complex.

The prospects for increases in the quality of weather forecasts in the future have been discussed in Chapter 1 of this volume, thereby providing a partial justification for the hypothetical increases in forecast quality that are assumed in the present chapter. Various aspects of forecast quality (e.g., bias and accuracy) have been described in Chapter 2. A simplified form of weather information is treated here in which a one-dimensional measure of quality can be defined that is synonymous with the concept of sufficiency (also described in Chapter 2). The Bayesian decision-theoretic approach to assessing the economic value of imperfect information is adopted, an approach that has been introduced in Chapter 3. Analytical results obtained for prototype decision-making models are compared with those for real-world case studies that have also employed this prescriptive approach to real-world decision-making problems (reviewed in Chapter 4). Reasons for the underutilization (or nonoptimal use) of existing weather forecasts have been outlined in Chapter 5, and some explanations for such behavior based on theoretical considerations are presented here.

In Section 2, first the concepts of scientific quality, sufficiency, economic value, and optimal policy in a general Bayesian decision-analytic setting are briefly reviewed. The few relationships among these concepts that always hold are identified. Then the prototype form of weather information on which the present chapter focuses is introduced. Only two states of weather are allowed, and only two conditional probabilities are required to characterize fully the quality of the imperfect weather forecasting system. Several forms of prototype decision-making models are described in Section 3, including the static cost–loss ratio (or "umbrella") problem and generalizations to dynamic situations, that treat both finite-horizon, undiscounted and infinite-horizon, discounted problems. For each model, the structure of the optimal policy and the shape of the quality/value curve can be expressed in closed form. In an attempt to broaden the scope, extensions of these prototype decision-making models are considered in Section 4. These extensions primarily involve allowing for more complex, and consequently more realistic, forms of information about weather. Finally, Section 5 deals with the implications of these results, including their relevance for real-world decision-making problems. Some technical results are relegated to an appendix. We note that Ehrendorfer and Murphy (1992b) have also reviewed quality/value relationships for imperfect weather forecasting systems.

2. Concepts

2.1. General setting

As defined in Chapter 2 of this volume, the scientific quality of a forecasting system is the totality of the statistical characteristics embodied in the joint distribution of forecasts and observations (equivalent definitions can be formulated in terms of conditional and marginal distributions). In general, more than one number is required to describe forecast quality completely; that is, forecast quality is inherently a multidimensional concept. Thus, scalar measures of the correspondence between forecasts and observations, such as the mean square error or a correlation coefficient, represent measures of particular aspects of quality (i.e., accuracy or linear association). These measures are usually based solely on the forecasts and observations, and they do not make explicit use of any of the economic parameters associated with a specific decision-making problem. Typically, such measures can be scaled so that they range from zero for a forecasting system with no predictive ability to one for perfect information, with the score for imperfect forecasts falling between these two limits. The relative merits of common measures of various aspects of forecast quality have been discussed in Chapter 2.

The Bayesian decision-analytic concept of the *economic value* of imperfect weather forecasts is predicated upon the existence of an individual decision maker who considers making use of the forecasts. For a particular decision-making problem, this individual is assumed to select the action that maximizes his (or her) expected utility. Here the expectation is taken with respect to the information available about the future weather. A rule that specifies which action the decision maker should take as a function of the information received is termed the *optimal policy.* This information could be in the form of a genuine forecast. On the other hand, even if no such forecast were available, it is reasonable to assume that the decision maker would still be aware of the long-run statistical behavior of the weather variable (termed "climatological information"). Economic value is thus measured by comparing the expected utilities with and without the forecast. This concept of information value has been described in more detail in Chapter 3 of this volume.

Measures of aspects of quality do not necessarily have any straightforward connection to the economic value of weather forecasts. In fact, it is possible for one forecasting system to have higher accuracy (according to some reasonable one-dimensional measure) than another system, and yet the inferior system (in terms of accuracy) still produces forecasts of higher economic value for certain decision-making problems or decision makers (e.g., Murphy and Ehrendorfer, 1987). Another concept, namely *sufficiency*, needs to be introduced in order to produce any consistent connection between the scientific quality of forecasts and their economic value in general. Essentially, one forecasting system is sufficient for a second one, if the forecast information generated by the second can be obtained by a stochastic transformation of the information generated by the first system. Here the forecast information is characterized by the conditional distribution of forecasts given the weather observation, and the stochastic transformation involves a conditioning (or "nesting") operation, termed the "sufficiency relation." The precise definition of sufficiency has been given in Chapter 2 of this volume (also see Ehrendorfer and Murphy, 1988, 1992a; Krzysztofowicz and Long, 1991). Here the important point to note is that if one forecasting system is sufficient for another, then it is guaranteed to result in at least as much economic value as the other system, no matter what the decision maker's loss or payoff function (Blackwell, 1953).

For some special forms of weather information, it is possible to define a one-dimensional measure that satisfies the sufficiency relation, and thereby measures forecast quality in its entirety. In this case, for any given decision-making problem, economic value must be a nondecreasing function of quality. However, nothing can still be said in general about the rate at which such a quality/value curve increases (e.g., is it convex or concave?). Moreover, the sufficiency concept provides just about the only instance in which a change in one of the attributes of a decision-making problem results in a monotonic relationship with economic value (Hilton, 1981).

2.2. Prototype form of weather information

A simplified form of weather information is treated in the remainder of the present chapter. The weather variable, a random variable, denoted by Θ, has only two possible states:

(i) *adverse weather* $(\Theta = 1)$;
(ii) *no adverse weather* $(\Theta = 0)$.

Climatological information consists of a single probability of adverse weather

$$p_\Theta = \Pr\{\Theta = 1\}, \tag{6.1}$$

perhaps derived from historical weather records. From a Bayesian perspective, the parameter p_Θ can be viewed as the "prior probability" of adverse weather.

Imperfect information about Θ consists of a random variable Z, indicating a forecast of adverse weather $(Z = 1)$ or of no adverse weather $(Z = 0)$. This simple forecasting system is completely characterized by the two conditional probabilities of adverse weather

$$p_1 = \Pr\{\Theta = 1|Z = 1\}, \quad p_0 = \Pr\{\Theta = 1|Z = 0\}. \tag{6.2}$$

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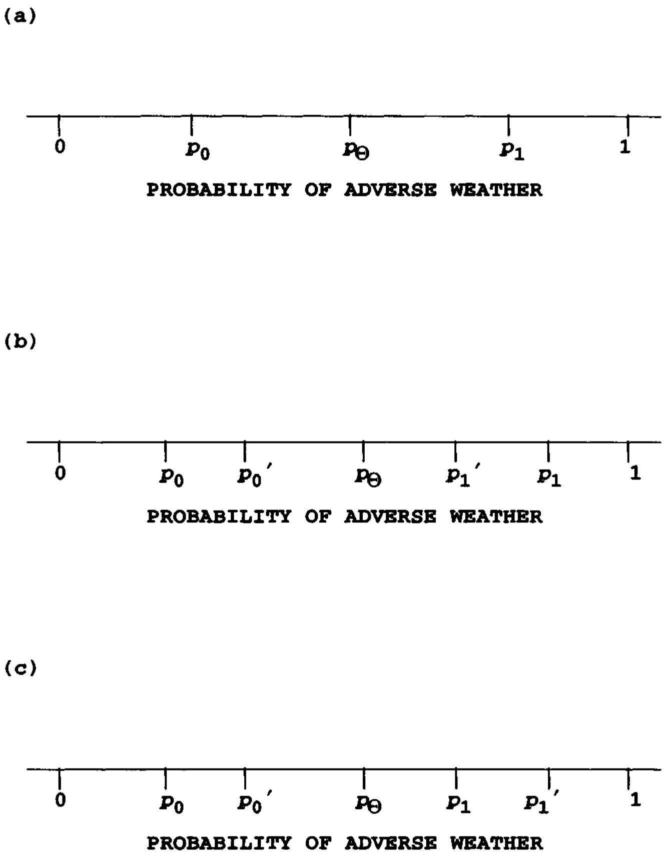
Without loss in generality, it is assumed that these conditional probabilities satisfy the ordering $0 \leq p_0 \leq p_\Theta \leq p_1 \leq 1$. This parameter configuration is shown in Figure 6.1a. The limiting circumstance of no forecasting ability (i.e., climatological information) corresponds to $p_0 = p_1 = p_\Theta$, whereas the other extreme of perfect information corresponds to $p_0 = 0$ and $p_1 = 1$. This form of imperfect forecasts has been termed "categorical," when viewed from an *ex post* perspective, whereas it is a "primitive" case of probabilistic forecasts in which only two possible probabilities of adverse weather are ever issued when viewed from an *ex ante* perspective (see Chapter 3 of this volume for an explanation of the terms *ex post* and *ex ante*). In Bayesian terminology, the parameters, p_0 and p_1, would be termed "posterior probabilities" of adverse weather.

For this prototype form of weather information, the sufficiency relation reduces to a very simple condition on the two conditional probabilities of the forecasting system. Consider another forecasting system with possibly different conditional probabilities, denoted by p_0' and p_1'. Then the original forecasting system is *sufficient* for the other system if and only if

$$p_0 \leq p_0' \leq p_\Theta \leq p_1' \leq p_1. \tag{6.3}$$

This condition (6.3) is illustrated in Figure 6.1b, and has appeared either in this form or in an equivalent form in DeGroot (1970, p. 444), Ehrendorfer and Murphy (1988), and Krzysztofowicz and Long (1990). In order to guarantee that one forecasting system produces at least as great an economic value for all decision makers as the other no matter what the decision makers' loss or payoff functions, both conditional probabilities must be at least as far away from the climatological probability p_Θ as the corresponding ones for the other system, an intuitively appealing requirement. The intermediate case (i.e., either $p_0 \leq p_0'$ and $p_1 \leq p_1'$, or $p_0' \leq p_0$ and $p_1' \leq p_1$) is indeterminate with respect to economic value (see Figure 6.1c). Specifically, for some loss or payoff functions the first system will be superior to the second in terms of economic value, whereas for other loss or payoff functions the second system will be superior to the first (e.g., Murphy and Ehrendorfer, 1987).

To simplify the characterization of the forecasting system further, one additional requirement will be imposed. The plausible assumption is made that forecasts of adverse weather are issued with the same long-run relative frequency as the occurrence of



adverse weather (termed "overall reliability" or "unconditionally unbiased"); that is,

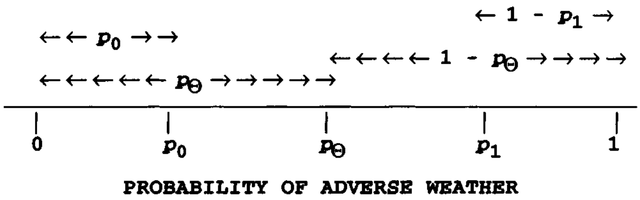
$$\Pr\{Z = 1\} = p_\Theta. \tag{6.4}$$

This requirement is equivalent to constraining the conditional probability p_0 to move from p_Θ toward zero at the same relative rate at which the conditional probability p_1 moves from p_Θ toward one (see Figure 6.1d). In particular, equation (6.4) implies that

$$p_0 = \frac{(1 - p_1)p_\Theta}{1 - p_\Theta}. \tag{6.5}$$

That is, p_0 is simply $1 - p_1$ weighted by the climatological "odds" of adverse weather, $p_\Theta/(1 - p_\Theta)$. The forecasting system is then completely characterized by either one of the two conditonal probabilities in equation (6.2), say p_1, for a fixed climatological prob-

(d)



(e)

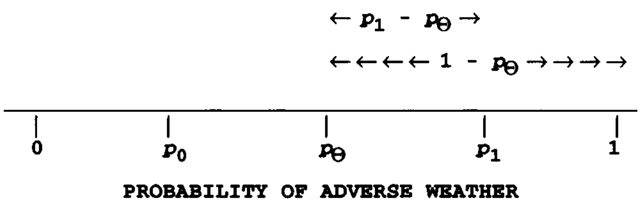


Figure 6.1. Relationship among parameters of imperfect weather forecasting system: (a) basic parameter configuration; (b) example in which sufficiency condition (6.3) is satisfied; (c) example in which sufficiency condition (6.3) is *not* satisfied; (d) constraint (6.5) imposed by requirement of overall reliability, equation (6.4); (e) measure of forecast quality, equation (6.6).

ability of adverse weather p_Θ. This condition (6.5) of overall reliability of the forecasts means that the sufficiency relation (6.3) is automatically satisfied as p_1 is increased between p_Θ and one.

It is convenient to rescale the conditional probability of adverse weather p_1, defining the measure of *quality* for this prototype form of imperfect weather information as

$$q = \frac{p_1 - p_\Theta}{1 - p_\Theta}, \tag{6.6}$$

just the relative distance of p_1 between p_Θ and one (see Figure 6.1e). Note that $0 \leq q \leq 1$, with $q = 0$ for climatological information and $q = 1$ for perfect information. It is easy to show that q is equal to the ordinary correlation between the forecast variable Z and the weather variable Θ [i.e., $q = \text{Corr}(Z, \Theta)$, where "Corr" denotes the coefficient of correlation]. Because the unconditional and conditional biases have been assumed away (i.e., condition 6.4 and the fact that the probabilities in equation 6.2 are "reliable" in an *ex ante* sense), the correlation coefficient turns out to be a reasonable measure of quality (see Chapter 2 of this volume).

Finally, it is important to reiterate the implications for the economic value of the imperfect weather forecasting system of the fact that the quality measure q satisfies the sufficiency relation. No matter what the particular form of decision maker's loss or payoff function, the economic value, denoted by $V(q)$, must be a nondecreasing function of the quality q of the forecasting system. The graph of $V(q), 0 \leq q \leq 1$, is referred to as the *quality/value curve* [note, by definition, $V(0) = 0$]. What remains to be done is to obtain analytical expressions for $V(q)$ for certain prototype decision-making models. Because the optimal policy plays a crucial role in determining the shape of the quality/value curve, attention is also focused on deriving analytical expressions for its structure. In particular, $V(q) = 0$ unless the optimal policy based on the forecasts differs from that based on climatological information alone.

3. Prototype decision-making models

All of the prototype decision-making models treated in this section are versions of the so-called cost–loss ratio model. This problem has essentially the simplest possible form of decision maker's loss or payoff function. Nevertheless, it has a long tradition of study, especially within the meteorological community. The cost–loss ratio model was apparently introduced by Thompson (1952). Note that, in this chapter, the term "cost–loss" refers to a more specific class of decision-making models than does the same term in Chapter 3 of this volume.

3.1. Static cost–loss ratio model

The decision maker must choose between two possible actions:

Table 6.1. Expense matrix for cost–lost ratio decision-making model

	Weather state	
Action	Adverse ($\Theta = 1$)	Not adverse ($\Theta = 0$)
Protect	C	C
Do not protect	L	0

(i) *protect*;
(ii) *do not protect.*

If protective action is taken, then the decision maker incurs a *cost* C, no matter what the weather. If protective action is not taken and adverse weather does occur (i.e., $\Theta = 1$), then the decision maker incurs a *loss* L, $0 < C < L$. The complete expense matrix for this decision-making problem is given in Table 6.1. The so-called umbrella problem refers to the situation in which the protective action is to "take an umbrella" and the adverse weather is "rain" (Katz and Murphy, 1987).

For now, the static version of the cost–loss ratio model is considered. This situation could be viewed as a "one-shot" problem, in which the decision maker need only choose an action to take on a single occasion. On the other hand, it could be viewed as a repetitive problem in which the decision maker faces the identical situation on a sequence of occasions, with no carryover effects from one occasion to the next (i.e., no "dynamics"). In particular, the fact that a loss L has been incurred on one occasion does not preclude the decision maker suffering the same loss again on some future occasion. Further, the state of weather on one occasion is taken, for the time being, to be independent of that on any subsequent occasion.

The decision maker must select the action (either protect or do not protect) that achieves the desired goal of minimizing the expected expense (sometimes termed "Bayes risk"). This criterion is equivalent to maximizing expected return, which is in turn a special case of maximizing expected utility when the decision maker's utility function is linear in monetary expense (see Chapter 3 of this volume). The economic value of the imperfect weather forecasting system is the reduction in expected expense associated with the forecasts, as compared to the expected expense when only the climatological probability of adverse weather, equation (6.1), is available. Formally, the minimal expected expense, expressed as a function of the quality q of the weather forecasting system, equation (6.6) in Section 2.2, is denoted by $E(q)$. The corresponding economic value of the forecasts is

$$V(q) = E(0) - E(q), \quad 0 \leq q \leq 1. \tag{6.7}$$

As stated in Section 2.2, $V(q)$ must be a nondecreasing function of q. For the static cost–loss ratio problem, a simple analytical expression for this quality/value curve can be obtained.

Example [*Climatological information* (*i.e.*, $q = 0$) *with* $C = 0.25$ *and* $L = 1$]. When protective action is taken, the decision maker incurs a cost $C = 0.25$, no matter what the climatological probability of adverse weather. If this probability $p_\Theta = 0.3$, for example, then the expected expense when protective action is not taken is $p_\Theta L = (0.3)(1) = 0.3$. Because $0.25 < 0.3$, the decision maker should take protective action. On the other hand, if $p_\Theta = 0.2$, for example, then $p_\Theta L = (0.2)(1) = 0.2 < 0.25$ and the decision maker should not take protective action. Evidently, for fixed economic parameters C and L, whether the decision maker should take protective action is governed by the likelihood of occurrence of adverse weather.

This example serves to motivate the following general results for the case of climatological information alone. The structure of the optimal policy is to take protective action provided $C < p_\Theta L$; that is,

(i) *protect* if $p_\Theta > C/L$;
(ii) *do not protect* if $p_\Theta < C/L$.

The fact that the optimal policy depends on the two economic parameters only through the ratio C/L explains the origin of the name "cost–loss ratio" decision-making situation (e.g., Murphy, 1977). The corresponding minimal expected expense for climatological information is

$$E(0) = \min\{C, \ p_\Theta L\}. \tag{6.8}$$

This expected expense expression provides one of the two inputs required in equation (6.7) to determine the economic value of an imperfect weather forecasting system.

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Example (*Imperfect forecasts with* $C = 0.25$, $L = 1$, *and* $p_\Theta = 0.2$). It has already been established that this probability of adverse weather, $p_\Theta = 0.2$, is too low for protection to be optimal with climatological information alone. But suppose that an imperfect weather forecasting system exists, for example, with quality $q = 0.05$ (i.e., the conditional probabilities of adverse weather are $p_1 = 0.24$ and $p_0 = 0.19$ from equations 6.5 and 6.6). According to assumption (6.4), the likelihood that a forecast of $Z = 1$ is issued is $p_\Theta = 0.2$ (i.e., the conditional probability of adverse weather is $p_1 = 0.24$). The likelihood that a forecast of $Z = 0$ is issued is $1 - p_\Theta = 0.8$ (i.e., the conditional probability of adverse weather is $p_0 = 0.19$). First, suppose that the decision maker receives a forecast of $Z = 1$. If protective action is taken, then the expected expense is again $C = 0.25$. If protective action is not taken, the expected expense is now $p_1 L = (0.24)(1) = 0.24$. Because $0.24 < 0.25$, the decision maker should still not take protective action. Second, suppose that $Z = 0$. The expected expense if protective action is not taken is $p_0 L = (0.19)(1) = 0.19 < 0.25$. So the optimal policy remains never to take protective action, in spite of the availability of the forecasts. Consequently, forecasts of this particular quality level are of no economic value for this specific decision maker [as a check, $E(0.05) = (0.2)(0.24) + (0.8)(0.19) = 0.2 = E(0)$].

On the other hand, suppose that a weather forecasting system exists with a higher quality, for example, $q = 0.5$ (i.e., $p_1 = 0.6$ and $p_0 = 0.1$). If $Z = 1$, then the expected expense when protective action is not taken increases to $p_1 L = (0.6)(1) = 0.6 > 0.25$, and protective action should be taken. If $Z = 0$, then this expected expense is $p_0 L = (0.1)(1) = 0.1 < 0.25$, and protective action should not be taken. So, weighting by the likelihood of each possible forecast being issued, the minimal expected expense associated with this imperfect weather forecasting system is $E(0.5) = p_\Theta C + (1 - p_\Theta)(p_0 L) = (0.2)(0.25) + (0.8)(0.1) = 0.13$. The economic value of the forecasts is $V(0.5) = E(0) - E(0.5) = 0.2 - 0.13 = 0.07$. In other words, the decision maker would save an average of 7/100th of the possible loss L by relying on the forecasting system instead of climatological information. Evidently, the quality must exceed a certain threshold for the weather forecasting system to be of any economic value to the decision maker.

Again, this example motivates the following general results for the imperfect weather forecasting system. First, as in the example, suppose that $0 < p_\Theta < C/L$. Provided $0 \le q < q^*$, where

$$q^* = \frac{C/L - p_\Theta}{1 - p_\Theta} \tag{6.9}$$

(i.e., $p_\Theta \le p_1 < C/L$), it is never optimal to protect no matter what the weather forecast Z. The minimal expected expense is $E(q) = E(0) = p_\Theta L$, and $V(q) = 0, 0 \le q < q^*$. On the other hand, provided $q^* < q \le 1$ (i.e., $C/L < p_1 \le 1$), the optimal policy is of the form

(i) *protect* if $Z = 1$;
(ii) *do not protect* if $Z = 0$.

Using equations (6.5) and (6.6), the minimal expected expense is

$$\begin{aligned} E(q) &= p_\Theta C + (1 - p_\Theta) p_0 L = p_\Theta [C + (1 - p_1) L] \\ &= p_\Theta [C + (1 - p_\Theta)(1 - q) L], \quad q^* \le q \le 1. \end{aligned} \tag{6.10}$$

Using equations (6.7) and (6.10), the economic value of the imperfect weather forecasting system is

$$V(q) = E(0) - E(q) = p_\Theta \{[p_\Theta + (1 - p_\Theta) q] L - C\}, \quad q^* \le q \le 1. \tag{6.11}$$

In other words, the quality/value curve is zero below the quality threshold q^* defined by equation (6.9), and rises linearly above this threshold. Interestingly, the slope of this linear function of q is proportional to the climatological variance, $\mathrm{Var}(\Theta) = p_\Theta(1 - p_\Theta)$, which can be viewed as the "uncertainty" in the underlying weather variable. Figure 6.2 illustrates the piecewise linear shape (in particular, convex — the derivative, provided it exists, of a convex function is itself a nondecreasing function) of this quality/value curve. For the example of $C = 0.3$, $L = 1$, and $p_\Theta = 0.2$, the quality threshold is $q^* = 0.125$ (see equation 6.9) and the slope of the straight line above the threshold is $\mathrm{Var}(\Theta) = 0.16$. Analogous expressions hold for the alternative case in which $C/L < p_\Theta < 1$ (see Katz and Murphy, 1987, for further details).

3.2. Dynamic cost–loss ratio model (finite horizon)

As noted previously, most real-world decision-making problems are dynamic, rather than static, in nature. That is, current and

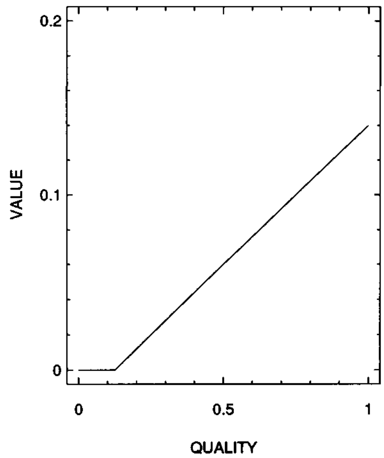

Figure 6.2. Quality/value curve for static cost–loss ratio decision-making model, where $C = 0.3$, $L = 1$, and $p_{\Theta} = 0.2$. (From Katz and Murphy, 1987)

previous actions and their economic consequences influence future actions and consequences. For instance, the so-called fruit-frost problem is an inherently dynamic decision-making problem with substantial economic ramifications (Baquet, Halter, and Conklin, 1976; Katz, Murphy, and Winkler, 1982). An orchardist must decide whether or not to protect (e.g., by employing heaters, sprinklers, or wind machines) fruit trees during the spring when the buds are especially vulnerable to damage from freezing temperatures. Because the buds damaged or killed cannot recover, it is a dynamic problem. To aid the orchardist in making this decision, each evening during the frost-protection season a minimum temperature forecast is provided. The goal of the orchardist is to minimize the expected expense, not over a single night, but totaled over the entire season (i.e., a finite number of nights) (see Chapter 4 of this volume and Katz et al., 1982 for more details). This type of dynamic decison-making problem is termed "finite horizon."

It is our intent to mimic the dynamic nature of the fruit-frost problem in generalizing the cost–loss ratio model from the static situation, already treated in Section 3.1, to a dynamic situation. Just as buds already killed cannot recover, it is assumed that the loss L can be incurred once at most. The goal of the decision maker is to minimize the expected expense totaled over a finite horizon, with the number of occasions denoted by n. Otherwise, the prob-

Table 6.2. Expected expense for possible strategies for two-stage dynamic cost–loss ratio decision-making model with climatological information

Strategy		Expected expense		
First occasion	Second occasion	First occasion	Second occasion	Total
(i) Protect	Protect	C	C	$2C$
(ii) Protect	Do not protect	C	$p_\Theta L$	$C + p_\Theta L$
(iii) Do not protect	Protect	$p_\Theta L$	$(1 - p_\Theta)C$	$C + p_\Theta(L - C)$
(iv) Do not protect	Do not protect	$p_\Theta L$	$(1 - p_\Theta)p_\Theta L$	$(2 - p_\Theta)p_\Theta L$

lem is unchanged from the static cost–loss ratio model, retaining the same economic parameters, C and L, and the same form of climatological information (with probability of adverse weather p_Θ) and imperfect weather forecasting system (with quality q). Nevertheless, the introduction of this very simplified form of dynamics into the decision-making model results in a much more complex structure of the optimal policy and shape of the quality/value curve. To motivate the dynamic version of the problem, first the simplest case of only a two-occasion (i.e., $n = 2$) situation is considered as an example.

Example ($n = 2$). Table 6.2 lists the total expected expenses for the four possible strategies (i.e., the combinations of either protect or do not protect on the first and second occasions) when climatological information alone is available to the decision maker. To see how the dynamics enters into this problem, we consider the third strategy in this table (i.e., do not protect on the first occasion, but protect on the second occasion). The expected expense on the second occasion is $(1 - p_\Theta)C$, *not* C, because protective action needs to be taken on the second occasion only if the loss L has *not* been incurred on the first occasion (i.e., with probability $1 - p_\Theta$).

As a numerical example, suppose that $C = 0.25$, $L = 1$, and $p_\Theta = 0.3$. Substituting these numerical values into the expressions in Table 6.2, the total expected expenses are:

strategy (i): 0.5;
strategy (ii): 0.55;
strategy (iii): 0.475;

strategy (iv): 0.51.

Strategy (iii), of protecting only on the second occasion, is optimal. Recall that the optimal policy for the static model (i.e., $n = 1$) with the same numerical values of the parameters C, L, and p_Θ, is to take protective action (established in a previous example). If this strategy were naively followed on both occasions [i.e., strategy (i) in Table 6.2], an additional, unnecessary expected expense of $0.5 - 0.475 = 0.025$ (or 1/40th of the possible loss L) would be incurred. The lesson is that following the strategy that minimizes the immediate expected expense on a given occasion does not necessarily correspond to minimizing the total expected expense over all the occasions (two, in this example), because the dynamics of the decision-making model have been ignored.

Returning to the general two-occasion model, it is evident from Table 6.2 that strategy (iii) is always superior to strategy (ii). The motivation for this result is that, given that the decision maker can afford to protect on at most only one of the two occasions, it is preferable to postpone protection until the second occasion, allowing for the possibility that the loss L will be incurred on the first occasion (making protection unnecessary). Upon reaching the second (and last) stage of the decision-making process, it is also evident from this table that the prescribed action is identical to the optimal policy for the static ($n = 1$) problem; namely, protect on the second occasion if $p_\Theta > C/L$. It is only on the first occasion that the prescribed action differs from that for the static situation. Murphy et al. (1985) and Winkler and Murphy (1985) have treated this two-occasion version of the dynamic cost–loss ratio model in more detail.

It should be evident from this example that enumerating all possible strategies for the general n-occasion model with climatological information alone, or even for the two-occasion model when an imperfect forecasting system is available, would be quite tedious. An alternative approach, based on the idea of "backward induction," is suggested by the example just treated. First, the one-occasion (i.e., $n = 1$) or static problem is solved (corresponding to the last occasion). Then, expressing the two-occasion problem as a function of the one-occasion problem and making use of the solution already obtained for the last occasion, a complete solution to the two-occasion problem can be obtained. Continuing to work backward until the first of the n occasions is reached, and expressing the n-occasion problem as a function of the $(n-1)$-occasion problem already solved, produces a solution to the general n-occasion problem. This method of determining the optimal policy on each of the n occasions and the associated minimal total expected expense over the n occasions involves a recursive equation, and is referred to as "stochastic dynamic programming" (e.g., Ross, 1983; White, 1978).

For relatively complex dynamic decision-making models, the recursive equation is employed purely to generate numerical solutions. The use of stochastic dynamic programming for this purpose has been cited in Chapter 4 of this volume. For models whose structure is relatively simple, such as the dynamic cost–loss ratio model, the recursion can be solved analytically. In particular, Murphy et al. (1985) obtained certain analytical results concerning the structure of the optimal policy for the general n-occasion problem, whereas Krzysztofowicz and Long (1990) extended this approach to produce complete analytical solutions for both the structure of the optimal policy and the corresponding minimal total expected expense over the n occasions. White (1966) was apparently the first researcher to advocate the approach of stochastic dynamic programming in assessing the economic value of forecasts, although he did not explicitly consider weather forecasts.

Rather than reproduce these solutions here, we shall be content to describe the essential features of the results. First, a description is provided of how the stochastic dynamic programming recursions for the n-occasion model are derived, expressions from which all the analytical results originate. In keeping with the notation introduced in Section 3.1, let $E_n(q)$, $n = 1, 2, \ldots$, denote the minimal expected expense totaled over the n occasions when the imperfect weather forecasting system with quality q (see Section 2.2) is available to the decision maker. As in equation (6.7), the corresponding economic value of the forecasting system is

$$V_n(q) = E_n(0) - E_n(q), \; 0 \le q \le 1. \tag{6.12}$$

To simplify matters, first consider the situation in which only climatological information is available to the decision maker (i.e., quality $q = 0$). For the one-occasion (or static) cost–loss ratio model, the expression (6.8) already derived in Section 3.1 can be restated with this new notation as

$$E_1(0) = \min\{C, \; p_\Theta L\}. \tag{6.13}$$

For the two-occasion model (i.e., $n = 2$), the minimal total expected expense $E_2(0)$ can be expressed as a function of the minimal expected expense $E_1(0)$ for the static model as follows:

$$E_2(0) = \min\{C + E_1(0),\ p_\Theta L + (1 - p_\Theta)E_1(0)\}. \qquad (6.14)$$

The first (second) term on the right-hand side of equation (6.14) represents the total expected expense over the last two occasions when protective action is (is not) taken on the next to last occasion. The first term consists of the cost C of protecting on the first occasion, as well as the minimal expected expense on the second (last) occasion. The second term consists of the loss L on the first occasion, which is incurred with probability p_Θ, as well as the minimal expected expense on the second occasion, which is incurred with probability $1 - p_\Theta$. It is also easy to verify that the expressions already listed for $E_2(0)$ in Table 6.2 satisfy equation (6.14).

Similarly, for the general n-occasion model, the minimal total expected expense with climatological information $E_n(0)$ is related to $E_{n-1}(0)$ by

$$E_n(0) = \min\{C + E_{n-1}(0),\ p_\Theta L + (1 - p_\Theta)E_{n-1}(0)\}, \qquad (6.15)$$

$n = 2, 3, \ldots$. This expression can be obtained simply by substituting $E_n(0)$ for $E_2(0)$ and $E_{n-1}(0)$ for $E_1(0)$ in equation (6.14). Utilizing equations (6.13) and (6.15), the minimal total expected expense with climatological information can be determined for any desired number of occasions n.

For the situation in which a weather forecasting system is available to the decision maker, the stochastic dynamic programming recursion for $E_n(q), 0 < q \leq 1$, can be derived in an analogous fashion (Murphy et al., 1985). For completeness, this more complex expression is included in the Appendix (see equation 6.A1).

Figure 6.3 shows the structure of the optimal policy for three types of information: climatological (i.e., $q = 0$), imperfect forecasts (i.e., $0 < q < 1$), and perfect information (i.e., $q = 1$). When only climatological information is available to the decision maker, the structure of the optimal policy is of the following form:

(i) *do not protect* on the first $n - k_0$ occasions;
(ii) *protect* on the last k_0 occasions;

for some constant $0 \leq k_0 \leq n$ (see Figure 6.3a). Here k_0 is a function of the cost–loss ratio C/L and the climatological probability

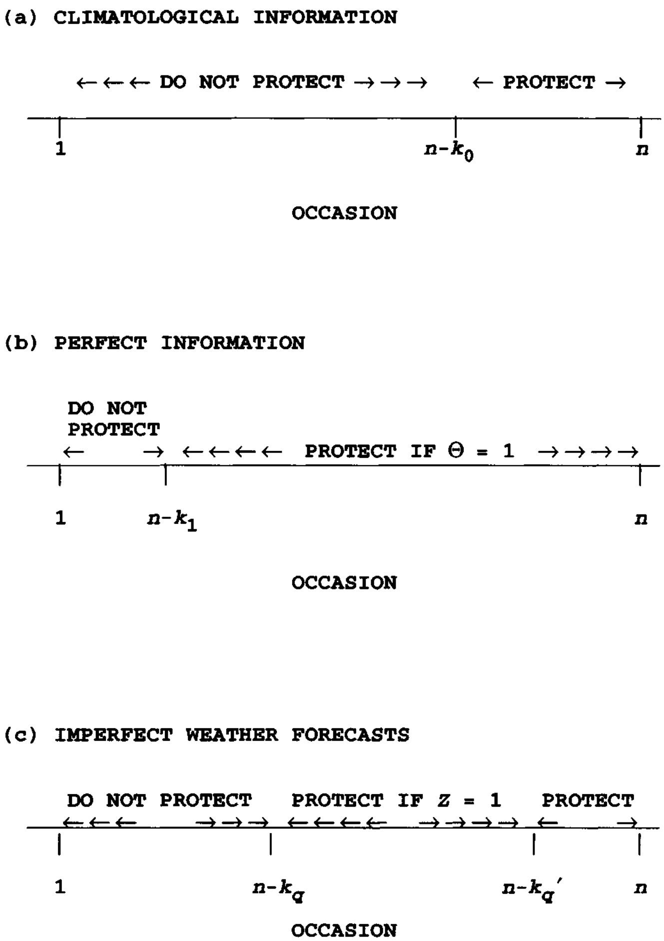

Figure 6.3. Structure of optimal policy for finite-horizon version of dynamic cost–loss ratio decision-making model for various forms of information: (a) climatological; (b) perfect; (c) imperfect weather forecasting system. (From Murphy et al., 1985)

of adverse weather p_Θ (see Krzysztofowicz and Long, 1990). This result is consistent with the phenomenon already observed for the two-occasion example, namely, that it is preferable to postpone protecting as long as possible, avoiding unnecessary protection in the event that the loss L is incurred.

Figure 6.3b shows the structure of the optimal policy given perfect information (i.e., $q = 1$):

(i) *do not protect* on the first $n - k_1$ occasions;
(ii) *protect* on the last k_1 occasions whenever $\Theta = 1$;

for some constant $k_1, 1 \leq k_1 \leq n$. Here k_1 again depends on the parameters C/L and p_Θ (see Krzysztofowicz and Long, 1990), and $k_1 \geq 1$ because it is always optimal to protect given $\Theta = 1$ in the static model (recall that $C/L < 1$). In this case of perfect information, the decision maker is assumed to be clairvoyant (but only concerning the next occasion, *not* the remaining periods), implying that protection needs be taken only on those occasions when it is known that adverse weather will occur. Hence, the structure of the optimal policy is of the same form as that for climatological information, except that protection can be initiated earlier (i.e., $k_0 \leq k_1$).

Figure 6.3c shows the structure of the optimal policy given the imperfect forecasting system (i.e., $0 < q < 1$):

(i) *do not protect* on the first $n - k_q$ occasions;
(ii) *protect* on occasions $n - k_q + 1$ through $n - k'_q$ whenever $Z = 1$;
(iii) *protect* on the last k'_q occasions;

for some constants k_q and $k'_q, 0 \leq k'_q \leq k_q \leq n$. The exact numerical values of k_q and k'_q depend on the parameters C/L and p_Θ, as well as on the forecast quality q (see Krzysztofowicz and Long, 1990). As in the case of climatological information, it is always optimal to protect on the last k'_q occasions. Moreover, as in the case of perfect information, it is optimal to protect given a forecast of adverse weather on the last k_q occasions. These constants are related by

$$k'_q \leq k_0 \leq k_q \leq k_1. \tag{6.16}$$

In particular, as the forecast quality q increases from zero to one, k_q increases from k_0 to k_1 and k'_q decreases from k_0 to zero. These results concerning the structure of the optimal policy follow directly from the corresponding stochastic dynamic programming recursion (see the Appendix and Krzysztofowicz and Long, 1990; Murphy et al., 1985).

Figure 6.4 shows some examples of the quality/value curves for the dynamic cost–loss ratio decision-making model when the cost $C = 0.3$, the loss $L = 1$, and the probability of adverse weather $p_\Theta = 0.2$, for several different lengths of finite horizon n, ranging from $n = 2$ to $n = 16$. Like the static (i.e., $n = 1$) model

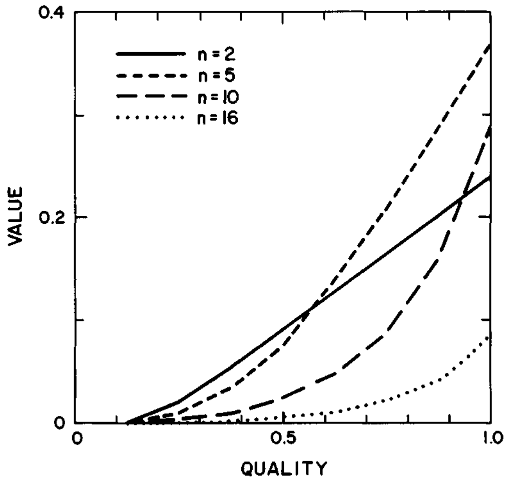

Figure 6.4. Quality/value curves for finite-horizon version of dynamic cost–loss ratio decision-making model, where $C = 0.3$, $L = 1$, and $p_\Theta = 0.2$, for $n = 2$, 5, 10, and 16. (From Murphy et al., 1985)

(see Figure 6.2), a quality threshold is still present below which the forecasting system has no economic value for the decision maker. Above this threshold, the economic value of the imperfect weather forecasting system increases in a piecewise linear fashion. The quality/value curve is convex, since the straight-line segments have increasing slope as quality q increases. The number of linear segments increases, resulting in curves that gradually become smoother in appearance, as the horizon n increases (Murphy et al., 1985).

3.3. Dynamic cost–loss ratio model (infinite horizon, discounted)

Although some decision-making applications, such as the fruit-frost problem already mentioned in Section 3.2, are inherently finite horizon, others involve a process that continues indefinitely into the future. For instance, the so-called fallowing/planting problem refers to the decision faced each year by a wheat farmer as to whether to plant a crop or to let the land lie fallow and accumulate soil moisture until planting time the following year (see Chapter 4 of this volume and Brown, Katz, and Murphy, 1986; Katz, Brown, and Murphy, 1987). It is reasonable to assume that such a decision maker wants to maximize the expected return totaled over a relatively large number of years (i.e., effectively, an *infinite* horizon).

Consequently, an infinite-horizon version of the dynamic cost–loss ratio decision-making model will now be treated. The nature of the dynamics and the form of imperfect weather forecasts remains the same as for the finite-horizon model previously considered (Section 3.2). However, when dealing with an economic optimization problem over a relatively long period of time, one additional complication does arise. Future expenses (or returns) need to be *discounted*, the rationale being that a dollar today is worth more than a dollar in the future because of uncertainties as to whether or not future expenses actually will be incurred and because of opportunities for investment and consumption. Specifically, an expense E incurred on the next occasion has a "present value" of only αE, where α, $0 < \alpha < 1$, is the *discount factor*. In relative terms, this "future value" E is diminished (or discounted) at the rate

$$r = \frac{E - \alpha E}{\alpha E} = \frac{1 - \alpha}{\alpha}, \tag{6.17}$$

called the *discount rate* (for a more detailed explanation, see Alchian and Allen, 1972). In summary, a version of the dynamic cost–loss ratio model will be considered in which the decision maker minimizes expected expense, totaled and discounted (with discount factor α) over an infinite horizon.

In some respects, the infinite-horizon model is actually conceptually simpler to deal with than the finite-horizon model. Because an infinitely long future always remains, the decision-making process can be viewed, at least in a probabilistic sense, as starting over and over again. Therefore, it is straightforward to derive a stochastic dynamic programming recursion for $E(q)$, now representing the minimal expected expense, totaled and discounted over an infinite horizon. One might question why an infinite horizon is realistic when the loss L will surely be incurred within some finite number of occasions (unless protection is always taken). But it must be recognized that a positive probability always remains of not incurring the loss L over the first n occasions, no matter how large n is. Further, the infinite-horizon problem can often serve as a convenient approximation to the corresponding finite-horizon problem.

First, the situation is treated in which only climatological information is available to the decision maker. In this case, $E(0)$

satisfies the recursion

$$E(0) = \min\{C + \alpha E(0),\ p_\Theta L + (1 - p_\Theta)\alpha E(0)\}. \tag{6.18}$$

The first (second) term within the brackets represents the total, discounted expected expense when protection is (is not) taken on the initial occasion. This recursion resembles the one for the finite-horizon situation depicted in equation (6.15), except that now the length of the horizon is always infinite (as opposed to finite length n or $n-1$) and the discount factor α has been introduced. Turning to the case in which the imperfect weather forecasting system is available to the decision maker, a recursion analogous to equation (6.18) can be derived for $E(q)$, $0 < q < 1$ (Katz and Murphy, 1990). This more complex expression is included in the Appendix (see equation 6.A2).

Using the recursion (6.18), Katz and Murphy (1990) showed that the optimal policy with climatological information alone is of the form:

(i) *protect* if $p_\Theta > p_\Theta^*$;
(ii) *do not protect* if $p_\Theta < p_\Theta^*$.

Here the threshold p_Θ^* for the climatological probability of adverse weather is given by

$$p_\Theta^* = \frac{C/L}{1 - [(C/L)/r]}. \tag{6.19}$$

For this threshold to be well defined, expenses must be discounted at a fast enough rate (i.e., $\alpha < 1 - C/L$). Otherwise, protection is never optimal with only climatological information. The structure of the optimal policy resembles that for the static cost–loss ratio model (Section 3.1), with the discount rate r also entering into the threshold, depicted in equation (6.19). Similar, but more complex results can be obtained for the structure of the optimal policy when the imperfect forecasting system is available to the decision maker (Katz and Murphy, 1990).

Like the static and finite-horizon versions of the cost–loss ratio model, a threshold in quality exists below which the economic value of the forecasts is zero. This threshold arises because the optimal policy with the imperfect weather forecasting system does not deviate from that for climatological information alone, unless the quality q is sufficiently greater than zero. For example, suppose that $p_\Theta < p_\Theta^*$ (i.e., the optimal policy is not to protect for

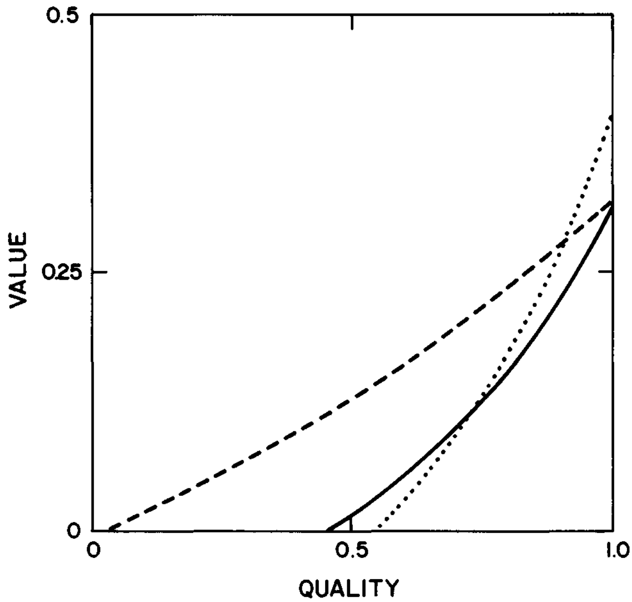

Figure 6.5. Quality/value curves for infinite-horizon version of dynamic cost–loss ratio decision-making model, where $\alpha = 0.9$: $C = 0.05$, $L = 1$, and $p_\Theta = 0.05$ (dashed line); $C = 0.2$, $L = 1$, and $p_\Theta = 0.2$ (solid line); and $C = 0.05$, $L = 1$, and $p_\Theta = 0.2$ (dotted line). (From Katz and Murphy, 1990)

climatological information). Then the quality threshold can be expressed as

$$q^* = (C/L)\left[\frac{1 - \alpha(1 - p_\Theta)}{1 - \alpha}\right] - p_\Theta \tag{6.20}$$

(Katz and Murphy, 1990). This threshold is identical to the one for the static situation depicted in equation (6.9), except for the term involving the discount factor α. An analogous expression holds for the alternative case of $p_\Theta > p_\Theta^*$ (Katz and Murphy, 1990).

Above the quality threshold, it can be shown (Katz and Murphy, 1990) that the economic value of the imperfect weather forecasting system, $V(q)$, must increase at an increasing rate as the quality q increases (i.e., a convex curve). Figure 6.5 shows three examples of these quality/value curves. Their shape resembles the curves for the finite-horizon situation (Figure 6.4), but they are smooth rather than piecewise linear.

4. Extensions

4.1. Autocorrelated weather variables

All the results presented so far concerning the structure of the optimal policy and the shape of the quality/value curve have been predicated upon the simplifying assumption that the states of the weather variable are temporally independent. Of course, it is well known that, depending on the time scale, most weather variables may actually possess a substantial degree of temporal dependence, usually exhibiting a tendency for a given weather state to persist (i.e., positive autocorrelation). If the decision-making situation being modeled were static (e.g., the static cost–loss ratio model of Section 3.1), then the issue of autocorrelation could be treated in a relatively straightforward manner. Knowing that a weather variable is autocorrelated could be viewed simply as a special case of having forecasts of higher quality than those based on the assumption of independence. In particular, the same quality/value curve would apply (e.g., Figure 6.2 for the static cost–loss ratio model).

With dynamic decision-making situations, however, the presence of autocorrelation makes the problem inherently more complex. For this reason, only a few attempts have been made to allow for temporal dependence when assessing the economic value of imperfect weather forecasts (Epstein and Murphy, 1988; Katz, 1993; Wilks, 1991). Nevertheless, given the recent interest in the economic value of forecasts of ENSO (El Niño–Southern Oscillation), a phenomenon with substantial dependence on monthly, seasonal, and even annual time scales, this issue is of practical importance (Adams et al., 1995; Kite-Powell and Solow, 1994).

Katz (1993) has attempted to develop a more general analytical framework in which forecasts of an autocorrelated weather variable could be treated naturally. This framework is briefly outlined. Exactly the same prototype form of weather information introduced in Section 2.2 is retained. But now the sequence of weather states (i.e., adverse or no adverse weather) $\{\Theta_t\colon\ t = 1,\ 2,\ \ldots\}$ is assumed to constitute a first-order Markov chain (e.g., Gabriel and Neumann, 1962). Besides the probability of adverse weather p_Θ, one new parameter needs to be incorporated into the model, namely, the first-order autocorrelation or *persistence parameter* $d = \text{Corr}(\Theta_t,\ \Theta_{t+1})$. To allow for the persistence typical of weather variables, we take $0 < d < 1$. The previous sections of this chapter have tacitly assumed that $d = 0$ (i.e., an independent climate). The other extreme of $d = 1$ represents a perfectly persistent climate.

A major complication for any dynamic decision-making problem arises because the assumed Markovian property implies that all future weather states are correlated with the present one; that is,

$$\mathrm{Corr}(\Theta_t,\ \Theta_{t+k}) = d^k,\ k = 1,\ 2,\ \ldots\ . \tag{6.21}$$

In effect, the decision maker has available on a given occasion a sequence of climatological "forecasts" for the indefinite future, whose skill cascades down to zero. In selecting the action for the present occasion, the decision maker should not ignore this information about the weather on future occasions. As an extreme example, if an orchardist knew for sure (or with a high probability) that the weather tomorrow night would destroy the fruit crop, then there would be no point in taking protective action tonight, no matter what the forecast for the present occasion.

The infinite-horizon, discounted version of the dynamic cost–loss ratio decision-making model, just treated in Section 3.3 for $d = 0$, is again considered, but now with an autocorrelated weather variable (i.e., $d > 0$). First, the situation in which only climatological information is available to the decision maker is discussed. A stochastic dynamic programming recursion that is a straightforward generalization of equation (6.18) can be derived (Katz, 1993). For a given strategy, the minimal expected expense, totaled and discounted over an infinite horizon, naturally depends on the persistence parameter d. Consequently, the optimal policy for climatological information with d sufficiently greater than zero might well differ from that for $d = 0$. A decision maker who naively believed that the states of the weather variable were temporally independent would in some circumstances actually be following a "suboptimal policy," incurring unnecessary additional expense. Katz (1993) described specific instances in which this phenomenon actually occurs.

With an autocorrelated weather variable, care must be taken in making any assumptions about the stochastic properties of the sequence of imperfect weather forecasts (introduced in Section 2.2). Recall that these forecasts are for the next occasion only, not for the remaining occasions. In this context, Z_t denotes a forecast of Θ_t available at time $t-1$. It is assumed that this sequence of forecast states $\{Z_t : t = 1,\ 2,\ \ldots\}$ has the same probabilistic structure as the weather variable, that is, a first-order Markov chain with the identical persistence parameter d. Because this requirement must hold in the two limiting situations of $d = 0$ and $d = 1$, it is a reasonable condition to impose in general.

It is necessary to make additional assumptions about the structure of the bivariate process of weather and forecast states $\{(\Theta_t, Z_t): t = 1, 2, \ldots\}$. Katz (1992) identified the exact conditions. The basic idea is that the present forecast (i.e., for the next occasion) should subsume any predictive information contained in past weather and forecast states, and that the next forecast (i.e., for the occasion after next) need only be based on the present weather state. In other words, the forecast variable can be thought of as "leading" the weather variable by one occasion.

As in the case of an independent weather variable, the quality of the imperfect weather forecasting system is specified through the conditional probability of adverse weather given a forecast of adverse weather

$$p_1 = \Pr\{\Theta_t = 1 | Z_t = 1\}, \text{ for } p_\Theta + d(1 - p_\Theta) \leq p_1 \leq 1. \quad (6.22)$$

Except that the time index t has been made explicit, equation (6.22) is identical to the original definition of p_1 in equation (6.2). The lower bound on p_1 is no longer p_Θ; instead, it reflects the temporal dependence of the weather variable, being $\Pr\{\Theta_{t+1} = 1 | \Theta_t = 1\}$. The measure of forecast quality q is defined in the same manner as previously in equation (6.6), with p_1 now specified by equation (6.22). This quality measure still represents the correlation between the forecast and weather variables [i.e., $q = \text{Corr}(Z_t, \Theta_t)$, $d \leq q \leq 1$]. The lower bound on q is d, rather than zero, because of the predictive capability of the Markov chain model itself.

Despite the fact that the weather forecasting system provides only one-occasion-ahead forecasts, all future weather states are correlated with the present forecast. In particular, it can be shown that the cross correlation function between the forecast and weather variables is of the form

$$\text{Corr}(Z_t, \Theta_{t+k-1}) = qd^{k-1}, \; k = 2, 3, \ldots. \quad (6.23)$$

Here k denotes the lead time because Z_t is a forecast of Θ_t available at time $t - 1$. In other words, the combination of one-occasion-ahead forecasting quality and temporal dependence induces "forecasts" two occasions or more ahead, whose quality exceeds that based on the autocorrelation of the weather variable alone [i.e., $qd^{k-1} \geq d^k = \mathrm{Corr}(\Theta_{t-1}, \Theta_{t+k-1})$, since $q \geq d$].

Because the requirement of overall calibration in equation (6.4) has been retained, the quality measure q still satisfies the sufficiency relation (see Section 2.2). Therefore, economic value $V(q)$ must be a nondecreasing function of q, $d \leq q \leq 1$. As already observed for the situation of climatological information alone, the presence of autocorrelation in a dynamic decision-making problem affects the expressions for the minimal expected expenses, totaled and discounted over an infinite horizon. Consequently, the economic value of a forecasting system with a fixed level of quality q might well differ depending on the degree of persistence d. Katz (1992) relied again on the approach of stochastic dynamic programming to determine these expenses.

Some numerical examples for the infinite-horizon, discounted version of the dynamic cost–loss ratio decision-making model are provided to illustrate how the shape and magnitude of the quality/value curve change as a function of the autocorrelation d. Figure 6.6 (top) shows an example in which the economic value of the imperfect weather forecasting system is relatively sensitive to the degree of autocorrelation of the weather variable. The quality/value curve remains convex for $d > 0$. Holding quality q constant, the economic value is substantially reduced as d increases, with the reduction being roughly linear in d. Moreover, the curve rises at a less rapid rate the greater the degree of persistence.

Figure 6.6 (bottom) relates to an example in which the economic value of the imperfect weather forecasting system is less sensitive to the degree of autocorrelation of the weather variable. Again, the curve is convex for $d > 0$. But the curve does not change much unless the degree of persistence is relatively high. Holding quality q constant, the reduction in economic value is a highly nonlinear function of d. On the other hand, the curve rises at nearly the same rate, no matter what the degree of persistence d.

It would be natural to make a further extension to the situation in which the weather forecasting system produces forecasts not only for the next occasion, but also for subsequent occasions. The quality of such forecasts would be assumed to decay with lead time, because of limits to predictability (see Chapter 1 of this volume). Since the situation just treated already involves, in effect, "forecasts" for subsequent occasions, this extension would be straightforward.

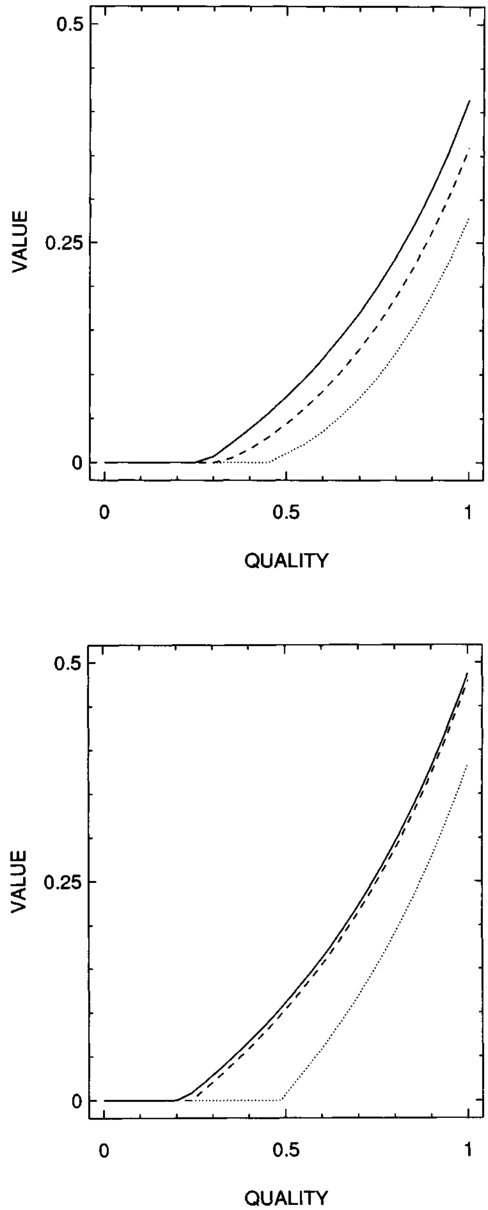

Figure 6.6. Quality/value curves for infinite-horizon version of dynamic cost–loss ratio decision-making model with an autocorrelated weather variable (solid line represents persistence parameter $d = 0$, dashed line represents $d = 0.25$, and dotted line represents $d = 0.5$): (top) $\alpha = 0.9$, $C = 0.15$, $L = 1$, and $p_\Theta = 0.2$; (bottom) $\alpha = 0.98$, $C = 0.01$, $L = 1$, and $p_\Theta = 0.025$. (From Katz, 1992)

4.2. Other extensions

There are several other respects in which one could attempt to generalize upon the results presented in this chapter. For instance, Katz (1987) examined the implications for the shape of the quality/value curve of relaxing the requirement of overall reliability of the imperfect weather forecasting system in equation (6.4). The constraint (6.5) requires that the conditional probability of adverse weather p_0 move from the climatological probability of adverse weather p_Θ toward zero at the same rate at which the other conditional probability p_1 moves from p_Θ toward one (see Figure 6.1d). Instead, these two parameters, p_0 and p_1, are now free to move toward their respective limits at rates that are independent of one another. The sufficiency condition (6.3) is satisfied, and economic value must be a nondecreasing function of p_1 and a nonincreasing function of p_0. We do not treat the most general situation in which p_0 might increase as p_1 decreases or vice versa, because the sufficiency condition would not be satisfied.

Because the original definition of the measure of forecast quality, equation (6.6), is predicated upon the overall reliability condition (6.4) being in force, first this measure needs to be extended to the more general situation now being considered. The quality of the forecasting system now depends on both p_0 and p_1, and it can be completely characterized only by a two-dimensional measure. Nevertheless, one natural way to generalize the forecast quality measure is to retain its property of being the correlation between the forecast and weather variables [i.e., $q = \mathrm{Corr}(Z, \Theta)$]. In general, this correlation coefficient is given by

$$q = \left[\frac{(p_\Theta - p_0)(p_1 - p_\Theta)}{p_\Theta(1 - p_\Theta)}\right]^{1/2}. \tag{6.24}$$

The generalized measure of quality q depends explicitly on both p_0 and p_1, reflecting the distance of these two parameters from p_Θ. It is easy to show that the original quality measure (6.6) is obtained in the special case in which the constraint (6.5) is substituted into equation (6.24). Of course, the generalized measure still has the inherent limitation of being one-dimensional.

Katz (1987) studied the shape of the quality/value curve for the static cost–loss ratio decision-making model (Section 3.1) under these weaker conditions being imposed on the weather forecasting system. If no assumptions are made about the relative rates at which p_0 and p_1 move toward their respective limits, then the shape of the quality/value curve cannot be characterized in a simple manner. All that is guaranteed is that economic value remains a nondecreasing function of q, because the sufficiency condition (6.3) is still satisfied. In particular, its shape is no longer necessarily convex, but may be concave or even locally convex for certain ranges of forecast quality and locally concave for other ranges of quality. Of course, it could be argued that the constraint (6.5) represents a plausible way for improvements in a weather forecasting system to be realized in practice.

As part of their study of the finite-horizon version of the dynamic cost–loss ratio decision-making model (Section 3.2), Krzysztofowicz and Long (1990) also dealt with a more complex form of weather forecast information than the prototype form treated in this chapter (Section 2.2). In addition to the case of a two-state forecast variable, the situation is analyzed in which probability forecasts [i.e., a continuous variable on the interval (0,1)] are available to the decision maker. By making use of a particular form of parametric model (i.e., the beta distribution), analytical solutions for the structure of the optimal policy can be derived for this more realistic form of forecast information.

The structure of the cost–loss ratio decision-making model (Section 3.1) also can be generalized in several respects. For instance, it would be natural to allow for more than two actions and more than two states of weather (Murphy, 1985). Nevertheless, it is difficult to obtain tractable, analytical results concerning the structure of the optimal policy and the shape of the quality/value curve in such situations. Perhaps, a more conceptually appealing approach would be to generalize the model directly to the situation in which the weather variable is continuous. In particular, a starting point could be the case in which the joint distribution of the forecast and weather variables is bivariate normal. Such an assumption is reasonable for temperature and was employed in the fruit-frost problem (Katz et al., 1982). One could allow for only two possible actions, or also permit the action to be a continuous variable (e.g., Gandin, Murphy, and Zhukovsky, 1992). A treatment of a continuous weather variable that allows for autocorrelation could follow Krzysztofowicz (1985).

5. Implications

The relationship between the scientific quality and economic value of imperfect weather forecasts has been examined for various forms of prototype decision-making models. Although all these models are simpler than most real-world decision-making problems, they do retain some essential features of such situations, including the fact that many of these situations are dynamic in nature. A virtually ubiquitous result is the convex shape of the quality/value curve. Economic value is zero for forecasting systems whose quality falls below a threshold. Above this threshold, economic value rises at an increasing rate as forecast quality increases toward that of perfect information. A quality threshold also arises in the fallowing/planting problem (Brown et al., 1986). However, neither this problem nor the fruit-frost problem (Katz et al., 1982) necessarily possess a quality/value curve whose shape is convex.

These results based on prototype decision-making models have some important implications for research both on weather forecasting and on the economic value of forecasts. In particular, the existence of a quality threshold may explain why current long-range (i.e., monthly or seasonal) weather forecasts, which are necessarily of relatively low quality, are apparently ignored by many decision makers (Changnon, Changnon, and Changnon, 1995; Easterling, 1986; also see Chapter 5 of this volume). Moreover, the convexity of the quality/value curve is somewhat discouraging with respect to the potential benefits of realistic improvements in the quality of weather forecasts, at least for those cases in which forecast quality is now relatively far from that of perfect information. However, it must be kept in mind that the rate at which economic value actually increases will depend on the rate at which quality improves, and quality might well be a concave function of monetary investment in meteorological research.

The prototype forms of decision-making models that have been treated were motivated in part by case studies of real-world decision-making situations, such as the fruit-frost and fallowing/planting problems. A vital need exists for more such case studies of the economic value of imperfect weather and climate forecasts in real-world applications (as observed in Chapter 4 of this volume). Studies of the analytical properties of prototype decision-making models like those presented in this chapter play an important complementary role. Specifically, they help to distinguish those features of case studies that are truly novel from others that ought to have been anticipated from theoretical value-of-information studies based on decision making under uncertainty.

Appendix

Stochastic dynamic programming

Finite-horizon dynamic cost–loss ratio model. The stochastic dynamic programming recursion for the minimal total expected expense $E_n(q)$ for the imperfect weather forecasting system is

$$\begin{aligned} E_n(q) = p_\Theta \min\{C + E_{n-1}(q),\ p_1 L + (1-p_1)E_{n-1}(q)\} \\ +\ (1-p_\Theta)\min\{C + E_{n-1}(q),\ p_0 L + (1-p_0)E_{n-1}(q)\}, \end{aligned} \tag{6.A1}$$

$n = 1,\ 2,\ \ldots\ ;\ 0 \leq q \leq 1$, with the convention that $E_0(q) = 0$ (Murphy et al., 1985). The conditional probabilities, p_0 and p_1, that appear in equation (6.A1) can be expressed as functions of the forecast quality q through equations (6.5) and (6.6). The first term in curly brackets on the right-hand side of equation (6.A1) represents the minimal total expected expense over the n occasions when adverse weather is forecast (i.e., $Z = 1$) on the first of the n occasions, whereas the second term in curly brackets represents the corresponding expense for the case of a forecast of no adverse weather (i.e., $Z = 0$) on the first occasion.

Infinite-horizon, discounted dynamic cost–loss ratio model. The stochastic dynamic programming recursion for the total, discounted expected expense $E(q)$ for the imperfect weather forecasting system is

$$\begin{aligned} E(q) = p_\Theta \min\{C + \alpha E(q),\ p_1 L + (1-p_1)\alpha E(q)\} \\ +\ (1-p_\Theta)\min\{C + \alpha E(q),\ p_0 L + (1-p_0)\alpha E(q)\}, \end{aligned} \tag{6.A2}$$

$0 \leq q \leq 1$ (Katz and Murphy, 1990). Again, p_0 and p_1 are related to q through equations (6.5) and (6.6). The first term in curly brackets on the right-hand side of equation (6.A2) represents the total discounted expected expense given a forecast of adverse weather (i.e., $Z = 1$) on the initial occasion, whereas the second term in curly brackets represents the corresponding expense for the case of a forecast of no adverse weather (i.e., $Z = 0$) on the initial occasion.

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