Optimality With Hydropower System

Jussi Keppo, *Member, IEEE*

***Abstract*—This paper studies an electricity producer's long-term optimality in the case of multireservoir hydropower system. The model solves the optimal production process and trading strategy of electricity and weather derivatives by maximizing the utility from production and terminal water reservoir level. The optimal trading strategy hedges the rainfall and electricity price uncertainties.**

***Index Terms*—Derivatives, hedging, hydropower, production optimization.**

I. Introduction

The PURPOSE of this paper is to calculate the optimal long-term electricity production and trading strategy that hedges the production uncertainties in the case of hydropower system with stochastic water inflows. Traditionally hydropower system scheduling is based on the minimization of operational costs. In this paper, the problem is solved by employing the financial theory of optimal consumption and portfolio selection. In the hydropower system, there are multiple reservoirs that are connected. Thus, each reservoir has its own optimal production process and these processes depend on each other.

If electricity in the market is mainly produced by using hydropower systems, then the rainfalls affect the electricity price because water reservoirs can be understood as energy stocks. For instance, the lack of water increases the electricity spot price and the future prices of electricity future contracts. Therefore, by trading the electricity derivatives the uncertainties in water inflows can be hedged and in this case there is no need to use weather derivatives. Fig. 1 illustrates the correlation between the electricity spot price and reservoir content in Norway. The correlation is -0.7 for the time period February 1997–February 1999. In Norway, about 99% of the electricity is produced by hydropower systems, which explains the high negative correlation.

If the correlation between spot price and reservoir content was equal to -1, then the energy companies would be exposed to little if any rainfall uncertainty. This is due to the fact that during dry seasons, electricity production would be low but the electricity price high, and during wet periods, the production would be high but the price low. Therefore, the cash flows would be almost constant and the situation would be similar to a monopoly market where the producer can roll its increased production costs into prices. In practice there does not exist this kind of automatic hedging, because the correlation significantly differs from -1. Thus, energy companies that use hydropower systems are exposed to rainfall uncertainty, a risk that can be hedged using weather derivatives (for the pricing of weather derivatives see [1]).

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The author is with the Department of Industrial and Operations Engineering, University of Michigan, Ann Arbor, MI 48109-2117 USA.

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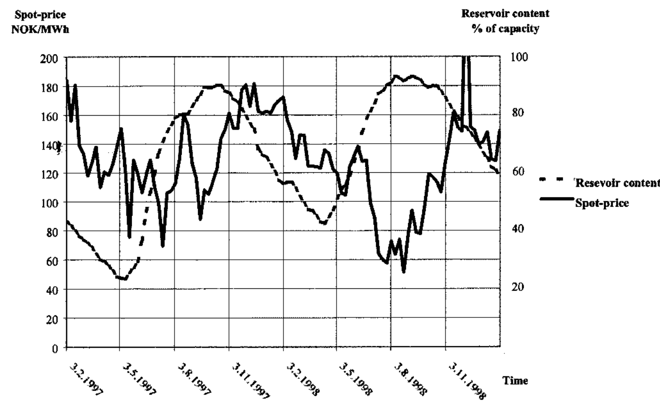

Fig. 1. Electricity spot price and reservoir content in Norway during February 1997–February 1999.

Many other papers have studied the single agent's optimality. Merton [2], [3], Cox and Huang [4], and Karatzas *et al.* [5] have solved the optimal portfolio and consumption choice problem in complete financial markets. The optimization problem in the case of electricity markets is studied, for example, in [6]–[8]. The model presented in [7] can be applied to production scheduling but not directly to the case of hydropower system. Gjelsvik *et al.* [9] considered the long-term generation scheduling of hydropower system by a stochastic dynamic programming. Pereira [10] introduced an approach called *stochastic dual dynamic programming* that can be used with multireservoir system. The main idea in Pereira's method is that in the Bellman equation the cost-to-go function can be represented by a set of hyperplanes. In the calculation of the optimal production we use a different approach based on a martingale method and on a different objective function. The advantage of the martingale method is that the production and hedging problems separate, because in our model, if a production strategy satisfies the given budget equation then there exists a hedging strategy for that production. Furthermore, in contrast to [10], our framework results in a two-dimensional (2-D) dynamic programming problem because with each reservoir we have usual stochastic control problem, which is solved backward in the reservoir chain starting from the lowest reservoir and ending at the highest reservoir.

The rest of the paper is organized as follows: Section II defines the framework used in the paper and Section III derives the optimal strategies of a single producer. Section IV illustrates the model with an example and Section V concludes.

II. Model

We explore an economy where financial instruments, e.g., electricity and weather derivatives, are traded continuously within a finite time horizon $[0, T]$. In describing the probabilistic structure of the economy, we refer to an underlying probability space (Ω, F, P), along with the standard filtration $\{F_t: t \in [0, T]\}$. Here, Ω is a set, F is a σ-algebra of subsets of Ω generated by an n-dimensional Brownian motion $B = (B_1, \ldots, B_n)$, and P is a probability measure on F.

We consider an electricity producer whose actions cannot affect the market prices, i.e., its production and trading strategy of financial instruments do not change the market prices that are assumed to be exogenous variables. This proves the fact that, in this paper, we do not consider game models and we can study a single agent's optimality without analyzing the behavior of the other market participants. The energy producer of this paper is defined by nonzero instantaneous water endowment processes $\{e_1, \ldots, e_m\}$ and the corresponding initial wealth $\{x_1, \ldots, x_m\}$, where m is the number of water reservoirs. The endowment process e_i refers to water inflow to reservoir $i \in \{1, \ldots, m\}$ due to rain and x_i is the corresponding initial water amount in reservoir $i \in \{1, \ldots, m\}$. Both these variables are measured in energy units and, therefore, they reflect the water volume and the conversion factor of reservoir i. The reservoirs are connected to each other such that the total endowment process of a reservoir $i \in \{1, \ldots, m\}$ is $e_i + p_{i-1}$, where p_i is the instantaneous electricity production with reservoir $i \in \{1, \ldots, m\}$ and p_0 is a production process that equals zero. Thus, the conversion factors of all the reservoirs are the same and the amount that is produced with the upper reservoir is a part of the next reservoir's total endowment. This assumption of constant conversion factors is made in order to simplify the presentation and one can take the differing factors into account by defining a new multiplier in the calculation of the total endowment. The serial connection assumption can be extended to consider more complex structures by dividing the initial reservoir structure into several simple serial structures.

The following assumptions characterize our economy in more detail.

Assumption 1: The stochastic variables of the economy follow an Itô stochastic differential equation

$$dS(t) = S(t)\alpha_S(t)\,dt + S(t)\sigma_S(t)\,dB(t) \qquad \text{for all} \quad t \in [0, T] \quad (1)$$

where the initial value $S(0) \in \mathbf{R}_+$, at time t the drift $\alpha(t) \in \mathbf{R}$ and the volatility $\sigma(t) \in \mathbf{R}^n$ and these variables are deterministic and bounded, $B(\cdot)$ is an n-dimensional Brownian motion on the probability space (Ω, F, P), along with the standard filtration $\{F_t: t \in [0, T]\}$.

The drift and volatility parameters are assumed to be deterministic but they are functions of time. We assume that there exist $n+1$ financial instruments in the market. Because our budget equation will be in energy units we divide the financial instrument prices by electricity spot price. That is, from now on all the variables are described in energy units and, therefore, we can hedge the uncertainties in the production processes and terminal water levels by using these real financial instruments.

The real risk-free asset is the water level of a reservoir without production and water inflow and, therefore, the real risk-free rate is assumed to be zero. The remaining n securities are risky real assets. They correspond to nominal risk-free asset and different maturity electricity and weather derivatives. Their real price processes are given as follows

$$dS_k(t) = S_k(t)\alpha_k(t)\,dt + S_k(t)\sigma_k(t)\,dB(t) \qquad \text{for all} \quad k \in \{1, \ldots, n\} \quad (2)$$

where the price S_k is in energy units. Because, as illustrated in Fig. 1, there exist huge cycles in the electricity price, the drift and volatility parameters in (2) are not constant. For the modeling and estimation of electricity and electricity forward price processes, see [11]–[14].

Assumption 1 implies also that the rainfall processes are given by the Itô stochastic differential equation and we assume that the instantaneous rainfall corresponding to reservoir i is given as follows:

$$de_i(t) = e_i(t)\mu_i(t)\,dt + e_i(t)\nu_i(t)\,dB(t) \qquad \text{for all} \quad i \in \{1, \ldots, m\} \quad (3)$$

where the rainfall is measured in energy units. Note that our real risk-free asset is a water reservoir level without the stochastic endowment and, thus, the endowment process makes the water level risky. However, by trading the real risky assets, we are able to hedge the uncertainties in the water reservoir levels.

Assumption 2: Financial markets are complete and there is no arbitrage.

Assumption 2 implies that the following $n \times n$-dimensional matrix:

$$\sigma(t) = \begin{bmatrix} \sigma_1(t) \\ \vdots \\ \sigma_n(t) \end{bmatrix} \quad (4)$$

is nonsingular almost surely for all $t \in [0, T]$. Given (4), there exists a unique state-price deflator (see [15])

$$H(t) = \exp\left(-\int_0^t \theta(s)^\tau\,dB(s) - \int_0^t \tfrac{1}{2}\,\theta(s)^\tau\theta(s)\,ds\right) \qquad \text{for all} \quad t \in [0, T] \quad (5)$$

where $\theta(t) = \sigma^{-1}(t)(\alpha(t) - r\mathrm{I})$, $\mathrm{I} = [1 \cdots 1]^\tau$, $\alpha(t) = [\alpha_1(t) \cdots \alpha_n(t)]^\tau$, and θ^τ is the transpose of θ. We will refer to θ as the market price of risk and it satisfies $E[\int_0^T \theta(t)^\tau\theta(t)\,dt] < \infty$.

The electricity producer of this paper desires to maximize its utility from electricity production and from terminal water reservoir. A utility function is concave, nondecreasing, and upper semicontinuous function $U : \mathbf{R} \to [-\infty, \infty)$. The function is continuous on the half-line $dom(U) = \{y \in \mathbf{R} | U(y) > -\infty\}$ that is a nonempty subset of $[0, \infty)$. $U'(y) = (\partial U(y))/\partial y$ is continuous, positive, and strictly decreasing on the interior of $dom(U)$, $U'(\infty) = \lim_{y\to\infty} U'(y) = 0$ and $U'(\overline{y}+) = \lim_{y\downarrow\overline{y}} U'(y) \in (0, \infty]$, where $\overline{y} = \inf\{y \in \mathbf{R} | U(y) > -\infty\}$ and $\overline{y} \in [0, \infty)$. Thus, $dom(U) = [\overline{y}, \infty)$ or $dom(U) = (\overline{y}, \infty)$.

Assumption 3: The objective function of the agent is to maximize the utility from electricity production and from terminal reservoir level in the following way:

$$\sum_{i\in\{1,\ldots,m\}} E\left[\int_0^T S(t)u_i(p_i(t),\,t)\,dt + S(T)U_i(X_i(T))\right] \quad (6)$$

where $S(\cdot)$ is the electricity spot price, $p_i(\cdot)$ is an adapted nonnegative instantaneous production of reservoir $i \in \{1,\ldots,m\}$, $u_i(\cdot,t)$: $\mathbf{R} \rightarrow [-\infty,\infty)$ is a utility function for all $t \in [0,T]$ and $i \in \{1,\ldots,m\}$ with subsistence production $\overline{p}_i(t) = \inf\{p \in \mathbf{R} | u_i(p,t) > -\infty\}$, and U_i is a utility function for all $i \in \{1,\ldots,m\}$ with subsistence terminal water reservoir defined by $\overline{x}_i = \inf\{x \in \mathbf{R} | U_i(x) > -\infty\}$.

Assumption 3 implies that the agent does not only consider the amount of sold electricity during $[0,T]$, but also the electricity that is sold after this period. Therefore, the producer usually does not convert all its water reservoirs into the production of electricity on $[0,T]$. The utility functions are multiplied by the electricity price in order to get high marginal utility when the electricity price is high and low when the electricity price is low. The production lower bound of reservoir $i \in \{1,\ldots,m\}$ corresponds to the physical size of the reservoir since the reservoir level can be modeled as follows:

$$X_i(t+\Delta t) = X(t) + [e_i(t) + p_{i-1}(t) - p_i(t)]\Delta t \le \hat{x}_i \quad (7)$$

where $\hat{x}_i$ is the upper bound of the reservoir i and $\hat{x}_i > \overline{x}_i$. We have the reservoir upper boundary $\hat{x}_i$ because of the physical size of the reservoir and the lower boundary $\overline{x}_i$ because of environmental reasons. Equation (7) gives

$$\max\left\{\frac{X_i(t) + [e_i(t) + p_{i-1}(t)]\Delta t - \hat{x}_i}{\Delta t},\, 0\right\} \le p_i(t) \quad \text{for all} \quad t \in [0,T],\, i \in \{1,\ldots,m\}. \quad (8)$$

The maximum is in the equation because the production cannot be negative and the left-hand side (LHS) is strictly positive if the expected reservoir level is higher than the upper bound. The LHS of (8) is the production lower boundary $\overline{p}_i(t)$.

Assumption 4: The electricity productions and the terminal water reservoirs cannot exceed the given upper boundaries.

We denote the production upper bound of reservoir $i \in \{1,\ldots,m\}$ by $\hat{p}_i$, where $\hat{p}_i(t) > \overline{p}_i(t)$ for all $t \in [0,T]$. From *Assumption 3* and (7) we get

$$\min\left\{\frac{X_i(t) + [e_i(t) + p_{i-1}(t)]\Delta t - \overline{x}_i}{\Delta t},\, p_i^{\max}\right\} \ge p_i(t) \quad \text{for all} \quad t \in [0,T],\, i \in \{1,\ldots,m\} \quad (9)$$

where $p_i^{\max}$ is the maximum production due to the production capacity of reservoir $i \in \{1,\ldots,m\}$. We have the minimum in (9) because the production cannot exceed the maximum production capacity. The LHS of (9) is the production upper boundary $\hat{p}_i(t)$.

III. Single Agent's Optimality

In this section, we consider the single producer's optimality. The agent has to solve the optimal production processes, terminal reservoir levels, and the portfolio strategy of financial instruments that hedges the uncertainties in the production processes and terminal reservoir levels.

Given the initial water reservoir x_i, the instantaneous inflow e_i and production process of the upper reservoir p_{i-1} there exists a portfolio strategy that finances an adapted production process p_i if

$$\begin{aligned} H(t)X_i(t) &= x_i + \int_0^t H(s)[e_i(s) + p_{i-1}(s) - p_i(s)]\,ds \\ &\quad + \int_0^t H(s)(\sigma(s)^\tau \Pi_i(s) - X_i(s)\theta(s))^\tau\, dB(s) \\ &\quad \text{for all} \quad t \in [0,T] \end{aligned} \quad (10)$$

where $X(\cdot) = \sum_{i\in\{1,\ldots,m\}} X_i(\cdot)$, $p_0(t) = 0$ for all t, $\Pi_i(t) = [\pi_i^1(t)S_1(t)\cdots\pi_i^n(t)S_n(t)]$, and $\pi_i^k(t)$ is the number of financial instruments $k \in \{1,\ldots,n\}$ in the portfolio corresponding to reservoir $i \in \{1,\ldots,m\}$ at time $t \in [0,T]$. We have the representation of (10) because all the prices are deflated by the electricity price, markets are complete, and because the total endowment process of a reservoir $i \in \{1,\ldots,m\}$ is $e_i + p_{i-1}$. Since all the prices are deflated by the electricity price, the prices of financial instruments, water inflows, and productions are measured in energy units. The first integral of (10) is the discounted total endowment minus the production and the second integral is the discounted gains and losses from the financial instruments.

We write $(p,\pi) \in A(e,x)$, where $p = [p_1\cdots p_m]$, $\pi = \sum_{i\in\{1,\ldots,m\}} \pi_i$, $\pi_i = [\pi_i^1\cdots\pi_i^n]$, $e = [e_1\cdots e_m]$, and $x = \sum_{i\in\{1,\ldots,m\}} x_i$, if the wealth process $X_i^{e,x,p,\pi}(\cdot)$ corresponding to e, x, p, π satisfies

$$X_i^{e,x,p,\pi}(T) \in [\overline{x}_i,\, \hat{x}_i] \quad \text{for all} \quad i \in \{1,\ldots,m\} \quad (11)$$

$$E[\min\{0,\, S(T)U_i(X^{e_i,x_i,p_i,\pi_i}(T))\}] > -\infty \quad \text{for all} \quad i \in \{1,\ldots,m\} \quad (12)$$

and if the production strategy $p_i(\cdot)$ satisfies

$$p_i(t) \in [\overline{p}_i(t),\, \hat{p}_i(t)] \quad \text{for all} \quad t \in [0,T],\, i \in \{1,\ldots,m\} \quad (13)$$

$$E\left[\int_0^T \min\{0,\, S(t)u_i(p_i(t),\,t)\}\,dt\right] > -\infty \quad \text{for all} \quad i \in \{1,\ldots,m\}. \quad (14)$$

Equations (11) and (13) imply that if $X_i(T) \notin [\overline{x}_i,\,\hat{x}_i]$ or $p_i(t) \notin [\overline{p}_i(t),\,\hat{p}_i(t)]$ for all production and trading strategies, then $A(e,x) = \varnothing$ and we set the supremum over the empty set equal to $-\infty$. According to (12) and (14), the preference structure forces $p_i(t) \ge \overline{p}_i(t)$ and $X_i(T) \ge \overline{x}_i$. We denote by $A_i(e_i + p_{i-1},\, x_i)$ the set of production and trading strategies corresponding to reservoir $i \in \{1,\ldots,m\}$, i.e., the strategies that satisfy (11)–(14) for the reservoir i.

According to *Assumption 3*, the electricity producer maximizes the utility from production and terminal reservoir level by selecting optimal production strategies, terminal reservoir levels, and portfolio process of financial instruments. That is

$$V(e, x) = \sup_{(p, \pi)\in A(e, x)} \sum_{i\in\{1, \ldots, m\}} \cdot E\left[\int_0^T S(t)u_i(p_i(t), t)\, dt + S(T)U_i(X_i(T))\right] \quad (15)$$

subject to the budget constraint

$$E\left[\int_0^T H(t)p_i(t)\, dt + H(T)X_i(T)\right] = x_i + E\left[\int_0^T H(t)[e_i(t) + p_{i-1}(t)]\, dt\right] \quad \text{for all} \quad i \in \{1, \ldots, m\}. \quad (16)$$

Equation (16) is derived from (10) by integrating from initial time to terminal time. Because of the chain structure of the water reservoirs the corresponding subproblem of reservoir $i \in \{1, \ldots, m\}$ is

$$V_i(e_i + p_{i-1}, x_i) = \sup_{(p_i, \pi_i)\in A_i(e_i+p_{i-1}, x_i)} \cdot \left\{E\left[\int_0^T S(t)u_i(p_i(t), t)\, dt + S(T)U_i(X_i(T))\right] + V_{i+1}(e_{i+1} + p_i, x_{i+1})\right\} \quad (17)$$

subject to

$$E\left[\int_0^T H(t)p_i(t)\, dt + H(T)X_i(T)\right] = x_i + E\left[\int_0^T H(t)[e_i(t) + p_{i-1}(t)]\, dt\right]. \quad (18)$$

Remark: Equations (15)–(18) imply that the problem of the electricity producer has to be solved backward in the following way. First, V_m and the corresponding production and trading strategies are solved. These solutions depend on p_{m-1}. Second, the combined problem of V_m and V_{m-1} is calculated because V_m is a function of p_{m-1}. This gives the optimal production and trading strategies corresponding to reservoir $m-1$. These solutions depend on p_{m-2}. Repeating this many times, finally the combined problem of V_1 and V_2 is solved. This leads to the solution of the whole problem. Because in each reservoir we have a dynamic optimization problem and these single problems are solved backward in the reservoir chain, the described solution technique can be viewed as a 2-D dynamic programming.

Now we can state the main theorem of this paper.

Theorem 1: Suppose that $V(e, x) < \infty$ then the optimal production process is given as follows:

$$p_i^*(\lambda_i, t) = \min\left\{y_i\left(\lambda_i \frac{H(t)}{S(t)}, t\right), \hat{p}_i(t)\right\} \quad \text{for all} \quad t \in [0, T], i \in \{1, \ldots, m\} \quad (19)$$

and the terminal wealth

$$X_i^*(\lambda_i, T) = \min\left\{Y_i\left(\lambda_i \frac{H(T)}{S(T)}\right), \hat{x}_i\right\} \quad \text{for all} \quad i \in \{1, \ldots, m\} \quad (20)$$

where $y_i(\cdot, t)$ inverts

$$\frac{\partial u_i(\cdot, t)}{\partial p_i} + \frac{\partial V_{i+1}(e_{i+1} + \cdot, x_{i+1})}{\partial p_i}$$

meaning that

$$y_i\left(\frac{\partial u_i(s, t)}{\partial p_i} + \frac{\partial V_{i+1}(e_{i+1} + s, x_{i+1})}{\partial p_i}, t\right) = s$$

for all s and t, and $V_{m+1}(e_{m+1} + p_m, x_{m+1}) = 0$. $Y_i(\cdot)$ inverts $(\partial U_i(\cdot))/\partial X_i$ and $\lambda_i > 0$ is a Lagrange multiplier satisfying

$$E\left[\int_0^T H(t)p_i^*(\lambda_i, t)\, dt + H(T)X_i^*(\lambda_i, T)\right] = x_i + E\left[\int_0^T H(t)[e_i(t) + p_{i-1}^*(\lambda_{i-1}, t)]\, dt\right] \quad \text{for all} \quad i \in \{1, \ldots, m\}. \quad (21)$$

The optimal asset holding of instrument $k \in \{1, \ldots, n\}$ is

$$\frac{\Pi_k^*(t)}{S_k(t)}$$

where

$$\Pi^*(t) = \sigma^{-1}(t)^\tau \cdot \left[\theta(t) \sum_{i\in\{1, \ldots, m\}} X_i(t) + \frac{1}{H(t)} \sum_{i\in\{1, \ldots, m\}} \psi_i(t)\right] \quad (22)$$

and $\psi_i(\cdot)$ is given by

$$E\left(\int_0^T H(s)[p_i^*(\lambda_i, s) - e_i(s) - p_{i-1}^*(\lambda_{i-1}, s)]\, ds + H(T)X_i^*(\lambda_i, T)|F_t\right) = x_i + \int_0^t \psi_i(s)^\tau\, dB(s). \quad (23)$$

Proof: By the saddle point theorem (see [16]) and strict monotonity of u_i and U_i, the optimal production process and terminal wealth solve the unconstrained problem

$$\sup E\left[\int_0^T S(t)u_i(p_i(t),\, t)\,dt + V_{i+1}(e_{i+1}+p_i,\, x_{i+1}) + S(T)U_i(X_i(T))\right] - \lambda_i E\left\{\int_0^T H(t)[p_i(t) - e_i(t) - p_{i-1}^*(\lambda_{i-1},\, t)]\,dt - x_i + H(T)X_i(T)\right\} \tag{24}$$

where $p_{i-1}^*(\lambda_{i-1},\, \cdot)$ is the optimal production of reservoir $i-1$, $V_{m+1}(e_{m+1}+p_m,\, x_{m+1}) = 0$, $p_i(\cdot) \le \hat{p}_i(\cdot)$, and $X_i(T) \le \hat{x}_i$ for all $i \in \{1,\, \ldots,\, m\}$. From (24) and *Assumptions 3* and *4* we get (19) and (20). There exists a Lagrange multiplier such that

$$\Theta(\lambda_i) = E\left\{\int_0^T H(t)p_i^*(\lambda_i,\, t)\,dt + H(T)X_i^*(\lambda_i,\, T)\right\} = x_i + E\left\{\int_0^T H(t)[e_i(t) + p_{i-1}^*(\lambda_{i-1},\, t)]\,dt\right\}$$

holds since $\Theta(\cdot)$ is continuous and decreasing with

$$\Theta(0+) = E\left\{\int_0^T H(t)\hat{p}_i(t)\,dt + H(T)\hat{x}_i\right\}$$

and

$$\Theta(\infty) = E\left\{\int_0^T H(t)\overline{p}_i(t)\,dt + H(T)\overline{x}_i\right\}$$

and since we assume that $A_i(e_i + p_{i-1},\, x_i) \ne \emptyset$. This gives the optimal production process and terminal wealth.

From (10) and (23), we get (22). This gives the optimal asset holding. *Q.E.D.*

Remark: Theorem 1 implies that the job of the Lagrange multiplier is to set the optimal production and terminal wealth into such levels that the budget constraint, (21), is satisfied. Given the optimal production and terminal wealth the optimal portfolio process of financial instruments, (22), eliminates the uncertainties associated with rainfall and electricity price. This means that if the energy company makes unpredictable production losses it will win the same amount in the financial market, and in the same way if it gets unpredictable production gains it will lose in the financial markets. Therefore, the total gains are constant.

The budget equation dynamics, (23), is derived from (1), (3), (5), (19), and (20), and by using the martingale representation theorem (see [17]). Thus, the volatility ψ_i corresponds to the volatilities of electricity spot price, state-price deflator, and rainfall.

In practice, Theorem 1 cannot be applied directly and there are a few steps that the energy company has to go through before the optimal solutions are obtained. The following list illustrates the implementation of the model.

1) Production model:
   a) selection of the planning horizon, $[0,\, T]$;
   b) identification of objective function, *Assumption 3*;
   c) description of the reservoirs' serial connections;
   d) estimation of rainfall models, (3);
   e) calculation of production upper and lower boundaries, (8) and (9).
2) Financial market model:
   a) selection of the set of financial instruments, i.e., the set $\{1,\, \ldots,\, n\}$;
   b) estimation of the price processes, (2), and calculation of the state-price deflator, (5).
3) Optimization:
   a) calculation of optimal production process and terminal wealth for each reservoir, Theorem 1;
   b) calculation of optimal portfolio process of financial instruments, Theorem 1.

As can be seen from the list, the main problem in the implementation is the production modeling. Due to the seasonal effects in the electricity spot price and weather derivative prices the real price processes include cycles. Therefore, also the parameter estimation of the financial models is demanding. The optimization means employing of the production and financial models as described in Theorem 1.

IV. Example

In this section, we illustrate our framework with a simple example. Let $T = 1344$ h, $\theta = 0$, $m = 2$, and $n = 2$, i.e., the planning horizon is eight weeks, the market price of risk is equal to zero, there are two water reservoirs and two independent sources of uncertainties. The first Brownian motion refers to the rainfall uncertainty and the second to the electricity price uncertainty. We assume that the cumulative rainfalls are given as follows:

$$G(t) = E\left[\int_0^{1344} e_1(s)\,ds \,\middle|\, F_t\right] = E\left[\int_0^{1344} e_2(s)\,ds \,\middle|\, F_t\right] = 10^6 + \int_0^t (s-1344)\cdot 100\cdot dB_1(s) \quad \text{for all} \quad t \in [0,\, 1344] \tag{25}$$

i.e., $e_1(t) = e_2(t)$ for all t, the expected cumulative energy is equal to 10^6 MWh with both the reservoirs, and the cumulative rainfall volatility decreases with time.

In the market there exist weather derivatives that can be used to hedge the uncertainties in the rainfall. The payoff of the 1344-maturity weather forward is given as follows:

$$G(1344) - E[G(1344)|F_t] = \int_t^{1344} (s-1344)\cdot 100\cdot dB_1(s) \tag{26}$$

where time t is assumed to be the purchase time of the forward. Comparing (25) and (26), we can realize that the weather uncertainties can be hedged by trading the forward. Using the notation of (2), we get $S_1(t)\sigma_1(t) = (t-1344)\cdot 100$ for all $t \in [0,\, 1344]$.

For simplicity, we assume that there are no lower and upper boundaries for the production and terminal reservoir level. Both the initial reservoir levels are 10^5 MWh. The utility function for production is defined as

$$u_i(c, t) = \log(c) \quad \text{for all} \quad t \in [0, 1344], i \in \{1, 2\} \tag{27}$$

and therefore $y_1(x, t) = y_2(x, t) = 1/x$. This utility function satisfies the conditions of *Assumption 3*. Furthermore, we assume that the utility from terminal water reservoirs is equal to zero, which gives the fact that the terminal levels are equal to zero.

The cumulative electricity uncertainty is given as follows:

$$\begin{aligned} F(t) &= E\left[\int_0^{1344} S(y)\,dy \middle| F_t\right] \\ &= 4 \cdot 10^4 + \int_0^t (1344 - s) \cdot dB_2(s) \\ &\qquad \text{for all} \quad t \in [0, 1344] \end{aligned} \tag{28}$$

where $S(t)$ is the electricity spot price at time t.

We assume that there exist 1344-maturity forwards on F. These forwards can be used in the hedging of the electricity price uncertainty and their payoff is given as follows:

$$E[F(1344)|F_t] - F(1344) = \int_t^{1344} (s - 1344)\,dB_2(s) \tag{29}$$

where time t is assumed to be the purchase time of the forward. From (28) and (29), we see that the forward contracts can be used in the hedging of the electricity uncertainty. Equation (29) implies that the volatility process of the second risky asset is given by $S_2(t)\sigma_2(t) = (t - 1344)$ for all $t \in [0, 1344]$. Note that the correlation between this asset and electricity spot price is -1 and, therefore, the second real financial asset can be viewed as a nominal risk-free asset (or portfolio of risk-free assets). This is because the real process of the nominal risk-free asset is stochastic and negatively correlating with the spot price.

From Theorem 1, we now get $p_i^*(\lambda_i, t) = (1/\lambda_i)S(t)$, because $y_i(x, t) = 1/x$ and because due to the zero market price of risk the state price deflator $H(t) = 1$ for all t. Since the total endowment of reservoir 1 is equal to the rainfall by using the budget equation of Theorem 1 we get

$$\lambda_1 = \frac{40\,000}{10^5 + 10^6} \tag{30}$$

which implies $p_1^*(\lambda_1, t) = 27.5 \cdot S(t)$. According to (28) the expected spot price is close to \$30/MWh and, therefore, the expected production is close to 825 MW. Furthermore, the higher the electricity spot price the higher the production.

With the lower reservoir the total endowment is equal to the rainfall plus the production of reservoir 1. Therefore, we get from the budget equation of Theorem 1

$$\lambda_2 = \frac{20\,000}{10^5 + 10^6} \tag{31}$$

which gives $p_2^*(\lambda_2, t) = 55 \cdot S(t)$. Thus, the production in the lower reservoir is double of that in the higher reservoir. This is because the production of the higher reservoir is part of the lower reservoir's endowment.

From Theorem 1, we get that the dynamics of reservoir 1's budget equation is given as follows:

$$\begin{aligned} &E\left(\int_0^T (p_1^*(\lambda_1, s) - e_1(s))\,ds \middle| F_t\right) \\ &\quad = x_1 - \int_0^t (s - 1344) \cdot 100 \cdot dB_1(s) \\ &\qquad + \int_0^t 27.5 \cdot (1344 - s) \cdot dB_2(s) \end{aligned} \tag{32}$$

and the dynamics of reservoir 2's budget equation is given by

$$\begin{aligned} &E\left(\int_0^T [p_2^*(\lambda_i, s) - e_2(s) - p_1^*(\lambda_{i-1}, s)]\,ds \middle| F_t\right) \\ &\quad = x_2 - \int_0^t (s - 1344) \cdot 100 \cdot dB_1(s) \\ &\qquad + \int_0^t 27.5 \cdot (1344 - s) \cdot dB_2(s). \end{aligned} \tag{33}$$

Now from Theorem 1 and (26), (29), (32), and (33), we get that the hedging strategy satisfies

$$\begin{aligned} &-\int_0^t (s - 1344) \cdot 200 \cdot dB_1(s) + \int_0^t 55 \cdot (1344 - s) \cdot dB_2(s) \\ &\quad = \int_0^t \pi_1(s)(s - 1344) \cdot 100 \cdot dB_1(s) \\ &\qquad + \int_0^t \pi_2(s)(s - 1344)\,dB_2(s) \end{aligned} \tag{34}$$

which proves that the optimal asset holdings are $\pi_1 = -2$ and $\pi_2 = -55$. Thus, in our example, by selling two real weather derivatives and 55 real electricity assets, the company can hedge its electricity price and rainfall uncertainties. The weather derivatives hedge the unpredictable movements in the cumulative rainfall amount, and the real electricity instruments, that can be seen as nominal risk-free assets, hedge the changes in the optimal production due to the changes in the cumulative electricity price.

V. Summary

In this paper, we derived a new optimization model for a multireservoir hydropower system. The objective of the electricity company was to maximize the expected utility of production over the planning horizon and the expected utility of water reservoir at the end of the horizon. The producer acted also in the financial market, and by trading financial instruments the agent was able to hedge the rainfall and electricity price uncertainties.

The implementation of the model into everyday industry practice can be divided into three parts: 1) production modeling; 2) financial market modeling; and 3) optimization. The most difficult part is the production modeling. The framework was illustrated with an example that gives explicit solutions for production and trading strategies.

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Jussi Keppo (M'00) received the M.Sc. degree in industrial economics in 1995, and the Dr.Tech. degree in applied mathematics in 1998, both from Helsinki University of Technology, Helsinki, Finland.

He continued as a Postdoctoral Fellow at the Department of Statistics at Columbia University, New York, NY, where he conducted research on financial engineering. In 2001, he started as an Assistant Professor at the Department of Industrial and Operations Engineering in University of Michigan, Ann Arbor. Currently, he is working on pricing of bandwidth instruments and on applying results from the game theory to determine optimal investment entry times and hedging strategies for large companies.