Thèse de Doctorat

présentée par

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sous la direction de Denis Benois
et co-direction de Nicola Mazzari

pour obtenir le grade de

Docteur de L'Université de Bordeaux

École Doctorale de Mathématiques et Informatique

Spécialité : Théorie des Nombres

Finite height representations and syntomic complex

Soutenance prévue le 15 Novembre 2021 à l'Institut de Mathématiques de Bordeaux
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To my parents.

Finite height representations and syntomic complex

Abstract: The aim of this thesis is to study finite height crystalline representations in relative p-adic Hodge theory, and apply the results thus obtained towards the computation of continuous Galois cohomology of these representations via syntomic methods.

In 1980's, Fontaine initiated a program for classifying p-adic representations of the absolute Galois group of a p-adic local field by means of certain linear-algebraic objects functorially attached to the representations. One of the aspects of his program was to classify all p-adic representations of the Galois group in terms of étale (φ, Γ)-modules. On the other hand, Fontaine showed that crystalline representations can be classified in terms of filtered φ-modules. Therefore, it is a natural question to ask for crystalline representations: Does there exist some direct relation between the filtered φ-module and the étale (φ, Γ)-module? Fontaine explored this question himself, where he considered finite height represenations (defined in terms of (φ, Γ)-modules) and examined their relationship with crystalline representations. This line of thought was further explored by Wach, Colmez, and Berger. In particular, Wach gave a description of finite height crystalline representations in terms of (φ, Γ)-modules.

In the relative case, the theory of (φ, Γ)-modules has been developed by the works of Andreatta, Brinon and Iovita. Further, the analogous notion of crystalline representations was studied by Brinon.

The first main contribution of our work is the notion of relative Wach modules. Motivated by the theory of Fontaine, Wach and Berger, we define and study some properties of relative Wach modules. Further, we explore their relation with Brinon's theory of relative crystalline representations and associated F-isocrystals.

The second result is concerned with the computation of Galois cohomology using syntomic complex with coefficients. This idea was utilized in a recent work of Colmez and Nizioł, where they carry out the computation for cyclotomic twists of the trivial representation. Under certain technical assumptions, we show that for finite height crystalline representations, one can essentially generalize the local result of Colmez and Nizioł.

Keywords: p-adic Hodge theory, p-adic representations, (φ, Γ)-modules, finite height, Wach modules, syntomic complex.

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Représentations de hauteur finie et complexe syntomique

Résumé : Le but de cette thèse est d'étudier les représentations cristallines de hauteur finie en théorie de Hodge p-adique relative, et d'appliquer les résultats ainsi obtenus au calcul de la cohomologie galoisienne continue de telles représentations via des méthodes syntomiques.

Dans les années 1980, Fontaine a lancé un programme pour classer les représentations p-adiques du groupe de Galois absolu d'un corps local p-adique au moyen de certains objets algébriques linéaires attachés fonctoriellement aux représentations. Un aspect de son programme consistait à classer toutes les représentations p-adiques du groupe de Galois en termes de (φ, Γ)-modules étales. D'autre part, Fontaine a montré que les représentations cristallines peuvent être classées en termes de φ-modules filtrés admissibles. Par conséquent, c'est une question naturelle de demander pour des représentations cristallines : existe-t-il une relation directe entre le φ-module filtré et le (φ, Γ)-module étale ? Fontaine a exploré cette question lui-même, où il a considéré les représentations de hauteur finie (définies en termes de (φ, Γ)-modules) et examiné leur relation avec les représentations cristallines. Ce point de vue a été exploré plus avant par Wach, Colmez et Berger. En particulier, Wach a donné une description des représentations cristallines de hauteur finie en termes de (φ, Γ)-modules.

Dans le cas relatif, la théorie des (φ, Γ)-modules a été développée par les travaux d'Andreatta, Brinon et Iovita. De plus, la notion analogue de représentations cristallines a été étudiée par Brinon.

La première contribution de notre travail est la notion de modules de Wach relatifs. Motivés par la théorie de Fontaine, Wach et Berger, nous définissons et étudions quelques propriétés des modules de Wach relatifs. De plus, nous explorons le lien avec la théorie de Brinon des représentations cristallines relatives et F-isocristaux associé.

Le deuxième résultat concerne le calcul de la cohomologie galoisienne à l'aide de complexes syntomiques à coefficients. Cette idée a été utilisée dans un travail récent de Colmez et Nizioł, où ils effectuent le calcul pour les représentations associées aux puissances du caractère cyclotomique. Sous certaines hypothèses techniques, nous montrons que pour des représentations cristallines de hauteur finie, on peut essentiellement généraliser le résultat local de Colmez et Nizioł.

Mots-clés : Théorie de Hodge p-adiques, représentations p-adiques, (φ, Γ)-modules, hauteur finie, modules de Wach, complexe syntomique.

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Acknowledgements

The thesis presented here owes an immense intellectual debt to the mathematical wisdom of my advisor Denis Benois. His generosity with time and ideas, immense patience with my work, and eloquent explanation of several themes in mathematics have played a pivotal role in my understanding and progression. The work in this thesis germinated from our discussions about classical Wach module and Galois cohomology almost two years ago and it would not have evolved to its current form without his constant input.

Further, I am thankful to my co-advisor Nicola Mazzari for his guidance, support and encouragement during my stay in Bordeaux. He introduced me to the world of p-adic Hodge theory during my masters and has continued to guide me through its exploration. His expertise in syntomic cohomology and our enlightening discussions on coefficients have been an indispensable source of motivation for obtaining the results presented here.

Next, I would like to thank the referees Fabrizio Andreatta and Wiesława Nizioł for reading, evaluating and sharing their insights on the results obtained in the thesis. I would also like to thank members of the jury Olivier Brinon, Xavier Caruso, Gabriel Dospinescu and Adriano Marmora for their support with the defense.

Now, I would like to thank my mathematical friends in Bordeaux: Yulin Cai, Thuong Dang, Jean Kieffer, Emauele Tron and Luming Zhao with whom I had the pleasure of discussing several mathematical ideas. Moreover, the help provided by Emanuele on numerous occasions while dealing with French bureaucratic systems cannot be understated.

My stay in Bordeaux was made easier with the help and support of friends and colleagues. I would like to thank Jared, Pierre, Quentin and Paul for the conversations and laughters we shared during lunch. Moreover, I am thankful to Dimitrios, Rolando, Yiye, Martin, Paul, Pei, Thanh, and Badreddine for their support and encouragement during these three years. Also, many thanks to my friend Maneesha for her kindness and support during the pandemic lockdowns. Lastly, I am thankful to Gaston for his support and cheers during these three years, as well as, many meals and conversations that I shared with him, his wife Monica and their wonderful cat Culebra.

Next, I would like to thank my friends Mohit and Sunil in India who have been supportive of me for over a decade and with whom I can always share a laugh, even under difficult circumstances. Further, I am thankful for the support and encouragement that I have received from several friends and colleagues in my life who are now dispersed across the world.

Finally, I would like to thank my parents, my brother and my sister for always believing in me, for their unconditional love, for providing the much needed support in my life, and numerous other things which would take ages to describe.

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Preface

The aim of this thesis is to study finite height crystalline representations in relative p-adic Hodge theory, and apply the results thus obtained towards the computation of continuous Galois cohomology of these representations via syntomic methods.

Our first main contribution is the notion of relative Wach modules. Motivated by the theory of Fontaine [Fon90], Wach [Wac96] and Berger [Ber04], we define and study some properties of relative Wach modules. Further, we explore its relation with relative crystalline representations and the associated F-isocrystal, in the sense of Brinon [Bri08] (see Theorem 3.24).

The second result is concerned with the computation of Galois cohomology using syntomic complex with coefficients. This idea was utilized by Colmez and Nizioł in [CN17] where they carry out the computation for cyclotomic twists of the trivial representation using which they were able to prove the semistable comparison theorem for formal log-schemes. Under certain technical assumptions, we show that for finite height crystalline representations, one can essentially generalize the local result of Colmez and Nizioł (see Theorem 5.6).

Following is a brief description of different chapters of this thesis:

- **p-adic Hodge theory** : In this chapter we provide the setup, recall the basic definitions and the theory of relative de Rham and crystalline p-adic Galois representations following [Bri08].

- **(φ, Γ)-modules and crystalline coordinates** : The aim of this chapter is two fold. First, we introduce the theory of (φ, Γ)-modules following [And06, AB08, AI08], using which we generalize a result of Berger on regularization by Frobenius (see §2.2.1). Next, we introduce certain rings of analytic functions, study their properties as well as several operators on them, and prove a version of Poincaré lemma to be utilised in Chapter 5.

- **Finite height crystalline representations** : This chapter consists of our first main result. We begin by introducing classical Wach modules [Wac96] and its refinement worked out by Berger [Ber04]. Then we introduce the notion of Wach modules in the relative setting and prove several useful properties. Finally, we provide the necessary constructions to state and prove the main statement (see Theorem 3.24). In the last section we give an example illustrating the key ideas behind Theorem 3.24.

- **Cohomological complexes** : In this chapter we recall the theory of Fontaine-Herr complex computing continuous Galois cohomology of p-adic representations, in classical p-adic Hodge theory, as well as its generalization to the relative setting by Andreatta and Iovita [AI08]. Further, we introduce Koszul complexes and relate it to relative Fontaine-Herr complex. Finally, we study the action of the Lie algebra Lie Γ_R over certain rings of analytic functions from

§2.3 and introduce Koszul complexes computing the Lie algebra cohomology of modules over these rings.

- **Syntomic complex and Galois cohomology** : This chapter contains our second main result. We commence the chapter by providing the motivation behind our result which comes from the main technical part of the work by Colmez and Nizioł [CN17]. Then we introduce the necessary setup to introduce the statement of the main result (see Theorem 5.6). Rest of the chapter is devoted to proving this result. First part of the proof concerns working and manipulating syntomic complexes, while the second part is concerned with Koszul complexes. Both these parts are connected via Poincaré lemma from Chapter 2 which is applicable due to the comparison result of Theorem 3.24.

- **Galois cohomology and classical Wach modules** : In this appendix chapter, we work with Wach modules in classical p-adic Hodge theory and Fontaine-Herr complex to study crystalline extension classes of the trivial representation by a crystalline representation. The computation done in this chapter served as the original motivation for pursuing Theorem 5.6, the proof of which persuaded us to investigate Theorem 3.24.

Introduction

Over the course of last century, the *modus operandi* for mathematicians trying to understand spaces has been to investigate natural invariants attached to those spaces. This approach has proven to be a very fruitful one. An example of this comes from topology where one constructs *singular homology* groups attached to a topological space X. Concretely, it is a collection of abelian groups $\{H_k(X, \mathbb{Z})\}_{k\in\mathbb{N}}$, where these groups are computed as the homology of the singular complex attached to X and the k-th homology group describes equivalence classes of k-dimensional holes in X. In terms of application, vanishing statements about homology establishes claims such as Brouwer's fixed point theorem, among others.

Dualizing the construction of singular chain complexes, one can define a contravariant theory, aptly named, *singular cohomology* groups $\{H^k(X, \mathbb{Z})\}_{k\in\mathbb{N}}$ attached to X. Further developments in mathematics have led to the construction (co)homology theories in a myriad of different contexts. For example, de Rham cohomology for differential forms on manifolds, (continuous) group (co)homology, Lie algebra cohomology, étale cohomology for algebraic varieties, etc.

Comparison in complex algebraic geometry

In analytic and algebraic geometry, study of cohomology theories compared to homology has turned out to be a more natural one. Moreover, under amicable circumstances, certain cohomology theories tend to interact with each other. One of the first observations made in this direction was due to de Rham [DR31]. In 1931, he showed that for a smooth manifold M, the pairing of differential forms and singular chains, via integration, gives a homomorphism from de Rham cohomology groups $H^k_{\mathrm{dR}}(M)$ to singular cohomology groups $H^k_{\mathrm{sing}}(M, \mathbb{R})$, which is in fact an isomorphism (see [Sam01] for a historical survey).

In 1966, this result was further extended to the context of complex algebraic geometry by Grothendieck. More precisely, let X be a smooth complex algebraic variety and let X^{an} denote the complex manifold obtained from the complex rational points $X(\mathbb{C})$ of the algebraic variety X. In [Gro66], Grothendieck defined the algebraic de Rham cohomology groups for X and showed that these are canonically isomorphic to de Rham cohomology groups of X^{an}. In conclusion, we have

Theorem A (de Rham, Grothendieck)**.** *Let X be a smooth complex algebraic variety. For each $k \in \mathbb{N}$, there exists a canonical isomorphism of complex vector spaces*

$$H^k_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\simeq} H^k_{\mathrm{dR}}(X^{\mathrm{an}}/\mathbb{C}) \xrightarrow{\simeq} H^k_{\mathrm{dR}}(X/\mathbb{C}).$$

The two sides of this isomorphism contribute complementary information on X; namely, singular cohomology supplies an integral structure for $H^k_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{R})$ (the lattice of periods) and de Rham cohomology gives the Hodge filtration: neither of these two structures are reducible to each other.

In complex algberaic geometry, one can do better. Let us assume that X is a smooth and projective scheme over $\mathbb{C}$ and let X^{an} denote the associated complex manifold. Then X^{an} is a compact Kähler manifold equipped with a Kähler metric. If we let $\Omega^j_{X^{\mathrm{an}}}$ denote the sheaf of holomorphic differential forms on X^{an}, then we have the *Hodge decomposition*

$$H^k_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \bigoplus_{i+j=k} H^i\big(X^{\mathrm{an}}, \Omega^j_{X^{\mathrm{an}}}\big).$$

Further, let $\Omega^1_{X/\mathbb{C}}$ denote the sheaf of Kähler differentials on X and set $\Omega^j_{X/\mathbb{C}} = \bigwedge^j \Omega^1_{X/\mathbb{C}}$. Then combining Hodge decomposition with Serre's GAGA principle, we obtain that

$$H^k_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \bigoplus_{i+j=k} H^i\big(X^{\mathrm{an}}, \Omega^j_{X^{\mathrm{an}}}\big) \simeq \bigoplus_{i+j=k} H^i\big(X, \Omega^j_{X/\mathbb{C}}\big).$$

One of the primary goals of p-adic Hodge theory is to explicate similar phenomenon for p-adic cohomology theories of algebraic varieties defined over p-adic fields.

p-adic comparison theorems

In this section let p denote a fixed prime, K a mixed characteristic discrete valuation field with ring of integers O_K and residue field κ perfect of characteristic p.

In the context of algebraic geometry the Zariski topology on algebraic varieties is too coarse to obtain a meaningful notion of singular cohomology. Therefore, in 1963-64 a replacement in the form of étale cohomology was provided by Grothendieck in [AGV71], where he defined p-adic étale cohomology groups attached to a scheme defined over any field (in particular, finite extensions of $\mathbb{Q}_p$), whereas the definition of algebraic de Rham cohomology carries over for smooth schemes. Again, mathematicians observed that in this setting, these two cohomology theories interact with each other.

The origin of comparing p-adic cohomology theories, collectively termed as *p-adic comparison theorems*, can be attributed to the work of Tate on p-divisible groups in [Tat67]. Tate showed that for an abelian scheme A defined over O_K, the first étale cohomology group of A with coefficients in $\mathbb{Z}_p$ determines the p-divisible group A_{p^∞}, i.e. the p-primary torsion subgroup of A, and vice versa. Further, let $\overline{K}$ denote a fixed algebraic closure of K with $\mathbb{C}_p$ as its p-adic completion. Then the Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$ acts linearly and continuously on the $\mathbb{Z}_p$-module $H^1_{\text{ét}}\big(A_{\overline{K}}, \mathbb{Z}_p\big)$. As a consequence of his general study of p-divisible groups, Tate showed that for $k \leq 2\dim A$, there exists a natural G_K-equivariant isomorphism

$$H^k_{\text{ét}}\big(A_{\overline{K}}, \mathbb{Z}_p\big) \otimes_{\mathbb{Z}_p} \mathbb{C}_p \simeq \bigoplus_{i+j=k} H^i\big(A, \Omega^j_A\big) \otimes_K \mathbb{C}_p(-j), \tag{0.1}$$

where for $j \in \mathbb{Z}$, we define $\mathbb{C}_p(j) := \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(j)$ and $\mathbb{Q}_p(j)$ is the j-th tensor power of the one-dimensional p-adic representation $\mathbb{Q}_p(1)$ on which G_K acts via the p-adic cyclotomic character. Tate conjectured that a G_K-equivariant decomposition as above should exist for any smooth projective variety defined over K.

On the other hand, in [Gro74], Grothendieck showed that the de Rham cohomology groups of an abelian scheme carry extra information as well. Using his crystalline Dieudonné theory, he determined that $H^1_{\mathrm{dR}}(A/K)$ is a K-vector space acquiring a canonical basis over F, where $F = \mathrm{Fr}\, W$ for $W = W(\kappa)$ the ring of p-typical Witt vectors with coefficients in κ. The F-vector-space admits a Frobenius-semilinear automorphism φ, and has a Hodge filtration after extending scalars to K. Further, he showed that A_{p^∞} is determined, up to isogeny, by $H^1_{\mathrm{dR}}(A/K)$ together with its Hodge filtration, basis over F which is equipped with an automorphism φ.

Considering both these phenomena, Grothendieck was led to ask the question of describing an algebraic procedure that would allow one to pass directly from $H^1_{\mathrm{dR}}(A/K)$ to $H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$, without a detour to the p-divisible group A_{p^∞}; he also suspected that such a procedure should exist in arbitrary cohomology degrees (the well known problem of Grothendieck's *mysterious functor*).

This question was resolved by Fontaine in degree one and for arbitrary degree he proposed a precise conjecture in [Fon82, Fon83]. Fontaine's *crystalline conjecture* for an O_K-scheme, examines the relationship between the p-adic étale cohomology of the generic fiber and the crystalline cohomology of the special fiber. This conjecture has now been fully proven by the works of many authors. Before stating the crystalline conjecture, let us mention the work of Faltings generalizing the Hodge-Tate decomposition in (0.1):

Theorem B ([Fal88, Faltings])**.** *Let X be a smooth and proper K-scheme. Then for each $k \in \mathbb{N}$, there exists a canonical G_K-equivariant isomorphism*

$$H^k_{\text{ét}}\big(X_{\overline{K}}, \mathbb{Z}_p\big) \otimes_{\mathbb{Z}_p} \mathbb{C}_p \simeq \bigoplus_{i+j=k} H^i\big(X, \Omega^j_X\big) \otimes_K \mathbb{C}_p(-j).$$

One of the first comparison theorems to be proven in the p-adic setting, the proof of Theorem B relies on Faltings' idea of almost mathematics.

Now we come back to the crystalline conjecture: Let X be a proper and smooth scheme defined over O_K, let $i : X_K \rightarrowtail X$ denote its generic fiber and $j : X_\kappa \rightarrowtail X$ denote its special fiber. For the generic fiber, we will consider the usual p-adic étale cohomology groups $H^i_{\text{ét}}\big(X_{\overline{K}}, \mathbb{Q}_p\big)$, whereas for schemes in characteristic p, i.e. X_κ, we will consider a variant of de Rham cohomology provided by Grothendieck, which is again a p-adic cohomology known as *crystalline cohomology* $H^k_{\mathrm{cris}}(X_\kappa/W(\kappa))$. Then we have,

Theorem C ([FM87, Fontaine-Messing], [Fal89, Faltings], [KM92, Kato-Messing], [Tsu99, Tsuji])**.** *For each $k \in \mathbb{N}$ there exists a natural isomorphism*

$$H^k_{\text{ét}}\big(X_{\overline{K}}, \mathbb{Q}_p\big) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{cris}} \xrightarrow{\simeq} H^k_{\mathrm{cris}}\big(X_\kappa/W(\kappa)\big) \otimes_{W(\kappa)} \mathbf{B}_{\mathrm{cris}},$$

compatible with the action of G_K, the Frobenius, filtration (and Poincaré duality, Künneth formula, cycle class and Chern class maps) on each side.

Here $\mathbf{B}_{\mathrm{cris}}$ denotes the crystalline period ring constructed by Fontaine (see [Fon94a]), and it is equipped with a continuous action of G_K, the Frobenius and a filtration.

In [FM87] Fontaine and Messing initiated a program for proving the crystalline conjecture via *syntomic* methods and managed to prove the claim in the case $K = F$ and $\dim X_K < p$. In [KM92] Kato and Messing proved the conjecture under the assumption $\dim X_K < (p-1)/2$ but without any assumption on K. Further, this program was generalized to the semistable case by Fontaine and Janssen. The semistable conjecture was shown by Fontaine for abelian varieties and then proved by Kato in [Kat94] in the case $\dim X_K < (p-1)/2$, generalizing the methods of [KM92]. Finally, this program was concluded by Tsuji in [Tsu99] completing the proof of crystalline and semistable conjectures.

Over the course of four decades, many mathematicians have worked on p-adic comparison theorems. In [Fal89], Faltings proved the crystalline conjecture and also generalized his methods to non-trivial coefficients. He further showed the semistable comparison theorem using his theory of almost étale extensions in [Fal02]. In [Niz98] Nizioł gave another proof of the crystalline conjecture using K-theory. Yamashita proved the non-proper case in [Yam11]. Employing completely different constructions Beilinson proved all incarnations of p-adic comparison theorems in [Bei12, Bei13]. Further, Scholze proved the de Rham comparison theorem for rigid analytic varieties in [Sch13], where he works completely over the generic fiber and considers non-trivial p-adic local systems on the étale side. Generalizing Faltings' ideas, Andreatta and Iovita proved the crystalline comparison for smooth formal schemes in [AI13], where their proof works for non-trivial coefficients as well. Further, Andreatta and Iovita generalized their proof to the semistable case in [AI12].

In [CN17] using syntomic methods and techniques from the theory of (φ, Γ)-modules, Colmez and Nizioł have proved the semistable comparison for formal log-schemes. The major part of [CN17] consists of local computations, i.e. over affinoids covering the scheme X. In the case of smooth proper scheme X, the covering can be given by an étale algebra over a formal torus over O_K. The motivation for our cohomological results with coefficients in this thesis stems from this article (see Theorem H). The pursuit of the cohomological statement led to our exploration of finite height crystalline representations in the relative setting (see Theorem E). We will come back to these connections later.

An integral version of comparison theorems was obtained by Bhatt, Morrow and Scholze in [BMS18], where they have defined a new cohomology theory over Fontaine's infinitesimal ring $\mathbf{A}_{\mathrm{inf}}$. The work of [BMS18] was generalized to the semistable case by Česnavičius and Koshikawa in [ČK19]. Finally, further generalizing their work, Bhatt and Scholze have put forward the theory of prismatic cohomology in [BS19] which unifies all known p-adic cohomology theories.

p-adic representations and linear algebra

Since the age of Galois, mathematicians have been interested in understanding Galois groups of field extensions. While some finite and profinite cases are simple and explicit to state, in general these groups are quite complex to decipher, for example, the absolute Galois group G_K in the previous section is as far away from being explicit as possible. To understand such groups, a general approach is to study their representations, i.e. the action of such groups on certain modules. This is another common theme in p-adic Hodge theory, i.e. studying p-adic representations of Galois groups such as G_K.

The p-adic étale cohomology groups $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$, appearing in Theorem C, are $\mathbb{Q}_p$-vector spaces equipped with a linear and continuous action of the Galois group G_K. In other words, we have obtained p-adic representations of the Galois group G_K. On the other hand, the crystalline cohomology groups $F \otimes_W H^i_{\mathrm{cris}}(X_\kappa / W)$ are F-vector spaces equipped with a Frobenius-semilinear automorphism φ and a filtration after extending scalars to K. Theorem C states that these two objects are related to each other.

In 1980s-90s Fontaine stated and carried out several programs in order to study p-adic representations of G_K. In [Fon79, Fon82, Fon94a, Fon94b], Fontaine describes the subcategories of crystalline, semi-stable and de Rham representations. For example, the étale cohomology groups appearing in Theorem C are crystalline representations of G_K. Fontaine's theory is rich and an incredible journey to take, however we will content ourselves with a description of crystalline representations. Moreover, for the sake of simplicity, we will work under the assumption that $K = F$ is unramified over $\mathbb{Q}_p$, however some of the results are true in more general settings.

Crystalline representations

In order to classify crystalline representations, Fontaine came up with a general formalism. He constructs a period ring $\mathbf{B}_{\mathrm{cris}}$ which is a p-adically complete F-algebra equipped with a Frobenius and a filtration (see [Fon94a], we will recall the construction in a more general setting in §1.3). Now let V be a p-adic representation of G_F, and set

$$\mathbf{D}_{\mathrm{cris}}(V) := (\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_F}.$$

It is a finite-dimensional F-vector space such that $\dim_F \mathbf{D}_{\mathrm{cris}}(V) \leq \dim_{\mathbb{Q}_p} V$, and it is equipped with a Frobenius-semilinear endomorphism φ, and a filtration coming from the filtration on $\mathbf{B}_{\mathrm{cris}}$. Moreover, this construction is functorial in V and it takes values in the category filtered φ-modules over F.

The representation V is said to be *crystalline* if and only if it is $\mathbf{B}_{\text{cris}}$-admissible, or equivalenty, $\dim_F \mathbf{D}_{\text{cris}}(V) = \dim_{\mathbb{Q}_p} V$. In particular, the p-adic periods of V belong to $\mathbf{B}_{\text{cris}}$. The functor $\mathbf{D}_{\text{cris}}$ is exact and fully faithful as well as establishes an equivalence between the category of crystalline representations and its essential image under the functor, compatible with exact sequences, tensor products and taking duals.

The terminology *crystalline* accentuates the fact that if the representation "comes from geometry", i.e. computed as étale cohomology of generic fiber of a smooth and proper W-scheme, then there exists a comparison with the crystalline cohomology of the special fiber. For example, if we let $V_i := H^i_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)$ in Theorem C, then we have $\mathbf{D}_{\text{cris}}(V_i) = F \otimes_W H^i_{\text{cris}}(X_\kappa/W)$. Moreover, given $H^i_{\text{cris}}(X_\kappa/W)$ with its complimentary structures, one can recover the $\mathbb{Q}_p$-vector space $H^i_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)$ with its Galois action, and vice versa. This is quite a surprising result in contrast with the complex case (see Theorem A).

(φ, Γ)-modules and finite height representations

A different perspective on p-adic representations is the theory of (φ, Γ)-modules. Morally, such a theory is an attempt to describe p-adic representations of G_F in terms of modules over complicated base rings, admitting a Frobenius-semilinear endomorphism and simpler action of a piece of the Galois group.

More precisely, let $F_\infty = \cup_{n \in \mathbb{N}} F(\zeta_{p^n})$ where $\zeta_{p^n} \in \overline{F}$ denotes a primitive p^n-th root of unity, and let $\mathbb{C}_p^\flat$ denote the tilt of $\mathbb{C}_p$ (see §1.2 for a precise definition). Let $H_F = \mathrm{Gal}(\overline{F}/F_\infty)$ and $\Gamma_F = \mathrm{Gal}(F_\infty/F)$, then we have an exact sequence

$$1 \longrightarrow H_F \longrightarrow G_F \longrightarrow \Gamma_F \longrightarrow 1.$$

Using the *field-of-norms* construction in [FW79b, FW79a, Win83], Fontaine and Wintenberger defined a non-archimedean complete discrete valuation field $\mathbf{E}_F \subset \mathbb{C}_p^\flat$ of characteristic p with residue class field κ, and functorial in F. In [Fon90], Fontaine utilised the theory from fields-of-norms construction to classify mod-p representations of G_F in terms of étale (φ, Γ_F)-modules over $\mathbf{E}_F$. By some technical considerations one can lift this to characteristic 0, i.e. classify $\mathbb{Z}_p$-representations of G_F in terms of *étale* (φ, Γ_F)*-modules* over a two dimensional regular local ring $\mathbf{A}_F \subset W\big(\widehat{F}_\infty^\flat\big)$. In particular, the p-adic periods of any $\mathbb{Z}_p$-representation of G_F belong to the ring $\mathbf{A} \subset W(\mathbb{C}_p^\flat)$. Similar equivalence of categories can be obtained for p-adic representations and étale (φ, Γ_F)-modules over $\mathbf{B}_F = \mathbf{A}_F\left[\frac{1}{p}\right]$, i.e. p-adic periods of p-adic representations of G_F belong to $\mathbf{B} = \mathbf{A}\left[\frac{1}{p}\right] \subset \mathrm{Fr}\, W(\mathbb{C}_p^\flat)$.

The theory of (φ, Γ)-modules was further refined by Cherbonnier and Colmez in [CC98]. They showed that all $\mathbb{Z}_p$-representations (resp. p-adic representations) are *overconvergent*, the p-adic periods belog to a subring $\mathbf{A}^\dagger \subset \mathbf{A}$ (resp. $\mathbf{B}^\dagger \subset \mathbf{B}$). Many applications of (φ, Γ)-modules make use of the result of Cherbonnier-Colmez (see [CC99], [Ber02, Ber03], etc.).

The field-of-norms functor was further generalized to higher-dimensional local fields by Abrashkin in [Abr07]. A vast generalization of the theory of Fontaine and Wintenberger, also known as the *tilting correspondence*, was done by Scholze in [Sch12].

Finite height crystalline representations

So far we have seen the classification of p-adic crystalline representations of G_F in terms of filtered φ-modules over F, and all p-adic representations of G_K in terms of étale (φ, Γ)-modules over $\mathbf{B}_F$. By the latter equivalence of categories, it becomes a natural question to ask : Is it possible to describe crystalline representations intrinsically in the category of étale (φ, Γ)-modules? To answer this question, Fontaine initiated a program relating p-adic crystalline representations and finite height representations.

A p-adic representation V of G_F is said to be of *finite height* if the p-adic periods of V belong to the "integral" subring $\mathbf{B}^+ \subset \mathbf{B}$ (see §3.1). In other words, the associated (φ, Γ_F)-module over $\mathbf{B}_F$ admits a basis in a lattice, i.e. has a basis over the period ring $\mathbf{B}_F^+ \subset \mathbf{B}_F$. For crystalline representations there exist lattices over which the action of Γ_F is simpler. Finite height and crystalline representations of G_F are related by the following result:

Theorem D ([Wac96, Wach], [Col99, Colmez], [Ber02, Berger]). *Let V be a p-adic representation of G_F. Then V is crystalline if and only if it is of finite height and there exists $r \in \mathbb{Z}$ and a $\mathbf{B}_F^+$-submodule $N \subset \mathbf{D}(V)$ of rank $= \dim_{\mathbb{Q}_p} V$, stable under the action of Γ_F, such that Γ_F acts trivially over $(N/\pi N)(-r)$.*

In the situation of Theorem D, the module N is not unique. A functorial construction was given by Berger in [Ber04] using which he established an equivalence of categories between the crystalline representations of G_F and Wach modules over $\mathbf{B}_F^+$. Moreover, for a crystalline representation V, there exists a bijection between $\mathbb{Z}_p$-lattices inside V and Wach modules over the integral subring $\mathbf{A}_F^+ \subset \mathbf{B}_F^+$, and contained in the rational Wach module $\mathbf{N}(V)$. Finally, given $\mathbf{N}(V)$ one can canonically recover the other linear algebraic object attached to V, i.e. $\mathbf{D}_{\mathrm{cris}}(V)$ (see [Ber04, Propositions II.2.1 & III.4.4]).

The theory and construction of Wach modules has witnessed many applications, for example, Iwasawa theory of crystalline representations in [Ben00, BB08], Berger's proof of p-adic monodromy conjecture [Ber02], as well as, in the study of p-adic local Langlands program [BB10]. The notion of Wach modules was generalized as Breuil-Kisin modules for mixed characteristic discretely valued (possibly ramified) extension $K/\mathbb{Q}_p$ (see [Bre99, Bre02, Kis06]). The existence of Wach modules also served as a motivation for Scholze's idea of q-deformations [Sch17], which paved the way for Bhatt-Scholze theory of prisms and prismatic cohomology [BS19]. Moreover, similar to Berger's classification in the finite unramified case, Bhatt and Scholze have shown that for any mixed characteristic discretely valued extension $K/\mathbb{Q}_p$, the catgeory of prismatic F-crystals on $\mathrm{Spf}\,(O_K)$ is equivalent to the category of $\mathbb{Z}_p$-lattices inside crystalline representations of G_K (see [BS21, Theorem 1.2]).

Relative finite height crystalline representations

As indicated before, we are interested in the local version of relative p-adic Hodge theory. So let us introduce the setup briefly: Let us now fix $p \geq 3$, and let $d \in \mathbb{N}$ with $X = (X_1, X_2, \ldots, X_d)$ some indeterminates. We set $W\{X\} := \big\{ \sum_{\mathbf{k} \in \mathbb{N}^d} a_{\mathbf{k}} X^{\mathbf{k}}$, where $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d, X^{\mathbf{k}} = X_1^{k_1} \cdots X_d^{k_d}, a_{\mathbf{k}} \in W$, and $a_{\mathbf{k}} \longrightarrow 0$ as $\mathbf{k} \longrightarrow \infty \big\}$, to be a p-adically complete algebra over W. Similarly we define $R_0 := W\{X^{\pm 1}\}$. Let $K = F(\zeta_{p^m})$, where $m \in \mathbb{N}_{\geq 1}$, ζ_{p^m} is a primitive p^m-th root of unity, let O_K denote the ring of integers of K and set $R := O_K\{X^{\pm 1}\}$.

Note. In the main body of the thesis, we will work in a more general setup, i.e. over the p-adic completion of an étale algebra over $W\{X^{\pm 1}\}$ and corresponding extension of R_0 and R above (see §1.1). However, for the sake of lucidity of the exposition, we introduce the results under simplified assumptions.

Crystalline representations

Akin to Fontaine's formalism, in [Bri08] Brinon studied the p-adic representations of G_R, the étale fundamental group of $R\big[\frac{1}{p}\big]$. In the relative setting there are two notions of crystalline representations: horizontal crystalline and (big) crystalline representations. We are interested in the latter category of representations.

To classify crystalline representations, Brinon constructs a period ring $\mathcal{O}\mathbf{B}_{\mathrm{cris}}$ which is a p-adically complete $R_0\big[\frac{1}{p}\big]$-algebra equipped with a Frobenius, a filtration and a $\mathbf{B}_{\mathrm{cris}}$-linear connection satisfying Griffiths transversality (see [Bri08], note that these are relative version of Fontaine's construction, we recall the details in §1.3). Now let V be a p-adic representation of G_{R_0}, and let

$$\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) := (\mathcal{O}\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_{R_0}}.$$

It is a finite projective $R_0\left[\frac{1}{p}\right]$-module of rank $\leq \dim_{\mathbb{Q}_p} V$, and it is equipped with a Frobenius-semilinear endomorphism φ, a filtration arising from the filtration on $\mathcal{O}\mathbf{B}_{\mathrm{cris}}$ and a quasi-nilpotent integrable connection satisfying Griffiths transversality and stemming from the connection on $\mathcal{O}\mathbf{B}_{\mathrm{cris}}$ (see §1.5 for details). Moreover, this construction is functorial in V and it takes values in the category of filtered (φ, ∂)-modules over $R_0\left[\frac{1}{p}\right]$. The representation V is said to be *crystalline* if and only if it is $\mathcal{O}\mathbf{B}_{\mathrm{cris}}$-admissible (see §1.5.2). In particular, the p-adic periods of V belong to $\mathcal{O}\mathbf{B}_{\mathrm{cris}}$. The functor $\mathcal{O}\mathbf{D}_{\mathrm{cris}}$ is exact and fully faithful as well as establishes an equivalence between the category of (big) crystalline representations and its essential image under the functor, compatible with exact sequences, tensor products and taking duals.

(φ, Γ)-modules and finite height representations

Parallel to the arithmetic case, in the relative setting we can again classify all p-adic representations in terms of (φ, Γ)-modules. For $n \in \mathbb{N}$, let $F_n = F(\zeta_{p^n})$ where ζ_{p^n} is a primitive p^n-th root of unity. Let R_n denote the integral closure of $R_0 \otimes O_{F_n}\left[X_1^{p^{-n}}, \ldots X_d^{p^{-n}}\right]$ inside $\overline{R}\left[\frac{1}{p}\right]$, and let $R_\infty := \cup_n R_n$. We set $G_{R_0} := \mathrm{Gal}\left(\overline{R}\left[\frac{1}{p}\right]/R_0\left[\frac{1}{p}\right]\right)$, $\Gamma_{R_0} := \mathrm{Gal}\left(R_\infty\left[\frac{1}{p}\right]/R_0\left[\frac{1}{p}\right]\right)$, and $H_{R_0} := \mathrm{Ker}\,(G_{R_0} \longrightarrow \Gamma_{R_0})$. The ring $R_\infty\left[\frac{1}{p}\right]$ is a Galois extension of $R_0\left[\frac{1}{p}\right]$ with Galois group Γ_{R_0} fitting into an exact sequence

$$1 \longrightarrow \Gamma'_{R_0} \longrightarrow \Gamma_{R_0} \longrightarrow \Gamma_F \longrightarrow 1, \tag{0.2}$$

where, for $1 \leq i \leq d$ we have $\Gamma'_{R_0} = \mathrm{Gal}\left(R_\infty\left[\frac{1}{p}\right]/F_\infty R_0\left[\frac{1}{p}\right]\right) \simeq \mathbb{Z}_p^d$, and $\Gamma_F = \mathrm{Gal}(F_\infty/F) \simeq \mathbb{Z}_p^\times$.

Fontaine's classification was generalized by Andreatta in [And06] to the relative setting. Andreatta constructs an analogue of $\mathbf{E}_F$ in the relative setting, i.e. to R_0, he associates a Noetherian regular domain $\mathbf{E}_{R_0}^+$. Further, he lifts this ring to characteristic 0, i.e. we have $\mathbf{A}_{R_0}^+$ equipped with a Frobenius endomorphism and a continuous action of Γ_{R_0}. Finally, we have $\mathbf{A}_{R_0}$ as the p-adic completion of $\mathbf{A}_{R_0}^+\left[\frac{1}{\pi}\right]$.

Next, an *étale* (φ, Γ_{R_0})*-module* is a finitely generated $\mathbf{A}_{R_0}$-module equipped with a Forbenius-semilinear automorphism φ and a semilinear and continuous action of Γ_{R_0}. Andreatta shows that there is an equivalence of categories between $\mathbb{Z}_p$-representations of G_{R_0} and étale (φ, Γ_{R_0})-modules over $\mathbf{A}_{R_0}$. In particular, the p-adic periods of any $\mathbb{Z}_p$-representation of G_{R_0} live in the ring $\mathbf{A} \subset W(\mathbb{C}(R)^\flat)$ (see §2.1). Similar equivalence of categories can be obtained for p-adic representations and étale (φ, Γ_{R_0})-modules over $\mathbf{B}_{R_0} := \mathbf{A}_{R_0}\left[\frac{1}{p}\right]$, i.e. the p-adic periods of p-adic representations of G_{R_0} belong to $\mathbf{B} = \mathbf{A}\left[\frac{1}{p}\right] \subset W(\mathbb{C}(R)^\flat)\left[\frac{1}{p}\right]$. Note that the discussion above is true in a more general setting, in particular for R (see §2.1 which is an adaptation of [And06]).

In [AB08], Andreatta-Brinon have generalized the result of Cherbonnier-Colmez to the relative setting, i.e. they have shown that all $\mathbb{Z}_p$-representations (resp. p-adic representations) of G_{R_0} are overconvergent (see §2.2 for details), i.e. the p-adic periods belong to a subring $\mathbf{A}^\dagger \subset \mathbf{A}$ (resp. $\mathbf{B}^\dagger \subset \mathbf{B}$).

Wach representations

So far we have discussed crystalline representations and (φ, Γ)-modules in the relative setting. Parallel to the arithmetic case, we are now interested in understanding finite height representations and Wach modules in the relative setting. Further, we expect that there should be a connection between finite height and crystalline representations.

Let V be a p-adic representation of the Galois group G_{R_0}. It is said to be of *finite height* if the p-adic periods of V belong to the subring $\mathbf{B}^+ \subset \mathbf{B}$ (see §3.2). In other words, the $\mathbf{B}_{R_0}^+ = \mathbf{A}_{R_0}^+\left[\frac{1}{p}\right]$-submodule $\mathbf{D}^+(V) \subset \mathbf{D}(V)$ (defined functorially in V) is a finitely generated (φ, Γ_{R_0})-module such that $\mathbf{B}_{R_0} \otimes_{\mathbf{B}_{R_0}^+} \mathbf{D}^+(V) \simeq \mathbf{D}(V)$.

Now we take V to be a p-adic de Rham representation with non-positive Hodge-Tate weights, $T \subset V$ a free $\mathbb{Z}_p$-lattice of rank $= \dim_{\mathbb{Q}_p} V$, stable under the action of G_{R_0}. We say that V is a *positive Wach representation* if it is of finite height and there exists $\mathbf{N}(T) \subset \mathbf{D}^+(T)$, a finite projective

(φ, Γ_{R_0})-module over $\mathbf{A}^+_{R_0}$ satisfying certain technical conditions describing the action of φ and Γ_{R_0} (see Definition 3.8). We set $\mathbf{N}(V) := \mathbf{N}(T)\left[\frac{1}{p}\right]$, and the uniqueness of these modules follows from the definition (see Lemma 3.14). Further, these modules are equipped with a natural filtration.

The aim of Chapter 3 is to show that Wach representations are crystalline. Further, for a positive Wach representation V the $\mathbf{B}^+_{R_0}$-module $\mathbf{N}(V)$ and the $R_0\left[\frac{1}{p}\right]$-module $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ are related in a precise manner and the latter can be recovered from the former. To relate these objects we construct a fat relative period ring $\mathcal{O}\mathbf{A}^{\mathrm{PD}}_R \subset \mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0)$ equipped with compatible Frobenius, filtration, connection and the action of Γ_{R_0} (see §3.2).

Theorem E (see Theorem 3.24)**.** *Let V be a positive Wach representation of G_{R_0}, then V is a positive crystalline representation. Further, let $M\left[\frac{1}{p}\right] := \left(\mathcal{O}\mathbf{A}^{\mathrm{PD}}_R \otimes_{\mathbf{A}^+_{R_0}} \mathbf{N}(V)\right)^{\Gamma_{R_0}}$, then we have an isomorphism of $R_0\left[\frac{1}{p}\right]$-modules $M\left[\frac{1}{p}\right] \simeq \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ compatible with Frobenius, filtration, and connection on each side. Moreover, after extending scalars to $\mathcal{O}\mathbf{A}^{\mathrm{PD}}_R$, we obtain natural isomorphisms*

$$\mathcal{O}\mathbf{A}^{\mathrm{PD}}_R \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xleftarrow{\simeq} \mathcal{O}\mathbf{A}^{\mathrm{PD}}_R \otimes_{R_0} M\left[\tfrac{1}{p}\right] \xrightarrow{\simeq} \mathcal{O}\mathbf{A}^{\mathrm{PD}}_R \otimes_{\mathbf{A}^+_{R_0}} \mathbf{N}(V),$$

compatible with Frobenius, filtration, connection and the action of Γ_{R_0} on each side.

The proof of the theorem proceeds in three steps: First, we explicitly state the structure of Wach module attached to a one-dimensional Wach representation, we also show that all one-dimensional crystalline representations are Wach representations and one can recover $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ starting with the Wach module $\mathbf{N}(V)$. Next, in higher dimensions and under the conditions of the statement above, we will describe a process (successive approximation) by which we can recover a submodule of $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ starting from the Wach module, here we establish a comparison by passing to the one-dimensinal case. Finally, the claims made in the theorem are shown by exploiting some properties of Wach modules and the comparison obtained in the second step. In the second step, approximating for the action of geometric part of Γ_{R_0} turns out to be non-trivial and most of our work goes into showing this part; the arithmetic part of Γ_{R_0} follows from the work of Wach [Wac96].

Syntomic complex and Galois cohomology

Having introduced an interesting class of representations, we come back to our discussion of crystalline conjecture in Theorem C. Let $K = F(\zeta_{p^m})$ for $m \geq 1$, let X be a smooth proper scheme over O_K, such that $j : X_K := X \otimes_{O_K} K \rightarrowtail X$ denotes the inclusion of its generic fiber and $i : X_\kappa := X \otimes_{O_K} \kappa \rightarrowtail X$ denotes the inclusion of its special fiber. To attack the crystalline conjecture, Fontaine and Messing initiated a program for proving it via syntomic methods (see [FM87]). For $r \geq 0$, let $\mathcal{S}_n(r)_X$ denote the syntomic sheaf modulo p^n on $X_{\kappa,\text{ét}}$. It can be thought of as a derived Frobenius and filtration eigenspace of crystalline cohomology. Then, Fontaine and Messing constructed a period morphism

$$\alpha^{\mathrm{FM}}_{r,n} : \mathcal{S}_n(r)_X \longrightarrow i^*\mathbf{R}j_*\mathbb{Z}/p^n(r)'_{X_K},$$

from syntomic cohomology to p-adic nearby cycles, where $\mathbb{Z}_p(r)' := \frac{1}{p^{a(r)}}\mathbb{Z}_p(r)$, for $r = (p-1)a(r) + b(r)$ with $0 \leq b(r) \leq p-1$.

In [CN17], Colmez and Nizioł have shown that the Fontaine-Messing period map $\alpha^{\mathrm{FM}}_{r,n}$, after a suitable truncation, is essentially a quasi-isomorphism. More precisely,

Theorem F ([CN17, Theorem 1.1])**.** *For $0 \leq k \leq r$, the map*

$$\alpha^{\mathrm{FM}}_{r,n} : \mathcal{H}^k(\mathcal{S}_n(r)_X) \longrightarrow i^*\mathbf{R}^k j_*\mathbb{Z}/p^n(r)'_{X_K},$$

is a p^N-isomorphism, i.e. there exists $N = N(e,p,r) \in \mathbb{N}$ depending on r and the absolute ramification index e of K but not on X or n, such that the kernel and cokernel of the map is killed by p^N.

In fact, for $k \leq r \leq p-1$, the map $\alpha_{r,n}^{\mathrm{FM}}$ was shown to be an isomorphism by Kato [Kat89, Kat94], Kurihara [Kur87], and Tsuji [Tsu99]. In [Tsu96], Tsuji generalized this result to some suitable étale local systems.

Theorem F also holds for base change of smooth and proper schemes. In particular, after passing to the limit and inverting p above, for each $0 \leq k \leq r$ we obtain an isomorphism

$$\alpha_r^{\mathrm{FM}} : H_{\mathrm{syn}}^k\big(X_{O_{\overline{K}}}, r\big)_{\mathbb{Q}} \xrightarrow{\simeq} H_{\text{ét}}^k\big(X_{\overline{K}}, \mathbb{Q}_p(r)\big). \tag{0.3}$$

The isomorphism displayed above is the most important step in proving the crystalline conjecture via syntomic methods. These ideas have been used in [FM87], [KM92], [Kat87], [Kat94], and [Tsu99]. However, all these proofs have been worked out directly over $\overline{K}$, but with no restrictions on r.

The proof of Colmez and Nizioł is different from earlier approaches. They prove Theorem F first, and deduce the comparison in (0.3) via base change. To prove their claim, they construct another local period map $\alpha_r^{\mathcal{L}az}$, employing techniques from the theory of (φ, Γ)-modules and a version of integral Lazard isomorphism between Lie algebra cohomology and continuous group cohomology. Then they proceed to show that this map is a quasi-isomorphism and coincides with Fontaine-Messing period map up to some constants. Moreover, all of their results have been worked out in the general setting of log-schemes.

Local computation of Colmez and Nizioł

As specified earlier, the major part of [CN17] consists of local computations, i.e. over affinoids covering a formal scheme. In the case of a smooth proper formal scheme, the covering can be given by an étale algebra over $R = O_K\{X^{\pm 1}\}$, where $X = (X_1, \dots, X_d)$ are some indeterminates (see §1.1 for notations). To state the local result, we will restrict ourselves to the familiar setting of R, however the results also hold for an étale algebra over R (Colmez and Nizioł work with log structures as well).

Let R_{ϖ}^+ denote the (p, X_0)-adic completion of $W[X_0, X^{\pm 1}]$, and let $S = R_{\varpi}^{\mathrm{PD}}$ denote the p-adic completion of the divided power envelope with respect to the kernel of the map $R_{\varpi} \twoheadrightarrow R$ sending X_0 to $\zeta_{p^m} - 1$. Further, let Ω_S^1 denote the p-adic completion of the module of differentials of S relative to $\mathbb{Z}$ and $\Omega_S^k = \wedge^k \Omega_S^1$ for $k \in \mathbb{N}$. The syntomic cohomology of R can be computed by the complex

$$\mathrm{Syn}(R, r) := \mathrm{Cone}\big(F^r\Omega_S^{\bullet} \xrightarrow{p^r - p^{\bullet}\varphi} \Omega_S^{\bullet}\big)[-1],$$

such that we have $H_{\mathrm{syn}}^i(R, r) = H^i(\mathrm{Syn}(R, r))$. If K contains enough roots of unity, i.e. for m large enough, Colmez and Nizioł have shown that,

Theorem G ([CN17, Theorem 1.6])**.** *The maps*

$$\begin{aligned} \alpha_r^{\mathcal{L}az} &: \tau_{\leq r}\mathrm{Syn}(R, r) \longrightarrow \tau_{\leq r}\mathbf{R}\Gamma_{\mathrm{cont}}(G_R, \mathbb{Z}_p(r)), \\ \alpha_{r,n}^{\mathcal{L}az} &: \tau_{\leq r}\mathrm{Syn}(R, r)_n \longrightarrow \tau_{\leq r}\mathbf{R}\Gamma_{\mathrm{cont}}(G_R, \mathbb{Z}/p^n(r)) \longrightarrow \tau_{\leq r}\mathbf{R}\Gamma\big(\big(\mathrm{Sp}\ R\big[\tfrac{1}{p}\big]\big)_{\text{ét}}, \mathbb{Z}/p^n(r)\big), \end{aligned} \tag{0.4}$$

are p^{Nr}-quasi-isomorphisms for a universal constant N.

Finally, using Galois descent one can obtain the result over K (not necessarily having enough roots of unity, with N depending on K, p and r, see [CN17, Theorem 5.4]). Note that the truncation here denotes the canonical truncation in literature. The proof of Colmez and Nizioł relies of comparing the syntomic complex with the complex of (φ, Γ)-modules computing the continuous G_R-cohomology of $\mathbb{Z}_p(r)$. This is achieved using a version of Poincaré lemma. Further, note that they work with log structures, i.e. all defintitions above should be replaced with their log analogues (without log structures one should truncate in degree $\leq r-1$, see Theorem H below).

Fontaine-Herr complex

The right side of the map in the p-adic version of the result of Colmez and Nizioł, i.e. the first isomorphism in (0.4), is concerned with the computation of continuous G_R-cohomology of $\mathbb{Z}_p(r)$. This computation can be carried out with complexes made up of (φ, Γ)-modules, the origins of which lie in the work of Herr (see [Her98]).

Let V be a p-adic representation (resp. $\mathbb{Z}_p$-representation) of G_F, and let $\mathbf{D}(V)$ denote the associated étale (φ, Γ_F)-module over $\mathbf{B}_F$ (resp. $\mathbf{A}_F$). Let $\gamma \in \Gamma_F$ denote a topological generator of Γ_F, then we have a complex

$$C^\bullet : \mathbf{D}(V) \xrightarrow{(1-\varphi, \gamma-1)} \mathbf{D}(V) \oplus \mathbf{D}(V) \xrightarrow{\left(\begin{smallmatrix} \gamma-1 \\ 1-\varphi \end{smallmatrix}\right)} \mathbf{D}(V),$$

where the second map is $(x, y) \mapsto (\gamma - 1)x - (1 - \varphi)y$. The Fontaine-Herr complex $C^\bullet$ computes the continuous G_F-cohomology of V in each cohomological degree, i.e. for $k \in \mathbb{N}$, we have natural isomorphisms $H^k(C^\bullet) \simeq H^k_{\mathrm{cont}}(G_F, V)$.

The continuous G_F-cohomology groups are useful invariants attached to V. For example, the first continuous cohomology group of V, i.e. $H^1_{\mathrm{cont}}(G_F, V)$ classifies equivalent classes of extensions of the trivial representation $\mathbb{Q}_p$ by V in $\mathrm{Rep}_{\mathbb{Q}_p}(G_F)$, and which can be represented by a pair $x, y \in \mathbf{D}(V)$ satisfying the equation $(\gamma - 1)x = (1 - \varphi)y$. Further, if V is crystalline then any crystalline extension of $\mathbb{Q}_p$ by $V(r)$ (cyclotomic twist of V) can be represented by a pair (x, y) with $x \in \mathbf{N}(V)(r)$ and $y \in \mathbf{N}(V(r))$ such that $(\gamma - 1)x = (1 - \varphi)y$ (see Lemma A.2 and Proposition A.4). In fact, this statement combined with the computation carried out by Colmez and Nizioł served as the original motivation for obtaining Theorem H.

In the relative setting, we have the relative version of Fontaine-Herr complex which computes the continuous G_R-cohomology of a p-adic representation (see [AI08, Theorem 3.3], we recall the description in §4.1). Explicit complexes computing the continuous G_R-cohomology of T can also be obtained, which we collectively refer to as *Koszul complexes* (see §4.2). Further, Koszul complexes play a central role in the proof of Theorem H.

Syntomic complex with coefficients

In Theorem G, we are interested in the p-adic result, i.e. the first isomorphism in (0.4). Our objective is to replace the representation $\mathbb{Z}_p(r)$ there by a more general representation $T(r)$, and adapt the method of Colmez and Nizioł to obtain a relation between syntomic complex with coefficients and continuous G_R-cohomology of $T(r)$. The interesting class of representations for us are the crystalline Wach representations of G_{R_0}. In the notation of Theorem E, for the coefficient of syntomic complex, we will choose a lattice inside the filtered (φ, ∂)-module $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$, whereas to compute the Galois cohomology we will exploit the properties of the associated Wach module $\mathbf{N}(V)$. The two sides will then be compared using a version of Poincaré lemma, where a crucial input is the comparison obtained in Theorem E.

More precisely, let V be a p-adic Wach representation of G_{R_0} with non-positive Hodge-Tate weights and let $s \in \mathbb{N}$ denote the maximum among the absolute value of Hodge-Tate weights of V. Let $T \subset V$ be a free $\mathbb{Z}_p$-lattice of rank $= \dim_{\mathbb{Q}_p} V$, stable under the action of G_{R_0}. Assume that $\mathbf{N}(T)$ is a free $\mathbf{A}^+_{R_0}$-module of rank $= \dim_{\mathbb{Q}_p} V$, and there exists a free R_0-submodule $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ of rank $= \dim_{\mathbb{Q}_p} V$, such that $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)\left[\frac{1}{p}\right] = \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ and the induced connection over $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$ is quasi-nilpotent, integrable and satisfies Griffiths transversality with respect to the induced filtration. Let $r \in \mathbb{N}$ and we set $V(r) := V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(r)$ and $T(r) := T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(r)$.

Note. The choice of $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$ is not canonical and we discuss some ways to obtain such a module in Proposition 3.31, Remark 3.42 and Remark 5.4. However, we fix such a choice for the rest of the discussion.

Define

$$D^{\mathrm{PD}} := R^{\mathrm{PD}}_{\varpi} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T).$$

There is a Frobenius-semilinear endomorphism on D^{PD} given by the diagonal action of the Frobenius on each component of the tensor product, a filtration coming from the product of filtration on each component of the tensor product and a connection induced from the connection on $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$ and the differential operator on R_{ϖ}^{PD}. Further, this connection is integrable and satisfies Griffiths transversality with respect to the filtration (see Chapter 5 for precise definitions). In particular, let $S = R_{\varpi}^{\mathrm{PD}}$ and we have a filtered de Rham complex for $k \in \mathbb{Z}$,

$$\mathrm{Fil}^k \mathcal{D}^{\bullet} := \mathrm{Fil}^k D^{\mathrm{PD}} \otimes_S \Omega_S^1 \longrightarrow \mathrm{Fil}^{k-1} D^{\mathrm{PD}} \otimes_S \Omega_S^1 \longrightarrow \cdots.$$

Let $D_R := R \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$. Define the *syntomic complex* $\mathrm{Syn}(D_R, r)$ and the *syntomic cohomology* of R with coefficients in D_R as

$$\begin{aligned} \mathrm{Syn}(D_R, r) &:= \left[\mathrm{Fil}^r \mathcal{D}^{\bullet} \xrightarrow{p^r - p^{\bullet}\varphi} \mathcal{D}^{\bullet} \right]; \\ H^*_{\mathrm{syn}}(D_R, r) &:= H^*(\mathrm{Syn}(D_R, r)). \end{aligned}$$

We will relate this complex to Fontaine-Herr complex computing the continuous G_R-cohomology of $T(r)$. The main result of Chapter 5 is:

Theorem H (see Theorem 5.6)**.** *Let V be a positive Wach representation of G_{R_0}, $T \subset V$ a free G_{R_0}-stable $\mathbb{Z}_p$-lattice, $s \in \mathbb{N}$ the maximum among the absolute value of the Hodge-Tate weights of V and $r \in \mathbb{N}$ such that $r \geq s+1$. Then there exists a p^N-quasi-isomorphism*

$$\tau_{\leq r-s-1}\mathrm{Syn}(D_R, r) \simeq \tau_{\leq r-s-1}\mathbf{R}\Gamma_{\mathrm{cont}}(G_R, T(r)),$$

where $N = N(T, e, r) \in \mathbb{N}$ depends on the representation T, ramification index e, and r. In particular, we have p^N-isomorphisms

$$H^k_{\mathrm{syn}}(D_R, r) \xrightarrow{\simeq} H^k(G_R, T(r)),$$

for $0 \leq k \leq r-s-1$.

The proof of Theorem H proceeds in two main steps: First, we modify the syntomic complex with coefficients in D_R to relate it to a "differential" Koszul complex with coefficients in $\mathbf{N}(T)$. Next, in the second step we modify the Koszul complex from the first step and use a version of Poincaré lemma to obtain Koszul complex computing continuous G_R-cohomology of $T(r)$.

As alluded to before, for $T = \mathbb{Z}_p$, the result was proven in [CN17]. However, direct generalizations did not seem to work and the technical issues tend to amplify when dealing with the case of $\dim_{\mathbb{Q}_p} V \geq 1$. In order to prove the statement of the theorem we will write down explicit complexes with suitable modifications at each step. The key to the connection between syntomic complexes with coefficients and "(φ, Γ)-module Koszul complexes" is provided by the comparison isomorphism in Theorem E. In fact, an attempt to relate these two steps led to our search and discovery of the comparison result in Theorem E in the first place.

What lies ahead?

The world of relative p-adic Hodge theory, though extensively studied in certain directions, remains much less explored and no less challenging than its arithmetic counterpart. Therefore, several natural questions have emerged which remain unaswered.

The very first question that could be asked is whether all crystalline representations are of finite height? This is certainly true for 1-dimensional representations. However, the higher dimensional case remains quite mysterious. An answer to this question would possibly involve recovering the module $\mathbf{N}(V)$ given $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$.

In his recent work [Tsu20], Tsuji has used Wach's ideas (see [Wac97]) and Faltings' generalization of Fontaine-Laffaille modules (see [Fal89]) to construct generalized representations of G_R. His theory has ties to the remarkable work of Bhatt, Morrow and Scholze on $\mathbf{A}_{\text{inf}}$-cohomology in [BMS18]. Tsuji and Morrow in [MT20], have developed a theory of coefficients in integral p-adic Hodge theory. Tsuji's objects are closely related to the definition of Wach modules in the relative case. It would be interesting to explore these relations and obtain some concrete statements on cohomology. Also, it would be interesting to understand the relation between relative Wach modules and coefficients in integral p-adic Hodge theory, as well as, their relation to Bhatt-Scholze theory of prisms and prismatic cohomology in [BS19].

The globalization of the approach of Colmez and Nizioł, helped them in proving the semistable comparison theorem for formal log-schemes. On the other hand, in [Tsu96] Tsuji considered a system of coefficients for syntomic cohomology and obtained similar results under certain restrictions. The result in Theorem H is of similar flavour (at least locally), where we only consider the case of good reduction. It would be interesting to sheafify the notion of finite height representations or Wach modules as in the work of Colmez and Nizioł and in the spirit of crystalline sheaves of Andreatta and Iovita (see [AI13]). Carrying out such a program would yield a comparison isomorphism for proper smooth formal schemes and non-trivial coefficients via syntomic methods.

As mentioned before, for a mixed characteristic discretely valued (possibly ramified) extension of $\mathbb{Q}_p$, Wach modules have been generalized in the form of Breuil-Kisin modules (see [Bre99, Bre02, Kis06]). In the relative setting, Kim has given a certain generalization of Kisin's theory (see [Kim15]). On the other hand, there also exists classification of classical p-adic representations by Caruso in terms of (φ, τ)-modules (see [Car13]). Then it is natural to ask whether there exists an explicit complex (akin to Fontaine-Herr complex) of (relative) Breuil-Kisin modules or (relative) (φ, τ)-modules which computes Galois cohomology of a crystalline representation? Further, in that case it would also be possible to work with semistable representations and log-syntomic complex with coefficients.

A positive answer to the questions above, also opens the door for many applications. One such application could be into Iwasawa theory. In [Ben00], Benois has used Wach modules to study the Iwasawa theory of crystalline representations, in the classical case. One could hope to carry out a similar program in the relative setting.

Présentation en français

Au cours du siècle dernier, le *modus operandi* pour les mathématiciens essayant de comprendre les espaces a été d'étudier les invariants naturels attachés à ces espaces. Cette approche s'est avérée très fructueuse. Un exemple en vient de la topologie où l'on construit des groupes d'homologie singuliers attachés à un espace topologique X. Concrètement, il s'agit d'une collection de groupes abéliens $\{H_k(X,\mathbb{Z})\}_{n\in\mathbb{N}}$ calculés comme l'homologie du complexe singulier attaché à X et le k-ème groupe d'homologie décrit les classes d'équivalence de trous k-dimensionnels dans X. En termes d'application, les annulations des groupes d'homologie établissent des résultats que le théorème du point fixe de Brouwer, entre autres.

En dualisant la construction des complexes de chaînes singulières, on peut définir une théorie contravariante, bien nommée, des groupes de cohomologie singulière $\{H^k(X,\mathbb{Z})\}_{k\in\mathbb{N}}$ attachés à X. De nouveaux développements en mathématiques ont conduit à la construction de théories de (co)homologiques dans une myriade de contextes différents. Par exemple, la cohomologie de de Rham pour les formes différentielles sur les variétés, la (co)homologie de groupe (continue), la cohomologie d'algèbre de Lie, la cohomologie étale pour les variétés algébriques, etc.

Comparaison en géométrie algébrique complexe

En géométrie analytique et algébrique, l'étude de la cohomologique s'est avérée plus naturelle par rapport à l'homologie . De plus, dans des circonstances convenables, certaines théories differentes ont tendance à interagir les unes avec les autres. Une des premières observations faites dans ce sens est due à de Rham [DR31]. En 1931, il montra que pour une variété lisse M, l'accouplement des formes différentielles et de chaînes singulières, via l'intégration, donne un homomorphisme des groupes de cohomologie de de Rham $H^k_{\mathrm{dR}}(M,\mathbb{R})$ aux groupes de cohomologie singuliére $H^k_{\mathrm{sing}}(M,\mathbb{R})$, qui est en fait un isomorphisme (voir [Sam01] pour une étude historique).

En 1966, ce résultat a été étendu au contexte de la géométrie algébrique complexe par Grothendieck. Plus précisément, soit X une variété algébrique complexe lisse et soit X^{an} la variété complexe obtenue à partir des points rationnels complexes $X(\mathbb{C})$ de la variété algébrique X. Dans [Gro66], Grothendieck a défini les groupes de cohomologie de de Rham algébrique pour X et a montré que ceux-ci sont canoniquement isomorphes aux groupes de cohomologie de de Rham analytique de X^{an}. En conclusion, nous avons

Théorèm A (de Rham, Grothendieck)**.** *Soit X une variété algébrique complexe et lisse. Pour chaque $k\in\mathbb{N}$, il existe un isomorphisme canonique d'espaces vectoriels complexes*

$$H^k_{\mathrm{sing}}(X^{\mathrm{an}},\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C} \xrightarrow{\simeq} H^k_{\mathrm{dR}}(X^{\mathrm{an}}/\mathbb{C}) \xrightarrow{\simeq} H^k_{\mathrm{dR}}(X/\mathbb{C}).$$

Les deux côtés de cet isomorphisme apportent des informations complémentaires sur X; à savoir, la cohomologie singulière fournit une structure intégrale pour $H^k_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{R})$ (le réseau des périodes) et la cohomologie de de Rham donne la filtration de Hodge.

En géométrie algébrique complexe, on peut faire mieux. Supposons que X soit un schéma lisse et projectif sur $\mathbb{C}$ et soit X^{an} la variété complexe associée. Alors X^{an} est une variété compacte équipée d'une métrique Kähler. Si nous laissons $\Omega^j_{X^{\mathrm{an}}}$ désigner le faisceau de formes différentielles holomorphes sur X^{an}, alors nous avons la *décomposition de Hodge*

$$H^k_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \bigoplus_{i+j=k} H^i\big(X^{\mathrm{an}}, \Omega^j_{X^{\mathrm{an}}}\big).$$

De plus, soit $\Omega^1_{X/\mathbb{C}}$ le faisceau de différentiels de Kähler sur X et défini $\Omega^j_{X/\mathbb{C}} = \bigwedge^j \Omega^1_{X/\mathbb{C}}$. Puis en combinant la décomposition de Hodge avec le principe GAGA de Serre, on obtient que

$$H^k_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \bigoplus_{i+j=k} H^i\big(X^{\mathrm{an}}, \Omega^j_{X^{\mathrm{an}}}\big) \simeq \bigoplus_{i+j=k} H^i\big(X, \Omega^j_{X/\mathbb{C}}\big).$$

L'un des principaux objectifs de la théorie de Hodge p-adique est d'expliquer un phénomène similaire pour les cohomologies p-adiques de variétés algébriques définies sur un corps p-adiques.

Théorèmes de comparaison p-adiques

Dans cette section, soit p un nombre premier fixe, K un corps d'évaluation discret caractéristique mixte avec un anneau d'entiers O_K et un corps résiduel κ parfait de caractéristique p.

Dans le contexte de la géométrie algébrique, la topologie de Zariski sur les variétés algébriques est trop grossière pour obtenir une notion significative de cohomologie singulière. Par conséquent, en 1963-64, Grothendieck dans [AGV71] a défini des groupes de cohomologie étale attachés à un schéma défini sur n'importe quel corps (en particulier, les extensions finies de $\mathbb{Q}_p$), alors que la définition de la cohomologie algébrique de de Rham s'applique aux schémas lisses. Encore une fois, les mathématiciens ont observé que dans ce cadre, ces deux théories interagissent l'une avec l'autre.

L'origine de la comparaison des théories de cohomologie p-adiques, appelées *théorèmes de comparaison p-adiques*, peut être attribuée aux travaux de Tate sur les groupes p-divisibles dans [Tat67]. Tate a montré que pour un schéma abélien A défini sur O_K, le premier groupe de cohomologie étale de A avec des coefficients dans $\mathbb{Z}_p$ détermine le groupe p-divisible A_{p^∞}, c'est-à-dire le sous-groupe de torsion p-primaire de A, et vice versa. De plus, soit $\overline{K}$ une clôture algébrique fixe de K avec $\mathbb{C}_p$ comme complétion p-adique. Alors le groupe de Galois $G_K := \mathrm{Gal}(\overline{K}/K)$ agit linéairement et continûment sur le $\mathbb{Z}_p$-module $H^1_{\text{ét}}\big(A_{\overline{K}}, \mathbb{Z}_p\big)$. En conséquence de son étude générale des groupes p-divisibles, Tate a montré que pour $k \leq 2 \dim A$, il existe un isomorphisme G_K-équivariant naturel

$$H^k_{\text{ét}}\big(A_{\overline{K}}, \mathbb{Z}_p\big) \otimes_{\mathbb{Z}_p} \mathbb{C}_p \simeq \bigoplus_{i+j=k} H^i\big(A, \Omega^j_A\big) \otimes_K \mathbb{C}_p(-j), \tag{0.5}$$

où pour $j \in \mathbb{Z}$, on définit $\mathbb{C}_p(j) := \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(j)$ et $\mathbb{Q}_p(j)$ est le j-ième puissance tenseur de la représentation p-adique unidimensionnelle $\mathbb{Q}_p(1)$ sur laquelle G_K agit via le caractère cyclotomique p-adique. Tate a conjecturé qu'une décomposition G_K-équivariante comme ci-dessus devrait exister pour toute variété projective lisse définie sur K.

D'autre part, dans [Gro74], Grothendieck a montré que les groupes de cohomologie de de Rham d'un schéma abélien portent également des informations supplémentaires. En utilisant sa théorie cristalline de Dieudonné, il a déterminé que $H^1_{\mathrm{dR}}(A/K)$ est un K-espace vectoriel acquérant une base canonique sur F, où $F = \mathrm{Fr}\, W$ pour $W = W(\kappa)$ l'anneau de vecteurs de Witt p-typiques avec des coefficients dans κ. L'espace vectoriel sur F admet un automorphisme semi-linéaire de Frobenius φ, et possède une filtration de Hodge après extension des scalaires à K. De plus, il a montré que A_{p^∞} est déterminé, à isogénie près, par $H^1_{\mathrm{dR}}(A/K)$ avec sa filtration de Hodge, la base sur F qui est équipé l'automorphisme φ.

Considérant ces deux phénomènes, Grothendieck a été amené à se poser la question de décrire une procédure algébrique qui permettrait de passer directement de $H^1_{\mathrm{dR}}(A/K)$ à $H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$, sans détour par le groupe p-divisible A_{p^∞} ; il soupçonnait aussi qu'une telle procédure devrait exister dans des degrés de cohomologie arbitraires (le problème bien connu du *foncteur mystérieux* de Grothendieck).

Cette question a été résolue par Fontaine en degré un et pour en degré arbitraire il a proposé une conjecture précise dans [Fon82, Fon83]. La *conjecture cristalline* de Fontaine pour un O_K-schéma examine la relation entre la cohomologie p-adique étale de la fibre générique et la cohomologie cristalline de la fibre spéciale. Cette conjecture est maintenant pleinement prouvée par les travaux de nombreux auteurs. Avant d'énoncer la conjecture cristalline, mentionnons les travaux de Faltings généralisant la décomposition de Hodge-Tate dans (0.5):

Théorèm B ([Fal88, Faltings])**.** *Soit X un K-schéma lisse et propre. Alors pour chaque $k \in \mathbb{N}$, il existe un isomorphisme canonique G_K-équivariant*

$$H^k_{\text{ét}}\left(X_{\overline{K}}, \mathbb{Z}_p\right) \otimes_{\mathbb{Z}_p} \mathbb{C}_p \simeq \bigoplus_{i+j=k} H^i\left(X, \Omega^j_X\right) \otimes_K \mathbb{C}_p(-j).$$

L'un des premiers théorèmes de comparaison à être prouvé dans le cadre p-adique, la preuve du théorème B repose sur l'idée de Faltings de presque mathématique.

Revenons maintenant à la conjecture cristalline: Soit X un schéma propre et lisse défini sur O_K, soit $i : X_K \rightarrowtail X$ sa fibre générique et $j : X_\kappa \rightarrowtail X$ désigne sa fibre spéciale. Pour la fibre générique, nous considérerons les groupes de cohomologie p-adique étale usuels $H^i_{\text{ét}}\left(X_{\overline{K}}, \mathbb{Q}_p\right)$, tandis que pour les schémas en caractéristique p, c'est-à-dire X_κ, nous considérerons une variante de la cohomologie de de Rham fournie par Grothendieck, qui est encore une cohomologie p-adique connue sous le nom de *cohomologie cristalline* $H^k_{\mathrm{cris}}(X_\kappa/W(\kappa))$. Ensuite nous avons,

Théorèm C ([FM87, Fontaine-Messing], [Fal89, Faltings], [KM92, Kato-Messing], [Tsu99, Tsuji])**.** *Pour chaque $k \in \mathbb{N}$ il existe un isomorphisme naturel*

$$H^k_{\text{ét}}\left(X_{\overline{K}}, \mathbb{Q}_p\right) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{cris}} \xrightarrow{\simeq} H^k_{\mathrm{cris}}\left(X_\kappa/W(\kappa)\right) \otimes_{W(\kappa)} \mathbf{B}_{\mathrm{cris}},$$

compatible avec l'action de G_K, le Frobenius, la filtration (et la dualité de Poincaré, la formule Künneth, les morphismes de classe de cycle et de classe de Chern) de chaque côté.

Ici $\mathbf{B}_{\mathrm{cris}}$ désigne l'anneau des périodes cristalline construit par Fontaine (voir [Fon94a]), et il est doté d'une action continue de G_K, du Frobenius et d'une filtration .

Dans [FM87] Fontaine et Messing ont lancé un programme pour prouver la conjecture cristalline via des méthodes *syntomiques* et ont réussi à prouver l'affirmation dans le cas $K = F$ et $\dim X_K < p$. Dans [KM92], Kato et Messing ont prouvé la conjecture sous l'hypothèse $\dim X_K < (p-1)/2$ mais sans aucune hypothèse sur K. De plus, ce programme a été généralisé au cas semistable par Fontaine et Janssen. La conjecture semistable a été montrée par Fontaine pour les variétés abéliennes puis prouvée par Kato dans [Kat94] dans le cas $\dim X_K < (p-1)/2$, en généralisant les méthodes de [KM92]. Enfin, ce programme a été conclu par Tsuji dans [Tsu99] complétant la preuve des conjectures cristallines et semistables.

Au cours de quatre décennies, de nombreux mathématiciens ont travaillé sur des théorèmes de comparaison p-adiques. Dans [Fal89], Faltings a prouvé la conjecture cristalline et a également généralisé ses méthodes aux coefficients non triviaux. Il a en outre montré le théorème de comparaison semistable en utilisant sa théorie des extensions presque étales dans [Fal02]. Dans [Niz98] Nizioł a donné une autre preuve de la conjecture cristalline en utilisant la K-théorie. Yamashita a prouvé le cas non approprié dans [Yam11]. En utilisant des constructions complètement différentes, Beilinson a prouvé toutes les incarnations des théorèmes de comparaison p-adiques dans [Bei12, Bei13]. De plus, Scholze a prouvé le théorème de comparaison de Rham pour les variétés analytiques rigides dans [Sch13], où il travaille complètement sur la fibre générique et considère les systèmes locaux

p-adiques non triviaux du côté étale. En généralisant les idées de Faltings, Andreatta et Iovita ont prouvé la comparaison cristalline pour les schémas formels lisses dans [AI13], où leur preuve fonctionne également pour les coefficients non triviaux. De plus, Andreatta et Iovita ont généralisé leur preuve au cas semistable dans [AI12].

Dans [CN17] en utilisant des méthodes et techniques syntomiques de la théorie des (φ, Γ)-modules, Colmez et Nizioł ont prouvé la comparaison semistable pour les schémas logarithmiques formels. La majeure partie de [CN17] consiste en des calculs locaux, c'est-à-dire sur des affinoïdes couvrant le schéma X. Dans le cas du schéma propre et lisse X, le revêtement peut être donné par une algèbre étale sur un tore formel sur O_K. La motivation de nos résultats cohomologiques à coefficients dans cette thèse découle de cet article (voir le théorème H). La poursuite de l'énoncé cohomologique a conduit à notre exploration des représentations cristallines de hauteur finie dans le cadre relatif (voir le théorème E). Nous reviendrons plus tard sur ces connexions.

Une version intégrale des théorèmes de comparaison a été obtenue par Bhatt, Morrow et Scholze dans [BMS18], où ils ont défini une nouvelle théorie de cohomologie sur l'anneau infinitésimal de Fontaine $\mathbf{A}_{\mathrm{inf}}$. Le travail de [BMS18] a été généralisé au cas semistable par Česnavičius et Koshikawa dans [ČK19]. Enfin, généralisant davantage leurs travaux, Bhatt et Scholze ont avancé la théorie de la cohomologie prismatique dans [BS19] qui unifie toutes les théories de cohomologie p-adiques connues.

Représentations p-adiques et algèbre linéaire

Depuis l'époque galoisienne, les mathématiciens se sont intéressés à la compréhension des groupes galoisiens d'extensions de corp. Alors que certains cas finis et profinis sont simples et explicites à énoncer, en général ces groupes sont assez complexes à déchiffrer, par exemple, le groupe de Galois absolu G_K dans la section précédente est aussi loin d'être explicite que possible. Pour comprendre de tels groupes, une approche générale consiste à étudier leurs représentations, c'est-à-dire l'action de tels groupes sur certains modules. C'est un autre thème commun dans la théorie de Hodge p-adique, c'est-à-dire l'étude des représentations p-adiques des groupes de Galois tels que G_K.

Les groupes de cohomologie étale p-adique $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$, apparaissant dans le théorème C, sont $\mathbb{Q}_p$-espaces vectoriels dotés d'une action linéaire et continue du groupe de Galois G_K. En d'autres termes, nous avons obtenu des représentations p-adiques du groupe de Galois G_K. D'autre part, les groupes de cohomologie cristalline $F \otimes_W H^i_{\mathrm{cris}}(X_\kappa / W)$ sont des F-espaces vectoriels équipés d'un automorphisme de Frobenius-semilinéaire φ et une filtration après extension des scalaires à K. Le théorème C indique que ces deux objets sont liés l'un à l'autre.

Dans les années 1980-90, Fontaine a énoncé et réalisé plusieurs programmes afin d'étudier les représentations p-adiques de G_K. Dans [Fon79, Fon82, Fon94a, Fon94b], Fontaine décrit les sous-catégories de représentations cristallines, semi-stables et de Rham. Par exemple, les groupes de cohomologie étale apparaissant dans le théorème C sont des représentations cristallines de G_K. La théorie de Fontaine est riche et un voyage incroyable à parcourir, cependant nous nous contenterons d'une description des représentations cristallines. De plus, par souci de simplicité, nous travaillerons en supposant que $K = F$ n'est pas ramifié sur $\mathbb{Q}_p$, cependant certains des résultats sont vrais dans des contextes plus généraux.

Représentations cristallines

Pour classer les représentations cristallines, Fontaine propose un formalisme général. Il construit un anneau de périodes $\mathbf{B}_{\mathrm{cris}}$ qui est le completé p-adique d'une F-algèbre équipée d'un Frobenius et d'une filtration (voir [Fon94a], nous rappelons la construction dans un cadre plus général dans §1.3). Soit maintenant V une représentation p-adique de G_F, et définissons

$$\mathbf{D}_{\mathrm{cris}}(V) := (\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_F}.$$

C'est un espace vectoriel F de dimension finie tel que $\dim_F \mathbf{D}_{\mathrm{cris}}(V) \leq \dim_{\mathbb{Q}_p} V$, et il est muni d'un endomorphisme semi-linéaire de Frobenius φ, et une filtration venant de la filtration sur $\mathbf{B}_{\mathrm{cris}}$. De plus, cette construction est fonctorial en V et elle prend des valeurs dans la catégorie de φ-modules filtrée sur F. La représentation V est dite *cristalline* si et seulement si elle est $\mathbf{B}_{\mathrm{cris}}$-admissible, ou équivalent, $\dim_F \mathbf{D}_{\mathrm{cris}}(V) = \dim_{\mathbb{Q}_p} V$. En particulier, les périodes p-adiques de V appartiennent à $\mathbf{B}_{\mathrm{cris}}$. Le foncteur $\mathbf{D}_{\mathrm{cris}}$ est exact et pleinment fidèle et établit une équivalence entre la catégorie des représentations cristallines et son image essentielle sous le foncteur, compatible avec les suites exactes, les produits tensoriels et la prise de duals.

La terminologie *cristalline* accentue le fait que si la représentation "vient de la géométrie", c'est-à-dire calculée comme étale cohomologie de fibre générique d'un W-schéma lisse et propre, alors il existe une comparaison avec le cohomologie cristalline de la fibre spéciale. Par exemple, si nous laissons $V_i := H^i_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)$ dans le théorème C, alors nous avons $\mathbf{D}_{\mathrm{cris}}(V_i) = F \otimes_W H^i_{\mathrm{cris}}(X_\kappa / W)$. De plus, étant donné $H^i_{\mathrm{cris}}(X_\kappa/W)$ avec ses structures complémentaires, on peut récupérer $H^i_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p)$ l'espace vectoriel $\mathbb{Q}_p$ avec son action galoisienne, et vice versa. C'est un résultat assez surprenant en contraste avec le cas complexe (voir le théorème A).

(φ, Γ)-modules et représentations de hauteur finie

Une perspective différente sur les représentations p-adiques est la théorie des (φ, Γ)-modules. Moralement, une telle théorie est une tentative de décrire des représentations p-adiques de G_F en termes de modules sur des anneaux de base compliqués, admettant un endomorphisme semi-linéaire de Frobenius et une action plus simple d'un morceau du groupe de Galois.

Plus précisément, soit $F_\infty = \cup_{n\in\mathbb{N}} F(\zeta_{p^n})$ où $\zeta_{p^n} \in \overline{F}$ désigne une racine primitive p^n-ième de l'unité, et soit $\mathbb{C}_p^\flat$ l'inclinaison de $\mathbb{C}_p$ (voir §1.2 pour une définition précise). Soit $H_F = \mathrm{Gal}(\overline{F}/F_\infty)$ et $\Gamma_F = \mathrm{Gal}(F_\infty/F)$, alors on a une suite exacte

$$1 \longrightarrow H_F \longrightarrow G_F \longrightarrow \Gamma_F \longrightarrow 1.$$

En utilisant la construction *corps-des-normes* dans [FW79b, FW79a, Win83], Fontaine et Wintenberger ont défini un corps d'évaluation discret complet non archimédien $\mathbf{E}_F \subset \mathbb{C}_p^\flat$ de caractéristique p avec corp de classe de résidus κ, et fonctorial en F. Dans [Fon90], Fontaine a utilisé la théorie de la construction des corps des normes pour classer les représentations mod-p de G_F en termes des (φ, Γ_F)-modules étale sur $\mathbf{E}_F$. Par quelques considérations techniques, on peut élever cela à la caractéristique 0, c'est-à-dire classer les $\mathbb{Z}_p$-représentations de G_F en termes des (φ, Γ_F)*-modules étale* sur un anneau régulier local de dimension deux $\mathbf{A}_F \subset W\big(\widehat{F}_\infty^\flat\big)$. En particulier, les périodes p-adiques de toute $\mathbb{Z}_p$-représentation de G_F appartiennent à l'anneau $\mathbf{A} \subset W(\mathbb{C}_p^\flat)$. Une équivalence similaire des catégories peut être obtenue pour les représentations p-adiques et les (φ, Γ_F)-modules étale sur $\mathbf{B}_F = \mathbf{A}_F\left[\frac{1}{p}\right]$, i.e. les périodes p-adiques des représentations p-adiques de G_F appartiennent à $\mathbf{B} = \mathbf{A}\left[\frac{1}{p}\right] \subset \mathrm{Fr}\ W(\mathbb{C}_p^\flat)$.

La théorie des (φ, Γ)-modules a été affinée par Cherbonnier et Colmez dans [CC98]. Ils ont montré que toutes les $\mathbb{Z}_p$-représentations (resp. représentations p-adiques) sont *surconvergentes*, i.e. les périodes p-adiques appartiennent à un sous-anneau $\mathbf{A}^\dagger \subset \mathbf{A}$ (resp. $\mathbf{B}^\dagger \subset \mathbf{B}$). De nombreuses applications de (φ, Γ)-modules utilisent le résultat de Cherbonnier-Colmez (voir [CC99], [Ber02, Ber03], etc.).

Le foncteur de corps-des-normes a été ensuite généralisé aux corps locaux de dimension supérieure par Abrashkin dans [Abr07]. Une vaste généralisation de la théorie de Fontaine et Wintenberger, également connue sous le nom de *tilting correspondence*, a été faite par Scholze dans [Sch12].

Représentations cristallines de hauteur finie

Jusqu'ici nous avons vu la classification des représentations p-adiques cristallines de G_F en termes de φ-modules filtrés sur F, et toutes les représentations p-adiques de G_K en termes de (φ, Γ)-modules étale sur $\mathbf{B}_F$. Par cette dernière équivalence de catégories, il devient naturel de se poser la question : est-il possible de décrire des représentations cristallines intrinsèquement dans la catégorie des (φ, Γ)-modules étale? Pour répondre à cette question, Fontaine a lancé un programme reliant les représentations cristallines p-adiques et les représentations de hauteur finie.

Une représentation p-adique V de G_F est dite de *hauteur finie* si les périodes p-adiques de V appartiennent au sous-anneau "intégral" $\mathbf{B}^+ \subset \mathbf{B}$ (voir §3.1). En d'autres termes, le (φ, Γ_F)-module sur $\mathbf{B}_F$ admet une base dans un réseau, c'est-à-dire a une base sur l'anneau de période $\mathbf{B}_F^+ \subset \mathbf{B}_F$. Pour les représentations cristallines il existe des réseaux sur lesquels l'action de Γ_F est plus simple. La hauteur finie et les représentations cristallines de G_F sont liées par le résultat suivant :

Théorèm D ([Wac96, Wach], [Col99, Colmez], [Ber02, Berger])**.** *Soit V une représentation p-adique de G_F. Alors V est cristalline si et seulement s'il est de hauteur finie et il existe $r \in \mathbb{Z}$ et un $\mathbf{B}_F^+$-submodule $N \subset \mathbf{D}(V)$ de rang $= \dim_{\mathbb{Q}_p} V$, stable sous l'action de Γ_F, tel que Γ_F agit trivialement sur $(N/\pi N)(-r)$.*

Dans la situation du théorème D, le module N n'est pas unique. Une construction fonctorial a été donnée par Berger dans [Ber04] à l'aide de laquelle il a établi une equivalence de catégories entre les représentations cristallines de G_F et des modules de Wach sur $\mathbf{B}_F^+$. De plus, pour une représentation cristalline V, il existe une bijection entre $\mathbb{Z}_p$-réasaux à l'intrieur des modules V et modules de Wach sur le sous-anneau intégral $\mathbf{A}_F^+ \subset \mathbf{B}_F^+$, et contenue dans le module de Wach rationnel $\mathbf{N}(V)$. Enfin, étant donné $\mathbf{N}(V)$ on peut récupérer canoniquement l'autre objet algébrique linéaire attaché à V, soit $\mathbf{D}_{\mathrm{cris}}(V)$ (voir [Ber04, Propositions II.2.1 & III. 4.4]).

La théorie et la construction des modules de Wach ont connu de nombreuses applications, par exemple, la théorie d'Iwasawa des représentations cristallines dans [Ben00, BB08], la preuve de Berger de la conjecture de monodromie p-adique [Ber02], ainsi que, dans l'étude du programme de Langlands p-adique local [BB10]. La notion de modules de Wach a été généralisée au cas des modules de Breuil-Kisin sur K corps p-adqique (voir [Bre99, Bre02, Kis06]). L'existence de modules Wach a également servi de motivation pour l'idée de Scholze de q-déformations [Sch17], qui a ouvert la voie à la théorie de Bhatt-Scholze des prismes et à la cohomologie prismatique [BS19]. De plus, similaire à la classification de Berger dans le cas fini non ramifié, Bhatt et Scholze ont montré que pour toute extension finie $K/\mathbb{Q}_p$, la catégorie des F-cristaux prismatiques sur Spf (O_K) est équivalent à la catégorie des $\mathbb{Z}_p$-réseaux à l'intérieur des représentations cristallines de G_K (voir [BS21, Theorem 1.2]).

Représentations cristallines de hauteurs finies relatives

Comme indiqué précédemment, nous nous intéressons à la version locale de la théorie de Hodge p-adique relative. Alors, présentons brièvement la configuration: Fixons maintenant $p \geq 3$, et soit $d \in \mathbb{N}$ avec $X = (X_1, X_2, \ldots, X_d)$ quelques indéterminés. On définit $W\{X\} := \left\{ \sum_{\mathbf{k} \in \mathbb{N}^d} a_{\mathbf{k}} X^{\mathbf{k}}, \text{ où } \mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d, X^{\mathbf{k}} = X_1^{k_1} \cdots X_d^{k_d}, a_{\mathbf{k}} \in W, \text{ et } a_{\mathbf{k}} \longrightarrow 0 \text{ comme } \mathbf{k} \longrightarrow \infty \right\}$, pour être une algèbre p-adiquement complète sur W. De même, nous définissons $R_0 := W\{X^{\pm 1}\}$. Soit $K = F(\zeta_{p^m})$, où $m \in \mathbb{N}_{\geq 1}$, ζ_{p^m} est une racine primitive p^m-ième de l'unité, soit O_K l'anneau des entiers de K et soit $R := O_K\{X^{\pm 1}\}$.

Note. Dans le corps principal de la thèse, nous travaillerons dans une configuration plus générale, c'est-à-dire sur la complétion p-adique d'une algèbre étale sur $W\{X^{\pm 1}\}$ et l'extension correspondante de R_0 et R ci-dessus (voir §1.1). Cependant, par souci de lucidité de l'exposé, nous introduisons les résultats sous des hypothèses simplifiées.

Répresentations cristallines

Inspiré par le formalisme de Fontaine, dans [Bri08] Brinon a étudié les représentations p-adiques de G_R, le groupe fondamental étale de $R\left[\frac{1}{p}\right]$. Dans le cadre relatif, il y a deux notions de représentations cristallines: les représentations cristallines horizontales et les (grandes) représentations cristallines. Nous nous intéressons à cette dernière catégorie de représentations.

Pour classer les représentations cristallines, Brinon construit un anneau de périodes $\mathcal{O}\mathbf{B}_{\mathrm{cris}}$ qui est une $R_0\left[\frac{1}{p}\right]$-algèbre p-adicalement complète équipée de un Frobenius, une filtration et une connexion $\mathbf{B}_{\mathrm{cris}}$-linéaire satisfaisant la transversalité de Griffiths (voir [Bri08], notez que ce sont des versions relatives de la construction de Fontaine, nous rappelons les détails dans §1.3). Soit maintenant V une représentation p-adique de G_{R_0}, et soit

$$\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) := (\mathcal{O}\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_{R_0}}.$$

C'est un $R_0\left[\frac{1}{p}\right]$-module projectif fini de rang $\leq \dim_{\mathbb{Q}_p} V$, et il est muni d'un endomorphisme de Frobenius-semi-linéaire φ, une filtration issue de la filtration sur $\mathcal{O}\mathbf{B}_{\mathrm{cris}}$ et une connexion intégrable quasi-nilpotente satisfaisant la transversalité de Griffiths et issue de la connexion sur $\mathcal{O}\mathbf{B}_{\mathrm{cris}}$ (voir §1.5 pour plus de détails). De plus, cette construction est fonctorial dans V et elle prend des valeurs dans la catégorie des (φ, ∂)-modules filtrés sur $R_0\left[\frac{1}{p}\right]$. La représentation V est dite *cristalline* si et seulement si elle est $\mathcal{O}\mathbf{B}_{\mathrm{cris}}$-admissible (voir §1.5.2). En particulier, les périodes p-adiques de V appartiennent à $\mathcal{O}\mathbf{B}_{\mathrm{cris}}$. Le foncteur $\mathcal{O}\mathbf{D}_{\mathrm{cris}}$ est exact et pleinement fidèle et établit une équivalence entre la catégorie des (grandes) représentations cristallines et son image essentielle sous le foncteur, compatible avec les suites exactes, les produits tensoriels et la prise de duals .

(φ, Γ)-modules et représentations de hauteur finie

Parallèlement au cas arithmétique, dans le cadre relatif, nous pouvons à nouveau classer toutes les représentations p-adiques en termes de (φ, Γ)-modules. Pour $n \in \mathbb{N}$, soit $F_n = F(\zeta_{p^n})$ où ζ_{p^n} est une p^n-ième racine primitive de l'unité. Soit R_n la fermeture intégrale de $R_0 \otimes O_{F_n}\left[X_1^{p^{-n}}, \ldots X_d^{p^{-n}}\right]$ à l'intérieur de $\overline{R}\left[\frac{1}{p}\right]$, et soit $R_\infty := \cup_n R_n$. On définit $G_{R_0} := \mathrm{Gal}\left(\overline{R}\left[\frac{1}{p}\right]/R_0\left[\frac{1}{p}\right]\right)$, $\Gamma_{R_0} := \mathrm{Gal}\left(R_\infty\left[\frac{1}{p}\right]/R_0\left[\frac{1}{p}\right]\right)$, et $H_{R_0} := \mathrm{Ker}\,(G_{R_0} \longrightarrow \Gamma_{R_0})$. L'anneau $R_\infty\left[\frac{1}{p}\right]$ est une extension galoisienne de $R_0\left[\frac{1}{p}\right]$ avec groupe de Galois Γ_{R_0} s'insérant dans une séquence exacte

$$1 \longrightarrow \Gamma'_{R_0} \longrightarrow \Gamma_{R_0} \longrightarrow \Gamma_F \longrightarrow 1, \tag{0.6}$$

où, pour $1 \leq i \leq d$ on a $\Gamma'_{R_0} = \mathrm{Gal}\left(R_\infty\left[\frac{1}{p}\right]/F_\infty R_0\left[\frac{1}{p}\right]\right) \simeq \mathbb{Z}_p^d$, et $\Gamma_F = \mathrm{Gal}(F_\infty/F) \simeq \mathbb{Z}_p^\times$.

La classification de Fontaine a été généralisée par Andreatta dans [And06] au cas relatif. Andreatta construit un analogue de $\mathbf{E}_F$, c'est-à-dire à R_0 il associe un domaine régulier noetherien $\mathbf{E}_{R_0}^+$. De plus, il élève cet anneau à la caractéristique 0, c'est-à-dire que nous avons $\mathbf{A}_{R_0}^+$ équipé d'un endomorphisme de Frobenius et d'une action continue de Γ_{R_0}. Enfin, nous avons $\mathbf{A}_{R_0}$ comme complétion p-adique de $\mathbf{A}_{R_0}^+\left[\frac{1}{\pi}\right]$.

Ensuite, un (φ, Γ_{R_0})-*module étale* est un $\mathbf{A}_{R_0}$-module de génération finie équipé d'un automorphisme Forbenius-semi-linéaire φ et d'un action semi-linéaire et continue de Γ_{R_0}. Andreatta montre qu'il existe une équivalence de catégories entre les $\mathbb{Z}_p$-représentations de G_{R_0} et étale (φ, Γ_{R_0})-modules sur $\mathbf{A}_{R_0}$. En particulier, les périodes p-adiques de toute $\mathbb{Z}_p$-représentation de G_{R_0} vivent dans l'anneau $\mathbf{A} \subset W(\mathbb{C}(R)^\flat)$ (voir §2.1). Une équivalence similaire des catégories peut être obtenue pour les représentations p-adiques et les (φ, Γ_{R_0})-modules étale sur $\mathbf{B}_{R_0} := \mathbf{A}_{R_0}\left[\frac{1}{p}\right]$, c'est-à-dire que les périodes p-adiques des représentations p-adiques de G_{R_0} appartiennent à $\mathbf{B} = \mathbf{A}\left[\frac{1}{p}\right] \subset W(\mathbb{C}(R)^\flat)\left[\frac{1}{p}\right]$. Notez que la discussion ci-dessus est vraie dans un cadre plus général, en particulier pour R (voir §2.1 qui est une adaptation de [And06]).

Dans [AB08], Andreatta et Brinon a généralisé le résultat de Cherbonnier et Colmez au cadre relatif, c'est-à-dire qu'ils ont montré que toutes les $\mathbb{Z}_p$-représentations (resp. p-adiques) de G_{R_0} sont surconvergents (voir §2.2 pour plus de détails), c'est-à-dire que les périodes p-adiques appartiennent à un sous-anneau $\mathbf{A}^\dagger \subset \mathbf{A}$ (resp . $\mathbf{B}^\dagger \subset \mathbf{B}$).

Représentations de Wach

Jusqu'ici nous avons discuté des représentations cristallines et des (φ, Γ)-modules dans le cadre relatif. Parallèlement au cas arithmétique, nous nous intéressons maintenant à la compréhension des représentations à hauteur finie et des modules de Wach dans le cas relatif. De plus, nous nous attendons à ce qu'il y ait un lien entre la hauteur finie et les représentations cristallines.

Soit V une représentation p-adique du groupe de Galois G_{R_0}. On dit qu'elle est de *hauteur finie* si les périodes p-adiques de V appartiennent au sous-anneau $\mathbf{B}^+ \subset \mathbf{B}$ (voir §3.2) . En d'autres termes, le $\mathbf{B}_{R_0}^+ = \mathbf{A}_{R_0}^+\left[\frac{1}{p}\right]$-sous-module $\mathbf{D}^+(V) \subset \mathbf{D}(V)$ (fonctoriel en V) est un (φ, Γ_{R_0})-module de type fini tel que $\mathbf{B}_{R_0} \otimes_{\mathbf{B}_{R_0}^+} \mathbf{D}^+(V) \simeq \mathbf{D}(V)$.

Maintenant, nous prenons V une représentation de Rham p-adique avec des poids de Hodge-Tate non positifs, $T \subset V$ un $\mathbb{Z}_p$-réseau libre de rang $= \dim_{\mathbb{Q}_p} V$, stable sous l'action de G_{R_0}. On dit que V est une *représentation de Wach positive* s'il est de hauteur finie et il existe $\mathbf{N}(T) \subset \mathbf{D}^+(T)$, un (φ, Γ_{R_0})-module projectif fini sur $\mathbf{A}_{R_0}^+$ satisfaisant certaines conditions techniques décrivant l'action de φ et Γ_{R_0} (voir Définition 3.8) . On pose $\mathbf{N}(V) := \mathbf{N}(T)\left[\frac{1}{p}\right]$, et l'unicité de ces modules découle de la définition (voir le lemme 3.14). De plus, ces modules sont équipés d'une filtration naturelle.

Le but du chapitre 3 est de montrer que les représentations de Wach sont cristallines. De plus, pour une représentation de Wach positive V le $\mathbf{B}_{R_0}^+$-module $\mathbf{N}(V)$ et le $R_0\left[\frac{1}{p}\right]$-module $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ sont liés de manière précise et ce dernier peut être récupéré du premier. Pour relier ces objets nous construisons un gros anneau de période relative $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \subset \mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0)$ équipé de Frobenius, filtration, connexion et action de Γ_{R_0} (voir §3.2).

Théorèm E (voir Theorem 3.24)**.** *Soit V une représentation de Wach positive de G_{R_0}, alors V est une représentation cristalline positive. De plus, soit $M\left[\frac{1}{p}\right] := \left(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V)\right)^{\Gamma_{R_0}}$. Alors on a un isomorphisme de $R_0\left[\frac{1}{p}\right]$-modules $M\left[\frac{1}{p}\right] \simeq \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ compatible avec Frobenius, filtration et connexion de chaque côté. De plus, après avoir étendu les scalaires à $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$, on obtient des isomorphismes naturels*

$$\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xleftarrow{\simeq} \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} M\left[\tfrac{1}{p}\right] \xrightarrow{\simeq} \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V),$$

compatible avec Frobenius, filtration, connexion et l'action de Γ_{R_0} de chaque côté.

La preuve du théorème se déroule en trois étapes : Premièrement, nous énonçons explicitement la structure du module de Wach attaché à une représentation de Wach de dimension un, nous montrons également que toutes les représentations cristallines unidimensionnelles sont des représentations de Wach et on peut récupérer $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ en commençant avec le module de Wach $\mathbf{N}(V)$. Ensuite, dans des dimensions supérieures et dans les conditions du théorème E, nous décrirons un processus par lequel nous pouvons récupérer un sous-module de $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ à partir du module de Wach, on établit ici une comparaison en passant au cas unidimensionnel. Enfin, les affirmations faites dans le théorème sont montrées en exploitant certaines propriétés des modules de Wach et la comparaison obtenue dans la deuxième étape. Dans la deuxième étape, l'approximation pour l'action de la partie géométrique de Γ_{R_0} s'avère non triviale et la plupart de notre travail consiste à montrer cette partie ; la partie arithmétique de Γ_{R_0} découle des travaux de Wach [Wac96].

Complexe syntomique et cohomologie galoisienne

Après avoir introduit une classe intéressante de représentations, nous revenons à notre discussion de la conjecture cristalline dans le théorème C. Soit $K = F(\zeta_{p^m})$ pour $m \geq 1$, soit X un schéma formel propre et lisse sur O_K, tel que $j : X_K := X \otimes_{O_K} K \rightarrowtail X$ désigne l'inclusion de sa fibre générique et $i : X_\kappa := X \otimes_{O_K} \kappa \rightarrowtail X$ désigne l'inclusion de sa fibre spéciale. Pour attaquer la conjecture cristalline, Fontaine et Messing ont lancé un programme pour la prouver via des méthodes syntomiques (voir [FM87]). Pour $r \geq 0$, soit $\mathcal{S}_n(r)_X$ le faisceau syntomique modulo p^n sur $X_{\kappa,\text{ét}}$. Il peut être considéré comme un espace propre dérivé Frobenius sur un morceau de la cohomologie cristalline. Ensuite, Fontaine et Messing ont construit des morphismes de periodes

$$\alpha_{r,n}^{\mathrm{FM}} : \mathcal{S}_n(r)_X \longrightarrow i^*\mathbf{R}j_*\mathbb{Z}/p^n(r)'_{X_K}$$

de la cohomologie syntomique aux cycles proches p-adiques, où $\mathbb{Z}_p(r)' := \frac{1}{p^{a(r)}}\mathbb{Z}_p(r)$, pour $r = (p-1)a(r) + b(r)$ avec $0 \leq b(r) \leq p-1$.

Dans [CN17], Colmez et Nizioł ont montré que l'application des périodes Fontaine-Messing $\alpha_{r,n}^{\mathrm{FM}}$, après une troncature appropriée, est essentiellement un quasi-isomorphisme. Plus précisément,

Théorèm F ([CN17, Theorem 1.1])**.** *Pour $0 \leq k \leq r$, l'application*

$$\alpha_{r,n}^{\mathrm{FM}} : \mathcal{H}^k(\mathcal{S}_n(r)_X) \longrightarrow i^*\mathbf{R}^k j_*\mathbb{Z}/p^n(r)'_{X_K},$$

est un p^N-isomorphisme, c'est-à-dire qu'il existe $N = N(e,p,r) \in \mathbb{N}$ dépendant de r et de l'indice de ramification absolu e de K mais pas de X ou n, de sorte que le noyau et le conoyau du morphisme sont tués par p^N.

En fait, pour $k \leq r \leq p-1$, l'application $\alpha_{r,n}^{\mathrm{FM}}$ a été montrée être un isomorphisme par Kato [Kat89, Kat94], Kurihara [Kur87] et Tsuji [Tsu99]. Dans [Tsu96], Tsuji a généralisé ce résultat à certains systèmes locaux.

Le théorème F est également valable pour le changement de base des schémas lisses et propres. En particulier, après passage à la limite et inversion de p ci-dessus, pour chaque $0 \leq k \leq r$ on obtient un isomorphisme

$$\alpha_r^{\mathrm{FM}} : H^k_{\mathrm{syn}}\big(X_{O_{\overline{K}}}, r\big)_{\mathbb{Q}} \xrightarrow{\simeq} H^k_{\text{ét}}\big(X_{\overline{K}}, \mathbb{Q}_p(r)\big). \tag{0.7}$$

L'isomorphisme affiché ci-dessus est l'étape la plus importante pour prouver la conjecture cristalline via des méthodes syntomiques. Ces idées ont été utilisées dans [FM87], [KM92], [Kat87], [Kat94] et [Tsu99]. Cependant, toutes ces preuves ont été élaborées directement sur $\overline{K}$, mais sans aucune restriction sur r.

La preuve de Colmez et Nizioł est différente des approches précédentes. Ils prouvent d'abord le théorème F, et en déduisent la comparaison dans (0.7) via changement de base. Pour prouver leur affirmation, ils construisent une autre morphisme de période locale $\alpha_r^{\mathcal{L}az}$, en utilisant des techniques de la théorie des (φ,Γ)-modules et une version de l'isomorphisme intégral de Lazard entre la cohomologie de l'algèbre de Lie et la cohomologie de groupe continue. Ensuite, ils montrent que morphisme est un quasi-isomorphisme et coïncide avec le morphisme de Fontaine-Messing à quelques constantes près. De plus, tous leurs résultats ont été élaborés dans le cas général de schémas logarithmiques.

Calcul local de Colmez et Nizioł

Comme précisé précédemment, la majeure partie de [CN17] consiste en des calculs locaux, c'est-à-dire sur des affinoïdes couvrant un schéma formel. Dans le cas d'un schéma formel propre et lisse, le revêtement peut être donné par une algèbre étale sur $R = O_K\{X^{\pm 1}\}$ où $X = (X_1, \ldots, X_d)$ sont des indéterminés. Pour énoncer le résultat local, nous nous limiterons au cadre familier de R, mais les résultats sont également valables pour une algèbre étale sur R (Colmez et Nizioł travaillent également avec des structures log).

Soit R_ϖ^+ la complétion (p, X_0)-adique de $W[X_0, X^{\pm 1}]$, et soit $S = R_\varpi^{\mathrm{PD}}$ désigne la complétion p-adique de l'enveloppe de puissance divisée par rapport au noyau de la morphisme $R_\varpi \twoheadrightarrow R$ envoyant X_0 à $\zeta_{p^m} - 1$. De plus, soit Ω^1_S la complétion p-adique du module de différentiels de S par rapport à $\mathbb{Z}$ et $\Omega^k_S = \bigwedge^k \Omega^1_S$ pour $k \in \mathbb{N}$. La cohomologie syntomique de R peut être calculée par le complexe

$$\mathrm{Syn}(R,r) := \mathrm{Cone}\Big(F^r\Omega^\bullet_S \xrightarrow{p^r - p^\bullet\varphi} \Omega^\bullet_S\Big)[-1],$$

tel que nous avons $H^i_{\text{syn}}(R,r) = H^i(\text{Syn}(R,r))$. Si K contient suffisamment de racines d'unité, c'est-à-dire pour m assez grand, Colmez et Nizioł a montré que,

Théorèm G ([CN17, Theorem 1.6]). *Les morphismes*

$$\begin{aligned} \alpha_r^{\mathcal{L}az} &: \tau_{\leq r}\text{Syn}(R,r) \longrightarrow \tau_{\leq r}\mathbf{R}\Gamma_{\text{cont}}(G_R, \mathbb{Z}_p(r)), \\ \alpha_{r,n}^{\mathcal{L}az} &: \tau_{\leq r}\text{Syn}(R,r)_n \longrightarrow \tau_{\leq r}\mathbf{R}\Gamma_{\text{cont}}(G_R, \mathbb{Z}/p^n(r)) \longrightarrow \tau_{\leq r}\mathbf{R}\Gamma\big(\big(\text{Sp } R\big[\tfrac{1}{p}\big]\big)_{\text{ét}}, \mathbb{Z}/p^n(r)\big), \end{aligned} \tag{0.8}$$

sont des p^{Nr}-quasi-isomorphismes pour une constante universelle N.

Enfin, en utilisant la descente galoisienne on peut obtenir le résultat sur K (pas forcément ayant assez de racines d'unité, avec N dépendant de K, p et r, voir [CN17, Théorème 5.4]). Notez que la truncation désigne ici la truncation canonique dans la littérature. La preuve de Colmez et Nizioł consiste à comparer le complexe syntomique avec le complexe de (φ,Γ)-modules calculant la G_R-cohomologie continue de $\mathbb{Z}_p(r)$. Ceci est réalisé en utilisant une version du lemme de Poincaré. De plus, notez qu'ils fonctionnent avec des structures log, c'est-à-dire que toutes les définitions ci-dessus doivent être remplacées par leurs analogues log (sans structures log, il faut tronquer en degré $\leq r-1$, voir le théorème H ci-dessous).

Complexe de Fontaine-Herr

Le côté droit de l'application dans la version p-adique du résultat de Colmez et Nizioł, c'est-à-dire le premier isomorphisme dans (0.4), concerne le calcul de la G_R-cohomologie continue de $\mathbb{Z}_p(r)$. Ce calcul peut être effectué avec des complexes constitués de (φ,Γ)-modules, dont les origines se trouvent dans les travaux de Herr (voir [Her98]).

Soit V une représentation p-adique (resp. $\mathbb{Z}_p$-representation) de G_F, et soit $\mathbf{D}(V)$ le (φ,Γ_F)-module étale associé sur $\mathbf{B}_F$ (resp. $\mathbf{A}_F$). Soit $\gamma \in \Gamma_F$ un générateur topologique de Γ_F, alors on a un complexe

$$C^\bullet : \mathbf{D}(V) \xrightarrow{(1-\varphi,\gamma-1)} \mathbf{D}(V) \oplus \mathbf{D}(V) \xrightarrow{\begin{pmatrix} \gamma-1 \\ 1-\varphi \end{pmatrix}} \mathbf{D}(V),$$

où la deuxième application est $(x,y) \mapsto (\gamma-1)x - (1-\varphi)y$. Le complexe de Fontaine-Herr $C^\bullet$ calcule la G_F-cohomologie continue de V dans chaque degré cohomologique, c'est-à-dire que pour $k \in \mathbb{N}$, on a les isomorphes naturels $H^k(C^\bullet) \simeq H^k_{\text{cont}}(G_F, V)$.

Les groupes de G_F-cohomologie continus sont des invariants utiles attachés à V. Par exemple, le premier groupe de cohomologie continue de V, $H^1_{\text{cont}}(G_F, V)$ classifie les extensions de la représentation triviale $\mathbb{Q}_p$ par V dans $\text{Rep}_{\mathbb{Q}_p}(G_F)$, et qui peut être représenté par un couple $x, y \in \mathbf{D}(V)$ satisfaisant l'équation $(\gamma-1)x = (1-\varphi)y$. De plus, si V est cristalline alors toute extension cristalline de $\mathbb{Q}_p$ par $V(r)$ (torsion cyclotomique de V) peut être représentée par une paire (x,y) avec $x \in \mathbf{N}(V)(r)$ et $y \in \mathbf{N}(V(r))$ tels que $(\gamma-1)x = (1-\varphi)y$ (voir le lemme A.2 et Proposition A.4). En fait, cette affirmation combinée au calcul effectué par Colmez et Nizioł a servi de motivation originale pour l'obtention du théorème H.

Dans le cas relatif, nous avons la version relative du complexe de Fontaine-Herr qui calcule la G_R-cohomologie continue d'une représentation p-adique (voir [AI08, Théorème 3.3], on rappelle la description dans §4.1). Des complexes explicites calculant la G_R-cohomologie continue de T peuvent également être obtenus, que nous appelons collectivement *complexes de Koszul* (voir §4.2). De plus, les complexes de Koszul jouent un rôle central dans la preuve du théorème H.

Complexe syntomique à coefficients

Dans le théorème G, nous nous intéressons au résultat p-adique, c'est-à-dire le premier isomorphisme dans (0.8). Notre objectif est d'y remplacer la représentation $\mathbb{Z}_p(r)$ par une représentation plus générale $T(r)$, et d'adapter la méthode de Colmez et Nizioł pour obtenir une relation entre le complexe syntomique à coefficients et la cohomologie G_R-continue de $T(r)$. La classe de représentations qui nous intéresse est celle des représentations cristallines de Wach de G_{R_0}. Dans la notation du théorème E, pour les coefficients du complexe syntomique, nous choisirons un réseau à l'intérieur du (φ, ∂)-module filtré $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$, alors que pour calculer la cohomologie galoisienne nous exploiterons les propriétés du module de Wach associé $\mathbf{N}(V)$. Les deux côtés seront ensuite comparés en utilisant une version du lemme de Poincaré, où est cruciale la comparaison obtenue dans le théorème E.

Plus précisément, soit V une représentation p-adique de Wach de G_{R_0} avec des poids de Hodge-Tate non positifs et soit $s \in \mathbb{N}$ le maximum parmi les valeurs absolues poids de Hodge-Tate de V. Soit $T \subset V$ un $\mathbb{Z}_p$-réseau libre de rang $= \dim_{\mathbb{Q}_p} V$ stable sous l'action de G_{R_0}. Supposons que $\mathbf{N}(T)$ est un $\mathbf{A}_{R_0}^+$-module libre de rang $= \dim_{\mathbb{Q}_p} V$, et qu'il existe un R_0-sous-module libre $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ de rang $= \dim_{\mathbb{Q}_p} V$, tel que $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)\left[\frac{1}{p}\right] = \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ et la connexion induite sur $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$ est quasi-nilpotente, intégrable et satisfait la transversalité de Griffiths par rapport à la filtration induite. Soit $r \in \mathbb{N}$ et on pose $V(r) := V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(r)$ et $T(r) := T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(r)$.

Le choix de $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$ n'est pas canonique et nous discutons de quelques manières d'obtenir un tel module dans proposition 3.31, remarque 3.42 et remarque 5.4. Cependant, nous fixons un tel choix pour le reste de la discussion.

On pose

$$D^{\mathrm{PD}} := R_{\varpi}^{\mathrm{PD}} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T).$$

Il existe un endomorphisme semi-linéaire de Frobenius sur D^{PD} donné par l'action diagonale du Frobenius sur chaque composante du produit tensoriel, une filtration provenant du produit de filtration sur chaque composante du produit tensoriel et une connexion induite par la connexion sur $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$ et l'opérateur différentiel sur R_{ϖ}^{PD}. De plus, cette connexion est intégrable et satisfait la transversalité de Griffiths par rapport à la filtration (voir chapitre 5 pour des définitions précises). En particulier, soit $S = R_{\varpi}^{\mathrm{PD}}$ et nous avons un complexe de de Rham filtré pour $k \in \mathbb{Z}$,

$$\mathrm{Fil}^k \mathcal{D}^\bullet := \mathrm{Fil}^k D^{\mathrm{PD}} \otimes_S \Omega_S^1 \longrightarrow \mathrm{Fil}^{k-1} D^{\mathrm{PD}} \otimes_S \Omega_S^1 \longrightarrow \cdots.$$

Soit $D_R := R \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$. Définir le *complexe syntomique* $\mathrm{Syn}(D_R, r)$ et la *cohomologie syntomique* de R avec des coefficients dans D_R comme

$$\begin{aligned} \mathrm{Syn}(D_R, r) &:= \left[\ \mathrm{Fil}^r \mathcal{D}^\bullet \xrightarrow{p^r - p^\bullet \varphi} \mathcal{D}^\bullet\ \right]; \\ H^*_{\mathrm{syn}}(D_R, r) &:= H^*(\mathrm{Syn}(D_R, r)). \end{aligned}$$

Nous allons relier ce complexe au complexe de Fontaine-Herr calculant la G_R-cohomologie continue de $T(r)$. Le résultat principal du chapitre 5 est :

Théorèm H (voir Theorem 5.6)**.** *Soit V une représentation de Wach positive de G_{R_0}, $T \subset V$ un G_{R_0}-stable libre $\mathbb{Z}_p$-réseau, $s \in \mathbb{N}$ le maximum parmi les valeurs absolues des poids de Hodge-Tate de V et $r \in \mathbb{N}$ tels que $r \geq s+1$. Alors il existe un p^N-quasi-isomorphisme*

$$\tau_{\leq r-s-1} \mathrm{Syn}(D_R, r) \simeq \tau_{\leq r-s-1} \mathbf{R}\Gamma_{\mathrm{cont}}(G_R, T(r)),$$

où $N = N(T, e, r) \in \mathbb{N}$ dépend de la représentation T, de l'indice de ramification e, et r. En particulier, on a des p^N-isomorphismes

$$H^k_{\mathrm{syn}}(D_R, r) \xrightarrow{\simeq} H^k(G_R, T(r)),$$

pour $0 \leq k \leq r-s-1$.

La preuve du théorème H se déroule en deux étapes principales: dans une première étape, on modifie le complexe syntomique à coefficients dans D_R pour le relier à un complexe de Koszul “différentiel” à coefficients dans $\mathbf{N}(T)$. Ensuite, dans la deuxième étape, nous modifions le complexe de Koszul de la première étape et utilisons une version du lemme de Poincaré pour obtenir le complexe de Koszul calculant la G_R-cohomologie continue de $T(r)$.

Comme mentionné précédemment, pour $T = \mathbb{Z}_p$, le résultat a été prouvé dans [CN17]. Cependant, les généralisations directes ne semblent pas fonctionner et les problèmes techniques ont tendance à s'amplifier lorsqu'on traite le cas de $\dim_{\mathbb{Q}_p} V \geq 1$. Afin de prouver l'énoncé du théorème, nous écrirons des complexes explicites avec des modifications appropriées à chaque étape. La clé de la connexion entre les complexes syntomiques à coefficients et les "complexes de Koszul de (φ, Γ)-module" est fournie par l'isomorphisme de comparaison dans le théorème E. En fait, une tentative de relier ces deux étapes a conduit à notre recherche et à notre découverte du résultat de la comparaison dans le théorème E en premier lieu.

Qu'est-ce qui est devant?

Le monde de la théorie de Hodge p-adique relative, bien que largement étudié dans certaines directions, reste beaucoup moins exploré et non moins difficile que son pendant arithmétique. Par conséquent, plusieurs questions naturelles ont émergé qui restent sans réponse.

La toute première question qui pourrait être posée est de savoir si toutes les représentations cristallines sont de hauteur finie? Ceci est certainement vrai pour les représentations à une dimension. Cependant, le cas de dimension supérieure reste assez mystérieux. Une réponse à cette question impliquerait éventuellement de récupérer le module $\mathbf{N}(V)$ étant donné $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$.

Dans son travail récent [Tsu20], Tsuji a utilisé les idées de Wach (voir [Wac97]) et la généralisation de Faltings des modules de Fontaine-Laffaille (voir [Fal89]) pour construire des représentations généralisées de G_R. Sa théorie est liée aux travaux remarquables de Bhatt, Morrow et Scholze sur la $\mathbf{A}_{\mathrm{inf}}$-cohomologie dans [BMS18]. Tsuji et Morrow dans [MT20], ont développé une théorie des coefficients en théorie de Hodge p-adique intégrale. Les objets de Tsuji sont étroitement liés à la définition des modules Wach dans le cas relatif. Il serait intéressant d'explorer ces relations et d'obtenir des énoncés concrets sur la cohomologie. De plus, il serait intéressant de comprendre la relation entre les modules de Wach relatifs et les coefficients dans la théorie de Hodge p-adique intégrale, ainsi que leur relation avec la théorie de Bhatt-Scholze des prismes et la cohomologie prismatique dans [BS19].

La globalisation de l'approche de Colmez et Nizioł, les a aidés à prouver le théorème de comparaison semi-stable pour les log-schémas formels. D'autre part, dans [Tsu96], Tsuji a considéré un système de coefficients pour la cohomologie syntomique et a obtenu des résultats similaires sous certaines restrictions. Le résultat du théorème H est de même saveur (au moins localement), où l'on ne considère que le cas d'une bonne réduction. Il serait intéressant de structurer la notion de représentations à hauteurs finies ou modules de Wach comme dans les travaux de Colmez et Nizioł et dans l'esprit des faisceaux cristallins d'Andreatta et Iovita (voir [AI13]). La réalisation d'un tel programme produirait un isomorphisme de comparaison pour des schémas formels lisses appropriés et des coefficients non triviaux via des méthodes syntomiques.

Comme mentionné précédemment, pour une extension finie (éventuellement ramifiée) de $\mathbb{Q}_p$, les modules de Wach ont été généralisés sous la forme de modules de Breuil-Kisin (voir [Bre99, Bre02, Kis06]). Dans le cas relatif, Kim a donné une certaine généralisation de la théorie de Kisin (voir [Kim15]). D'autre part, il existe aussi une classification des représentations p-adiques classiques par Caruso en termes de (φ, τ)-modules (voir [Car13]). Il est alors naturel de se demander s'il existe un complexe explicite (apparenté au complexe de Fontaine-Herr) de modules (relatifs) de Breuil-Kisin ou de (φ, τ)-modules (relatifs) qui calcule la cohomologie galoisienne d'un représentation? De plus, dans ce cas, il serait également possible de travailler avec des représentations semi-stables et des complexes log-syntomiques à coefficients.

Une réponse positive aux questions ci-dessus, ouvre également la porte à de nombreuses applications. Une telle application pourrait être dans la théorie d'Iwasawa. Dans [Ben00], Benois a utilisé des modules de Wach pour étudier la théorie d'Iwasawa des représentations cristallines, dans le cas classique. On pourrait espérer réaliser un programme similaire dans le cas relatif.

CHAPTER 1

p-adic Hodge theory

Let K be a mixed characteristic non-archimedean complete discrete valuation field, with ring of integers O_K and residue field κ of characteristic p. For κ a perfect field, Fontaine established in [Fon94a] the theory of p-adic de Rham and crystalline representations of the absolute Galois group G_K of K. Moreover, he classified crystalline representations in terms of certain linear algebraic objects called filtered φ-modules over $F = W(\kappa)\left[\frac{1}{p}\right]$, where $W(\kappa)$ denotes the p-typical Witt vectors with coefficients in κ. Generalizing this approach in [Bri06], Brinon studied the p-adic crystalline and de Rham representations of G_K in the case when κ is a non-perfect field admitting a finite p-basis, i.e. $[\kappa : \kappa^p] < +\infty$ and gave a similar classification for crystalline representations of G_K. This theory was further extended by Brinon in [Bri08], to the *relative* case, where he again considers κ to be perfect but replaces K by $R\left[\frac{1}{p}\right]$ for certain integral, normal and p-adically complete O_K-algebra R. In this section our objective is to recall constructions and results in the relative case, albeit in a simpler setting compared to Brinon's book.

1.1. Setup and notations

In this section, we will describe the setup for the rest of the text and fix some notations. Our conventions and notations are by and large in agreement with the article of Colmez and Nizioł [CN17].

Convention. We will work under the convention that $0 \in \mathbb{N}$, the set of natural numbers.

Let $p \geq 3$ be a fixed prime number, κ a finite field of characteristic p, $W := W(\kappa)$ the ring of p-typical Witt vectors with coefficients in κ and $F := W\left[\frac{1}{p}\right]$, the fraction field of W. In particular, F is an unramified extension of $\mathbb{Q}_p$ with ring of integers $O_F = W$. For $n \in \mathbb{N}$, let ζ_{p^n} denote a primitive p^n-th root of unity, and we set $F_n := F(\zeta_{p^n})$ and $F_\infty := \cup_n F_n$. From now onwards, we will fix some $m \in \mathbb{N}$ and set $K := F_m$, with ring of integers O_K. Let $\overline{K} = \overline{F}$ be a fixed algebraic closure of K such that its residue field, denoted as $\overline{\kappa}$, is an algebraic closure of κ. Further, we denote by $G_K = \mathrm{Gal}(\overline{K}/K)$, the absolute Galois group of K. The element $\varpi = \zeta_{p^m} - 1 \in O_K$ is a uniformizer of K, and its minimal polynomial $P_\varpi(X) = \frac{(1+X)^{p^m}-1}{(1+X)^{p^{m-1}}-1}$ is an Eisenstein polynomial in $W[X]$ of degree $e := [K : F] = p^{m-1}(p-1)$.

Let $Z = (Z_1, \ldots, Z_s)$ denote a set of indeterminates and $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}^s$ be a multi-index, then we write $Z^{\mathbf{k}} := Z_1^{k_1} \cdots Z_s^{k_s}$. For $\mathbf{k} \to \infty$ we will mean that $\sum k_i \to \infty$. Now for a topological algebra

Λ we define
$$\Lambda\{Z\} := \Big\{ \sum_{\mathbf{k}\in\mathbb{N}^s} a_{\mathbf{k}} Z^{\mathbf{k}}, \text{ where } a_{\mathbf{k}} \in \Lambda \text{ and } a_{\mathbf{k}} \longrightarrow 0 \text{ as } \mathbf{k} \longrightarrow \infty \Big\}.$$

We are interested in the p-adic Hodge theory of an étale algebra over a formal torus defined over O_K. More precisely, let $d \in \mathbb{N}$ and $X = (X_1, X_2, \dots, X_d)$ be some indeterminates. Let R_0 denote the p-adic completion of an étale algebra over $W\{X^{\pm 1}\}$. In other words, we have a presentation

$$R_0 := W\{X, X^{-1}\}\{Z_1, \dots, Z_s\}/(Q_1, \dots, Q_s),$$

where $Q_i(Z_1, \dots, Z_s) \in W\{X, X^{-1}\}[Z_1, \dots, Z_s]$ for $1 \leq i \leq s$, are multivariate polynomials such that $\det\big(\frac{\partial Q_i}{\partial Z_j}\big)_{1\leq i,j\leq s}$ is invertible in R_0. Finally, we set $R = R_0[\varpi]$, which is absolutely ramified at the prime ideal $(p) \subset R_0$.

Next, we provide a system of coordinates for R, which we call a *framing*. Let

$$R_\square := O_K\{X, X^{-1}\},$$

and endow it with the spectral norm. Using the polynomials appearing in the definition of R_0, we can write
$$R := R_\square\{Z_1, \dots, Z_s\}/(Q_1, \dots, Q_s).$$
Therefore, we have a Cartesian diagram

$$\begin{tikzcd} \operatorname{Spf} R \arrow[r] \arrow[d] & \operatorname{Spf} R_0 \arrow[d] \\ \operatorname{Spf} R_\square \arrow[r] & \operatorname{Spf} W\{X, X^{-1}\}, \end{tikzcd}$$

and $R_\square$ provides a *system of coordinates* for R. From the assumptions on R, we have that R is *small* in the sense of Faltings (see [Fal88, §II 1(a)]).

The p-adic Hodge theory of R entails a study of p-adic representations of the étale fundamental group of $R\big[\frac{1}{p}\big]$, which we introduce next. We fix an algebraic closure of $\operatorname{Fr} R$ as $\overline{\operatorname{Fr} R}$ such that it contains $\overline{K}$. Let $\overline{R}$ denote the union of finite R-subalgebras $S \subset \overline{\operatorname{Fr} R}$, such that $S\big[\frac{1}{p}\big]$ is étale over $R\big[\frac{1}{p}\big]$. Let $\overline{\eta}$ denote the corresponding geometric point of the generic fiber $\operatorname{Spec} R\big[\frac{1}{p}\big]$ and let $G_R := \pi_1^{\text{ét}}(\operatorname{Spec} R\big[\frac{1}{p}\big], \overline{\eta})$ denote the étale fundamental group. By [Gro63, Exposé V, §8], we can write this étale fundamental group as the Galois group (of the fraction field of $\overline{R}\big[\frac{1}{p}\big]$ over the fraction field of $R\big[\frac{1}{p}\big]$)
$$G_R = \pi_1^{\text{ét}}\big(\operatorname{Spec} R\big[\tfrac{1}{p}\big], \overline{\eta}\big) = \operatorname{Gal}\big(\overline{R}\big[\tfrac{1}{p}\big]/R\big[\tfrac{1}{p}\big]\big).$$

Proposition 1.1 ([Bri08, Proposition 2.0.2])**.** *For any $q \in \mathbb{N}$, let Ω_R^q denote the p-adic completion of the module of differentials of R relative to $\mathbb{Z}$. Then we have*

$$\Omega^1_{R_0} = \bigoplus_{i=1}^{d} R_0 \, d\log X_i, \quad \textit{and} \quad \Omega^q_{R_0} = \bigwedge^q \Omega^1_{R_0}.$$

Moreover, the kernel and cokernel of the natural map $\Omega^q_{R_0} \otimes_{R_0} R \longrightarrow \Omega^q_R$ is killed by a power of p. In particular, we have
$$\Omega^q_R\big[\tfrac{1}{p}\big] = \bigwedge^q \Big(\bigoplus_{k=1}^{d} R_0\big[\tfrac{1}{p}\big] \, d\log X_i\Big).$$

For R_0 and R we have that $R = R_0[\varpi]$, $R_0/pR_0 \xrightarrow{\simeq} R/\varpi R$ and for any $n \in \mathbb{N}$, $R_0/p^n R_0$ is a formally smooth $\mathbb{Z}/p^n\mathbb{Z}$-algebra. Finally, we fix a lift $\varphi : R_0 \longrightarrow R_0$ of the absolute Frobenius $x \mapsto x^p$ over $R/\varpi R$.

Convention. While working with completion of tensor products, we would assume it to be the completion of the usual tensor product for the p-adic topology.

1.2. The de Rham period ring

In this section we will recall definitions and properties of the relative version of Fontaine's period ring $\mathbf{B}_{\mathrm{dR}}$. These rings will be useful in classifying de Rham representations of G_R. We begin by recalling some well-known constructions from [Fon94a].

Let us note that the field $\mathbb{C}_p = \widehat{\overline{K}}$, the p-adic completion of $\overline{K}$, is a perfectoid field and we denote its ring of integers as $O_{\mathbb{C}_p}$. We have the tilt of $O_{\mathbb{C}_p}$ as

$$O_{\mathbb{C}_p}^{\flat} := \lim_{x \mapsto x^p} O_{\mathbb{C}_p}/pO_{\mathbb{C}_p} = \lim_{x \mapsto x^p} O_{\overline{K}}/pO_{\overline{K}},$$

The element $p \in O_{\mathbb{C}_p}$ is a pseudo-uniformizer and therefore $p^{\flat} := (p, p^{1/p}, p^{1/p^2}, \ldots) \in O_{\mathbb{C}_p}^{\flat}$ is a pseudo-uniformizer. We set $\mathbb{C}_p^{\flat} := O_{\mathbb{C}_p}^{\flat}\big[\frac{1}{p^{\flat}}\big]$, which is a perfect field in characteristic p.

Next, we endow $\overline{R}\big[\frac{1}{p}\big]$ with the spectral valuation v_p, i.e. $v_p(x) = \sup\{v_p(z)$, for $z \in \mathbb{C}_p^{\times}$ such that $x \in z\overline{R}\}$. Denote by $\mathbb{C}(R)$ the completion of $\overline{R}\big[\frac{1}{p}\big]$ for v_p and $\mathbb{C}^+(R) := \{x \in \mathbb{C}(R)$, such that $v_p(x) \geq 0\}$, which is a subring of $\mathbb{C}(R)$. We define $\mathbb{C}^+(R)^{\flat}$ as the tilt of $\mathbb{C}^+(R)$, i.e.

$$\mathbb{C}^+(R)^{\flat} := \lim_{x \mapsto x^p} \mathbb{C}^+(R)/p\mathbb{C}^+(R) = \lim_{x \mapsto x^p} \overline{R}/p\overline{R},$$

and we set $\mathbb{C}(R)^{\flat} := \mathbb{C}^+(R)^{\flat}\big[\frac{1}{p^{\flat}}\big]$. An element $x \in \mathbb{C}(R)^{\flat}$ can be described as a sequence $(x_n)_{n\in\mathbb{N}}$, with $x_n \in \mathbb{C}(R)$ and $x_{n+1}^p = x_n$, for all $n \in \mathbb{N}$. We define $v^{\flat}$ on $\mathbb{C}(R)^{\flat}$ by setting $v^{\flat}(x) := v_p(x^{\sharp})$ where $x^{\sharp} := x_0$. This is a valuation on $\mathbb{C}(R)^{\flat}$ for which it is complete and we have that $\mathbb{C}^+(R)^{\flat}$ is the subring of elements $x \in \mathbb{C}(R)^{\flat}$ such that $v^{\flat}(x) \geq 0$. These rings admit an action of the Galois group G_R which is continuous for the valuation topology.

We will fix some choices of compatible p-power roots which will appear throughout the text. Let

$$\begin{aligned} \varepsilon &:= (1, \zeta_p, \zeta_{p^2}, \ldots) \in \mathbb{C}_p^{\flat}, \\ X_i^{\flat} &:= \big(X_i, X_i^{1/p}, X_i^{1/p^2}, \ldots\big) \in \mathbb{C}(R)^{\flat} \quad \text{for } 1 \leq i \leq d. \end{aligned}$$

We set $\mathbf{A}_{\mathrm{inf}}(R) := W\big(\mathbb{C}^+(R)^{\flat}\big)$ as the ring of p-typical Witt vectors with coefficients in $\mathbb{C}^+(R)^{\flat}$. For $x \in \mathbb{C}^+(R)^{\flat}$, let $[x] = (x, 0, 0, \ldots) \in \mathbf{A}_{\mathrm{inf}}(R)$ denote its Teichmüller representative. The absolute Frobenius on $\mathbb{C}^+(R)^{\flat}$ lifts to an endomorphism $\varphi : \mathbf{A}_{\mathrm{inf}}(R) \longrightarrow \mathbf{A}_{\mathrm{inf}}(R)$ and the action of G_R extends to $\mathbf{A}_{\mathrm{inf}}(R)$ which is continuous for the weak topology (see §2.1 for weak topology). Any element $x \in \mathbf{A}_{\mathrm{inf}}(R)$ can be uniquely written as $x = \sum_{k\in\mathbb{N}} p^k[x_k]$ for $x_k \in \mathbb{C}^+(R)^{\flat}$. We set

$$\pi := [\varepsilon] - 1, \quad \pi_1 := \varphi^{-1}(\pi) = [\varepsilon^{1/p}] - 1 \quad \text{and} \quad \xi := \tfrac{\pi}{\pi_1}.$$

The action of G_R and the Frobenius φ on these elements is given as,

$$\begin{aligned} &g([\varepsilon]) = [\varepsilon]^{\chi(g)} \quad \text{and} \quad g(\pi) = (1+\pi)^{\chi(g)} - 1 \quad \text{for} \quad g \in G_R, \\ &\varphi([\varepsilon]) = [\varepsilon]^p \qquad \text{and} \quad \varphi(\pi) = (1+\pi)^p - 1, \end{aligned}$$

where $\chi : G_R \longrightarrow \mathbb{Z}_p^{\times}$ is the p-adic cyclotomic character. Define the map

$$\begin{aligned} \theta : \mathbf{A}_{\mathrm{inf}}(R) &\longrightarrow \mathbb{C}^+(R) \\ \sum_{k\in\mathbb{N}} p^k[x_k] &\longmapsto \sum_{k\in\mathbb{N}} p^k x_k^{\sharp}. \end{aligned} \tag{1.1}$$

The map θ is a G_R-equivariant surjective ring homomorphism whose kernel is principal, and generated by any $x \in \mathrm{Ker}\,\theta$ such that its Witt vector expansion $x = (x_0, x_1, \dots,)$ has the property that x_1 is a unit in $\mathbb{C}^+(R)^\flat$, for example $p - [p^\flat]$ or ξ (see [Fon82, Proposition 2.4 (ii)]). By $\mathbb{Q}_p$-linearity, the map θ can be extended to $\theta : \mathbf{A}_{\mathrm{inf}}(R)\left[\frac{1}{p}\right] \longrightarrow \mathbb{C}(R)$.

Definition 1.2. Define

$$\mathbf{B}_{\mathrm{dR}}^+(R) := \lim_n \mathbf{A}_{\mathrm{inf}}(R)\left[\tfrac{1}{p}\right]/(\mathrm{Ker}\,\theta)^n,$$

as the $(\mathrm{Ker}\,\theta)$-adic completion of $\mathbf{A}_{\mathrm{inf}}(R)\left[\frac{1}{p}\right]$.

The ring $\mathbf{B}_{\mathrm{dR}}^+(R)$ is an F-algebra and the action of G_R on $\mathbf{A}_{\mathrm{inf}}(R)$ extends to an action on $\mathbf{B}_{\mathrm{dR}}^+(R)$ which is continuous for the $(\mathrm{Ker}\,\theta)$-adic topology. The map θ further extends to a G_R-equivariant surjective ring homomorphism $\theta : \mathbf{B}_{\mathrm{dR}}^+(R) \longrightarrow \mathbb{C}(R)$. The element

$$t := \log[\varepsilon] = \log(1 + \pi) = \sum_{k \in \mathbb{N}} (-1)^k \frac{\pi^{k+1}}{k+1} \in \mathbf{B}_{\mathrm{dR}}^+(R), \tag{1.2}$$

and we have that $\mathrm{Ker}\,\theta = \xi\mathbf{B}_{\mathrm{dR}}^+(R) = \pi\mathbf{B}_{\mathrm{dR}}^+(R) = t\mathbf{B}_{\mathrm{dR}}^+(R)$ (see [Bri08, Proposition 5.1.3]). Moreover, for $g \in G_R$ we have that $g(t) = \chi(g)t$. The ring $\mathbf{B}_{\mathrm{dR}}^+(R)$ is t-torsion free (see [Bri08, Proposition 5.1.4]).

Definition 1.3. Define the *de Rham period ring* as

$$\mathbf{B}_{\mathrm{dR}}(R) := \mathbf{B}_{\mathrm{dR}}^+(R)\left[\tfrac{1}{t}\right].$$

This construction is functorial in R but $\mathbf{B}_{\mathrm{dR}}^+(R)$ only depends on $\overline{R}$. The ring $\mathbf{B}_{\mathrm{dR}}(R)$ is an F-algebra equipped with a continuous action of G_R, for the $(\mathrm{Ker}\,\theta)$-adic topology.

Next, we will put a filtration on $\mathbf{B}_{\mathrm{dR}}(R)$ by setting $\mathrm{Fil}^r\mathbf{B}_{\mathrm{dR}}(R) := t^r\mathbf{B}_{\mathrm{dR}}^+(R)$ for $r \in \mathbb{Z}$, which is a decreasing, separated and exhaustive filtration on $\mathbf{B}_{\mathrm{dR}}(R)$. We equip $\mathbf{B}_{\mathrm{dR}}^+(R)$ with the induced filtration. For the associated graded pieces, we have the identification (see [Bri08, Proposition 5.2.1])

$$\mathrm{gr}^\bullet\mathbf{B}_{\mathrm{dR}}^+(R) \simeq \mathbb{C}(R)[t] \quad \text{and} \quad \mathrm{gr}^\bullet\mathbf{B}_{\mathrm{dR}}(R) \simeq \mathbb{C}(R)[t, t^{-1}],$$

where t denotes its image in $\mathrm{gr}^1\mathbf{B}_{\mathrm{dR}}^+(R)$.

We can extend the map $\theta : \mathbf{A}_{\mathrm{inf}}(R) \longrightarrow \mathbb{C}^+(R)$ by R-linearity to obtain a G_R-equivariant surjective ring homomorphism

$$\theta_R : R \otimes_{\mathbb{Z}} \mathbf{A}_{\mathrm{inf}}(R) \longrightarrow \mathbb{C}^+(R). \tag{1.3}$$

Let $\mathcal{O}\mathbf{A}_{\mathrm{inf}}(R)$ denote the $\theta_R^{-1}\big(p\mathbb{C}^+(R)\big)$-adic completion of $R \otimes_{\mathbb{Z}} \mathbf{A}_{\mathrm{inf}}(R)$ (the ideal $\theta_R^{-1}\big(p\mathbb{C}^+(R)\big)$ is generated by p and $\mathrm{Ker}\,\theta_R$). The morphism θ_R then extends to a G_R-equivariant surjective ring homomorphism

$$\theta_R : \mathcal{O}\mathbf{A}_{\mathrm{inf}}(R) \longrightarrow \mathbb{C}^+(R),$$

which can be extended by $\mathbb{Q}_p$-linearity to a G_R-equivariant surjective ring homomorphism

$$\theta_R : \mathcal{O}\mathbf{A}_{\mathrm{inf}}(R)\left[\tfrac{1}{p}\right] \longrightarrow \mathbb{C}(R).$$

Definition 1.4. Define

$$\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(R) := \lim_n \mathcal{O}\mathbf{A}_{\mathrm{inf}}(R)\left[\tfrac{1}{p}\right]/\big(\mathrm{Ker}\,\theta_R\big)^n,$$

as the $(\mathrm{Ker}\,\theta_R)$-adic completion of $\mathcal{O}\mathbf{A}_{\mathrm{inf}}(R)\left[\frac{1}{p}\right]$.

The ring $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(R)$ is an $R\left[\frac{1}{p}\right]$-algebra and the action of G_R on $\mathcal{O}\mathbf{A}_{\mathrm{inf}}(R)$ extends to an action on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(R)$ which is continuous for the $(\mathrm{Ker}\,\theta_R)$-adic topology. The homomorphism θ_R extends to a G_R-equivariant surjective ring homomorphism $\theta_R : \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(R) \longrightarrow \mathbb{C}(R)$. By funtoriality of the construction of $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(R)$, the homomorphism $W(\kappa) \longrightarrow R$ induces a morphism of rings $\mathbf{B}_{\mathrm{dR}}^+(R) \longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(R)$ which is injective (see Proposition 1.6). Finally,

Definition 1.5. Define the *(fat) de Rham period ring* as

$$\mathcal{O}\mathbf{B}_{\mathrm{dR}}(R) := \mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R)\left[\tfrac{1}{t}\right].$$

The ring $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(R)$ is an $R\left[\frac{1}{p}\right]$-algebra equipped with a continuous action of G_R for the $(\mathrm{Ker}\ \theta_R)$-adic topology.

We can give a more explicit description of the ring $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R)$. Note that we have $X_i \otimes 1 - 1 \otimes [X_i^{\flat}] \in \mathrm{Ker}\ \theta_R \subset R \otimes_{\mathbb{Z}} \mathbf{A}_{\mathrm{inf}}(R)$ for $1 \leq i \leq d$. Let z_i denote its image in $\mathcal{O}\mathbf{A}_{\mathrm{inf}}(R) \subset \mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R)$. Since $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R)$ is complete for the $(\mathrm{Ker}\ \theta_R)$-adic topology, the homomorphism $\mathbf{B}_{\mathrm{dR}}^{+}(R) \longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R)$, extends to a homomorphism

$$\begin{aligned} f \;:\; \mathbf{B}_{\mathrm{dR}}^{+}(R)[[T_1, \dots, T_d]] &\longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R) \\ T_i &\longmapsto z_i, \qquad \text{for } 1 \leq i \leq d. \end{aligned}$$

In fact, we have that

Proposition 1.6 ([Bri08, Proposition 5.2.2])**.** *f is an isomorphism and* $\mathrm{Ker}\ \theta_R = (t, z_1, \dots, z_d)$.

Remark 1.7. (i) By the previous proposition, we can identify $\mathbf{B}_{\mathrm{dR}}^{+}(R)$ as a subring of $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R)$.

(ii) The rings $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R)$ and $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(R)$ are G_R-equivariant $\overline{R}\left[\frac{1}{p}\right]$-algebras. Moreover, the map θ_R from $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(R)$ to $\mathbb{C}(R)$ restricts to the canonical inclusion of $\overline{R}\left[\frac{1}{p}\right]$ in $\mathbb{C}(R)$ (see [Bri08, Proposition 5.2.3]).

(iii) Let R^{ur} denote the union of finite étale R-subalgebras $S \subset \overline{R}$, and let $\widehat{R^{\mathrm{ur}}}$ denote its p-adic completion. It is an R-subalgebra of $\mathbb{C}(R)$ equipped with a continuous action of G_R, and $\left(\widehat{R^{\mathrm{ur}}}\left[\frac{1}{p}\right]\right)^{G_R} = R\left[\frac{1}{p}\right]$. Moreover, we have $R^{\mathrm{ur}}\left[\frac{1}{p}\right] \subset \overline{R}\left[\frac{1}{p}\right] \subset \mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R)$, and $R^{\mathrm{ur}}\left[\frac{1}{p}\right]$-algebra structure on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R)$ and $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(R)$ uniquely extends to a G_R-equivariant $\widehat{R^{\mathrm{ur}}}\left[\frac{1}{p}\right]$-algebra structure (see [Bri08, Proposition 5.2.4]).

Next, we equip $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R)$ with a filtration $\mathrm{Fil}^r \mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R) := (\mathrm{Ker}\ \theta_R)^r$ for $r \in \mathbb{N}$, which is a decreasing, separated and exhaustive filtration, stable under the action of G_R. For $n \in \mathbb{N}$ we have

$$t^{-n}\mathrm{Fil}^n \mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R) = \mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R) + \left(\tfrac{z_1}{t}, \dots, \tfrac{z_d}{t}\right)^n \mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R).$$

So we set

$$\begin{aligned} \mathrm{Fil}^0 \mathcal{O}\mathbf{B}_{\mathrm{dR}}(R) &:= \sum_{n=0}^{\infty} t^{-n}\mathrm{Fil}^n \mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R) = \mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R)\left[\tfrac{z_1}{t}, \dots, \tfrac{z_d}{t}\right], \\ \mathrm{Fil}^r \mathcal{O}\mathbf{B}_{\mathrm{dR}}(R) &:= t^r \mathrm{Fil}^0 \mathcal{O}\mathbf{B}_{\mathrm{dR}}(R) \text{ for } r \in \mathbb{Z}. \end{aligned}$$

This filtration is decreasing, separated, exhaustive and stable under the action of G_R. Moreover, the induced filtrations on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R)$, $\mathbf{B}_{\mathrm{dR}}^{+}(R)$ and $\mathbf{B}_{\mathrm{dR}}(R)$ match with the ones defined before (see [Bri08, Proposition 5.2.8, Corollaire 5.2.11]). For the associated graded pieces, we have identifications (see [Bri08, Propositions 5.2.5, 5.2.6])

$$\begin{aligned} \mathrm{gr}^{\bullet} \mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R) &\simeq \mathbb{C}(R)[t, \overline{z}_1, \dots, \overline{z}_d], \\ \mathrm{gr}^{0} \mathcal{O}\mathbf{B}_{\mathrm{dR}}(R) &\simeq \mathbb{C}(R)[w_1, \dots, w_d], \\ \mathrm{gr}^{\bullet} \mathcal{O}\mathbf{B}_{\mathrm{dR}}(R) &\simeq \mathbb{C}(R)[t, t^{-1}, w_1, \dots, w_d], \end{aligned} \tag{1.4}$$

where $\overline{z}_i$ is the image of z_i in $\mathrm{gr}^1 \mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R)$ and w_i is the image of $\frac{z_i}{t}$ in $\mathrm{gr}^0 \mathcal{O}\mathbf{B}_{\mathrm{dR}}(R)$. Finally, the elements $\overline{R}\left[\frac{1}{p}\right] \setminus \{0\} \subset \mathbf{B}_{\mathrm{dR}}(R)$ are non-zero-divisors and we have $\left(\mathcal{O}\mathbf{B}_{\mathrm{dR}}(R)\right)^{G_R} = R\left[\frac{1}{p}\right]$ (see [Bri08, Corollaire 5.2.9, Proposition 5.2.12]).

We can equip the rings $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R)$ and $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R)$ with some extra structure. Namely, we are going to define a formal connection on these rings. First, note that since R is étale over $R_{\square}$, the p-adic completion of module of differentials of R relative to $\mathbb{Z}$ is given by $\Omega_R^1 = R \otimes_{R_\square} \Omega_{R_\square}^1$ and we have $\Omega_R^1\big[\frac{1}{p}\big] = R \otimes_{R_0} \Omega_{R_0}^1\big[\frac{1}{p}\big]$ (see Proposition 1.1). Now, let N_i denote the unique (Ker θ_R)-adically continuous and $\mathbf{B}_{\mathrm{dR}}^{+}(R)$-linear derivation on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R)$ as

$$N_i(z_j) = \delta_{ij} X_j \quad \text{for } 1 \leq i, j \leq d,$$

where δ_{ij} denotes the Kronecker delta symbol. The derivation N_i extends to $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(R)$ since $N_i(t) = 0$.

Definition 1.8. Define a connection

$$\begin{aligned} \partial \,:\, \mathcal{O}\mathbf{B}_{\mathrm{dR}}(R) &\longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{dR}}(R) \otimes_{R[\frac{1}{p}]} \Omega_R^1\big[\tfrac{1}{p}\big] \\ x &\longmapsto \sum_{i=1}^{d} N_i(x) \otimes \, d\log X_i. \end{aligned}$$

The connection ∂ is G_R-equivariant and satisfies Griffiths transversality for the filtration $\mathrm{Fil}^{\bullet}\mathcal{O}\mathbf{B}_{\mathrm{dR}}(R)$, i.e.

$$\partial\big(\mathrm{Fil}^r \mathcal{O}\mathbf{B}_{\mathrm{dR}}(R)\big) \longrightarrow \mathrm{Fil}^{r-1}\mathcal{O}\mathbf{B}_{\mathrm{dR}}(R) \otimes_{R[\frac{1}{p}]} \Omega_R^1\big[\tfrac{1}{p}\big],$$

(see [Bri08, Propositions 5.3.1, 5.3.9]). Its restriction to $R\big[\frac{1}{p}\big]$ is the canonical differential operator. We also have

$$\big(\mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R)\big)^{\partial=0} = \mathbf{B}_{\mathrm{dR}}^{+}(R) \quad \text{and} \quad \big(\mathcal{O}\mathbf{B}_{\mathrm{dR}}(R)\big)^{\partial=0} = \mathbf{B}_{\mathrm{dR}}(R).$$

Finally, the canonical map

$$u_{\mathrm{dR}} \,:\, R\big[\tfrac{1}{p}\big] \otimes_K \mathbf{B}_{\mathrm{dR}}(R) \longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{dR}}(R),$$

is injective (see [Bri08, Propositions 5.3.3, 5.3.8]) and

Theorem 1.9 ([Bri08, Théorème 5.4.1])**.** *The rings $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(R)$ and $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(R)$ are faithfully flat as $R\big[\frac{1}{p}\big]$-algebras.*

1.3. The crystalline period ring

In this section, we will define crystalline period rings and study their properties following [Bri08]. Note that Brinon defines these rings under certain assumption on his base rings (see the condition (BR) on [Bri08, p. 9]). However, this assumption always holds in our setting.

Let us consider the map $\theta \,:\, \mathbf{A}_{\mathrm{inf}}(R) \longrightarrow \mathbb{C}^{+}(R)$ from (1.1). The kernel of this map is a principal ideal generated by ξ or $p - [p^{\flat}]$. Now let

$$x^{[k]} \,:=\, \frac{x^k}{k!} \quad \text{for } x \in \mathrm{Ker}\, \theta \subset \mathbf{A}_{\mathrm{inf}}(R) \ \text{ and } k \in \mathbb{N}.$$

The divided power envelope of $\mathbf{A}_{\mathrm{inf}}(R)$ with respect to Ker θ is given as

$$\mathbf{A}_{\mathrm{inf}}(R)\big[x^{[k]},\ x \in \mathrm{Ker}\, \theta\big]_{k \in \mathbb{N}} = \mathbf{A}_{\mathrm{inf}}(R)\big[\xi^{[k]}\big]_{k \in \mathbb{N}}. \tag{1.5}$$

Definition 1.10. Define

$$\mathbf{A}_{\mathrm{cris}}(R) \,:=\, p\text{-adic completion of } \mathbf{A}_{\mathrm{inf}}(R)\big[\tfrac{\xi^k}{k!}\big]_{k \in \mathbb{N}}.$$

Also, set $\mathbf{A}_{\max}(R)$ to be the p-adic completion of the $\mathbf{A}_{\mathrm{inf}}(R)$-subalgebra generated by $\frac{1}{p}\mathrm{Ker}\, \theta$ inside $\mathbf{A}_{\mathrm{inf}}(R)\big[\frac{1}{p}\big]$.

The $W(\kappa)$-algebras $\mathbf{A}_{\mathrm{cris}}(R)$ and $\mathbf{A}_{\max}(R)$ are functorial in R (depending only on $\overline{R}$) and equipped with a continuous action of G_R. Further, these rings are p-torsion free (see [Bri08, Proposition 6.1.3]). The Frobenius on $\mathbf{A}_{\inf}(R)$ can be extended to $\mathbf{A}_{\mathrm{cris}}(R)$ as follows: we know that $\varphi(\xi) = \xi^p + py$ for some $y \in \mathbf{A}_{\inf}(R)$. We write $\varphi(\xi) = p(y + (p-1)!\xi^{[p]})$ and therefore $\varphi(\xi^k) = p^k(y + (p-1)!\xi^{[p]})^k$ for $k \in \mathbb{N}$. Now it easily follows that $\varphi\big(\xi^{[k]}\big) = \frac{p^k}{k!}\big(y + (p-1)!\xi^{[p]}\big)^k \in \mathbf{A}_{\mathrm{cris}}(R)$, as desired. Similarly, the Frobenius φ extends to $\mathbf{A}_{\max}(R)$ as well.

Since $\mathrm{Ker}\ \theta \subset \mathbf{A}_{\inf}(R)$ has divided powers in $\mathbf{A}_{\inf}(R)\big[\frac{1}{p}\mathrm{Ker}\ \theta\big]$, the universal property of divided power envelope induces a canonical G_R and Frobenius-equivariant injection $\iota : \mathbf{A}_{\mathrm{cris}}(R) \longrightarrow \mathbf{A}_{\max}(R)$. The homomorphism θ of (1.1) extends to surjective homomorphisms (see [Bri08, p. 62]),

$$\theta : \mathbf{A}_{\mathrm{cris}}(R) \longrightarrow \mathbb{C}^+(R) \ \text{ and } \ \theta : \mathbf{A}_{\max}(R) \longrightarrow \mathbb{C}^+(R).$$

From (1.2) we have,

$$t = \log(1+\pi) = \sum_{k \in \mathbb{N}} (-1)^k \frac{\pi^{k+1}}{k+1} \in \mathbf{A}_{\mathrm{cris}}(R),$$

and the action of G_R and the Frobenius φ on this element is given as

$$g(t) = \chi(g)t \ \text{ for } g \in G_R \ \text{ and } \ \varphi(t) = pt.$$

We have that $t \in \mathrm{Ker}\ \theta \subset \mathbf{A}_{\mathrm{cris}}(R)$ and $\mathrm{Ker}\ \theta \subset \mathbf{A}_{\mathrm{cris}}(R)$ is a divided power ideal. Moreover $t^{p-1} \in p\mathbf{A}_{\mathrm{cris}}(\mathbb{Z}_p)$ (see [Fon94a, 2.3.4]) and the rings $\mathbf{A}_{\mathrm{cris}}(R)$ and $\mathbf{A}_{\max}(R)$ are t-torsion free (see [Bri08, Corollaire 6.2.2]). Finally, we set $\varphi\big(\frac{1}{t}\big) = \frac{1}{pt}$.

Definition 1.11. Define the *crystalline period rings* as

$$\begin{aligned} \mathbf{B}^+_{\mathrm{cris}}(R) &:= \mathbf{A}_{\mathrm{cris}}(R)\big[\tfrac{1}{p}\big] \ \text{ and } \ \mathbf{B}_{\mathrm{cris}}(R) := \mathbf{B}^+_{\mathrm{cris}}(R)\big[\tfrac{1}{t}\big],\\ \mathbf{B}^+_{\max}(R) &:= \mathbf{A}_{\max}(R)\big[\tfrac{1}{p}\big] \ \text{ and } \ \mathbf{B}_{\max}(R) := \mathbf{B}^+_{\max}(R)\big[\tfrac{1}{t}\big]. \end{aligned}$$

These are F-algebras equipped with a continuous action of G_R and the Frobenius φ.

Next, let us consider the map $\theta_{R_0} : R_0 \otimes_{\mathbb{Z}} \mathbf{A}_{\inf}(R) \longrightarrow \mathbb{C}^+(R)$ obtained by extending (1.1) R_0-linearly. This is a G_R-equivariant surjective ring homomorphism with kernel generated by $\{1 \otimes \xi, z_1, \ldots, z_d\}$, where $z_i = X_i \otimes 1 - 1 \otimes [X_i^\flat]$ for $1 \leq i \leq d$. As in (1.5) the divided power envelope of $R_0 \otimes_{\mathbb{Z}} \mathbf{A}_{\inf}(R)$ with respect to $\mathrm{Ker}\ \theta_{R_0}$ is given as

$$R_0 \otimes_{\mathbb{Z}} \mathbf{A}_{\inf}(R)\big[x^{[k]},\ x \in \mathrm{Ker}\ \theta_{R_0}\big]_{k \in \mathbb{N}}.$$

Definition 1.12. Define

$\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0) :=$ p-adic completion of the divided power envelope of $R_0 \otimes_{\mathbb{Z}} \mathbf{A}_{\inf}(R)$ with respect to $\mathrm{Ker}\ \theta_{R_0}$.

Also, set $\mathcal{O}\mathbf{A}_{\max}(R_0)$ to be the p-adic completion of the $R_0 \otimes_{\mathbb{Z}} \mathbf{A}_{\inf}(R)$-subalgebra generated by $\frac{1}{p}\mathrm{Ker}\ \theta_{R_0}$ inside $R_0 \otimes_{\mathbb{Z}} \mathbf{A}_{\inf}(R)\big[\frac{1}{p}\big]$.

The R_0-algebras $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R)$ and $\mathcal{O}\mathbf{A}_{\max}(R)$ are functorial in R_0 and equipped with a continuous action of G_R. Taking the diagonal action of the Frobenius on $R_0 \otimes_{\mathbb{Z}} \mathbf{A}_{\inf}(R)$, we take $\varphi\big(\xi^{[k]}\big)$ as above, and

$$\varphi\big(z_i^{[k]}\big) = \varphi\big((X_i \otimes 1 - 1 \otimes [X_i^\flat])^{[k]}\big) = \frac{\big(X_i^p \otimes 1 - 1 \otimes [X_i^\flat]^p\big)^k}{k!} \ \text{ for } 1 \leq i \leq d.$$

Therefore, we see that the Frobenius extends to $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0)$ as well as to $\mathcal{O}\mathbf{A}_{\max}(R_0)$ which we will again denote by φ. Since $\mathrm{Ker}\ \theta_{R_0} \subset R_0 \otimes \mathbf{A}_{\inf}(R)$ has divided powers in $R_0 \otimes \mathbf{A}_{\inf}(R)\big[\frac{1}{p}\mathrm{Ker}\ \theta_{R_0}\big]$, the universal property of divided power envelope induces a canonical G_R and Frobenius-equivariant injection $\iota : \mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0) \longrightarrow \mathcal{O}\mathbf{A}_{\max}(R_0)$. The ring $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0)$ is an $\mathbf{A}_{\mathrm{cris}}(R)$-algebra and the ring

$\mathcal{O}\mathbf{A}_{\max}(R_0)$ is an $\mathbf{A}_{\max}(R)$-algebra. The homomorphism θ_{R_0} from (1.3) extends to surjective homomorphisms (see [Bri08, pg. 65])

$$\theta_{R_0} : \mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0) \longrightarrow \mathbb{C}^+(R) \text{ and } \theta_{R_0} : \mathcal{O}\mathbf{A}_{\max}(R_0) \longrightarrow \mathbb{C}^+(R).$$

Let $T = (T_1, \ldots, T_d)$ be some indeterminates as in Proposition 1.6. Let $\mathbf{A}_{\mathrm{cris}}(R)\langle T\rangle^\wedge$ denote the p-adic completion of the divided power polynomial algebra in indeterminates T and coefficients in $\mathbf{A}_{\mathrm{cris}}(R)$. Then we have a homomorphism of $\mathbf{A}_{\mathrm{cris}}(R)$-algebras

$$\begin{aligned} f_{\mathrm{cris}} : \mathbf{A}_{\mathrm{cris}}(R)\langle T\rangle^\wedge &\longrightarrow \mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0) \\ T_i &\longmapsto z_i \quad \text{for } 1 \le i \le d. \end{aligned}$$

Similarly, we can define a homomorphism of $\mathbf{A}_{\max}(R)$-algebras

$$\begin{aligned} f_{\max} : \mathbf{A}_{\max}(R)\left\{\tfrac{T_1}{p}, \ldots, \tfrac{T_d}{p}\right\} &\longrightarrow \mathcal{O}\mathbf{A}_{\max}(R_0) \\ \tfrac{T_i}{p} &\longmapsto \tfrac{z_i}{p} \quad \text{for } 1 \le i \le d. \end{aligned}$$

Then, we have that

Proposition 1.13 ([Bri08, Proposition 6.1.5])**.** *The maps f_{cris} and $f_{\max}$ are isomorphisms.*

The rings $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R)$ and $\mathcal{O}\mathbf{A}_{\max}(R)$ are p-torsion free as well as t-torsion free (see [Bri08, Proposition 6.1.7, Corollaire 6.2.2]).

Definition 1.14. Define the *(fat) crystalline period rings* as

$$\begin{aligned} \mathcal{O}\mathbf{B}^+_{\mathrm{cris}}(R_0) &:= \mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0)\left[\tfrac{1}{p}\right] \text{ and } \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) := \mathcal{O}\mathbf{B}^+_{\mathrm{cris}}(R_0)\left[\tfrac{1}{t}\right], \\ \mathcal{O}\mathbf{B}^+_{\max}(R_0) &:= \mathcal{O}\mathbf{A}_{\max}(R_0)\left[\tfrac{1}{p}\right] \text{ and } \mathcal{O}\mathbf{B}_{\max}(R_0) := \mathcal{O}\mathbf{B}^+_{\max}(R_0)\left[\tfrac{1}{t}\right]. \end{aligned}$$

The rings defined above are $R_0\left[\frac{1}{p}\right]$-algebras equipped with a continuous action of G_R and Frobenius endomorphism which we again denote by φ. Moreover, this construction is functorial in R_0. Finally, the inclusion $\iota : \mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0) \to \mathcal{O}\mathbf{A}_{\max}(R_0)$ extends to an inclusion $\iota : \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \to \mathcal{O}\mathbf{B}_{\max}(R_0)$.

Next, we will relate crystalline period rings to de Rham period rings. Notice that for each $n \in \mathbb{N}$, $\mathcal{O}\mathbf{B}^+_{\mathrm{dR}}(R)/(\mathrm{Ker}\ \theta_R)^n$ admits divided powers with respect to the ideal $\mathrm{Ker}\ \theta_R/(\mathrm{Ker}\ \theta_R)^n$. Also, the grading of $\mathcal{O}\mathbf{B}^+_{\mathrm{dR}}(R)/(\mathrm{Ker}\ \theta_R)^n$ (for the filtration defined by the divided power of the ideal $\mathrm{Ker}\ \theta_R/(\mathrm{Ker}\ \theta_R)^n$) is a free $\mathbb{C}(R)$-module of finite rank by (1.4). So we obtain a homomorphism of rings (see [Bri08, §6.2.1])

$$\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0) \longrightarrow \mathcal{O}\mathbf{B}^+_{\mathrm{dR}}(R)/(\mathrm{Ker}\ \theta_R)^n.$$

These morphisms are compatible for all $n \in \mathbb{N}$, therefore we have an induced homomorphism of rings

$$\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0) \longrightarrow \mathcal{O}\mathbf{B}^+_{\mathrm{dR}}(R).$$

Similarly, since p is invertible in $\mathcal{O}\mathbf{B}^+_{\mathrm{dR}}(R)/(\mathrm{Ker}\ \theta_R)^n$, we get an induced homomorphism of rings

$$\mathcal{O}\mathbf{A}_{\max}(R_0) \longrightarrow \mathcal{O}\mathbf{B}^+_{\mathrm{dR}}(R).$$

Further, these homomorphisms extend to

$$\begin{aligned} &\mathbf{B}^+_{\mathrm{cris}}(R) \longrightarrow \mathbf{B}^+_{\max}(R) \longrightarrow \mathbf{B}^+_{\mathrm{dR}}(R) \text{ and } \mathcal{O}\mathbf{B}^+_{\mathrm{cris}}(R_0) \longrightarrow \mathcal{O}\mathbf{B}^+_{\max}(R_0) \longrightarrow \mathcal{O}\mathbf{B}^+_{\mathrm{dR}}(R), \\ &\mathbf{B}_{\mathrm{cris}}(R) \longrightarrow \mathbf{B}_{\max}(R) \longrightarrow \mathbf{B}_{\mathrm{dR}}(R) \text{ and } \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \longrightarrow \mathcal{O}\mathbf{B}_{\max}(R_0) \longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{dR}}(R). \end{aligned}$$

All these homomorphisms are injective and G_R-equivariant (see [Bri08, Proposition 6.2.1, Corollaire 6.2.3]). The natural map

$$u_{\mathrm{cris}} : R_0\left[\tfrac{1}{p}\right] \otimes_F \mathbf{B}_{\mathrm{cris}}(R) \longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0),$$

is injective as well (see [Bri08, Proposition 6.2.4]). Using the injections described above, we get an induced filtration on crystalline period rings as

$$\mathrm{Fil}^r\mathbf{B}_{\mathrm{cris}}(R) := \mathbf{B}_{\mathrm{cris}}(R) \cap \mathrm{Fil}^r\mathbf{B}_{\mathrm{dR}}(R), \text{ and } \mathrm{Fil}^r\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) := \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \cap \mathrm{Fil}^r\mathcal{O}\mathbf{B}_{\mathrm{dR}}(R) \text{ for } r \in \mathbb{Z},$$

which is decreasing, separated and exhaustive.

The inclusion of (fat) crystalline period ring into (fat) de Rham period ring enables us to equip the former ring with a connection induced from the connection on the latter ring. More precisely, for $n \in \mathbb{N}$ we have

$$\partial\big(z_i^{[n]}\big) = z_i^{[n-1]} \otimes dX_i \text{ for } 1 \leq i \leq d,$$

and we get that for any $x \in \mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0) = \mathbf{A}_{\mathrm{cris}}(R)\langle T\rangle^\wedge$, we have $\partial(x) \in \mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0) \otimes_{R_0} \Omega^1_{R_0}$. This gives us an induced connection

$$\partial : \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \longrightarrow \mathbf{B}_{\mathrm{cris}}(R) \otimes_{R_0[\frac{1}{p}]} \Omega^1_{R_0}\big[\tfrac{1}{p}\big].$$

The connection ∂ is G_R-equivariant and satisfies Griffiths transversality for the filtration $\mathrm{Fil}^\bullet\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)$ since the same is true for the filtration on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(R)$. Its restriction to $R_0\big[\frac{1}{p}\big]$ is the canonical differential operator. Moreover,

$$\big(\mathcal{O}\mathbf{A}^+_{\mathrm{cris}}(R_0)\big)^{\partial=0} = \mathbf{A}_{\mathrm{cris}}(R), \quad \big(\mathcal{O}\mathbf{B}^+_{\mathrm{cris}}(R_0)\big)^{\partial=0} = \mathbf{B}^+_{\mathrm{cris}}(R) \text{ and } \big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)\big)^{\partial=0} = \mathbf{B}_{\mathrm{cris}}(R).$$

Over $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)$, the Frobenius operator commutes with the connection, i.e. $\varphi\partial = \partial\varphi$ (see [Bri08, Proposition 6.2.5]). In our setting we have $R = R_0[\varpi]$, therefore the natural morphism

$$R\big[\tfrac{1}{p}\big] \otimes_{R_0[\frac{1}{p}]} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{dR}}(R), \tag{1.6}$$

is injective (see [Bri08, Proposition 6.2.7]). Moreover, we have $\big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)\big)^{G_R} = R_0\big[\frac{1}{p}\big]$ (see [Bri08, Proposition 6.2.9]) and $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)$ is a faithfully flat $R_0\big[\frac{1}{p}\big]$-algebra (see [Bri08, Théorème 6.3.8]). Finally, in the relative setting we have the fundamental exact sequence:

Proposition 1.15 ([Bri08, Proposition 6.2.24])**.** *The sequence*

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow (\mathbf{B}_{\mathrm{cris}}(R))^{\varphi=1} \longrightarrow \mathbf{B}_{\mathrm{dR}}(R)/\mathbf{B}^+_{\mathrm{dR}}(R) \longrightarrow 0,$$

is exact, where the second non-trivial map is the canonical projection.

1.4. Filtered (φ, ∂)-modules

In [Fon94a] Fontaine used some categories of linear algebra data to classify de Rham and crystalline representations of the Galois group G_K. In case of de Rham representations these are finite dimensional K-vector spaces equipped with a decreasing, separated and exhaustive filtration, whereas in the case of crystalline representations these are finite dimensional F-vector spaces equipped with a Frobenius-semilinear automorphism and which acquire a decreasing, separated and exhaustive filtration after extending scalars along $F \longrightarrow K$. In the relative setting, Brinon introduced analogous categories of linear algebra data in [Bri08, Chapitre 7]. In this section, we will recall definitions and results useful in our case.

Let D be an R-module. A *connection* on D is defined as a continuous O_F-linear map

$$\partial_D : D \longrightarrow D \otimes_R \Omega^1_R,$$

such that $\partial(a \otimes x) = a \otimes \partial_D(x) + \partial_R(a) \otimes x$ for $a \in R$ and $x \in D$. The connection ∂_D is said to be *integrable* if $\partial_D \circ \partial_D = 0$. To simplify notations, below we will write ∂ instead of ∂_D.

Definition 1.16. A finitely generated $R\left[\frac{1}{p}\right]$-module D is said to be a *∂-module* if it is equipped with an integrable connection, i.e. $\partial \circ \partial = 0$, where

$$\partial : D \longrightarrow D \otimes_{R[\frac{1}{p}]} \Omega^1_R\left[\tfrac{1}{p}\right].$$

A morphism between ∂-modules is a morphism of $R\left[\frac{1}{p}\right]$-modules compatible with connection on each side. We denote this category by $\mathbf{M}_R(\partial)$.

Remark 1.17. If D is of finite type, then it is projective (see [Bri08, Proposition 7.1.2]). This observation makes it easy to deduce that $\mathbf{M}_R(\partial)$ is in fact an abelian category.

Now we will impose some restrictions over the connection ∂. The connection ∂ over the $R_0\left[\frac{1}{p}\right]$-module D is said to be *quasi-nilpotent* if there exists a finite and p-adically complete R_0-submodule $D_0 \subset D$, stable under ∂, such that $D = D_0\left[\frac{1}{p}\right]$ and the connection induced on the reduction of D_0 modulo p is quasi-nilpotent, i.e. for $1 \leq i \leq d$ there exist integers a_i such that $\prod_{i=1}^n N_i^{pa_i}$ sends D_0 into pD_0, where N_i are the derivations associated to ∂.

Definition 1.18. A *(φ, ∂)-module* over $R_0\left[\frac{1}{p}\right]$ is a ∂-module D over $R_0\left[\frac{1}{p}\right]$ such that ∂ is quasi-nilpotent and D is equipped with a Frobenius-semilinear endomorphism $\varphi : D \longrightarrow D$ such that the induced $R_0\left[\frac{1}{p}\right]$-linear map

$$1 \otimes \varphi : R_0\left[\tfrac{1}{p}\right] \otimes_{R_0[\frac{1}{p}],\varphi} D \longrightarrow D$$

is an isomorphism. A morphism between such modules is a $R_0\left[\frac{1}{p}\right]$-linear map compatible with respective structures on each side. These modules form an abelian category which we denote by $\mathbf{M}_{R_0}(\varphi, \partial)$ (see [Bri08, Proposition 7.1.9]).

Remark 1.19. The category $\mathbf{M}_{R_0}(\varphi, \partial)$ is, in fact, Tannakian in the sense of [DM82].

Next, we will study $R\left[\frac{1}{p}\right]$-modules equipped with a filtration.

Definition 1.20. A *filtered ∂-module* over $R\left[\frac{1}{p}\right]$ is a ∂-module D over $R\left[\frac{1}{p}\right]$ equipped with a decreasing, separated and exhaustive filtration by $R\left[\frac{1}{p}\right]$-submodules $\mathrm{Fil}^r D \subset D$ for $r \in \mathbb{Z}$, satisfying Griffiths transversality, i.e.

$$\partial(\mathrm{Fil}^r D) \subset \mathrm{Fil}^{r-1} D \otimes_{R[\frac{1}{p}]} \Omega^1_R\left[\tfrac{1}{p}\right],$$

and such that the associated graded $R\left[\frac{1}{p}\right]$-modules $\mathrm{gr}^\bullet D$ are projective. A morphism between such modules are morphisms of ∂-modules respecting filtration. These modules form an additive non-abelian category $\mathbf{MF}_R(\partial)$.

We can combine the previous two definitions to define,

Definition 1.21. A *filtered (φ, ∂)-module* over $R\left[\frac{1}{p}\right]$ relative to $R_0\left[\frac{1}{p}\right]$ is a (φ, ∂)-module D over $R_0\left[\frac{1}{p}\right]$ such that $D_R = R\left[\frac{1}{p}\right] \otimes_{R_0\left[\frac{1}{p}\right]} D$ is a filtered ∂-module over $R\left[\frac{1}{p}\right]$. A morphism between such modules is a morphism of (φ, ∂)-modules such that the induced morphism, after extension of scalars to $R\left[\frac{1}{p}\right]$, is a mophism of filtered ∂-modules. These modules form an additive tensor non-abelian category $\mathbf{MF}_{R/R_0}(\varphi, \partial)$.

Note that R_0/pR_0 admits a p-basis $(X_1, \ldots, X_d)$, which enables us to identify the category $\mathbf{M}_{R_0}(\varphi, \partial)$ with the category of *F-isocrystals* over R_0/pR_0 (see [BM90, Proposition 1.3.3]). Let D be an F-isocrystal over R_0/pR_0 and consider a "test-object", i.e. a quadruple (B, I, δ, s) such that B is a p-adically complete $\mathbb{Z}_p$-algebra, $I \subset B$ is an ideal admitting δ-divided powers compatible with the canonical divided powers over pB and $s : R_0/pR_0 \longrightarrow B/I$ is a ring homomorphism giving B/I an R_0/pR_0-algebra structure. Then by "evaluation" of D at such a test-object we will mean that there exists a projective B-module $D_{(B,I,\delta,s)}$ and a map $1 \otimes \varphi : D_{(B,I,\delta,s\circ\varphi)} \longrightarrow D_{(B,I,\delta,s)}$.

By the equivalence described, we can also speak of evaluating (φ, ∂)-modules at a test-object. More precisely, let D be a (φ, ∂)-module of rank h over $R_0\left[\frac{1}{p}\right]$, k a perfect field of characteristic p and $f : R_0/pR_0 \longrightarrow k$ a homomorphism. Then we have the test-object $(W(k), pW(k), \delta, f)$ and evaluating D at this test-object gives us a $W(k)\left[\frac{1}{p}\right]$-vector space $D_f := D_{(W(k), pW(k), \delta, f)}$ of dimension h which is further equipped with a Frobenius-semilinear endomorphism φ. Let z denote a nonzero vector in the (φ, ∂)-module $\bigwedge^h D_f$ over $W(k)\left[\frac{1}{p}\right]$, such that we have $\bigwedge^h D_f = W(k)\left[\frac{1}{p}\right]z$. Then there exists $\lambda \in W(k)\left[\frac{1}{p}\right]$ such that $\varphi(z) = \lambda z$. The p-adic valuation of $v_p(\lambda)$ is independent of the choice of z and depends only on $\mathfrak{p} = \operatorname{Ker} f \in \operatorname{Spec}(R_0/pR_0)$. We define this quantity as the *Newton number* of D at the prime $\mathfrak{p} \in R_0/pR_0$, i.e. $t_N(D, \mathfrak{p}) := v_p(\lambda)$.

Newton numbers satisfy some nice properties. If D is a (φ, ∂)-module of rank h over $R_0\left[\frac{1}{p}\right]$ and $r \in \mathbb{Z}$ then we have $t_N(D^\vee, \mathfrak{p}) = -t_N(D, \mathfrak{p})$ and $t_N(D(r), \mathfrak{p}) = t_N(D, \mathfrak{p}) - rh$ for all $\mathfrak{p} \in \operatorname{Spec}(R_0/pR_0)$. Also, by the specialization theorem of Grothendieck (see [Kat79, Theorem 2.3.1]), the function $\mathfrak{p} \mapsto t_N(D, \mathfrak{p})$ is increasing for specializations. The function $t_N(-, \mathfrak{p})$ is additive for $\mathfrak{p} \in \operatorname{Spec}(R_0/pR_0)$, i.e. for an exact sequence of (φ, ∂)-modules

$$0 \longrightarrow D' \longrightarrow D \longrightarrow D'' \longrightarrow 0,$$

we have $t_N(D, \mathfrak{p}) = t_N(D', \mathfrak{p}) + t_N(D'', \mathfrak{p})$ (see [Bri08, Proposition 7.1.12]).

Next, let us consider D to be a filtered ∂-module over $R\left[\frac{1}{p}\right]$ of rank h. The $R\left[\frac{1}{p}\right]$-module $\bigwedge^h D$ is projective of rank 1 and the associated graded module is projective over $R\left[\frac{1}{p}\right]$. There exists $n \in \mathbb{Z}$ such that $\operatorname{gr}^n \bigwedge^h D \simeq \bigwedge^h D$ and $\operatorname{gr}^m \bigwedge^h D = 0$ for $m \neq n$. We define the *Hodge number* of D as $t_H(D) := n$.

Similar to above, Hodge numbers satisfy some nice properties as well. If D is a filtered ∂-module of rank h over $R\left[\frac{1}{p}\right]$ and $r \in \mathbb{Z}$ then we have $t_H(D^\vee) = -t_H(D)$ and $t_H(D(r)) = t_H(D) - rh$. Moreover, the function $t_H(-)$ is additive, i.e. for an exact sequence of filtered ∂-modules

$$0 \longrightarrow D' \longrightarrow D \longrightarrow D'' \longrightarrow 0,$$

we have $t_H(D) = t_H(D') + t_H(D'')$ (see [Bri08, Proposition 7.1.15]).

An admissibilty criterion based on Newton and Hodge numbers of D can be described:

Definition 1.22. A filtered (φ, ∂)-module D over $R\left[\frac{1}{p}\right]$ relative to $R_0\left[\frac{1}{p}\right]$ is said to be *pointwise weakly admissible* if for each $\mathfrak{p} \in \operatorname{Spec}(R_0/pR_0)$ the following conditions are satisfied:

(i) $t_H(D) = t_N(D, \mathfrak{p})$;

(ii) For any subobject $D' \subset D$ (in the category $\mathbf{MF}_{R/R_0}(\varphi, \partial)$), we have that $t_H(D') \leq t_N(D', \mathfrak{p})$.

We denote by $\mathbf{MF}^{pwa}_{R/R_0}(\varphi, \partial)$ the full subcategory of $\mathbf{MF}_{R/R_0}(\varphi, \partial)$ consisting of filtered (φ, ∂)-modules over $R\left[\frac{1}{p}\right]$ relative to $R_0\left[\frac{1}{p}\right]$ that are pointwise weakly admissible.

Remark 1.23. (i) In the arithemtic setting, i.e. $R_0 = O_F$ weakly admissible objects in the category of filtered φ-modules over F were first studied in [Fon79, Kat79].

(ii) In [Bri08, Définition 7.1.11], Brinon calls the modules in Definition 1.22 as *ponctuellement faiblement admissible.*

1.5. *p*-adic representations

In this section we will study p-adic representations of the Galois group G_R and associate some linear algebra data to de Rham and crystalline representations. We begin with some formal definitions. Let E denote a topological field and G a topological group. We denote by $\operatorname{Rep}_E(G)$ the category of E-representations of G, whose objects are finite dimensional E-vector spaces equipped with a linear and continuous action of G and a morphism between the objects of $\mathrm{Rep}_E(G)$ is a G-equivariant E-linear map.

Let B be a reduced commutative topological E-algebra equipped with a continuous E-linear action of G. Let V be an E-representation of G and we set

$$\mathbf{D}_B(V) := (B \otimes_E V)^G.$$

This is a B^G-module and we have a natural morphism of B-modules, functorial in V

$$\begin{aligned}\alpha_B(V) : B \otimes_{B^G} \mathbf{D}_B(V) &\longrightarrow B \otimes_E V\\ b \otimes d &\longmapsto bd.\end{aligned}$$

The representation V is said to be B-*admissible* if α_B is an isomorphism. Moreover, the E-algebra B is said to be G-*regular* if it satisfies the following properties:

(i) B is faithfully flat over B^G;

(ii) For all $V \in \mathrm{Rep}_E(G)$, the homomorphism $\alpha_B(V)$ is injective;

(iii) B^G is Noetherian;

(iv) If V is a B-admissible E-representation of G of dimension 1, then the dual representation $V^\vee$ is B-admissible as well.

Below we will consider $G := G_R$, $E = \mathbb{Q}_p$, and vary B depending on the class of representations we are interested in studying. We first look at the unramified representations of G_R. Recall that R^{ur} denotes the union of finite étale R-subalgebras $S \subset \overline{R}$. Let us set

$$G_R^{\mathrm{ur}} := \mathrm{Gal}\big(R^{\mathrm{ur}}\big[\tfrac{1}{p}\big]/R\big[\tfrac{1}{p}\big]\big).$$

It is a quotient of G_R.

Definition 1.24. A p-adic representation $\rho : G_R \longrightarrow \mathrm{GL}(V)$ is said to be *unramified* if ρ factorizes through $G_R \longrightarrow G_R^{\mathrm{ur}}$.

From Remark 1.7 (iii) we have that $\widehat{R^{\mathrm{ur}}}\big[\tfrac{1}{p}\big]$ is a subring of $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(R)$ and $\big(\widehat{R^{\mathrm{ur}}}\big[\tfrac{1}{p}\big]\big)^{G_R} = R\big[\tfrac{1}{p}\big]$. Moreover, the ring $\widehat{R^{\mathrm{ur}}}\big[\tfrac{1}{p}\big]$ is faithfully flat over $R\big[\tfrac{1}{p}\big]$ (see [Bri08, Proposition 8.1.3]). Also note that R_0^{ur} is the union of finite étale R_0-subalgebras $S \subset \overline{R}$, and $\widehat{R_0^{\mathrm{ur}}}$ is complete for the p-adic topology. Therefore, from the proof of [Bri08, Proposition 6.1.5], it follows that $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0)$ is an R_0^{ur}-algebra and since the foremer is also p-adically complete, it is an $\widehat{R_0^{\mathrm{ur}}}$-algebra. In particular, $\mathcal{O}\mathbf{B}_{\mathrm{cris}}^+(R_0)$ and $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)$ are $\widehat{R_0^{\mathrm{ur}}}\big[\tfrac{1}{p}\big]$-algebras.

Now let V be a p-adic representation of G_R, then we set

$$\mathbf{D}_{\mathrm{ur}}(V) := \big(\widehat{R^{\mathrm{ur}}}\big[\tfrac{1}{p}\big] \otimes_{\mathbb{Q}_p} V\big)^{G_R}.$$

It is an $R\big[\tfrac{1}{p}\big]$-module and we have a homomorphism

$$\alpha_{\mathrm{ur}}(V) : \widehat{R^{\mathrm{ur}}}\big[\tfrac{1}{p}\big] \otimes_{R[\frac{1}{p}]} \mathbf{D}_{\mathrm{ur}}(V) \longrightarrow \widehat{R^{\mathrm{ur}}}\big[\tfrac{1}{p}\big] \otimes_{\mathbb{Q}_p} V. \tag{1.7}$$

The homomorphism $\alpha_{\mathrm{ur}}(V)$ in (1.7) is injective. Moreover, V is unramified if and only if V is $\widehat{R^{\mathrm{ur}}}\big[\tfrac{1}{p}\big]$-admissible, i.e. if and only if the map $\alpha_{\mathrm{ur}}(V)$ in (1.7) is bijective (see [Bri08, Propositions 8.1.2, 8.1.3]).

Remark 1.25. Let V be a p-adic representation of G_R and $T \subset V$ a free $\mathbb{Z}_p$-lattice stable under the action of G_R. Consider the associated continuous cocycle $f : G_R^{\mathrm{ur}} \longrightarrow \mathrm{GL}_h(\widehat{R^{\mathrm{ur}}})$ describing the action of G_R^{ur} over $\widehat{R^{\mathrm{ur}}} \otimes_{\mathbb{Z}_p} T$. Since V is unramified f is trivial and from [Bri08, Proposition 8.1.2], there exists $b \in 1 + p \cdot Mat(h, R^{\mathrm{ur}})$ such that f is cohomologous to the trivial cocycle $g \mapsto g(b^{-1})f(g)b = 1$. In this case we say that f is *trivialised* by $b \in 1 + p \cdot Mat(h, R^{\mathrm{ur}})$.

1.5.1. De Rham representations

In this section we will describe de Rham representations of G_R as well as the associated linear algebra object equipped with supplementary structures. We first note that the algebra $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(R)$ is a G_R-regular $R\left[\frac{1}{p}\right]$-algebra and $\mathbf{B}_{\mathrm{dR}}(R)$ is a G_R-regular K-algebra. We set

$$\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V) := \left(\mathcal{O}\mathbf{B}_{\mathrm{dR}}(R) \otimes_{R[\frac{1}{p}]} V\right)^{G_R} \text{ and } \mathbf{D}_{\mathrm{dR}}(V) := \left(\mathbf{B}_{\mathrm{dR}}(R) \otimes_{R[\frac{1}{p}]} V\right)^{G_R}.$$

We will denote the category of de Rham representations ($\mathcal{O}\mathbf{B}_{\mathrm{dR}}(R)$-admissible) as $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathcal{O}\mathrm{dR}}(G_R)$ and the category of horizontal de Rham representations ($\mathbf{B}_{\mathrm{dR}}(R)$-admissible) as $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_R)$.

There are several supplementary structures on the $R\left[\frac{1}{p}\right]$-module $\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ (resp. K-vector space $\mathbf{D}_{\mathrm{dR}}(V)$) (see [Bri08, §8.3]). It is equipped with a decreasing, separated and exhaustive filtration induced from the filtration on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(R) \otimes_{\mathbb{Q}_p} V$ (resp. $\mathbf{B}_{\mathrm{dR}}(R) \otimes_{\mathbb{Q}_p} V$), where we consider the G_R-stable filtration on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(R)$ (resp. $\mathbf{B}_{\mathrm{dR}}(R)$) from §1.2. Moreover, the module $\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ is equipped with an integrable connection, induced from the G_R-equivariant integrable connection

$$\begin{aligned} \partial : V \otimes_{\mathbb{Q}_p} \mathcal{O}\mathbf{B}_{\mathrm{dR}}(R) &\longrightarrow (V \otimes_{\mathbb{Q}_p} \mathcal{O}\mathbf{B}_{\mathrm{dR}}(R)) \otimes_{R[\frac{1}{p}]} \Omega^1_R\left[\tfrac{1}{p}\right] \\ v \otimes b &\longmapsto v \otimes \partial(b). \end{aligned}$$

We denote the induced connection on $\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ again by ∂. Since the connection ∂ on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(R)$ satisfies Griffiths transversality, therefore the same is true for $\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$, i.e.

$$\partial(\mathrm{Fil}^r \mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)) \subset \mathrm{Fil}^{r-1} \mathcal{O}\mathbf{D}_{\mathrm{dR}}(V) \otimes_{R_0[\frac{1}{p}]} \Omega^1_R\left[\tfrac{1}{p}\right].$$

Further, the module $\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ is projective of rank $\leq \dim(V)$ and $\mathbf{D}_{\mathrm{dR}}(V)$ is free of rank $\leq \dim(V)$. If V is de Rham then for all $r \in \mathbb{Z}$, the $R\left[\frac{1}{p}\right]$-modules $\mathrm{Fil}^r \mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ and $\mathrm{gr}^r \mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ are projective of finite type and therefore $\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ is an object of $\mathbf{MF}_R(\partial)$ (see [Bri08, Propositions 8.3.1, 8.3.2, 8.3.4]). For a de Rham representation V, the collection of integers r_i for $1 \leq i \leq \dim_{\mathbb{Q}_p}(V)$ such that $\mathrm{gr}^{-r_i} \mathcal{O}\mathbf{D}_{\mathrm{dR}}(V) \neq 0$ are called *Hodge-Tate weights* of V. Moreover, we say that V is positive if and only if $r_i \leq 0$ for all $1 \leq i \leq \dim_{\mathbb{Q}_p}(V)$.

Next, from [Bri08, §8.2] we have that the homomorphism

$$\alpha_{\mathcal{O}\mathrm{dR}}(V) : \mathcal{O}\mathbf{B}_{\mathrm{dR}}(R) \otimes_{R_0[\frac{1}{p}]} \mathcal{O}\mathbf{D}_{\mathrm{dR}}(V) \longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{dR}}(R) \otimes_{\mathbb{Q}_p} V,$$

is injective. The module $\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ is equipped with a connection ∂ coming from the connection on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(R)$ and we have $\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)^{\partial=0} = \mathbf{D}_{\mathrm{dR}}(V)$. The natural map

$$\beta_{\mathrm{dR}}(V) : R\left[\tfrac{1}{p}\right] \otimes_K \mathbf{D}_{\mathrm{dR}}(V) \longrightarrow \mathcal{O}\mathbf{D}_{\mathrm{dR}}(V),$$

as well as the homomorphism

$$\alpha_{\mathrm{dR}}(V) : \mathbf{B}_{\mathrm{dR}}(R) \otimes_K \mathbf{D}_{\mathrm{dR}}(V) \longrightarrow \mathbf{B}_{\mathrm{dR}}(R) \otimes_{\mathbb{Q}_p} V,$$

are injective. The latter map is bijective if and only if $\alpha_{\mathcal{O}\mathrm{dR}}(V)$ and $\beta_{\mathrm{dR}}(V)$ are bijective (see [Bri08, Propositions 8.2.10]).

Theorem 1.26 ([Bri08, Théorème 8.4.2]). *The category $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathcal{O}\mathrm{dR}}(G_R)$ is a Tannakian subcategory of $\mathrm{Rep}_{\mathbb{Q}_p}(G_R)$ and the restriction of the functor $\mathcal{O}\mathbf{D}_{\mathrm{dR}}$ to $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathcal{O}\mathrm{dR}}(G_R)$ is an $R\left[\frac{1}{p}\right]$-fiber functor. If*

$V \in \mathrm{Rep}^{\mathcal{O}\mathrm{dR}}_{\mathbb{Q}_p}(G_R)$ the isomorphism $\alpha_{\mathcal{O}\mathrm{dR}}(V)$ is compatible with the supplementary structures described above. In the horizontal de Rham case, the category $\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_R)$ is a Tannakian subcategory of $\mathrm{Rep}_{\mathbb{Q}_p}(G_R)$ and the restriction of the functor $\mathbf{D}_{\mathrm{dR}}$ to $\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_R)$ is a K-fiber functor. If $V \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_R)$, the isomorphism $\alpha_{\mathrm{dR}}(V)$ is compatible with the supplementary structures.

Note that for $\mathcal{C}$ a Tannakian subcategory of $\mathrm{Rep}_{\mathbb{Q}_p}(G_R)$ and Λ a commutative ring, a Λ-*fiber functor* is a faithful, exact, $\otimes$-functor from $\mathcal{C}$ to the category of Λ-modules such that the essential image of the functor lies in the subcategory of finitely generated projective Λ-modules.

1.5.2. Crystalline representations

In this section we will describe crystalline representations of G_R and the associated linear algebra object equipped with complementary structures. Note that the algebra $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)$ is a G_R-regular $R_0\left[\frac{1}{p}\right]$-algebra and $\mathbf{B}_{\mathrm{cris}}(R)$ is a G_R-regular F-algebra. We set

$$\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) := \left(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R) \otimes_{R[\frac{1}{p}]} V\right)^{G_R} \text{ and } \mathbf{D}_{\mathrm{cris}}(V) := \left(\mathbf{B}_{\mathrm{cris}}(R) \otimes_{R[\frac{1}{p}]} V\right)^{G_R}.$$

We will denote the category of crystalline representations ($\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R)$-adimissible) as $\mathrm{Rep}^{\mathcal{O}\mathrm{cris}}_{\mathbb{Q}_p}(G_R)$ and the category of horizontal crystalline representations ($\mathbf{B}_{\mathrm{cris}}(R)$-adimissible) as $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_R)$. There are several complimentary structures on the $R_0\left[\frac{1}{p}\right]$-module $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ (resp. F-vector space $\mathbf{D}_{\mathrm{cris}}(V)$) (see [Bri08, §8.3]). It is equipped with a Frobenius-semilinear operator φ induced from the Frobenius on $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \otimes_{\mathbb{Q}_p} V$ (resp. $\mathbf{B}_{\mathrm{cris}}(R_0) \otimes_{\mathbb{Q}_p} V$), where we consider the G_R-equivariant Frobenius on $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)$ (resp. $\mathbf{B}_{\mathrm{cris}}(R_0)$). Since $R = R_0[\varpi]$, therefore $R\left[\frac{1}{p}\right] \otimes_{R_0[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is an $R\left[\frac{1}{p}\right]$-submodule of $\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ (resp. K-subvector space $\mathbf{D}_{\mathrm{dR}}(V)$) and we equip it with the induced filtration and connection which satisfies Griffiths transversality with respect to the filtration. Additionally, we have $\partial\varphi = \varphi\partial$ over $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \otimes_{\mathbb{Q}_p} V$.

The module $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is projective of rank $\leq \dim(V)$ (see [Bri08, Propositions 8.3.1]). If V is crystalline, then the $R_0\left[\frac{1}{p}\right]$-linear homomorphism

$$1 \otimes \varphi : R_0\left[\tfrac{1}{p}\right] \otimes_{R_0[\frac{1}{p}],\varphi} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \longrightarrow \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V),$$

is an isomorphism and $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is an object of $\mathbf{MF}_{R/R_0}(\varphi, \partial)$ (see [Bri08, Propositions 8.3.3, 8.3.4]). Similarly, if V is horizontal crystalline, then the $R_0\left[\frac{1}{p}\right]$-linear homomorphism

$$1 \otimes \varphi : F \otimes_{F,\varphi} \mathbf{D}_{\mathrm{cris}}(V) \longrightarrow \mathbf{D}_{\mathrm{cris}}(V),$$

is an isomorphism. Finally, the inclusions $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \rightarrowtail \mathcal{O}\mathbf{B}_{\mathrm{dR}}(R)$ and $\mathbf{B}_{\mathrm{cris}}(R_0) \rightarrowtail \mathbf{B}_{\mathrm{dR}}(R)$ induce respective inclusions $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \rightarrowtail \mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ and $\mathbf{D}_{\mathrm{cris}}(V) \rightarrowtail \mathbf{D}_{\mathrm{dR}}(V)$, and the induced homomorphisms

$$R\left[\tfrac{1}{p}\right] \otimes_{R_0[\frac{1}{p}]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \longrightarrow \mathcal{O}\mathbf{D}_{\mathrm{dR}}(V) \text{ and } K \otimes_F \mathbf{D}_{\mathrm{cris}}(V) \longrightarrow \mathbf{D}_{\mathrm{dR}}(V).$$

are injective (see [Bri08, Proposition 8.2.1]).

Next, from [Bri08, §8.2] we have that the homomorphism

$$\alpha_{\mathcal{O}\mathrm{cris}}(V) : \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \otimes_{R_0[\frac{1}{p}]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \otimes_{\mathbb{Q}_p} V,$$

is injective. The module $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is equipped with a connection ∂ coming from the connection on $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)$ and we have $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)^{\partial=0} = \mathbf{D}_{\mathrm{cris}}(V)$. The natural map

$$\beta_{\mathrm{cris}}(V) : R\left[\tfrac{1}{p}\right] \otimes_K \mathbf{D}_{\mathrm{cris}}(V) \longrightarrow \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V),$$

as well as the homomorphism

$$\alpha_{\mathrm{cris}}(V) : \mathbf{B}_{\mathrm{cris}}(R) \otimes_K \mathbf{D}_{\mathrm{cris}}(V) \longrightarrow \mathbf{B}_{\mathrm{cris}}(R) \otimes_{\mathbb{Q}_p} V,$$

are injective. It is bijective if and only if $\alpha_{\mathcal{O}\mathrm{cris}}(V)$ and $\beta_{\mathrm{cris}}(V)$ are bijective. Finally, the natural map of K-vector spaces

$$K \otimes_F \mathbf{D}_{\mathrm{cris}}(V) \longrightarrow \mathbf{D}_{\mathrm{dR}}(V),$$

is injective.

Let $V \in \mathrm{Rep}^{\mathcal{O}\mathrm{cris}}_{\mathbb{Q}_p}(G_R)$ (resp. $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_R)$), then V is crystalline (resp. horizontal crystalline) and the natural map

$$R\big[\tfrac{1}{p}\big] \otimes_{R_0[\frac{1}{p}]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \longrightarrow \mathcal{O}\mathbf{D}_{\mathrm{dR}}(V) \quad \big(\text{resp. } K \otimes_F \mathbf{D}_{\mathrm{cris}}(V) \longrightarrow \mathbf{D}_{\mathrm{dR}}(V)\big),$$

is an isomorphism (see [Bri08, Proposition 8.2.1]).

Theorem 1.27 ([Bri08, Théorème 8.4.2])**.** *The category $\mathrm{Rep}^{\mathcal{O}\mathrm{cris}}_{\mathbb{Q}_p}(G_R)$ is a Tannakian subcategory of $\mathrm{Rep}_{\mathbb{Q}_p}(G_R)$ and the restriction of the functor $\mathcal{O}\mathbf{D}_{\mathrm{cris}}$ to $\mathrm{Rep}^{\mathcal{O}\mathrm{cris}}_{\mathbb{Q}_p}(G_R)$ is an $R_0\big[\frac{1}{p}\big]$-fiber functor. For $V \in \mathrm{Rep}^{\mathcal{O}\mathrm{cris}}_{\mathbb{Q}_p}(G_R)$, the isomorphism $\alpha_{\mathcal{O}\mathrm{cris}}(V)$ is compatible with the supplementary structures described above. In the horizontal crystalline case, the category $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_R)$ is a Tannakian subcategory of $\mathrm{Rep}_{\mathbb{Q}_p}(G_R)$ and the restriction of the functor $\mathbf{D}_{\mathrm{cris}}$ to $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_R)$ is an F-fiber functor. For $V \in \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_R)$, then the isomorphism $\alpha_{\mathrm{cris}}(V)$ is compatible with the supplementary structures.*

Let $\mathbf{MF}^{\mathrm{ad}}_{R/R_0}(\varphi, \partial)$ denote the essential image of the functor

$$\mathcal{O}\mathbf{D}_{\mathrm{cris}} : \mathrm{Rep}^{\mathcal{O}\mathrm{cris}}_{\mathbb{Q}_p}(G_R) \longrightarrow \mathbf{MF}_{R/R_0}(\varphi, \partial).$$

These objects are called *admissible* filtered (φ, ∂)-modules. As it turns out the essential image of $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ forms a category with rich structures (see [Bri08, Théorème 8.5.1]): The category $\mathbf{MF}^{\mathrm{ad}}_{R/R_0}(\varphi, \partial)$ is abelian. If D_1 and D_2 are two admissible filtered (φ, ∂)-modules over $R\big[\frac{1}{p}\big]$ then the same is true for $D_1 \otimes D_2$. Similarly, if D is an admissible filtered (φ, ∂)-module over $R\big[\frac{1}{p}\big]$ then the same is true for $D^\vee$. Equipped with these structures, the category $\mathbf{MF}^{\mathrm{ad}}_{R/R_0}(\varphi, \partial)$ is Tannakian.

Theorem 1.28 ([Bri08, Théorème 8.5.1])**.** *The functor $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ induces an equivalence of Tannakian categories*

$$\mathcal{O}\mathbf{D}_{\mathrm{cris}} : \mathrm{Rep}^{\mathcal{O}\mathrm{cris}}_{\mathbb{Q}_p}(G_R) \longrightarrow \mathbf{MF}^{\mathrm{ad}}_{R/R_0}(\varphi, \partial),$$

with a quasi-inverse given by the functor

$$\begin{aligned} \mathcal{O}\mathbf{V}_{\mathrm{cris}} : \mathbf{MF}^{\mathrm{ad}}_{R/R_0}(\varphi, \partial) &\longrightarrow \mathrm{Rep}^{\mathcal{O}\mathrm{cris}}_{\mathbb{Q}_p}(G_R) \\ D &\longmapsto \Big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \otimes_{R_0[\frac{1}{p}]} D\Big)^{\varphi=1, \partial=0} \cap \mathrm{Fil}^0\Big(\mathcal{O}\mathbf{B}_{\mathrm{dR}}(R) \otimes_{R[\frac{1}{p}]} D_R\Big)^{\partial=0}. \end{aligned}$$

Further, we have that the module $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \in \mathbf{MF}_{R/R_0}(\varphi, \partial)$ is pointwise weakly admissible in the sense of Definition 1.22 (see [Bri08, Proposition 8.5.2]).

Remark 1.29. In the arithmetic setting, Fontaine showed that admissible objects in the category of filtered φ-modules are weakly admissible and conjectured that converse holds as well. This conjecture was resolved by Fontaine-Colmez in [Fon94a]. Since then several different proofs have been given in [Col02, Colmez], [Ber08, Berger] and [Kis06, Kisin].

In the relative setting, Brinon calls a crystalline representation V *weakly admissible* if it is pointwise weakly admissible and the module $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ becomes free over a finite étale extension of R_0 (see [Bri08, p. 136]). For 1-dimensional crystalline representations, Brinon has shown that they are weakly admissible (see below). However, in higher dimensions it is not known whether all crystalline representations of G_R are weakly admissible? Further, the converse statement is also open, i.e. does weakly admissibility imply admissibility?

In the 1-dimensional case, it is possible to classify all de Rham and crystalline representations as in the following result:

Proposition 1.30 ([Bri08, Propositions 8.4.1, 8.6.1])**.** *Let $\eta : G_{R_0} \longrightarrow \mathbb{Z}_p^\times$ be a continuous character.*

(i) *The character η is de Rham if and only if we can write $\eta = \eta_{\mathrm{f}}\eta_{\mathrm{ur}}\chi^n$ where η_{f} is a finite character, η_{ur} is an unramified character which takes values $1 + p\mathbb{Z}_p$ and it is trivialized by an element $\alpha \in 1 + p\widehat{R_0^{\mathrm{ur}}}$ (see Remark 1.25), χ is the p-adic cyclotomic character and $n \in \mathbb{Z}$.*

(ii) *The character η is crystalline if and only if we can write $\eta = \eta_{\mathrm{f}}\eta_{\mathrm{ur}}\chi^n$ where η_{f} is a finite unramified character, η_{ur} is an unramified character which takes values in $1 + p\mathbb{Z}_p$ and it is trivialized by an element $\alpha \in 1 + p\widehat{R_0^{\mathrm{ur}}}$ (see Remark 1.25), χ is the p-adic cyclotomic character and $n \in \mathbb{Z}$.*

In particular, a 1-dimensional de Rham representation is potentially crystalline.

(iii) *Let $V = \mathbb{Q}_p(\eta)$ be a one-dimensional crystalline representation. Then there exists a finite étale extension $R_0 \longrightarrow R_0'$ such that the $R_0'\left[\frac{1}{p}\right]$-module $R_0'\left[\frac{1}{p}\right] \otimes_{R_0\left[\frac{1}{p}\right]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is free. In particular, if η_{f} is trivial then $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is a free $R_0\left[\frac{1}{p}\right]$-module of rank 1.*

CHAPTER 2

(φ, Γ)-modules and crystalline coordinates

Let K be a mixed characteristic non-archimedean complete discrete valuation field with residue field κ of characteristic p. In [Fon90] Fontaine gave a classification of p-adic representations of the absolute Galois group G_K in terms of étale (φ, Γ_K)-modules over a certain two dimensional local field $\mathbf{B}_K$. In the same article, Fontaine also considered finite height representations, i.e. representations whose periods live in a smaller ring $\mathbf{B}_K^+ \subset \mathbf{B}_K$. Moreover, he conjectured some relations between finite height representations and crystalline representations in case K is unramified over $\mathbb{Q}_p$. We will explore this line of thought in the relative setting in Chapter 3.

Studying p-adic representations from the point of view of (φ, Γ)-modules has proven to be very fruitful. Carrying forward Fontaine's point of view on the classification of all p-adic representations, Cherbonnier-Colmez in [CC98] showed that one can consider étale (φ, Γ)-modules over a subring $\mathbf{B}_K^\dagger \subset \mathbf{B}_K$ and classify all p-adic representations of G_K in terms of such modules. More succinctly, one can say that all p-adic representations of G_K are *overconvergent*. Embedding the overconvergent ring into the Robba ring, Berger in [Ber02] classified p-adic rerpesentations in terms of (φ, Γ)-modules over the Robba ring. As an application Berger in [Ber02] and Kedlaya in [Ked04] were able to connect the theory of (φ, Γ)-modules to the semilinear-algebraic objects stemming from Fontaine's classification of de Rham and crystalline representations.

On the other hand, following Fontaine's classification, in [Her98] Herr gave a three term complex in terms of (φ, Γ)-modules computing the Galois cohomology of the associated representation. Herr's complex was adapted to overconvergent setting by Cherbonnier-Colmez in [CC98]. An appropriate generalization of these results to the relative case has been done in [And06, AB08, AI08]. We will come back to the computation of Galois cohomology and study some explicit complexes in Chapter 4.

The current chapter consists of two parts, in the first part we will recall definitions and results on (φ, Γ)-modules in the relative setting, whereas in the second part we will study several analytic rings and some of their properties which will be useful in the next chapters. In the rest of the chapter we will work in the setting described in §1.1.

2.1. Relative (φ, Γ)-modules

Recall that F is a finite unramified extension of $\mathbb{Q}_p$ and $K = F(\zeta_{p^m})$ for some fixed $m \geq 1$. Let $K_n = K(\zeta_{p^n})$ where ζ_{p^n} is a primitive p^n-th root of unity, for $n \in \mathbb{N}$ and $n \geq m$. We take R_n to be the integral closure of $R \otimes_{O_K} O_{K_n}\left[X_1^{p^{-n}}, \dots X_d^{p^{-n}}\right]$ inside $\overline{R}\left[\frac{1}{p}\right]$. Let us set $R_\infty := \cup_{n \geq m} R_n$. Note that $K_\infty = \cup_n K_n \subset R_\infty\left[\frac{1}{p}\right]$. The ring R_∞ is an integral domain and a subring of $\overline{R}$.

Definition 2.1. Define $G_R := \mathrm{Gal}\left(\overline{R}\left[\frac{1}{p}\right]/R\left[\frac{1}{p}\right]\right), \Gamma_R := \mathrm{Gal}\left(R_\infty\left[\frac{1}{p}\right]/R\left[\frac{1}{p}\right]\right)$ and $H_R := \mathrm{Ker}\,(G_R \longrightarrow \Gamma_R)$.

Next, we will define certain rings useful in the theory of (φ,Γ)-modules. Recall that $\mathbb{C}(R)$ denotes the p-adic completion of $\overline{R}$ and $\mathbb{C}^+(R) \subset \mathbb{C}(R)$ is the subring of x's such that $v_p(x) \geq 0$. Since $\mathbb{C}(R)$ is a perfectoid algebra, its tilt $\mathbb{C}(R)^\flat$ is a perfect ring in characteristic p and we set

$$\mathbf{A}_{\overline{R}} := W(\mathbb{C}(R)^\flat),$$

the ring of p-typical Witt vectors with coefficients in $\mathbb{C}(R)^\flat$. The absolute Frobenius over $\mathbb{C}(R)^\flat$ lifts to an endomorphism $\varphi : \mathbf{A}_{\overline{R}} \longrightarrow \mathbf{A}_{\overline{R}}$, which we again call the Frobenius. The action of G_R on $\mathbb{C}(R)^\flat$ extends to a continuous action on $\mathbf{A}_{\overline{R}}$ which commutes with the Frobenius. The inclusion $K \subset R\left[\frac{1}{p}\right]$ induces inclusions

$$\overline{K} \subset \overline{R}\left[\tfrac{1}{p}\right], \quad \mathbb{C}_p^\flat \subset \mathbb{C}(R)^\flat \quad \text{and} \quad \mathbf{A}_{\overline{K}} \subset \mathbf{A}_{\overline{R}}.$$

Recall from §1.2 that an element $x \in \mathbb{C}(R)^\flat$ can be described as the set of sequences $(x_n)_{n\in\mathbb{N}}$ with $x_n \in \mathbb{C}(R)$ and $x_{n+1}^p = x_n$ for all $n \in \mathbb{N}$. We defined a valuation $v^\flat$ on $\mathbb{C}(R)^\flat$ by setting $v^\flat(x) := v_p(x^\sharp)$ where $x^\sharp := x_0$. The field $\mathbb{C}(R)^\flat$ is complete for this valuation. Moreover, $\mathbb{C}^+(R)$ is perfectoid and it can be shown that

$$\mathbb{C}^+(R)^\flat = \left\{x \in \mathbb{C}(R)^\flat, \text{ such that } v^\flat(x) \geq 0\right\}.$$

Further, recall that we set

$$\mathbf{A}_{\mathrm{inf}}(R) := W\left(\mathbb{C}^+(R)^\flat\right).$$

The inclusion $O_K \subset R$ induces inclusions

$$O_{\mathbb{C}_K}^\flat \subset \mathbb{C}^+(R)^\flat \quad \text{and} \quad \mathbf{A}_{\mathrm{inf}}(O_K) \subset \mathbf{A}_{\mathrm{inf}}(R).$$

Moreover, we fixed some elements in these rings as

$$\varepsilon := (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathbb{C}^+(R)^\flat, \quad \pi := [\varepsilon] - 1 \in \mathbf{A}_{\mathrm{inf}}(O_K) \quad \text{and} \quad \xi := \tfrac{\pi}{\varphi^{-1}(\pi)} = \tfrac{\pi}{\pi_1}.$$

Next, we will describe the weak topology on $\mathbf{A}_{\overline{R}}$. On $\mathbb{C}(R)^\flat$ consider its natural valuation topology (as described above), where the collection of ideals $\left\{\overline{\pi}^n\mathbb{C}^+(R)^\flat\right\}_{n\in\mathbb{N}}$ serve as a fundamental system of neighborhoods of 0. On the truncated Witt vectors $W_r\left(\mathbb{C}(R)^\flat\right)$ consider the product topology via the isomorphism $W_r\left(\mathbb{C}(R)^\flat\right) \simeq \left(\mathbb{C}(R)^\flat\right)^r$ (via the ghost map in theory of Witt vectors). *The weak topology* on $\mathbf{A}_{\overline{R}}$ is defined as the projective limit topology on

$$W\left(\mathbb{C}(R)^\flat\right) = \lim_r W_r\left(\mathbb{C}(R)^\flat\right).$$

Alternatively, for $i, j \in \mathbb{N}$ and

$$U_{i,j} = \pi^i \mathbf{A}_{\mathrm{inf}}(R) + p^j \mathbf{A}_{\overline{R}},$$

the weak topology can also be described by taking $\{U_{i,j}\}_{i,j\in\mathbb{N}}$ as a fundamental system of neighborhoods for $\mathbf{A}_{\overline{R}}$.

In the description above, if we endow the truncated Witt vectors $W_r(\mathbb{C}(R)^\flat)$ with discrete topology, then the projective limit topology on $\mathbf{A}_{\overline{R}}$ is the usual p-adic topology which is of course stronger than the topology considered above, hence the terminology.

Remark 2.2. Note that Γ_{R_0} is isomorphic to the semidirect product of Γ_F and Γ'_{R_0}, where $\Gamma_F = \mathrm{Gal}(K_\infty/F)$ and $\Gamma'_{R_0} = \mathrm{Gal}\left(R_\infty\left[\frac{1}{p}\right]/K_\infty R_0\left[\frac{1}{p}\right]\right)$. In particular, we have an exact sequence

$$1 \longrightarrow \Gamma'_{R_0} \longrightarrow \Gamma_{R_0} \longrightarrow \Gamma_F \longrightarrow 1, \tag{2.1}$$

where, for $1 \leq i \leq d$ we have (see [Bri08, p. 9] and [And06, §2.4])

$$\Gamma'_{R_0} = \mathrm{Gal}\big(R_\infty\big[\tfrac{1}{p}\big]/K_\infty R_0\big[\tfrac{1}{p}\big]\big) \simeq \mathbb{Z}_p^d,$$
$$\chi : \Gamma_F = \mathrm{Gal}(K_\infty/F) \simeq \mathbb{Z}_p^\times.$$

The group Γ_F can be viewed as a subgroup of Γ_{R_0}, i.e. we can take a section of the projection map in (2.1) such that for $\gamma \in \Gamma_F$ and $g \in \Gamma'_{R_0}$, we have $\gamma g \gamma^{-1} = g^{\chi(\gamma)}$. In particular, we can choose topological generators $\{\gamma, \gamma_1, \dots, \gamma_d\}$ of Γ_{R_0} such that

$$\begin{aligned} \gamma(\pi) &= (1+\pi)^{\chi(\gamma)} - 1, \quad \gamma_i(\pi) = \pi && \text{for } 1 \leq i \leq d, \\ \gamma_i([X_i^\flat]) &= (1+\pi)[X_i^\flat], \quad \gamma_i([X_j^\flat]) = [X_j^\flat] && \text{for } i \neq j \text{ and } 1 \leq j \leq d, \end{aligned}$$

and that $\gamma_0 = \gamma^e$ is a topological generator of $\Gamma_K = \mathrm{Gal}(K_\infty/K)$ with $\chi(\gamma_0) = \exp(p^m)$, and where $e = [K : F]$. It follows that $\{\gamma_1, \dots, \gamma_d\}$ are topological generators of Γ'_{R_0}, γ is a lift of a topological generator of Γ_F, and γ_0 is a lift of a topological generator of Γ_K.

Next, we have $G_R = \mathrm{Gal}\big(\overline{R}\big[\frac{1}{p}\big]/R\big[\frac{1}{p}\big]\big)$ and we define $\Gamma_R = \mathrm{Gal}\big(R_\infty\big[\frac{1}{p}\big]/R\big[\frac{1}{p}\big]\big)$ and $H_R = \mathrm{Ker}\,(G_R \longrightarrow \Gamma_R)$. So we have that Γ_R is isomorphic to the semidirect product of Γ_K and $\Gamma'_R = \Gamma'_{R_0}$. In particular, for $1 \leq i \leq d$ we have

$$\Gamma'_R = \mathrm{Gal}\big(R_\infty\big[\tfrac{1}{p}\big]/K_\infty R\big[\tfrac{1}{p}\big]\big) \simeq \mathbb{Z}_p^d,$$
$$\chi : \Gamma_K = \mathrm{Gal}(K_\infty/K) \simeq 1 + p^m\mathbb{Z}_p.$$

In [FW79b, FW79a, Win83], using the field-of-norms functor, Fontaine and Wintenberger constructed a non-archimedean complete discrete valuation field $\mathbf{E}_K \subset \mathbb{C}_p^\flat$ of characteristic p, with residue class field κ and functorial in K. One of the remarkable results in their theory is the isomorphism of certain Galois groups, which can be stated as follows,

Theorem 2.3 (Fontaine-Wintenberger)**.** *Let $\mathbf{E}_K^{\mathrm{sep}}$ denote the separable closure of $\mathbf{E}_K$ inside $\mathbb{C}_p^\flat$. Then we have a natural isomorphism of Galois groups*

$$\mathrm{Gal}(\overline{K}/K_\infty) \xrightarrow{\simeq} \mathrm{Gal}\big(\mathbf{E}_K^{\mathrm{sep}}/\mathbf{E}_K\big).$$

Remark 2.4. (i) In modern language, we also have that the completion of the perfect closure of $\mathbf{E}_K$ is $\widehat{K}_\infty^\flat$ and there is a natural isomorphism of Galois groups,

$$\mathrm{Gal}\big(\overline{K}/K_\infty\big) \xrightarrow{\simeq} \mathrm{Gal}\big(\mathbb{C}_p/\widehat{K}_\infty\big) \xrightarrow{\simeq} \mathrm{Gal}\big(\mathbb{C}_p^\flat/\widehat{K}_\infty^\flat\big) \xrightarrow{\simeq} \mathrm{Gal}\big(\mathbf{E}_K^{\mathrm{sep}}/\mathbf{E}_K\big).$$

A vast generalization of the above isomorphism for perfectoid algebras, also known as the tilting correspondence, was done by Scholze in [Sch12] and Fontaine-Fargues in [FF18].

(ii) The field-of-norms functor was further generalized to higher-dimensional local fields by Abrashkin in [Abr07], as well as in another direction by Scholl in [Sch06].

In [Fon90], Fontaine utilised the isomorphism of Galois groups to classify mod-p representations of G_K in terms of étale (φ, Γ_K)-modules over $\mathbf{E}_K$. By some technical considerations one can lift this to characteristic 0, i.e. classify $\mathbb{Z}_p$-representations of G_K in terms of étale (φ, Γ_K)-modules over a two dimensional local ring $\mathbf{A}_K \subset W\big(\widehat{K}_\infty^\flat\big)$ (see [Fon90] for details).

We are interested in an analogous theory in the relative setting. To describe such a theory we need to consider generically étale algebras over any finite extension of R in the cyclotomic tower R_∞/R. More precisely, let S be an R_n-algebra such that S is finite as an R_n-module and $S\big[\frac{1}{p}\big]$ is étale over $R_n\big[\frac{1}{p}\big]$. Let $k \geq n$ and we denote by S_k the integral closure of $S \otimes_{R_n} R_k$ in $S \otimes_{R_n} R_k\big[\frac{1}{p}\big]$ and set $S_\infty := \cup_{k \geq n} S_k$. For S as described, S_∞ is a normal R_∞-algebra and an integral domain as a subring of $\overline{R}$. As in the case of R, for S we define $G_S := \mathrm{Gal}\big(\overline{R}\big[\frac{1}{p}\big]/S\big[\frac{1}{p}\big]\big)$, $\Gamma_S := \mathrm{Gal}\big(S_\infty\big[\frac{1}{p}\big]/S\big[\frac{1}{p}\big]\big)$ and

$H_S := \mathrm{Ker}\,(G_S \longrightarrow \Gamma_S)$. Again, Γ_S is isomorphic to the semidirect product of Γ_K and Γ'_S, where $\Gamma'_S = \mathrm{Gal}\big(S_\infty\big[\frac{1}{p}\big]/K_\infty S\big[\frac{1}{p}\big]\big)$ is a finite index subgroup of $\Gamma'_R \simeq \mathbb{Z}_p^d$.

In [And06], Andreatta constructs an analogue of $\mathbf{E}_K$ viewed as a subfield of $\widehat{K}_\infty^\flat$; to any S in Definition 2.1, he functorially associates a ring $\mathbf{E}_S \subset \mathrm{Fr}\,\widehat{S}_\infty^\flat$. We will recall his constructions below.

Let $\mathbf{E}_K^+$ denote the valuation ring of $\mathbf{E}_K$ and let $\overline{\pi}_K \in \widehat{K}_\infty^\flat$ be a uniformizer which is the reduction of $\pi_K \in W\big(\widehat{K}_\infty^\flat\big)$ modulo p (see Remark 2.6 for the choice of π_K).

Definition 2.5. Let $\delta \in \mathbb{Q}$, $0 < \delta < 1$ and $N \in \mathbb{N}$. For δ small enough and N large enough, depending on S (see [And06, Definition 4.2] for precise formulation of δ and N), we define the ring

$$\mathbf{E}_S^+ := \Big\{(a_0, \dots, a_k, \dots) \in \widehat{S}_\infty^\flat, \text{ such that } a_k \in S_k/p^\delta S_k \text{ for all } k \geq N\Big\},$$

and set

$$\mathbf{E}_S := \mathbf{E}_S^+\big[\tfrac{1}{\overline{\pi}_K}\big].$$

In [And06, Proposition 4.5, Corollaries 5.3, 5.4], Andreatta shows that the ring $\mathbf{E}_S^+$ is finite and torsion free as an $\mathbf{E}_R^+$-module. It is a reduced Noetherian ring and $\overline{\pi}$-adically complete. By construction, it is endowed with a $\overline{\pi}$-adically continuous action of Γ_S and a Frobenius endomorphism φ, which commute with each other and are compatible with the respective structures on $\widehat{S}_\infty^\flat$. Moreover, $\mathbf{E}_S^+$ is a normal extension of $\mathbf{E}_R^+$, étale after inverting $\overline{\pi}_K$ and of degree equal to the generic degree of $R_m \subset S$. The set of elements $\{\overline{\pi}_K, X_1^\flat, \dots, X_d^\flat\}$ form an absolute p-basis of $\mathbf{E}_R^+$. Further, the ring $\widehat{S}_\infty^\flat$ is normal and coincides with the $\overline{\pi}_K$-adic completion of the perfect closure of $\mathbf{E}_S^+$. The extension $\mathbf{E}_S^+ \longrightarrow \widehat{S}_\infty^\flat$ is faithfully flat. For every finitely generated $\mathbf{E}_S^+$-module M, the base change of M via the above extension is $\overline{\pi}_K$-adically complete.

We have liftings of these rings to characteristic 0. From [And06, Appendix C, Proposition 7.8], we have that there exists a Noetherian regular domain, complete for the weak topology (induced from the weak topology on the ring of Witt vectors),

$$\mathbf{A}_R \subset W\big(\widehat{R}_\infty^\flat\big[\tfrac{1}{\overline{\pi}}\big]\big),$$

endowed with continuous and commuting actions of Γ_R and φ, lifting those defined on $\mathbf{E}_R$. Moreover, it contains a p-adically complete subring $\mathbf{A}_R^+$ lifting $\mathbf{E}_R^+$ and it contains $\{\pi, [X_1^\flat], \dots, [X_d^\flat]\}$.

Let S be an R-algebra as in Definition 2.1. By the equivalence between the categories of almost étale R_∞-algebras and almost étale $\mathbf{E}_R$-algebras (see [And06, Theorem 6.3, Proposition 7.2]), let $\mathbf{A}_S$ be the unique finite étale $\mathbf{A}_R$-algebra lifting the finite étale extension $\mathbf{E}_R \subset \mathbf{E}_S$. It is a Noetherian regular domain, complete for the weak topology, endowed with a continuous action of Γ_S and the Frobenius operator φ, lifting those defined on $\mathbf{E}_S$ and commuting with each other. Moreover, it contains subring $\mathbf{A}_S^+$ lifting $\mathbf{E}_S^+$ such that the former is complete for the weak topology.

Remark 2.6. Specializing the definition of $\mathbf{A}_S$ above for $S = O_K$ gives us that $\mathbf{A}_K^+$ is the ring of power series $\sum_{i \in \mathbb{N}} a_i \pi_K^i$ (see also [Fon90]), where $a_i \in O_F$ goes to 0 as $i \longrightarrow +\infty$ and $\pi_K \in W\big(\widehat{K}_\infty^\flat\big)$.

Next, we will take the union of $\mathbf{E}_S$ above which will produce a ring helpful in the classification of mod-p representations of G_R, in terms of étale (φ, Γ_R)-module over $\mathbf{E}_R$.

Definition 2.7. Define

$$\mathbf{E}^+ := \bigcup_S \mathbf{E}_S^+,$$

where the union runs over all R_n-subalgebras $S \subset \overline{R}$, for some $n \in \mathbb{N}$ such that S is normal and finite as an R_n-module and $S\big[\frac{1}{p}\big]$ is étale over $R_n\big[\frac{1}{p}\big]$. Also, we set

$$\mathbf{E} := \mathbf{E}^+\big[\tfrac{1}{\overline{\pi}}\big].$$

These rings are complete for the $\overline{\pi}$-adic topology, and equipped with Frobenius and a continuous action of G_R. Further, from [AI08, Proposition 2.9], we have that $\mathbb{C}(R)^{H_R} = \widehat{R}_\infty$, $\big(\mathbb{C}^+(R)^\flat\big)^{H_R} = \widehat{R}_\infty^\flat$,

$\left(\mathbb{C}(R)^\flat\right)^{H_R} = \widehat{R}^\flat_\infty\left[\frac{1}{\bar{\pi}}\right]$, $\left(\mathbf{E}^+\right)^{H_R} = \mathbf{E}^+_R$, and $\mathbf{E}^{H_R} = \mathbf{E}_R$.

Next, in characteristic 0 we set

$$\mathbf{B}_{\overline{R}} := \mathbf{A}_{\overline{R}}\left[\tfrac{1}{p}\right] = \bigcup_{j \in \mathbb{N}} p^{-j}\mathbf{A}_{\overline{R}},$$

equipped with the direct limit topology.

Definition 2.8. Define

$$\mathbf{A} := \text{completion of } \bigcup_S \mathbf{A}_S \subset \mathbf{A}_{\overline{R}} \text{ for the } p\text{-adic topology},$$

where the union is taken over all R_n-subalgebras $S \subset \overline{R}$, for some $n \in \mathbb{N}$ such that S is normal and finite as an R_n-module and $S\left[\frac{1}{p}\right]$ is étale over $R_n\left[\frac{1}{p}\right]$. We also equip $\mathbf{A}$ with the weak topology induced by the inclusion $\mathbf{A} \subset \mathbf{A}_{\overline{R}}$. Next, we set

$$\mathbf{A}^+ := \mathbf{A} \cap \mathbf{A}_{\inf}(R), \quad \mathbf{B}^+ := \mathbf{A}^+\left[\tfrac{1}{p}\right], \quad \text{and} \quad \mathbf{B} := \mathbf{A}\left[\tfrac{1}{p}\right],$$

and equip them with the topology induced from the weak topology on $\mathbf{A}$.

These rings are stable under φ and are equipped with an action of G_R, continuous for the weak topology. Moreover, from [AI08, Lemma 2.11], we have $\mathbf{A}^{H_R} = \mathbf{A}_R$, $(\mathbf{A}^+)^{H_R} = \mathbf{A}^+_R$ and $\mathbf{A}_R/p\mathbf{A}_R = \mathbf{E}_R$.

Having introduced all the necessary rings, finally we come to (φ, Γ_R)-modules.

Definition 2.9. A (φ, Γ_R)-module D over $\mathbf{A}_R$ is a finitely generated module equipped with

(i) A semilinear action of Γ_R, continuous for the weak topology (see Remark 2.12);

(ii) A Frobenius-semilinear homomorphism φ commuting with Γ_R.

These modules are called *étale* if the natural map,

$$1 \otimes \varphi : \mathbf{A}_R \otimes_{\mathbf{A}_R, \varphi} D \longrightarrow D,$$

is an isomorphism of $\mathbf{A}_R$-modules.

Denote by $(\varphi, \Gamma_R)\text{-Mod}^{\text{ét}}_{\mathbf{A}_R}$ the category of étale (φ, Γ_R)-modules over $\mathbf{A}_R$ with morphisms between objects being continuous, φ-equivariant and Γ_R-equivariant morphisms of $\mathbf{A}_R$-modules. Next, denote by $\text{Rep}_{\mathbb{Z}_p}(G_R)$ the category of finitely generated $\mathbb{Z}_p$-modules equipped with a linear and continuous action of G_R, with morphisms between objects being continuous and G_R-equivariant morphisms of $\mathbb{Z}_p$-modules.

Let $T \in \text{Rep}_{\mathbb{Z}_p}(G_R)$ then we have that,

Proposition and Definition 2.10. *The module*

$$\mathbf{D}(T) := (\mathbf{A} \otimes_{\mathbb{Z}_p} T)^{H_R},$$

is equipped with a semilinear action of φ and a continuous and semilinear action of Γ_R, which commute with each other. The functor $\mathbf{D}$ takes values in the category $(\varphi, \Gamma_R)\text{-Mod}^{\text{ét}}_{\mathbf{A}_R}$, i.e. $\mathbf{D}(T)$ is an étale (φ, Γ_R)-module over $\mathbf{A}_R$. Further, if T is free of finite rank, then $\mathbf{D}(T)$ is projective of rank $= \text{rk}_{\mathbb{Z}_p} T$.

Theorem 2.11 ([And06, Theorem 7.11])**.** *The functor*

$$\mathbf{D} : \text{Rep}_{\mathbb{Z}_p}(G_R) \longrightarrow (\varphi, \Gamma_R)\text{-Mod}^{\text{ét}}_{\mathbf{A}_R},$$

defines an equivalence of categories. For D an étale (φ, Γ_R)-module over $\mathbf{A}_R$, a quasi-inverse is given as

$$\mathbf{V}(D) := \left(\mathbf{A} \otimes_{\mathbf{A}_R} D\right)^{\varphi=1}.$$

Let T be a $\mathbb{Z}_p$-representation of G_R, then the natural map

$$\mathbf{A} \otimes_{\mathbf{A}_R} \mathbf{D}(T) \xrightarrow{\simeq} \mathbf{A} \otimes_{\mathbb{Z}_p} T$$

is an isomorphism of $\mathbf{A}$-modules compatible with Frobenius and the action of G_R on each side.

Remark 2.12. Let T be a $\mathbb{Z}_p$-module equipped with a continuous and linear action of G_R. Suppose that

$$T \simeq \mathbb{Z}_p^n \times \prod_{j=1}^{k} \mathbb{Z}/p^{r_j}\mathbb{Z},$$

as a $\mathbb{Z}_p$-module. Then

$$T \otimes_{\mathbb{Z}_p} \mathbf{A}_{\overline{R}} \xrightarrow{\simeq} \mathbf{A}_{\overline{R}}^n \times \prod_{j=1}^{k} \mathbf{A}_{\overline{R}}/p^{r_j}\mathbf{A}_{\overline{R}}$$

as $\mathbf{A}_{\overline{R}}$-module and, in particular, considering the weak topology on $\mathbf{A}_{\overline{R}}$, the product topology defines a topology on $T \otimes_{\mathbb{Z}_p} \mathbf{A}_{\overline{R}}$. It is independent of the choice of the presentation of T as a $\mathbb{Z}_p$-module and the action of G_R is continuous for such a topology. By construction, $\mathbf{D}(T)$ are submodules of $T \otimes_{\mathbb{Z}_p} \mathbf{A}_{\overline{R}}$ and therefore are endowed with induced topology. This topology is called *the weak topology* on (φ, Γ_R)-modules.

On the other hand, given a finitely generated $\mathbf{A}_R$-module D, we can equip D with a *weak topology* induced as the quotient topology from the surjection $\mathbf{A}_R^n \twoheadrightarrow D$, for some $n \in \mathbb{N}$ and where we consider the product of weak topology on $\mathbf{A}_R^n$.

The operator ψ

Next, we will define a left inverse ψ of the Frobenius operator φ on the ring $\mathbf{A}$. Let S be an R-algebra as in Definition 2.5. Then, from [AB08, Corollaire 4.10] we note that the $\mathbf{A}_S$-module $\varphi^{-1}(\mathbf{A}_S)$ is free with a basis given as

$$u_{\alpha/p} = (1+\pi)^{\alpha_0/p}[X_1^\flat]^{\alpha_1/p} \cdots [X_d^\flat]^{\alpha_d/p} \quad \text{for} \quad \alpha = (\alpha_0, \ldots, \alpha_d) \in \{0, 1, \ldots, p-1\}^{[0,d]}.$$

Considering the union over all such S we get that $\varphi^{-1}(\mathbf{A})$ is a free $\mathbf{A}$-module with a basis given as above (slight caveat is that we should replace $\varphi^{-1}(\mathbf{A}_S)$ by $\mathbf{A}_S$ and take p-th root of all the basis elements in loc. cit.).

Define the operator

$$\begin{aligned} \psi : \mathbf{A} &\longrightarrow \mathbf{A} \\ x &\longmapsto \tfrac{1}{p^{d+1}} \circ \mathrm{Tr}_{\varphi^{-1}(\mathbf{A})/\mathbf{A}} \circ \varphi^{-1}(x). \end{aligned}$$

Proposition 2.13 ([AB08, §4.8])**.** *The operator ψ satisfies the following properties:*

(i) *$\psi \circ \varphi = id$; let $x \in \mathbf{A}$ and write $\varphi^{-1}(x) = \sum_\alpha x_\alpha u_{\alpha/p}$, then we have $\psi(x) = x_0$;*

(ii) *ψ commutes with the action of G_R;*

(iii) *$\psi(\mathbf{A}^+) \subset \mathbf{A}^+$.*

2.2. Overconvergence

In the article [CC98], Cherbonnier-Colmez have shown that all $\mathbb{Z}_p$-representations (resp. p-adic representations) of G_K are overconvergent. Generalizing this to the relative case, in [AB08], Andreatta-Brinon have shown that all $\mathbb{Z}_p$-representations (resp. p-adic representations) of G_R are overconvergent. In this section we will recall some of these results. We begin by defining overconvergent subrings of $\mathbf{A}_{\overline{R}}$. Let $v > 0$ and let $\alpha \in O_{\mathbb{C}_p}^{\flat}$ such that $v_{\mathbf{E}}(\alpha) = 1/v$. Set

$$\begin{aligned}\mathbf{A}_{\overline{R}}^{(0,v]} &:= \Big\{ \sum_{k\in\mathbb{N}} p^k[x_k],\ v v_{\mathbf{E}}(x_k) + k \longrightarrow +\infty \text{ when } k \longrightarrow +\infty \Big\} \\ \mathbf{A}_{\overline{R}}^{(0,v]+} &:= \Big\{ \sum_{k\in\mathbb{N}} p^k[x_k] \in \mathbf{A}_{\overline{R}}^{(0,v]} \text{ with } v v_{\mathbf{E}}(x_k) + k \geq 0 \Big\} \\ &= p\text{-adic completion of } \mathbf{A}_{\mathrm{inf}}(R)\big[\tfrac{p}{[\alpha]}\big].\end{aligned}$$

Note that we have $\mathbf{A}_{\overline{R}}^{(0,v]} = \mathbf{A}_{\overline{R}}^{(0,v]+}\big[\frac{1}{[p^\flat]}\big]$. The action of G_R on $\mathbf{A}_{\mathrm{inf}}(R)$ extends to a continuous action of G_R on these rings which commutes with the induced Frobenius φ. For the homomorphism φ, we have

$$\varphi\big(\mathbf{A}_{\overline{R}}^{(0,v]+}\big) = \mathbf{A}_{\overline{R}}^{(0,v/p]+} \text{ and } \varphi\big(\mathbf{A}_{\overline{R}}^{(0,v]}\big) = \mathbf{A}_{\overline{R}}^{(0,v/p]}.$$

Moreover, we have injections (see [CN17, §2.4.2])

$$\mathbf{A}_{\overline{R}}^{(0,v]+} \rightarrowtail \mathbf{B}_{\mathrm{dR}}^{+}(R) \text{ and } \mathbf{A}_{\overline{R}}^{(0,v]} \rightarrowtail \mathbf{B}_{\mathrm{dR}}^{+}(R) \text{ if } v \geq 1.$$

Definition 2.14. Define the ring of *overconvergent coefficients* as

$$\mathbf{A}_{\overline{R}}^{\dagger} := \bigcup_{v\in\mathbb{Q}_{>0}} \mathbf{A}_{\overline{R}}^{(0,v]} \text{ and } \mathbf{B}_{\overline{R}}^{\dagger} := \bigcup_{v\in\mathbb{Q}_{>0}} \mathbf{B}_{\overline{R}}^{(0,v]} = \bigcup_{v\in\mathbb{Q}_{>0}} \mathbf{A}_{\overline{R}}^{(0,v]}\big[\tfrac{1}{p}\big].$$

Next, set

$$\mathbf{A}_{R}^{(0,v]} := \mathbf{A}_R \cap \mathbf{A}_{\overline{R}}^{(0,v]} \text{ and } \mathbf{A}^{(0,v]} := \mathbf{A} \cap \mathbf{A}_{\overline{R}}^{(0,v]},$$

and define

$$\mathbf{A}_{R}^{\dagger} := \mathbf{A}_R \cap \mathbf{A}_{\overline{R}}^{\dagger} = \bigcup_{v\in\mathbb{Q}_{>0}} \mathbf{A}_{R}^{(0,v]} \text{ and } \mathbf{A}^{\dagger} := \mathbf{A} \cap \mathbf{A}_{\overline{R}}^{\dagger} = \bigcup_{v\in\mathbb{Q}_{>0}} \mathbf{A}^{(0,v]}.$$

Now, let us describe the topology on the rings defined above. For $x = \sum_{k\in\mathbb{Z}} p^k[x_k] \in \mathbf{B}_{\overline{R}}^{(0,v]+}$, we set

$$w_v(z) := \inf_{k\in\mathbb{Z}} (v v^\flat(x_k) + k).$$

This induces a valuation on $\mathbf{A}_{\overline{R}}^{(0,v]+}$ and it is complete for the topology induced by the valuation (see [AB08, Proposition 4.2]). We will equip $\mathbf{A}_{\overline{R}}^{\dagger}$ with the topology induced by the inductive limit of the topology described above. Further, $\mathbf{A}^{\dagger}$ is also endowed with a Frobenius endomorphism φ and a continuous action of G_R which commutes with φ (see [And06, Proposition 7.2]). These actions are induced from the inclusion $\mathbf{A}_{\overline{R}}^{\dagger} \subset \mathbf{A}_{\overline{R}}$. Further, all subrings of $\mathbf{A}_{\overline{R}}^{\dagger}$ appearing above induce these structures as well.

Lemma 2.15. (i) *The restriction of the operator ψ from Proposition 2.13 to $\mathbf{A}^{\dagger}$ gives us that $\psi(\mathbf{A}^{\dagger}) \subset \mathbf{A}^{\dagger}$ (see [AB08, §4.8]).*

(ii) *We have $\big(\mathbf{A}^{(0,v]}\big)^{H_R} = \mathbf{A}_R^{(0,v]}$, $(\mathbf{A}^{\dagger})^{H_R} = \mathbf{A}_R^{\dagger}$ and $\mathbf{A}_R^{\dagger}/p\mathbf{A}_R^{\dagger} = \mathbf{E}_R$ (see [AI08, Lemma 2.11]).*

Now we come to overconvergent (φ, Γ_R)-modules.

Definition 2.16. A (φ, Γ_R)-module D over $\mathbf{A}_R^{\dagger}$ is a finitely generated module equipped with

(i) A semilinear action of Γ_R, continuous for the weak topology (see §2.1);

(ii) A Frobenius-semilinear homomorphism φ commuting with Γ_R.

These modules are called *étale* if the natural map,

$$1 \otimes \varphi : \mathbf{A}_R^{\dagger} \otimes_{\mathbf{A}_R^{\dagger}, \varphi} D \longrightarrow D,$$

is an isomorphism of $\mathbf{A}_R^\dagger$-modules. Let $(\varphi, \Gamma_R)\text{-Mod}_{\mathbf{A}_R^\dagger}^{\text{ét}}$ denote the category of such modules.

Denote by $(\varphi, \Gamma_R)\text{-Mod}_{\mathbf{A}_R^\dagger}^{\text{ét}}$ the category of étale (φ, Γ_R)-modules over $\mathbf{A}_R^\dagger$ with morphisms between objects being continuous, φ-equivariant and Γ_R-equivariant morphisms of $\mathbf{A}_R^\dagger$-modules. Recall that $\text{Rep}_{\mathbb{Z}_p}(G_R)$ is the category of finitely generated $\mathbb{Z}_p$-modules equipped with a linear and continuous action of G_R, with morphisms between objects being continuous and G_R-equivariant morphisms of $\mathbb{Z}_p$-modules.

Let $T \in \text{Rep}_{\mathbb{Z}_p}(G_R)$ then we have that,

Proposition and Definition 2.17. *The module*

$$\mathbf{D}^\dagger(T) := (\mathbf{A}^\dagger \otimes_{\mathbb{Z}_p} T)^{H_R},$$

is equipped with a semilinear action of φ and a continuous and semilinear action of Γ_R, which commute with each other. The functor $\mathbf{D}^\dagger$ takes values in the category $(\varphi, \Gamma_R)\text{-Mod}_{\mathbf{A}_R^\dagger}^{\text{ét}}$, i.e. $\mathbf{D}^\dagger(T)$ is an étale (φ, Γ_R)-module over $\mathbf{A}_R^\dagger$. Further, if T is free of finite rank, then $\mathbf{D}^\dagger(T)$ is projective of rank $= \text{rk}_{\mathbb{Z}_p} T$.

Theorem 2.18 ([AB08, Théorèm 4.35])**.** (i) *The functor*

$$\mathbf{D}^\dagger : \text{Rep}_{\mathbb{Z}_p}(G_R) \longrightarrow (\varphi, \Gamma_R)\text{-Mod}_{\mathbf{A}_R^\dagger}^{\text{ét}},$$

defines an equivalence of categories. For D an étale (φ, Γ_R)-module over $\mathbf{A}_R^\dagger$, a quasi-inverse is given as

$$\mathbf{V}^\dagger(D) := \left(\mathbf{A}^\dagger \otimes_{\mathbf{A}_R^\dagger} D\right)^{\varphi=1}.$$

(ii) *Let T be a $\mathbb{Z}_p$-representation of G_R, then the scalar extension along $\mathbf{A}_R^\dagger \rightarrowtail \mathbf{A}_R$ gives an isomorphism of (φ, Γ_R)-modules over $\mathbf{A}_R$,*

$$\mathbf{A}_R \otimes_{\mathbf{A}_R^\dagger} \mathbf{D}^\dagger(T) \xrightarrow{\simeq} \mathbf{D}(T).$$

Moreover, the natural map

$$\mathbf{A}^\dagger \otimes_{\mathbf{A}_R} \mathbf{D}^\dagger(T) \xrightarrow{\simeq} \mathbf{A}^\dagger \otimes_{\mathbb{Z}_p} T$$

is an isomorphism of $\mathbf{A}^\dagger$-modules compatible with Frobenius and the action of G_R on each side.

(iii) *If T is free of rank h, then there exists an R-algebra S such that S is normal and finite over R, $S\left[\frac{1}{p}\right]$ is Galois over $R\left[\frac{1}{p}\right]$ and $\mathbf{A}_S^\dagger \otimes_{\mathbf{A}_R^\dagger} \mathbf{D}^\dagger(T)$ is a free $\mathbf{A}_S^\dagger$-module of rank h.*

Remark 2.19. By construction, $\mathbf{D}^\dagger(T)$ is a submodule of $T \otimes_{\mathbb{Z}_p} \mathbf{A}_{\overline{R}}$ and therefore endowed with induced weak topology. On the other hand, given a finitely generated $\mathbf{A}_R^\dagger$-module D, we can equip D with a *weak topology* induced as the quotient topology from the surjection $\mathbf{A}_R^{\dagger n} \twoheadrightarrow D$, for some $n \in \mathbb{N}$ and where we consider the product of weak topology on $\mathbf{A}_R^{\dagger n}$.

2.2.1. Regularization by Frobenius

In this section we will introduce certain analytic rings. These rings will be useful in generalizing certain technical results of Berger (see Proposition 2.23) and at the same time it will set the stage for introducing certain variants of these rings in the next section which we will be useful for Chapters 3 & 5. Let $0 < u \leq v$ and let $\alpha, \beta \in O_{\mathbb{C}_p}^\flat$ such that $v_{\mathbf{E}}(\alpha) = 1/v$ and $v_{\mathbf{E}}(\beta) = 1/u$. Set

$$\mathbf{A}_R^{[u]} := p\text{-adic completion of } \mathbf{A}_{\text{inf}}(R)\left[\tfrac{[\beta]}{p}\right],$$
$$\mathbf{A}_R^{[u,v]} := p\text{-adic completion of } \mathbf{A}_{\text{inf}}(R)\left[\tfrac{p}{[\alpha]}, \tfrac{[\beta]}{p}\right].$$

The action of G_R on $\mathbf{A}_{\mathrm{inf}}(R)$ extends to a continuous action of G_R on these rings which commutes with the induced Frobenius φ. For the homomorphism φ, we have

$$\varphi\big(\mathbf{A}_{\overline{R}}^{[u]}\big) = \mathbf{A}_{\overline{R}}^{[u/p]} \text{ and } \varphi\big(\mathbf{A}_{\overline{R}}^{[u,v]}\big) = \mathbf{A}_{\overline{R}}^{[u/p,v/p]}.$$

Moreover, we have injections (see [CN17, §2.4.2])

$$\mathbf{A}_{\overline{R}}^{[u]} \rightarrowtail \mathbf{B}_{\mathrm{dR}}^{+}(R) \text{ if } u \leq 1 \text{ and } \mathbf{A}_{\overline{R}}^{[u,v]} \rightarrowtail \mathbf{B}_{\mathrm{dR}}^{+}(R) \text{ if } u \leq 1 \leq v.$$

We set

$$\mathbf{B}_{\overline{R},\mathrm{rig}}^{\dagger} := \bigcup_{v>0} \bigcap_{u \leq v} \mathbf{B}_{\overline{R}}^{[u,v]} = \bigcup_{v>0} \bigcap_{u \leq v} \mathbf{A}_{\overline{R}}^{[u,v]}\big[\tfrac{1}{p}\big].$$

The ring $\mathbf{B}_{\overline{R},\mathrm{rig}}^{\dagger}$ induces a continuous action of G_R, as well as a Frobenius endomorphism $\varphi : \mathbf{B}_{\overline{R},\mathrm{rig}}^{\dagger} \longrightarrow \mathbf{B}_{\overline{R},\mathrm{rig}}^{\dagger}$. It follows from the definitions that $\mathbf{B}_{\overline{R}}^{\dagger} \subset \mathbf{B}_{\overline{R},\mathrm{rig}}^{\dagger}$ compatible with the action of G_R and the Frobenius endomorphism. Finally, let $\mathbf{B}_{\overline{R},\mathrm{rig}}^{+} := \cap_{n\in\mathbb{N}} \varphi^n(\mathbf{B}_{\mathrm{max}}^{+})$ which is stable under the action of G_R and the Frobenius homomorphism.

Remark 2.20. Let $[\overline{\pi}] \in \mathbf{A}_{\mathrm{inf}}$ denote the Teichmüller lift of the reduction modulo p of π, written as $\overline{\pi} \in O_{\mathbb{C}_K}^{\flat}$. Let us take $r, s \in \mathbb{Q}$ such that $r = \frac{p-1}{pv}$ and $s = \frac{p-1}{pu}$. Then it can easily be checked that $\mathbf{A}_{\mathrm{inf}}(R)\big\{\frac{p}{[\overline{\pi}^r]}, \frac{[\overline{\pi}^s]}{p}\big\} = \mathbf{A}_{\overline{R}}^{[u,v]}$. This is the translation between Berger's notation and ours (see [Ber02, §2.1]).

Using the remark above, it is straightforward to check that the results of [Ber02, §2.1] hold in our case as well. In particular, we have

Lemma 2.21 ([Ber02, Lemme 2.5, Exemple 2.8])**.** (i) *For $u_1 \leq u_2 \leq v_2 \leq v_1$, we have a natural inclusion $\mathbf{A}_{\overline{R}}^{[u_1,v_1]} \rightarrowtail \mathbf{A}_{\overline{R}}^{[u_2,v_2]}$.*

(ii) *We have equalities $\mathbf{A}_{\mathrm{max}}(R) = \mathbf{A}_{\overline{R}}^{[1,+\infty]}$, $\mathbf{B}_{\mathrm{max}}^{+} = \mathbf{B}_{\overline{R}}^{[1,+\infty]}$ and $\mathbf{B}_{\overline{R},\mathrm{rig}}^{+} = \cap_{u>0} \mathbf{B}_{\overline{R}}^{[u,+\infty]}$.*

Using Lemma 2.21 (i), we can define for any interval $I \subset \mathbb{R} \cup \{+\infty\}$ the rings $\mathbf{A}_{\overline{R}}^{I} := \cap_{[u,v]\subset I} \mathbf{A}_{\overline{R}}^{[u,v]}$ and $\mathbf{B}_{\overline{R}}^{I} := \cap_{[u,v]\subset I} \mathbf{B}_{\overline{R}}^{[u,v]}$. Next, we define a p-adic valuation $V_{[u,v]}$ on $\mathbf{B}_{\overline{R}}^{[u,v]}$ by setting $V_{[u,v]}(x) = 0$ if and only if $x \in \mathbf{A}_{\overline{R}}^{[u,v]} - p\mathbf{A}_{\overline{R}}^{[u,v]}$ and such that the image of $V_{[u,v]}$ is $\mathbb{Z}$. Further, let us set $\mathbf{B}_{\overline{R}}^{[v]} := \cap_{u\leq v} \mathbf{B}_{\overline{R}}^{[u,v]}$, then we have $\mathbf{B}_{\overline{R},\mathrm{rig}}^{\dagger} = \cup_{v>0} \mathbf{B}_{\overline{R}}^{[v]}$. We equip $\mathbf{B}_{\overline{R}}^{[v]}$ with the Fréchet topology defined by the set of V_I where $I \subset (0, v]$ runs through all closed intervals. Finally, we see that $\mathbf{A}_{\overline{R}}^{[v]}$ is the ring of integers of $\mathbf{B}_{\overline{R}}^{[v]}$ for the valuation $V_{[v,v]}$.

Lemma 2.22. (i) *Let $u_0 = v_0 = 1$, then the natural inclusion of $\mathbf{A}_{\mathrm{max}}^{+}(R)$ and $\mathbf{A}_{\overline{R}}^{(0,v_0]+}$ in $\mathbf{A}_{\overline{R}}^{[u_0,v_0]}$ induces an exact sequence*

$$0 \longrightarrow \mathbf{A}_{\mathrm{inf}}(R) \longrightarrow \mathbf{A}_{\overline{R}}^{(0,v_0]+} \oplus \mathbf{A}_{\mathrm{max}}(R) \longrightarrow \mathbf{A}_{\overline{R}}^{[u_0,v_0]} \longrightarrow 0.$$

(ii) *Let $\mathbf{B}_{\mathrm{inf}}(R) = \mathbf{A}_{\mathrm{inf}}(R)\big[\frac{1}{p}\big]$, then for $v \in \mathbb{Q}_{>0}$ we have an exact sequence*

$$0 \longrightarrow \mathbf{B}_{\mathrm{inf}}(R) \longrightarrow \mathbf{B}_{\overline{R}}^{(0,v]} \oplus \mathbf{B}_{\overline{R},\mathrm{rig}}^{+} \longrightarrow \mathbf{B}_{\overline{R}}^{[v]} \longrightarrow 0.$$

Proof. (i) The proof essentially follows from the proof of [Ber02, Lemme 2.15]. The map $\mathbf{A}_{\overline{R}}^{(0,v_0]+} \oplus \mathbf{A}_{\mathrm{max}}(R) \longrightarrow \mathbf{A}_{\overline{R}}^{[u_0,v_0]}$ is surjective because it suffices to write an element of the right hand side as a sum of elements of objects on the left hand side. This is clear from the definitions and Lemma 2.21. Next, $\mathbf{A}_{\mathrm{inf}}(R)$ is contained both in $\mathbf{A}_{\overline{R}}^{(0,v_0]+}$ as well as in $\mathbf{A}_{\mathrm{max}}(R)$, therefore in their intersection. So we need to show that the map $\mathbf{A}_{\mathrm{inf}}(R) \longrightarrow \mathbf{A}_{\overline{R}}^{(0,v_0]+} \cap \mathbf{A}_{\mathrm{max}}(R)$ is surjective as well. We are going to show this modulo $p\mathbf{A}_{\overline{R}}^{[u_0,v_0]}$ and conclude the general case by dévissage (note that modulo p this map is not injective anymore). Let X, Y be two indeterminates, then from [Ber02, Lemmes 2.1 & 2.9] we have identifications $\mathbf{A}_{\max}(R) = \mathbf{A}_{\inf}(R)\{X\}/(pX - [p^\flat])$, $\mathbf{A}_{\overline{R}}^{(0,v_0]+} = \mathbf{A}_{\inf}(R)\{Y\}/([p^\flat]Y - p)$ and $\mathbf{A}_{\overline{R}}^{[u_0,v_0]}/(p) = \mathbb{C}^+(R)^\flat/(p^\flat)[X, X^{-1}]$. The image of $\mathbf{A}_{\max}(R)$ in the latter ring is identified with $\mathbb{C}^+(R)^\flat/(p^\flat)[X]$ and the image of $\mathbf{A}_{\overline{R}}^{(0,v_0]+}$ gets identified with $\mathbb{C}^+(R)^\flat/(p^\flat)[X^{-1}]$. This shows that the image of their intersection (which is a subset of the intersection of their respective images) is a subring of $\mathbb{C}^+(R)^\flat/(p^\flat)$ and therefore the map $\mathbf{A}_{\inf}(R) \longrightarrow \mathbf{A}_{\overline{R}}^{(0,v_0]+}/\mathbf{A}_{\max}(R)$ is surjective modulo $p\mathbf{A}_{\overline{R}}^{[u_0,v_0]}$. So if $x \in \mathbf{A}_{\max}(R) \cap \mathbf{A}_{\overline{R}}^{(0,v_0]+}$, then there exists $y \in \mathbf{A}_{\inf}(R)$ such that $x - y \in p\mathbf{A}_{\overline{R}}^{[u_0,v_0]}$. This means that $x - y$ is an element of $p\mathbf{A}_{\max}(R)$ as well as of $p\mathbf{A}_{\overline{R}}^{(0,v_0]+} + [p^\flat]\mathbf{A}_{\inf}(R)$ (this follows from the discussion above and [Ber02, Lemme 2.9]). Since p divides $[p^\flat]$ in $\mathbf{A}_{\max}(R)$, there exists $z \in [p^\flat]\mathbf{A}_{\inf}(R)$ such that $x - y - z \in p(\mathbf{A}_{\max}(R) \cap \mathbf{A}_{\overline{R}}^{(0,v_0]+})$. Since $\mathbf{A}_{\inf}(R)$ is p-adically complete, we can iterate this process to conclude the claim.

(ii) The proof essentially follows from the proof of [Ber02, Lemme 2.18]. Let $u_n = p^{-n}$ for $n \in \mathbb{N}$. First, for $u_n \leq v$ we will show that the sequence

$$0 \longrightarrow \mathbf{B}_{\inf}(R) \longrightarrow \mathbf{B}_{\overline{R}}^{(0,v]} \oplus \mathbf{B}_{\overline{R}}^{[u_n,+\infty]} \longrightarrow \mathbf{B}_{\overline{R}}^{[u_n,v]} \longrightarrow 0,$$

is exact. It is clear that any element of $\mathbf{B}_{\overline{R}}^{[u_n,v]}$ can be written as a sum of elements of $\mathbf{B}_{\overline{R}}^{(0,v]}$ and $\mathbf{B}_{\overline{R}}^{[u_n,+\infty]}$ and we need to show that two such expressions differ by an element of $\mathbf{B}_{\inf}(R)$. This amounts to showing that $\mathbf{B}_{\overline{R}}^{(0,v]} \cap \mathbf{B}_{\overline{R}}^{[u_n,+\infty]} = \mathbf{B}_{\inf}(R)$, which can be deduced directly or by applying φ^{-n} to $\mathbf{B}_{\overline{R}}^{(0,p^nv]} \cap \mathbf{B}_{\overline{R}}^{[u_0,+\infty]} = \mathbf{B}_{\inf}(R)$ where the latter expression is true from (i).

Next, we will show the claim. For each $n \in \mathbb{N}$, we have $\mathbf{B}_{\overline{R}}^{[v]} \subset \mathbf{B}_{\overline{R}}^{[u_n,v]}$, therefore any $x \in \mathbf{B}_{\overline{R}}^{[v]}$ can be written as $x = a_n + b_n$ with $a_n \in \mathbf{B}_{\overline{R}}^{(0,v]}$ and $b_n \in \mathbf{B}_{\overline{R}}^{[u_n,+\infty]}$. Note that if we have another expression $x = a_{n+1} + b_{n+1}$ with $a_{n+1} \in \mathbf{B}_{\overline{R}}^{(0,v]}$ and $b_{n+1} \in \mathbf{B}_{\overline{R}}^{[u_n,+\infty]}$ such that $b_{n+1} - b_n \in \mathbf{B}_{\inf}(R)$, then up to modifying a_{n+1} and b_{n+1} by elements of $\mathbf{B}_{\inf}(R)$, we can suppose that $a_n = a_{n+1}$ and $b_n = b_{n+1}$. Therefore, $x = a + b$ with $a \in \mathbf{B}_{\overline{R}}^{(0,v]}$ and $b \in \cap_{n\in\mathbb{N}} \mathbf{B}_{\overline{R}}^{[u_n,+\infty]} = \mathbf{B}_{\overline{R},\mathrm{rig}}^+$ (see Lemma 2.21 (ii)).

■

Now we come to the main result of this section: *regularization by Frobenius*,

Proposition 2.23. *Let $h \in \mathbb{N}$ and matrices $A \in Mat\big(h, \mathbf{B}_{\overline{R},\mathrm{rig}}^\dagger\big)$ and $Y, Z \in Mat\big(h, \mathbf{B}_{\overline{R},\mathrm{rig}}^+\big)$ such that $\varphi(A) = YAZ$, then $A \in Mat\big(h, \mathbf{B}_{\overline{R},\mathrm{rig}}^+\big)$.*

Proof. The proof essentially follows from the proof of [Ber04, Proposition I.4.1]. Note that there exists $v > 0$ such that $A \in Mat\big(h, \mathbf{B}_{\overline{R}}^{[v]}\big)$, so there exists $c \in \mathbb{N}$ such that $A \in Mat\big(h, p^{-c}\mathbf{A}_{\overline{R}}^{[v]}\big)$. By the definition of $\mathbf{B}_{\overline{R},\mathrm{rig}}^+$ we have that $\varphi^{-1}(Y), \varphi^{-1}(Z) \in Mat\big(h, \mathbf{B}_{\overline{R},\mathrm{rig}}^+\big)$. Since $\mathbf{B}_{\overline{R},\mathrm{rig}}^+ \subset \mathbf{B}_{\overline{R}}^{[v]}$ (see Lemma 2.21), from [Ber02, Corollaire 2.20] we get that there exists $m \in \mathbb{N}$ such that $\varphi^{-1}(Y), \varphi^{-1}(Z) \in Mat\big(h, p^{-m}\mathbf{A}_{\overline{R}}^{[w]}\big)$ for all $w \geq v$. Next, we know that $\varphi^{-1}\big(\mathbf{A}_{\overline{R}}^{[v]}\big) = \mathbf{A}_{\overline{R}}^{[pv]}$. Therefore, by induction over $n \in \mathbb{N}$ and using the equation $A = \varphi^{-1}(YAZ)$, we get that $A \in Mat\big(h, p^{-c-2mn}\mathbf{A}_{\overline{R}}^{[p^nv]}\big)$. Now using Lemma 2.24 with $k = 2m$, we have that $\cap_{n\in\mathbb{N}} p^{-kn}\mathbf{A}_{\overline{R}}^{[p^nv]} \subset \mathbf{B}_{\overline{R},\mathrm{rig}}^+$. Hence, we get the claim. ■

Following observation was used above:

Lemma 2.24. *Let $k \in \mathbb{N}_{>0}$. Then*

$$\bigcap_{n\in\mathbb{N}} p^{-kn}\mathbf{A}_{\overline{R}}^{(0,p^nv]+} = \mathbf{A}_{\inf}(R) \quad \textit{and} \quad \bigcap_{n\in\mathbb{N}} p^{-kn}\mathbf{A}_{\overline{R}}^{[p^nv]} \subset \mathbf{B}_{\overline{R},\mathrm{rig}}^+.$$

Proof. The proof essentially follows from the proof of [Ber02, Lemme 3.1]. Let $x \in \cap_{n\in\mathbb{N}} p^{-kn}\mathbf{A}_{\overline{R}}^{(0,v]+}$. Since $x \in \mathbf{A}_{\overline{R}}^{(0,v]+}$, we can write $x = \sum_{i\in\mathbb{N}} p^i[x_i]$ uniquely and so we have $p^{kn}x = \sum_{i\in\mathbb{N}} p^{i+kn}[x_i]$. Now $p^{kn}x \in \mathbf{A}_{\overline{R}}^{(0,p^n v]+}$, so we get $p^n v v^\flat(x_i) + i + kn \geq 0$, which implies that $v^\flat(x_i) \geq -\frac{i+kn}{p^n v}$. The right hand side of the latter inequality goes to 0 as n approaches $+\infty$, therefore $v^\flat(x_i) \geq 0$ for each $i \in \mathbb{N}$, i.e. $x \in \mathbf{A}_{\text{inf}}(R)$.

Next, let $x \in \cap_{n\in\mathbb{N}} p^{-kn}\mathbf{A}_{\overline{R}}^{[p^n v]}$. For each $n \in \mathbb{N}$, we can write $x = a_n + b_n$ with $a_n \in p^{-kn}\mathbf{A}_{\overline{R}}^{(0,p^n v]+}$ and $b_n \in \mathbf{B}^+_{\overline{R},\text{rig}}$. By Lemma 2.22 (ii), we obtain that $a_n - a_{n+1} \in \mathbf{B}_{\text{inf}}(R)$, whereas we already have that $a_n - a_{n+1} \in p^{-k(n+1)}\mathbf{A}_{\overline{R}}^{(0,p^n v]+}$. So this implies $a_n - a_{n+1} \in p^{-k(n+1)}\mathbf{A}_{\text{inf}}(R)$ and therefore, up to modifying a_{n+1} by an element of $p^{-k(n+1)}\mathbf{A}_{\text{inf}}$, we can assume that $a_n = a_{n+1} = a$. This implies that $a \in \cap_{n\in\mathbb{N}} p^{-kn}\mathbf{A}_{\overline{R}}^{(0,p^n v]+} = \mathbf{A}_{\text{inf}}(R)$, hence $x \in \mathbf{B}^+_{\overline{R},\text{rig}}$. ■

The following statement will be useful for the proof of Lemma 3.12:

Corollary 2.25. *Let $h \in \mathbb{N}$ and matrices $A \in Mat(h, \mathbf{B}^\dagger)$ and $Y, Z \in Mat(h, \mathbf{B}^+)$ such that $\varphi(A) = YAZ$, then $A \in Mat(h, \mathbf{B}^+)$.*

Proof. The proof essentially follows from the proof of [Ber04, Corollaire I.4.3]. From Proposition 2.23 we have that $A \in Mat\big(h, \mathbf{B}^+_{\overline{R},\text{rig}}\big)$. So we only need to show that $\mathbf{B}^\dagger \cap \mathbf{B}^+_{\overline{R},\text{rig}} = \mathbf{B}^+$. But this follows from the fact that $\mathbf{B}^\dagger_{\overline{R}} \cap \mathbf{B}^+_{\overline{R},\text{rig}} = \mathbf{B}_{\text{inf}}(R)$ and $\mathbf{B}^\dagger \cap \mathbf{B}_{\text{inf}}(R) = \mathbf{B}^+$ (see Lemma 2.22). ■

2.3. Rings of analytic functions

Recall that R_0 is the p-adic completion of an étale algebra over $W\{X, X^{-1}\}$, i.e. we wrote

$$R_0 := W\{X, X^{-1}\}\{Z_1, \dots, Z_s\}/(Q_1, \dots, Q_s),$$

with $Q_i \in W\{X, X^{-1}\}\{Z_1, \dots, Z_s\}$ for $1 \leq i \leq s$, some multivariate polynomials such that $\det\big(\frac{\partial Q_i}{\partial Z_j}\big)_{1\leq i,j\leq s}$ is invertible in R_0. Next, we defined $R_\square := O_K\{X, X^{-1}\}$ and using the definition of R_0, we set

$$R := R_\square\{Z_1, \dots, Z_s\}/(Q_1, \dots, Q_s),$$

so that $R_\square$ provides a *system of coordinates* for R and the latter is totally ramified at the prime ideal $(p) \subset R_0$.

Let r^+_ϖ and r_ϖ denote the algebras $O_F[[X_0]]$ and $O_F[[X_0]]\{X_0^{-1}\}$. Sending X_0 to ϖ induces a surjective homomorphism $r^+_\varpi \twoheadrightarrow O_K$. Let $R^+_{\varpi,\square}$ denote the completion of $O_F[X_0, X, X^{-1}]$ for the (p, X_0)-adic topology. Sending X_0 to ϖ induces a surjective homomorphism $R^+_{\varpi,\square} \twoheadrightarrow R_\square$, whose kernel is generated by $P = P_\varpi(X_0)$. This provides a closed embedding of Spf $R_\square$ into a formal scheme Spf $R^+_{\varpi,\square}$, which is smooth over O_F. Since R is étale over $R_\square$, we have that $\det\big(\frac{\partial Q_i}{\partial Z_j}\big)$ is invertible in R. As Q_j's have coefficients in $W\{X, X^{-1}\}$, we can set R^+_ϖ to be the quotient by $(Q_1, \dots, Q_s)$ of the completion of $R^+_{\varpi,\square}[Z_1, \dots, Z_s]$ for the (p, X_0)-adic topology. Again, we have that $\det\big(\frac{\partial Q_i}{\partial Z_j}\big)$ is invertible in R^+_ϖ (since it is modulo P). Hence, R^+_ϖ is étale over $R^+_{\varpi,\square}$ and smooth over O_F. Sending X_0 to ϖ induces a surjective homomorphism $R^+_\varpi \twoheadrightarrow R$ whose kernel is generated by $P = P_\varpi(X_0)$. This can be summarized with a commutative diagram of rings

$$\begin{tikzcd}
\operatorname{Spf} R \arrow[rr, tail] \arrow[dr] \arrow[ddd] & & \operatorname{Spf} R_{\varpi}^{+} \arrow[dl] \arrow[ddd] \\
& \operatorname{Spf} R_0 \arrow[d] & \\
& \operatorname{Spf} W(\kappa)\{X, X^{-1}\} & \\
\operatorname{Spf} R_{\square} \arrow[rr, tail] \arrow[ur] & & \operatorname{Spf} R_{\varpi,\square}^{+}, \arrow[ul]
\end{tikzcd}$$

where the vertical arrows are étale extensions and the horizontal maps are obtained by sending $X_0 \mapsto \varpi$ and the rest are natural maps. Since $P = X_0^e \bmod p$, we have

$$R_{\varpi}^{+}\left[\tfrac{P^k}{k!}\right]_{k\in\mathbb{N}} = R_{\varpi}^{+}\left[\tfrac{X_0^k}{[k/e]!}\right]_{k\in\mathbb{N}}.$$

So, we set

$$R_{\varpi}^{\mathrm{PD}} := p\text{-adic completion of } R_{\varpi}^{+}\left[\tfrac{P^k}{k!}\right]_{k\in\mathbb{N}}.$$

In summary, we have a diagram of formal schemes where the horizontal arrows are closed embeddings into formal schemes smooth over O_F, obtained by sending $X_0 \mapsto \varpi$ on the level of algebras,

$$\begin{tikzcd}
& \operatorname{Spf} R_{\varpi}^{\mathrm{PD}} \arrow[dr] & \\
\operatorname{Spf} R \arrow[ur, tail] \arrow[rr, tail] \arrow[d] & & \operatorname{Spf} R_{\varpi}^{+} \arrow[d] \\
\operatorname{Spf} R_{\square} \arrow[rr, tail] \arrow[d] & & \operatorname{Spf} R_{\varpi,\square}^{+} \arrow[d] \\
\operatorname{Spf} O_K \arrow[rr, tail] \arrow[d] & & \operatorname{Spf} r_{\varpi}^{+} \arrow[dll] \\
\operatorname{Spf} O_F. & &
\end{tikzcd}$$

Let Ω_R^q denote the p-adic completion of the modules of differential of R relative to $\mathbb{Z}$. We have that

$$\Omega_{R_0}^1 = \bigoplus_{i=1}^{d} R_0 \, d\log X_i \quad\text{and}\quad \Omega_{R_0}^k = \bigwedge^k \Omega_{R_0}^1,$$

and the cokernel of the natural map $\Omega_{R_0}^k \otimes_{R_0} R \longrightarrow \Omega_R^k$ is killed by a power of p (see Proposition 1.1). In particular,

$$\Omega_R^k\left[\tfrac{1}{p}\right] = \bigwedge^k \Big(\bigoplus_{i=1}^{d} R\left[\tfrac{1}{p}\right] \, d\log X_i\Big).$$

Moreover, since R_{ϖ}^{+} is étale over $R_{\varpi,\square}^{+}$, for $S = R_{\varpi}^{+}, R_{\varpi,\square}^{+}$ we have that

$$\Omega_S^1 = S\frac{dX_0}{1+X_0} \oplus \Big(\bigoplus_{i=1}^{d} S \, d\log X_i\Big).$$

Definition 2.26. For $0 < u \leq v$ define the rings,

$$\begin{aligned} R_\varpi^{(0,v]+} &:= p\text{-adic completion of } R_\varpi^+\big[\tfrac{p^{\lfloor vk/e \rfloor}}{X_0^k}\big]_{k\in\mathbb{N}}, \quad R_\varpi^{(0,v]} := R_\varpi^{(0,v]+}\big[\tfrac{1}{X_0}\big], \\ R_\varpi^{[u]} &:= p\text{-adic completion of } R_\varpi^+\big[\tfrac{X_0^k}{p^{\lfloor uk/e \rfloor}}\big]_{k\in\mathbb{N}}, \\ R_\varpi^{[u,v]} &:= p\text{-adic completion of } R_\varpi^+\big[\tfrac{X_0^k}{p^{\lfloor uk/e \rfloor}}, \tfrac{p^{\lfloor vk/e \rfloor}}{X_0^k}\big]_{k\in\mathbb{N}}, \\ R_\varpi &:= p\text{-adic completion of } R_\varpi^+\big[\tfrac{1}{X_0}\big]. \end{aligned}$$

We will write $R_\varpi^\star$ for $\star \in \{\ , +, \mathrm{PD}, [u], (0,v]+, [u,v]\}$ and for $R = O_K$, we write $r_\varpi^\star$ instead. Going from R_ϖ^+ to $R_\varpi^\star$ involves only the arithmetic variable X_0, so we have isomorphisms

$$R_\varpi^\star = r_\varpi^\star \widehat{\otimes}_{r_\varpi^+} R_\varpi^+,$$

where $\widehat{\otimes}$ is the completion of tensor product for the p-adic topology.

Definition 2.27. We define a filtration on the rings in Definition 2.26 by order of vanishing at $X_0 = \varpi = \zeta_{p^m} - 1$.

(a) Let $S = R_\varpi^{(0,v]+}$ $(v < 1)$, $R_\varpi^{(0,v]}$ $(v < 1)$, $R_\varpi^{[u,v]}$ $(1 \notin [u,v])$ or R_ϖ. As P is invertible in $S\big[\frac{1}{p}\big]$, we put the trivial filtration on S.

(b) Let S be the placeholder for all other rings occuring in Definition 2.26, such that P is not invertible in $S\big[\frac{1}{p}\big]$. Then there is a natural embedding $S \longrightarrow R\big[\frac{1}{p}\big][[P]]$ by completing $S\big[\frac{1}{p}\big]$ for the P-adic topology. We use this embedding to endow S with the natural filtration $\mathrm{Fil}^k S = S \cap P^k R\big[\frac{1}{p}\big][[P]]$ for $k \in \mathbb{Z}$.

Next, we note a lemma that will be useful in Chapter 5.

Lemma 2.28 ([CN17, Lemma 2.6])**.** *Let $r \in \mathbb{N}$.*

(i) *For $f \in R_\varpi^{\mathrm{PD}}$ we can write $f = f_1 + f_2$ with $f_1 \in \mathrm{Fil}^r R_\varpi^{\mathrm{PD}}$ and $f_2 \in \frac{1}{(r-1)!} R_\varpi^+$.*

(ii) *For $f \in R_\varpi^{[u]}$ we can write $f = f_1 + f_2$ with $f_1 \in \mathrm{Fil}^r R_\varpi^{[u]}$ and $f_2 \in \frac{1}{p^{\lfloor ru \rfloor}} R_\varpi^+$.*

Proof. First we note that from the definitions an element $f \in r_\varpi^{\mathrm{PD}}$ $\big($resp. $f \in r_\varpi^{[u]}\big)$ can be written (uniquely) in the form $f = f^+ + f^-$ with $f^+ \in \mathrm{Fil}^r r_\varpi^{\mathrm{PD}}$ and $f^- \in \frac{1}{(r-1)!} O_F[X_0]$ $\big($resp. $f^- \in \frac{1}{p^{\lfloor ru \rfloor}} O_F[X_0]\big)$ of degree $\leq re - 1$. Next, from the equality $R_\varpi^{\mathrm{PD}} = r_\varpi^{\mathrm{PD}} \widehat{\otimes}_{r_\varpi^+} R_\varpi^+$ (resp. $R_\varpi^{\mathrm{PD}} = r_\varpi^{[u]} \widehat{\otimes}_{r_\varpi^+} R_\varpi^+$), it follows that we can write any $f \in R_\varpi^{\mathrm{PD}}$ as $f_1 + f_2$ with $f_1 \in \mathrm{Fil}^r R_\varpi^{\mathrm{PD}}$ and $f_2 \in \frac{1}{(r-1)!} R_\varpi^+$ and we have the same statement for $R_\varpi^{[u]}$ with $f_1 \in \mathrm{Fil}^r R_\varpi^{[u]}$ and $f_2 \in \frac{1}{p^{\lfloor ru \rfloor}} R_\varpi^+$. ∎

Notation. Let S be a $\mathbb{Z}_p$-algebra. A homomorphism $f : M \longrightarrow N$ between two S-modules is said to be a *p^n-isomorphism*, for some $n \in \mathbb{N}$ if the kernel and the cokernel of the map f are killed by p^n.

Lemma 2.29 ([CN17, Lemma 2.11])**.** *Let $t := p^m \log(1 + X_0)$. If $\frac{p-1}{p} \leq u \leq \frac{v}{p} < 1 < v$ and $\frac{1}{p} < u$, then*

(i) *t belongs to $pr_\varpi^{[u,v]}$ and to $pr_\varpi^{[u,v/p]}$;*

(ii) *$\frac{t}{p} \in p^{-1} r_\varpi^{[u,v]}$ and $t \in p^{-2} r_\varpi^{[u,v/p]}$;*

(iii) *$x \mapsto t^r x$ induces a p^r-isomorphism $r_\varpi^{[u,v]} \simeq \mathrm{Fil}^r r_\varpi^{[u,v]}$ and a p^{2r}-isomorphism $r_\varpi^{[u,v/p]} \simeq r_\varpi^{[u,v/p]}$.*

We note an important fact from [CN17], the *implicit function theorem*, which would help us lift certain maps over étale extensions. Let $\lambda : R_{\varpi,\square}^+ \longrightarrow \Lambda$ be a continuous morphism of topological rings. We have $R_\varpi^+ = R_{\varpi,\square}^+\{Z\}/(Q)$, where $Z = (Z_1, \ldots, Z_s)$ and $Q = (Q_1, \ldots, Q_s)$. We would like to extend λ to R_ϖ^+ which amounts to solving the equation $Q^\lambda(Y) = 0$ in Λ, where if $F \in R_{\varpi,\square}^+\{Z\}$, we note $F^\lambda \in \Lambda\{Z\}$ the series obtained by applying λ to the coeffficients of F. Then,

Proposition 2.30 ([CN17, Proposition 2.1 & Remark 2.2]). *The equation $Q^{\lambda}(Y)$ has a unique solution in $Z_{\lambda} + I^s$.*

Proof. Let $J = \left(\frac{\partial Q_i}{\partial Z_j}\right)_{1\leq i,j\leq s} \in Mat\big(s, R^+_{\varpi,\square}\{Z_1, \dots, Z_s\}\big)$. Suppose that there exists an ideal $I \subset \Lambda$ such that Λ is complete with respect to the I-adic topology, $Z_{\lambda} = (Z_{1,\lambda}, \dots, Z_{s,\lambda}) \in \Lambda^s$ and $H_{\lambda} \in Mat(s, \Lambda)$, such that the entries of $Q^{\lambda}(Z_{\lambda})$ belong to I. Now, since R^+_{ϖ} is étale over Λ, so $\det J$ is invertible in $R^+_{\varpi,\square}$ and therefore there exists $H \in Mat\big(s, R^+_{\varpi,\square}\{Z_1, \dots, Z_s\}\big)$ such that $HJ - 1$ has its entries in $(Q_1, \dots, Q_s)$. But $Q^{\lambda}(Z_{\lambda})$ has coordinates in the ideal I, therefore $H^{\lambda}J^{\lambda} - 1$ has entries in I. Thus, we can apply [CN17, Proposition 2.1], by taking (in the notation of loc. cit.) $z = 1$ and $H_{\lambda} = H^{\lambda}(Z_{\lambda})$. Hence, the equation $Q^{\lambda}(Y)$ has a unique solution in $Z_{\lambda} + I^s$. ■

2.3.1. Cyclotomic Frobenius

Definition 2.31. Over $R^+_{\varpi,\square}$ we can define a lift of the absolute Frobenius on $R^+_{\varpi,\square}/p$ by

$$\begin{aligned} \varphi \, : \, R^+_{\varpi,\square} &\longrightarrow R^+_{\varpi,\square} \\ X_0 &\longmapsto (1 + X_0)^p - 1 \\ X_i &\longmapsto X_i^p \ \text{ for } \ i \leq i \leq d, \end{aligned}$$

which we will call the (cyclotomic) Frobenius. Clearly, $\varphi(x) - x^p \in pR^+_{\varpi,\square}$ for $x \in R^+_{\varpi,\square}$. Using Proposition 2.30 with $\Lambda_1 = R^+_{\varpi,\square}$, $\Lambda_1' = \Lambda_2 = R^+_{\varpi}$, $\lambda = \varphi$, $I = (p)$ and $Z_{\lambda} = Z^p$, we can extend the Frobenius homomorphism to $\varphi \, : \, R^+_{\varpi} \longrightarrow R^+_{\varpi}$. By continuity, the Frobenius endomorphism φ admits unique extensions

$$R^{\mathrm{PD}}_{\varpi} \longrightarrow R^{\mathrm{PD}}_{\varpi}, \quad R^{[u]}_{\varpi} \longrightarrow R^{[u]}_{\varpi}, \quad R^{(0,v]+}_{\varpi} \longrightarrow R^{(0,v/p]+}_{\varpi}, \quad R^{[u,v]}_{\varpi} \longrightarrow R^{[u,v/p]}_{\varpi} \quad \text{and} \quad R_{\varpi} \longrightarrow R_{\varpi}.$$

Explicitly, we can write

$$\begin{aligned} r^{\mathrm{PD}}_{\varpi} &= \Big\{ f = \sum_{k\in\mathbb{N}} a_k \tfrac{X_0^k}{[k/e]!}, \text{ such that } a_k \in O_F \text{ goes to } 0 \text{ as } i \longrightarrow \infty \Big\}, \\ r^{[u]}_{\varpi} &= \Big\{ f = \sum_{k\in\mathbb{N}} a_k \tfrac{X_0^k}{p^{\lfloor \frac{ku}{e} \rfloor}}, \text{ such that } a_k \in O_F \text{ goes to } 0 \text{ as } i \longrightarrow \infty \Big\}. \end{aligned}$$

Let $S = r^{\mathrm{PD}}_{\varpi}$ or $r^{[u]}_{\varpi}$. Denote by $v_{X_0} \, : \, S \longrightarrow \mathbb{N} \cup \{+\infty\}$ the valuation relative to X_0, i.e. if $f = \sum a_k X_0^k$, then $v_{X_0}(f) = \inf\{i \in \mathbb{N}, a_i \neq 0\}$. For $N \in \mathbb{N}$, we define $S_N = \{f \in S, \ v_{X_0}(f) \geq N\}$. Define $R^{\mathrm{PD}}_{\varpi,N}$ and $R^{[u]}_{\varpi,N}$ as the topological closures of $r^{\mathrm{PD}}_{\varpi,N} \otimes_{r^+_{\varpi}} R^+_{\varpi} \subset R^{\mathrm{PD}}_{\varpi}$ and $r^{[u]}_{\varpi,N} \otimes_{r^+_{\varpi}} R^+_{\varpi} \subset R^{[u]}_{\varpi}$, respectively.

Lemma 2.32. (i) *Let $N \in \mathbb{N}_{>0}$, $s \in \mathbb{Z}$ and $N \geq se$ (resp. $N \geq se/u(p-1)$), then $1 - p^{-s}\varphi$ is bijective on $R^{\mathrm{PD}}_{\varpi,N}$ $\big($resp. $R^{[u]}_{\varpi,N}\big)$.*

(ii) *The maps*

$$\begin{aligned} 1 - \varphi \, &: \, R^{\mathrm{PD}}_{\varpi}/R^+_{\varpi} \longrightarrow R^{\mathrm{PD}}_{\varpi}/R^+_{\varpi}; \\ 1 - \varphi \, &: \, R^{[u]}_{\varpi}/R^+_{\varpi} \longrightarrow R^{[u]}_{\varpi}/R^+_{\varpi}, \end{aligned}$$

are bijective.

Proof. For (i), see [CN17, Proposition 3.1]. In (ii), we will only treat the case of $R^{\mathrm{PD}}_{\varpi}/R^+_{\varpi}$, the other case follows similarly (an application of (i)). Write $x \in R^{\mathrm{PD}}_{\varpi}$ as

$$x = \sum_{k\geq 0} a_k \frac{X_0^k}{[k/e]!},$$

where $a_k \in R_\varpi^+$ goes to 0 as $k \longrightarrow \infty$. By (i), we know that the series of operators $1 + \varphi + \varphi^2 + \cdots$ converge as an inverse to $1 - \varphi$, i.e. there exists $y \in R_\varpi^{\mathrm{PD}}$ such that $(1 - \varphi)y = x - a_0 \in R_{\varpi,1}^{\mathrm{PD}}$. Since $a_0 \in R_\varpi^+$, we get that $1 - \varphi$ is bijective on $R_\varpi^{\mathrm{PD}}/R_\varpi^+$. ■

2.3.2. The operator ψ

In this section we will define a left inverse of the cyclotomic Frobenius φ, which we will denote by ψ. This operator is closely related to the operator defined in Proposition 2.13 (this will become clear in §2.4). However, we prefer to give an explicit definition here. Let

$$u_\alpha = (1 + X_0)^{\alpha_0} X_1^{\alpha_1} \cdots X_d^{\alpha_d} \quad \text{for} \quad \alpha = (\alpha_0, \dots, \alpha_d) \in \{0, 1, \dots, p-1\}^{[0,d]}.$$

Remark 2.33. Note that R_ϖ is X_0-adically complete, therefore $1 + X_0$ is invertible in it. Moreover, by definition $X_1, \dots, X_d$ are invertible in R_ϖ, therefore u_α is invertible in R_ϖ^+ for $\alpha = (\alpha_0, \dots, \alpha_d) \in \{0, 1, \dots, p-1\}^{[0,d]}$.

Also, set

$$\partial_0 = (1 + X_0)\frac{d}{dX_0}, \qquad \partial_i = X_i \frac{d}{dX_i} \quad \text{for} \quad 1 \le i \le d.$$

Therefore, for $0 \le i \le d$ we have

$$\partial_i u_\alpha = \alpha_i u_\alpha \ \text{ and } \ \varphi(u_\alpha) = u_\alpha^p.$$

Lemma 2.34 ([CN17, Proposition 2.15])**.** (i) *Any $x \in R_\varpi/p$ can be written uniquely as $x = \sum_\alpha c_\alpha(x)$, with $\partial_i \circ c_\alpha(x) = \alpha_i c_\alpha(x)$ for $0 \le i \le d$.*

(ii) *There exists a unique $x_\alpha \in R_\varpi/p$ such that $c_\alpha(x) = x_\alpha^p u_\alpha$.*

(iii) *If $x \in R_\varpi^+/p$, then $c_\alpha(x) \in R_\varpi^+/p$.*

Proof. Let $S = R_\varpi/p$, $S^+ = R_\varpi^+/p$. Then $\partial_i(\partial_i - 1)\cdots(\partial_i - (p-1))$ is identically 0 on $R_{\varpi,\square}/p$, hence also on S by étaleness. It follows that ∂_i is diagonalizable for all i and since these operators commute pairwise, we can decompose S and S^+ into the direct sum of common eigenspaces. This shows (i) and (iii). Now, differentials of the elements in the set $\{1 + X_0, X_1, \dots, X_d\}$ form a basis of the module of differentials of $R_{\varpi,\square}/p$, hence also of S, since it is obtained as the completion of an étale algebra over $R_{\varpi,\square}/p$. From [Tyc88, §III, Theorem 1], it follows that $\{1 + X_0, X_1, \dots, X_d\}$ is a p-basis of S which can be rephrased by saying that any element x of S can be written uniquely as $x = \sum_\alpha x_\alpha^p u_\alpha$. Since $\partial_i(x_\alpha^p u_\alpha) = \alpha_i x_\alpha^p u_\alpha$ for $1 \le i \le d$, this proves (ii). ■

Proposition 2.35. (i) *Any $x \in R_\varpi$ can be written uniquely as $x = \sum_\alpha c_\alpha(x)$, with $c_\alpha(x) \in \varphi\big(R_\varpi\big) u_\alpha$.*

(ii) *If $x \in R_\varpi^+$ and if $c_\alpha(x) = \varphi(x_\alpha)u_\alpha$, then $c_\alpha(x) \in R_\varpi^+$ for all α and*

$$\partial_i c_\alpha(x) - \alpha_i c_\alpha(x) \in pR_\varpi^+ \quad \textit{for} \ \ 0 \le i \le d.$$

(iii) *For $x \in R_\varpi^{(0,v]+}$, we have $c_\alpha(x) \in R_\varpi^{(0,v]+}$ for all α.*

Proof. (i) and (ii) follow from the lemma above. (iii) follows from [CN17, Proposition 2.15]. ■

Definition 2.36. Define the left inverse ψ of the Frobenius φ on $S = R_\varpi^+$ or $S = R_\varpi$, by the formula

$$\psi(x) = \varphi^{-1}\big(c_0(x)\big).$$

Since R_ϖ is an extension of degree p^{d+1} of $\varphi(R_\varpi)$ with basis the u_α's and since $\varphi(u_\alpha) = u_\alpha^p$ for all α, we have

$$\mathrm{Tr}_{R_\varpi/\varphi(R_\varpi)}\big(u_\alpha\big) = 0 \ \text{ if } \ \alpha \ne 0,$$

and we can define ψ intrinsically, by the formula

$$\psi(x) := \tfrac{1}{p^{d+1}}\varphi^{-1}\circ \mathrm{Tr}_{R_\varpi/\varphi(R_\varpi)}(x).$$

Note that ψ is not a ring morphism; it is a left inverse to φ and more generally, we have $\psi(\varphi(x)y) = x\psi(y)$. Also,

$$\partial_i\circ\varphi = p\varphi\circ\partial_i \ \text{ and } \ \partial_i\circ\psi = p^{-1}\psi\circ\partial_i \ \text{ for } \ i = 0,1,\ldots,d.$$

The first equality can be obtained by checking on the basis elements u_α. For the second equality, note that for $x \in R_\varpi$ and in the notation of Proposition 2.35 we have

$$\partial_i(\varphi(x_\alpha)u_\alpha) = \partial_i\circ\varphi(x_\alpha)u_\alpha + \varphi(x_\alpha)\partial_i(u_\alpha) = (p\varphi\circ\partial_i(x_\alpha) + \alpha_i\varphi(x_\alpha))u_\alpha = \varphi(p\partial_i(x_\alpha) + \alpha_i x_\alpha)u_\alpha.$$

Applying ψ to the latter expression we note that it is nonzero only if $\alpha = 0$, in which case we get that $\psi\circ\partial_i \in pR_\varpi$ for all $0 \leq i \leq d$, the equality follows from this.

For any $k \in \mathbb{N}$, we can write $X_0^k = \sum_{j=0}^{p-1}\varphi(a_{j,k})(1+X_0)^j$ for $a_{j,k} \in R_\varpi^+$. Therefore, by continuity

Lemma 2.37. (i) *The explicit formula for ψ extends to maps $R_\varpi^{[u]} \longrightarrow R_\varpi^{[pu]}$ and $R_\varpi^{[u,v]} \longrightarrow R_\varpi^{[pu,pv]}$.*

(ii) *For the same reasons, the maps $x \mapsto c_\alpha(x)$ also extend and lead to decompositions $S = \oplus_\alpha S_\alpha$, where $S_\alpha = Su_\alpha$ for $S = R_\varpi^\star$ with $\star \in \{,+,[u],(0,v]+,[u,v]\}$. Since $\psi(x) = \varphi^{-1}\big(c_0(x)\big)$, we have*

$$S^{\psi=0} = \bigoplus_{\alpha\neq 0} S_\alpha.$$

Lemma 2.38. *If $S = R_\varpi^\star$ for $\star \in \{\ ,+,[u],(0,v]+,[u,v]\}$, then for $0 \leq i \leq d$ the operator ∂_i on $S_\alpha^\star/pS_\alpha^\star$ is given by multiplication by α_i, where α_i is the i-th entry in $\alpha = (\alpha_0,\ldots,\alpha_d)$.*

Proof. If $\star \in \{\ ,+\}$, this is part of Proposition 2.35. For $\star \in \{[u],(0,v]+,[u,v]\}$, elements of $S_\alpha^\star$ are those of the form $\sum_{k\in\mathbb{Z}} p^{r_k}X_0^k x_k$, where $x_k \in S^+$ goes to 0 when $k \longrightarrow +\infty$ and r_k is determined by "$\star$". Let $x = \sum_{k\in\mathbb{Z}} p^{r_k}X_0^k x_k$. For $1 \leq i \leq d$, we have

$$\partial_i(X_0^k a_k) - \alpha_i X_0^k a_k = X_0^k\big(\partial_i(a_k) - \alpha_i a_k\big) \in pS^+,$$

by Proposition 2.35.

For $i = 0$, first we look at $S^{[u]}$ and write

$$x = \sum_{k\in\mathbb{N}} p^{r_k}x_k\sum_{j=0}^{p-1}\varphi(a_{j,k})(1+X_0)^j \quad \text{for} \quad a_{j,k} \in S^+.$$

Then

$$c_\alpha(x) = \sum_{j=0}^{p-1}\sum_{k\in\mathbb{N}} p^{r_k}\varphi(a_{j,k})c_{(\alpha_0-j,\alpha_1,\cdots,\alpha_d)}(x_k)(1+X_0)^j,$$

where $\alpha_0 - j$ is to be understood as its representative modulo p between 0 and $p-1$. Since $\partial_0\big(c_{(\alpha_0-j,\alpha_1,\cdots,\alpha_d)}(x_k)\big) - (\alpha_0-j)c_{(\alpha_0-j,\alpha_1,\cdots,\alpha_d)}(x_k) \in pS^+$ and $\partial_0\circ\varphi = p\varphi\circ\partial_0$, we get the desired conclusion for $S^{[u]}$. Next, for $S^{(0,v]+}$ using the result for S we get that $\partial_0(x) - \alpha_0 x \in pS \cap S^{(0,v]+} = pS^{(0,v]+}$. Finally, combining the results for $S^{[u]}$ and $S^{(0,v]+}$ we get the conclusion for $S^{[u,v]}$. ∎

Next, we note a lemma which will be useful in the proof of the next claim and Proposition 5.41.

Lemma 2.39. *Let $x \in R_\varpi^{\psi=0}$, then $X_0^k\psi(x) = \psi(\varphi(X_0)^k x)$ for $k \in \mathbb{Z}$.*

Proof. Note that it is enough to prove the statement for $k = 1$. Indeed, $k \geq 2$ case immediately follows from this, whereas for $k = -1$ we observe that since X_0 is invertible in R_ϖ, we have $X_0\psi(\varphi(X_0^{-1})x) = \psi(\varphi(X_0)\varphi(X_0^{-1})x) = \psi(x)$.

Now, to show the case $k = 1$, we recall that $\varphi(X_0) = (1 + X_0)^p - 1$. Next, from Proposition 2.35 let us write $x = \sum_\alpha c_\alpha$, then we have $\psi(x) = \varphi^{-1}(c_0)$. Now it follows that,

$$\psi(\varphi(X_0)x) = \psi(((1 + X_0)^p - 1)x) = \psi((1 + X_0)^p x) - \psi(x) = (1 + X_0)\varphi^{-1}(c_0) - \varphi^{-1}(c_0) = X_0\psi(x),$$

as desired. ∎

Proposition 2.40 ([CN17, Proposition 2.16]). *Let $v < p$.*

(i) $\psi\left(X_0^{-pN} R_\varpi^{(0,v/p]+}\right) \subset X_0^{-N} R_\varpi^{(0,v]+}$;

(ii) *If $\ell = p^m$, then $X_0^{-\ell} R_\varpi^{(0,v]+}$ is stable under ψ;*

(iii) *The natural map*

$$\bigoplus_{\alpha \neq 0} \varphi\left(R_\varpi^{(0,v]+}\right) u_{\mathrm{cycl},\alpha} \longrightarrow \left(R_\varpi^{(0,v/p]+}\right)^{\psi=0}$$

is an isomorphism.

Proof. (i) follows from Proposition 2.35 (ii) and (iii), and taking into account the facts that $\psi\left(\varphi(X_0)^{-N} x\right) = X_0^{-N}\psi(x)$ and $\frac{\varphi(X_0)}{X_0^p}$ is a unit in $R_\varpi^{(0,v/p]+}$. (ii) is an immediate consequence of (i) and the inclusion $R_\varpi^{(0,v]+} \subset R_\varpi^{(0,v/p]+}$. Finally, if $x \in \left(R_\varpi^{(0,v/p]+}\right)^{\psi=0}$, using Proposition 2.35 (ii), we can write $x = \sum_{\alpha \neq 0} \varphi(x_\alpha)u_\alpha$ with $\varphi(x_\alpha)u_\alpha \in R_\varpi^{(0,v/p]+}$. But, u_α is invertible in $R_\varpi^{(0,v/p]+}$ (see Remark 2.33), hence $\varphi(x_\alpha) \in R_\varpi^{(0,v/p]+}$. From [CN17, Lemma 2.14], we have that if $x_\alpha \in R_\varpi$ such that $\varphi(x_\alpha) \in R_\varpi^{(0,v/p]+}$, then $x_\alpha \in R_\varpi^{(0,v]+}$. This gives us (iii). ∎

2.4. Cyclotomic embeddings

In this section, we will describe the (cyclotomic) embeddings of $R_\varpi^\bigstar$ into various period rings discussed in Chapter 1 and previous sections. Define an embedding

$$\begin{aligned} \iota_{\mathrm{cycl}} : R_{\varpi,\square}^+ &\longrightarrow \mathbf{A}_{\mathrm{inf}}(R) \\ X_0 &\longmapsto \pi_m = \varphi^{-m}(\pi), \\ X_i &\longmapsto [X_i^\flat] \quad \text{for } 1 \leq i \leq d. \end{aligned}$$

Lemma 2.41. *The map ι_{cycl} has a unique extension to an embedding $R_\varpi^+ \longrightarrow \mathbf{A}_{\mathrm{inf}}(R)$ such that $\theta \circ \iota_{\mathrm{cycl}}$ is the projection $R_\varpi^+ \longrightarrow R$.*

Proof. We can apply Proposition 2.30 with $\Lambda_1 = R_{\varpi,\square}^+$, $\Lambda_2 = \mathbf{A}_{\mathrm{inf}}(R)$, $\Lambda_1' = R_\varpi^+$, $\lambda = \iota_{\mathrm{cycl}}$, $I = (\xi)$ and $Z_\lambda = ([Z_1^\flat], \ldots, [Z_s^\flat])$. Next, from defintions we already have that $\theta \circ \iota_{\mathrm{cycl}} : R_{\varpi,\square}^+ \twoheadrightarrow R_\square$ coincides with the canonical projection and R_ϖ^+ is étale over $R_{\varpi,\square}^+$, hence the second claim follows. ∎

This embedding commutes with Frobenius on either side, i.e. $\iota_{\mathrm{cycl}} \circ \varphi_{\mathrm{cycl}} = \varphi \circ \iota_{\mathrm{cycl}}$. By continuity, the morphism ι_{cycl} extends to embeddings

$$R_\varpi^{\mathrm{PD}} \rightarrowtail \mathbf{A}_{\mathrm{cris}}(R), \quad R_\varpi^{[u]} \rightarrowtail \mathbf{A}_R^{[u]}, \quad R_\varpi^{(0,v]+} \rightarrowtail \mathbf{A}_R^{(0,v]+}, \quad R_\varpi^{[u,v]} \rightarrowtail \mathbf{A}_R^{[u,v]} \quad \text{and} \quad R_\varpi \rightarrowtail \mathbf{A}_{\overline{R}}.$$

Denote by $\mathbf{A}_R^\bigstar$ the image of $R_\varpi^\bigstar$ under ι_{cycl}. These rings are stable under the action of G_R. Moreover, this embedding induces a filtration on $\mathbf{A}_R^\bigstar$ for $\bigstar \in \{+, \mathrm{PD}, [u], [u, v], (0, v]+\}$ and $r \in \mathbb{Z}$ (use Definition 2.27).

Remark 2.42. From [CN17, §2.4.2], we have an inclusion of rings $\mathbf{A}_R^{[u']} \subset \mathbf{A}_R^{\mathrm{PD}} \subset \mathbf{A}_R^{[u]}$ for $u \geq \frac{1}{p-1}$ and $u' \leq \frac{1}{p}$.

Lemma 2.43. *For $\frac{p-1}{p} \leq u \leq \frac{v}{p} < 1 < v$, we have that $\frac{t}{\pi}$ is a unit in $\mathbf{A}_K^{\mathrm{PD}} \subset \mathbf{A}_K^{[u]} \subset \mathbf{A}_K^{[u,v]}$.*

Proof. We can write the fraction

$$\frac{t}{\pi} = \frac{\log(1+\pi)}{\pi} = \sum_{k\geq 0}(-1)^k \tfrac{\pi^k}{k+1}.$$

Formally, we can write

$$\frac{\pi}{t} = \frac{\pi}{\log(1+\pi)} = 1 + b_1\pi + b_2\pi^2 + b_3\pi^3 + \cdots,$$

where $v_p(b_k) \geq -\frac{k}{p-1}$ for all $k \geq 1$. Since $\pi = (1+\pi_m)^{p^m} - 1$, we get that $\pi \in (p, \pi_m^{p^m})\mathbf{A}_K^+$ (as $m \geq 1$). By induction over k, we can easily conclude that $\pi^k \in \big(p, \pi_m^{p^m}\big)^k \mathbf{A}_K^{\mathrm{PD}}$. Using this, we can re-express the series $\sum_k b_k \pi^k$ as a power series in π_m, written as $\sum_i c_i \pi_m^i$. We need to check that this re-expressed series converges in $\mathbf{A}_K^{\mathrm{PD}}$. To do this, we collect the terms with coefficients having the smallest p-adic valuation for each power of $\pi_m^{p^m}$ in the re-expressed series. For $k \geq 1$, b_k has the smallest p-adic valuation among the coefficients of $\pi_m^{p^m k}$, therefore it has the least p-adic valuation among coefficients of π_m^i for $p^m k \leq i < p^m(k+1)$. We write the collection of these terms as

$$\sum_{k\geq 1}(-1)^{k+1} b_k \pi_m^{p^m k} = \sum_{k\geq 1}(-1)^{k+1} b_k \left\lfloor \tfrac{p^m k}{e}\right\rfloor! \tfrac{\pi_m^{p^m k}}{\lfloor p^m k/e\rfloor!}, \tag{2.2}$$

and by the preeceding discussion it is sufficient to show that these coefficients go to 0 as k goes to $+\infty$. Moreover, for (2.2) it would suffice to check the estimate for $k = (p-1)j$ as j goes to $+\infty$ (this gets rid of the floor function above). With the observation in Remark 2.44, we have

$$v_p\Big(b_k \left\lfloor \tfrac{p^m k}{e}\right\rfloor!\Big) = v_p(b_k) + v_p((pj)!) \geq -\tfrac{(p-1)j}{p-1} + \tfrac{pj - s_p(pj)}{p-1} = \tfrac{j - s_p(j)}{p-1} = v_p(j!),$$

which goes to $+\infty$ as $j \longrightarrow +\infty$. Hence, $\frac{\pi}{t}$ converges in $\mathbf{A}_K^{\mathrm{PD}}$ and is an inverse to $\frac{t}{\pi}$. ∎

The following elementary observation was used above,

Remark 2.44. Let $n \in \mathbb{N}$, so we can write $n = \sum_{i=0}^k n_i p^i$ for some $k \in \mathbb{N}$, where $0 \leq n_i \leq p-1$ for $0 \leq i \leq k$. Let us set $s_p(n) = \sum_{i=0}^k n_i$. Then we have

$$\begin{aligned} v_p(n!) &= \sum_{j\geq 1}\left\lfloor \tfrac{n}{p^j}\right\rfloor = \sum_{j\geq 0}\left\lfloor \tfrac{\sum_{i=0}^k n_i p^i}{p^j}\right\rfloor = \sum_{j=1}^k \sum_{i=j}^k n_i p^{i-j} \\ &= \sum_{i=1}^k n_i \sum_{j=1}^i p^j = \sum_{i=1}^k n_i \tfrac{p^i - 1}{p-1} = \tfrac{n - s_p(n)}{p-1}. \end{aligned}$$

Also, note that we have $s_p(pn) = s_p(n)$ for any $n \in \mathbb{N}$.

Next, we prove some claims for the action of Γ_R on the analytic rings introduced above. These results will be useful when studying Koszul complexes computing Lie Γ_R-cohomology in §4.3.

Lemma 2.45. *Let $k \in \mathbb{N}$ and $i \in \{0, 1, \ldots, d\}$. Then $(\gamma_i - 1)^k \mathbf{A}_R^\star \subset \big(p^m, \pi_m^{p^m}\big)^k \mathbf{A}_R^\star$ for $\star \in \{+, \mathrm{PD}, [u]\}$;*

Proof. We will only consider the case of $\mathbf{A}_R^{\mathrm{PD}}$ as the estimates in other cases is easier. First, let $i = 0$. Then we have

$$\begin{aligned} (\gamma_0 - 1)\pi_m &= (1+\pi_m)\big((1+\pi_m)^{\chi(\gamma_0)-1} - 1\big) = (1+\pi_m)\big((1+\pi_m)^{p^m a} - 1\big) \\ &= (1+\pi_m)((1+\pi)^a - 1) = (1+\pi_m)\Big(a\pi + \tfrac{a(a-1)}{2!}\pi^2 + \tfrac{a(a-1)(a-2)}{3!}\pi^3 + \cdots\Big) = \pi x, \end{aligned}$$

for some $x \in \mathbf{A}_K^+$. Since $\pi = (1+\pi_m)^{p^m} - 1 = \pi_m^{p^m} + p^m\pi_m^{p^m-1} \cdots + p^m\pi_m$, we have that $\pi \in \big(p^m, \pi_m^{p^m}\big)\mathbf{A}_K^+$ (recall that we have $m \geq 1$), therefore $(\gamma_0 - 1)\pi_m \in \big(p^m, \pi_m^{p^m}\big)\mathbf{A}_K^+$. Next, we observe that

$$\begin{aligned}(\gamma_0 - 1)\pi_m^{p^m} &= \gamma_0(\pi_m)^{p^m} - \pi_m^{p^m} = (\pi x + \pi_m)^{p^m} - \pi_m^{p^m}\\ &= \pi^{p^m}x^{p^m} + \cdots + p^m\pi x\pi_m^{p^m-1} \in \big(p^m, \pi_m^{p^m}\big)^2\mathbf{A}_K^+.\end{aligned}$$

Therefore, $(\gamma_0 - 1)^2\pi_m \in \big(p, \pi_m^{p^m}\big)^2\mathbf{A}_K^+$. Proceeding by induction on $k \geq 0$, we conclude that

$$(\gamma_0 - 1)^k\pi_m \subset (\gamma_0 - 1)\big(p^m, \pi_m^{p^m}\big)^{k-1}\mathbf{A}_K^+ \subset \big(p^m, \pi_m^{p^m}\big)^k\mathbf{A}_K^+.$$

Now any $f \in \mathbf{A}_K^{\mathrm{PD}}$ can be written as $f = \sum_{n\in\mathbb{N}} f_n \frac{\pi_m^n}{[n/e]!}$ such that $f_n \in O_F$ goes to 0 as $n \longrightarrow +\infty$. So we want to show that $(\gamma_0 - 1)^k \frac{\pi_m^n}{[n/e!]} \subset \big(p^m, \pi_m^{p^m}\big)^k\mathbf{A}_K^{\mathrm{PD}}$. For notational convenience, we take $n = je$ for some $j \in \mathbb{N}$ and see that

$$\begin{aligned}\frac{(\gamma_0 - 1)\pi_m^{je}}{j!} &= \frac{\gamma_0(\pi_m)^{je} - \pi_m^{je}}{j!} = \frac{(\pi x + \pi_m)^{je} - \pi_m^{je}}{j!} = \frac{(\pi x)^{je} + je(\pi x)^{je-1}\pi_m + \cdots + je(\pi x)\pi_m^{je-1}}{j!}\\ &= \frac{(\pi x)^{je}}{j!} + \pi\frac{\pi_m^{je-1}z}{(j-1)!} \in \frac{1}{j!}\big(p^m, \pi_m^{p^m}\big)^{je}\mathbf{A}_K^{\mathrm{PD}} + \big(p^m, \pi_m^{p^m}\big)\mathbf{A}_K^{\mathrm{PD}} \subset \big(p^m, \pi_m^{p^m}\big)\mathbf{A}_K^{\mathrm{PD}}.\end{aligned}$$

Proceeding by induction on $k \geq 0$, we conclude that

$$(\gamma_0 - 1)^k\mathbf{A}_K^{\mathrm{PD}} \subset (\gamma_0 - 1)\big(p^m, \pi_m^{p^m}\big)^{k-1}\mathbf{A}_K^{\mathrm{PD}} \subset \big(p^m, \pi_m^{p^m}\big)^k\mathbf{A}_K^{\mathrm{PD}}.$$

Next, for $i \in \{1, \ldots, d\}$ we have $(\gamma_i - 1)[X_i^\flat] = \pi[X_i^\flat] \in \big(p^m, \pi_m^{p^m}\big)\mathbf{A}_R^+$ and $(\gamma_i - 1)\big([X_i^\flat]^{-1}\big) = -\pi(1+\pi)^{-1}[X_i^\flat]^{-1} \in \big(p^m, \pi_m^{p^m}\big)\mathbf{A}_R^+$. Proceeding by induction on $k \geq 0$, we conclude that

$$(\gamma_i - 1)^k\mathbf{A}_R^+ \subset (\gamma_i - 1)\big(p^m, \pi_m^{p^m}\big)^{k-1}\mathbf{A}_R^+ \subset \big(p^m, \pi_m^{p^m}\big)^k\mathbf{A}_R^+.$$

Since any $f \in \mathbf{A}_R^{\mathrm{PD}}$ can be written as $f = \sum_{j\in\mathbb{N}} f_j \frac{\pi_m^j}{\lfloor j/e\rfloor!}$ such that $f_j \in \mathbf{A}_R^+$ goes to 0 as $j \longrightarrow +\infty$, from the discussion for $\mathbf{A}_K^{\mathrm{PD}}$ and $\mathbf{A}_R^+$, we conclude that

$$(\gamma_i - 1)^k\mathbf{A}_R^{\mathrm{PD}} \subset (\gamma_i - 1)\big(p^m, \pi_m^{p^m}\big)^{k-1}\mathbf{A}_R^{\mathrm{PD}} \subset \big(p^m, \pi_m^{p^m}\big)^k\mathbf{A}_R^{\mathrm{PD}}.$$

■

The next claim will be useful in analyzing Koszul complexes for Γ_R-cohomology in Proposition 5.41 and Proposition 5.46.

Lemma 2.46. *Let $k \in \mathbb{N}$ and $i \in \{0, 1, \ldots, d\}$. Then $(\gamma_i - 1)^k\mathbf{A}_R^\bigstar \subset \big(p^m, \pi_m^{p^m}\big)^k\mathbf{A}_R^\bigstar$ for $\bigstar \in \{\ , (0, v]+, [u, v]\}$.*

Proof. First, we observe that

$$\gamma_0(\pi_m) = (1+\pi_m)^{\chi(\gamma_0)} - 1 = \chi(\gamma_0)\pi_m\big(1 + \tfrac{\chi(\gamma_0)-1}{2}\pi_m + \cdots\big) = \chi(\gamma_0)\pi_m f,$$

where $\chi(\gamma_0) = \exp(p) \in \mathbb{Z}_p^*$ and f is a unit in $\mathbf{A}_K^+$. From the expression above we also have that $1 - \chi(\gamma_0)f = p^m z$ for some $z \in \mathbf{A}_K^+$. So we can write

$$(\gamma_0 - 1)\pi_m^{-1} = \gamma_0(\pi_m)^{-1} - \pi_m^{-1} = (\chi(\gamma_0)f\pi_m)^{-1} - \pi_m^{-1} = \frac{1 - \chi(\gamma_0)f}{\chi(\gamma_0)f\pi_m} = \frac{p^m z}{\chi(\gamma_0)f\pi_m}$$

Now from the definitions we know that $\frac{p}{\pi_m} \in \mathbf{A}_K^{(0,v]+}$, therefore $(\gamma_0 - 1)\frac{p}{\pi_m} \in \big(p^m, \pi_m^{p^m}\big)\mathbf{A}_K^{(0,v]+}$.

Proceeding by induction on $k \geq 0$, we conclude that

$$(\gamma_0 - 1)^k \tfrac{p}{\pi_m} \subset (\gamma_0 - 1)\left(p^m, \pi_m^{p^m}\right)^{k-1} \mathbf{A}_K^{(0,v]+} \subset \left(p^m, \pi_m^{p^m}\right)^k \mathbf{A}_K^{(0,v]+}.$$

From Lemma 2.45 we already have that $(\gamma_i - 1)^k \mathbf{A}_R^+ \in \left(p^m, \pi_m^{p^m}\right)^k \mathbf{A}_R^+$ for $i \in \{1, \dots, d\}$. Therefore, we conclude that

$$(\gamma_0 - 1)^k \mathbf{A}_R^{(0,v]+} \subset \left(p^m, \pi_m^{p^m}\right)^k \mathbf{A}_K^{(0,v]+}.$$

The analysis of $\mathbf{A}_R^{[u,v]}$ and $\mathbf{A}_R$ follow in a similar manner (note that π_m is invertible in $\mathbf{A}_R$). ■

Finally, we show a claim which will be useful for changing the annulus of convergence in §5.2.

Lemma 2.47 ([CN17, Lemma 2.35])**.** *If $v \leq p$, then*

(i) *$\pi_m^{-p^{m-1}} \pi_1$ is a unit in $\mathbf{A}_R^{(0,v]+}$;*

(ii) *p is divisible by $\pi_m^{\lfloor (p-1)p^{m-1}/v \rfloor}$, hence also by $\pi_m^{(p-1)p^{m-2}}$;*

(iii) *$\frac{p^2}{\pi_1} \in \mathbf{A}_R^{(0,v]+}$ and is divisible by $\pi_m^{(2(p-1)-v)p^{m-2}}$;*

(iv) *$\frac{\pi}{\pi_1} \in \left(p, \pi_m^{(p-1)p^{m-1}}\right) \mathbf{A}_R^{(0,v]+}$ and is divisible by $\pi_m^{(p-1)p^{m-2}}$;*

(v) *Let $v = p - 1$ for $p \geq 3$ and $v = \frac{3}{2}$ for $p = 2$, then $\pi_m^{-p^m} \pi$ is a unit $\mathbf{A}_R^{(0,v/p]+}$ and $\frac{p}{\pi} \in \mathbf{A}_R^{(0,v/p]+}$.*

Proof. We can work in $r_\varpi^{(0,v]+}$, in which case π_m becomes X_0 and π_1 becomes $(1+X_0)^{p^{m-1}} - 1$ and we are looking at the annulus $0 < v_p(T) \leq \frac{v}{p^{m-1}(p-1)}$ on which $(1+X_0)^{p^{m-1}} - 1$ has no zero and $v_p((1+X_0)^{p^{m-1}} - 1) = p^{m-1} v_p(X_0)$ since $v < p$. This shows (i). The claim in (ii) comes from the definition of $R_\varpi^{(0,v]+}$. (iii) follows from (i) and (ii) since $2\left\lfloor \frac{(p-1)p^{m-1}}{v} \right\rfloor - p^{m-1} \geq (2(p-1) - v)p^{m-2}$. The claim in (iv) follows from (i), (ii) and the identity

$$\frac{\pi}{\pi_1} = \pi_1^{p-1} + p\pi_1^{p-2} + \cdots + p.$$

For (v), replacing π by $(1+X_0)^{p^m} - 1$, we see that $v_p((1+X_0)^{p^m} - 1) = p^m v_p(X_0)$. Using arguments similar to (i) gives us first part of (v). The second half of (v) follows from the first part and (ii) since $\left\lfloor \frac{(p-1)p^{m-1}}{(p-1)/p} \right\rfloor = p^m$. ■

2.5. Fat period rings

In this section we will introduce fat rings and give a version of PD-Poincaré lemma. The Poinaré lemma will be useful for relating complexes computing Galois cohomology and syntomic complex with coefficients in Chapter 5. Let S and Λ be p-adically complete filtered W-algebras, where W is the ring of integers of F. Let $\iota : S \longrightarrow \Lambda$ be a continuous injective morphism of filtered W-algebras and let $f : S \otimes \Lambda \longrightarrow \Lambda$ be the morphism sending $x \otimes y \mapsto \iota(x)y$.

Definition 2.48. Let $S\Lambda$ denote the p-adic completion of the PD-envelope of $S \otimes \Lambda \longrightarrow \Lambda$ with respect to $\operatorname{Ker} f$.

In the rest of this section we will take $S = R_\varpi^\star$ for $\star \in \{\mathrm{PD}, [u], [u,v]\}$.

Remark 2.49. (i) The ring $S\Lambda$ is the p-adic completion of $S \otimes \Lambda$ adjoined $(x \otimes 1 - 1 \otimes \iota(x))^{[k]}$, for $x \in S$ and $n \in \mathbb{N}$.

(ii) The morphism $f : S \otimes \Lambda \longrightarrow \Lambda$ extends uniquely to a continuous morphism $f : S\Lambda \longrightarrow \Lambda$.

(iii) We can filter $S\Lambda$ by defining $\mathrm{Fil}^r S\Lambda$ to be the topological closure of the ideal generated by the products of the form $x_1 x_2 \prod (V_i - 1)^{[k_i]}$, where $x_1 \in \mathrm{Fil}^{r_1} S$, $x_2 \in \mathrm{Fil}^{r_2}\Lambda$ and $V_i = \frac{X_i \otimes 1}{1 \otimes \iota(X_i)}$ for $1 \le i \le d$ such that $r_1 + r_2 + \sum k_i \ge r$.

Lemma 2.50 ([CN17, Lemma 2.36]). (i) *Any element $x \in S\Lambda$ can be uniquely written as*

$$x = \sum_{\mathbf{k} \in \mathbb{N}^{d+1}} x_{\mathbf{k}} \prod_{i=0}^{d} (1 - V_i)^{[k_i]},$$

with $x_{\mathbf{k}} \in \Lambda$ for all $\mathbf{k} = (k_0, \dots, k_d) \in \mathbb{N}^{d+1}$ and $x_{\mathbf{k}} \to 0$ when $\mathbf{k} \to \infty$.

(ii) *An element $x \in \mathrm{Fil}^r S\Lambda$, if and only if $x_{\mathbf{k}} \in \mathrm{Fil}^{r-|\mathbf{k}|}\Lambda$ for all $\mathbf{k} \in \mathbb{N}^{d+1}$.*

We set $\Omega^1 := \mathbb{Z}\frac{dX_0}{1+X_0} \oplus \left(\oplus_{i=1}^d \mathbb{Z}\frac{dX_i}{X_i}\right)$ and $\Omega^k := \wedge^k \Omega^1$. Therefore, we have $\Omega^k_{S\Lambda/\Lambda} = S\Lambda \otimes \Omega^k$. We filter the de Rham complex of $S\Lambda$ by subcomplexes

$$\mathrm{Fil}^r \Omega^{\bullet}_{S\Lambda/\Lambda} : \mathrm{Fil}^r S\Lambda \longrightarrow \mathrm{Fil}^{r-1} S\Lambda \otimes \Omega^1 \longrightarrow \mathrm{Fil}^{r-2} S\Lambda \otimes \Omega^2 \longrightarrow \cdots.$$

Let D be a finitely generated filtered Λ-module. We set $\Xi := S\Lambda \otimes_\Lambda D$ and define a filtration on Ξ by $\mathrm{Fil}^r \Xi := \sum_{a+b=r} \mathrm{Fil}^a S\Lambda \widehat{\otimes}_\Lambda \mathrm{Fil}^b D$. Then Ξ is a finitely generated filtered $S\Lambda$-module equipped with an integrable connection $\partial : \Xi \to \Xi \otimes_{S\Lambda} \Omega^1_{S\Lambda/\Lambda}$. For the differential operator on $S\Lambda$ we have $\partial(\mathrm{Fil}^k S\Lambda) \subset \mathrm{Fil}^{k-1} S\Lambda$, therefore the connection on Ξ satisfies Griffiths transversality with respect to the filtration on it. We can filter the de Rham complex with coefficients in Ξ as

$$\begin{aligned} \mathrm{Fil}^r \Xi \otimes \Omega^{\bullet}_{S\Lambda/\Lambda} : \mathrm{Fil}^r \Xi &\longrightarrow \mathrm{Fil}^{r-1} \Xi \otimes_{S\Lambda} \Omega^1_{S\Lambda/\Lambda} \longrightarrow \mathrm{Fil}^{r-2} \Xi \otimes_{S\Lambda} \Omega^2_{S\Lambda/\Lambda} \longrightarrow \cdots \\ = \mathrm{Fil}^r \Xi &\longrightarrow \mathrm{Fil}^{r-1} \Xi \otimes_{\mathbb{Z}} \Omega^1 \longrightarrow \mathrm{Fil}^{r-2} \Xi \otimes_{\mathbb{Z}} \Omega^2 \longrightarrow \cdots. \end{aligned}$$

Since $\mathrm{Fil}^r D = (\mathrm{Fil}^r \Xi)^{\partial = 0}$, we get a filtered Poincaré Lemma:

Lemma 2.51 ([CN17, Lemma 2.37]). *The natural map*

$$\mathrm{Fil}^r D \longrightarrow \mathrm{Fil}^r \Xi \otimes \Omega^{\bullet}_{S\Lambda/\Lambda}$$

is a quasi-isomorphism.

Proof. We have a natural injection $\epsilon : \mathrm{Fil}^r D \to \mathrm{Fil}^r \Xi$. We give a contracting (Λ-linear) homotopy. Define

$$\begin{aligned} h^0 : \mathrm{Fil}^r \Xi &\longrightarrow \mathrm{Fil}^r D \\ \sum_{j+k=r} x \otimes a &\longmapsto \sum_{l+m=r} x_0 \otimes a, \end{aligned}$$

where $x \in \mathrm{Fil}^j S\Lambda$, $a \in \mathrm{Fil}^k D$ and x_0 is the projection to the 0-th component (see Lemma 2.50). Clearly, $h^0 \epsilon = id$. For $q > 0$, define the map

$$h^q : \mathrm{Fil}^{j-q} \Xi \otimes \Omega^q \longrightarrow \mathrm{Fil}^{j-q+1} \Xi \otimes \Omega^{q-1}$$

by the formula

$$\begin{aligned} & x \otimes a \prod_{i=0}^{d} (V_i - 1)^{[k_i]} V_{i_1} \frac{dX_{i_1}}{X_{i_1}} \wedge \cdots \wedge V_{i_q} \frac{dX_{i_q}}{X_{i_q}} \\ \longmapsto & \begin{cases} x \otimes a \prod_{i=0}^d (V_i - 1)^{[k_i + \delta_{j i_1}]} V_{i_2} \frac{dX_{i_2}}{X_{i_2}} \wedge \cdots \wedge V_{i_q} \frac{dX_{i_q}}{X_{i_q}} & \text{if } k_j = 0 \text{ for } 0 \le j \le i_1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We have $\epsilon h^0 + h^1 d = id$ and $dh^q + h^{q+1} d = id$, as required. ∎

Next, let $R_1 = R_{\varpi}^{\star}$, $R_2 = \mathbf{A}_R^{\star}$ for $\star \in \{\mathrm{PD}, [u], [u,v]\}$, such that $\iota = \iota_{\mathrm{cycl}}$ is an isomorphism of filtered W-algebras, and $R_3 = S\Lambda$. We set $X_{0,1} = X_0$, $X_{0,2} = \pi_m$ and for $1 \leq i \leq d$, we set $X_{i,1} = X_i$ and $X_{i,2} = [X_i^{\flat}]$. Now for $j = 1, 2$, we set

$$\Omega_j^1 := \mathbb{Z}\frac{dX_{0,j}}{1+X_{0,j}} \bigoplus_{i=1}^{d} \mathbb{Z}\frac{dX_{i,j}}{X_{i,j}},$$

and $\Omega_3^1 := \Omega_1^1 \oplus \Omega_2^1$. For $j = 1, 2, 3$, let $\Omega_i^k = \bigwedge^k \Omega_j$. Therefore, $\Omega_{R_j}^k = R_j \otimes \Omega_j^k$.

Let Ξ be a finitely generated filtered R_3-module equipped with a quasi-nilpotent integrable connection satisfying Griffiths transversality with respect to the filtration. In other words, for each $k \in \mathbb{N}$, we have a complex

$$\mathrm{Fil}^k\Xi \otimes \Omega_3^{\bullet} : \mathrm{Fil}^k\Xi \xrightarrow{\partial_{R_3}} \mathrm{Fil}^{k-1}\Xi \otimes \Omega_3^1 \xrightarrow{\partial_{R_3}} \mathrm{Fil}^{k-2}\Xi \otimes \Omega_3^2 \xrightarrow{\partial_{R_3}} \cdots.$$

Now, let $D_1 = \Xi^{\partial_2=0}$ be a finitely generated R_1-module equipped with a filtration $\mathrm{Fil}^k D_1 = (\mathrm{Fil}^k\Xi)^{\partial_2=0}$, and a quasi-nilpotent integrable connection satisfying Griffiths transversality with respect to the filtration, i.e. for $k \in \mathbb{Z}$, we have

$$\partial_{R_1} : \mathrm{Fil}^k D_1 \longrightarrow \mathrm{Fil}^{k-1} D_1 \otimes_{\mathbb{Z}} \Omega_1^1,$$

In other words, we obtain a filtered de Rham complex

$$\mathrm{Fil}^k D_1 \otimes \Omega_1^{\bullet} : \mathrm{Fil}^k D_1 \xrightarrow{\partial_{R_1}} \mathrm{Fil}^{k-1} D_1 \otimes \Omega_1^1 \xrightarrow{\partial_{R_1}} \mathrm{Fil}^{k-2} D_1 \otimes \Omega_1^2 \xrightarrow{\partial_{R_1}} \cdots,$$

Similarly, let $D_2 = \Xi^{\partial_1=0}$ be a finitely generated R_2-module equipped with a filtration $\mathrm{Fil}^k D_2 = (\mathrm{Fil}^k\Xi)^{\partial_1=0}$, and a quasi-nilpotent integrable connection satisfying Griffiths transversality with respect to the filtration, i.e. for $k \in \mathbb{Z}$, we have

$$\partial_{R_2} : \mathrm{Fil}^k D_2 \longrightarrow \mathrm{Fil}^{k-1} D_2 \otimes_{\mathbb{Z}} \Omega_2^1,$$

In other words, we obtain a filtered de Rham complex

$$\mathrm{Fil}^k D_2 \otimes \Omega_2^{\bullet} : \mathrm{Fil}^k D_2 \xrightarrow{\partial_{R_2}} \mathrm{Fil}^{k-1} D_2 \otimes \Omega_2^1 \xrightarrow{\partial_{R_2}} \mathrm{Fil}^{k-2} D_2 \otimes \Omega_2^2 \xrightarrow{\partial_{R_2}} \cdots,$$

Proposition 2.52. *The natural maps*

$$\mathrm{Fil}^k D_1 \otimes \Omega_1^{\bullet} \longrightarrow \mathrm{Fil}^k\Xi \otimes \Omega_3^{\bullet} \longleftarrow \mathrm{Fil}^k D_2 \otimes \Omega_2^{\bullet}$$

are quasi-isomorphism of complexes.

Proof. Note that the claim is symmetric in R_1 and R_2, so we only prove the quasi-isomorphism for the map on the left. Since we have $\mathrm{Fil}^k D_1 = (\mathrm{Fil}^k\Xi)^{\partial_{R_2}=0}$, from Lemma 2.51 we obtain that the sequence

$$0 \longrightarrow \mathrm{Fil}^k D_1 \longrightarrow \mathrm{Fil}^k\Xi \xrightarrow{\partial_{R_2}} \mathrm{Fil}^{k-1}\Xi \otimes \Omega_2^1 \xrightarrow{\partial_{R_2}} \cdots,$$

is exact. We can extend the sequence above to a sequence of maps of de Rham complexes

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Fil}^k D_1 \arrow[r] \arrow[d, "\partial_{R_1}"] & \mathrm{Fil}^k \Xi \arrow[r, "\partial_{R_2}"] \arrow[d, "\partial_{R_1}"] & \mathrm{Fil}^{k-1} \Xi \otimes \Omega_2^1 \arrow[r, "\partial_{R_2}"] \arrow[d, "\partial_{R_1}"] & \cdots \\ 0 \arrow[r] & \mathrm{Fil}^k D_1 \otimes \Omega_1^1 \arrow[r] \arrow[d, "\partial_{R_1}"] & \mathrm{Fil}^k \Xi \otimes \Omega_1^1 \arrow[r, "\partial_{R_2}"] \arrow[d, "\partial_{R_1}"] & \mathrm{Fil}^{k-1} \Xi \otimes \left(\Omega_2^1 \wedge \Omega_1^1\right) \arrow[r, "\partial_{R_2}"] \arrow[d, "\partial_{R_1}"] & \cdots \\ & \vdots & \vdots & \vdots & . \end{tikzcd}$$

The contracting homotopy in the proof of Lemma 2.51 is R_1-linear, so it extends as well, which shows that the rows of the double complex above are exact. The total complex of the double complex

$$\mathrm{Fil}^k \Xi \otimes \Omega_1^{\bullet} \xrightarrow{\partial_{R_2}} \mathrm{Fil}^{k-1} \Xi \otimes \left(\Omega_2^1 \wedge \Omega_1^{\bullet}\right) \xrightarrow{\partial_{R_2}} \cdots,$$

is equal to the de Rham complex $\mathrm{Fil}^k \Xi \otimes \Omega_3^{\bullet}$. This allows us to conclude. ■

Lemma 2.51 and Proposition 2.52 play a key role in connecting syntomic complex with coefficients to “Koszul (φ, ∂)-complexes” (see Lemmas 5.26 & 5.27 and Proposition 5.30).

CHAPTER 3

Finite height crystalline representations

In [Fon90], Fontaine initiated a program on the classification of p-adic representations of the absolute Galois group of a p-adic local field by means of certain linear-algebraic objects attached to these representations. One of the aspects of his program was to classify all p-adic representations of the Galois group in terms of étale (φ, Γ)-modules. On the other hand, in [Fon82] Fontaine had already proposed that representations "coming from geometry" give rise to another class of linear-algebraic objects, for example in the case of good reduction, i.e. crystalline representations, these objects are called filtered φ-modules. Therefore, it is a natural question to ask for crystalline representations: Does there exist some direct relation between the filtered φ-module and the étale (φ, Γ)-module? Fontaine explored this question in [Fon90] where he considered a certain class of (φ, Γ)-modules, for which he called the associated representations to be of finite height and examined their relationship with crystalline representations. This line of thought was further explored by Wach [Wac96, Wac97], Colmez [Col99], and Berger [Ber02, Ber04]. In particular, Wach gave a description of finite height crystalline representations in terms of (φ, Γ)-modules. In this chapter, we will recall some definitions and results from these articles and construct analogous objects in the relative setting.

3.1. The arithmetic case

Recall that we have $G_F = \mathrm{Gal}(\overline{F}/F)$ as the absolute Galois group of F, $\Gamma_F := \mathrm{Gal}(F_\infty/F)$ and $H_F := \mathrm{Gal}(\overline{F}/F_\infty)$, where $F_\infty = \cup_n F(\zeta_{p^n})$. From the theory of (φ, Γ_F)-modules, we have a two dimensional local ring $\mathbf{A}_F$ given by the p-adic completion of $W[[\pi]]\left[\frac{1}{\pi}\right]$ and $\mathbf{B}_F := \mathbf{A}_F\left[\frac{1}{p}\right]$ which is a complete discrete valuation field with uniformizer p and residue field $\kappa((\overline{\pi}))$, the field of Laurent series with uniformizer $\overline{\pi}$, the reduction of π modulo p.

Next, we have certain subrings $\mathbf{A}_F^+ := W[[\pi]] \subset \mathbf{A}_F$ and $\mathbf{B}_F^+ = \mathbf{A}_F^+\left[\frac{1}{p}\right] \subset \mathbf{B}_F$, stable under the action of φ and Γ_F. Let V be a p-adic representation of G_F, then $\mathbf{D}^+(V) = (\mathbf{B}^+ \otimes_{\mathbb{Q}_p} V)^{H_F}$ is a free module over the local ring $\mathbf{B}_F^+$ of rank $\leq h$, equipped with a Frobenius-semilinear endomorphism φ and a continuous and semilinear action of Γ_F. Further, let $\mathbf{D}(V) = (\mathbf{B} \otimes_{\mathbb{Q}_p} V)^{H_F}$ be the associated (φ, Γ_F)-module which is a $\mathbf{B}_F$-vector space of dimension $h = \dim_{\mathbb{Q}_p} V$, equipped with a Frobenius-semilinear endomorphism φ and a continuous and semilinear action of Γ_F. We have a $\mathbf{B}_F^+$-linear inclusion $\mathbf{D}^+(V) \subset \mathbf{D}(V)$ compatible with the action of φ and Γ_F. Similarly, if $T \subset V$ is a free $\mathbb{Z}_p$-lattice of rank $h = \dim_{\mathbb{Q}_p} V$, stable under the action of G_F, then $\mathbf{D}^+(T) = (\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T)^{H_F}$ is a free $\mathbf{A}_F^+$-module of rank $\leq h$, stable under the action of φ and Γ_F (see [Fon90, §B.1.2]). Moreover, $\mathbf{D}(T) = (\mathbf{A} \otimes_{\mathbb{Z}_p} T)^{H_F}$ is a free $\mathbf{A}_F$-module of rank h equipped with a Frobenius-semilinear operator φ and a continuous and semilinear action of Γ_F, and we have $\mathbf{D}^+(T) \subset \mathbf{D}(T)$. We say that V is of *finite height* if $\mathbf{D}(V)$ has a basis over $\mathbf{B}_F$ made of elements of $\mathbf{D}^+(V)$.

Fontaine showed that V is of finite height if and only if there exists a finite free $\mathbf{B}_F^+$-submodule of $\mathbf{D}(V)$ of rank $h = \dim_{\mathbb{Q}_p} V$, stable under the operator φ (see [Fon90, §B.2.1] and [Col99, §III.2]). Moreover, if $T \subset V$ is a free $\mathbb{Z}_p$-lattice as above and V of finite height, then $\mathbf{D}^+(T)$ is a free $\mathbf{A}_F^+$-module of rank $h = \dim_{\mathbb{Q}_p} V$ such that $\mathbf{A}_F \otimes_{\mathbf{A}_F^+} \mathbf{D}^+(T) \simeq \mathbf{D}(T)$ (see [Fon90, Théorème B.1.4.2]).

For crystalline representations there exist submodules of $\mathbf{D}^+(V)$ over which the action of Γ_F is simpler. Finite height and crystalline representations of G_F are related by the following result:

Theorem 3.1 ([Wac96], [Col99], [Ber02])**.** *Let V be a p-adic representation of G_F of dimension d. Then V is crystalline if and only if it is of finite height and there exists $r \in \mathbb{Z}$ and a $\mathbf{B}_F^+$-submodule $N \subset \mathbf{D}^+(V)$ of rank $h = \dim_{\mathbb{Q}_p} V$, stable under the action of Γ_F, such that Γ_F acts trivially over $(N/\pi N)(-r)$.*

In the situation of Theorem 3.1, the module N is not unique. A functorial construction was given by Berger:

Proposition 3.2 ([Ber04, Proposition II.1.1])**.** *Let V be a positive crystalline representation of G_F of dimension h, i.e. all Hodge-Tate weights of V are ≤ 0. Let $T \subset V$ be a free $\mathbb{Z}_p$-lattice of rank h, stable under the action of G_F. Then there exists a unique $\mathbf{A}_F^+$-module $\mathrm{N}(T) \subset \mathbf{D}^+(T)$, which is free of rank h, stable under the action of φ and Γ_F, and the action of Γ_F is trivial over $\mathrm{N}(T)/\pi\mathrm{N}(T)$. Moreover, there exists $s \in \mathbb{N}$ such that $\pi^s\mathbf{D}^+(T) \subset \mathrm{N}(T)$. Finally, if we set $\mathrm{N}(V) := \mathbf{B}_F^+ \otimes_{\mathbf{A}_F^+} \mathrm{N}(T)$, then $\mathrm{N}(V)$ is a unique $\mathbf{B}_F^+$-submodule of $\mathbf{D}^+(V)$ satisfying analogous conditions.*

Notation. For an algebra S admitting an action of the Frobenius and an S-module M admitting a Frobenius-semilinear endomorphism $\varphi : M \longrightarrow M$, we denote by $\varphi^*(M) \subset M$ the S-submodule generated by the image of φ.

Remark 3.3. (i) In Proposition 3.2, Berger uses the existence of N in Theorem 3.1 to define $\mathrm{N}(V) := \mathbf{D}^+(V) \cap N\left[\frac{1}{\varphi^{n-1}(q)}\right]_{n\geq 1}$, where $q = \frac{\varphi(\pi)}{\pi}$. Using this one can take $\mathrm{N}(T) := \mathrm{N}(V) \cap \mathbf{D}(T)$ and it can be shown to satisfy the desired properties.

(ii) Berger further showed that in the setup of Proposition 3.2, if we take s to be the maximum among the absolute values of Hodge-Tate weights of V, then $\mathrm{N}(T)/\varphi^*(\mathrm{N}(T))$ is killed by q^s and we have that $\pi^s\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T \subset \mathbf{A}^+ \otimes_{\mathbf{A}_F^+} \mathrm{N}(T)$ (see [Ber04, Théorème III.3.1]).

Definition 3.4. Let $a, b \in \mathbb{Z}$ with $b \geq a$. A *Wach module* with weights in the interval $[a, b]$ is an $\mathbf{A}_F^+$-module or a $\mathbf{B}_F^+$-module N which is free of rank h, equipped with a continuous and semilinear action of Γ_F such that its action is trivial on $N/\pi N$ and a Frobenius-semilinear operator $\varphi : N\left[\frac{1}{\pi}\right] \longrightarrow N\left[\frac{1}{\varphi(\pi)}\right]$ which commutes with the action of Γ_F, $\varphi(\pi^b N) \subset \pi^b N$ and $\pi^b N/\varphi^*(\pi^b N)$ is killed by q^{b-a}.

Remark 3.5. The definition of the functor $\mathbf{N}$ can be extended to crystalline representations of arbitrary Hodge-Tate weights quite easily. Indeed, let $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_F)$ with Hodge-Tate weights in the interval $[a, b]$ and let $T \subset V$ a free $\mathbb{Z}_p$-lattice of rank $= \dim_{\mathbb{Q}_p} V$, stable under the action of G_F. Then $\mathrm{N}(T) = \pi^{-b}\mathrm{N}(T(-b)) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(b)$ is a Wach module over $\mathbf{A}_F^+$ with weights in the interval $[a, b]$.

As it turns out, one can recover the crystalline representation from a given Wach module:

Proposition 3.6 ([Ber04, Proposition III.4.2])**.** *The functor*

$$\begin{aligned} \mathrm{N} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_F) &\longrightarrow \text{Wach modules over } \mathbf{B}_F^+ \\ V &\longmapsto \mathrm{N}(V), \end{aligned}$$

establishes an equivalence of categories with a quasi-inverse given by $N \mapsto (\mathbf{B} \otimes_{\mathbf{B}_F^+} N)^{\varphi=1}$. These functors are compatible with tensor products, duality and preserve exact sequences. Moreover, for a crystalline representation V, the map $T \mapsto \mathrm{N}(T)$ induces a bijection between $\mathbb{Z}_p$-lattices inside V and Wach modules over $\mathbf{A}_F^+$ contained in $\mathrm{N}(V)$.

We have a natural filtration on the Wach modules given as

$$\mathrm{Fil}^k\mathbf{N}(V) = \{x \in \mathbf{N}(V) \text{ such that } \varphi(x) \in q^k\mathbf{N}(V)\} \text{ for } k \in \mathbb{Z}.$$

If V is positive crystalline, i.e. all its Hodge-Tate weights are ≤ 0, then for $r \in \mathbb{N}$ we have

$$\mathrm{Fil}^k\mathbf{N}(V(r)) = \mathrm{Fil}^k\pi^{-r}\mathbf{N}(V)(r) = \pi^{-r}\mathrm{Fil}^{k+r}\mathbf{N}(V)(r).$$

Using this filtration on $\mathbf{N}(V)$, one can also recover the other linear algebraic object associated to V, i.e. the filtered φ-module $\mathbf{D}_{\mathrm{cris}}(V)$: Let $\mathbf{B}^+_{\mathrm{rig},F} \subset F[[\pi]]$ denote the subring of convergent power series over the open unit disc. Then we have $\mathbf{D}_{\mathrm{cris}}(V) \subset \mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{N}(V)$ and this gives $\mathbf{D}_{\mathrm{cris}}(V) = \big(\mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{N}(V)\big)^{\Gamma_F}$ (see [Ber04, Proposition II.2.1]). Moreover, the induced map

$$\mathbf{D}_{\mathrm{cris}}(V) \longrightarrow \big(\mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{N}(V)\big)/\pi\big(\mathbf{B}^+_{\mathrm{rig},F} \otimes_{\mathbf{B}^+_F} \mathbf{N}(V)\big) = \mathbf{N}(V)/\pi\mathbf{N}(V),$$

is an isomorphism of filtered φ-modules (see [Ber04, Proposition III.4.4]).

3.2. The relative case

Recall that we fixed $m \geq 1$ and we have $K = F_m = F(\zeta_{p^m})$. The element $\varpi = \zeta_{p^m} - 1$ is a uniformizer of K. We have $X = (X_1, \ldots, X_d)$ a set of indeterminates and we defined R_0 to be the p-adic completion of an étale algebra over $W(\kappa)\{X, X^{-1}\}$; similarly, R to be the p-adic completion of an étale algebra over $R_\square = O_K\{X, X^{-1}\}$ (defined using the same equations as in the definition of R_0). For R_0 and R, we can use the (φ, Γ)-module theory discussed in §2.1, as well as the constructions in §2.3 and §2.4. In particular, we will use rings $r^\star_\varpi$ and $R^\star_\varpi$ for $\star \in \{+, \mathrm{PD}\}$.

In the relative setting, we define an analog of Wach modules using the formulation in Definition 3.4:

Definition 3.7. Let $a, b \in \mathbb{Z}$ with $b \geq a$. A *Wach module* over $\mathbf{A}^+_{R_0}$ (resp. $\mathbf{B}^+_{R_0}$) with weights in the interval $[a, b]$ is a finite projective $\mathbf{A}^+_{R_0}$-module (resp. $\mathbf{B}^+_{R_0}$-module) N, equipped with a continuous and semilinear action of Γ_{R_0} and a Frobenius-semilinear operator $\varphi : N\big[\frac{1}{\pi}\big] \longrightarrow N\big[\frac{1}{\varphi(\pi)}\big]$ which commutes with the action of Γ_{R_0}, such that the action of Γ_{R_0} is trivial on $N/\pi N$, $\varphi(\pi^b N) \subset \pi^b N$ and $\pi^b N/\varphi^*(\pi^b N)$ is killed by q^{b-a}.

Let V be an h-dimensional p-adic representation of the Galois group G_{R_0}. It is said to be of *finite height* if and only if the $\mathbf{B}^+_{R_0}$-module $\mathbf{D}^+(V) := (\mathbf{B}^+ \otimes_{\mathbb{Q}_p} V)^{H_{R_0}}$ is a finitely generated (φ, Γ_{R_0})-module such that $\mathbf{B}_{R_0} \otimes_{\mathbf{B}^+_{R_0}} \mathbf{D}^+(V) \simeq \mathbf{D}(V)$. Let $T \subset V$, be a G_R-stable $\mathbb{Z}_p$-lattice and we set $\mathbf{D}^+(T) := (\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T)^{H_{R_0}}$.

Definition 3.8. A positive *Wach representation* is a p-adic representation V of G_{R_0} admitting a free $\mathbb{Z}_p$-lattice $T \subset V$ of rank h, and satisfying the following conditions:

(i) V is a de Rham representation with non-positive Hodge-Tate weights (see §1.5 and [Bri08, Chapitre 4]). Let s be the maximum among the absolute value of these Hodge-Tate weights.

(ii) There exists a finite projective $\mathbf{A}^+_{R_0}$-submodule $\mathbf{N}(T) \subset \mathbf{D}^+(T)$ of rank h and let R'_0 be the p-adic completion of a finite étale algebra over R_0 such that

   a) $\mathbf{N}(T)$ is stable under the action of φ and Γ_{R_0}, and $\mathbf{A}_{R_0} \otimes_{\mathbf{A}^+_{R_0}} \mathbf{N}(T) \simeq \mathbf{D}(T)$;

   b) The $\mathbf{A}^+_{R_0}$-module $\mathbf{N}(T)/\varphi^*(\mathbf{N}(T))$ is killed by q^s;

   c) The action of Γ_{R_0} is trivial on $\mathbf{N}(T)/\pi\mathbf{N}(T)$;

   d) The $\mathbf{A}^+_{R'_0}$-module $\mathbf{A}^+_{R'_0} \otimes_{\mathbf{A}^+_{R_0}} \mathbf{N}(T)$ is free of rank h.

The module $\mathbf{N}(T)$ is a *Wach module* associated to T with weights in the interval $[-r_1, 0]$ and we set $\mathbf{N}(V) := \mathbf{N}(T)\left[\frac{1}{p}\right]$ which satisfies properties analogous to (a)-(d) above.

For $r \in \mathbb{Z}$, we set $V(r) := V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(r)$ and $T(r) := T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(r)$. We will call these twists as *Wach representations* and define

$$\mathbf{N}(T(r)) := \tfrac{1}{\pi^r}\mathbf{N}(T)(r) \ \text{ and } \ \mathbf{N}(V(r)) := \tfrac{1}{\pi^r}\mathbf{N}(V)(r).$$

Since $\mathbf{N}(V)$ and $\mathbf{N}(T)$ are Wach modules with weights in the interval $[-r_1, 0]$, twisting by r gives us Wach modules in the sense of Definition 3.7 with weights in the interval $[r - r_1, r]$.

Remark 3.9. In Definition 3.8 following Remark 3.3 (i), first one can define Wach module for the representation V and then consider the module $\mathbf{N}(T) = \mathbf{N}(V) \cap \mathbf{D}(T)$ associated to T. However, it is not clear whether the latter module, defined in this fashion, is a projective $\mathbf{A}_{R_0}^+$-module. Therefore, we impose the condition on $\mathbf{N}(T)$ to be projective, which is required in establishing several results in this section.

Definition 3.10. Let V and T as in Definition 3.8 and $r \in \mathbb{N}$, then there is a natural filtration on the associated Wach modules, given by

$$\mathrm{Fil}^k\mathbf{N}(V(r)) := \{x \in \mathbf{N}(V(r)), \text{ such that } \varphi(x) \in q^k\mathbf{N}(V(r))\} \ \text{ for } k \in \mathbb{Z},$$

and we set $\mathrm{Fil}^k\mathbf{N}(T(r)) := \mathrm{Fil}^k\mathbf{N}(V(r)) \cap \mathbf{N}(T(r))$, where the intersection is taken inside $\mathbf{N}(V(r))$.

Lemma 3.11. *We have*

$$\mathrm{Fil}^k\mathbf{N}(V(r)) = \mathrm{Fil}^k\pi^{-r}\mathbf{N}(V)(r) = \pi^{-r}\mathrm{Fil}^{k+r}\mathbf{N}(V)(r),$$

and similarly for $\mathrm{Fil}^k\mathbf{N}(T(r))$.

Proof. Note that the inclusion $\pi^{-r}\mathrm{Fil}^{k+r}\mathbf{N}(V)(r) \subset \mathrm{Fil}^k\pi^{-r}\mathbf{N}(V)(r)$ is obvious. To show the converse let $\pi^{-r}x \otimes \epsilon^{\otimes r} \in \mathrm{Fil}^k\pi^{-r}\mathbf{N}(V)(r)$, with $x \in \mathbf{N}(V)$ and $\epsilon^{\otimes r}$ being a basis of $\mathbb{Q}_p(r)$. Then we have that $\varphi(\pi^{-r}x \otimes \epsilon^{\otimes r}) = q^{-r}\pi^{-r}\varphi(x) \otimes \epsilon^{\otimes r} \in q^k\pi^{-r}\mathbf{N}(V)(r)$. Therefore, we obtain that $\varphi(x) \in q^{k+r}\mathbf{N}(V)$, i.e. $x \in \mathrm{Fil}^{k+r}\mathbf{N}(V)$. ∎

Lemma 3.12. *Let V be a positive Wach representation and $T \subset V$ a $\mathbb{Z}_p$-lattice as above. Then for $s = r_1$, we have $\pi^s\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T \subset \mathbf{A}^+ \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)$.*

Proof. To show the claim, we can assume that $\mathbf{N}(T)$ is free by base changing to the finite étale extension R_0' of R_0. Then $\mathbf{A}^+ \otimes_{\mathbf{A}_{R_0'}^+} \left(\mathbf{A}_{R_0'}^+ \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)\right) = \mathbf{A}^+ \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)$ is free. Since the discussion of previous chapters hold for the p-adic completion of a finite étale extension of R_0 (see [Bri08, Chapitre 2] and [AI08, §2] for more on this), base changing to R_0' is harmless. So with a slight abuse of notation, below we will replace R_0' obtained in this manner by R_0 and assume $\mathbf{N}(T)$ to be free of rank h over $\mathbf{A}_{R_0}^+$.

Rest of the proof follows the techniques of [Ber04, Théorème III.3.1]. First notice that we have $\left(\mathbf{A}^+ \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)\right) \cap p^n(\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T) = p^n\left(\mathbf{A}^+ \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)\right)$. To see this let $\{e_i\}_{1 \le i \le h}$ be an $\mathbf{A}_{R_0}^+$-basis of $\mathbf{N}(T)$, then since $\mathbf{A}_{R_0} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T) \simeq \mathbf{D}(T)$, it is also an $\mathbf{A}_{R_0}$-basis of $\mathbf{D}(T)$ and therefore an $\mathbf{A}$-basis of $\mathbf{A} \otimes_{\mathbf{A}_{R_0}} \mathbf{D}(T) = \mathbf{A} \otimes_{\mathbb{Z}_p} T$. Now writing $x \in \left(\mathbf{A}^+ \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)\right) \cap p^n(\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T)$ in the chosen basis we have $x = \sum_{i=1}^h x_i e_i$ and therefore $x_i \in p^n\mathbf{A}$. The claimed equality now follows from the fact that $p^n\mathbf{A} \cap \mathbf{A}^+ = p^n\mathbf{A}^+$.

From the discussion above and the fact that $\mathbf{B}^+ = \mathbf{A}^+\left[\frac{1}{p}\right]$, we conclude that showing $\pi^s\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T \subset \mathbf{A}^+ \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)$ is equivalent to showing that $\pi^s\mathbf{B}^+ \otimes_{\mathbb{Q}_p} V \subset \mathbf{B}^+ \otimes_{\mathbf{B}_{R_0}^+} \mathbf{N}(V)$. So let $A \in Mat(h, \mathbf{B}^+)$ be the matrix obtained by expressing a basis of $\mathbf{N}(V)$ in the basis of V. Also, let $P \in Mat(h, \mathbf{B}_{R_0}^+)$ be the matrix of φ in the basis of $\mathbf{N}(V)$. Then we have $\varphi(A) = AP$ and therefore $\varphi(\pi^s A^{-1}) = (q^s P^{-1})(\pi^s A^{-1})$. The fact that $\mathbf{N}(V)/\varphi^*(\mathbf{N}(V))$ is killed by q^s implies that $q^s P^{-1} \in Mat(h, \mathbf{B}_{R_0}^+)$, therefore from Corollary 2.25 we obtain that $\pi^s A^{-1} \in Mat(h, \mathbf{B}^+)$. Hence, we conclude that $\pi^s\mathbf{B}^+ \otimes_{\mathbb{Q}_p} V \subset \mathbf{B}^+ \otimes_{\mathbf{B}_{R_0}^+} \mathbf{N}(V)$. ∎

Corollary 3.13. *By taking H_{R_0}-invariants in Lemma 3.12 it follows that $\pi^s\mathbf{D}^+(T) \subset \mathbf{N}(T)$.*

Lemma 3.14. *Let V be a Wach representation G_{R_0}. The Wach module $\mathbf{N}(V)$ over $\mathbf{B}^+_{R_0}$ is unique. Same holds true for the $\mathbf{A}^+_{R_0}$-module $\mathbf{N}(T)$.*

Proof. The argument carries over from the classical case [Ber04, p. 13]. First note that we can assume that all Hodge-Tate weights of V are ≤ 0, since by definition the uniquess of Wach module for such a representation is equivalent to uniqueness for all its Tate twists. In this case, let N_1 and N_2 be two $\mathbf{A}^+_{R_0}$-modules satisfying the conditions of Definition 3.8 (the proof stays the same for $\mathbf{N}(V)$). By symmetry, it is enough to show that $N_1 \subset N_2$. Since we have $\pi^s N_1 \subset \pi^s\mathbf{D}^+(T) \subset N_2$ (see Corollary 3.13) and N_2 is π-torsion free, therefore for any $x \in N_1$ there exists $k \leq s$ such that $\pi^k x \in N_2$ but $\pi^k x \notin \pi N_2$. Varying over all $x \in N_1 \setminus \pi N_1$, we can take $k \leq s$ to be the minimal integer such that $\pi^k N_1 \subset N_2$. Since $\pi^k x \in N_2$ and Γ_{R_0} acts trivially on $N_2/\pi N_2$, we have that $(\gamma_0 - 1)(\pi^k x) \in \pi N_2$. So we can write

$$(\gamma_0 - 1)(\pi^k x) = \gamma_0(\pi^k)(\gamma_0(x) - x) + (\gamma_0(\pi^k) - \pi^k)x.$$

Since Γ_{R_0} also acts trivially on $N_1/\pi N_1$ and $\pi^k N_1 \subset N_2$, we see that $\gamma_0(\pi^k)(\gamma_0(x) - x) \in \pi N_2$, therefore $(\gamma_0(\pi^k) - \pi^k)x \in \pi N_2$, which means that $(\chi(\gamma_0)^k - 1)\pi^k x \in \pi N_2$. But $\pi \nmid \chi(\gamma_0)^k - 1$ if $k \geq 1$, and $\pi^k x \notin \pi N_2$. Hence, we must have $k = 0$, i.e. $N_1 \subset N_2$. ∎

The uniqueness of Wach modules helps us in establishing compatibility with usual operations:

Lemma 3.15. *Let V and V' be two Wach representations of G_{R_0}. Then we have that $\mathbf{N}(V \oplus V') = \mathbf{N}(V) \oplus \mathbf{N}(V')$ and $\mathbf{N}(V \otimes V') = \mathbf{N}(V) \otimes \mathbf{N}(V')$. Similar statements hold for $\mathbf{N}(T)$ and $\mathbf{N}(T')$.*

Proof. We note similar to previous lemma that it is enough to show the statement for V and V' such that both representations have non-positive Hodge-Tate weights. By uniqueness of Wach modules proved in Lemma 3.14, it is enough to show that direct sum and tensor product of Wach representations are again Wach representations.

First, it is straightforward to see that $\mathbf{N}(T) \oplus \mathbf{N}(T') \subset \mathbf{D}^+(T \oplus T')$ is a projective $\mathbf{A}^+_{R_0}$-module of rank $\mathrm{rk}_{\mathbb{Z}_p}(T \oplus T')$ such that $\mathbf{A}_{R_0} \otimes_{\mathbf{A}^+_{R_0}} (\mathbf{N}(T) \oplus \mathbf{N}(T')) \simeq \mathbf{D}(T) \oplus \mathbf{D}(T')$. Similarly, we have that $\mathbf{N}(T) \otimes \mathbf{N}(T') \subset \mathbf{D}^+(T \otimes T')$ is a projective $\mathbf{A}^+_{R_0}$-module of rank $\mathrm{rk}_{\mathbb{Z}_p}(T \otimes T')$ such that $\mathbf{A}_{R_0} \otimes_{\mathbf{A}^+_{R_0}} (\mathbf{N}(T) \otimes \mathbf{N}(T')) \simeq \mathbf{D}(T) \otimes \mathbf{D}(T')$.

Next, let s and s' denote the maximum among the absolute value of Hodge-Tate weights of V and V' respectively and let $i := \max(s, s')$. Then we see that $(\mathbf{N}(T) \oplus \mathbf{N}(T'))/\varphi^*(\mathbf{N}(T) \oplus \mathbf{N}(T'))$ is killed by q^i and $(\mathbf{N}(T) \otimes \mathbf{N}(T'))/\varphi^*(\mathbf{N}(T) \otimes \mathbf{N}(T'))$ is killed by $q^{s+s'}$. Further, Γ_{R_0} acts trivially modulo π on $\mathbf{N}(T) \oplus \mathbf{N}(T')$ and $\mathbf{N}(T) \otimes \mathbf{N}(T')$. This verifies conditions (i), (ii) and (iii) for these modules. Hence, we get the claim. ∎

Corollary 3.16. *Let V be a Wach representation of G_{R_0} and $T \subset V$ a G_{R_0}-stable free $\mathbb{Z}_p$-lattice of rank $= \dim_{\mathbb{Q}_p} V$. Then, $\mathrm{Sym}^k(V)$ and $\wedge^k V$ are Wach representations for $k \in \mathbb{N}$.*

Proof. Note that the compatibility with tensor products in Lemma 3.15 is enough to establish the compatibility with symmetric powers and exterior powers because then we can set

$$\mathbf{N}\big(\mathrm{Sym}^k(T)\big) := \mathrm{Sym}^k(\mathbf{N}(T)), \quad \text{and} \quad \mathbf{N}\Big(\overset{k}{\wedge} T\Big) := \bigwedge^k \mathbf{N}(T).$$

We have $\mathbf{N}\big(\mathrm{Sym}^k(T)\big) \subset \mathrm{Sym}^k(\mathbf{D}^+(T)) \subset \mathbf{D}^+\big(\mathrm{Sym}^k(T)\big)$, since $\mathbf{A}^+ \otimes_{\mathbf{A}^+_{R_0}} \mathrm{Sym}^k(\mathbf{D}^+(T)) \subset \mathbf{A}^+ \otimes_{\mathbf{A}^+_{R_0}} \mathbf{D}^+\big(\mathrm{Sym}^k(T)\big)$. Similarly, $\mathbf{N}\big(\wedge^k T\big) \subset \mathbf{D}^+\big(\wedge^k T\big)$. Rest of the assumptions of Definition 3.8 follows in a same manner as in the proof of Lemma 3.15. This establshes that $\mathrm{Sym}^k(V)$ and $\wedge^k V$ are Wach representations and gives us the corresponding Wach modules. ∎

Following result will be useful while studying complexes with coefficients in Wach modules in Chapter 5.

Lemma 3.17. *Let V be a Wach representation of G_{R_0}, such that the associated Wach module $\mathbf{N}(T)$ over $\mathbf{A}_{R_0}^+$ is free of rank $= \dim_{\mathbb{Q}_p}(V)$. Then for $j \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have*

$$\pi^j \mathrm{Fil}^k \mathbf{N}(T) \cap \pi^{j+1}\mathbf{N}(T) = \pi^{j+1}\mathrm{Fil}^{k-1}\mathbf{N}(T).$$

Same holds true for the $\mathbf{B}_{R_0}^+$-module $\mathbf{N}(V)$.

Proof. The claim is obvious if $\mathrm{Fil}^{k-1}\mathbf{N}(T) = \mathbf{N}(T)$. So we assume that $\mathrm{Fil}^{k-1}\mathbf{N}(T) \subsetneq \mathbf{N}(T)$, and let $x \in \mathrm{Fil}^k\mathbf{N}(T)$ such that

$$\pi^j x \in \pi^j \mathrm{Fil}^k \mathbf{N}(T) \cap \pi^{j+1}\mathbf{N}(T).$$

Then we must have $x = \pi y$ for some $y \in \mathbf{N}(T)$. Since $\varphi(x) \in q^k\mathbf{N}(V) \cap \mathbf{N}(T)$, where $q = \frac{\varphi(\pi)}{\pi} = p + \pi w$ for $w \in \mathbf{A}_F^+$. So we get that $\pi\varphi(y) \in q^{k-1}\mathbf{N}(T) \cap \mathbf{N}(T)$, i.e. $\pi\varphi(y) = q^{k-1}z$ for some $z \in \mathbf{N}(V)$. Since $\mathbf{N}(T)$ is free of rank h and p does not divide q in $\mathbf{A}_{R_0}^+$, we obtain that $z \in \mathbf{N}(T)$.

Now let $\{e_1, \ldots, e_h\}$ be an $\mathbf{A}_{R_0}^+$-basis of the scalar extension and we write $\varphi(y) = \sum_{i=1}^h y_i e_i$ and $z = \sum_{i=1}^h z_i e_i$ for $y_i, z_i \in \mathbf{A}_{R_0}^+$. Further we have an embedding $\iota_{\mathrm{cycl}} : R_0 \rightarrowtail \mathbf{A}_{R_0}^+$, so we can write the coefficients above as power series in π. In particular, we have $y_i = \sum_{j\in\mathbb{N}} y_{ij}\pi^j$ such that the constant term $y_{i0} \in \iota_{\mathrm{cycl}}(R_0)$ and $y_{ij} \in \mathbf{A}_{R_0}^+$ go to zero p-adically as $j \longrightarrow +\infty$. Similarly, we can write $z_i = \sum_{j\in\mathbb{N}} z_{ij}\pi^j$, such that constant $z_{i0} \in \iota_{\mathrm{cycl}}(R_0)$ and $z_{ij} \in \mathbf{A}_{R_0}^+$ go to zero p-adically as $j \longrightarrow +\infty$. Now, from $\pi\varphi(y) = q^{k-1}z$, we obtain that $\pi y_i = q^{k-1}z_i$ for $1 \leq i \leq h$. But looking at the constant term on each side (coefficient of π^0), we obtain $p^{k-1}z_{i0} = 0$. Since $\mathbf{A}_{R_0'}^+$ is p-torsion free, we obtain that $z_{i0} = 0$ for $1 \leq i \leq h$, i.e. π divides z_i. Therefore, $y_i \in q^{k-1}\mathbf{A}_{R_0}^+$, for $1 \leq i \leq h$, i.e. $y \in \mathrm{Fil}^{k-1}\mathbf{N}(T)$.

The other inclusion is obvious, since we have that $\pi x \in \mathrm{Fil}^k\mathbf{N}(T)$ for $x \in \mathrm{Fil}^{k-1}\mathbf{N}(T)$. So we get the claim. ∎

3.2.1. Statement of the main result

In this section, we will relate the notion of crystalline and Wach representations. As we will see, we can recover the $R_0\left[\frac{1}{p}\right]$-module $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ from the $\mathbf{A}_{R_0}^+$-module $\mathbf{N}(V)$ after passing to a sufficiently large period ring. We begin by introducing this ring below.

Recall from §1.1 that we have F as a finite unramified extenion of $\mathbb{Q}_p$ with ring of integers W and we take $K = F(\zeta_{p^m})$ for $m \geq 1$. Note that the formulation of the results and proofs depend on m and it is necessary to have $m > 0$ for the discussion below to make sense.

In this section, we will work with the ring $\mathbf{A}_R^+$ defined in §2.4, equipped with an action of the Frobenius φ and a continuous action of Γ_{R_0}. Since we have a natural injection $\mathbf{A}_R^+ \rightarrowtail \mathbf{A}_{\mathrm{inf}}(R)$, we obtain a G_{R_0}-equivariant commutative diagram

$$\begin{tikzcd} \mathbf{A}_R^+ \arrow[r, two heads, "\theta"] \arrow[d, tail] & R \arrow[d, tail] \\ \mathbf{A}_{\mathrm{inf}}(R) \arrow[r, two heads, "\theta"] & \mathbb{C}^+(R). \end{tikzcd}$$

By R_0-linearlity, extending scalars for the map θ above, we obtain a ring homomorphism

$$\theta_{R_0} : R_0 \otimes_W \mathbf{A}_R^+ \longrightarrow R,$$

sending $X_i \otimes 1 \mapsto X_i$, $1 \otimes [X_i^\flat] \mapsto X_i$ for $1 \leq i \leq d$ and $1 \otimes \pi_n \mapsto \zeta_{p^n} - 1$. Note that we have inclusion of ideals $\big(\xi, X_i \otimes 1 - 1 \otimes [X_i^\flat], \text{ for } 1 \leq i \leq d\big) \subset \mathrm{Ker}\ \theta_{R_0} \subset R_0 \otimes_W \mathbf{A}_R^+$, where $\xi = \frac{\pi}{\pi_1}$.

Definition 3.18. Let $x^{[n]} := x^n/n!$ for $x \in \mathrm{Ker}\ \theta_{R_0}$. Define $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ to be the p-adic completion of the divided power envelope of $R_0 \otimes_W \mathbf{A}_R^+$ with respect to $\mathrm{Ker}\ \theta_{R_0}$.

We have $\mathbf{A}_R^+ \subset \mathbf{A}_{\text{inf}}(R)$ and θ_{R_0} above is the restriction of $\theta_{R_0} : R_0 \otimes_W \mathbf{A}_{\text{inf}}(R) \twoheadrightarrow \mathbb{C}^+(R)$ (see §1.3). Taking the divided power envelope of θ_{R_0}/p^n, we notice that $\mathcal{O}\mathbf{A}_R^{\text{PD}}/p^n \rightarrowtail \mathcal{O}\mathbf{A}_{\text{cris}}(R_0)/p^n$. Since $\mathcal{O}\mathbf{A}_R^{\text{PD}} = \lim_n \mathcal{O}\mathbf{A}_R^{\text{PD}}/p^n$ and $\mathcal{O}\mathbf{A}_{\text{cris}}(R_0) = \lim_n \mathcal{O}\mathbf{A}_{\text{cris}}(R_0)/p^n$, and (projective) limit is left exact, it follows that for the p-adic completion of divided power envelope of θ_{R_0}, we have $\mathcal{O}\mathbf{A}_R^{\text{PD}} \subset \mathcal{O}\mathbf{A}_{\text{cris}}(R_0)$. Now, over the ring $\mathcal{O}\mathbf{A}_R^{\text{PD}}$ we can consider the induced action of Γ_{R_0} under which it is stable, and it admits a Frobenius endomorphism arising from the Frobenius on each component of the tensor product. In particular, from the diagram above we obtain a Frobenius and G_R-equivariant commutative diagram

$$\begin{tikzcd}
\mathcal{O}\mathbf{A}_R^{\text{PD}} \arrow[r, two heads, "\theta_{R_0}"] \arrow[d, tail] & R \arrow[d, tail] \\
\mathcal{O}\mathbf{A}_{\text{cris}}(R_0) \arrow[r, two heads, "\theta_{R_0}"] & \mathbb{C}^+(\overline{R}).
\end{tikzcd}$$

Next, we will give an alternative description of the ring $\mathcal{O}\mathbf{A}_R^{\text{PD}}$. Let $T = (T_1, \ldots, T_d)$ denote a set of indeterminates and let $\mathbf{A}_{\text{cris}}(R)\langle T\rangle^\wedge$ denote the p-adic completion of the divided power polynomial algebra $\mathbf{A}_{\text{cris}}(R)\langle T\rangle = \mathbf{A}_{\text{cris}}(R)[T_i^{[n]},\ n \in \mathbb{N},\ 1 \leq i \leq d]$. Recall from §1.3 that we have an isomorphism of rings

$$\begin{aligned} f_{\text{cris}} : \mathbf{A}_{\text{cris}}(R)\langle T\rangle^\wedge &\xrightarrow{\simeq} \mathcal{O}\mathbf{A}_{\text{cris}}(R_0) \\ T_i &\longmapsto X_i \otimes 1 - 1 \otimes [X_i^\flat], \quad \text{for } 1 \leq i \leq d. \end{aligned}$$

Now recall that $\mathbf{A}_R^{\text{PD}}$ is the p-adic completion of the divided power envelope of the surjective map $\theta : \mathbf{A}_R^+ \twoheadrightarrow R$ with respect to its kernel (see §2.3). Next, let $\mathbf{A}_R^{\text{PD}}\langle T\rangle^\wedge$ denote the p-adic completion of the divided power polynomial algebra $\mathbf{A}_R^{\text{PD}}\langle T\rangle = \mathbf{A}_R^{\text{PD}}[T_i^{[n]},\ n \in \mathbb{N},\ 1 \leq i \leq d]$. Then via the isomorphism f^{PD} (see Lemma 3.19 below), we will show that the preimage of $\mathcal{O}\mathbf{A}_R^{\text{PD}}$, under f_{cris} is exactly $\mathbf{A}_R^{\text{PD}}\langle T\rangle^\wedge$. In other words,

Lemma 3.19. *The morphism of rings*

$$\begin{aligned} f^{\text{PD}} : \mathbf{A}_R^{\text{PD}}\langle T\rangle^\wedge &\longrightarrow \mathcal{O}\mathbf{A}_R^{\text{PD}} \\ T_i &\longmapsto X_i \otimes 1 - 1 \otimes [X_i^\flat], \quad \textit{for } 1 \leq i \leq d, \end{aligned}$$

is an isomorphism.

Proof. The proof follows [Bri08, Proposition 6.1.5] closely.

Recall that we have a surjective ring homomorphism $\theta : \mathbf{A}_R^{\text{PD}} \twoheadrightarrow R$, which is the restriction of the map $\theta : \mathbf{A}_{\text{cris}}(R) \twoheadrightarrow \overline{R}$ defined in §1.3. This can be extended in a unique manner into the homomorphism $\theta : \mathbf{A}_{\text{cris}}(R)\langle T\rangle^\wedge \twoheadrightarrow \overline{R}$. Restriction of the latter map gives us $\theta : \mathbf{A}_R^{\text{PD}}\langle T\rangle^\wedge \twoheadrightarrow R$ such that $\theta(T_i^{[n]}) = 0$ for $1 \leq i \leq d$ and $n \geq 1$.

First, we will show that the $W\{X^{\pm 1}\}$-algebra structure on $\mathbf{A}_R^{\text{PD}}\langle T\rangle^\wedge$ given by $X_i \mapsto [X_i^\flat] + T_i$, extends uniquely to an R_0-algebra structure. Let $\mathcal{A} := (\mathbf{E}_R^+/\overline{\pi}^{p-1}\mathbf{E}_R^+)[T_1, \ldots, T_d]/(T_1^p, \ldots, T_d^p)$. We have a surjective map $\theta : \mathbf{A}_R^+ \twoheadrightarrow R$ and its reduction modulo p is given as $\overline{\theta} : \mathbf{E}_R^+ \twoheadrightarrow R/pR$. Since $\xi^p \equiv \overline{\pi}^{p-1}$ mod p, where $\xi = \frac{\pi}{\pi_1}$ is a generator of $\text{Ker }\theta \subset \mathbf{A}_R^+$, we obtain that $\overline{\theta}$ factors as $\overline{\theta} : \mathbf{E}_R^+/\overline{\pi}^{p-1}\mathbf{E}_R^+ \twoheadrightarrow R/pR$. This can be extended to a map $\overline{\theta} : \mathcal{A} \twoheadrightarrow R/pR$ by setting $\overline{\theta}(T_i) = 0$ for $1 \leq i \leq d$. The kernel $\mathcal{I} = \text{Ker }\overline{\theta} \subset \mathcal{A}$ is generated by $\overline{\pi}^{p-1}$ and $\{T_i\}_{1 \leq i \leq d}$. Now from the natural inclusion $R_0/pR_0 \rightarrowtail R/pR$ and the isomorphism $\mathcal{A}/\mathcal{I} \simeq R/pR$ via $\overline{\theta}$, we obtain a map $\overline{g} : R_0/pR_0 \longrightarrow \mathcal{A}/\mathcal{I}$ such that $\overline{g}(X_i) = X_i$, which is the image of $X_i^\flat \in \mathcal{A}$ under the map $\overline{\theta}$. So we obtain a commutative diagram

$$\begin{tikzcd} \kappa[X^{\pm 1}] \arrow[r] \arrow[d] & \mathcal{A} \arrow[d, two heads] \\ R_0/pR_0 \arrow[r, tail] \arrow[ur, dashed, "\overline{g}"] & \mathcal{A}/\mathcal{I} \end{tikzcd}$$

where the top horizontal arrow is the map $X_i \mapsto X_i^\flat + T_i$. Note that $\mathcal{I}^{(d+1)p} = 0$. Since R_0/pR_0 is étale over $\kappa[X^{\pm 1}]$, there exists a unique lift of $\overline{g} : R_0/pR_0 \longrightarrow \mathcal{A}/\mathcal{I}$ to a homomorphism $\overline{g} : R_0/pR_0 \longrightarrow \mathcal{A}$ (which we again denote by $\overline{g}$ by slight abuse of notations).

Further, by the description of divided power envelope in [Bri08, Proposition 6.1.1] we have that

$$\begin{aligned} \mathbf{A}_R^+[Y_0, Y_1, \ldots]/(pY_0 - \xi^p, pY_{n+1} - Y_n^p)_{n\geq 1} &\xrightarrow{\simeq} \mathbf{A}_R^{\mathrm{PD}} \\ Y_n &\longmapsto \frac{\xi^{p^{n+1}}}{p^{n+1}}. \end{aligned}$$

Therefore,

$$\mathbf{A}_R^{\mathrm{PD}}/p\mathbf{A}_R^{\mathrm{PD}} \simeq (\mathbf{E}_R^+/\overline{\pi}^{p-1}\mathbf{E}_R^+)[Y_0, Y_1, \ldots]/(Y_n^p)_{n\geq 1}$$

Similarly, we have

$$\mathbf{A}_R^{\mathrm{PD}}\langle T\rangle \simeq (\mathbf{A}_R^{\mathrm{PD}}[T_1, \ldots, T_d])[T_{i,0}, T_{i,1}, \ldots]/(pT_{i,0} - T_i^p, pT_{i,n+1} - T_{i,n}^p)_{1\leq i\leq d,\, n\in\mathbb{N}}.$$

Therefore,

$$\mathbf{A}_R^{\mathrm{PD}}\langle T\rangle/p\mathbf{A}_R^{\mathrm{PD}}\langle T\rangle \simeq (\mathbf{A}_R^{\mathrm{PD}}/p\mathbf{A}_R^{\mathrm{PD}})[T_1, \ldots, T_d][T_{i,0}, T_{i,1}, \ldots]/(T_i^p, T_{i,n}^p)_{1\leq i\leq d,\, n\in\mathbb{N}}.$$

In conclusion, we have

$$\mathbf{A}_R^{\mathrm{PD}}\langle T\rangle/p\mathbf{A}_R^{\mathrm{PD}}\langle T\rangle \simeq \mathcal{A}[Y_0, Y_1, \ldots, T_{i,0}, T_{i,1}, \ldots]/(Y_n^p, T_{i,n}^p)_{1\leq i\leq d,\, n\in\mathbb{N}}.$$

Therefore, from the discussion above we obtain a natural map of $\kappa[X^{\pm 1}]$-algebras by composition $\overline{g}_1 : R_0/pR_0 \longrightarrow \mathcal{A} \longrightarrow \mathbf{A}_R^{\mathrm{PD}}\langle T\rangle/p\mathbf{A}_R^{\mathrm{PD}}\langle T\rangle$.

Now let $n \in \mathbb{N}$, then modulo p^n we have the natural map $W\{X^{\pm 1}\}/p^nW\{X^{\pm 1}\} \longrightarrow \mathbf{A}_R^{\mathrm{PD}}\langle T\rangle/p^n\mathbf{A}_R^{\mathrm{PD}}\langle T\rangle$. Again, since R_0/p^nR_0 is étale over $W\{X^{\pm 1}\}/p^nW\{X^{\pm 1}\}$, we have a unique lift of $\overline{g}_n : R_0/p^nR_0 \longrightarrow \mathbf{A}_R^{\mathrm{PD}}\langle T\rangle/p^n\mathbf{A}_R^{\mathrm{PD}}\langle T\rangle$ in the commutative diagram

$$\begin{tikzcd} W\{X^{\pm 1}\}/p^nW\{X^{\pm 1}\} \arrow[r] \arrow[d] & \mathbf{A}_R^{\mathrm{PD}}\langle T\rangle/p^n\mathbf{A}_R^{\mathrm{PD}}\langle T\rangle \arrow[d, two heads] \\ R_0/p^nR_0 \arrow[r] \arrow[ur, dashed, "\overline{g}_n"] & \mathbf{A}_R^{\mathrm{PD}}\langle T\rangle/p\mathbf{A}_R^{\mathrm{PD}}\langle T\rangle. \end{tikzcd}$$

Via this lifting, the following diagram commutes

$$\begin{tikzcd} R_0/p^{n+1}R_0 \arrow[r] \arrow[d] & \mathbf{A}_R^{\mathrm{PD}}\langle T\rangle/p^{n+1}\mathbf{A}_R^{\mathrm{PD}}\langle T\rangle \arrow[d] \\ R_0/p^nR_0 \arrow[r] & \mathbf{A}_R^{\mathrm{PD}}\langle T\rangle/p^n\mathbf{A}_R^{\mathrm{PD}}\langle T\rangle, \end{tikzcd}$$

where the vertical arrows are natural projection maps. From the universal property of inverse limit of the right side of the diagram, we obtain a natural map of $W\{X^{\pm 1}\}$-algebras

$$g : R_0 \longrightarrow \lim_n \mathbf{A}_R^{\mathrm{PD}}\langle T\rangle/p^n\mathbf{A}_R^{\mathrm{PD}}\langle T\rangle = \mathbf{A}_R^{\mathrm{PD}}\langle T\rangle^\wedge.$$

Now, let $\overline{\theta} : \mathbf{A}_R^{\mathrm{PD}}\langle T\rangle/p\mathbf{A}_R^{\mathrm{PD}}\langle T\rangle \longrightarrow R/pR$ denote the reduction of θ modulo p. Recall that by construction, $\overline{\theta}\circ\overline{g}$ is the inclusion of R_0/pR_0 in R/pR. Therefore, the reduction modulo p of $\theta\circ g$ and the natural inclusion $R_0 \rightarrowtail R$ coincide. Since R is p-torsion free, arguing as above we obtain that for each $n \in \mathbb{N}$, the natural inclusion and $\theta\circ g$ coincide modulo p^n.

Next, by $\mathbf{A}_R^+$-linearity, g can be extended to a map $g : R_0 \otimes_W \mathbf{A}_R^+ \longrightarrow \mathbf{A}_R^{\mathrm{PD}}\langle T\rangle^\wedge$. From the discussion above and the definition of θ_{R_0}, we have that it coincides with the homomorphism $\theta\circ g : R_0\otimes_W \mathbf{A}_R^+ \longrightarrow R$. In particular, $g(\mathrm{Ker}\,\theta_{R_0}) \subset \mathrm{Ker}\,\theta \subset \mathbf{A}_R^{\mathrm{PD}}\langle T\rangle^\wedge$. Since $\mathrm{Ker}\,\theta$ contains divided powers, the map g extends to a map

$$g : (R_0\otimes_W \mathbf{A}_R^+)[x^{[n]}, x\in \mathrm{Ker}\,\theta_{R_0}, n\in\mathbb{N}] \longrightarrow \mathbf{A}_R^{\mathrm{PD}}\langle T\rangle^\wedge.$$

Finally, since $\mathbf{A}_R^{\mathrm{PD}}\langle T\rangle^\wedge$ is p-adically complete, g extends to a map $g : \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \longrightarrow \mathbf{A}_R^{\mathrm{PD}}\langle T\rangle^\wedge$.

Now by uniqueness of $g : R_0 \longrightarrow \mathbf{A}_R^{\mathrm{PD}}\langle T\rangle^\wedge$, the composition

$$\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \xrightarrow{g} \mathbf{A}_R^{\mathrm{PD}}\langle T\rangle^\wedge \xrightarrow{f^{\mathrm{PD}}} \mathcal{O}\mathbf{A}_R^{\mathrm{PD}},$$

coincides with the identity over $R_0 \subset \mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$. Since it also coincides with identity on the image of $\mathbf{A}_R^+$ (by $\mathbf{A}_R^+$-linearity), we obtain that $f^{\mathrm{PD}}\circ g = \mathrm{id}$ over $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$. Similarly, the homomorphism $g\circ f^{\mathrm{PD}}$ coincides with identity over $\mathbf{A}_R^+$ as well as over $W\{X^{\pm 1}\}$ (since g lifts the map $W\{X^{\pm1}\} \longrightarrow \mathbf{A}_R^{\mathrm{PD}}\langle T\rangle^\wedge$), therefore it is identity over $\mathbf{A}_R^{\mathrm{PD}}\langle T\rangle^\wedge$. This establishes that f^{PD} is an isomorphism of rings. ■

Remark 3.20. We can give an alternative construction of the ring $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$. Note that we have a ring homomorphism $\iota : R_0 \longrightarrow \mathbf{A}_R^{\mathrm{PD}}$, where $X_i \longmapsto [X_i^\flat]$ for $1\le i\le d$. As in Definition 2.48, we define a map $g : R_0\otimes_W \mathbf{A}_R^{\mathrm{PD}} \longrightarrow \mathbf{A}_R^{\mathrm{PD}}$, where $x\otimes y \longmapsto \iota(x)y$. We obtain that $\mathrm{Ker}\, g = \big(X_i\otimes 1 - 1\otimes[X_i^\flat],\ \text{for } 1\le i\le d\big) \subset \mathrm{Ker}\,\theta_{R_0} \subset \mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0)$. Since $R_0\otimes \mathbf{A}_R^{\mathrm{PD}}$ already contains divided powers of ξ, from Definition 3.18 we obtain that the p-adic completion of the divided power envelope of $R_0\otimes_W \mathbf{A}_R^{\mathrm{PD}}$ with respect to $\mathrm{Ker}\, g$ is the same as $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$.

There is a natural filtration over the ring $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ by Γ_{R_0}-stable submodules:

Definition 3.21. Let $V_i := \frac{1\otimes[X_i^\flat]}{X_i\otimes 1}$ for $1\le i\le d$, then we define the filtration over $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ as

$$\mathrm{Fil}^r\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} := \Big\langle (a\otimes b)\prod_{i=1}^{d}(V_i-1)^{[k_i]} \in \mathcal{O}\mathbf{A}_R^{\mathrm{PD}},\ \text{such that } a\in R_0, b\in \mathrm{Fil}^j\mathbf{A}_R^{\mathrm{PD}},\ \text{and } j+\sum_i k_i \ge r\Big\rangle \ \text{ for } r\in\mathbb{Z}.$$

Remark 3.22. The filtration over $\mathbf{A}_R^{\mathrm{PD}}$ (via its identification with R_ϖ^{PD}, see §2.4 and Definition 2.27) coincides with the filtration induced from its embedding in $\mathbf{A}_{\mathrm{cris}}(R)$. Indeed, in both cases we have $\mathrm{Fil}^r\mathbf{A}_R^{\mathrm{PD}} = \big(\xi^{[k]},\ k\le r\big) \subset \mathbf{A}_R^{\mathrm{PD}}$ for $r\ge 0$, whereas $\mathrm{Fil}^r\mathbf{A}_R^{\mathrm{PD}} = \mathbf{A}_R^{\mathrm{PD}}$ for $r<0$. Now the filtration on $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0)$ is defined as the induced filtration from its embedding inside $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^+(R)$, where the filtration on the latter ring is given by powers of $\mathrm{Ker}\,\theta_R$ (see §1.2 & §1.3 for definition and notation). The induced filtration over $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0)$ is therefore given by divided powers of the ideal $\mathrm{Ker}\,\theta_{R_0}\subset \mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0)$. Since the filtration over $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ in Definition 3.21 is again defined by divided powers of the generators of the ideal $\mathrm{Ker}\,\theta_{R_0}\subset\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$, we infer that this filtration coincides with the one induced by its embedding into $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0)$.

Lemma 3.23. (i) *The action of Γ_R is trivial on $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}/\pi$, whereas Γ_{R_0}/Γ_R acts trivially over $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}/\pi_m$.*

(ii) *The Γ_{R_0}-invariants of $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ are given by R_0.*

Proof. (i) The first part follows from the definition of $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ and the action of Γ_R on $\mathbf{A}_R^{\mathrm{PD}}$ (see Lemma 2.45). The second part follows from observing that Γ_{R_0}/Γ_R is a finite cyclic group of order $[K : F] = p^{m-1}(p-1)$, and a lift $g\in\Gamma_{R_0}$ of a generator of Γ_{R_0}/Γ_R acts as $g(\pi_m) = (1+\pi_m)^{\chi(g)}-1$.

(ii) This is straightforward, since

$$R_0 \subset \left(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}\right)^{\Gamma_{R_0}} \subset \left(\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0)\right)^{G_{R_0}} = R_0.$$

∎

Next we consider a connection over $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ induced by the connection on $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0)$,

$$\partial : \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \longrightarrow \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes \Omega^1_{R_0},$$

where we have $\partial\big(X_i \otimes 1 - 1 \otimes [X_i^\flat]\big)^{[n]} = \big(X_i \otimes 1 - 1 \otimes [X_i^\flat]\big)^{[n-1]}\, dX_i$. This connection over $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ satisfies Griffiths transversality with respect to the filtration since it does so over $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0)$.

The main result of this section is as follows:

Theorem 3.24. *With notations as above let V be an h-dimensional positive Wach representation of G_{R_0}, then V is a positive crystalline representation. Further, let $M := \left(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)\right)^{\Gamma_{R_0}}$, then we have an isomorphism of $R_0\left[\frac{1}{p}\right]$-modules $M\left[\frac{1}{p}\right] \simeq \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ compatible with Frobenius, filtration, and connection on each side. Moreover, after extending scalars to $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$, we obtain natural isomorphisms*

$$\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xleftarrow{\simeq} \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} M\left[\tfrac{1}{p}\right] \xrightarrow{\simeq} \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V),$$

compatible with Frobenius, filtration, connection and the action of Γ_{R_0} on each side.

Remark 3.25. The statement of Theorem 3.24 can be seen an analogue of the result of Berger [Ber04, Proposition II.2.1] (see the discussion after Proposition 3.6).

Recall that from Definition 3.8 any Wach representation is a twist of a positive Wach representation by $\mathbb{Q}_p(r)$, for $r \in \mathbb{N}$. Since twist by $\mathbb{Q}_p(r)$ of crystalline representations are again crystalline, we obtain that:

Corollary 3.26. *All Wach representations of G_{R_0} are crystalline.*

The proof of Theorem 3.24 will proceed in three steps: First, we explicitly state the structure of Wach module attached to a one-dimensional Wach representation, we will also show that all one-dimensional crystalline representations are Wach representations and one can recover $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ starting with the Wach module $\mathbf{N}(V)$. Next, in higher dimensions and under the conditions of the theorem, we will describe a process (successive approximation) by which we can recover a submodule of $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ starting from the Wach module, here we establish a comparison by passing to the one-dimensinal case. Finally, the claims made in the theorem are shown by exploiting some properties of Wach modules and the comparison obtained in the second step.

3.2.2. One-dimensional representations

In this section we are going to study one-dimensional crystalline representations as well as one-dimensional Wach representations. We will show that all one-dimensional crystalline representations are Wach representations. Moreover, for Wach representations we will prove a technical statement which will be used in the proof of Proposition 3.31.

One-dimensional crystalline representations

In this section our goal is to show the following claim:

Proposition 3.27. *All one-dimensional crystalline representations of G_{R_0} are Wach representations. Furthermore, for a one-dimensional crystalline representation V we have an isomorphism of $R_0\left[\frac{1}{p}\right]$-modules*

$$\left(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V)\right)^{\Gamma_{R_0}} \xrightarrow{\simeq} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V).$$

Therefore, there exists natural isomorphisms

$$\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xleftarrow{\simeq} \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V)\big)^{\Gamma_{R_0}} \xrightarrow{\simeq} \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V),$$

compatible with Frobenius, filtration and the action of Γ_{R_0}.

Proof. The structure of one-dimensional crystalline representations of G_{R_0} is well-known (see [Bri08, §8.6]). First, recall that a p-adic representation of G_{R_0} is unramified if the action of G_{R_0} factorizes through the quotient $G_{R_0}^{\mathrm{ur}}$ (see §1.5). Now from Proposition 1.30 we have that for $\eta : G_{R_0} \longrightarrow \mathbb{Z}_p^\times$, a continuous character, $V = \mathbb{Q}_p(\eta)$ is crystalline if and only if we can write $\eta = \eta_{\mathrm{f}}\eta_{\mathrm{ur}}\chi^n$ with $n \in \mathbb{Z}$, and where η_{f} is a finite unramified character, η_{ur} is an unramified character taking values in $1 + p\mathbb{Z}_p$ and trivialized by an element $\alpha \in 1 + p\widehat{R_0^{\mathrm{ur}}}$, and χ is the p-adic cyclotomic character. Moreover, if η_{f} is trivial then $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is a free $R_0\big[\frac{1}{p}\big]$-module of rank 1.

In Lemma 3.28 below, we show that crystalline representations $V_1 := \mathbb{Q}_p(\eta_{\mathrm{f}}\eta_{\mathrm{ur}})$ and $V_2 := \mathbb{Q}_p(\chi^n)$ are Wach represenations. For a one-dimensional crystalline representation $V := \mathbb{Q}_p(\eta) = \mathbb{Q}_p(\eta_{\mathrm{f}}\eta_{\mathrm{ur}}) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\chi^n) = V_1 \otimes_{\mathbb{Q}_p} V_2$ as above, by compatibility of tensor products in Lemma 3.15 we get that V is a Wach representation as well with $\mathbf{N}(V) = \mathbf{N}(V_1) \otimes_{\mathbf{B}_{R_0}^+} \mathbf{N}(V_2)$.

Now, from the isomorphisms of $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$-modules in Lemma 3.28 and compatibility of tensor product of Wach modules in Lemma 3.15 and compatibility of the functor $\mathcal{O}\mathbf{D}_{\mathrm{cris}}$ with tensor products in §1.5 (see also [Bri08, Théorème 8.4.2]), we get a string of isomorphisms of $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$-modules compatible with Frobenius, filtration and the action of Γ_{R_0},

$$\begin{aligned}
\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) &\simeq \big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V_1)\big) \otimes_{\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}} \big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V_2)\big)\\
&\simeq \Big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V_1)\big)^{\Gamma_{R_0}}\Big) \otimes_{\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}} \Big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V_2)\big)^{\Gamma_{R_0}}\Big)\\
&\simeq \big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V_1)\big) \otimes_{\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}} \big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V_2)\big)\\
&\simeq \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V_1) \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V_2)\\
&\simeq \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V_1 \otimes_{\mathbb{Q}_p} V_2) \simeq \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V).
\end{aligned}$$

Taking Γ_{R_0}-invariants of the first and the last term gives us that $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \simeq \big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V)\big)^{\Gamma_{R_0}}$, compatible with Frobenius and filtration. Hence, we obtain the claim. ∎

Following claim was used above:

Lemma 3.28. (i) *Let* $\eta : G_{R_0} \longrightarrow \mathbb{Z}_p^\times$ *be a continuous unramified character. Then the p-adic representation* $\mathbb{Q}_p(\eta)$ *is a Wach representation.*

(ii) *Let* χ *be the p-adic cyclotomic character then for* $n \in \mathbb{Z}$, *the p-adic representation* $\mathbb{Q}_p(n)$ *is a Wach representation.*

Further, for $V = \mathbb{Q}_p(\eta), \mathbb{Q}_p(n)$ *we have an isomorphism of* $R_0\big[\frac{1}{p}\big]$*-modules*

$$\big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V)\big)^{\Gamma_{R_0}} \xrightarrow{\simeq} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V).$$

Therefore, there exists natural isomorphisms

$$\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xleftarrow{\simeq} \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V)\big)^{\Gamma_{R_0}} \xrightarrow{\simeq} \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V),$$

compatible with Frobenius, filtration and the action of Γ_{R_0}.

Proof. Let $\eta = \eta_{\mathrm{f}}\eta_{\mathrm{ur}}$, where η_{f} is an unramified character of finite order and η_{ur} is an unramified character taking values in $1 + p\mathbb{Z}_p$ and trivialised by an element $\alpha \in 1 + p\widehat{R_0^{\mathrm{ur}}}$ (see Proposition 1.30).

First, let us consider the finite unramified character η_{f}. Set $T = \mathbb{Z}_p(\eta_{\mathrm{f}}) = \mathbb{Z}_p e$, such that $g(e) = \eta_{\mathrm{f}}(g)e$. We have

$$\mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big) = \big(\mathbf{A}^+ \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\eta_{\mathrm{f}})\big)^{H_{R_0}} \simeq \big\{ a \otimes e, \text{ with } a \in \mathbf{A}^+ \text{ such that } g(a) = \eta_{\mathrm{f}}^{-1}(g)a, \text{ for } g \in H_{R_0} \big\}.$$

Since η_{f} is a finite unramified character, it trivializes over a finite Galois extension S_0 over R_0 (see [Bri08, Proposition 8.6.1]), and we have that $\mathrm{Gal}\big(S_0\big[\frac{1}{p}\big]/R_0\big[\frac{1}{p}\big]\big) = G_{R_0}/G_{S_0} = H_{R_0}/H_{S_0} = \Gamma_{R_0}/\Gamma_{S_0}$. As S_0 is étale over R_0 the construction of previous chapters apply and we obtain that the $\mathbf{A}^+_{S_0}$-module $\mathbf{D}^+_{S_0}\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big) = \big(\mathbf{A}^+ \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\eta_{\mathrm{f}})\big)^{H_{S_0}} = \mathbf{A}^+_{S_0}(\eta_{\mathrm{f}}) = \mathbf{A}^+_{S_0} e$ is free of rank 1. Further, we know that $\mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big) = \mathbf{D}^+_{S_0}\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big)^{H_{R_0}/H_{S_0}}$, which implies that the natural inclusion

$$\mathbf{A}^+_{S_0} \otimes_{\mathbf{A}^+_{R_0}} \mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big) \longrightarrow \mathbf{D}^+_{S_0}\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big),$$

is bijective. Now, since $\mathbf{A}^+_{R_0} \longrightarrow \mathbf{A}^+_{S_0}$ is faithfully flat, we obtain that $\mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big)$ is projective of rank 1. Moreover, $\mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big)$ admits a Frobenius-semilinear endomorphism φ such that $\mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big) \simeq \varphi^*\big(\mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big)\big)$ (one can obtain this after faithfully flat scalar extension $\mathbf{A}^+_{R_0} \longrightarrow \mathbf{A}^+_{S_0}$ and applying descent as above). The action of Γ_{R_0} is trivial on $\mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big)$. Now, note that unramified representations are crystalline of Hodge-Tate weight 0, so we can take $\mathbf{N}\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big) = \mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big)$. From the discussion above, $\mathbf{N}\big(\mathbb{Z}_p(\eta_{\mathrm{f}})\big)$ clearly satisfies the conditions of Definition 3.8. Also, we have that $\mathbf{N}(\mathbb{Q}_p(\eta_{\mathrm{f}})) = \mathbf{D}^+(\mathbb{Q}_p(\eta_{\mathrm{f}}))$. On the other hand, we have

$$\mathcal{O}\mathbf{D}_{\mathrm{cris}}\big(\mathbb{Q}_p(\eta_{\mathrm{f}})\big) = \big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\eta_{\mathrm{f}})\big)^{G_{R_0}} = \big\{ b \otimes e, \text{ with } b \in \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \text{ such that } g(b) = \eta_{\mathrm{f}}(g)b \big\}.$$

Since η_{f} trivializes over the finite Galois extension S_0 over R_0, we set $S = S_0(\zeta_{p^m})$ and we have

$$\big(\mathcal{O}\mathbf{A}^{\mathrm{PD}}_S \otimes_{\mathbf{A}^+_{R_0}} \mathbf{N}\big(\mathbb{Q}_p(\eta_{\mathrm{f}})\big)\big)^{\Gamma_{S_0}} = S_0\big[\tfrac{1}{p}\big] e = \big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(S_0) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\eta_{\mathrm{f}})\big)^{G_{S_0}},$$

where the rings $\mathcal{O}\mathbf{A}^{\mathrm{PD}}_S$ and $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(S_0)$ are defined for S_0 over which all the construction of previous sections apply (since S_0 is étale over R_0). Now taking invariants under the finite Galois group $\mathrm{Gal}\big(S_0\big[\frac{1}{p}\big]/R_0\big[\frac{1}{p}\big]\big) = G_{R_0}/G_{S_0}$, gives us

$$\big(\mathcal{O}\mathbf{A}^{\mathrm{PD}}_R \otimes_{\mathbf{A}^+_{R_0}} \mathbf{N}\big(\mathbb{Q}_p(\eta_{\mathrm{f}})\big)\big)^{\Gamma_{R_0}} = \mathcal{O}\mathbf{D}_{\mathrm{cris}}\big(\mathbb{Q}_p(\eta_{\mathrm{f}})\big).$$

Clearly, the natural maps

$$\mathcal{O}\mathbf{A}^{\mathrm{PD}}_R \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}\big(\mathbb{Q}_p(\eta_{\mathrm{f}})\big) \xleftarrow{\simeq} \mathcal{O}\mathbf{A}^{\mathrm{PD}}_R \otimes_{R_0} \big(\mathcal{O}\mathbf{A}^{\mathrm{PD}}_R \otimes_{\mathbf{A}^+_{R_0}} \mathbf{N}\big(\mathbb{Q}_p(\eta_{\mathrm{f}})\big)\big)^{\Gamma_{R_0}} \xrightarrow{\simeq} \mathcal{O}\mathbf{A}^{\mathrm{PD}}_R \otimes_{\mathbf{A}^+_{R_0}} \mathbf{N}\big(\mathbb{Q}_p(\eta_{\mathrm{f}})\big),$$

are isomorphisms compatible with Frobenius, filtration and the action of Γ_{R_0}.

Next, let us consider the unramified character η_{ur} which takes values in $1 + p\mathbb{Z}_p$ and trivialised by an element $\alpha \in 1 + p\widehat{R_0^{\mathrm{ur}}}$ (see Proposition 1.30). Set $T = \mathbb{Z}_p(\eta_{\mathrm{ur}}) = \mathbb{Z}_p e$, such that $g(e) = \eta_{\mathrm{ur}}(g)e$. We have

$$\mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{ur}})\big) = \big(\mathbf{A}^+ \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\eta_{\mathrm{ur}})\big)^{H_{R_0}} = \mathbf{A}^+_{R_0}\alpha e.$$

Since unramified representations are crystalline of Hodge-Tate weight 0, we can take $\mathbf{N}\big(\mathbb{Z}_p(\eta_{\mathrm{ur}})\big) = \mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{ur}})\big) = \mathbf{A}^+_{R_0}\alpha e$. This clearly satisfies the conditions of Definition 3.8. Also, we have that $\mathbf{N}(\mathbb{Q}_p(\eta_{\mathrm{ur}})) = \mathbf{D}^+(\mathbb{Q}_p(\eta_{\mathrm{ur}}))$. On the other hand, we have

$$\begin{aligned}\mathcal{O}\mathbf{D}_{\mathrm{cris}}\big(\mathbb{Q}_p(\eta_{\mathrm{ur}})\big) &= \big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\eta_{\mathrm{ur}})\big)^{G_{R_0}} \\ &= \big\{ b \otimes e, \text{ with } b \in \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \text{ such that } g(b) = \eta_{\mathrm{ur}}(g)b \big\} = R_0\big[\tfrac{1}{p}\big]\alpha e.\end{aligned}$$

Therefore, we obtain

$$\left(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}\big(\mathbb{Q}_p(\eta_{\mathrm{f}}\eta_{\mathrm{ur}})\big)\right)^{\Gamma_{R_0}} = R_0\left[\tfrac{1}{p}\right]\alpha e = \left(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\eta_{\mathrm{f}}\eta_{\mathrm{ur}})\right)^{G_{R_0}}.$$

Clearly, the natural maps

$$\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}\big(\mathbb{Q}_p(\eta_{\mathrm{ur}})\big) \xleftarrow{\simeq} \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \left(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}\big(\mathbb{Q}_p(\eta_{\mathrm{ur}})\big)\right)^{\Gamma_{R_0}} \xrightarrow{\simeq} \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}\big(\mathbb{Q}_p(\eta_{\mathrm{ur}})\big),$$

are isomorphisms compatible with Frobenius, filtration and the action of Γ_{R_0}.

Finally, let $T = \mathbb{Z}_p(n) = \mathbb{Z}_p e_n$ such that $g(e_n) = \chi(g)^n e_n$, then $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$ is a crystalline representation with single Hodge-Tate weight n. In this case, we can take $\mathbf{N}\big(\mathbb{Z}_p(n)\big) = \mathbf{A}_{R_0}^+ \pi^{-n} e_n$. Note that for $n \leq 0$, we have that $\mathbf{N}\big(\mathbb{Z}_p(n)\big)/\varphi^*\big(\mathbf{N}\big(\mathbb{Z}_p(n)\big)\big)$ is killed by q^n, where $q = \frac{\varphi(\pi)}{\pi}$. It can easily be verified that Γ_R acts trivially modulo π on $\mathbf{N}(T)$. So, we set $\mathbf{N}\big(\mathbb{Q}_p(n)\big) = \mathbf{B}_{R_0}^+ \pi^{-n} e_n$. Similarly,

$$\mathcal{O}\mathbf{D}_{\mathrm{cris}}\big(\mathbb{Q}_p(n)\big) = \left(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n)\right)^{G_{R_0}} = R_0\left[\tfrac{1}{p}\right] t^{-n} e_n,$$

and $\left(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}\big(\mathbb{Q}_p(n)\big)\right)^{\Gamma_{R_0}} = R_0\left[\frac{1}{p}\right] t^{-n} e_n = \mathcal{O}\mathbf{D}_{\mathrm{cris}}\big(\mathbb{Q}_p(n)\big)$ compatible with Frobenius, filtration and connection on each side. Finally, the map

$$\begin{aligned} \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}\big(\mathbb{Q}_p(n)\big) &\longrightarrow \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}\big(\mathbb{Q}_p(n)\big) \\ t^{-n} e_n &\longmapsto \tfrac{\pi^n}{t^n} \pi^{-n} e_n. \end{aligned}$$

is trivially an isomorphism compatible with Frobenius, filtration and the action of Γ_{R_0}, since $\frac{\pi^n}{t^n} \in \mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ are units for $n \in \mathbb{Z}$ (see Lemma 2.43). This proves the lemma. ∎

Remark 3.29. Note that for $T = \mathbb{Z}_p\big(\eta_{\mathrm{f}}\eta_{\mathrm{ur}}\big)$ or $\mathbb{Z}_p(n)$, we even have an isomorphism on the integral level

$$\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \left(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)\right)^{\Gamma_{R_0}} \xrightarrow{\simeq} \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T).$$

One-dimensional Wach representations

In this section we will explicitly state Wach module associated to a one-dimensional representation, and prove a statement useful for the proof of Proposition 3.31. Recall from Defintion 3.8 that a Wach representation is a de Rham representation with additional structure.

Note that the structure of one-dimensional de Rham representations of G_{R_0} is well-known (see [Bri08, §8.6]). From Proposition 1.30 we have that given $\eta : G_{R_0} \longrightarrow \mathbb{Z}_p^\times$, a continuous character, the p-adic representation $V = \mathbb{Q}_p(\eta)$ is de Rham if and only if we can write $\eta = \eta_{\mathrm{f}}\eta_{\mathrm{ur}}\chi^n$ for $n \in \mathbb{Z}$, where η_{f} is a finite character, η_{ur} is an unramified character taking values in $1 + p\mathbb{Z}_p$ and trivialized by an element $\alpha \in 1 + p\widehat{R_0^{\mathrm{ur}}}$, and χ is the p-adic cyclotomic character. We recall that a p-adic representation of G_{R_0} is unramified if the action of G_{R_0} factorizes through the quotient $G_{R_0}^{\mathrm{ur}}$ (see §1.5).

First, let $\eta_{\mathrm{f}} : G_{R_0} \longrightarrow \mathbb{Z}_p^\times$ be a finite Wach character, i.e. a finite de Rham character satisfying the properties of Definition 3.8. Let $T = \mathbb{Z}_p(\eta_{\mathrm{f}})$ and $V = \mathbb{Z}_p(\eta_{\mathrm{f}})$. Then V has single Hodge-Tate weight which is equal to 0. Furthermore, we have the Wach module $\mathbf{N}(T)$, and from Corollary 3.13 we obtain that $\mathbf{N}(T) = \mathbf{D}^+(T) = (\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T)^{H_{R_0}}$. From the conditions of Definition 3.8, we have an isomorphism of projective $\mathbf{A}_{R_0}^+$-modules $\mathbf{N}(T) \simeq \varphi^*(\mathbf{N}(T))$. Finally, the action of Γ_{R_0} is trivial over $\mathbf{N}(T)/\pi\mathbf{N}(T)$ and there exists a finite étale algebra R_0' over R_0 such that $\mathbf{A}_{R_0'}^+ \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)$ is a free $\mathbf{A}_{R_0'}^+$-module of rank 1.

Next, let $\eta_{\mathrm{ur}} : G_{R_0} \longrightarrow \mathbb{Z}_p^\times$ be an unramified character taking values in $1 + p\mathbb{Z}_p$ and trivialized by an element $\alpha \in 1 + p\widehat{R_0^{\mathrm{ur}}}$. Set $T = \mathbb{Z}_p(\eta_{\mathrm{ur}}) = \mathbb{Z}_p e$ and $V = \mathbb{Q}_p(\eta_{\mathrm{ur}})$. Then from Lemma 3.28, we have that V is a Wach representation of Hodge-Tate weight 0, and we can take $\mathbf{N}\big(\mathbb{Z}_p(\eta_{\mathrm{ur}})\big) = \mathbf{D}^+\big(\mathbb{Z}_p(\eta_{\mathrm{ur}})\big) = \mathbf{A}_{R_0}^+ \alpha e$, which is a free $\mathbf{A}_{R_0}^+$-module of rank 1.

Finally, let $\eta = \chi$, the p-adic cyclotomic character and $T = \mathbb{Z}_p(\chi^n) = \mathbb{Z}_p(n) = \mathbb{Z}_p e_n$ and $V =$

$\mathbb{Q}_p(\chi^n) = \mathbb{Q}_p(n)$. Then V is a Wach representation with single Hodge-Tate weight $n \in \mathbb{Z}$. In this case, from Lemma 3.28 we have $\mathbf{N}(\mathbb{Z}_p(n)) = \mathbf{A}_{R_0}^+ \pi^{-n} e_n$.

Lemma 3.30. *Let $\eta : G_{R_0} \longrightarrow \mathbb{Z}_p^\times$ be a continuous character such that the p-adic representation $V = \mathbb{Q}_p(\eta)$ is a Wach representation, with $\mathbf{N}(V)$ the associated Wach module over $\mathbf{A}_{R_0}^+$. Then we have an isomorphism of $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}\left[\frac{1}{p}\right]$-modules $\varphi^*\left(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V)\right) \simeq \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V)$.*

Proof. From the discussion above we can write $\eta = \eta_{\mathrm{f}}\eta_{\mathrm{ur}}\chi^n$ for some $n \in \mathbb{Z}$, and where η_{f} is a finite character, η_{ur} is an unramified character taking values in $1 + p\mathbb{Z}_p$ and trivialized by an element $\alpha \in 1 + p\widehat{R_0^{\mathrm{ur}}}$, and χ is a the p-adic cyclotomic character. In particular, we have $T = \mathbb{Z}_p(\eta) = \mathbb{Z}_p(\eta_{\mathrm{f}}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\eta_{\mathrm{ur}}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\chi^n)$, therefore by Lemma 3.15 we obtain that $\mathbf{N}(T) = \mathbf{N}(\mathbb{Z}_p(\eta_{\mathrm{f}})) \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(\mathbb{Z}_p(\eta_{\mathrm{ur}})) \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(\mathbb{Z}_p(\chi^n))$. So it is enough to show the claim for η_{f}, η_{ur} and χ^n separately.

Now $T = \mathbb{Z}_p(\eta_{\mathrm{f}})$ and $\mathbb{Z}_p(\eta_{\mathrm{ur}})$ the claim is trivial as we have $\varphi^*(\mathbf{N}(T)) \simeq \mathbf{N}(T)$ as $\mathbf{A}_{R_0}^+$-modules from the discussion above.

For $T = \mathbb{Z}_p(\chi^n)$, we see that $\varphi^*(\mathbf{N}(T)) = q^{-n}\mathbf{N}(T)$, where $q = \frac{\varphi(\pi)}{\pi}$. Recall that we have $q = p\varphi\left(\frac{\pi}{t}\right)\frac{t}{\pi}$ and $\frac{t}{\pi}$ is a unit in $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ (see Lemma 2.43). Therefore, for $V = \mathbb{Q}_p(\chi^n)$ we obtain that $\varphi^*\left(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V)\right) \simeq \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V)$, proving the claim. ■

3.2.3. From (φ, Γ)-modules to (φ, ∂)-modules

The objective of this section is to prove the following statement:

Proposition 3.31. *Let V be an h-dimensional positive Wach representation of G_{R_0}, $T \subset V$ a free $\mathbb{Z}_p$-lattice of rank h stable under the action of G_{R_0} and $\mathbf{N}(T)$ the associated Wach module. Then $M := \left(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)\right)^{\Gamma_{R_0}}$ is a finitely generated R_0-module contained in $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$. Moreover, $M\left[\frac{1}{p}\right]$ is a finitely generated projective $R_0\left[\frac{1}{p}\right]$-module of rank h and the natural inclusion*

$$\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} M\left[\tfrac{1}{p}\right] \longrightarrow \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V),$$

is an isomorphism compatible with Frobenius, filtration, connection and the action of Γ_{R_0}. Finally, if we assume $\mathbf{N}(T)$ to be free over $\mathbf{A}_{R_0}^+$ then there exists a free R_0-module $M_0 \subset M$ such that $M_0\left[\frac{1}{p}\right] = M\left[\frac{1}{p}\right]$ are free modules of rank h over $R_0\left[\frac{1}{p}\right]$.

Proof. We will use the notation of Definition 3.8 without repeating them. The first claim is easy to establish. Since we have $H_{R_0} = \mathrm{Gal}\left(\overline{R}\left[\frac{1}{p}\right]/R_\infty\left[\frac{1}{p}\right]\right)$, therefore we can write

$$\begin{aligned}
M = \left(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)\right)^{\Gamma_{R_0}} &\subset \left(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{D}^+(T)\right)^{\Gamma_{R_0}} \subset \left(\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0)^{H_{R_0}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{D}^+(T)\right)^{\Gamma_{R_0}} \\
&\subset \left(\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0)^{H_{R_0}} \otimes_{\mathbf{A}_{R_0}^+} \left(\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T\right)^{H_{R_0}}\right)^{\Gamma_{R_0}} \subset \left(\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0) \otimes_{\mathbb{Z}_p} T\right)^{G_{R_0}} \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V).
\end{aligned} \tag{3.1}$$

The module $\left(\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0) \otimes_{\mathbb{Z}_p} T\right)^{G_{R_0}}$ is finitely generated over R_0. Since R_0 is Noetherian, M is finitely generated.

Independently, we have that $R_0\left[\frac{1}{p}\right]$ is Noetherian and $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is a finitely generated $R_0\left[\frac{1}{p}\right]$-module, therefore $M\left[\frac{1}{p}\right] \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is finitely generated over $R_0\left[\frac{1}{p}\right]$. Moreover, the module $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)$ is equipped with an $\mathbf{A}_R^{\mathrm{PD}}$-linear and integrable connection $\partial_N = \partial \otimes 1$, where ∂ is the connection on $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ described after Lemma 3.23. Therefore, we can consider the induced connection on $M\left[\frac{1}{p}\right]$, which is integrable since it is integrable over $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)$. This connection is compatible with the one on $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ since the connection over $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ is induced from the connection over $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0)$. So by [Bri08, Proposition 7.1.2] we obtain that $M\left[\frac{1}{p}\right]$ must be projective of rank $\leq h$. Further, the inclusion $M\left[\frac{1}{p}\right] \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is compatible with natural Frobenius on each module since all the inclusions in (3.1) are compatible with Frobenius.

Next, we will show that the rank of $M\big[\frac{1}{p}\big]$ as a projective $R_0\big[\frac{1}{p}\big]$-module is exactly h. It is enough to show that the rank is h after a finite étale extension of R_0. Let us consider R_0' to be the p-adic completion of a finite étale extension of R_0 such that the corresponding scalar extension $\mathbf{A}_{R_0'}^+ \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)$ is a free module of rank h (see Definition 3.8) and $R_0'\big[\frac{1}{p}\big]/R_0\big[\frac{1}{p}\big]$ is Galois. The discussion of previous chapters hold for R_0' (see [Bri08, Chapitre 2] and [AI08, §2] for more on this). In particular, let $R' = R_0'(\zeta_{p^m})$ and we have rings $\mathbf{A}_{R_0'}^+$, $\mathbf{A}_R^+$, $\mathbf{A}_{R'}^{\mathrm{PD}}$ and $\mathcal{O}\mathbf{A}_{R'}^{\mathrm{PD}}$. Let $R_\infty'\big[\frac{1}{p}\big]$ denote the cyclotomic tower over $R_0'\big[\frac{1}{p}\big]$ and

$$\Gamma_{R_0'} = \mathrm{Gal}\big(R_\infty'\big[\tfrac{1}{p}\big]/R_0'\big[\tfrac{1}{p}\big]\big) \ \text{ and } \ H_{R_0'} = \mathrm{Ker}\,(G_{R_0'} \longrightarrow \Gamma_{R_0'}).$$

Similarly, we have Galois groups $\Gamma_{R'}$ and $H_{R'}$. Let

$$G' := \mathrm{Gal}\big(R_\infty'\big[\tfrac{1}{p}\big]/R_\infty\big[\tfrac{1}{p}\big]\big) = \mathrm{Gal}\big(R'\big[\tfrac{1}{p}\big]/R\big[\tfrac{1}{p}\big]\big) = \mathrm{Gal}\big(R_0'\big[\tfrac{1}{p}\big]/R_0\big[\tfrac{1}{p}\big]\big),$$

then we have that $H_R/H_{R'} = H_{R_0}/H_{R_0'} = G'$. So we obtain that

$$\begin{aligned} \mathbf{A}_{R_0}^+ &= (\mathbf{A}^+)^{H_{R_0}} = \big((\mathbf{A}^+)^{H_{R_0'}}\big)^{H_{R_0}/H_{R_0'}} = \big(\mathbf{A}_{R_0'}^+\big)^{G'} \\ \mathbf{A}_R^+ &= (\mathbf{A}^+)^{H_R} = \big((\mathbf{A}^+)^{H_{R'}}\big)^{H_R/H_{R'}} = \big(\mathbf{A}_{R'}^+\big)^{G'}. \end{aligned}$$

From these equalities and the description of the action of Γ_{R_0} on $\xi = \frac{\pi}{\pi_1}$, it is clear that

$$\mathbf{A}_R^{\mathrm{PD}} = \big(\mathbf{A}_{R'}^{\mathrm{PD}}\big)^{G'}, \ \text{ and therefore } \ \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} = \big(\mathcal{O}\mathbf{A}_{R'}^{\mathrm{PD}}\big)^{G'}.$$

Now, since $\mathbf{N}(T)$ is projective and G' acts trivially on it, we obtain that

$$\begin{aligned} \big(\mathcal{O}\mathbf{A}_{R'}^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0'}^+} \big(\mathbf{A}_{R_0'}^+ \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)\big)\big)^{G'} &= \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T) \\ \big(\mathcal{O}\mathbf{A}_{R'}^{\mathrm{PD}} \otimes_{R_0'} \big(R_0' \otimes_{R_0} M\big[\tfrac{1}{p}\big]\big)\big)^{G'} &= \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} M\big[\tfrac{1}{p}\big]. \end{aligned}$$

In particular, base changing to $\mathbf{A}_{R_0'}^+$ to obtain $\mathbf{N}(T)$ as a free module is harmless. For the convenience in notation, below we will replace R_0' obtained in this manner by R_0 and assume $\mathbf{N}(T)$ to be free over $\mathbf{A}_{R_0}^+$.

In order to show that the rank of $M\big[\frac{1}{p}\big]$ is at least h, we will successively approximate a basis of $\mathbf{N}(T)$ (after scalar extension to $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$) to linearly independent elements of $M\big[\frac{1}{p}\big]$. To carry this out, first we will define several new rings following [Wac96, §B.1] and examine their relation with $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$. After extending scalars, we will approximate the elements of $\mathbf{N}(T)$ with elements invariant under the geometric action of Γ_{R_0}, i.e. Γ_{R_0}'. Finally, we will approximate the elements obtained from the previous step to elements which are invariant under the arithmetic action of Γ_{R_0}, i.e. Γ_F. Note that whereas Γ_{R_0}' is a commutative group, Γ_{R_0} is not. Further, the action of Γ_{R_0}' on the geometric variables involves the element π on which Γ_F acts (see §2.1), therefore it is imperative that we carry out the approximation steps in the order mentioned above.

Auxiliary rings and modules

For $n \in \mathbb{N}$, let us define a p-adically complete ring

$$S_n^{\mathrm{PD}} := \mathbf{A}_{R_0}^+\Big\{\tfrac{\pi}{p^n}, \tfrac{\pi^2}{2!p^{2n}}, \ldots, \tfrac{\pi^k}{k!p^{kn}}, \ldots\Big\}.$$

Let $I_n^{[i]}$ denote the ideal of S_n^{PD} generated by $\frac{\pi^k}{k!p^{kn}}$ for $k \geq i$ and we set

$$\hat{S}_n^{\mathrm{PD}} := \lim_i S_n^{\mathrm{PD}}/I_n^{[i]}.$$

Note that $\widehat{S}_n^{\mathrm{PD}}$ is p-adically complete as well. Further, note that we can write $\varphi(\pi) = (1+\pi)^p - 1 = \pi^p + p\pi x$ for some $x \in \mathbf{A}_F^+$, therefore

$$\begin{aligned}\frac{\varphi(\pi^k)}{k!p^{kn}} &= \frac{(\pi^p + p\pi x)^k}{k!p^{kn}} = \frac{\sum_{i=0}^k \binom{k}{i}\pi^{pi}\cdot(p\pi x)^{k-i}}{k!p^{kn}} \\ &= \sum_{i=0}^k \frac{(k+(p-1)i)!p^{i(n(p-1)-p)}}{i!(k-i)!} \cdot \frac{\pi^{k+(p-1)i}x^{k-i}}{(k+(p-1)i)!p^{(k+(p-1)i)(n-1)}} \in \widehat{S}_{n-1}^{\mathrm{PD}}\end{aligned}$$

Using this, the Frobenius operator on S can be extended to a map $\varphi : \widehat{S}_n^{\mathrm{PD}} \longrightarrow \widehat{S}_{n-1}^{\mathrm{PD}}$, which we will again call Frobenius. The ring $\widehat{S}_n^{\mathrm{PD}}$ readily admits a continuous action of Γ_{R_0} which commutes with the Frobenius.

Lemma 3.32. *The ring $\widehat{S}_0^{\mathrm{PD}}$ is a subring of $\mathbf{A}_R^{\mathrm{PD}}$, and therefore $\varphi^n\big(\widehat{S}_n^{\mathrm{PD}}\big) \subset \mathbf{A}_R^{\mathrm{PD}}$.*

Proof. The first claim is true because we have

$$\pi_1^p \equiv \pi \bmod p\mathbf{A}_K^+, \quad \text{which gives} \quad \pi_1^{p^i} \equiv \pi^{p^{i-1}} \bmod p^i\mathbf{A}_K^+.$$

So for $k \geq p^i$ we can write

$$\frac{\pi^k}{k!} = \frac{\xi^k\pi_1^k}{k!} = \frac{\xi^k}{k!}\pi_1^{k-p^i}\big(\pi^{p^{i-1}} + p^i a\big) = p^i a\pi_1^{k-p^i}\frac{\xi^k}{k!} + p^{i-1}\pi_1^{p^{i-1}}\frac{(k+p^{i-1})!}{k!p^{i-1}}\frac{\xi^{k+p^{i-1}}}{(k+p^{i-1})!} \in p^{i-1}\mathbf{A}_K^{\mathrm{PD}},$$

for some $a \in \mathbf{A}_K^+$. Therefore, we get that $I_0^{[p^i]} \subset p^{i-1}\mathbf{A}_R^{\mathrm{PD}}$ and hence $\widehat{S}_0^{\mathrm{PD}} \subset \mathbf{A}_R^{\mathrm{PD}}$. The second claim is obvious. ■

In the relative setting, we need slightly larger rings. Let us consider the W-linear homomorphism of rings

$$\begin{aligned}\iota : R_0 &\longrightarrow \widehat{S}_n^{\mathrm{PD}} \\ X_j &\longmapsto [X_j^\flat] \ \text{ for } 1 \leq j \leq d.\end{aligned}$$

Using ι we can define a W-linear morphism of rings

$$\begin{aligned}f : R_0 \otimes_W \widehat{S}_n^{\mathrm{PD}} &\longrightarrow \widehat{S}_n^{\mathrm{PD}} \\ a \otimes b &\longmapsto \iota(a)b.\end{aligned}$$

Let $\mathcal{O}\widehat{S}_n^{\mathrm{PD}}$ denote the p-adic completion of the divided power envelope of $R_0 \otimes_W \widehat{S}_n^{\mathrm{PD}}$ with respect to Ker f. Further, the morphism f extends uniquely to a continuous morphism $f : \mathcal{O}\widehat{S}_n^{\mathrm{PD}} \longrightarrow \widehat{S}_n^{\mathrm{PD}}$. Now, it easily follows from the discussion in §2.5 that the kernel of the morphism f is generated by $(V_1 - 1, \ldots, V_d - 1)$, where $V_j = \frac{1\otimes[X_j^\flat]}{X_j \otimes 1}$ for $1 \leq j \leq d$.

The Frobenius operator extends to $\mathcal{O}\widehat{S}_n^{\mathrm{PD}}$ as well as the continuous action of Γ_{R_0}. From the discussion above we have $\varphi^n(\widehat{S}_n^{\mathrm{PD}}) \subset \widehat{S}_0^{\mathrm{PD}} \subset \mathbf{A}_R^{\mathrm{PD}}$, and following the description of $\mathcal{O}\widehat{S}_0^{\mathrm{PD}}$ §2.5 and of $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ from Remark 3.20, we obtain that

$$\mathcal{O}\widehat{S}_0^{\mathrm{PD}} \subset \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \quad \text{and} \quad \varphi^n\big(\mathcal{O}\widehat{S}_n^{\mathrm{PD}}\big) \subset \mathcal{O}\mathbf{A}_R^{\mathrm{PD}}.$$

Moreover, we have a canonical inclusion of $\widehat{S}_n^{\mathrm{PD}} \subset \mathcal{O}\widehat{S}_n^{\mathrm{PD}}$ compatible with all the structures.

Recall that we have $m \in \mathbb{N}_{\geq 1}$ such that $K = F(\zeta_{p^m})$, so below we will consider the ring $\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$. Now consider the ideal

$$J := \big(\tfrac{\pi}{p^m}, V_1 - 1, \ldots, V_d - 1\big) \subset \mathcal{O}\widehat{S}_m^{\mathrm{PD}},$$

and its divided power

$$J^{[i]} := \left\langle \tfrac{\pi^{[k_0]}}{p^{nk_0}} \prod_{j=1}^{d} (V_j - 1)^{[k_j]},\ \mathbf{k} = (k_0, k_1, \dots, k_d) \in \mathbb{N}^{d+1} \text{ such that } \sum_{j=0}^{d} k_j \geq i \right\rangle \subset \mathcal{O}\hat{S}_m^{\mathrm{PD}}.$$

By the construction of $\mathcal{O}\hat{S}_m^{\mathrm{PD}}$, it is clear that this ring is p-adically complete with respect to the PD-ideal $J^{[i]}$. In other words, the series $\sum_{\mathbf{k} \in \mathbb{N}^{d+1}} x_{\mathbf{k}} \frac{\pi^{[k_0]}}{p^{nk_0}} \prod_{j=1}^{d} (V_j - 1)^{[k_j]}$, where $x_{\mathbf{k}} \in \hat{S}_m^{\mathrm{PD}}$ goes to 0 as $|\mathbf{k}| = \sum_j k_j \longrightarrow +\infty$, converges in $\mathcal{O}\hat{S}_m^{\mathrm{PD}}$.

Next, we set

$$\mathcal{O}N^{\mathrm{PD}} := \mathcal{O}\hat{S}_m^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T).$$

Again, $\mathcal{O}N^{\mathrm{PD}}$ is p-adically complete and it is equipped with a Frobenius-semilinear operator φ and a continuous and semilinear action of Γ_{R_0}. Also, we take

$$M' := \left(\mathcal{O}N^{\mathrm{PD}}\right)^{\Gamma'_{R_0}} \text{ and } M'' := (M')^{\Gamma_F} = \left(\mathcal{O}N^{\mathrm{PD}}\right)^{\Gamma_{R_0}}.$$

Since we assumed $\mathbf{N}(T)$ to be free, we have that $\mathcal{O}N^{\mathrm{PD}}$ is a free $\mathcal{O}\hat{S}_m^{\mathrm{PD}}$-module of rank h. Since $\varphi^m\left(\mathcal{O}\hat{S}_m^{\mathrm{PD}}\right) \subset \mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$, we get that $\varphi^m(M'') \subset \left(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)\right)^{\Gamma_{R_0}}$. Therefore, it is enough to successively approximate an element of $\mathbf{N}(T)$ to an element of M''. Let $\{\gamma, \gamma_1, \dots, \gamma_d\}$ be a set of topological generators of Γ_{R_0} such that $\{\gamma_1, \dots, \gamma_d\}$ generate Γ'_{R_0} topologically, and γ is a lift of a topological generator of Γ_F such that $\gamma^e = \gamma_0$ is a lift of a topological generator of Γ_K and $e = [K : F]$ (see §2.1).

Geometric part of Γ_{R_0}

Lemma 3.33. *For any $x \in \mathbf{N}(T)$, there exists $x' \in \mathcal{O}N^{\mathrm{PD}}$ such that*

$$\begin{aligned} x' &\equiv x \mod J^{[1]}\mathcal{O}N^{\mathrm{PD}}, \\ \gamma_i(x') &= x' \quad \text{for } 1 \leq i \leq d. \end{aligned}$$

In particular, $x' \in M'$.

Proof. We will successively approximate $x \in \mathbf{N}(T)$ to an element $x' \in M'$ by adding elements from $J^{[n]}\mathcal{O}N^{\mathrm{PD}}$, for $n \geq 1$ converging for the p-adic topology and such that the action of Γ'_{R_0} converges to identity.

We start by setting $x_1 := x \in \mathbf{N}(T) \subset \mathcal{O}N^{\mathrm{PD}}$ so that we have $\gamma_s(x_1) = x_1 + \pi y_s$ for some $y_s \in \mathbf{N}(T) \subset J^{[0]}\mathcal{O}N^{\mathrm{PD}}$. Next, let

$$x_2 := x_1 + (V_1 - 1)z_1 + \cdots + (V_d - 1)z_d,$$

where $z_s \in \mathbf{N}(T)$ for $1 \leq s \leq d$, which we need to determine. Clearly, we have that $x_2 \equiv x_1 \mod J^{[1]}\mathcal{O}N^{\mathrm{PD}}$. Now note that

$$\gamma_s(V_s - 1) = (1 + \pi)(V_s - 1 + 1) - 1 \equiv V_s - 1 + \pi \mod \pi J^{[1]}\mathcal{O}\hat{S}_m^{\mathrm{PD}}, \tag{3.2}$$

and since we must have $\gamma_s(z_t) = z_t \mod \pi\mathbf{N}(T)$ for $1 \leq s, t \leq d$, therefore the action of γ_s on x_2 can be given as

$$\begin{aligned} \gamma_s(x_2) &= \gamma_s(x_1) + (V_1 - 1)\gamma_s(z_1) + \cdots + \gamma_s(V_s - 1)\gamma_s(z_s) + \cdots + (V_d - 1)\gamma_s(z_d) \\ &\equiv x_1 + \pi y_s + (V_1 - 1)z_1 + \cdots + (V_s - 1 + \pi)z_s + \cdots + (V_d - 1)z_d \mod \pi J^{[1]}\mathcal{O}N^{\mathrm{PD}} \\ &\equiv x_2 + \pi(y_s + z_s) \mod \pi J^{[1]}\mathcal{O}N^{\mathrm{PD}}. \end{aligned}$$

Setting $z_s \equiv -y_s$ for $1 \leq s \leq d$, we obtain that $\gamma_s(x_2) \equiv x_2 \mod \pi J^{[1]}\mathcal{O}N^{\mathrm{PD}}$.

Now we will proceed inductively over n, i.e. we will show that for $n \geq 2$, if there exists $x_n \in \mathcal{O}N^{\mathrm{PD}}$ such that

$$\begin{aligned} x_n &\equiv x_{n-1} \mod J^{[n-1]}\mathcal{O}N^{\mathrm{PD}}, \\ \gamma_s(x_n) &\equiv x_n \mod \pi J^{[n-1]}\mathcal{O}N^{\mathrm{PD}} \text{ for } 1 \leq s \leq d, \end{aligned}$$

then there exists $x_{n+1} \in \mathcal{O}N^{\mathrm{PD}}$ such that

$$\begin{aligned} x_{n+1} &\equiv x_n \mod J^{[n]}\mathcal{O}N^{\mathrm{PD}}, \\ \gamma_s(x_{n+1}) &\equiv x_{n+1} \mod \pi J^{[n]}\mathcal{O}N^{\mathrm{PD}} \text{ for } 1 \leq s \leq d. \end{aligned}$$

For $n \in \mathbb{N}$, let us define the multi index set $\Lambda_n := \{\mathbf{k} = (k_0, k_1, \dots, k_d) \in \mathbb{N}^{d+1},$ such that $k_0 + \cdots + k_d = n\}$. We set

$$x_{n+1} := x_n + \sum_{\mathbf{i} \in \Lambda_n} \tfrac{\pi^{[i_0]}}{p^{mi_0}}(V_1 - 1)^{[i_1]} \cdots (V_d - 1)^{[i_d]} z_{\mathbf{i}}, \tag{3.3}$$

for some $z_{\mathbf{i}} \in \mathbf{N}(T)$, which we need to determine. We will solve for $z_{\mathbf{i}}$ by studying the action of γ_s on x_{n+1} for $1 \leq s \leq d$. For the action of γ_s on x_n, we have

$$\gamma_s(x_n) = x_n + \pi \sum_{|\mathbf{k}| \geq n-1} \tfrac{\pi^{[k_0]}}{p^{mk_0}}(V_1 - 1)^{[k_1]} \cdots (V_d - 1)^{[k_d]} y_{\mathbf{k}}^{(s)},$$

where $|\mathbf{k}| = \sum_j k_j$ and $y_{\mathbf{k}}^{(s)} \in \mathbf{N}(T)$ goes to zero p-adically as $|\mathbf{k}| \longrightarrow +\infty$. Truncating the equation above for $|\mathbf{k}| \geq n$, we obtain

$$\gamma_s(x_n) \equiv x_n + \pi \sum_{\mathbf{k} \in \Lambda_{n-1}} \tfrac{\pi^{[k_0]}}{p^{mk_0}}(V_1 - 1)^{[k_1]} \cdots (V_d - 1)^{[k_d]} y_{\mathbf{k}}^{(s)} \mod \pi J^{[n]}\mathcal{O}N^{\mathrm{PD}}. \tag{3.4}$$

To determine x_{n+1}, we begin with $s = 1$. From (3.2), recall that

$$\gamma_1(V_1 - 1) \equiv (V_1 - 1 + \pi) \mod \pi J^{[1]}\mathcal{O}\hat{S}_m^{\mathrm{PD}},$$

and since we must have $\gamma_1(z_{\mathbf{i}}) = z_{\mathbf{i}} \mod \pi\mathbf{N}(T)$, therefore the action of γ_1 on x_{n+1} using (3.4) can be given as

$$\begin{aligned} \gamma_1(x_{n+1}) \equiv{}& x_n + \pi \sum_{\mathbf{k} \in \Lambda_{n-1}} \tfrac{\pi^{[k_0]}}{p^{mk_0}}(V_1 - 1)^{[k_1]} \cdots (V_d - 1)^{[k_d]} y_{\mathbf{k}}^{(1)} \\ &+ \sum_{\mathbf{i} \in \Lambda_n} \tfrac{\pi^{[i_0]}}{p^{mi_0}}(V_1 - 1 + \pi)^{[i_1]}(V_2 - 1)^{[i_2]} \cdots (V_d - 1)^{[i_d]} z_{\mathbf{i}} \mod \pi J^{[n]}\mathcal{O}N^{\mathrm{PD}} \\ \equiv{}& x_{n+1} + \pi \sum_{\mathbf{k} \in \Lambda_{n-1}} \tfrac{\pi^{[k_0]}}{p^{mk_0}}(V_1 - 1)^{[k_1]} \cdots (V_d - 1)^{[k_d]} y_{\mathbf{k}}^{(1)} \\ &+ \sum_{\mathbf{i} \in \Lambda_n} \tfrac{\pi^{[i_0]}}{p^{mi_0}}\big((V_1 - 1 + \pi)^{[i_1]} - (V_1 - 1)^{[i_1]}\big)(V_2 - 1)^{[i_2]} \cdots (V_d - 1)^{[i_d]} z_{\mathbf{i}} \mod \pi J^{[n]}\mathcal{O}N^{\mathrm{PD}}. \end{aligned}$$

For $\mathbf{k} = (k_0, \dots, k_d) \in \Lambda_{n-1}$, i.e. $k_0 + \cdots + k_d = n - 1$, the coefficient of $\pi^{k_0+1}(V_1 - 1)^{k_1} \cdots (V_d - 1)^{k_d}$ in the right side of the congruence above is given by the expression

$$\begin{aligned}&\frac{y^{(1)}_{\mathbf{k}}}{\mathbf{k}!p^{mk_0}}+\binom{k_1+k_0+1}{k_1}\frac{z_{(0,k_1+k_0+1,k_2,\dots,k_d)}}{(k_1+k_0+1)!k_2!\cdots k_d!}+\binom{k_1+k_0}{k_1}\frac{z_{(1,k_1+k_0,k_2,\dots,k_d)}}{(k_1+k_0)!k_2!\cdots k_d!p^m}\\&\quad+\binom{k_1+k_0-1}{k_1}\frac{z_{(2,k_1+k_0-1,k_2,\dots,k_d)}}{2!(k_1+k_0-1)!k_2!\cdots k_d!p^{2m}}+\cdots+\\&\quad+\cdots+\binom{k_1+2}{k_1}\frac{z_{(k_0-1,k_1+2,k_2,\dots,k_d)}}{(k_0-1)!(k_1+2)!k_2!\cdots k_d!p^{m(k_0-1)}}+\binom{k_1+1}{k_1}\frac{z_{(k_0,k_1+1,k_2,\dots,k_d)}}{k_0!(k_1+1)!k_2!\cdots k_d!p^{mk_0}},\end{aligned}\tag{3.5}$$

where $\mathbf{k}! = k_0!\cdots k_d!$. To write more succinctly, we set

$$\Lambda^{(1)}_{n,\mathbf{k}} := \left\{\mathbf{j}=(j_0,j_1,\dots,j_d),\ \text{such that } 0\le j_0\le k_0,\ j_1=k_1+k_0+1-j_0,\ j_2=k_2,\ \dots,\ j_d=k_d\right\}\subset\Lambda_n,$$

and therefore, the summation in (3.5) can be expressed as

$$\frac{y^{(1)}_{\mathbf{k}}}{\mathbf{k}!p^{mk_0}}+\sum_{j_0=0}^{k_0}\binom{j_1}{k_1}\frac{z_{\mathbf{j}}}{\mathbf{j}!p^{mj_0}},\tag{3.6}$$

where we have $\mathbf{j}! = j_0!\cdots j_d!$ and the summation runs over indices in $\Lambda^{(1)}_{n,\mathbf{k}}$. To get $\gamma_1(x_{n+1})\equiv x_{n+1}$ mod $\pi J^{[n]}\mathcal{O}N^{\mathrm{PD}}$, it is enough to have the summation in (3.6) belong to $\pi J^{[n]}\mathcal{O}N^{\mathrm{PD}}$ for each $\mathbf{k}\in\Lambda_{n-1}$. We take $\mathbf{i}=(k_0,k_1+1,k_2,\dots,k_d)\in\Lambda^{(1)}_{n,\mathbf{k}}$ and putting (3.6) congruent to 0 modulo $\pi J^{[n]}\mathcal{O}N^{\mathrm{PD}}$ and simplifying the expression, we get a congruence relation

$$z_{\mathbf{i}}\equiv-\left(y^{(1)}_{\mathbf{k}}+\sum_{j_0=0}^{k_0-1}\frac{k_0!p^{m(k_0-j_0)}}{j_0!(k_0+1-j_0)!}z_{\mathbf{j}}\right)\mod \pi\mathbf{N}(T),\tag{3.7}$$

where the summation runs over indices in $\Lambda^{(1)}_{n,\mathbf{k}}\setminus\{\mathbf{i}\}$. Since $j_0<k_0$ in (3.7), we see that the coefficients of $z_{\mathbf{j}}$ appearing in the summation above can be re-written as $\binom{k_0}{j_0}\frac{p^{m(k_0-j_0)}}{k_0+1-j_0}$ which has non-negative p-adic valuation (positive p-adic valuation for $p\ge3$). So from (3.7), we obtain an expression for $z_{\mathbf{i}}$ in terms of $z_{\mathbf{j}}$ such that $\mathbf{j}<\mathbf{i}$ lexicographically. Here by lexicographic ordering we mean that for $\mathbf{j},\mathbf{j}'\in\Lambda_n$, we have $\mathbf{j}<\mathbf{j}'$ if and only if $j_0<j_0'$, or $j_0=j_0'$ and $j_1<j_1'$, or $j_0=j_0'$, $j_1=j_1'$ and $j_2<j_2'$, and so on.

To determine $z_{\mathbf{i}}$ modulo $\pi\mathbf{N}(T)$ for $\mathbf{i}\in\Lambda_n$ such that $i_1\neq0$, we will proceed by lexicographic induction over the index $\mathbf{i}$. For the base case we have $\mathbf{i}=(0,i_1,i_2,\dots,i_d)$ for $1\le i_1\le n$ and $i_1+\cdots+i_d=n$, so taking $\mathbf{k}=(0,i_1-1,i_2,\dots,i_d)$, from (3.7) we obtain

$$z_{(0,i_1,i_2,\dots,i_d)}\equiv-y^{(1)}_{(0,i_1-1,i_2,\dots,i_d)}\mod \pi\mathbf{N}(T).$$

Lexicographically, next we have $\mathbf{i}=(1,i_1,i_2,\dots,i_d)$ for $1\le i_1\le n-1$ and $1+i_1+\cdots+i_d=n$. Then we take $\mathbf{k}=(1,i_1-1,i_2,\dots,i_d)$ and obtain that $\Lambda^{(1)}_{n,\mathbf{k}}=\{(0,i_1+1,i_2,\dots,i_d),\ (1,i_1,i_2,\dots,i_d)\}$. Since $(0,i_1+1,i_2,\dots,i_d)<(1,i_1,i_2,\dots,i_d)$, from (3.7) we obtain the value of $z_{(1,i_1,i_2,\dots,i_d)}$. For the induction step, let $\mathbf{i}=(i_0,i_1,i_2,\dots,i_d)\in\Lambda_n$, such that $i_1\neq0$ and $i_0+\cdots+i_d=n$. Then we take $\mathbf{k}=(i_0,i_1-1,i_2,\dots,i_d)\in\Lambda_{n-1}$ so that we have $\mathbf{i}\in\Lambda^{(1)}_{n,\mathbf{k}}$ and $\mathbf{j}<\mathbf{i}$ for all $\mathbf{j}\in\Lambda^{(1)}_{n,\mathbf{k}}\setminus\{\mathbf{i}\}$ as $j_0<k_0=i_0$. Plugging this value of $\mathbf{k}$ in the computation above and in particular, from (3.7) we obtain the value of $z_{\mathbf{i}}$ modulo $\pi\mathbf{N}(T)$ by induction.

Next, we will repeat the computation above for the action of γ_s on x_n in (3.4) for $2\le s\le d$. Let $\mathbf{i}=(i_0,\dots,i_s,\dots,i_d)$ such that $i_s\neq0$, $\mathbf{k}=(i_0,\dots,i_s-1,\dots,i_d)$,

$$\Lambda^{(s)}_{n,\mathbf{k}} := \{\mathbf{j}=(j_0,\dots,j_d),\ \text{such that } 0\le j_0\le k_0,\ j_1=k_1,\ \dots,\ j_s=k_s+k_0+1-j_0,\ \dots,\ j_d=k_d\}\subset\Lambda_n,$$

and we set

$$z_{\mathbf{i}} \equiv -\Big(y_{\mathbf{k}}^{(s)} + \sum_{j_0=0}^{k_0-1} \frac{k_0!p^{m(k_0-j_0)}}{j_0!(k_0+1-j_0)!} z_{\mathbf{j}}\Big) \mod \pi\mathbf{N}(T), \tag{3.8}$$

where the summation runs over indices in $\Lambda_{n,\mathbf{k}}^{(s)}$. For the base case we have $\mathbf{i} = (0, i_1, \dots, i_s, \dots, i_d)$ for $1 \leq i_s \leq n$ and $i_1 + \cdots + i_d = n$. So taking $\mathbf{k} = (0, i_1, \dots, i_s - 1, \dots, i_d)$, from (3.8) we obtain

$$z_{(0,i_1,\dots,i_s,\dots,i_d)} \equiv -y_{(0,i_1,\dots,i_s-1,\dots,i_d)}^{(s)} \mod \pi\mathbf{N}(T).$$

Lexicographically, next we have $\mathbf{i} = (1, i_1, \dots, i_s, \dots, i_d)$ for $1 \leq i_s \leq n-1$ and $1 + i_1 + \dots + i_d = n$. Then we take $\mathbf{k} = (1, i_1, \dots, i_s - 1, \dots, i_d)$ and obtain that $\Lambda_{n,\mathbf{k}}^{(s)} = \{(0, i_1, \dots, i_s + 1, \dots, i_d),\ (1, i_1, \dots, i_s, \dots, i_d)\}$. Since $(0, i_1, \dots, i_s - 1, \dots, i_d) < (1, i_1, \dots, i_s, \dots, i_d)$, from (3.8) we obtain the value of $z_{(1,i_1,\dots,i_s,\dots,i_d)}$. For the induction step, let $\mathbf{i} = (i_0, \dots, i_d) \in \Lambda_n$, such that $i_s \neq 0$ and $i_0 + \dots + i_d = n$. Then we take $\mathbf{k} = (i_0, i_1, \dots, i_s - 1, \dots, i_d) \in \Lambda_{n-1}$ so that we have $\mathbf{i} \in \Lambda_{n,\mathbf{k}}^{(s)}$ and $\mathbf{j} < \mathbf{i}$ for all $\mathbf{j} \in \Lambda_{n,\mathbf{k}}^{(s)} \setminus \{\mathbf{i}\}$ as $j_0 < k_0 = i_0$. Plugging this value of $\mathbf{k}$ in the computation above and in particular in (3.8), we obtain the value of $z_{\mathbf{i}}$ modulo $\pi\mathbf{N}(T)$ by induction.

From the computation above we obtain solutions for $z_{\mathbf{i}}$ and only when $i_s \neq 0$ for some $s \in \{1, \dots, d\}$. So we set $z_{(n,0,\dots,0)} = 0 \mod \pi\mathbf{N}(T)$. Note that we have

(i) unique value for $z_{\mathbf{i}}$ modulo $\pi\mathbf{N}(T)$ when $i_s \neq 0$ for exactly one $s \in \{1, \dots, d\}$,

(ii) more than one value for $z_{\mathbf{i}}$ modulo $\pi\mathbf{N}(T)$ when $i_s \neq 0$ for more than one $s \in \{1, \dots, d\}$.

Note that our procedure of obtaining a value for $z_{\mathbf{i}}$ modulo $\pi\mathbf{N}(T)$ involves fixing some s such that $i_s \neq 0$, and solving some equations arising from the action of γ_s. For an index $\mathbf{i} \in \Lambda_n$, if $s \neq s'$ such that $i_s \neq 0$ and $i_{s'} \neq 0$, then we obtain more than one value for $z_{\mathbf{i}}$. But from Lemma 3.34 below, we see that these values are in fact, equivalent modulo $\pi\mathbf{N}(T)$. Therefore, the value of x_{n+1} in (3.3) is uniquely determined modulo $\pi J^{[n]}\mathcal{O}N^{\mathrm{PD}}$. Moreover, from the expression obtained for $z_{\mathbf{i}}$ in (3.7), it is clear that p-adically $z_{\mathbf{i}} \longrightarrow 0$ as $|\mathbf{i}| \longrightarrow +\infty$. In conclusion, the sequence x_n converges p-adically to some $x' \in M' = \big(\mathcal{O}N^{\mathrm{PD}}\big)^{\Gamma'_{R_0}}$. ∎

Following conclusion was applied above:

Lemma 3.34. *For each $\mathbf{j} \in \Lambda_n$, multiple values of $z_{\mathbf{j}}$ obtained are congruent modulo $\pi\mathbf{N}(T)$.*

Proof. For a fixed $\mathbf{j} \in \Lambda_n$, we need to show that in case of multiple solutions for $z_{\mathbf{j}}$, we must have that these solutions are equivalent modulo $\pi\mathbf{N}(T)$. To do this, we need to work with all indices at once. So we will consider two sets of solutions $\{z_{\mathbf{i}},\ \mathbf{i} \in \Lambda_n\}$ such that entries in these sets are distinct for indices $\mathbf{i} \in \Lambda_n$ for which we have multiple solutions. Further, our proof will exploit the commutativity of Γ'_{R_0}.

For simplicity in the presentation of the argument, out of d generators of Γ'_{R_0}, we will fix two generators say γ_1 and γ_2. Now, let us denote the first set of solutions $\big\{z_{\mathbf{i}}^{(1)},\ \text{for } \mathbf{i} \in \Lambda_n\big\}$, where for $\mathbf{i} \in \Lambda_n$ such that if $i_1 \neq 0$ we take the solutions obtained from trivializing the action of γ_1 (see (3.7)) and if $i_1 = 0$ we take solutions obtained from trivializing the action of γ_s (see (3.8)) for some $s \in \{2, \dots, d\}$ such that $i_s \neq 0$. Next, we take another set of solution $\big\{z_{\mathbf{i}}^{(2)},\ \text{for } \mathbf{i} \in \Lambda_n\big\}$, where for $\mathbf{i} \in \Lambda_n$ such that if $i_2 \neq 0$ we take the solutions obtained from trivializing the action of γ_2 and if $i_2 = 0$ we take the solutions obtained from trivializing the action of γ_s for some $s \in \{1, \dots, d\}$ such that $i_s \neq 0$. In the second set of solutions, we also impose the condition that in case $i_2 = 0$ and there exist multiple solutions for $z_{\mathbf{i}}$, then we will choose the value of $z_{\mathbf{i}}^{(2)}$ such that it is not the same as $z_{\mathbf{i}}^{(1)}$. Since the only relation between these set of solutions obtained and x_n is given by (3.3), we will construct two different values for x_{n+1}. More precisely, we set

$$\begin{aligned} z^{(1)} &:= \sum_{\mathbf{i}\in\Lambda_n} \frac{\pi^{[i_0]}}{p^{mi_0}}(V_1-1)^{[i_1]}\cdots(V_d-1)^{[i_d]} z_{\mathbf{i}}^{(1)}, \\ z^{(2)} &:= \sum_{\mathbf{i}\in\Lambda_n} \frac{\pi^{[i_0]}}{p^{mi_0}}(V_1-1)^{[i_1]}\cdots(V_d-1)^{[i_d]} z_{\mathbf{i}}^{(2)}. \end{aligned} \tag{3.9}$$

We have $z^{(1)}, z^{(2)} \in J^{[n]}\mathcal{O}N^{\mathrm{PD}}$. This allows us to set

$$\begin{aligned} x_{n+1}^{(1)} &:= x_n + z^{(1)}, \\ x_{n+1}^{(2)} &:= x_n + z^{(2)}. \end{aligned}$$

and we get that

$$\begin{aligned} \gamma_1\left(x_{n+1}^{(1)}\right) &\equiv x_{n+1}^{(1)} \mod \pi J^{[n]}\mathcal{O}N^{\mathrm{PD}}, \\ \gamma_2\left(x_{n+1}^{(2)}\right) &\equiv x_{n+1}^{(2)} \mod \pi J^{[n]}\mathcal{O}N^{\mathrm{PD}}. \end{aligned} \tag{3.10}$$

Further, we simplify the notations and write (3.4) as

$$\begin{aligned} \gamma_1(x_n) &\equiv x_n + \pi y^{(1)} \mod \pi J^{[n]}\mathcal{O}N^{\mathrm{PD}}, \\ \gamma_2(x_n) &\equiv x_n + \pi y^{(2)} \mod \pi J^{[n]}\mathcal{O}N^{\mathrm{PD}}, \end{aligned} \tag{3.11}$$

where it is obvious that $y^{(1)}, y^{(2)} \in J^{[n-1]}\mathcal{O}N^{\mathrm{PD}}$ replace the summations occuring in (3.4). Therefore, from (3.11) we can write

$$\begin{aligned} \gamma_2\gamma_1(x_n) &\equiv x_n + \pi y^{(2)} + \pi\gamma_2\left(y^{(1)}\right) \mod \pi J^{[n]}\mathcal{O}N^{\mathrm{PD}}, \\ \gamma_1\gamma_2(x_n) &\equiv x_n + \pi y^{(1)} + \pi\gamma_1\left(y^{(2)}\right) \mod \pi J^{[n]}\mathcal{O}N^{\mathrm{PD}}. \end{aligned}$$

Since Γ'_R is commutative, we have $\gamma_1\gamma_2 = \gamma_2\gamma_1$, therefore

$$(\gamma_2 - 1)\pi y^{(1)} \equiv (\gamma_1 - 1)\pi y^{(2)} \mod \pi J^{[n]}\mathcal{O}N^{\mathrm{PD}}.$$

Next, combining (3.11) and (3.10), we obtain

$$\begin{aligned} (\gamma_1 - 1)z^{(1)} &\equiv -\pi y^{(1)} \mod \pi J^{[n]}\mathcal{O}N^{\mathrm{PD}}, \\ (\gamma_2 - 1)z^{(2)} &\equiv -\pi y^{(2)} \mod \pi J^{[n]}\mathcal{O}N^{\mathrm{PD}}. \end{aligned}$$

Again, since γ_1 and γ_2 commute, we obtain

$$(\gamma_2 - 1)(\gamma_1 - 1)\left(z^{(1)} - z^{(2)}\right) \equiv 0 \mod \pi J^{[n]}\mathcal{O}N^{\mathrm{PD}}.$$

As $J^{[n]}\mathcal{O}N^{\mathrm{PD}}$ is stable under the action of Γ_{R_0}, applying Corollary 3.36 twice we obtain that

$$z_{\mathbf{i}}^{(1)} \equiv z_{\mathbf{i}}^{(2)} \mod \pi\mathbf{N}(T),$$

for $\mathbf{i} \in \Lambda_n$ such that $i_1, i_2 \neq 0$.

By repeating this argument for each pair of $r, s \in \{1, \dots, d\}$, we conclude that the multiple solutions of $z_{\mathbf{i}}$ for $\mathbf{i} \in \Lambda_n$, are equivalent modulo $\pi\mathbf{N}(T)$. ∎

Let us note a general result, which will be useful later and whose special case (see Corollary 3.36) was used above. Let $x \in J^{[n]}\mathcal{O}N^{\mathrm{PD}}$ and $s \in \{1, \ldots, d\}$. We write

$$x = \sum_{\mathbf{i} \in \Lambda_n} \frac{\pi^{[i_0]}}{p^{m i_0}} (V_1 - 1)^{[i_1]} \cdots (V_d - 1)^{[i_d]} x_{\mathbf{i}},$$

for $x_{\mathbf{i}} \in J^{[1]}\mathcal{O}N^{\mathrm{PD}}$, and $\Lambda_n = \{\mathbf{i} = (i_0, \ldots, i_d),\ \text{such that } i_0 + \cdots + i_d = n\}$. Then,

Lemma 3.35. *Let $s \in \{1, \ldots, d\}$ such that $i_s \neq 0$, then $(\gamma_s - 1)x \equiv 0 \mod J^{[n+1]}\mathcal{O}N^{\mathrm{PD}}$ if and only if $x_{\mathbf{i}} \in J^{[1]}\mathcal{O}N^{\mathrm{PD}}$.*

Proof. First note that $J^{[1]}\mathcal{O}N^{\mathrm{PD}}$ is stable under the action of γ_s for $1 \leq s \leq d$, so we get the "if" statement. For the converse, without loss of generality, we take $s = 1$ and set $\gamma := \gamma_1$. Then we have

$$\begin{aligned}
(\gamma - 1)x &\equiv \sum_{\mathbf{i} \in \Lambda_n} \frac{\pi^{[i_0]}}{p^{m i_0}} \left((V_1 - 1 + \pi)^{[i_1]} - (V_1 - 1)^{[i_1]}\right) \cdots (V_d - 1)^{[i_d]} x_{\mathbf{i}} \mod J^{[n+1]}\mathcal{O}N^{\mathrm{PD}} \\
&\equiv \sum_{\mathbf{i} \in \Lambda_n} \frac{\pi^{[i_0]}}{i_1! p^{m i_0}} (V_2 - 1)^{[i_2]} \cdots (V_d - 1)^{[i_d]} \sum_{k=0}^{i_1 - 1} \binom{i_1}{k} (V_1 - 1)^k \pi^{i_1 - k} x_{\mathbf{i}} \mod J^{[n+1]}\mathcal{O}N^{\mathrm{PD}} \\
&\equiv \sum_{\mathbf{i} \in \Lambda_n} \sum_{k=0}^{i_1 - 1} \binom{i_1}{k} \frac{\pi^{i_0 + i_1 - k}}{i_0! i_1! p^{m i_0}} (V_1 - 1)^k (V_2 - 1)^{[i_2]} \cdots (V_d - 1)^{[i_d]} x_{\mathbf{i}} \mod J^{[n+1]}\mathcal{O}N^{\mathrm{PD}} \\
&\equiv \sum_{\mathbf{i} \in \Lambda_n} \sum_{k=0}^{i_1 - 1} \binom{i_0 + i_1 - k}{i_0} p^{m(i_1 - k)} \frac{\pi^{[i_0 + i_1 - k]}}{p^{m(i_0 + i_1 - k)}} (V_1 - 1)^{[k]} (V_2 - 1)^{[i_2]} \cdots (V_d - 1)^{[i_d]} x_{\mathbf{i}} \mod J^{[n+1]}\mathcal{O}N^{\mathrm{PD}}.
\end{aligned}$$

Let $\mathbf{j} \in \Lambda_n$, then we set $\Lambda_{n,\mathbf{j}} := \{\mathbf{r} = (r_0, \ldots, r_d),\ \text{such that } j_1 \leq r_1 \leq j_0 + j_1, r_0 = j_0 + j_1 - r_1, r_2 = j_2, \ldots, r_d = j_d\}$. So we can write

$$(\gamma - 1)x \equiv \sum_{\substack{\mathbf{j} \in \Lambda_n \\ j_1 < n}} \frac{\pi^{[j_0]}}{p^{m j_0}} (V_1 - 1)^{[j_1]} (V_2 - 1)^{[j_2]} \cdots (V_d - 1)^{[j_d]} \sum_{\mathbf{r} \in \Lambda_{n,\mathbf{j}}} \binom{j_0}{r_0} p^{m(r_1 - j_1)} x_{\mathbf{r}} \mod J^{[n+1]}\mathcal{O}N^{\mathrm{PD}}.$$

To get $(\gamma - 1)x \in J^{[n+1]}\mathcal{O}N^{\mathrm{PD}}$, we can write each $x_{\mathbf{r}} = \sum_{i=1}^{h} x_{\mathbf{r}}^{(i)} e_i$ with $x_{\mathbf{r}}^{(i)} \in \mathcal{O}\hat{S}_m^{\mathrm{PD}}$ and $\{e_1, \ldots, e_h\}$ a chosen basis of $\mathbf{N}(T)$ to obtain for each $1 \leq i \leq h$, the congruence

$$\sum_{\substack{\mathbf{j} \in \Lambda_n \\ j_1 < n}} \frac{\pi^{[j_0]}}{p^{m j_0}} (V_1 - 1)^{[j_1]} (V_2 - 1)^{[j_2]} \cdots (V_d - 1)^{[j_d]} \sum_{\mathbf{r} \in \Lambda_{n,\mathbf{j}}} \binom{j_0}{r_0} p^{m(r_1 - j_1)} x_{\mathbf{r}}^{(i)} \equiv 0 \mod J^{[n+1]}\mathcal{O}\hat{S}_m^{\mathrm{PD}}.$$

Note that in the equation above we have that the first part of the left hand side is in $J^{[n+1]}\mathcal{O}\hat{S}_m^{\mathrm{PD}}$. Now for any two $\mathbf{j}, \mathbf{j}' \in \Lambda_n$, we have that $\mathbf{j} \neq \mathbf{j}'$, so the first part of the congruence for each term is different. Therefore to obtain the congruence above, we must have

$$\sum_{\mathbf{r} \in \Lambda_{n,\mathbf{j}}} \binom{j_0}{r_0} p^{m(r_1 - j_1)} x_{\mathbf{r}}^{(i)} \equiv 0 \mod J^{[1]}\mathcal{O}\hat{S}_m^{\mathrm{PD}}. \tag{3.12}$$

Combining (3.12) for each $1 \leq i \leq h$, we obtain that

$$\sum_{\mathbf{r} \in \Lambda_{n,\mathbf{j}}} \binom{j_0}{r_0} p^{m(r_1 - j_1)} x_{\mathbf{r}} \equiv 0 \mod J^{[1]}\mathcal{O}N^{\mathrm{PD}}. \tag{3.13}$$

From this set of equations, we see that for any $\mathbf{j} \in \Lambda_n$ it is enough to show that $x_{\mathbf{r}} \equiv 0 \mod J^{[1]}\mathcal{O}N^{\mathrm{PD}}$ for each $\mathbf{r} \in \Lambda_{n,\mathbf{j}}$.

We will proceed by lexicographic induction. First note that in the base case we have $\mathbf{j} = (0, j_1, j_2, \ldots, j_d)$ for $0 \leq j_1 \leq n - 1$ and $\Lambda_{n,\mathbf{j}} = \{(0, j_1, j_2, \ldots, j_d)\}$. So from (3.13) we obtain $x_{\mathbf{r}} \equiv 0$

mod $J^{[1]}\mathcal{O}N^{\mathrm{PD}}$. Lexicographically, next we have $\mathbf{j} = (1, j_1, j_2, \ldots, j_d)$ for $0 \leq j_1 \leq n-1$ and $\Lambda_{n,\mathbf{j}} = \{(1, j_1, j_2, \ldots, j_d), (0, j_1+1, j_2, \ldots, j_d)\}$. Let $\mathbf{r} = (0, j_1+1, j_2, \ldots, j_d)$, then from the previous step we have $x_{\mathbf{r}} \equiv 0 \mod J^{[1]}\mathcal{O}N^{\mathrm{PD}}$. Combining this with (3.13) we get that $x_{\mathbf{j}} \equiv 0 \mod J^{[1]}\mathcal{O}N^{\mathrm{PD}}$. For the induction step, let $\mathbf{j} = (j_0, j_1, j_2, \ldots, j_d)$ for $0 \leq j_1 \leq n-1$. Since we have $\mathbf{j} > \mathbf{r}$ for any $\mathbf{r} \in \Lambda_{n,\mathbf{j}} \setminus \{\mathbf{j}\}$, from (3.13) and induction we obtain that $x_{\mathbf{j}} \equiv 0 \mod J^{[1]}\mathcal{O}N^{\mathrm{PD}}$. This finishes the proof. ∎

The result above can be specialized to the following statement:

Corollary 3.36. *Let us assume that $x_{\mathbf{i}} \in \mathbf{N}(T)$ for all $\mathbf{i} \in \Lambda_n$. For $s \in \{1, \ldots, d\}$ such that $i_s \neq 0$, we have $(\gamma_s - 1)x \in \pi J^{[n]}\mathcal{O}N^{\mathrm{PD}}$ if and only if $x_{\mathbf{i}} \in \pi\mathbf{N}(T)$.*

Proof. First note that $\pi\mathbf{N}(T)$ is stable under the action of γ_s for $1 \leq s \leq d$, so we get the "if" statement. For the converse, let $\{e_1, \ldots, e_h\}$ denote an $\mathbf{A}^+_{R_0}$-basis of $\mathbf{N}(T)$, and we write $x_{\mathbf{i}} = \sum_{k=1}^h x_{\mathbf{i}}^{(k)} e_k$. Now, using Lemma 3.35 and the assumption in the claim, we obtain that $x_{\mathbf{i}} \in \mathbf{N}(T) \cap J^{[1]}\mathcal{O}N^{\mathrm{PD}} \subset \mathcal{O}N^{\mathrm{PD}}$. Therefore, we must have $x_{\mathbf{i}}^{(k)} \in \mathbf{A}^+_{R_0} \cap J^{[1]}\mathcal{O}\hat{S}_m^{\mathrm{PD}} \subset \mathcal{O}\hat{S}_m^{\mathrm{PD}}$. By definitions, we have that $\mathbf{A}^+_{R_0} \cap J^{[1]}\mathcal{O}\hat{S}_m^{\mathrm{PD}} = \pi\mathbf{A}^+_{R_0}$. Hence, $x_{\mathbf{i}} = \sum_{k=1}^h x_{\mathbf{i}}^{(k)} e_k \in \pi\mathbf{N}(T)$. ∎

Arithmetic part of Γ_{R_0}

Recall that we have γ as a topological generator of Γ_F such that $\gamma_0 = \gamma^e$ is a topological generator of Γ_K, where $e = [K : F]$. As a second step, we will successively approximate for the action of γ_0 and then obtain an element fixed by γ.

Let us consider the ideal and its divided powers for $n \geq 1$

$$H = \left(\tfrac{\pi}{p^m}\right) \subset \left(\mathcal{O}\hat{S}_m^{\mathrm{PD}}\right)^{\Gamma'_{R_0}},$$
$$H^{[n]} = \left\langle \tfrac{\pi^k}{k!p^{mk}}, k \geq n \right\rangle \subset \left(\mathcal{O}\hat{S}_m^{\mathrm{PD}}\right)^{\Gamma'_{R_0}}.$$

Recall that $M' = \left(\mathcal{O}N^{\mathrm{PD}}\right)^{\Gamma'_{R_0}}$ and $M'' = \left(\mathcal{O}N^{\mathrm{PD}}\right)^{\Gamma_{R_0}}$. Note that since $\left(\mathcal{O}\hat{S}_m^{\mathrm{PD}}\right)^{\Gamma'_{R_0}}$ is PD-complete with respect to the ideal H and $\mathcal{O}N^{\mathrm{PD}}$ is a finite free $\mathcal{O}\hat{S}_m^{\mathrm{PD}}$-module, we get that M' is PD-complete with respect to the ideal H.

Lemma 3.37. *For $x' \in M'$, there exists a unique $x'' \in M'$, such that*

$$x'' \equiv x' \mod H^{[1]}M',$$
$$\gamma(x'') = x''.$$

In particular, $x'' \in M''$.

Proof. The proof essentially follows the technique of [Wac96, §B.1.2, Lemme 1]. For uniqueness, we want to show that if $x'', y'' \in M'$ satisfy the conditions of the lemma then we must have $x'' = y''$. If x'' and y'' are distinct, then $x'' - y''$ is nonzero in $H^{[n-1]}M'/H^{[n]}M'$ for some smallest $n \geq 2$, i.e. $x'' - y'' = \pi^{n-1}\alpha \mod H^{[n]}M'$, with $\alpha \in M'$. Moreover, we have $\gamma_0 = \gamma^e$ where $e = [K : F]$, therefore $\gamma_0(x'') = x''$, $\gamma_0(y'') = y''$, and $\gamma_0(\alpha) \equiv \alpha \mod \pi M'$ since γ_0 acts trivially modulo π on $\mathbf{N}(T)$ and $\mathcal{O}\hat{S}_m^{\mathrm{PD}}$. So we obtain

$$\pi^{n-1}\alpha = x'' - y'' = \gamma_0(x'' - y'') = \gamma_0(\pi^{n-1}\alpha) \equiv \gamma_0(\pi^{n-1})\alpha \equiv \chi(\gamma_0)^{n-1}\pi^{n-1}\alpha \mod H^{[n]}M'.$$

Since $\chi(\gamma_0) = \exp(p^m)$ and $n \geq 2$, we conclude from the congruence above that $\alpha = 0$, i.e. $x'' = y''$.

Before proceeding to show the existence of x'', let us show that it is enough to approximate for the action of γ_0. Let $g \in \Gamma_F$ be a lift of a generator of the cyclic group Γ_F/Γ_K. Then we have that $y'' = \frac{1}{e}\sum_{k=0}^{e-1} g^k(x'') \in \left(\mathcal{O}N^{\mathrm{PD}}\right)^{\Gamma_{R_0}} = M''$. But by the claim of uniqueness proved above, we must have that $y'' = x''$, i.e. $x'' \in \left(\mathcal{O}N^{\mathrm{PD}}\right)^{\Gamma_{R_0}} = M''$.

For existence, we start by setting $x'_1 := x'$ and using successive approximation we will show that if there exists $x'_n \in M'$ such that

$$\begin{aligned} x'_n &\equiv x'_{n-1} \mod H^{[n-1]}M', \\ \gamma_0(x'_n) &\equiv x'_n \mod \pi H^{[n-1]}M', \end{aligned}$$

then there exists $x'_{n+1} \in M'$ such that

$$\begin{aligned} x'_{n+1} &\equiv x'_n \mod H^{[n]}M', \\ \gamma_0(x'_{n+1}) &\equiv x'_{n+1} \mod \pi H^{[n]}M'. \end{aligned}$$

To find such an x'_{n+1}, first we write

$$\gamma_0(x'_n) \equiv x'_n + \pi y'_n \mod \pi H^{[n]}M',$$

with $y'_n \in H^{[n-1]}M'$. Next, we set

$$x'_{n+1} := x'_n + z'_n,$$

for some $z'_n = \frac{\pi^n}{n!p^n} w_n \in H^{[n]}M'$, which we need to determine. Note that we have

$$\gamma_0(z'_n) = \frac{\gamma_0(\pi^n)}{n!p^n}\gamma_0(w_n) \equiv \chi(\gamma_0)^n \frac{\pi^n}{n!p^n} w_n \equiv \chi(\gamma_0)^n z'_n \mod \pi H^{[n]}M'.$$

Now, the action of γ_0 on x'_{n+1} can be given as

$$\begin{aligned} \gamma_0(x'_{n+1}) &\equiv x'_n + \pi y'_n + \chi(\gamma_0)^n z'_n \mod \pi H^{[n]}M' \\ &\equiv x'_{n+1} + \pi y'_n + (\chi(\gamma_0)^n - 1) z'_n \mod \pi H^{[n]}M'. \end{aligned}$$

Since $\chi(\gamma_0) = \exp(p^m)$, we have $\chi(\gamma_0)^n - 1 = np^m u$ with $u \in 1 + p\mathbb{Z}_p$. So, to get $\gamma_0(x'_{n+1}) \equiv x'_{n+1} \mod \pi H^{[n]}M'$, we can take $z'_n = -y'_n \frac{\pi}{np^m u}\frac{np^m}{\chi(\gamma_0)^n - 1} \in H^{[n]}M'$. Hence, we conclude that the sequence x'_n converges to some $x'' \in \left(\mathcal{O}N^{\mathrm{PD}}\right)^{\Gamma_{R_0}} = M''$. ■

Unique lift by successive approximation

From Lemmas 3.33 & 3.37 we get that for any $x \in \mathbf{N}(T)$ we can find $x'' \in \mathcal{O}N^{\mathrm{PD}}$ such that

$$\begin{aligned} x'' &\equiv x \mod J^{[1]}\mathcal{O}N^{\mathrm{PD}}, \\ \gamma_s(x'') &= x'' \quad \text{for } 0 \leq s \leq d. \end{aligned}$$

In particular, $x'' \in M'' = \left(\mathcal{O}N^{\mathrm{PD}}\right)^{\Gamma_{R_0}}$. Moreover, this solution is unique. Indeed, let x'', y'' be two such solutions. Then we must have that $x'' - y''$ is nonzero in $J^{[n]}\mathcal{O}N^{\mathrm{PD}} / J^{[n+1]}\mathcal{O}N^{\mathrm{PD}}$ for some smallest $n \geq 1$, i.e.

$$x'' - y'' \equiv \sum_{\mathbf{i} \in \Lambda_n} \frac{\pi^{[i_0]}}{p^{mi_0}} (V_1 - 1)^{[i_1]} \cdots (V_d - 1)^{[i_d]} \beta_{\mathbf{i}} \mod J^{[n+1]}\mathcal{O}N^{\mathrm{PD}}.$$

Let

$$\beta := \sum_{\mathbf{i} \in \Lambda_n} \frac{\pi^{[i_0]}}{p^{mi_0}} (V_1 - 1)^{[i_1]} \cdots (V_d - 1)^{[i_d]} \beta_{\mathbf{i}} \in J^{[n]}\mathcal{O}N^{\mathrm{PD}},$$

then because we have $(\gamma_s - 1)(x'' - y'') = 0$ for $s \in \{0, \ldots, d\}$, we obtain that $(\gamma_s - 1)\beta \equiv 0 \mod J^{[n+1]}\mathcal{O}N^{\mathrm{PD}}$. But from Lemma 3.35 this is only possible when $\beta_{\mathbf{i}} \equiv 0 \mod J^{[1]}\mathcal{O}N^{\mathrm{PD}}$ for $\mathbf{i} \in \Lambda_n \setminus \{\mathbf{k}\}$ where $\mathbf{k} = (n, 0, \ldots, 0)$.

Next, applying γ_0 to the reduced expression we obtain

$$\tfrac{\pi^{[n]}}{p^{mn}}\beta_{\mathbf{k}} = x'' - y'' = \gamma_0(x'' - y'') \equiv \chi(\gamma_0)^n \tfrac{\pi^n}{p^{mn}}\beta_{\mathbf{k}} \mod J^{[n+1]}\mathcal{O}N^{\mathrm{PD}}.$$

Again, this is only possible when $\beta_{\mathbf{k}} = 0 \mod J^{[1]}\mathcal{O}N^{\mathrm{PD}}$. Therefore, we obtain that we must have $x'' = y''$.

Finishing the proof of Proposition 3.31

Recall that at the beginning of the proof we assumed $\mathbf{N}(T)$ to be free of rank h (after extension of scalars to $\mathbf{A}_{R_0'}^+$ which we again wrote as $\mathbf{A}_{R_0}^+$ by abusing notations), therefore $\mathcal{O}N^{\mathrm{PD}}$ is free of rank h. Further, we have $M = \big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)\big)^{\Gamma_{R_0}}$ and since $M\big[\frac{1}{p}\big]$ is equipped with an integrable connection, it is projective of rank $\leq h$ (see the beginning of the proof). So by the successive approximation argument above, we obtain that the rank of $M\big[\frac{1}{p}\big]$ as an $R_0\big[\frac{1}{p}\big]$-module is exactly h.

Finally, we want to show that the natural inclusion $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} M\big[\frac{1}{p}\big] \rightarrowtail \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V)$ is bijective. Since we assumed $\mathbf{N}(T)$ to be a free module, let $\{e_1, \ldots, e_h\}$ be its $\mathbf{A}_{R_0}^+$-basis. Let $P \in \mathrm{Mat}(h, \mathbf{A}_{R_0}^+)$ denote the matrix for the action of Frobenius on $\mathbf{N}(T)$ in the basis $\{e_1, \ldots, e_h\}$. We want to show that $\varphi^*\big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V)\big) \simeq \big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V)\big)$. Note that we have $\wedge^h \mathbf{N}(V) = \mathbf{N}\big(\wedge^h V\big)$, which follows from the compatibility between exterior power of representations and exterior power of their respective Wach modules in Corollary 3.16. Since $\wedge^h V$ is again a positive Wach representation and taking exterior powers commutes with scalar extension (see [Bou98, Chapter III, §7.5, Proposition 8]), therefore passing to h-th exterior power we obtain that

$$\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}\big(\textstyle\bigwedge^h V\big) = \bigwedge^h \big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V)\big).$$

Now from Lemma 3.30 for one-dimensional representations we have that $\varphi^*\big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(\wedge^h V)\big) \simeq \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(\wedge^h V)$, and therefore

$$\varphi^*\Big(\textstyle\bigwedge^h \big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V)\big)\Big) \simeq \varphi^*\big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(\bigwedge^h V)\big) \simeq \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}\big(\bigwedge^h V\big) \simeq \bigwedge^h \big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V)\big).$$

Since the action of φ is diagonal and taking exterior powers commutes with scalar extension (see [Bou98, Chapter III, §7.5, Proposition 8]), we obtain that

$$\textstyle\bigwedge^h \varphi^*\big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V)\big) \simeq \varphi^*\big(\bigwedge^h \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V)\big) \simeq \bigwedge^h \big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V)\big).$$

In particular, we have obtained that $\det P$ is invertible in $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}\big[\frac{1}{p}\big]$.

Next, recall that $\mathcal{O}N^{\mathrm{PD}} = \mathcal{O}\widehat{S}_m^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)$ and $M'' = \big(\mathcal{O}N^{\mathrm{PD}}\big)^{\Gamma_{R_0}}$. So we consider the following commutative diagram

$$\begin{tikzcd}
\mathcal{O}\widehat{S}_m^{\mathrm{PD}} \otimes_{R_0} M'' \arrow[r] \arrow[d, "\varphi^m \otimes \varphi^m"'] & \mathcal{O}N^{\mathrm{PD}} \arrow[d, "\varphi^m"] \\
\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} M \arrow[r] & \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T),
\end{tikzcd}$$

where all arrows are injective. We also have that $\{e_1, \ldots, e_h\}$ is an $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$-basis of $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)$ as well as an $\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$-basis of $\mathcal{O}N^{\mathrm{PD}}$. From Lemmas 3.33 & 3.37 and the discussion above, we have $f_i \in M''$ for $1 \leq i \leq h$ such that $f_i = e_i + \sum_{i=1}^h a_{ij}e_j$ for $a_{ij} \in J^{[1]}\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$. So we let $A := id_h + (a_{ij}) \in \mathrm{Mat}(h, \mathcal{O}\widehat{S}_m^{\mathrm{PD}})$ denote the $h \times h$ matrix obtained in this manner. Since $\det A \in 1 + J^{[1]}\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$ and $\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$ is p-adically complete with respect to the PD-ideal $J^{[i]}$, we obtain that $\det A$ is invertible in $\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$.

Now let $g_i = (\varphi^m \otimes \varphi^m) f_i = \varphi^m(e_i) + \sum_{j=1}^h \varphi^m(a_{ij})\varphi^m(e_j) \in M$ and let M_0 be the R_0-submodule of M generated by $\{g_1, \ldots, g_h\}$. From the expression of $\{g_1, \ldots, g_h\}$ in the basis of $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)$, we obtain that the determinant of the inclusion $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} M_0 \rightarrowtail \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)$ is given by $\varphi^m(\det A)\det(P^m)$. Since $\det A$ is invertible in $\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$, we get that $\varphi(\det A)$ is invertible in $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ and from above we already have that $\det P$ is invertible in $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}\big[\frac{1}{p}\big]$. Therefore, the natural inclusions

$$\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} M_0\big[\tfrac{1}{p}\big] \longrightarrow \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} M\big[\tfrac{1}{p}\big] \longrightarrow \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V),$$

are bijective. These inclusions are compatible with Frobenius, filtration, connection and the action of Γ_{R_0} on each side.

Note that we assumed $\mathbf{N}(T)$ to be free of rank h, therefore we obtain a free R_0-lattice $M_0 \subset M$ such that

$$M_0\big[\tfrac{1}{p}\big] = \big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} M_0\big[\tfrac{1}{p}\big]\big)^{\Gamma_{R_0}} \simeq \big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} M\big[\tfrac{1}{p}\big]\big)^{\Gamma_{R_0}} = M\big[\tfrac{1}{p}\big],$$

which are free of rank h over $R_0\big[\frac{1}{p}\big]$. In general, when $\mathbf{N}(T)$ is projective of rank h, we obtain that $M\big[\frac{1}{p}\big]$ is projective of rank h. ∎

Finally, under simplified assumptions we make an observation which will be useful in Chapter 5.

Proposition 3.38. *Let V be an h-dimensional positive Wach representation of G_{R_0} and $T \subset V$ a free $\mathbb{Z}_p$-lattice of rank h stable under the action of G_{R_0}. Suppose $m = 1$ and let us assume that $\mathbf{N}(T)$ is a free $\mathbf{A}_{R_0}^+$-module and let $M_0 \subset \big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)\big)^{\Gamma_{R_0}}$ be the free R_0-module obtained in Proposition 3.31. Then, the R_0-module $M_0/\varphi^*(M_0)$ is killed by p^{2s}, where s is maximum among the absolute value of Hodge-Tate weights of V.*

Proof. Let $\mathbf{e} = \{e_1, \ldots, e_d\}$ be an $\mathbf{A}_{R_0}^+$-basis of $\mathbf{N}(T)$. Then in the notation of the proof of Proposition 3.31, we obtain that M_0 is a free R_0-module with a basis given as $\mathbf{g} = \{g_1, \ldots, g_d\}$, where $\mathbf{g} = \varphi(\mathbf{e})\varphi(A)$ for $A \in \mathit{Mat}(h, \mathcal{O}\widehat{S}_m^{\mathrm{PD}})$.

Now note that $q = \frac{\varphi(\pi)}{\pi} = p\varphi\big(\frac{\pi}{t}\big)\frac{t}{\pi}$ and since $\frac{\pi}{t}$ is a unit in $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ (see Lemma 2.43) we obtain that q and p are associates in the ring $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$. Further, we have that $\mathbf{N}(T)/\varphi^*(\mathbf{N}(T))$ is killed by q^s, where s is maximum among the absolute value of the Hodge-Tate weights of V. So, over $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ we obtain that $\big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)\big)/\varphi^*\big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)\big)$ is killed by p^s. Next, we have that $\det A$ is a unit in $\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$, therefore $\varphi(\det A)$ is a unit in $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ and $\varphi(A)$ is invertible over $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$. This implies that $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} M_0 \simeq \varphi^*\big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)\big)$. Thus, the cokernel of the natural inclusion $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} M_0 \rightarrowtail \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)$ is killed by p^s.

Now consider the commutative diagram with exact rows

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[r] \arrow[d] & M_1 \arrow[d] & \\
0 \arrow[r] & \varphi^*\big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} M_0\big) \arrow[r] \arrow[d] & \varphi^*\big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)\big) \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} M_0 \arrow[r] \arrow[d] & \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T) \arrow[r] \arrow[d] & M_3 \arrow[r] & 0 \\
 & M_4 \arrow[r] \arrow[d] & M_5 \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

By the discussion above, M_3 and M_5 are p^s-torsion modules over $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$. An argument similar to the case of M_3 shows that M_2 is p^s-torsion as well. This implies that the submodule $M_1 \subset M_2$ is

p^s-torsion. Next, an application of snake lemma, gives the exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_4 \longrightarrow M_5.$$

Since M_1 and M_5 are p^s-torsion, we conclude that M_4 is p^{2s}-torsion. In other words, the module $\big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} M_0\big)/\varphi^*\big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} M_0\big)$ is killed by p^{2s}.

Finally, we note that the action of Frobenius commutes with the action of Γ_{R_0}, therefore taking Γ_{R_0}-invariants, we obtain that the module $M_0/\varphi^*(M_0)$ is killed by p^{2s}. This proves the corollary. ■

Remark 3.39. Note that we fixed a choice of $m \in \mathbb{N}_{\geq 1}$ in the beginning. The R_0-modules that we have obtained depend on this choice. In particular, let $1 \leq m \leq n$ and $R = R_0(\zeta_{p^m})$ and $R' = R_0(\zeta_{p^{m'}})$. Then we have that $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \subset \mathcal{O}\mathbf{A}_{R'}^{\mathrm{PD}}$ with $M = \big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)\big)^{\Gamma_{R_0}}$ and $M' = \big(\mathcal{O}\mathbf{A}_{R'}^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)\big)^{\Gamma_{R_0}}$. Further, let M_0 and M_0' be the R_0-modules obtained for m and m' respectively in Proposition 3.31, then we have that $\varphi^{m'-m}(M_0') \subset M_0$ (this essentially follows from the fact that $\varphi^{m'-m}\big(\mathcal{O}\widehat{S}_{m'}^{\mathrm{PD}}\big) \subset \mathcal{O}\widehat{S}_m^{\mathrm{PD}}$).

3.2.4. Proof of Theorem 3.24

Let $M = \big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)\big)^{\Gamma_{R_0}}$ and we have a natural inclusion of projective $R_0\big[\frac{1}{p}\big]$-modules of rank h from Proposition 3.31, $M\big[\frac{1}{p}\big] \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$. First, we will show that V is crystalline and the inclusion described above is in fact bijective. Recall that from Proposition 3.31, we have an isomorphism of $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}\big[\frac{1}{p}\big]$-modules

$$\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} M\big[\tfrac{1}{p}\big] \xrightarrow{\simeq} \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V),$$

compatible with Frobenius, filtration, connection and the action of Γ_{R_0} on each side. Since both sides are projective modules, extending scalars along $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \rightarrowtail \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)$ we obtain an isomorphism of $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)$-modules

$$\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \otimes_{R_0[\frac{1}{p}]} M\big[\tfrac{1}{p}\big] \xrightarrow{\simeq} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \otimes_{\mathbf{B}_{R_0}^+} \mathbf{N}(V),$$

compatible with Frobenius, filtration, connection and the action of G_{R_0} on each side. Further recall that $\mathbf{A}^+ \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V) \rightarrowtail \mathbf{A}^+ \otimes_{\mathbf{A}_{R_0}^+} V$ and the cokernel is killed by π^s (see Lemma 3.12). Since π is invertible in $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)$, extending scalars along $\mathbf{A}^+ \rightarrowtail \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)$, we obtain an isomorphism of $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)$-modules

$$\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \otimes_{\mathbf{B}_{R_0}^+} \mathbf{N}(V) \xrightarrow{\simeq} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \otimes_{\mathbb{Q}_p} V,$$

compatible with Frobenius, filtration, connection and the action of G_{R_0}. Finally, since $R_0\big[\frac{1}{p}\big] \rightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)$ is faithfully flat (see [Bri08, Théorème 6.3.8]), we obtain an inclusion of $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)$-modules

$$\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \otimes_{R_0[\frac{1}{p}]} M\big[\tfrac{1}{p}\big] \subset \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \otimes_{R_0[\frac{1}{p}]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V),$$

compatible with Frobenius, filtration, connection and the action of G_{R_0} on each side. In particular, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \otimes_{R_0[\frac{1}{p}]} M\big[\frac{1}{p}\big] \arrow[r, "\simeq"] \arrow[d, tail] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \otimes_{\mathbf{B}_{R_0}^+} \mathbf{N}(V) \arrow[d, "\simeq"] \\
\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \otimes_{R_0[\frac{1}{p}]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \arrow[r, tail] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \otimes_{\mathbb{Q}_p} V,
\end{tikzcd}$$

compatible with Frobenius, filtration, connection and the action of G_{R_0} on each side. It is immediately clear from the diagram that the left vertical arrow and bottom horizontal arrow must be bijective as well. The bijection of bottom horizontal arrow implies that V is a crystalline representation of G_{R_0}.

Moreover, since $R_0\left[\frac{1}{p}\right] \longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)$ is faithfully flat (see [Bri08, Théorème 6.3.8]), we obtain an isomorphism of $R_0\left[\frac{1}{p}\right]$-modules $M\left[\frac{1}{p}\right] \xrightarrow{\simeq} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$.

Next, we will check the compatibility of the isomorphism obtained with supplementary structures on both sides. From Proposition 3.31 it is clear that this isomorphism is compatible with the action of Frobenius and connection on each side. So we are only left to check the compatibility with natural filtrations on each side. For this, first we see that using Definition 3.10 and Remark 3.20 (ii), the filtration on $M\left[\frac{1}{p}\right]$ is given as

$$\mathrm{Fil}^k M\left[\tfrac{1}{p}\right] = \Big(\sum_{i\in\mathbb{N}} \mathrm{Fil}^i \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \widehat{\otimes}_{\mathbf{A}_{R_0}^+} \mathrm{Fil}^{k-i}\mathbf{N}(V)\Big)^{\Gamma_{R_0}}.$$

Lemma 3.40. *In the notations already described, we have* $\mathrm{Fil}^k M\left[\frac{1}{p}\right] = \mathrm{Fil}^k \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ *for* $k \in \mathbb{Z}$

Proof. First, let $x \in \mathrm{Fil}^k M\left[\frac{1}{p}\right]$, then we can write it as a sum

$$x = \sum_{i\in\mathbb{N}} (a_i \otimes b_i)\xi^{[i_0]}(V_1-1)^{[i_1]}\cdots(V_d-1)^{[i_d]} \otimes y_{k-i} \;\text{ for } a_i \in R_0,\; b_i \in \mathbf{A}_R^+ \text{ and } y_{k-i} \in \mathrm{Fil}^{k-i}\mathbf{N}(V),$$

where $i_0 + \cdots + i_d = i$ and $\xi = \frac{\pi}{\pi_1}$. Writing $\varphi(y_{j-i}) = q^{k-i}z_{k-i} = \varphi(\xi^{k-i})z_{k-i}$ for some $z_{k-i} \in \mathbf{N}(V)$, we obtain

$$\varphi(x) = \sum_{i\in\mathbb{N}} \varphi(a\otimes b)\varphi\big(\xi^j\xi^{[i_0]}(V_1-1)^{[i_1]}\cdots(V_d-1)^{[i_d]}\big) \otimes z_{k-i}.$$

Since the action of φ and Γ_{R_0} commute, we get

$$\varphi(x) \in \varphi\big(\mathrm{Fil}^k \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(V)\big)^{\Gamma_{R_0}} \subset \varphi\big(\mathrm{Fil}^k\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)\otimes_{\mathbb{Q}_p} V\big)^{G_{R_0}} \subset \varphi\big(\mathrm{Fil}^k\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)\big).$$

As φ is injective, we must have $x \in \mathrm{Fil}^k\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$. This shows $\mathrm{Fil}^k M\left[\frac{1}{p}\right] \subset \mathrm{Fil}^k\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$

Conversely, let $\{e_1, \ldots, e_h\}$ denote a $\mathbb{Q}_p$-basis of V and let $x \in \mathrm{Fil}^k\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \setminus \mathrm{Fil}^{k+1}\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$. Then we can write $x = \sum_{i=1}^h b_ie_i$, with $b_i \in \mathrm{Fil}^k\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)$. Since $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \simeq M\left[\frac{1}{p}\right]$, we take $r \leq k$ to be the largest integer such that $x \in \mathrm{Fil}^r M\left[\frac{1}{p}\right]$. So we can also express $x = \sum_{j\in\mathbb{N}} c_j \otimes f_{r-j}$, for $c_j \in \mathrm{Fil}^j\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \setminus \mathrm{Fil}^{j+1}\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ and $f_{r-j} \in \mathrm{Fil}^{r-j}\mathbf{N}(V)\setminus\mathrm{Fil}^{r-j+1}\mathbf{N}(V)$. Note that using Lemma 3.41, we have that $\mathrm{Fil}^{r-j}\mathbf{N}(V) = (\xi^{r-j}\mathbf{B}^+ \otimes_{\mathbb{Q}_p} V)\cap\mathbf{N}(V)$. Therefore, $f_{r-j} \in (\xi^{r-j}\mathbf{B}^+\otimes_{\mathbb{Q}_p} V)\setminus(\xi^{r-j+1}\mathbf{B}^+\otimes_{\mathbb{Q}_p} V)$. So, in the basis of V, we can write $f_{r-j} = \sum_{i=1}^h \xi^{r-j}f_{r-j,i}\otimes e_i$, with $f_{r-j,i} \in \mathbf{B}^+\setminus\xi\mathbf{B}^+$ for $1 \leq i \leq h$. In conclusion, we obtain that

$$x = \sum_{j\in\mathbb{N}} c_j \otimes \Big(\sum_{i=1}^h \xi^{r-j}f_{r-j,i}\otimes e_i\Big) = \sum_{i=1}^h \Big(\sum_{j\in\mathbb{N}} c_j\otimes\xi^{r-j}f_{j-r,i}\Big)\otimes e_i.$$

Comparing the two expressions for x thus obtained in the basis of V, we get

$$\sum_{j\in\mathbb{N}} c_j\otimes\xi^{r-j}f_{j-r,i} = b_i \in \mathrm{Fil}^k\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \;\text{ for } 1 \leq i \leq h.$$

Now, recall that filtrations on $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ and $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)$ are compatible (see Remark 3.22). Further, let us equip $\mathbf{B}^+$ with the induced filtration from $\mathbf{B}_{\mathrm{cris}}(R)$. Since the filtration on $\mathbf{B}_{\mathrm{cris}}(R)$ is given by divided powers of ξ, we obtain that $\mathrm{Fil}^{k-j}\mathbf{B}^+ = \mathbf{B}^+\cap\mathrm{Fil}^{k-j}\mathbf{B}_{\mathrm{cris}}(R) = \xi^{k-j}\mathbf{B}^+$. In particular, we obtain that

$$\big(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}\otimes_{\mathbf{A}_{R_0}^+}\mathbf{B}^+\big)\cap\mathrm{Fil}^k\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) = \sum_{j\in\mathbb{N}}\mathrm{Fil}^j\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}\otimes_{\mathbf{A}_{R_0}^+}(\mathbf{B}^+\cap\mathrm{Fil}^{k-j}\mathbf{B}_{\mathrm{cris}}(R)) = \sum_{j\in\mathbb{N}}\mathrm{Fil}^j\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}\otimes_{\mathbf{A}_{R_0}^+}\xi^{k-j}\mathbf{B}^+.$$

Recall that $x \in \mathrm{Fil}^r M\left[\frac{1}{p}\right] \setminus \mathrm{Fil}^{r+1}M\left[\frac{1}{p}\right]$ and in the expression $\sum_{j\in\mathbb{N}} c_j\otimes\xi^{r-j}f_{j-r,i}$, we have that $c_j \in \mathrm{Fil}^j\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}\setminus\mathrm{Fil}^{j+1}\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ and $f_{r-j,i} \in \mathbf{B}^+\setminus\xi\mathbf{B}^+$ for $1 \leq i \leq h$. Therefore, $\sum_{j\in\mathbb{N}} c_j\otimes\xi^{r-j}f_{j-r,i} \in \mathrm{Fil}^r\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)\setminus\mathrm{Fil}^{r+1}\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)$. But then $\sum_{j\in\mathbb{N}} c_j\otimes\xi^{r-j}f_{j-r,i} \in \mathrm{Fil}^k\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)$ if and only if $r \geq k$.

Since $r \leq k$ by assumption, therefore we must have $r = k$. Hence $x \in \mathrm{Fil}^k M\left[\frac{1}{p}\right]$. This proves the claim. ∎

Following observation was used above:

Lemma 3.41. *Let $b \in \mathbb{N}$ and $q = \frac{\varphi(\pi)}{\pi}$, then we have $(q^b \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V) \cap \mathbf{N}(V) = q^b \mathbf{N}(V)$.*

Proof. First, let us assume that $\mathbf{N}(V)$ is free with $\{f_1, f_2, \dots, f_h\}$ as an $\mathbf{A}^+_{R_0}$-basis, and let $\{e_1, \dots, e_h\}$ be a $\mathbb{Q}_p$-basis of V. Now let $q^b x \in (\mathbf{B}^+ \otimes_{\mathbb{Q}_p} V) \cap \mathbf{N}(V)$ for $x = \sum_{i=1}^h x_i e_i \in \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V$. We can also write $q^b x = \sum_{i=1}^h y_i f_i$ for $y_i \in \mathbf{B}^+_{R_0}$. Next, from Lemma 3.12 we have $\pi^s \mathbf{B}^+ \otimes_{\mathbb{Q}_p} V \subset \mathbf{B}^+ \otimes_{\mathbf{B}^+_{R_0}} \mathbf{N}(V)$, so we can write $q^b x = \pi^{-s} q^b \sum_{i=1}^h x_i \pi^s e_i = \pi^{-s} q^b \sum_{i=1}^h x_i \sum_{j=1}^h z_{ij} f_j = \pi^{-s} \sum_{i=1}^h (\sum_{j=1}^h x_j z_{ji}) f_i$, with $z_{ij} \in \mathbf{B}^+$. But then we must have $\pi^{-s} q^b \sum_{j=1}^h x_j z_{ji} = y_i$ for $1 \leq i \leq h$. Since H_{R_0} acts trivially on π, q and y_i, we get that $w_i = \sum_{j=1}^h x_j z_{ji} \in \mathbf{B}^+_{R_0}$. But $y_i \in \mathbf{B}^+_{R_0}$ and π does not divide q in $\mathbf{B}^+_{R_0}$, therefore we obtain that $w_i \in \pi^s \mathbf{B}^+_{R_0}$. In particular, $y_i \in q^b \mathbf{B}^+_{R_0}$, therefore $q^b x = \sum_{i=1}^h y_i g_i \in q^b \mathbf{N}(V)$.

In the case when $\mathbf{N}(V)$ is projective (and not free) over $\mathbf{B}^+_{R_0}$, let R_0' be the p-dic completion of a finite étale algebra over R_0 such that the scalar extension $\mathbf{B}^+_{R_0'} \otimes_{\mathbf{B}^+_{R_0}} \mathbf{N}(V)$ is a free module over $\mathbf{B}^+_{R_0'}$ and $R_0'\left[\frac{1}{p}\right] / R_0\left[\frac{1}{p}\right]$ is Galois (see Definition 3.8). Then we can argue as above and conclude by taking $\mathrm{Gal}\big(R_0'\left[\frac{1}{p}\right] / R_0\left[\frac{1}{p}\right]\big)$-invariants of $q^b \mathbf{B}^+_{R_0'} \otimes_{\mathbf{B}^+_{R_0}} \mathbf{N}(V)$. ∎

Combining Lemma 3.40 with observations made before, we obtain that the isomorphism of $R_0\left[\frac{1}{p}\right]$-modules $M\left[\frac{1}{p}\right] \xrightarrow{\simeq} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is compatible with Frobenius, filtration and connection on each side.

Finally, we can compose these natural maps as

$$\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xleftarrow{\simeq} \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \left(\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}^+_{R_0}} \mathbf{N}(V)\right)^{\Gamma_{R_0}} \xrightarrow{\simeq} \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}^+_{R_0}} \mathbf{N}(V),$$

where the second map is compatible with the Frobenius, filtration, connection and the action of Γ_{R_0} on each side (see Proposition 3.31). This proves the theorem. ∎

Remark 3.42. In the case when $\mathbf{N}(T)$ is a free $\mathbf{A}^+_{R_0}$- module of rank h, from Proposition 3.31 we obtain that $M\left[\frac{1}{p}\right] \simeq \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is a free $R_0\left[\frac{1}{p}\right]$-module of rank h. In particular, for Wach representations there exists a finite étale extension R_0' over R_0 such that $R_0'\left[\frac{1}{p}\right] \otimes_{R_0\left[\frac{1}{p}\right]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is a free module of rank h.

3.3. The false Tate curve

In this section we will construct a set of examples of representations satisfying the conditions of Definition 3.8. These examples will arise from the Tate module attached to the false Tate curve.

Let $Y := X_1$, $G = \mathrm{Spec}\, R_0[Z, Z^{-1}]$ and $\mathcal{E} := G/Y^{\mathbb{Z}}$ denote the false Tate curve over R_0. The $\overline{R_0}\left[\frac{1}{p}\right]$-rational points of $\mathcal{E}$ form an abelian group and we consider its p^n-torsion points. In other words, there is an attached $\mathbb{Z}_p$-representation of G_{R_0} given by the Tate module of $\mathcal{E}$ as

$$T_p\mathcal{E} = \lim_n \mathcal{E}\left(\overline{R_0}\left[\tfrac{1}{p}\right]\right)[p^n] = \lim_n \left\{\zeta^i_{p^n}(Y^{(n)})^j,\ 1 \leq i, j < p^n\right\},$$

where $Y^{(1)} = Y$ and $Y^{(n)}$ is a compatible system of p^n-th roots of Y such that $(Y^{(n+1)})^p = Y^{(n)}$ for $n \geq 1$. Let $d_1 = (\zeta_{p^n})_{n \geq 0}$ and $d_2 = (Y^{(n)})_{n \geq 0}$, then in the additive notation, we have $T_p\mathcal{E} = \mathbb{Z}_p d_1 + \mathbb{Z}_p d_2$.

Next, let T denote the $\mathbb{Z}_p$-dual of $T_p\mathcal{E}$ with the dual basis $\{e_1, e_2\}$ such that $e_i(d_j) = \delta_{ij}$ for $i, j = 1, 2$. Then $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a 2-dimensional p-adic representation of G_{R_0}. The action of G_{R_0} on V is given by the matrix

$$\begin{pmatrix} \chi(g)^{-1} & 0 \\ -c(g)\chi(g)^{-1} & 1 \end{pmatrix},$$

where $\chi(g)$ is the usual p-adic cyclotomic character and $c : G_{R_0} \longrightarrow \mathbb{Z}_p$ is a 1-cocycle such that $g(Y^{(n)}) = \zeta_{p^n}^{c(g)} Y^{(n)}$ for $g \in G_{R_0}$. For $k \geq 1$, let us set $V_k := \mathrm{Sym}^k(V)$ as the $\mathbb{Q}_p$-linear k-th symmetric power of V, in particular, $V_1 = V$. Since $\dim_{\mathbb{Q}_p} V = 2$, we get that $\dim_{\mathbb{Q}_p} V_k = k+1$. An explicit basis for V_k can be given as $\big\{ e_1^{\otimes j} \otimes e_2^{\otimes(k-j)} \big\}_{0 \leq j \leq k}$. We set $T_k = \sum_{j=0}^k \mathbb{Z}_p e_1^{\otimes j} \otimes e_2^{\otimes(k-j)}$ which is a G_{R_0}-stable $\mathbb{Z}_p$-lattice inside V_k.

Next, we compute the crystalline modules associated to the representations described above.

Proposition 3.43 (The crystalline module). *The R_0-module $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V_k)$ is a free filtered (φ, ∂)-module of rank $k+1$ over $R_0\big[\frac{1}{p}\big]$ (see Definition 1.18). Moreover, there exists an R_0-submodule of $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V_k)$ satisfying analogous properties.*

Proof. For $k = 1$, this was worked out in [Bri08, Example, p. 120]. To get the module $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V_k)$, we will construct some Galois invariant elements in $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \otimes_{\mathbb{Q}_p} V$ and extrapolate a basis for $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V_k)$.

Let $f_1 = te_1 + \beta e_2$ and $f_2 = e_2$, where $\beta = \log([Y^\flat]/Y) \in \mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0)$. The element β is well-defined and converges in $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_0)$ because

$$\beta = \log\left(\tfrac{[Y^\flat]}{Y}\right) = \sum_{n \geq 1} \tfrac{(-1)^{n+1}}{n}\left(\tfrac{[Y^\flat]}{Y} - 1\right)^n = \sum_{n \geq 1} (-1)^{n+1}(n-1)!\left(\tfrac{[Y^\flat]}{Y} - 1\right)^{[n]},$$

where we have that $[Y^\flat]/Y \in R_0 \otimes_{O_F} \mathbf{A}_{\mathrm{inf}}(R)$ and $\theta_R([Y^\flat]/Y - 1) = 0$. Also, we conclude that for any $g \in G_{R_0}$, we have $g(\beta) = c(g)t + \beta$ since $g(Y^{(n)}) = \zeta_{p^n}^{c(g)} Y^{(n)}$. Clearly,

$$g(f_1) = t(e_1 - c(g)e_2) + (c(g)t + \beta)e_2 = te_1 + \beta e_2 = f_1, \quad \text{and} \quad g(f_2) = f_2.$$

So we get that $f_1, f_2 \in \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) = \big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0) \otimes_{\mathbb{Q}_p} V\big)^{G_{R_0}}$.

On the other hand, let $x \in \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ which we write as $x = ae_1 + be_2$ for $a, b \in \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)$. For any $g \in G_{R_0}$, we must have $g(x) = \chi(g)^{-1}g(a)e_1 - c(g)\chi(g)^{-1}g(a)e_2 + g(b)e_2 = x$, i.e.,

$$\chi(g)^{-1}g(a) = a, \quad \text{and} \quad g(b) - c(g)\chi(g)^{-1}g(a) = b.$$

Therefore, $t^{-1}a \in \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)^{G_{R_0}} = R_0\big[\frac{1}{p}\big]$. Moreover, we can write $x = ae_1 + be_2 = t^{-1}af_1 + (b - t^{-1}a\beta)f_2$. Now, for any $g \in G_{R_0}$ we get

$$\begin{aligned} g(b - t^{-1}a\beta) &= g(b) - \chi(g)^{-1}t^{-1}g(a)(c(g)t + \beta) \\ &= g(b) - \chi(g)^{-1}g(a)\beta - t^{-1}\chi(g)^{-1}g(a)\beta = b - t^{-1}a\beta \in \mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_0)^{G_{R_0}} = R_0\big[\tfrac{1}{p}\big]. \end{aligned}$$

Hence, (f_1, f_2) form a basis of $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$.

For $i \in \mathbb{Z}$, the filtration on $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is given as

$$\mathrm{Fil}^i \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) = \begin{cases} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) & \text{if } i \leq 0, \\ R_0\big[\frac{1}{p}\big]f_1 & \text{if } i = 1, \\ 0 & \text{if } i > 1. \end{cases}$$

From the filtration, we deduce that the Hodge-Tate weights of V are $(-1, 0)$. Also we have $\varphi(f_1) = pf_1$ and $\varphi(f_2) = f_2$. Further, the module $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is equipped with a quasi-nilpotent and integrable connection, given as

$$\begin{aligned} \partial : \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) &\longrightarrow \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \otimes_{\mathbb{Z}} \Omega^1_{R_0} \\ f_1 &\longmapsto -f_2 \otimes \tfrac{dY}{Y} \\ f_2 &\longmapsto 0. \end{aligned}$$

This connection on $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ satisfies Griffiths transversality with respect to the filtration above.

Let $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) := R_0 f_1 + R_0 f_2$ be an R_0-lattice inside $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$, which is stable under the Frobenius homomorphism. Moreover, this modules has an induced connection from $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ which sastisfies Griffiths transversality with respect to the induced filtration.

For $k \geq 2$, first we note that the functor $\mathcal{O}\mathbf{D}_{\mathrm{cris}}$ from p-adic Galois representations of G_{R_0} to (φ, ∂)-modules over $R_0\left[\frac{1}{p}\right]$ is compatible with symmetric powers (see Theorem 1.27). Therefore, we get that $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V_k) := \mathcal{O}\mathbf{D}_{\mathrm{cris}}(\mathrm{Sym}^k V) = \mathrm{Sym}^k \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$. An explicit $R_0\left[\frac{1}{p}\right]$-basis of $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V_k)$ can be given as $\left\{f_1^{\otimes j} \otimes f_2^{\otimes(k-j)}\right\}_{0 \leq j \leq k}$. By abuse of notation we will write $f_1^j f_2^{k-j} = f_1^{\otimes j} \otimes f_2^{\otimes(k-j)}$. We equip $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V_k)$ with a filtration induced from the natural filtration on k-th tensor power of V. Explicitly, for $i \in \mathbb{Z}$ we have

$$\mathrm{Fil}^i \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V_k) = \begin{cases} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V_k) & \text{if } i \leq 0, \\ \sum_{j=i}^k R_0\left[\frac{1}{p}\right] f_1^j f_2^{k-j} & \text{if } 1 \leq i \leq k, \\ 0 & \text{if } i > k. \end{cases}$$

From the filtration, we deduce that the Hodge-Tate weights of V_k are $(-k, -k+1, \ldots, -1, 0)$. Also, we have $\varphi(f_1^j f_2^{k-j}) = p^j f_1^j f_2^{k-j}$ for $0 \leq j \leq k$. Further, the module $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V_k)$ is equipped with a quasi-nilpotent, integrable connection, given as

$$\begin{aligned} \partial : \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V_k) &\longrightarrow \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V_k) \otimes \Omega^1_{R_0} \\ f_1^j f_2^{k-j} &\longmapsto -j f_1^{j-1} f_2^{k-j+1} \otimes \tfrac{dY}{Y}. \end{aligned}$$

This connection on $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V_k)$ satisfies Griffiths transversality with respect to the filtration above. Let $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T_k) := \sum_{j=0}^k R_0 f_1^k f_2^{k-j}$ be an R_0-lattice inside $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V_k)$, which is stable under the Frobenius homomorphism. Moreover, it induces a connection from $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V_k)$ and sastisfies Griffiths transversality with respect to the induced filtration. ■

Proposition 3.44 (An analogue of Wach module)**.** *There exists an $\mathbf{A}^+_{R_0}$-submodule $\mathbf{N}(T_k) \subset \mathbf{D}^+(T_k)$ satisfying the conditions of Definition 3.8.*

Proof. First we discuss the case $k = 1$ and let $T := T_1$ and $V := V_1$. Note that the action of G_{R_0} on T factors through Γ_{R_0}, so the (φ, Γ_{R_0})-module over $\mathbf{A}_{R_0}$ associated to T can be given as $\mathbf{D}(T) = \mathbf{A}_{R_0} \otimes_{\mathbb{Z}_p} T = \mathbf{A}_{R_0} e_1 + \mathbf{A}_{R_0} e_2$. An analogous reasoning gives us that $\mathbf{D}^+(T) = \mathbf{A}^+_{R_0} e_1 + \mathbf{A}^+_{R_0} e_2$. For the Wach module of T, we see that we can take $\mathbf{N}(T) := \mathbf{A}^+_{R_0} h_1 + \mathbf{A}^+_{R_0} h_2$, where $h_1 = \pi e_1$ and $h_2 = e_2$. Clearly, $\mathbf{N}(T) \subset \mathbf{D}^+(T)$, and it is endowed with a Frobenius-semilinear endormorphism $\varphi : \mathbf{N}(T) \longrightarrow \mathbf{N}(T)$ such that $\mathbf{N}(T)/\varphi^*(\mathbf{N}(T))$ is killed by q. such that we have an $\mathbf{A}^+_{R_0}$-lattice inside $\mathbf{D}(T_k)$ stable under the action of φ and Γ_{R_0}. We have $\{\gamma_0, \gamma_1, \ldots, \gamma_d\}$ as topological generators of Γ_R such that γ_0 generates Γ_K topologically, whereas $\{\gamma_1, \ldots, \gamma_d\}$ are topological generators of Γ'_R satisfying some compatibility conditions (see Remark 2.2). We have $\chi(\gamma_0) \in 1 + p^m \mathbb{Z}_p$, whereas $\chi(\gamma_i) = 1$, for $1 \leq i \leq d$ and therefore,

$$\gamma_0(\pi) = (1+\pi)^{\chi(\gamma_0)} - 1 \quad \text{and} \quad \gamma_i(\pi) = \pi, \quad \text{for} \quad i \neq 0.$$

Moreover,

$$\gamma_1([Y^\flat]) = (1+\pi)[Y^\flat] \quad \text{and} \quad \gamma_i([Y^\flat]) = [Y^\flat], \quad \text{for} \quad i \neq 1.$$

From this description, it is straightforward to check that the action of Γ_R is trivial over $\mathbf{N}(T)/\pi\mathbf{N}(T)$. As the Hodge-Tate weights of V are $(-1, 0)$, we conclude that V is a positive Wach representation in the sense of Definition 3.8.

Next, for $k \geq 2$, we know that the action of G_{R_0} on T_k factors through Γ_{R_0}, so the (φ, Γ_{R_0})-module over $\mathbf{A}_{R_0}$ associated to T_k can be given as $\mathbf{D}(T_k) = \mathbf{A}_{R_0} \otimes_{\mathbb{Z}_p} T_k = \sum_{j=0}^k \mathbf{A}_{R_0} e_1^j e_2^{k-j}$. An analogous reasoning gives us that $\mathbf{D}^+(T_k) = \sum_{j=0}^k \mathbf{A}^+_{R_0} e_1^j e_2^{k-j}$. Now, by compatibility of tensor products in Corollary 3.16 (see the proof of Proposition 3.43), we obtain that $\mathbf{N}(T_k) = \mathrm{Sym}^k(\mathbf{N}(T))$. Therefore, we have

$$\mathbf{N}(T_k) = \sum_{j=0}^{k} \mathbf{A}_{R_0}^{+} h_1^{\otimes j} \otimes h_2^{\otimes(k-j)}, \quad \text{where} \quad h_1 = \pi e_1, \quad \text{and} \quad h_2 = e_2,$$

Clearly, $\mathbf{N}(T_k) \subset \mathbf{D}^{+}(T_k)$, and it is endowed with a Frobenius-semilinear endormorphism $\varphi : \mathbf{N}(T_k) \longrightarrow \mathbf{N}(T_k)$ such that $\mathbf{N}(T_k)/\varphi^*(\mathbf{N}(T_k))$ is killed by q^k. With this definition, we see that $\mathbf{N}(T_k)$ satisfies all the assumptions of Definition 3.8. As the Hodge-Tate weights of V_k are $(-k, k+1, \ldots, -1, 0)$ (see the proof of Proposition 3.43), we conclude that V_k is a positive Wach representation in the sense of Definition 3.8. ■

Now that we have constructed the Wach module $\mathbf{N}(T_k)$, we would like to study some complimentary structures on it, and compare it to the R_0-module $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$ in the sense of Theorem 3.24. First we recall that there is a filtration over Wach modules given as

$$\mathrm{Fil}^r\mathbf{N}(T_k) = \{x \in \mathbf{N}(T_k) \text{ such that } \varphi(x) \in q^r\mathbf{N}(T_k)\} \quad \text{for} \quad r \in \mathbb{Z}.$$

In our case we can write the filtration more explicitly. Indeed, we have $\mathrm{Fil}^r\mathbf{N}(T_k) = \mathbf{N}(T_k)$ for $r \leq 0$, whereas $\mathrm{Fil}^r\mathbf{N}(T_k) = \sum_{j=0}^{k} \pi^{m_j} \mathbf{A}_{R_0}^{+} h_1^j h_2^{k-j}$, for $r > 0$ and where $m_j = \max\{r-j, 0\}$.

Proposition 3.45. *Let $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ be the ring as in Definition 3.18. There exists a bijective $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$-linear map*

$$\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T_k) \longrightarrow \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^{+}} \mathbf{N}(T_k)$$

compatible with Frobenius, filtration, connection and the action of Γ_R on each side.

Proof. Using the ring $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$, we extend scalars and set

$$P = \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^{+}} \mathbf{N}(T_k) = \sum_{j=0}^{k} \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} h_1^j \otimes h_2^{k-j},$$

which is equipped with a filtration, Frobenius, a connection given as $\partial_P = \partial_A \otimes 1$, where ∂_A denotes the connection on $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ mentioned in the discussion following Lemma 3.23. Note that the connection ∂_P satisfies Griffiths transversality with respect to the filtration. Moreover, P is equipped with a continuous action of Γ_R. Next, let

$$Q = \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T_k) = \sum_{j=0}^{k} \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} f_1^j \otimes f_2^{k-j},$$

which is equipped with a filtration, Frobenius and a connection given by $\partial_Q = \partial_A \otimes 1 + 1 \otimes \partial_D$ which satisfies Griffiths transversality with respect to the filtration.

We have an $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$-linear map between these two modules, given as

$$\begin{aligned} \lambda : \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T_k) &\longrightarrow \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^{+}} \mathbf{N}(T_k) \\ f_1^j \otimes f_2^{k-j} &\longmapsto \left(\tfrac{t}{\pi} h_1 + \beta h_2\right)^j \otimes h_2^{k-j}. \end{aligned} \tag{3.14}$$

It is straightforward to see that this map is bijective. The induced Frobenius on both modules are the same because for $0 \leq j \leq k$ we have

$$\begin{aligned} \varphi_P \circ \lambda\left(f_1^j \otimes f_2^{k-j}\right) &= \varphi_P\left(\left(\tfrac{t}{\pi} h_1 + \beta h_2\right)^j \otimes h_2^{k-j}\right) \\ &= \left(\tfrac{pt}{q\pi} q h_1 + p\beta h_2\right)^j \otimes h_2^{k-j} \\ &= \lambda\left(p^j f_1^j \otimes f_2^{k-j}\right) = \lambda \circ \varphi_Q\left(f_1^j \otimes f_2^{k-j}\right). \end{aligned}$$

The induced filtration also matches, since we can write any element of $\mathrm{Fil}^r \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T_k)$ as

$\sum_{j=0}^{k} a_j f_1^j \otimes f_2^{k-j}$, with $a_j \in \mathrm{Fil}^{r-j}\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$. In this case, we have

$$\sum_{j=0}^{k} \lambda\big(a_j f_1^j \otimes f_2^{k-j}\big) = \sum_{j=0}^{k} a_j\big(\tfrac{t}{\pi}h_1 + \beta h_2\big)^j \otimes h_2^{k-j} \in \mathrm{Fil}^r\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T_k),$$

since $\frac{t}{\pi}$ is a unit, $\beta \in \mathrm{Fil}^1\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ and $h_1^j \otimes h_2^{k-j} \in \mathrm{Fil}^j\mathbf{N}(T_k)$. Similarly, for any $\sum_{j=0}^{k} b_j h_1^j \otimes h_2^{k-j} \in \mathrm{Fil}^r\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T_k)$, with $b_j \in \mathrm{Fil}^{r-j}\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$, we have

$$\sum_{j=0}^{k} \lambda^{-1}\big(b_j h_1^j \otimes h_2^{k-j}\big) = \sum_{j=0}^{k} b_j\big(\tfrac{\pi}{t}f_1 - \tfrac{\beta\pi}{t}f_2\big)^j \otimes f_2^{k-j} \in \mathrm{Fil}^r\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T_k).$$

Next, the way the respective connections on P and Q are defined, for $0 \leq j \leq k$, we can write

$$\begin{aligned}
\partial_P \circ \lambda\big(f_1^j \otimes f_2^{k-j}\big) &= \partial_P\Big(\big(\tfrac{t}{\pi}h_1 + \beta h_2\big)^j\Big) \otimes h_2^{k-j} + \big(\tfrac{t}{\pi}h_1 + \beta h_2\big)^j \otimes \partial_P\big(h_2^{k-j}\big)\\
&= j\partial_P\big(\tfrac{t}{\pi}h_1 + \beta h_2\big) \otimes \big(\tfrac{t}{\pi}h_1 + \beta h_2\big)^{j-1} \otimes h_2^{k-j}\\
&= j\big(\partial_A\big(\tfrac{t}{\pi}\big) \otimes h_1 \partial_A(\beta) \otimes h_2\big) \otimes \big(\tfrac{t}{\pi}h_1 + \beta h_2\big)^{j-1} \otimes h_2^{k-j}\\
&= -j\big(\tfrac{t}{\pi}h_1 + \beta h_2\big)^{j-1} \otimes h_2^{k-j+1} \otimes \tfrac{dX}{X}\\
&= \lambda \circ \partial_Q\big(f_1^j \otimes f_2^{k-j}\big).
\end{aligned}$$

Finally, Γ_{R_0} acts trivially on $f^j \otimes f_2^{k-j}$ and the same is true for $\big(\frac{t}{\pi}h_1 + \beta h_2\big)^j \otimes h_2^{k-j}$. So we see that the isomorphism (3.14) is compatible with all the structures. This proves the proposition. ■

CHAPTER 4

Cohomological complexes

Let K be a mixed characteristic complete discrete valuation field with perfect residule field, G_K its absolute Galois group and V a p-adic representation of G_K. The continuous G_K-cohomology groups are useful invariants attached to V. For example, the first continuous cohomology group of V, i.e. $H^1_{\mathrm{cont}}(G_K, V)$ classifies equivalent classes of extensions of the trivial representation $\mathbb{Q}_p$ by V in $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. Further, by the equivalence between the category of p-adic representations of G_K and étale (φ, Γ_K)-modules over $\mathbf{B}_K$ (see Theorem 2.11), it is natural to ask if the continuous cohomology of a representation could be computed using a complex of the attached (φ, Γ_K)-module. This question was first answered in the article of Herr (see [Her98]). He gave a three term complex in terms of (φ, Γ_K)-module which computes the continuous cohomology of the representation in each cohomological degree. More precisely,

Theorem 4.1 (Fontaine-Herr)**.** *Let V be a p-adic representation (resp. $\mathbb{Z}_p$-representation) of G_K, and let $\mathbf{D}(V)$ denote the associated étale (φ, Γ_K)-module over $\mathbf{B}_K$ (resp. $\mathbf{A}_K$). Then we have a complex*

$$C^\bullet : \mathbf{D}(V) \xrightarrow{(1-\varphi, \gamma_0 - 1)} \mathbf{D}(V) \oplus \mathbf{D}(V) \xrightarrow{\left(\begin{smallmatrix} \gamma_0 - 1 \\ 1-\varphi \end{smallmatrix}\right)} \mathbf{D}(V),$$

where the second map is $(x, y) \mapsto (\gamma_0 - 1)x - (1 - \varphi)y$. The complex $C^\bullet$ computes the continuous cohomology of V in each cohomological degree, i.e. for $k \in \mathbb{N}$, we have natural isomorphims

$$H^k(C^\bullet) \xrightarrow{\simeq} H^k_{\mathrm{cont}}(G_K, V).$$

Before discussing the relative case, let us introduce some shorthand notation for writing certain complexes.

Notation. Let $f : C_1 \longrightarrow C_2$ be a morphism of complexes. The *mapping cone* of f is the complex $\mathrm{Cone}(f)$ whose degree n part is given as $C_1^{n+1} \oplus C_2^n$ and the differential is given by $d(c_1, c_2) = (-d(c_1), d(c_2) - f(c_1))$. Next, we denote the *mapping fiber* of f by

$$\left[C_1 \xrightarrow{f} C_2\right] := \mathrm{Cone}(f)[-1].$$

We also set

$$\left[\begin{array}{ccc} C_1 & \xrightarrow{f} & C_2 \\ \downarrow & & \downarrow \\ C_3 & \xrightarrow{g} & C_4 \end{array}\right] := \left[\left[C_1 \xrightarrow{f} C_2\right] \longrightarrow \left[C_3 \xrightarrow{g} C_4\right]\right].$$

In other words, this amounts to taking the total complex of the associated double complex.

Using the notation introduced above, we can also write the quasi-isomorphism of complexes in Theorem 4.1 as

$$\left[\mathbf{R}\Gamma_{\mathrm{cont}}(\Gamma_K, \mathbf{D}(V)) \xrightarrow{1-\varphi} \mathbf{R}\Gamma_{\mathrm{cont}}(\Gamma_K, \mathbf{D}(V))\right] \xrightarrow{\simeq} \mathbf{R}\Gamma_{\mathrm{cont}}(G_K, V).$$

4.1. Relative Fontaine-Herr complex

Now we turn our attention towards the relative case. We will keep the notations of Chapters 1 & 2. Similar to Theorem 4.1, we have results in the relative case where a complex of (φ, Γ)-modules computes the continuous G_R-cohomology of a p-adic representation. For this reason, we consider the continuous cohomology (for the weak topology) of (φ, Γ_R)-modules over $\mathbf{A}_R$ and $\mathbf{A}_R^\dagger$.

Definition 4.2. Let D be a continuous (φ, Γ_R)-module over $\mathbf{A}_R$ or $\mathbf{A}_R^\dagger$. Define $\mathcal{C}^\bullet(\Gamma_R, D)$ to be the complex of continuous cochains with values in D and let $\mathbf{R}\Gamma_{\mathrm{cont}}(\Gamma_R, D)$ denote this complex in the derived category of abelian groups.

Let T be a $\mathbb{Z}_p$-module, equipped with a continuous and linear action of G_R. Let $\mathbf{D}(T)$ and $\mathbf{D}^\dagger(T)$ denote the associated (φ, Γ)-module over $\mathbf{A}_R$ and $\mathbf{A}_R^\dagger$, respectively. Then we have that,

Theorem 4.3 ([AI08, Theorem 7.10.6])**.** *The natural maps*

$$\mathbf{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{D}(T)) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}\left(G_R, T \otimes_{\mathbb{Z}_p} \mathbf{A}_{\overline{R}}\right),$$
$$\mathbf{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{D}^\dagger(T)) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}\left(G_R, T \otimes_{\mathbb{Z}_p} \mathbf{A}_{\overline{R}}^\dagger\right),$$

are isomorphisms.

Moreover, from [AI08, Proposition 8.1] we have that the sequence

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbf{A}_{\overline{R}} \xrightarrow{1-\varphi} \mathbf{A}_{\overline{R}} \longrightarrow 0$$

is exact and remains exact if we replace $\mathbf{A}_{\overline{R}}$ above with $\mathbf{A}_{\overline{R}}^\dagger$, $\mathbf{A}$ or $\mathbf{A}^\dagger$. Furthermore, the exact sequence above admits a continuous right splitting $\sigma : \mathbf{A}_{\overline{R}} \longrightarrow \mathbf{A}_{\overline{R}}$ such that $\sigma\left(\mathbf{A}_{\overline{R}}^\dagger\right) \subset \mathbf{A}_{\overline{R}}^\dagger$, $\sigma(\mathbf{A}) \subset \mathbf{A}$ and $\sigma(\mathbf{A}^\dagger) \subset \mathbf{A}^\dagger$. Combining the short exact sequence above with Theorem 4.3 and by explicit computations, Andreatta and Iovita have shown that

Theorem 4.4 ([AI08, Theorem 3.3])**.** *There are isomorphisms of δ-functors from the category* $\mathrm{Rep}_{\mathbb{Z}_p}(G_R)$ *to the category of abelian groups*

$$\beta : \left[\mathbf{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{D}(-)) \xrightarrow{1-\varphi} \mathbf{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{D}(-))\right] \xrightarrow{\simeq} \mathbf{R}\Gamma_{\mathrm{cont}}(G_R, -),$$
$$\beta^\dagger : \left[\mathbf{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{D}^\dagger(-)) \xrightarrow{1-\varphi} \mathbf{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{D}^\dagger(-))\right] \xrightarrow{\simeq} \mathbf{R}\Gamma_{\mathrm{cont}}(G_R, -).$$

Furthermore, for $T \in \mathrm{Rep}_{\mathbb{Z}_p}(G_R)$, the natural inclusion of (φ, Γ_R)-modules $\mathbf{D}^\dagger(T) \subset \mathbf{D}(T)$ induces a natural isomorphism

$$\left[\mathbf{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{D}^\dagger(-)) \xrightarrow{1-\varphi} \mathbf{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{D}^\dagger(-))\right] \xrightarrow{\simeq} \left[\mathbf{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{D}(-)) \xrightarrow{1-\varphi} \mathbf{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{D}(-))\right],$$

compatible with β and $\beta^\dagger$.

4.2. Koszul complexes

In this section, we will introduce Koszul complexes which will be used to compute continuous Γ_R-cohomology of topological modules admitting a continuous action of Γ_R, in particular (φ, Γ_R)-modules. Koszul complexes have the advantage of being explicit and therefore easier to manipulate. We will follow the exposition in [CN17, §4.2].

Recall that we set $K = F(\zeta_{p^m})$ for some $m \geq 1$. From §2.1 that the ring $R_\infty\left[\frac{1}{p}\right]$ is a Galois extension of $R\left[\frac{1}{p}\right]$, with Galois group Γ_R fitting into an exact sequence

$$1 \longrightarrow \Gamma'_R \longrightarrow \Gamma_R \longrightarrow \Gamma_K \longrightarrow 1,$$

and we have topological generators $\{\gamma_0, \gamma_1, \dots, \gamma_d\}$ of Γ_R such that $\{\gamma_1, \dots, \gamma_d\}$ are topological generators of Γ'_R and γ_0 is a lift of a topological generator of Γ_K (see Remark 2.2). Further, let χ denote the p-adic cyclotomic character and recall the convention that $c = \chi(\gamma_0) = \exp(p^m)$.

Let $\tau_i = \gamma_i - 1$ for $0 \leq i \leq d$. We consider the case of an Iwasawa algebra $A = \mathbb{Z}_p[[\tau_1, \dots, \tau_d]]$.

Definition 4.5. The *Koszul complex* associated to $(\tau_1, \dots, \tau_d)$ is the complex

$$K(\tau_1, \dots, \tau_d) = K(\tau_1)\widehat{\otimes}_{\mathbb{Z}_p} K(\tau_2)\widehat{\otimes}_{\mathbb{Z}_p} \cdots \widehat{\otimes}_{\mathbb{Z}_p} K(\tau_d),$$

where $K(\tau_i)$ is the complex

$$0 \longrightarrow \mathbb{Z}_p[[\tau_i]] \xrightarrow{\tau_i} \mathbb{Z}_p[[\tau_i]] \longrightarrow 0,$$

where the non-trivial map is multiplication by τ_i and the right-hand term is placed in degree 0.

Remark 4.6. The Koszul complex defined above, in degree q, equals the exterior power $\wedge^q A^d$. In the standard basis $\{e_{i_1 \cdots i_q}\}$ of $\wedge^q A^d$ for $1 \leq i_1 < \cdots < i_q \leq d$, the differential $d_q^1 : \wedge^q A^d \longrightarrow \wedge^{q-1} A^d$ is given by the formula

$$d_q^1\left(a_{i_1 \cdots i_q}\right) = \sum_{k=1}^{q} (-1)^{k+1} a_{i_1 \cdots \widehat{i_k} \cdots i_q} \tau_{i_k}. \tag{4.1}$$

The augmentation map $A \longrightarrow \mathbb{Z}_p$ makes $K(\tau_1, \dots, \tau_d)$ into a resolution of $\mathbb{Z}_p$ in the category of topological A-modules.

We can use this to define Koszul complex for modules equipped with an action of Γ'_R or Γ_R. Let $\mathbb{Z}_p[[\Gamma'_R]]$ denote the Iwasawa algebra of Γ'_R, i.e. the completed group ring

$$\mathbb{Z}_p[[\Gamma'_R]] := \lim_{H \trianglelefteq \Gamma'_R} \mathbb{Z}_p[\Gamma'_R/H],$$

where the (projective) limit runs over all open normal subgroups H of Γ'_R and every group ring $\mathbb{Z}_p[\Gamma'_R/H]$ is equipped with the p-adic topology. We have $\mathbb{Z}_p[[\Gamma'_R]] \simeq \mathbb{Z}_p[[\tau_1, \dots, \tau_d]]$, $\tau_i = \gamma_i - 1$ for $i \in \{1, \dots, d\}$.

Definition 4.7. The Koszul complex $K(\tau_1, \dots, \tau_d)$ is given as

$$0 \longrightarrow \mathbb{Z}_p[[\Gamma'_R]]^{I'_d} \xrightarrow{d_{d-1}^1} \cdots \xrightarrow{d_1^1} \mathbb{Z}_p[[\Gamma'_R]]^{I'_1} \xrightarrow{d_0^1} \mathbb{Z}_p[[\Gamma'_R]] \longrightarrow 0,$$

where $I'_j = \{(i_1, \dots, i_j),\ 1 \leq i_1 < \cdots < i_j \leq d\}$ and differentials as in (4.1). Similarly, for $c = \chi(\gamma_0) = \exp(p^m)$ we can define the Koszul complex $K(\tau_1^c, \dots, \tau_d^c)$ (with differentials d_q^c), where $\tau_i^c := \gamma_i^c - 1$.

Both $K(\tau_1, \dots, \tau_d)$ and $K(\tau_1^c, \dots, \tau_d^c)$ are resolutions of $\mathbb{Z}_p$ in the category of $\mathbb{Z}_p[[\Gamma'_R]]$-modules.

Definition 4.8. Let $\Lambda := \mathbb{Z}_p[[\Gamma_R]]$, and define the complex

$$K(\Lambda) : 0 \longrightarrow \Lambda^{I'_d} \xrightarrow{d^1_{d-1}} \cdots \xrightarrow{d^1_1} \Lambda^{I'_1} \xrightarrow{d^1_0} \Lambda \longrightarrow 0.$$

We have an isomorphism

$$\lim_m \mathbb{Z}_p\left[\Gamma_K/(\Gamma_K)^{p^m}\right] \otimes_{\mathbb{Z}_p} K(\tau_1, \dots, \tau_d) \simeq K(\Lambda),$$

of left Λ- and right $\mathbb{Z}_p[[\tau_1, \dots, \tau_d]]$-modules (see [Mor08, Lemma 4.3]). Therefore, the complex $K(\Lambda)$ is a resolution of $\mathbb{Z}_p[[\Gamma_K]]$ in the category of topological left Λ-modules. Similarly, we have the complex $K^c(\Lambda)$, obtained from $K(\tau_1^c, \dots, \tau_d^c)$, which is again a resolution of $\mathbb{Z}_p[[\Gamma_K]]$.

Definition 4.9. Define a map

$$\tau_0 : K^c(\Lambda) \longrightarrow K(\Lambda),$$

by the commutative diagram of topological left Λ-modules

$$\begin{tikzcd}
0 \arrow[r] & \Lambda^{I'_d} \arrow[r, "d^c_{d-1}"] \arrow[d, "\tau_0^d"] & \cdots \arrow[r, "d^c_1"] & \Lambda^{I'_1} \arrow[r, "d^c_0"] \arrow[d, "\tau_0^1"] & \Lambda \arrow[r] \arrow[d, "\tau_0^0"] & \mathbb{Z}_p[[\Gamma_K]] \arrow[r] \arrow[d, "\gamma_0-1"] & 0 \\
0 \arrow[r] & \Lambda^{I'_d} \arrow[r, "d^1_{d-1}"] & \cdots \arrow[r, "d^1_1"] & \Lambda^{I'_1} \arrow[r, "d^1_0"] & \Lambda \arrow[r] & \mathbb{Z}_p[[\Gamma_K]] \arrow[r] & 0,
\end{tikzcd}$$

where the vertical maps are defined as

$$\begin{aligned}
\tau_0^0 &= \gamma_0 - 1 \\
\tau_0^q &: \left(a_{i_1 \cdots i_q}\right) \mapsto \left(a_{i_1 \cdots i_q}\left(\gamma_0 - \delta_{i_1 \cdots i_q}\right)\right),
\end{aligned}$$

for $1 \leq q \leq d$, $1 \leq i_1 < \cdots < i_q \leq d$, and $\delta_{i_1 \cdots i_q} = \delta_{i_q} \cdots \delta_{i_1}$, where $\delta_{i_j} = \left(\gamma_{i_j}^c - 1\right)\left(\gamma_{i_j} - 1\right)^{-1}$.

Let M be a topological $\mathbb{Z}_p$-module admitting a continuous action of Γ_R.

Definition 4.10. Define the complexes

$$\begin{aligned}
\mathrm{Kos}(\Gamma'_R, M) &:= \mathrm{Hom}_{\Lambda,\mathrm{cont}}(K(\Lambda), M) = \mathrm{Hom}_\Lambda(K(\Lambda), M), \\
\mathrm{Kos}^c(\Gamma'_R, M) &:= \mathrm{Hom}_{\Lambda,\mathrm{cont}}(K^c(\Lambda), M) = \mathrm{Hom}_\Lambda(K^c(\Lambda), M).
\end{aligned}$$

Remark 4.11. Using Definition 4.8, we can write the complexes in Definition 4.10 as

$$\mathrm{Kos}(\Gamma'_R, M) : M \xrightarrow{(\tau_i)} M^{I'_1} \longrightarrow \cdots \longrightarrow M^{I'_d},$$

$$\mathrm{Kos}^c(\Gamma'_R, M) : M \xrightarrow{(\tau_i^c)} M^{I'_1} \longrightarrow \cdots \longrightarrow M^{I'_d}.$$

The map $\tau_0 : K^c(\Lambda) \longrightarrow K(\Lambda)$ induces a map of complexes

$$\tau_0 : \mathrm{Kos}(\Gamma'_R, M) \longrightarrow \mathrm{Kos}^c(\Gamma'_R, M),$$

which can be represented by the commutative diagram

$$\begin{tikzcd}
M \arrow[r, "(\tau_i)"] \arrow[d, "\tau_0^0"] & M^{I'_1} \arrow[r] \arrow[d, "\tau_0^1"] & \cdots \arrow[r] & M^{I'_d} \arrow[d, "\tau_0^d"] \\
M \arrow[r, "(\tau_i^c)"] & M^{I'_1} \arrow[r] & \cdots \arrow[r] & M^{I'_d}.
\end{tikzcd}$$

Let $K(\Lambda,\tau) := [K^c(\Lambda) \xrightarrow{\tau_0} K(\Lambda)]$. This complex is a resolution of $\mathbb{Z}_p$ in the category of topological left Λ-modules.

Definition 4.12. Define the Γ_R-*Koszul complex* with values in M as

$$\mathrm{Kos}(\Gamma_R, M) := \mathrm{Hom}_{\Lambda,\mathrm{cont}}(K(\Lambda,\tau), M) = \big[\mathrm{Kos}(\Gamma_R', M) \xrightarrow{\tau_0} \mathrm{Kos}^c(\Gamma_R', M)\big].$$

By the general theory of continuous group cohomology for p-adic Lie groups, we have the following conclusion:

Proposition 4.13 ([Laz65, Lazard])**.** *There exists a natural quasi-isomorphism*

$$\mathrm{Kos}(\Gamma_R, M) \xrightarrow{\simeq} \mathbf{R}\Gamma_{\mathrm{cont}}(\Gamma_R, M).$$

Definition 4.14. Let D be a (φ,Γ_R)-module over $\mathbf{A}_R$ from Definition 2.9. Define the complex

$$\mathrm{Kos}(\varphi,\Gamma_R, D) := \left[\begin{tikzcd} \mathrm{Kos}(\Gamma_R', D) \arrow[r, "1-\varphi"] \arrow[d, "\tau_0"] & \mathrm{Kos}(\Gamma_R', D) \arrow[d, "\tau_0"] \\ \mathrm{Kos}^c(\Gamma_R', D) \arrow[r, "1-\varphi"] & \mathrm{Kos}^c(\Gamma_R', D) \end{tikzcd}\right].$$

Therefore, from Proposition 4.13 we have a natural quasi-isomorphism

$$\mathrm{Kos}(\varphi,\Gamma_R, D) \xrightarrow{\simeq} \big[\mathbf{R}\Gamma_{\mathrm{cont}}(\Gamma_R, D) \xrightarrow{1-\varphi} \mathbf{R}\Gamma_{\mathrm{cont}}(\Gamma_R, D)\big].$$

Using the definition above, we have the following conclusion for p-adic representations of G_R:

Proposition 4.15. *Let T be a $\mathbb{Z}_p$-representation of G_R and $\mathbf{D}(T)$ the associated (φ,Γ_R)-module over $\mathbf{A}_R$ (see Theorem 2.11). Then from the discussion above and Theorem 4.4, we get a natural quasi-isomorphism*

$$\mathrm{Kos}(\varphi,\Gamma_R,\mathbf{D}(T)) \xrightarrow{\simeq} \mathbf{R}\Gamma_{\mathrm{cont}}(G_R, T).$$

4.3. Lie algebra action and cohomology

In this section we will study the infinitesimal action of Γ_R on some of the rings constructed in previous sections. This will help us in computing continuous Lie algebra cohomology of certain $\mathbb{Z}_p[[\mathrm{Lie}\,\Gamma_R]]$-modules, which is roughly the same as continuous Lie group cohomology of these modules. Recall from the previous section that we have topological generators $\{\gamma_0, \gamma_1, \dots, \gamma_d\}$ of Γ_R such that $\{\gamma_1, \dots, \gamma_d\}$ are topological generators of Γ_R' and γ_0 is a lift of a topological generator of Γ_K.

In the rest of this section we will fix constants $u, v \in \mathbb{R}$ such that $\frac{p-1}{p} \leq u \leq \frac{v}{p} < 1 < v$, for example, one can fix $u = \frac{p-1}{p}$ and $v = p-1$. Recall from §2.4 that we have rings $\mathbf{A}_R^{\mathrm{PD}}$, $\mathbf{A}_R^{[u]}$ and $\mathbf{A}_R^{[u,v]}$ equipped with a continuous action of Γ_R.

Lemma 4.16. *For $i \in \{0,1,\dots,d\}$ the operators*

$$\nabla_i := \log \gamma_i = \sum_{k\in\mathbb{N}} (-1)^k \tfrac{(\gamma_i - 1)^{k+1}}{k+1},$$

converge as series of operators on $\mathbf{A}_R^{\mathrm{PD}}$, $\mathbf{A}_R^{[u]}$ and $\mathbf{A}_R^{[u,v]}$.

Proof. Recall that any $f \in \mathbf{A}_R^{\mathrm{PD}}$ can be written as $f = \sum_{n\in\mathbb{N}} f_n \frac{\pi_m^n}{[n/e]!}$ such that $f_n \in \mathbf{A}_R^+$ goes to 0 as $n \longrightarrow +\infty$. So it is enough to show that the series of operators $\log \gamma_0$ converge for π_m, i.e. $\nabla_0(\pi_m)$ converges in $\mathbf{A}_K^{\mathrm{PD}}$ and therefore in $\mathbf{A}_R^{\mathrm{PD}}$.

From Lemma 2.45, we already have that

$$(\gamma_0 - 1)^k \mathbf{A}_R^{\mathrm{PD}} \subset \big(p, \pi_m^{p^m}\big)^k \mathbf{A}_R^{\mathrm{PD}}. \tag{4.2}$$

So to check that the series $\nabla_0(\pi_m)$ converges over $\mathbf{A}_K^{\mathrm{PD}}$ we write it as $\sum_j c_j \pi_m^j$ and we collect the coefficients of $\pi_m^{p^m k}$ for $k \geq 1$, having the smallest p-adic valuation, which will also have the least p-adic valuation among the coefficients of π_m^j for $p^m k \leq j \leq p^m(k+1)$. We write the collection of these terms as

$$\sum_{k\geq 1} (-1)^{k+1} \frac{\pi_m^{p^m k}}{k} = \sum_{k \geq 1} (-1)^{k+1} \frac{\lfloor p^m k/e \rfloor !}{k} \frac{\pi_m^{p^m k}}{\lfloor p^m k/e \rfloor !},$$

and by the preceding discussion it is enough to show that these coefficients go to 0 as $k \longrightarrow +\infty$. Moreover, for this series it suffices to check the estimate of coefficients for $k = (p-1)r$ as $r \longrightarrow +\infty$ (this gets rid of the floor function above). With help of Remark 2.44, we have

$$v_p\Big(\frac{\lfloor p^m k/e \rfloor !}{k}\Big) = v_p\Big(\frac{(pr)!}{(p-1)r}\Big) = v_p((pr)!) - v_p((p-1)r) \geq \frac{pr - s_p(pr)}{p-1} - r = \frac{r - s_p(r)}{p-1} = v_p(r!),$$

which goes to $+\infty$ as $j \longrightarrow +\infty$. Hence, we conclude that $\nabla_0 = \log \gamma_0$ converges as a series of operators on $\mathbf{A}_K^{\mathrm{PD}}$.

Next, consider γ_i for $i \in \{1, \dots, d\}$. Again from Lemma 2.45 we have

$$(\gamma_i - 1)^k [X_i^\flat] = \pi [X_i^\flat] \in \big(p, \pi_m^{p^n}\big)^k \mathbf{A}_R^{\mathrm{PD}}. \tag{4.3}$$

By an argument similar to the case of γ_0 it follows that $\nabla_i = \log \gamma_i$ converges as a series of operator on $\mathbf{A}_R^{\mathrm{PD}}$. The arguments in the case of $\mathbf{A}_R^{[u]}$ and $\mathbf{A}_R^{[u,v]}$ follow similarly (the estimates of p-adic valuation of coefficients is easier). ∎

Next, note that formally we can write

$$\begin{aligned}\frac{\log(1+X)}{X} &= 1 + a_1 X + a_2 X^2 + a_3 X^3 + \cdots, \\ \frac{X}{\log(1+X)} &= 1 + b_1 X + b_2 X^2 + b_3 X^3 + \cdots,\end{aligned}$$

where $v_p(a_k) \geq -\frac{k}{p-1}$ for all $k \geq 1$ and therefore, $v_p(b_k) \geq -\frac{k}{p-1}$ for all $k \geq 1$. Setting $X = \gamma_i - 1$ for $i \in \{0, 1, \dots, d\}$, we make the following claim:

Lemma 4.17. *For $i \in \{0, 1, \dots, d\}$, the operators*

$$\frac{\nabla_i}{\gamma_i - 1} = \frac{\log \gamma_i}{\gamma_i - 1} \quad \textit{and} \quad \frac{\gamma_i - 1}{\nabla_i} = \frac{\gamma_i - 1}{\log \gamma_i}$$

converge as series of operators on $\mathbf{A}_R^{\mathrm{PD}}$, $\mathbf{A}_R^{[u]}$ and $\mathbf{A}_R^{[u,v]}$.

Proof. We will only show that these series converge on $\mathbf{A}_R^{\mathrm{PD}}$, the case of $\mathbf{A}_R^{[u]}$ and $\mathbf{A}_R^{[u,v]}$ follow similarly. Moreover, similar to Lemma 4.16 it suffices to check the convergence of these operators for their action on π_m.

So we will check that the series $\frac{\nabla_i}{\gamma_i - 1}(\pi_m)$ converges over $\mathbf{A}_K^{\mathrm{PD}}$ (the convergence of the other series follows similarly since $v_p(a_k) \geq -\frac{k}{p-1}$ and $v_p(b_k) \geq -\frac{k}{p-1}$ for $k \in \mathbb{N}$). From the description of the action of $\gamma_i - 1$ in (4.2) and (4.3), we can write the series $\frac{\nabla_i}{\gamma_i - 1}(\pi_m)$ as $\sum_j c_j \pi_m^j$. Next, we collect the coefficients of $\pi_m^{p^m k}$ for $k \geq 1$, having the smallest p-adic valuation, which will also have the least p-adic valuation among the coefficients of π_m^j for $p^m k \leq j \leq p^m(k+1)$. We write the collection of these terms as

$$\sum_{k\geq 1}(-1)^{k+1}a_k\pi_m^{p^m k} = \sum_{k\geq 1}(-1)^{k+1}a_k\lfloor p^m k/e\rfloor!\tfrac{\pi_m^{p^m k}}{\lfloor p^m k/e\rfloor!}.$$

By the preceding discussion it is enough to show that these coefficients go to 0 as $k \longrightarrow +\infty$. Moreover, for this series it suffices to check the estimate of coefficients for $k = (p-1)r$ as $r \longrightarrow +\infty$ (this gets rid of the floor function above). With help of Remark 2.44, we have

$$\upsilon_p\big(a_k\lfloor p^m k/e\rfloor!\big) = \upsilon_p\big(a_{(p-1)r}(pr)!\big) = \upsilon_p((pr)!) - \upsilon_p\big(a_{(p-1)r}\big) \geq \tfrac{pr-s_p(pr)}{p-1} - r = \tfrac{r-s_p(r)}{p-1} = \upsilon_p(r!),$$

which goes to $+\infty$ as $r \longrightarrow +\infty$. Hence, it follows that the series in the claim converge for $\mathbf{A}_R^{\mathrm{PD}}$, $\mathbf{A}_R^{[u]}$ and $\mathbf{A}_R^{[u,v]}$. ■

Example 4.18. Recall that in §3.3, we constructed Wach modules arising from symmetric powers of the p-adic Tate module of the false Tate curve. For $k \in \mathbb{N}$, we have T_k as a $\mathbb{Z}_p$-representation of G_{R_0} with a basis $\{e_1^j \otimes e_2^{k-j}\}_{0\leq j\leq k}$. The Wach module is given as $\mathbf{N}(T_k) = \sum_{j=0}^k \mathbf{A}_{R_0}^+ h_1^j \otimes h_2^{k-j}$. Our objective is to analyze the action of Lie Γ_R over $M^{\mathrm{PD}} := \mathbf{A}_R^{\mathrm{PD}} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T_k)$. Note that by Lemma 4.16, the operators $\nabla_i = \log\gamma_i$ for $i \in \{0,\ldots,d\}$, converge as a series of operators over $\mathbf{A}_R^{\mathrm{PD}}$.

It is straightforward to see that $\nabla_i(h_2) = 0$ for $0 \leq i \leq d$. Further, we have

$$\nabla_0(e_1) = \lim_{k\to\infty}\frac{\gamma_0^{p^k}(e_1) - e_1}{p^k} = \lim_{k\to\infty}\frac{\chi(\gamma_0)^{-p^k}e_1 - e_1}{p^k} = -\log\chi(\gamma_0)e_1,$$

and $\nabla_0(\pi) = t(1+\pi)$. So we get that $\nabla_0(\pi e_1) = (t(1+\pi) - \pi)e_1$. Therefore, for $0 \leq j \leq k$, we have

$$\nabla_0\big(h_1^j h_2^{k-j}\big) = \nabla_0\big(\pi^j e_1^j e_2^{k-j}\big) = j\big(t(1+\pi)\pi^{j-1} - \pi^j\big)e_1^j e_2^{k-j} = j\big(\tfrac{t}{\pi}(1+\pi) - 1\big)h_1^j h_2^{k-j}.$$

For the action of γ_1, we have

$$\nabla_1(e_1) = \lim_{n\to\infty}\frac{\gamma_1^{p^n}(e_1) - e_1}{p^n} = \lim_{n\to\infty} -\frac{p^n e_2}{p^n} = -e_2.$$

Since γ_1 has trivial action on π, we get that $\nabla_1(\pi e_1) = -\pi e_2$. Therefore, in this case, for $0 \leq j \leq k$, we have

$$\nabla_1\big(h_1^j h_2^{k-j}\big) = \nabla_1\big(\pi^j e_1^j e_2^{k-j}\big) = -j\pi^j e_1^{j-1}e_2^{k-j+1} = -j\pi h_1^{j-1}h_2^{k-j+1}.$$

Finally, for $2 \leq i \leq d$, we have $\nabla_i(\pi e_1) = 0$, therefore $\nabla_i\big(h_1^j h_2^{k-j}\big) = 0$. As we can see that $\nabla_i(M^{\mathrm{PD}}) \subset \pi M^{\mathrm{PD}}$ and since $\frac{\pi}{t}$ is a unit in $\mathbf{A}_R^{\mathrm{PD}}$ (see Lemma 2.43), we can introduce differential operators on M^{PD}. More precisely, in the basis $\big\{\frac{d\pi_m}{1+\pi_m}, d\log[X_1^\flat], \ldots, d\log[X_d^\flat]\big\}$ of $\Omega^1_{\mathbf{A}_R^{\mathrm{PD}}}$, the connection can be deduced by the relation $\nabla_i = t\partial_i$, for $1 \leq i \leq d$.

4.3.1. Koszul Complex

In this section, we turn our attention to the computation of Lie algebra cohomology using Koszul complexes. The Lie algebra Lie Γ_R' of the p-adic Lie group Γ_R' is a free $\mathbb{Z}_p$-module of rank d, i.e. Lie $\Gamma_R' = \mathbb{Z}_p[\nabla_i]_{1\leq i\leq d}$ with

$$\nabla_i := \log\gamma_i = \sum_{k\in\mathbb{N}}(-1)^k\tfrac{(\gamma_i-1)^{k+1}}{k+1} : M \longrightarrow M,$$

for any Lie Γ_R'-module M. Moreover, Lie Γ_R' is commutative. Similarly, the Lie algebra Lie Γ_R of the p-adic Lie group Γ_R is a free $\mathbb{Z}_p$-module of rank $d+1$, i.e. Lie $\Gamma_R = \mathbb{Z}_p[\nabla_i]_{0\leq i\leq d}$ (∇_i defined as above for $0 \leq i \leq d$). We have

$$\begin{aligned} [\nabla_i, \nabla_j] &= 0, && \text{for} \quad 1 \leq i, j \leq d, \\ [\nabla_0, \nabla_i] &= p^m \nabla_i, && \text{for} \quad 1 \leq i \leq d. \end{aligned} \tag{4.4}$$

It follows that $\mathrm{Lie}\,\Gamma'_R$ is not commutative.

Let M be a topological $\mathbb{Z}_p$-module admitting a continuous action of the Lie algebra $\mathrm{Lie}\,\Gamma_R$. Similar to the definition of Koszul complexes in the case of Γ_R (see §4.2), we define Koszul complexes for $\mathrm{Lie}\,\Gamma_R$.

Definition 4.19. Define the complex

$$\mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, M) \;:\; M \longrightarrow M^{I'_1} \longrightarrow \cdots \longrightarrow M^{I'_d},$$

with differentials dual to those in (4.1) (with τ_i replaced by ∇_i).

Now, consider the map

$$\nabla_0 \;:\; \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, M) \longrightarrow \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, M),$$

defined by the diagram

$$\begin{tikzcd} M \arrow[r, "(\nabla_i)"] \arrow[d, "\nabla_0"] & M^{I'_1} \arrow[r] \arrow[d, "\nabla_0 - p^m"] & \cdots \arrow[r] & M^{I'_q} \arrow[r] \arrow[d, "\nabla_0 - qp^m"] & \cdots \\ M \arrow[r, "(\nabla_i)"] & M^{I'_1} \arrow[r] & \cdots \arrow[r] & M^{I'_q} \arrow[r] & \cdots, \end{tikzcd}$$

which commutes since $\nabla_0\nabla_i - \nabla_i\nabla_0 = p^m\nabla_i$ for $1 \leq i \leq d$ (see (4.4)). Note that the k-th vertical arrow is $\nabla_0 - kp^m$ since the $(k-1)$-th vertical arrow is $\nabla_0 - (k-1)p^m$ and using (4.4) trivially we have $(\nabla_0 - kp^m)\nabla_i = \nabla_i(\nabla_0 - (k-1)p^m)$.

Definition 4.20. Define the $\mathrm{Lie}\,\Gamma_R$*-Koszul complex* for M as

$$\mathrm{Kos}(\mathrm{Lie}\,\Gamma_R, M) \;:=\; \big[\mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, M) \xrightarrow{\nabla_0} \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, M)\big].$$

Proposition 4.21 ([Laz65, Lazard]). *The Koszul complexes in Definitions 4.19 and 4.20 compute Lie algebra cohomology of* $\mathrm{Lie}\,\Gamma'_R$ *and* $\mathrm{Lie}\,\Gamma_R$ *respectively, with values in M. In other words, we have natural quasi-isomorphisms*

$$\begin{aligned} \mathbf{R}\Gamma_{\mathrm{cont}}(\mathrm{Lie}\,\Gamma'_R, M) &\simeq \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, M), \\ \mathbf{R}\Gamma_{\mathrm{cont}}(\mathrm{Lie}\,\Gamma_R, M) &\simeq \mathrm{Kos}(\mathrm{Lie}\,\Gamma_R, M). \end{aligned}$$

CHAPTER 5

Syntomic complex and Galois cohomology

Let K be a mixed characteristic complete discrete valuation field with ring of integers O_K and residue field κ of characteristic p. Let X be a smooth proper scheme over O_K, such that $j : X_K := X \otimes_{O_K} K \rightarrowtail X$ denotes the inclusion of its generic fiber and $i : X_0 := X \otimes_{O_K} \kappa \rightarrowtail X$ denotes the inclusion of its special fiber. For $r \geq 0$, let $\mathcal{S}_n(r)_X$ denote the syntomic sheaf modulo p^n on $X_{0,\text{ét}}$. In [FM87], Fontaine and Messing constructed period morphisms

$$\alpha_{r,n}^{\mathrm{FM}} : \mathcal{S}_n(r)_X \longrightarrow i^* \mathbf{R} j_* \mathbb{Z}/p^n(r)'_{X_K}, \quad r \geq 0,$$

from syntomic cohomology to p-adic nearby cycles, where $\mathbb{Z}_p(r)' := \frac{1}{p^{a(r)}} \mathbb{Z}_p(r)$, for $r = (p-1)a(r) + b(r)$ with $0 \leq b(r) \leq p-1$.

In [CN17], Colmez and Nizioł have shown that the Fontaine-Messing period map $\alpha_{r,n}^{\mathrm{FM}}$, after a suitable truncation, is essentially a quasi-isomorphism. More precisely,

Theorem 5.1 ([CN17, Theorem 1.1]). *For $0 \leq k \leq r$, the map*

$$\alpha_{r,n}^{\mathrm{FM}} : \mathcal{H}^k(\mathcal{S}_n(r)_X) \longrightarrow i^* \mathbf{R}^k j_* \mathbb{Z}/p^n(r)'_{X_K},$$

is a p^N-isomorphism, i.e. there exists $N = N(e, p, r) \in \mathbb{N}$ depending on r and the absolute ramification index e of K but not on X or n, such that the kernel and cokernel of the map is killed by p^N.

In fact, for $k \leq r \leq p-1$, the map $\alpha_{r,n}^{\mathrm{FM}}$ was shown to be an isomorphism by Kato [Kat89, Kat94], Kurihara [Kur87], and Tsuji [Tsu99]. Further, in [Tsu96] Tsuji generalized this result to some suitable étale local systems.

The proof of Colmez and Nizioł is different from earlier approaches. They construct another local period map $\alpha_r^{\mathcal{L}az}$, employing techniques from the theory of (φ, Γ)-modules and a version of integral Lazard isomorphism between Lie algebra cohomology and continuous group cohomology. Then they proceed to show that this map is a quasi-isomorphism and coincides with Fontaine-Messing period map up to some constants. Moreover, all of their results have been worked out in the general setting of log-schemes.

To state the local result, we will restrict ourselves to a familiar setting. We will assume the setup of Chapter 1, as well as notations from Chapter 2. Recall that we fixed κ to be a finite field of characteristic p; $F = \mathrm{Fr}\, W = \mathrm{Fr}\, W(\kappa)$; an integer $m \geq 1$ and $K = F(\zeta_{p^m})$, where ζ_{p^m} is a primitive p^m-th root of unity such that the element $\varpi = \zeta_{p^m} - 1$ is a uniformizer of K. Moreover, let $X = (X_1, \ldots, X_d)$ be a set of indeterminates, then we defined R_0 to be the p-adic completion of an étale algebra over $W(\kappa)\{X, X^{-1}\}$; similarly, R to be the p-adic completion of an étale algebra over $R_\square$ (defined using the same equations as in the definition of R_0). We also have rings $r_\varpi^\star$ and $R_\varpi^\star$ for $\star \in \{\ , +, \mathrm{PD}, [u], (0, v]+, [u, v]\}$. Recall that we assumed $p \geq 3$ and we take $u = \frac{p-1}{p}$ and $v = p-1$.The p-adic completion of the module of differentials of R_0 relative to $\mathbb{Z}$ is given as

$$\Omega^1_{R_0} = \bigoplus_{i=1}^d R_0 \ d\log X_i \ \text{ and } \ \Omega^k_{R_0} = \bigwedge^k \Omega^1_{R_0}, \ \text{ for } \ k \in \mathbb{N}.$$

Moreover, the kernel and cokernel of the natural map $\Omega^k_{R_0} \otimes_{R_0} R \longrightarrow \Omega^k_R$ is killed by a power of p (see Proposition 1.1). In particular,

$$\Omega^k_R\big[\tfrac{1}{p}\big] = \bigwedge^k \big(\bigoplus_{i=1}^d R\big[\tfrac{1}{p}\big] \ d\log X_i\big).$$

Also, for $S = R_\varpi^\star$ where $\star \in \{+, \mathrm{PD}, [u], [u, v]\}$, we have

$$\Omega^1_S = S\tfrac{dX_0}{1+X_0} \oplus \big(\bigoplus_{i=1}^d S \ d\log X_i\big).$$

The syntomic cohomology of R can be computed by the complex

$$\mathrm{Syn}(R, r) \ := \mathrm{Cone}\big(F^r\Omega^\bullet_{R_\varpi^{\mathrm{PD}}} \xrightarrow{p^r - p^\bullet\varphi} \Omega^\bullet_{R_\varpi^{\mathrm{PD}}}\big)[-1],$$

such that we have $H^i_{\mathrm{syn}}(R, r) = H^i(\mathrm{Syn}(R, r))$. For m large enough, Colmez and Nizioł have shown that,

Theorem 5.2 ([CN17, Theorem 1.6]). *The maps*

$$\begin{aligned} \alpha_r^{\mathcal{L}az} \ &: \ \tau_{\leq r}\mathrm{Syn}(R, r) \longrightarrow \tau_{\leq r}\mathbf{R}\Gamma_{\mathrm{cont}}(G_R, \mathbb{Z}_p(r)), \\ \alpha_{r,n}^{\mathcal{L}az} \ &: \ \tau_{\leq r}\mathrm{Syn}(R, r)_n \longrightarrow \tau_{\leq r}\mathbf{R}\Gamma_{\mathrm{cont}}(G_R, \mathbb{Z}/p^n(r)) \longrightarrow \tau_{\leq r}\mathbf{R}\Gamma\big(\big(\mathrm{Sp}\ R\big[\tfrac{1}{p}\big]\big)_{\text{ét}}, \mathbb{Z}/p^n(r)\big), \end{aligned} \tag{5.1}$$

are p^{Nr}-quasi-isomorphisms for a universal constant N.

Note that the truncation here denotes the canonical truncation in literature. Finally, using Galois descent one can obtain the result over K (not necessarily having enough roots of unity, with N depending on K, p and r, see [CN17, Theorem 5.4]).

Formulation of the main result

In Theorem 5.2, we are interested in the p-adic result, i.e. the first isomorphism in (5.1), where we would like to insert some representation on the right hand side and an appropriate syntomic object on the left. For this, we will introduce a certain class of representations: Let V be an h-dimensional p-adic Wach representation of G_{R_0} with non-positive Hodge-Tate weights $-s = -r_1 \leq -r_2 \leq \cdots \leq -r_h \leq 0$ and let $T \subset V$ a free $\mathbb{Z}_p$-lattice of rank h stable under the action of G_{R_0} (see Definition 3.8). Assume that $\mathbf{N}(T)$ is a free $\mathbf{A}^+_{R_0}$-module of rank h, and let $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ be a free R_0-submodule of rank h such that $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)\big[\frac{1}{p}\big] = \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ (see Remark 5.4 for conventions on $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$) and the induced connection over $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$ is quasi-nilpotent, integrable and satisfies Griffiths transversality with respect to the induced filtration.

Definition 5.3. For $r \in \mathbb{Z}$ we set $V(r) \ := V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(r)$ and $T(r) \ := T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(r)$ and call all such representations *free Wach representations* of G_{R_0}.

Remark 5.4. For our intended applications in this chapter, it would suffice to take $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) \ := \big(\mathcal{O}\mathbf{A}^{\mathrm{PD}}_R \otimes_{\mathbf{A}^+_{R_0}} \mathbf{N}(T)\big)^{\Gamma_R}$, with an additional assumption that it is free of rank h as an R_0-module (see

Remark 3.42). The module $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$ depends on the choice of $m \in \mathbb{N}_{\geq 1}$ (see Remark 3.39). On the other hand, using Proposition 3.31 we note that it would also suffice to take $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) = M_0$ (in the notation of the proposition), which also depends on m (see Remark 3.39). The reader should note that we do not assume the choice of $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$ to be "canonical". However, we fix this choice for the rest of the current chapter. The chosen notation is for the sake of consistency and being explanatory.

Our objective is to relate the (φ, Γ)-module complex computing the continuous G_R-cohomology of $T(r)$ (see Theorem 4.4), to syntomic complex with coefficient in the R_0-lattice $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$. Define

$$D^{\mathrm{PD}} := R_{\varpi}^{\mathrm{PD}} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T).$$

There is a Frobenius-semilinear endomorphism on D^{PD} given by the diagonal action of the Frobenius on each component of the tensor product, and a filtration $\{\mathrm{Fil}^k D^{\mathrm{PD}}\}_{k \in \mathbb{Z}}$ given as the sum of filtration on each component (see §5.1 for explicit formulas). Further, D^{PD} is equipped with a connection $\partial : D^{\mathrm{PD}} \longrightarrow D^{\mathrm{PD}} \otimes_{R_{\varpi}^{\mathrm{PD}}} \Omega^1_{R_{\varpi}^{\mathrm{PD}}}$ arising from the connection on $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$ and the differential operator on R_{ϖ}^{PD} (see §5.1 for details). Moreover, the connection on D^{PD} satistfies Griffiths transversality with respect to the filtration. In conclusion, we have a filtered de Rham complex for $k \in \mathbb{Z}$,

$$\mathrm{Fil}^k \mathcal{D}^\bullet := \mathrm{Fil}^k D^{\mathrm{PD}} \otimes_{R_{\varpi}^{\mathrm{PD}}} \Omega^1_{R_{\varpi}^{\mathrm{PD}}} \longrightarrow \mathrm{Fil}^{k-1} D^{\mathrm{PD}} \otimes_{R_{\varpi}^{\mathrm{PD}}} \Omega^2_{R_{\varpi}^{\mathrm{PD}}} \longrightarrow \cdots.$$

Definition 5.5. Let $r \in \mathbb{N}$ and $D_R := R \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$. Define the *syntomic complex* $\mathrm{Syn}(D_R, r)$ and the *syntomic cohomology* of R with coefficients in D_R as

$$\begin{aligned} \mathrm{Syn}(D_R, r) &:= \left[\ \mathrm{Fil}^r \mathcal{D}^\bullet \xrightarrow{p^r - p^\bullet \varphi} \mathcal{D}^\bullet\ \right]; \\ H^*_{\mathrm{syn}}(D_R, r) &:= H^*(\mathrm{Syn}(D_R, r)). \end{aligned}$$

We will relate this complex to Fontaine-Herr complex computing the continuous G_R-cohomology of $T(r)$. The key idea is to interpret all these complexes in terms of Koszul complexes, and by applying a version of Poincaré lemma, we can further relate the syntomic complexes to "(φ, Γ)-module Koszul complexes". The main result of this chapter is:

Theorem 5.6. *Let T be a free $\mathbb{Z}_p$-representation of G_{R_0} as in Definition 5.3 such that $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$ is a free positive Wach representation. Let s be the maximum among the absolute value of the Hodge-Tate weights of V and $r \in \mathbb{Z}$ such that $r \geq s+1$. Then there exists a p^N-quasi-isomorphism*

$$\tau_{\leq r-s-1} \mathrm{Syn}(D_R, r) \simeq \tau_{\leq r-s-1} \mathbf{R}\Gamma_{\mathrm{cont}}(G_R, T(r)),$$

where $N = N(T, e, p, r) \in \mathbb{N}$ depending on the representation T, ramification index e, the prime p, and r. In particular, we have p^N-isomorphisms

$$H^k_{\mathrm{syn}}(D_R, r) \xrightarrow{\simeq} H^k(G_R, T(r)),$$

for $0 \leq k \leq r-s-1$.

The proof of Theorem 5.6 will proceed in two main steps: First, we will modify the syntomic complex with coefficients in D_R to relate it to a "differential" Koszul complex with coefficients in $\mathbf{N}(T)$ (see Proposition 5.30). Next, in the second step we will modify the Koszul complex from the first step to obtain Koszul complex computing continuous G_R-cohomology of $T(r)$ (see Definition 5.6 and Proposition 5.31). The key to the connection between these two steps will be provided by the comparison isomorphism in Theorem 3.24.

5.1. Syntomic complex with coefficients

In this section we will carry out computations involving syntomic complexes in order to prove Theorem 5.6. More precisely, we will define syntomic complexes with coefficients in $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$, over various rings introduced in §2.3. Then, we will relate these complexes to differential Koszul complex with coefficients in $\mathbf{N}(T)$. Further computations clarifying relations between differential Koszul complex and Galois cohomology of $T(r)$ will be worked out in the next section.

We begin by fixing some notations for the rest of this section. For $\bigstar \in \{[u], [u, v], [u, v/p]\}$, we set

$$D^{\bigstar} := R_{\varpi}^{\bigstar} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T),$$

which is an $R_{\varpi}^{\bigstar}$-module. By considering the diagonal action of the Frobenius on each component of the tensor product, we can define Frobenius-semilinear operators $\varphi : D^{[u]} \longrightarrow D^{[u]}$ and $\varphi : D^{[u,v]} \longrightarrow D^{[u,v/p]}$. Let S be a placeholder notation for $R_{\varpi}^{\bigstar}$ and D a placeholder for $D^{\bigstar}$ below. We equip D with a filtration

$$\mathrm{Fil}^k D = \sum_{i+j=k} \mathrm{Fil}^i S \widehat{\otimes}_{R_0} \mathrm{Fil}^j \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T), \quad \text{for} \quad k \in \mathbb{Z}. \tag{5.2}$$

Further, if ∂_D denotes the connection on $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$ then we can equip D with a connection

$$\begin{aligned} \partial : D &\longrightarrow D \otimes_S \Omega_S^1 \\ a \otimes x &\longmapsto a \otimes \partial_D(x) + x da, \end{aligned}$$

which satistfies Griffiths transversality with respect to the filtration, since the differential operator on S as well as ∂_D satisfy this condition. So, we obtain a filtered de Rham complex,

$$\mathrm{Fil}^k \mathcal{D}^{\bullet} := \mathrm{Fil}^k D \otimes_S \Omega_S^1 \longrightarrow \mathrm{Fil}^{k-1} D \otimes_S \Omega_S^2 \longrightarrow \cdots, \quad \text{for} \quad k \in \mathbb{Z}.$$

Now, let $\bigstar \in \{\mathrm{PD}, [u], [u, v], [u, v/p]\}$. We fix a basis of Ω_S^1 as $\left\{\frac{dX_0}{1+X_0}, \frac{dX_1}{X_1}, \ldots, \frac{dX_d}{X_d}\right\}$. For $j \in \mathbb{N}$, let $I_j = \{0 \leq i_1 < \cdots < i_j \leq d\}$ and for $\mathbf{i} = (i_1, \ldots, i_j) \in I_j$, let

$$\omega_{\mathbf{i}} = \begin{cases} \frac{dX_0}{1+X_0} \wedge \frac{dX_{i_2}}{X_{i_2}} \wedge \cdots \wedge \frac{dX_{i_j}}{X_{i_j}} & \text{if } i_1 = 0, \\ \frac{dX_{i_1}}{X_{i_1}} \wedge \cdots \wedge \frac{dX_{i_j}}{X_{i_j}} & \text{otherwise.} \end{cases}$$

We define operators φ and ψ on Ω_S^j by

$$\varphi\Big(\sum_{\mathbf{i} \in I_j} x_{\mathbf{i}} \omega_{\mathbf{i}}\Big) = \sum_{\mathbf{i} \in I_j} \varphi(x_{\mathbf{i}}) \omega_{\mathbf{i}} \quad \text{and} \quad \psi\Big(\sum_{\mathbf{i} \in I_j} x_{\mathbf{i}} \omega_{\mathbf{i}}\Big) = \sum_{\mathbf{i} \in I_j} \psi(x_{\mathbf{i}}) \omega_{\mathbf{i}}. \tag{5.3}$$

Remark 5.7. Note that this is not the natural definition of Frobenius, as we have $d(\varphi(x)) = p\varphi(dx)$ for the natural Frobenius. But in order to define ψ integrally, we need to divide the usual Frobenius by powers of p.

Now we are ready to define syntomic cohomology. Let $\mathcal{D}^{\bullet}$ denote the de Rham complex with $\bigstar \in \{[u], [u, v]\}$ and $\mathcal{E}^{\bullet}$ denote the de Rham complex with coefficients in the module which are target under the Frobenius, i.e. $\bigstar \in \{[u], [u, v/p]\}$.

Definition 5.8. Define the *syntomic complex* $\mathrm{Syn}(D, r)$ and the *syntomic cohomology* of R with coefficients in D as

$$\begin{aligned} \mathrm{Syn}(D, r) &:= \left[\mathrm{Fil}^r \mathcal{D}^{\bullet} \xrightarrow{p^r - p^{\bullet}\varphi} \mathcal{E}^{\bullet}\right]; \\ H^*_{\mathrm{syn}}(D, r) &:= H^*(\mathrm{Syn}(D, r)). \end{aligned}$$

Remark 5.9. Note that for $\bigstar = [u]$, we have $\mathcal{D}^{\bullet} = \mathcal{E}^{\bullet}$.

5.1.1. Change of disk of convergence

In order to relate $\mathrm{Syn}(D^{\mathrm{PD}}, r)$ to Koszul complexes, we will first pass to the analytic ring $R_\varpi^{[u]}$ and then to $R_\varpi^{[u,v]}$. Recall that we have $D^{\mathrm{PD}} = R_\varpi^{\mathrm{PD}} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$ and $D^{[u]} = R_\varpi^{[u]} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$ equipped with supplementary structures described above.

Proposition 5.10. (i) *For $\frac{1}{p-1} \leq u \leq 1$, the morphism of complexes*

$$\mathrm{Syn}(D^{\mathrm{PD}}, r) \longrightarrow \mathrm{Syn}(D^{[u]}, r)$$

induced by the inclusion $D^{\mathrm{PD}} \subset D^{[u]}$ is a p^{2r}-isomorphism.

(ii) *For $u' \leq u \leq pu'$, the morphism of complexes*

$$\mathrm{Syn}(D^{[u']}, r) \longrightarrow \mathrm{Syn}(D^{[u]}, r)$$

induced by the inclusion $D^{[u']} \subset D^{[u]}$ is a p^{2r}-isomorphism.

The proposition follows from the following lemma by taking $k = r$.

Lemma 5.11. *Let $k \in \mathbb{N}$ and $S = R_\varpi^+$.*

(i) *If $\frac{1}{p-1} \leq u \leq 1$, the map*

$$p^k - p^j\varphi \,:\, \mathrm{Fil}^r D^{[u]} \otimes \Omega^j_{S^{[u]}} / \mathrm{Fil}^r D^{\mathrm{PD}} \otimes \Omega^j_{S^{\mathrm{PD}}} \longrightarrow D^{[u]} \otimes \Omega^j_{S^{[u]}} / D^{\mathrm{PD}} \otimes \Omega^j_{S^{\mathrm{PD}}},$$

is a p^{k+r}-isomorphism.

(ii) *If $u' \leq u \leq pu'$, the map*

$$p^k - p^j\varphi \,:\, \mathrm{Fil}^r D^{[u]} \otimes \Omega^j_{S^{[u]}} / \mathrm{Fil}^r D^{[u']} \otimes \Omega^j_{S^{[u']}} \longrightarrow D^{[u]} \otimes \Omega^j_{S^{[u]}} / D^{[u']} \otimes \Omega^j_{S^{[u']}},$$

is a p^{k+r}-isomorphism.

Proof. The proof follows in a manner similar to [CN17, Lemma 3.2].

(i) Note that we can decompose everything in the basis of the $\omega_{\mathbf{i}}$'s, where $\mathbf{i} \in I_j$. By the definition of Frobenius on $\omega_{\mathbf{i}}$ we are reduced to showing that

$$p^k - p^j\varphi \,:\, \mathrm{Fil}^r D^{[u]} / \mathrm{Fil}^r D^{\mathrm{PD}} \longrightarrow D^{[u]} / D^{\mathrm{PD}},$$

is a p^{k+r}-isomorphism. We have $D^{\mathrm{PD}} \subset D^{[u]}$ and $\varphi(D^{[u]}) \subset D^{\mathrm{PD}}$ since $\varphi(R_\varpi^{[u]}) \subset R_\varpi^{[u/p]} \subset R_\varpi^{\mathrm{PD}}$, for $\frac{1}{p-1} \leq u \leq 1$.

For p^k-injectivity, we note that we have $\mathrm{Fil}^r D^{[u]} = D^{[u]} \cap \mathrm{Fil}^r D^{\mathrm{PD}}$, so it suffices to show that if $(p^k - p^j\varphi)x \in D^{\mathrm{PD}}$ then $p^k x \in D^{\mathrm{PD}}$. But since we can write $p^k x = (p^k - p^j\varphi)x + p^j\varphi(x)$ and $\varphi(D^{[u]}) \subset D^{\mathrm{PD}}$, we get that $p^k x \in D^{\mathrm{PD}}$.

Now, let $\{f_1, \ldots, f_h\}$ be an R_0-basis of $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$. Then, to show p^{k+r}-surjectivity we write $x = \sum_{i=1}^h a_i \otimes f_i \in R_\varpi^{[u]} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) = D^{[u]}$. We will write $p^{k+r}x$ as a sum of elements in $(p^k - p^j\varphi)\mathrm{Fil}^r D^{[u]}$ and D^{PD}. Let $N = \frac{ke}{u(p-1)}$, then from the definition of $R_\varpi^{[u]}$ we can write

$$a_i = a_{i1} + a_{i2}, \quad \text{with} \quad a_{i2} \in R_{\varpi,N}^{[u]} \quad \text{and} \quad a_{i1} \in p^{-\lfloor Nu/e \rfloor} R_\varpi^+ \subset p^{-k} R_\varpi^{\mathrm{PD}},$$

where we write $R_{\varpi,N}^{[u]}$ as in the notation of Lemma 2.32 (it consists of power series in X_0 involving terms X_0^s for $s \geq N$). Now let $x_1 = \sum_{i=1}^h a_{i1} \otimes f_i$ and $x_2 = \sum_{i=1}^h a_{i2} \otimes f_i$, so that

$x = x_1 + x_2$. By Lemma 2.32, we can write

$$x_2 = (1 - p^{j-k}\varphi)z, \quad \text{for some} \quad z = \sum_{i=1}^{h} b_i \otimes f_i \in R_{\varpi}^{[u]} \otimes \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) = D^{[u]}.$$

Also, by Lemma 2.28 we can write $b_i = b_{i1} + b_{i2}$ with $b_{i1} \in \mathrm{Fil}^r R_{\varpi}^{[u]}$ and $b_{i2} \in p^{-\lfloor ru \rfloor} R_{\varpi}^{+}$. Let $z_1 = \sum_{i=1}^{h} b_{i1} \otimes f_i \in \mathrm{Fil}^r D^{[u]}$ and $z_2 = \sum_{i=1}^{h} b_{i2} \otimes f_i \in p^{-r} D^{\mathrm{PD}}$, then

$$(1 - p^{j-k}\varphi)z_2 = p^{-k}(p^k - p^j\varphi)z_2 \in p^{-k-r} D^{\mathrm{PD}},$$

and

$$x - (1 - p^{j-k}\varphi)z_1 = x_1 + x_2 - (1 - p^{j-k}\varphi)z_1 = x_1 + (1 - p^{j-k}\varphi)z_2 \in p^{-k} D^{\mathrm{PD}} + p^{-k-r} D^{\mathrm{PD}} \subset p^{-k-r} D^{\mathrm{PD}}.$$

Therefore, we obtain that

$$x \in p^{-k-r} D^{\mathrm{PD}} + p^{-k}(p^k - p^j\varphi)\mathrm{Fil}^r D^{[u]},$$

which allows us to conclude.

(ii) We can repeat the arguments in (i) by replacing D^{PD} with $D^{[u']}$, since $R_{\varpi}^{[u']} \subset R_{\varpi}^{[u]}$ and $\varphi\big(R_{\varpi}^{[u]}\big) \subset R_{\varpi}^{[u/p]} \subset R_{\varpi}^{[u']}$, for $u' \leq u \leq pu'$.

■

5.1.2. Change of annulus of convergence

Recall that our objective is to relate the syntomic complexes discussed in the last section to differential Koszul complexes. To realize this goal, we further base change our complex to the ring $R_{\varpi}^{[u,v]}$. Recall that we have $D^{[u]} = R_{\varpi}^{[u]} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$, and $D^{[u,v]} = R_{\varpi}^{[u,v]} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) = R_{\varpi}^{[u,v]} \otimes_{R_{\varpi}^{[u]}} D^{[u]}$.

Proposition 5.12. *For $pu \leq v$, there exists a p^{2r+4s}-quasi-isomorphism*

$$\tau_{\leq r-s-1}\mathrm{Syn}\big(D^{[u]}, r\big) \simeq \tau_{\leq r-s-1}\mathrm{Syn}\big(D^{[u,v]}, r\big),$$

i.e. we have p^{2r+4s}-isomorphisms

$$H^k_{\mathrm{syn}}\big(D^{[u]}, r\big) \simeq H^k_{\mathrm{syn}}\big(D^{[u,v]}, r\big),$$

for $0 \leq k \leq r - s - 1$.

Proof. Combining the results from Lemmas 5.13, 5.16 & 5.14, we get the claim. ■

From the definition of complexes displayed in the claim above, it is not at all immediate that we should expect them (before and after scalar extension) to be quasi-isomorphic. Adapting a technique used in the theory of (φ, Γ)-modules of passing to the corresponding (quasi-isomorphic) ψ-complex, we will establish a p-power quasi-isomorphism, between the complexes of interest. This motivates our next definition for an operator ψ over $R_{\varpi} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$, which would act as a left inverse to φ.

First of all, we know that $\varphi^*\big(\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)\big) \simeq \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$, or equivalently $\varphi(\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V))$ generates $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ as an $R_0\big[\frac{1}{p}\big]$-module. Let $\mathbf{f} = \{f_1, \ldots, f_h\}$ denote an R_0-basis of $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$, i.e. $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) = \oplus_{i=0}^{h} R_0 f_i$. Then $\mathbf{f}$ is also a basis of $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ over $R_0\big[\frac{1}{p}\big]$. Hence, $\varphi(\mathbf{f}) = \{\varphi(f_1), \ldots, \varphi(f_h)\}$ is also a basis of $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ over $R_0\big[\frac{1}{p}\big]$. From this we can write $\mathbf{f} = \varphi(\mathbf{f})X$ where $X = (x_{ij}) \in Mat\big(h, R_0\big[\frac{1}{p}\big]\big)$. For our choice of $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$ and Proposition 3.31 and Corollary 3.38, we conclude that $x_{ij} \in \frac{1}{p^s} R_0$ where

$i \leq i, j \leq h$ and s is maximum among the absolute values of Hodge-Tate weights of V. Therefore, we can define

$$\begin{aligned} \psi \; : \; R_{\varpi}^{[u]} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) &\longrightarrow \tfrac{1}{p^s} R_{\varpi}^{[pu]} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) \\ \sum_{i=1}^{h} y_i \otimes f_i = \mathbf{f}\mathbf{y}^{\intercal} &\longmapsto \mathbf{f}\,\psi(X\mathbf{y}^{\intercal}) = \sum_{j=1}^{h} \Big(\sum_{i=1}^{h} \psi\big(y_i x_{ij}\big) \Big) \otimes f_j, \end{aligned} \tag{5.4}$$

where we consider the operator ψ on $R_{\varpi}^{[u]}$ defined in §2.3.2. It is easy to show that this map is well-defined, i.e. independent of the choice of the basis for $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$.

Using the operator ψ on $D^{[u]} = R_{\varpi}^{[u]} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$ as above, we can define the complex

$$\mathrm{Syn}^{\psi}\big(D^{[u]}, r\big) \; := \Big[\mathrm{Fil}^r D^{[u]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[u]}} \xrightarrow{p^{r+s}\psi - p^{\bullet+s}} D^{[pu]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[pu]}}\Big],$$

where the operator ψ acts on $\Omega^{\bullet}_{R_{\varpi}^{[u]}}$ as in (5.3).

Lemma 5.13. *The commutative diagram*

$$\begin{tikzcd} \mathrm{Fil}^r D^{[u]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[u]}} \arrow[r, "p^r - p^{\bullet}\varphi"] \arrow[d, "id"] & D^{[u]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[u]}} \arrow[d, "p^s\psi"] \\ \mathrm{Fil}^r D^{[u]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[u]}} \arrow[r, "p^{r+s}\psi - p^{\bullet+s}"] & D^{[pu]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[pu]}}, \end{tikzcd}$$

defines a p^{2s}-quasi-isomorphism from $\mathrm{Syn}\big(D^{[u]}, r\big)$ *to* $\mathrm{Syn}^{\psi}\big(D^{[u]}, r\big)$, *where s is maximum among the absolute value of Hodge-Tate weights of V.*

Proof. First, let us look at the cokernel complex. Since the left vertical arrow is identity, we only need to look at the cokernel of the right vertical arrow. Now, by definition we have $\psi(D^{[u]}) \subset p^{-s} D^{[pu]}$ and in particular, $p^s\psi(D^{[u]}) \subset D^{[pu]}$. Moreover, note that the operator $\psi \,:\, R_{\varpi}^{[u]} \longrightarrow R_{\varpi}^{[pu]}$ is surjective and $p^s\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) \subset \varphi^*(\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T))$. Therefore, we have

$$D^{[pu]} = R_{\varpi}^{[pu]} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) \subset \psi(R_{\varpi}^{[u]} \otimes_{R_0} \varphi^*(\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T))) \subset \psi(R_{\varpi}^{[u]} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)) = \psi(D^{[u]})$$

Hence, we get that $p^s\psi(D^{[u]})$ is p^s-isomorphic to $D^{[pu]}$. In particular, the cokernel complex is killed by p^s.

Next, for the kernel complex, we proceed as follows: Let $S = R_{\varpi}^{[u]}$ and we take $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) = \oplus_{j=1}^{h} R_0 f_j$, so that we have $D^{[u]} = \oplus_{j=1}^{h} S f_j$. Now we know that $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)/\varphi^*(\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T))$ is killed by p^s, where s is maximum among the absolute values of Hodge-Tate weights of V (see Proposition 3.31 and Corollary 3.38). So by extending scalars to S, we obtain a p^s-isomorphism

$$S \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) \simeq \bigoplus_{j=1}^{h} S\varphi(f_j).$$

Note that an element

$$y = \sum_{j=1}^{h} y_j \varphi(f_j) \in \Big(\bigoplus_{j=1}^{h} S\varphi(f_j)\Big)^{\psi=0},$$

if and only if $y_j \in S^{\psi=0}$. Indeed, $\psi(y) = 0$ if and only if $\sum_{j=1}^{h} \psi(y_j) f_j = 0$. Since f_j are linearly independent over $R_0\big[\frac{1}{p}\big]$, we get that $\psi(y) = 0$ if and only if $\psi(y_j) = 0$ for all $1 \leq j \leq h$. In particular, we have a p^s-isomorphism

$$\big(D^{[u]}\big)^{\psi=0} = \big(S \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)\big)^{\psi=0} \simeq \Big(\bigoplus_{j=1}^{h} S\varphi(f_j)\Big)^{\psi=0} = \bigoplus_{j=1}^{h} S^{\psi=0}\varphi(f_j).$$

Next, recall from (5.3) that in the basis of Ω_S^k, the operator ψ is defined as $\psi\big(\sum_{\mathbf{i}\in I_k} x_{\mathbf{i}}\omega_{\mathbf{i}}\big) = \sum_{\mathbf{i}\in I_k} \psi(x_{\mathbf{i}})\omega_{\mathbf{i}}$. In particular, we obtain

$$\Big(\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) \otimes_{R_0} \Omega_S^k\Big)^{\psi=0} = \big(S \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)\big)^{\psi=0} \otimes_{\mathbb{Z}} \Omega^k, \tag{5.5}$$

where

$$\Omega^1 = \mathbb{Z}\tfrac{dX_0}{1+X_0} \bigoplus_{i=1}^{d} \mathbb{Z}\tfrac{dX_i}{X_i} \quad \text{and} \quad \Omega^k = \bigwedge^k \Omega^1.$$

From Lemma 2.37(ii), we have a decomposition $S^{\psi=0} = \oplus_{\alpha\neq 0} S_\alpha = Su_\alpha$, where $u_\alpha = (1+X_0)^{\alpha_0}X_1^{\alpha_1}\cdots X_d^{\alpha_d}$ for $\alpha = (\alpha_0, \ldots, \alpha_d) \in \{0, 1, \ldots, p-1\}^{[0,d]}$. Moreover, from §2.3.2, we have $\partial_i(u_\alpha) = \alpha_i u_\alpha$ for $0 \leq i \leq d$. In particular, $\partial_i(S_\alpha) \subset S_\alpha$.

Now, using the decomposition of $S^{\psi=0}$, we set $D_\alpha = \oplus_{j=1}^h S_\alpha\varphi(f_j)$ and obtain that $\big(D^{[u]}\big)^{\psi=0}$ is p^s-isomorphic to $\oplus_{\alpha\neq 0} D_\alpha$. From the differentials on S_α and the connection on $D^{[u]}$ we obtain an induced connection $\partial : D_\alpha \longrightarrow D_\alpha \otimes_S \Omega_S^1 = D_\alpha \otimes_{\mathbb{Z}} \Omega^1$, which is integrable. The decomposition of $\big(D^{[u]}\big)^{\psi=0}$ and (5.5) shows that the kernel complex in the claim is p^s-isomorphic to the direct sum of complexes

$$0 \longrightarrow D_\alpha \longrightarrow D_\alpha \otimes \Omega^1 \longrightarrow D_\alpha \otimes \Omega^2 \longrightarrow \cdots, \tag{5.6}$$

where $\alpha \neq 0$.

We will show that (5.6) is exact for each α. The idea for the rest of the proof is based on [CN17, Lemma 3.4]. Note that since everything is p-adically complete, we only need to show the exactness of (5.6) modulo p. For this we notice that for $y = \sum_{j=1}^h y_j\varphi(f_j) \in D_\alpha$, we have

$$\partial\Big(\sum_{j=1}^{h} y_j\varphi(f_j)\Big) = \sum_{j=1}^{h} y_j\partial_D(\varphi(f_j)) + \varphi(f_j)dy_j,$$

where ∂_D denotes the connection on $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$. By §1.5.2, we have $\varphi\partial_D = \partial_D\varphi$ over $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$. So, for $i \in \{1, \ldots, d\}$ we obtain that

$$\partial_D(\varphi(f_j)) = \varphi\big(\partial_D(f_j)\big) = \varphi\Big(\sum_{j=1}^{h} b_jf_j \otimes \tfrac{dX_j}{X_j}\Big) = p\sum_{j=1}^{h} \varphi(b_jf_j) \otimes \tfrac{dX_j}{X_j}.$$

Note that the operator φ in the equation above is the usual one (in (5.3) we replaced this operator by dividing out by powers of p). Moreover, by Lemma 2.38 we have that $\partial_i(y_j) - \alpha_i y_j \in pS_\alpha$. So we get that the complex (5.6) has a very simple shape modulo p: if $d = 0$, it is just $D_\alpha \xrightarrow{\alpha_0} D_\alpha$; if $d = 1$, it is the total complex attached to the double complex

$$\begin{tikzcd} D_\alpha \arrow[r, "\alpha_0"] \arrow[d, "\alpha_1"] & D_\alpha \arrow[d, "\alpha_1"] \\ D_\alpha \arrow[r, "\alpha_0"] & D_\alpha, \end{tikzcd}$$

and for general d, it is the total complex attached to a $(d+1)$-dimensional cube with all vertices equal to D_α and arrows in the i-th direction equal to α_i. As one of the α_i is invertible by assumption, this implies that the cohomology of the total complex is 0. This establishes that (5.6) is exact for each α and hence the kernel complex is p^s-acyclic. ■

Next, we will base change the complex to $R_\varpi^{[u,v]}$. As we will compare (ψ, ∂)-complexes, following (5.4), one can define an operator

$$\psi : R_\varpi^{[u,v]} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) \longrightarrow \tfrac{1}{p^s} R_\varpi^{[pu,pv]} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T),$$

as a left inverse to φ. Now using $D^{[u,v]} = R_\varpi^{[u,v]} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$, we can define the complex

$$\mathrm{Syn}^{\psi}\left(D^{[u,v]}, r\right) := \left[\mathrm{Fil}^r D^{[u,v]} \otimes \Omega^\bullet_{R_\varpi^{[u,v]}} \xrightarrow{p^{r+s}\psi - p^{\bullet+s}} D^{[pu,v]} \otimes \Omega^\bullet_{R_\varpi^{[pu,v]}}\right].$$

We can relate the two (ψ, ∂)-complexes discussed so far,

Lemma 5.14. *Let $u \leq 1 \leq v$. The natural morphism*

$$\mathrm{Syn}^{\psi}\left(D^{[u]}, r\right) \longrightarrow \mathrm{Syn}^{\psi}\left(D^{[u,v]}, r\right),$$

is a p^{2r}-quasi-isomorphism in degrees $k \leq r - s - 1$.

Proof. The map between complexes is induced by the diagram

$$\begin{array}{ccc} \mathrm{Fil}^r D^{[u]} \otimes \Omega^\bullet_{R_\varpi^{[u]}} & \xrightarrow{p^{r+s}\psi - p^{\bullet+s}} & D^{[pu]} \otimes \Omega^\bullet_{R_\varpi^{[pu]}} \\ \downarrow & & \downarrow \\ \mathrm{Fil}^r D^{[u,v]} \otimes \Omega^\bullet_{R_\varpi^{[u,v]}} & \xrightarrow{p^{r+s}\psi - p^{\bullet+s}} & D^{[pu,v]} \otimes \Omega^\bullet_{R_\varpi^{[pu,v]}}, \end{array}$$

where the vertical arrows are natural maps induced by the inclusion $R_\varpi^{[u]} \subset R_\varpi^{[u,v]}$. Therefore, it suffices to show that the mapping fiber

$$\left[\mathrm{Fil}^r D^{[u,v]} \otimes \Omega^\bullet_{R_\varpi^{[u,v]}} / \mathrm{Fil}^r D^{[u]} \otimes \Omega^\bullet_{R_\varpi^{[u]}} \xrightarrow{p^{r+s}\psi - p^{\bullet+s}} D^{[pu,v]} \otimes \Omega^\bullet_{R_\varpi^{[pu,v]}} / D^{[pu]} \otimes \Omega^\bullet_{R_\varpi^{[pu]}}\right],$$

is p^{2r}-acyclic. By Lemma 5.15, we can ignore the filtration and, working in the basis $\{\omega_{\mathbf{i}},\ \mathbf{i} \in I_k\}$ of Ω^k, it is enough to show that

$$p^{r+s}\psi - p^{k+s} : D^{[u,v]}/D^{[u]} \longrightarrow D^{[pu,v]}/D^{[pu]},$$

is a p^r-isomorphism for $k \leq r - s - 1$. But

$$D^{[u,v]}/D^{[u]} \simeq D^{[pu,v]}/D^{[pu]},$$

and therefore $1 - p^i\psi$ is an endomorphism of this quotient for $i = r - k$. Moreover, for $i \geq s + 1$ we get that $1 - p^i\psi$ is invertible on $D^{[u,v]}/D^{[u]}$ with inverse given as $1 + p^{i-s}(p^s\psi) + p^{2(i-s)}(p^s\psi)^2 + \cdots$. Therefore $p^{r+s}\psi - p^{k+s} = p^{k+s}(p^{r-k}\psi - 1)$ is a p^{k+s}-isomorphism. Since $k + s \leq r - 1$, we obtain that the complex in the claim is p^{2r}-acyclic. ■

Following observation was used above,

Lemma 5.15. *For $u \leq 1 \leq v$, the natural morphism*

$$\mathrm{Fil}^r D^{[u,v]} / \mathrm{Fil}^r D^{[u]} \longrightarrow D^{[u,v]}/D^{[u]},$$

is a p^r-isomorphism.

Proof. First we recall that

$$\mathrm{Fil}^r D^{[u,v]} = \sum_{a+b=r} \mathrm{Fil}^a R_\varpi^{[u,v]} \widehat{\otimes} \mathrm{Fil}^b \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T).$$

Now the map in the claim is clearly injective. For p^r-surjectivity, let $\{f_1, \ldots, f_h\}$ be an R_0-basis of $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$ and let $x = \sum_{i=1}^h b_i \otimes f_i \in R_\varpi^{[u,v]} \otimes \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$. By [CN17, Lemma 3.5], we have a p^r-isomorphism

$$\mathrm{Fil}^r R_\varpi^{[u,v]} / \mathrm{Fil}^r R_\varpi^{[u]} \longrightarrow R_\varpi^{[u,v]} / R_\varpi^{[u]},$$

so we can write $p^r b_i = b_{i1} + b_{i2}$, with $b_{i1} \in \mathrm{Fil}^r R_\varpi^{[u,v]}$ and $b_{i2} \in R_\varpi^{[u]}$. Since $\sum_{i=1}^h b_{i1} \otimes f_i \in \mathrm{Fil}^r D^{[u,v]}$, we get the desired conclusion. ∎

Finally, we can get back to the (φ, ∂)-complex,

Lemma 5.16. *The commutative diagram*

$$\begin{tikzcd}
\mathrm{Fil}^r D^{[u,v]} \otimes \Omega^\bullet_{R_\varpi^{[u,v]}} \arrow[r, "p^r - p^\bullet \varphi"] \arrow[d, "id"] & D^{[u,v/p]} \otimes \Omega^\bullet_{R_\varpi^{[u,v/p]}} \arrow[d, "p^s \psi"] \\
\mathrm{Fil}^r D^{[u,v]} \otimes \Omega^\bullet_{R_\varpi^{[u,v]}} \arrow[r, "p^{r+s}\psi - p^{\bullet+s}"] & D^{[pu,v]} \otimes \Omega^\bullet_{R_\varpi^{[pu,v]}},
\end{tikzcd}$$

defines a p^{2s}-quasi-isomorphism from $\mathrm{Syn}\big(D^{[u,v]}, r\big)$ to $\mathrm{Syn}^\psi\big(D^{[u,v]}, r\big)$.

Proof. We can repeat the arguments in the proof of Lemma 5.13 by replacing $D^{[u]}$ with $D^{[u,v]}$ and $R_\varpi^{[u]}$ with $R_\varpi^{[u,v]}$. We briefly sketch the argument. First, for the cokernel complex, we only need to look at the cokernel of the right vertical arrow. We have $\psi(D^{[u,v/p]}) \subset p^{-s} D^{[pu,v]}$, and in particular $p^s\psi(D^{[u,v/p]}) \subset D^{[pu,v]}$. Further, the operator $\psi : R_\varpi^{[u,v/p]} \longrightarrow R_\varpi^{[pu,v]}$ is surjective and $p^s \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) \subset \varphi^*(\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T))$. Therefore, we have

$$D^{[pu,v]} = R_\varpi^{[pu,v]} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) \subset \psi(R_\varpi^{[u,v/p]} \otimes_{R_0} \varphi^*(\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T))) \subset \psi(R_\varpi^{[u,v/p]} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)) = \psi(D^{[u,v/p]})$$

Hence, we get that $p^s\psi(D^{[u,v/p]})$ is p^s-isomorphic to $D^{[pu,v]}$. In particular, the cokernel complex is killed by p^s.

Next, we look at the kernel complex. Let $S = R_\varpi^{[u,v/p]}$ and arguing as in Lemma 5.13, we obtain a p^s-isomorphism

$$\big(D^{[u,v]}\big)^{\psi=0} = \big(S \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)\big)^{\psi=0} \simeq \Big(\bigoplus_{j=1}^h S\varphi(f_j)\Big)^{\psi=0} = \bigoplus_{j=1}^h S^{\psi=0}\varphi(f_j).$$

Now using (5.3), we can write

$$\Big(\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) \otimes_{R_0} \Omega_S^k\Big)^{\psi=0} = \big(S \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)\big)^{\psi=0} \otimes_{\mathbb{Z}} \Omega^k, \tag{5.7}$$

where

$$\Omega^1 = \mathbb{Z}\tfrac{dX_0}{1+X_0} \bigoplus_{i=1}^d \mathbb{Z}\tfrac{dX_i}{X_i} \quad \text{and} \quad \Omega^k = \bigwedge^k \Omega^1.$$

From Lemma 2.37(ii), we have a decomposition $S^{\psi=0} = \oplus_{\alpha\neq 0} S_\alpha = Su_\alpha$, where $u_\alpha = (1+X_0)^{\alpha_0} X_1^{\alpha_1} \cdots X_d^{\alpha_d}$ for $\alpha = (\alpha_0, \dots, \alpha_d) \in \{0, 1, \dots, p-1\}^{[0,d]}$. From §2.3.2, we have $\partial_i(u_\alpha) = \alpha_i u_\alpha$ for $0 \leq i \leq d$. In particular, $\partial_i(S_\alpha) \subset S_\alpha$. So using the decomposition of $S^{\psi=0}$, we set $D_\alpha = \oplus_{j=1}^h S_\alpha \varphi(f_j)$ and obtain that $\big(D^{[u,v]}\big)^{\psi=0}$ is p^s-isomorphic to $\oplus_{\alpha\neq 0} D_\alpha$. From the differentials on S_α and the connection on $D^{[u,v]}$ we obtain an induced connection $\partial : D_\alpha \longrightarrow D_\alpha \otimes_S \Omega_S^1 = D_\alpha \otimes_{\mathbb{Z}} \Omega^1$, which is integrable. The decomposition of $\big(D^{[u,v]}\big)^{\psi=0}$ and (5.7) shows that the kernel complex in the claim is p^s-isomorphic to the direct sum of complexes

$$0 \longrightarrow D_\alpha \longrightarrow D_\alpha \otimes \Omega^1 \longrightarrow D_\alpha \otimes \Omega^2 \longrightarrow \cdots, \tag{5.8}$$

where $\alpha \neq 0$. An analysis similar to Lemma 5.13 shows that the complex (5.8) has a very simple shape modulo p: if $d = 0$, it is just $D_\alpha \xrightarrow{\alpha_0} D_\alpha$; if $d = 1$, it is the total complex attached to the double complex

$$\begin{tikzcd} D_\alpha \arrow[r, "\alpha_0"] \arrow[d, "\alpha_1"] & D_\alpha \arrow[d, "\alpha_1"] \\ D_\alpha \arrow[r, "\alpha_0"] & D_\alpha, \end{tikzcd}$$

and for general d, it is the total complex attached to a $(d+1)$-dimensional cube with all vertices equal to D_α and arrows in the i-th direction equal to α_i. As one of the α_i is invertible by assumption, this implies that the cohomology of the total complex is 0. This establishes that (5.8) is exact for each α and hence the kernel complex is p^s-acyclic. ■

5.1.3. Differential Koszul Complex

In the previous sections we studied syntomic complexes over various base rings with coefficients in $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$. In this section, we will study differential Koszul complex over the base ring $\mathbf{A}_R^{[u,v]}$ with coefficients in the Wach module $\mathbf{N}(T)$. As we shall see the differential Koszul complex is very closely related to syntomic complexes. Such a relationship is to be expected, since we have an isomorphism of rings $\iota_{\mathrm{cycl}} : R_\varpi^{[u,v]} \xrightarrow{\simeq} \mathbf{A}_R^{[u,v]}$ in §2.4 and there exists a natural comparison between $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ and $\mathbf{N}(V)$ after extension of scalars to $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ on both sides (see Theorem 3.24). Note that from now onwards, we will be working under the assumption that $\frac{p-1}{p} \leq u \leq \frac{v}{p} < 1 < v$, for example, one can take $u = \frac{p-1}{p}$ and $v = p-1$.

The ring $R_\varpi^{[u,v]}$ is a p-adically complete $\mathbb{Z}_p$-algebra, equipped with a Frobenius $\varphi : R_\varpi^{[u,v]} \longrightarrow R_\varpi^{[u,v/p]}$, lifting the absolute Frobenius on $R_\varpi^{[u,v]}/p$. Let $\Omega^\bullet_{\mathbf{A}_R^{[u,v]}}$ denote the p-adic completion of the module of differentials of $\mathbf{A}_R^{[u,v]}$ relative to $\mathbb{Z}$. Recall from §2.3 that $\Omega^1_{R_\varpi^{[u,v]}}$ has a basis of differentials $\left\{\frac{dX_0}{1+X_0}, \frac{dX_1}{X_1}, \ldots, \frac{dX_d}{X_d}\right\}$. So via the identification $\iota_{\mathrm{cycl}} : R_\varpi^{[u,v]} \xrightarrow{\simeq} \mathbf{A}_R^{[u,v]}$ we obtain differential operators ∂_i over $\mathbf{A}_R^{[u,v]}$, for $0 \leq i \leq d$. Moreover, from Definition 2.27 we can endow $\mathbf{A}_R^{[u,v]}$ with a filtration $\{\mathrm{Fil}^k \mathbf{A}_R^{[u,v]}\}_{k \in \mathbb{Z}}$ and obtain filtered de Rham complex

$$\mathrm{Fil}^k \Omega^\bullet_{\mathbf{A}_R^{[u,v]}} : \mathrm{Fil}^k \mathbf{A}_R^{[u,v]} \longrightarrow \mathrm{Fil}^{k-1} \mathbf{A}_R^{[u,v]} \otimes \Omega^1_{\mathbf{A}_R^{[u,v]}} \longrightarrow \mathrm{Fil}^{k-2} \mathbf{A}_R^{[u,v]} \otimes \Omega^2_{\mathbf{A}_R^{[u,v]}} \longrightarrow \cdots, \quad \text{for } k \in \mathbb{Z}.$$

Further, the differential operators ∂_i can be related to the infinitesimal action of Γ_R by the relation

$$\nabla_i := \log \gamma_i = t\partial_i \ \text{ for } 0 \leq i \leq d,$$

where $\log \gamma_i = \sum_{k \in \mathbb{N}} (-1)^k \frac{(\gamma_i - 1)^{k+1}}{k+1}$. We will study similar operators over the $\mathbf{A}_R^{[u,v]}$-module arising from the Wach module $\mathbf{N}(T)$.

Note that for an indeterminate X we can formally write

$$\begin{aligned} \frac{\log(1+X)}{X} &= 1 + a_1 X + a_2 X^2 + a_3 X^3 + \cdots, \\ \frac{X}{\log(1+X)} &= 1 + b_1 X + b_2 X^2 + b_3 X^3 + \cdots, \end{aligned}$$

where $v_p(a_k) \geq -\frac{k}{p-1}$ for all $k \geq 1$ and therefore, $v_p(b_k) \geq -\frac{k}{p-1}$ for all $k \geq 1$. We have the following claim:

Lemma 5.17. *Let $M^{[u,v]} = \mathbf{A}_R^{[u,v]} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)$. Then, for $i \in \{0, 1, \ldots, d\}$ the operators*

$$\nabla_i = \log \gamma_i; \qquad \frac{\nabla_i}{\gamma_i - 1} = \frac{\log \gamma_i}{\gamma_i - 1}; \quad \textit{and} \quad \frac{\gamma_i - 1}{\nabla_i} = \frac{\gamma_i - 1}{\log \gamma_i}.$$

converge as series of operators on $M^{[u,v]}$.

Proof. For $0 \leq i \leq d$, observe that $\gamma_i - 1$ acts as a twisted derivation, i.e. for $a \in \mathbf{A}_R^{[u,v]}$ and $x \in \mathbf{N}(T)$, we have

$$(\gamma_i - 1)(ax) = (\gamma_i - 1)a \cdot x + \gamma_i(a)(\gamma_i - 1)x.$$

The action of Γ_R is trivial on $\mathbf{N}(T)/\pi\mathbf{N}(T)$, so we can write $(\gamma_i - 1)x = \pi y$, for some $y \in \mathbf{N}(T)$. Now, from the proof of Lemma 4.16 and (4.2), we have

$$(\gamma_i - 1)^k M^{[u,v]} \subset \big(p, \pi_m^{p^m}\big)^k M^{[u,v]}.$$

The same estimation of p-adic valuation of coefficients as in that proof helps us in concluding that $\log \gamma_i$ converges as a series of operators on $M^{[u,v]}$. The claim for the convergence of operators $\frac{\nabla_i}{\gamma_i - 1}$ and $\frac{\gamma_i - 1}{\nabla_i}$ follows in a manner similar to Lemma 4.17. ∎

Note that $M^{[u,v]}$ is a topological $\mathbf{A}_R^{[u,v]}$-module equipped with a filtration by $\mathbf{A}_R^{[u,v]}$-submodules

$$\mathrm{Fil}^k M^{[u,v]} = \sum_{i+j=k} \mathrm{Fil}^i \mathbf{A}_R^{[u,v]} \widehat{\otimes} \mathrm{Fil}^j \mathbf{N}(T), \quad \text{for } k \in \mathbb{Z}, \tag{5.9}$$

such that $\mathrm{Fil}^k M^{[u,v]}$ is stable under the action of Γ_R.

Remark 5.18. The results of Lemma 5.17 continue to hold if we replace $\mathbf{N}(T)$ with $\mathbf{N}(T(r))$ for $r \in \mathbb{Z}$, or $\mathrm{Fil}^k M^{[u,v]}$ for $k \in \mathbb{Z}$, or filtered pieces of $\mathbf{A}_R^{[u,v]} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T(r))$.

Lemma 5.19. *For the filtered modules and operators ∇_i defined above, we have*

$$\nabla_i\big(\mathrm{Fil}^k M^{[u,v]}\big) \subset \pi \mathrm{Fil}^{k-1} M^{[u,v]} = t \mathrm{Fil}^{k-1} M^{[u,v]} \quad \textit{for } 0 \leq i \leq d.$$

Proof. Note that the action of Γ_R is trivial on $\mathrm{Fil}^k M^{[u,v]}/\pi \mathrm{Fil}^k M^{[u,v]}$ and from this we infer that for $0 \leq i \leq d$, we have

$$\nabla_i\big(\mathrm{Fil}^k M^{[u,v]}\big) \subset \mathrm{Fil}^k M^{[u,v]} \cap \pi M^{[u,v]} = \pi \mathrm{Fil}^{k-1} M^{[u,v]},$$

where the last equality follows from Lemma 3.17. As $\frac{t}{\pi}$ is a unit in $S = \mathbf{A}_R^{[u,v]}$ (see Lemma 2.43), we can also write $\nabla_i\big(\mathrm{Fil}^k M^{[u,v]}\big) \subset t\mathrm{Fil}^{k-1} M^{[u,v]}$. ∎

The lemma above enables us to introduce differential operators ∂_i over $M^{[u,v]}$ by the formula

$$\nabla_i = \log \gamma_i = t\partial_i, \quad \text{for } 0 \leq i \leq d,$$

where the operators ∂_i are well-defined by dividing out the image under the operator ∇_i by t. Recall that via the identification $R_{\varpi}^{[u,v]} \xrightarrow{\simeq} \mathbf{A}_R^{[u,v]}$, we have a basis for $\Omega^1_{\mathbf{A}_R^{[u,v]}}$ given by $\big\{ \frac{dX_0}{1+X_0}, \frac{dX_1}{X_1}, \ldots, \frac{dX_d}{X_d} \big\}$. Therefore, by setting $\partial = (\partial_0, \ldots, \partial_d)$ we obtain a connection over $M^{[u,v]}$

$$\begin{aligned} \partial : M^{[u,v]} &\longrightarrow M^{[u,v]} \otimes \Omega^1_{\mathbf{A}_R^{[u,v]}} \\ ax &\longmapsto a\partial(x) + x \otimes d(a). \end{aligned}$$

Lemma 5.20. *The connection ∂ on $M^{[u,v]}$ is integrable and satisfies Griffiths transversality with respect to the filtration, i.e.*

$$\partial_i(\mathrm{Fil}^k M^{[u,v]}) \subset \mathrm{Fil}^{k-1} M^{[u,v]} \quad \textit{for } 0 \leq i \leq d.$$

Proof. Recall that from (4.4) we have $[\nabla_i, \nabla_j] = 0$ for $1 \leq i, j \leq d$, whereas $[\nabla_0, \nabla_i] = p^m \nabla_i$, for $1 \leq i \leq d$. So it follows that over $M^{[u,v]}$ we have the composition of operators

$$t^2(\partial_i \circ \partial_j - \partial_j \circ \partial_i) = t\partial_i(t\partial_j) - t\partial_j(t\partial_i) = \nabla_i \circ \nabla_j - \nabla_j \circ \nabla_i = 0, \quad \text{for } 1 \leq i, j \leq d.$$

Next, for $1 \leq i \leq d$, we have

$$\begin{aligned}\nabla_0 \circ \nabla_i - \nabla_i \circ \nabla_0 &= t\partial_0 \circ (t\partial_i) - t\partial_i \circ (t\partial_0)\\ &= tp^m\partial_i + t^2\partial_0 \circ \partial_i - t^2\partial_i \circ \partial_0 = p^m\nabla_i + t^2(\partial_0 \circ \partial_i - \partial_i \circ \partial_0).\end{aligned}$$

In particular, $\partial_0 \circ \partial_i - \partial_i \circ \partial_0 = 0$. Since $\partial \circ \partial = (\partial_i \circ \partial_j)_{i,j}$ for $0 \leq i \leq j \leq d$ and $M^{[u,v]}$ is t-torsion free, we conclude that the connection ∂ is integrable. Moreover, it satisfies Griffiths transversailty since $\partial_i\big(\mathrm{Fil}^k M^{[u,v]}\big) = t^{-1}\nabla_i\big(\mathrm{Fil}^k M^{[u,v]}\big) \subset \mathrm{Fil}^{k-1} M^{[u,v]}$, for $0 \leq i \leq d$. ■

Now we are in a position to write the filtered de Rham complex for $M^{[u,v]}$ as

$$\mathrm{Fil}^k M^{[u,v]} \otimes \Omega^{\bullet}_{\mathbf{A}_R^{[u,v]}} : \mathrm{Fil}^k M^{[u,v]} \longrightarrow \mathrm{Fil}^{k-1} M^{[u,v]} \otimes \Omega^1_{\mathbf{A}_R^{[u,v]}} \longrightarrow \mathrm{Fil}^{k-2} M^{[u,v]} \otimes \Omega^2_{\mathbf{A}_R^{[u,v]}} \longrightarrow \cdots. \tag{5.10}$$

Further, we know that $\Omega^1_{\mathbf{A}_R^{[u,v]}}$ has a basis $\{\omega_1, \ldots, \omega_d\}$, such that an element of $\Omega^q_{\mathbf{A}_R^{[u,v]}} = \wedge^q \Omega^1_{\mathbf{A}_R^{[u,v]}}$ can be uniquely written as $\sum x_{\mathbf{i}}\omega_{\mathbf{i}}$, with $x_{\mathbf{i}} \in \mathbf{A}_R^{[u,v]}$ and $\omega_{\mathbf{i}} = \omega_{i_1} \wedge \cdots \wedge \omega_{i_q}$ for $\mathbf{i} = (i_1, \ldots, i_q) \in I_q = \{0 \leq i_1 < \cdots < i_q \leq d\}$. In this case, the map involving differential operators becomes

$$(\partial_i) : \big(\mathrm{Fil}^{k-q} M^{[u,v]}\big)^{I_q} \longrightarrow \big(\mathrm{Fil}^{k-q-1} M^{[u,v]}\big)^{I_{q+1}}, \quad \text{for} \quad 0 \leq i \leq d.$$

Definition 5.21. Define the *∂-Koszul complex* for $\mathrm{Fil}^k M^{[u,v]}$ as

$$\mathrm{Kos}\big(\partial_A, \mathrm{Fil}^k M^{[u,v]}\big) : \mathrm{Fil}^k M^{[u,v]} \xrightarrow{(\partial_i)} \big(\mathrm{Fil}^{k-1} M^{[u,v]}\big)^{I_1} \longrightarrow \big(\mathrm{Fil}^{k-2} M^{[u,v]}\big)^{I_2} \longrightarrow \cdots.$$

Remark 5.22. (i) By definition, we have an ismorphism of complexes $\mathrm{Fil}^k M^{[u,v]} \otimes \Omega^{\bullet}_{\mathbf{A}_R^{[u,v]}} \simeq \mathrm{Kos}\big(\partial_A, \mathrm{Fil}^k M^{[u,v]}\big)$.

(ii) Let $I'_j = \{(i_1, \ldots, i_j)$, such that $1 \leq i_1 < \cdots < i_j \leq d\}$ and let $\partial' = (\partial_1, \ldots, \partial_d)$. We can also set

$$\mathrm{Kos}\big(\partial'_A, \mathrm{Fil}^k M^{[u,v]}\big) : \mathrm{Fil}^k M^{[u,v]} \xrightarrow{(\partial_i)} \big(\mathrm{Fil}^{k-1} M^{[u,v]}\big)^{I'_1} \longrightarrow \big(\mathrm{Fil}^{k-2} M^{[u,v]}\big)^{I'_2} \longrightarrow \cdots,$$

and therefore we get that

$$\mathrm{Kos}\big(\partial_A, \mathrm{Fil}^k M^{[u,v]}\big) = \big[\mathrm{Kos}\big(\partial'_A, \mathrm{Fil}^k M^{[u,v]}\big) \xrightarrow{\partial_0} \mathrm{Kos}\big(\partial'_A, \mathrm{Fil}^{k-1} M^{[u,v]}\big)\big].$$

(iii) The computation carried out in this section are true over the ring $\mathbf{A}_R^{[u,v/p]}$ as well.

5.1.4. Poincaré Lemma

Recall from §2.5 that given two p-adically complete W-algebras S and Λ, and $\iota : S \longrightarrow \Lambda$ a continuous injective morphism of filtered O_F-algebras. Then for $f : S \otimes \Lambda \longrightarrow \Lambda$ the morphism sending $x \otimes y \mapsto \iota(x)y$, we can define the ring $S\Lambda$ to be the p-adic completion of the PD-envelope of $S \otimes \Lambda \longrightarrow \Lambda$ with respect to $\mathrm{Ker}\, f$.

Definition 5.23. Let $\star \in \{\mathrm{PD}, [u], [u,v]\}$ and define $E_R^{\star} = S\Lambda$ for $S = R_{\varpi}^{\star}$, $\Lambda = \mathbf{A}_R^{\star}$, and $\iota = \iota_{\mathrm{cycl}}$ (see §2.4).

Note that we are working under the assumption that $\frac{p-1}{p} \leq u \leq \frac{v}{p} < 1 < v$, for example, one can take $u = \frac{p-1}{p}$ and $v = p-1$. These rings have desirable properties:

Lemma 5.24 ([CN17, Lemma 2.38]). (i) $E_R^{\mathrm{PD}} \subset E_R^{[u]} \subset E_R^{[u,v]}$.

(ii) *The Frobenius φ extends uniquely to continuous morphisms*

$$E_R^{\mathrm{PD}} \longrightarrow E_R^{\mathrm{PD}}, \qquad E_R^{[u]} \longrightarrow E_R^{[u]}, \qquad E_R^{[u,v]} \longrightarrow E_R^{[u,v/p]}.$$

(iii) *The action of G_R extends uniquely to continuous actions on E_R^{PD}, $E_R^{[u]}$, and $E_R^{[u,v]}$ which commutes with the Frobenius.*

Remark 5.25. (i) In Definition 5.23 if we reverse the roles of S and Λ, i.e. if we take $S = \mathbf{A}_R^{\bigstar}$, $\Lambda = R_\varpi^{\bigstar}$ and $\iota = \iota_{\mathrm{cycl}}^{-1}$, then we get an isomorphism $S\Lambda \simeq E_R^{\bigstar}$ with obvious commutativity of the action of Frobenius and the Galois group G_R on each side.

(ii) Let $V_i = \frac{X_i \otimes 1}{1 \otimes \iota(X_i)}$, for $0 \leq i \leq d$. We filter $E_R^{\bigstar}$ by defining $\mathrm{Fil}^r E_R^{\bigstar}$ to be the topological closure of the ideal generated by the products of the form $x_1 x_2 \prod (V_i - 1)^{[k_i]}$, with $x_1 \in \mathrm{Fil}^{r_1} R_\varpi^{\bigstar}$, $x_2 \in \mathrm{Fil}^{r_2} \mathbf{A}_R^{\bigstar}$, and $r_1 + r_2 + \sum k_i \geq r$.

From Definition 3.18, we have a p-adically complete ring $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ equipped with a Frobenius and a continuous action of Γ_R. In Remark 3.20, we mentioned an alternative construction of $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ using an embedding $\iota : R_0 \longrightarrow \mathbf{A}_R^{\mathrm{PD}}$ defined by sending $X_i \mapsto [X_i^\flat]$, for $1 \leq i \leq d$. Identifying R_0 as a subring of R_ϖ^{PD}, and extending the embedding ι to $R_\varpi^{\mathrm{PD}} \longrightarrow \mathbf{A}_R^{\mathrm{PD}}$ by sending $X_0 \mapsto \pi_m$, we get that the extended embedding is exactly ι_{cycl}. Since the action of the Frobenius and the Galois group G_R over $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ and E_R^{PD} can be given by their action on each component of the tensor product, we obtain a Frobenius and Galois-equivariant embedding $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \rightarrowtail E_R^{\mathrm{PD}}$. Moreover, the filtration on $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ (see Definition 3.21) coincides with the filtration induced from its embedding into E_R^{PD}. Note that since $R_\varpi^{\mathrm{PD}} \subset E_R^{\mathrm{PD}}$, the key difference between E_R^{PD} and $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ is that the former ring contains the indeterminate X_0 and its divided powers, whereas the latter ring does not.

Next, from the natural inclusion $R_0 \rightarrowtail R_\varpi^{\mathrm{PD}}$ we know that the differential operator on R_0 is compatible with the differential operator on R_ϖ^{PD}. Further, we have an identification $\iota_{\mathrm{cycl}}^{-1} : \mathbf{A}_R^{\mathrm{PD}} \xrightarrow{\simeq} R_\varpi^{\mathrm{PD}}$ (see §2.4) using which we obtain differential operators on $\mathbf{A}_{R^{\mathrm{PD}}}$. Also, over the ring $\mathbf{A}_R^{\mathrm{PD}}$, the operators $\nabla_i = \log \gamma_i$ converge for $0 \leq i \leq d$ (see Lemma 4.16), which are related to the differential operators by the relation $\nabla_i = t\partial_i$. Thus if we denote this differential operator over $\mathbf{A}_R^{\mathrm{PD}}$ as $\partial_A = (\partial_i)_{0 \leq i \leq d}$ and the differential operator over R_ϖ^{PD} (as well as over R_0) as ∂_R, then we see that the induced differential operator $\partial_R \otimes 1 + 1 \otimes \partial_A$ over $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ as well as E_R^{PD} are compatible. Note that E_R^{PD} is naturally contained in $E_R^{[u,v]}$ compatible with all the structures. Hence, below we will identify $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$ as a subring of $E_R^{[u,v]}$.

Now we turn to the comparison between $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$ and $\mathbf{N}(T)$ over the ring $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$. Recall from the proof of Proposition 3.31 that we have a natural map

$$\mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) \longrightarrow \mathcal{O}\mathbf{A}_R^{\mathrm{PD}} \otimes_{R_0} \mathbf{N}(T), \tag{5.11}$$

compatible with Frobenius, filtration, connection and the action of Γ_R on each side. Moreover, (5.11) is an injective map which is $p^{n(T,e)}$-surjective for some constant $n(T,e) \in \mathbb{N}$ (since it is an isomorphism after inverting p), depending on the representation T and the ramification index e of K/F (see Remarks 3.39 & 5.4). We can promote this comparison over $\mathcal{O}\mathbf{A}_R^{\mathrm{PD}}$, by extension of scalars, over to the ring $E_R^{[u,v]}$ such that the natural injection of modules

$$E_R^{[u,v]} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T) \longrightarrow E_R^{[u,v]} \otimes_{R_0} \mathbf{N}(T),$$

is a $p^{n(T,e)}$-surjection compatible with Frobenius, filtration, connection and the action of Γ_R on each side. Let $D^{[u,v]} = R_\varpi^{[u,v]} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$, and $M^{[u,v]} = \mathbf{A}_R^{[u,v]} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)$, then we can rephrase the comparison above as a $p^{n(T,e)}$-isomorphism

$$E_R^{[u,v]} \otimes_{R_\varpi^{[u,v]}} D^{[u,v]} \simeq E_R^{[u,v]} \otimes_{\mathbf{A}_R^{[u,v]}} M^{[u,v]}, \tag{5.12}$$

compatible with Frobenius, filtration, connection, and the action of Γ_R on each side.

Let $R_1 = R_\varpi^{[u,v]}$, $R_2 = \mathbf{A}_R^{[u,v]}$, and $R_3 = E_R^{[u,v]}$. We set $X_{0,1} = X_0$, $X_{0,2} = \pi_m$ and for $1 \leq i \leq d$, we set $X_{i,1} = X_i$ and $X_{i,2} = [X_i^\flat]$. Now for $j = 1, 2$, we set

$$\Omega_j^1 := \mathbb{Z}\tfrac{dX_{0,j}}{1+X_{0,j}} \bigoplus_{i=1}^{d} \mathbb{Z}\tfrac{dX_{i,j}}{X_{i,j}},$$

and $\Omega_3^1 := \Omega_1^1 \oplus \Omega_2^1$. For $j = 1, 2, 3$, let $\Omega_i^k = \wedge^k \Omega_j$. Therefore, $\Omega_{R_j}^k = R_j \otimes \Omega_j^k$.

Recall that we have $D^{[u,v]} = R_\varpi^{[u,v]} \otimes_{R_0} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(T)$ is a filtered $R_\varpi^{[u,v]}$-module equipped with a quasi-nilpotent integrable connection satisfying Griffiths transversality with respect to the filtration as defined above. In other words, for each $k \in \mathbb{N}$, we have a complex

$$\mathrm{Fil}^k D^{[u,v]} \otimes \Omega_1^\bullet : \mathrm{Fil}^k D^{[u,v]} \xrightarrow{\partial_{R_1}} \mathrm{Fil}^{k-1} D^{[u,v]} \otimes \Omega_1^1 \xrightarrow{\partial_{R_1}} \mathrm{Fil}^{k-2} D^{[u,v]} \otimes \Omega_1^2 \xrightarrow{\partial_{R_1}} \cdots,$$

Next, let $\Xi := E_R^{[u,v]} \otimes_{R_\varpi^{[u,v]}} D^{[u,v]}$ and define a filtration on Ξ using the filtrations on each factor of the tensor product. For $k \in \mathbb{Z}$, we have

$$\partial_{R_3} : \mathrm{Fil}^k E_R^{[u,v]} \longrightarrow \mathrm{Fil}^{k-1} E_R^{[u,v]} \otimes_{\mathbb{Z}} \Omega_3^1, \quad \text{and} \quad \partial_{R_1} : \mathrm{Fil}^k D^{[u,v]} \longrightarrow \mathrm{Fil}^{k-1} D^{[u,v]} \otimes_{\mathbb{Z}} \Omega_1^1,$$

therefore we obtain that $\partial_{R_3} : \mathrm{Fil}^k \Xi \longrightarrow \mathrm{Fil}^{k-1} \Xi \otimes_{\mathbb{Z}} \Omega_3^1$. Hence, we have the filtered de Rham complex

$$\mathrm{Fil}^k \Xi \otimes \Omega_3^\bullet : \mathrm{Fil}^k \Xi \xrightarrow{\partial_{R_3}} \mathrm{Fil}^{k-1} \Xi \otimes \Omega_3^1 \xrightarrow{\partial_{R_3}} \mathrm{Fil}^{k-2} \Xi \otimes \Omega_3^2 \xrightarrow{\partial_{R_3}} \cdots.$$

Lemma 5.26. *The natural map*

$$\mathrm{Fil}^k D^{[u,v]} \otimes \Omega_1^\bullet \longrightarrow \mathrm{Fil}^k \Xi \otimes \Omega_3^\bullet$$

is a quasi-isomorphism.

Proof. Note that we have assumed $R_1 = R_\varpi^{[u,v]}$. Since we have $\mathrm{Fil}^k D^{[u,v]} = (\mathrm{Fil}^k \Xi)^{\partial_{R_2}=0}$, from Lemma 2.51 and Proposition 2.52 we obtain that the claim. ■

Next, recall from (5.10) that for $R_2 = \mathbf{A}_R^{[u,v]}$ and the module $M^{[u,v]} = \mathbf{A}_R^{[u,v]} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)$, we have the filtered de Rham complex

$$\mathrm{Fil}^k M^{[u,v]} \otimes \Omega_2^\bullet : \mathrm{Fil}^k M^{[u,v]} \longrightarrow \mathrm{Fil}^{k-1} M^{[u,v]} \otimes \Omega_2^1 \longrightarrow \mathrm{Fil}^{k-2} M^{[u,v]} \otimes \Omega_2^2 \longrightarrow \cdots, \quad \text{for } k \in \mathbb{Z}.$$

Also, let $\Delta := E_R^{[u,v]} \otimes_{R_\varpi^{[u,v]}} M^{[u,v]}$ and define a filtration on Δ using the filtrations on each factor of the tensor product. Then similar to the case of Ξ, we have the de Rham complex

$$\mathrm{Fil}^k \Delta \otimes \Omega_3^\bullet : \mathrm{Fil}^k \Delta \xrightarrow{\partial_{R_3}} \mathrm{Fil}^{k-1} \Delta \otimes \Omega_3^1 \xrightarrow{\partial_{R_3}} \mathrm{Fil}^{k-2} \Delta \otimes \Omega_3^2 \xrightarrow{\partial_{R_3}} \cdots.$$

Now, since $\mathrm{Fil}^k M^{[u,v]} = (\mathrm{Fil}^k \Delta)^{\partial_1=0}$, in a manner similar to Lemma 5.26 one can show that,

Lemma 5.27. *The natural map*

$$\mathrm{Fil}^k M^{[u,v]} \otimes \Omega_2^\bullet \longrightarrow \mathrm{Fil}^k \Delta \otimes \Omega_3^\bullet,$$

is a quasi-isomorphism.

Remark 5.28. The computations above continue to hold if we replace the ring $R_\varpi^{[u,v]}$ (resp. $\mathbf{A}_R^{[u,v]}$) with the ring $R_\varpi^{[u,v/p]}$ (resp. $\mathbf{A}_R^{[u,v/p]}$).

Definition 5.29. Let $M^{[u,v]}$ as above such that it admits a Frobenius-semilinear morphism $\varphi : M^{[u,v]} \longrightarrow M^{[u,v/p]}$. Using Definition 5.21 and Remark 5.22, define the (φ, ∂)-complex

$$\mathrm{Kos}\big(\varphi, \partial_A, \mathrm{Fil}^k M^{[u,v]}\big) := \left[\begin{array}{ccc} \mathrm{Kos}\big(\partial'_A, \mathrm{Fil}^k M^{[u,v]}\big) & \xrightarrow{p^k - p^{\bullet}\varphi} & \mathrm{Kos}\big(\partial'_A, M^{[u,v/p]}\big) \\ \downarrow{\scriptstyle \partial_0} & & \downarrow{\scriptstyle \partial_0} \\ \mathrm{Kos}\big(\partial'_A, \mathrm{Fil}^{k-1} M^{[u,v]}\big) & \xrightarrow{p^k - p^{\bullet+1}\varphi} & \mathrm{Kos}\big(\partial'_A, M^{[u,v/p]}\big) \end{array}\right].$$

Proposition 5.30. *The complexes* $\mathrm{Syn}\big(D^{[u,v]}, r\big)$ *and* $\mathrm{Kos}\big(\varphi, \partial_A, \mathrm{Fil}^r M^{[u,v]}\big)$ *are* $p^{2n(T,e)}$*-quasi-isomorphic, where* $n(T,e) \in \mathbb{N}$ *is as described after* (5.11).

Proof. Using Lemma 5.26 with $R_1 = R_{\varpi}^{[u,v]}$, $R_3 = E_R^{[u,v]}$, $\Xi = E_R^{[u,v]} \otimes_{R_{\varpi}^{[u,v]}} D^{[u,v]}$, and $\Xi' = E_R^{[u,v/p]} \otimes_{R_{\varpi}^{[u,v/p]}} D^{[u,v/p]}$, we have a quasi-isomorphism

$$\mathrm{Syn}(D^{[u,v]}, r) \simeq \Big[\mathrm{Fil}^r D^{[u,v]} \otimes \Omega_1^{\bullet} \xrightarrow{p^r - p^{\bullet}\varphi} D^{[u,v/p]} \otimes \Omega_1^{\bullet}\Big] \simeq \Big[\mathrm{Fil}^r \Xi \otimes \Omega_3^{\bullet} \xrightarrow{p^r - p^{\bullet}\varphi} \Xi' \otimes \Omega_3^{\bullet}\Big].$$

Using Lemma 5.27 with $R_2 = \mathbf{A}_R^{[u,v]}$, $R_3 = E_R^{[u,v]}$, $\Delta = E_R^{[u,v]} \otimes_{\mathbf{A}_R^{[u,v]}} M^{[u,v]}$, and $\Delta' = E_R^{[u,v/p]} \otimes_{\mathbf{A}_R^{[u,v/p]}} M^{[u,v/p]}$, we have a quasi-isomorphism

$$\mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r M^{[u,v]}) \simeq \Big[\mathrm{Fil}^r M^{[u,v]} \otimes \Omega_2^{\bullet} \xrightarrow{p^r - p^{\bullet}\varphi} \mathrm{Fil}^r M^{[u,v/p]} \otimes \Omega_2^{\bullet}\Big] \simeq \Big[\mathrm{Fil}^r \Delta \otimes \Omega_3^{\bullet} \xrightarrow{p^r - p^{\bullet}\varphi} \Delta' \otimes \Omega_3^{\bullet}\Big].$$

Note that in the quasi-ismorphism we used Remark 5.22 to identify the complexes $\mathrm{Fil}^k M^{[u,v]} \otimes \Omega^{\bullet}_{\mathbf{A}_R^{[u,v]}} \simeq \mathrm{Kos}\big(\partial_A, \mathrm{Fil}^k M^{[u,v]}\big)$.

Now using (5.12) we have $p^{n(T,e)}$-isomorphisms $\mathrm{Fil}^r \Xi \simeq \mathrm{Fil}^r \Delta$ and $\Xi' \simeq \Delta'$. Combining this with the isomorphisms above, we obtain a $p^{2n(T,e)}$-quasi-isomorphism

$$\mathrm{Syn}\big(D^{[u,v]}, r\big) \simeq \mathrm{Kos}\big(\varphi, \partial_A, \mathrm{Fil}^r M^{[u,v]}\big).$$

∎

5.2. Wach representations and Galois cohomology

In this section, for free Wach $\mathbb{Z}_p$-representations $T(r)$ of G_{R_0}, we will carry out the second step of the proof of Theorem 5.6, i.e. study complexes computing continuous G_R-cohomology of $T(r)$. To state the main result of this section, we introduce some notations. Recall that we defined an $\mathbf{A}_R^{[u,v]}$-module as

$$M^{[u,v]} = \mathbf{A}_R^{[u,v]} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T).$$

Note that we are working under the assumption that $\frac{p-1}{p} \leq u \leq \frac{v}{p} < 1 < v$, for example, one can take $u = \frac{p-1}{p}$ and $v = p - 1$. From (5.9) we have a filtration on $M^{[u,v]}$ as

$$\mathrm{Fil}^k M^{[u,v]} = \sum_{i+j=k} \mathrm{Fil}^i \mathbf{A}_R^{[u,v]} \widehat{\otimes}_{\mathbf{A}_{R_0}^+} \mathrm{Fil}^j \mathbf{N}(T).$$

These submodules are stable under the action of Γ_R and from Definition 5.29, we have the complex

$$\mathrm{Kos}\big(\varphi, \partial_A, \mathrm{Fil}^r M^{[u,v]}\big) = \left[\begin{tikzcd} \mathrm{Kos}\big(\partial_A', \mathrm{Fil}^r M^{[u,v]}\big) \arrow[r, "p^r - p^{\bullet}\varphi"] \arrow[d, "\partial_0"'] & \mathrm{Kos}\big(\partial_A', M^{[u,v/p]}\big) \arrow[d, "\partial_0"] \\ \mathrm{Kos}\big(\partial_A', \mathrm{Fil}^{r-1} M^{[u,v]}\big) \arrow[r, "p^r - p^{\bullet+1}\varphi"] & \mathrm{Kos}\big(\partial_A', M^{[u,v/p]}\big). \end{tikzcd}\right]$$

From the theory of (φ, Γ_R)-modules in Chapter 2, we have $\mathbf{D}_R(T(r)) = (\mathbf{A} \otimes_{\mathbb{Z}_p} T(r))^{H_R} = \mathbf{A}_R \otimes_{\mathbf{A}_{R_0}} \mathbf{D}(T(r))$. Using Proposition 4.15, we have the complex

$$\mathrm{Kos}\big(\varphi, \Gamma_R, \mathbf{D}_R(T(r))\big) = \left[\begin{tikzcd} \mathrm{Kos}\big(\Gamma_R', \mathbf{D}_R(T(r))\big) \arrow[r, "1-\varphi"] \arrow[d, "\tau_0"'] & \mathrm{Kos}\big(\Gamma_R', \mathbf{D}_R(T(r))\big) \arrow[d, "\tau_0"] \\ \mathrm{Kos}^c\big(\Gamma_R', \mathbf{D}_R(T(r))\big) \arrow[r, "1-\varphi"] & \mathrm{Kos}^c\big(\Gamma_R', \mathbf{D}_R(T(r))\big) \end{tikzcd}\right].$$

By Proposition 4.13 and Theorem 4.4 we see that the Koszul complex defined above computes the continuous Galois cohomology of $T(r)$, i.e.

$$\mathrm{Kos}\big(\varphi, \Gamma_R, \mathbf{D}_R(T(r))\big) \simeq \mathbf{R}\Gamma_{\mathrm{cont}}(G_R, T(r)).$$

The main result of this section is the comparison between the Koszul complexes introduced above.

Proposition 5.31. *There exists a p^N-quasi-isomorphism*

$$\tau_{\leq r}\mathrm{Kos}\big(\varphi, \partial_A, \mathrm{Fil}^r M^{[u,v]}\big) \simeq \tau_{\leq r}\mathrm{Kos}\big(\varphi, \Gamma_R, \mathbf{D}_R(T(r))\big) \simeq \tau_{\leq r}\mathbf{R}\Gamma_{\mathrm{cont}}(G_R, T(r)),$$

where $N = N(T, r) \in \mathbb{N}$ depends on the representation T, and r.

5.2.1. Proof of Theorem 5.6

Using the results of previous section and Proposition 5.31, we will show Theorem 5.6. Let us recall the statement,

Theorem 5.32. *Let T be a free $\mathbb{Z}_p$-representation of G_{R_0} as in Definition 5.3, s the maximum among the absolute values of Hodge-Tate weights of $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$, and an integer $r \geq s+1$. Then there exists a p^N-quasi-isomorphism*

$$\tau_{\leq r-s-1}\mathrm{Syn}(D_R, r) \simeq \tau_{\leq r-s-1}\mathbf{R}\Gamma_{\mathrm{cont}}(G_R, T(r)),$$

i.e. we have p^N-isomorphisms

$$H^k_{\mathrm{syn}}(D_R, r) \xrightarrow{\simeq} H^k(G_R, T(r)),$$

for $0 \leq k \leq r-s-1$ and $N = N(T, e, r) \in \mathbb{N}$ depending on the representation T, ramification index e, and r.

Proof. Combining Proposition 5.10 and Proposition 5.12, we have p^{4r+4s}-quasi-isomorphisms

$$\tau_{\leq r-s-1}\mathrm{Syn}\big(D^{\mathrm{PD}}, r\big) \simeq \tau_{\leq r-s-1}\mathrm{Syn}\big(D^{[u]}, r\big) \simeq \tau_{\leq r-s-1}\mathrm{Syn}\big(D^{[u,v]}, r\big).$$

Next, from Proposition 5.30 we have a $p^{2n(T,e)}$-quasi-isomorphism

$$\mathrm{Syn}\big(D^{[u,v]}, r\big) \simeq \mathrm{Kos}\big(\varphi, \partial_A, \mathrm{Fil}^r M^{[u,v]}\big).$$

Finally, thanks to Proposition 5.31, we have a $p^{14r+3s+2}$-quasi-isomorphism (see the proof of the proposition for the explicit constant)

$$\tau_{\leq r}\mathrm{Kos}\big(\varphi, \partial_A, \mathrm{Fil}^r M^{[u,v]}\big) \simeq \tau_{\leq r}\mathrm{Kos}\big(\varphi, \Gamma_R, \mathbf{D}_R(T(r))\big).$$

Combining all these statement gives us the desired conclusion with $N = 2n(T,e) + 18r + 7s + 2$. ■

In the rest of this section, we will prove Proposition 5.31.

5.2.2. From differential forms to infinitesimal action of Γ_R

Note that we are working under the assumption that $\frac{p-1}{p} \leq u \leq \frac{v}{p} < 1 < v$, for example, one can take $u = \frac{p-1}{p}$ and $v = p-1$. From Definition 4.19 we have the complex $\mathrm{Kos}\big(\mathrm{Lie}\,\Gamma_R', \mathrm{Fil}^r M^{[u,v]}\big)$ and we consider a subcomplex, i.e. a complex made of submodules in each degree stable under the differentials of the former complex

$$\begin{aligned}\mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', \mathrm{Fil}^r M^{[u,v]}\big)\ :\ \mathrm{Fil}^r M^{[u,v]} &\xrightarrow{(\nabla_i)} \big(t\mathrm{Fil}^{r-1} M^{[u,v]}\big)^{I_1'} \longrightarrow \cdots \\ \cdots \longrightarrow \big(t^n\mathrm{Fil}^{r-n} M^{[u,v]}\big)^{I_n'} &\longrightarrow \big(t^{n+1}\mathrm{Fil}^{r-n-1} M^{[u,v]}\big)^{I_{n+1}'} \longrightarrow \cdots.\end{aligned}$$

Similarly, we define the complex $\mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', t\mathrm{Fil}^{r-1} M^{[u,v]}\big)$ as a subcomplex of $\mathrm{Kos}\big(\mathrm{Lie}\,\Gamma_R', \mathrm{Fil}^r M^{[u,v]}\big)$. Now, consider the map

$$\nabla_0\ :\ \mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', \mathrm{Fil}^r M^{[u,v]}\big) \longrightarrow \mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', t\mathrm{Fil}^{r-1} M^{[u,v]}\big),$$

defined by the diagram

$$\begin{tikzcd}
\mathrm{Fil}^r M^{[u,v]} \arrow[r, "(\nabla_i)"] \arrow[d, "\nabla_0"] & \big(t\mathrm{Fil}^{r-1} M^{[u,v]}\big)^{I_1'} \arrow[r] \arrow[d, "\nabla_0 - p^m"] & \cdots \arrow[r] & \big(t^n\mathrm{Fil}^{r-n} M^{[u,v]}\big)^{I_n'} \arrow[r] \arrow[d, "\nabla_0 - np^m"] & \cdots \\
t\mathrm{Fil}^{r-1} M^{[u,v]} \arrow[r, "(\nabla_i)"] & \big(t^2\mathrm{Fil}^{r-2} M^{[u,v]}\big)^{I_1'} \arrow[r] & \cdots \arrow[r] & \big(t^{n+1}\mathrm{Fil}^{r-n-1} M^{[u,v]}\big)^{I_n'} \arrow[r] & \cdots,
\end{tikzcd}$$

which commutes since $\nabla_0\nabla_i - \nabla_i\nabla_0 = p^m\nabla_i$ for $1 \leq i \leq d$ (see (4.4) and the discussion after Definition 4.19). We write the total complex of the diagram above as $\mathcal{K}\big(\mathrm{Lie}\,\Gamma_R, \mathrm{Fil}^r M^{[u,v]}\big)$, which is a subcomplex of $\mathrm{Kos}\big(\mathrm{Lie}\,\Gamma_R, \mathrm{Fil}^r M^{[u,v]}\big)$. In a similar manner, we can define complexes $\mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', M^{[u,v/p]}\big)$ and $\mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', tM^{[u,v/p]}\big)$, and a map ∇_0 from the former to the latter complex. Note that since the filtration on $\mathbf{A}_R^{[u,v/p]}$ is trivial (see Definition 2.27), therefore $\mathrm{Fil}^k M^{[u,v/p]} = M^{[u,v/p]}$ for all $k \in \mathbb{Z}$.

Next, from Definition 5.29 we have the complex $\mathrm{Kos}\big(\varphi, \partial_A, \mathrm{Fil}^r M^{[u,v]}\big)$. Since $\nabla_i = t\partial_i$, for $0 \leq i \leq d$, we consider the morphism of complexes $\mathrm{Kos}\big(\partial_A', \mathrm{Fil}^r M^{[u,v]}\big) \longrightarrow \mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', \mathrm{Fil}^r M^{[u,v]}\big)$ given by the diagram

$$\begin{tikzcd}
\mathrm{Fil}^r M^{[u,v]} \arrow[r, "(\partial_i)"] \arrow[d, "t^0 = id"] & \big(\mathrm{Fil}^{r-1} M^{[u,v]}\big)^{I_1'} \arrow[r] \arrow[d, "t^1"] & \cdots \arrow[r] & \big(M^{[u,v]}\big)^{I_r'} \arrow[r] \arrow[d, "t^r"] & \big(M^{[u,v]}\big)^{I_{r+1}'} \arrow[r] \arrow[d, "t^{r+1}"] & \cdots \\
\mathrm{Fil}^r M^{[u,v]} \arrow[r, "(\nabla_i)"] & (t\mathrm{Fil}^{r-1} M^{[u,v]})^{I_1'} \arrow[r] & \cdots \arrow[r] & (t^r M^{[u,v]})^{I_r'} \arrow[r] & (t^{r+1} M^{[u,v]})^{I_{r+1}'} \arrow[r] & \cdots.
\end{tikzcd}$$

Since the vertical maps are bijective, it is an isomorphism of complexes. Similarly, we can define maps from $\mathrm{Kos}\big(\partial_A', t\mathrm{Fil}^{r-1} M^{[u,v]}\big) \longrightarrow \mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', tM^{[u,v]}\big)$, $\mathrm{Kos}\big(\partial_A', M^{[u,v/p]}\big) \longrightarrow \mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', M^{[u,v/p]}\big)$ and $\mathrm{Kos}\big(\partial_A', M^{[u,v/p]}\big) \longrightarrow \mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', tM^{[u,v/p]}\big)$, which are isomorphisms as well. Since each term of these complexes admit a Frobenius-semilinear morphism $\varphi : t^j \mathrm{Fil}^{r-j} M^{[u,v]} \longrightarrow t^j M^{[u,v/p]}$, we obtain an induced morphism

$$\begin{bmatrix} \mathrm{Kos}\big(\partial'_A, \mathrm{Fil}^r M^{[u,v]}\big) \xrightarrow{p^r - p^\bullet \varphi} \mathrm{Kos}\big(\partial'_A, M^{[u,v/p]}\big) \\ \downarrow \partial_0 \qquad\qquad\qquad\qquad \downarrow \partial_0 \\ \mathrm{Kos}\big(\partial'_A, \mathrm{Fil}^{r-1} M^{[u,v]}\big) \xrightarrow{p^r - p^{\bullet+1}\varphi} \mathrm{Kos}\big(\partial'_A, M^{[u,v/p]}\big) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{K}\big(\mathrm{Lie}\,\Gamma'_R, \mathrm{Fil}^r M^{[u,v]}\big) \xrightarrow{p^r - \varphi} \mathcal{K}\big(\mathrm{Lie}\,\Gamma'_R, M^{[u,v/p]}\big) \\ \downarrow \nabla_0 \qquad\qquad\qquad\qquad \downarrow \nabla_0 \\ \mathcal{K}\big(\mathrm{Lie}\,\Gamma'_R, t\mathrm{Fil}^{r-1} M^{[u,v]}\big) \xrightarrow{p^r - \varphi} \mathcal{K}\big(\mathrm{Lie}\,\Gamma'_R, tM^{[u,v/p]}\big) \end{bmatrix}, \tag{5.13}$$

where the source complex in (5.13) above is $\mathrm{Kos}\big(\varphi, \partial_A, \mathrm{Fil}^r M^{[u,v]}\big)$. Tautologically, we have that

Lemma 5.33. *The map constructed in* (5.13) *is a quasi-isomorphism of complexes.*

Next, recall that s is maximum among the absolute values of the Hodge-Tate weights of V and $r \geq s+1$ is an integer. Let us set $N^{[u,v]}(T(r)) = \mathbf{A}_R^{[u,v]} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T(r))$, and we can define a filtration on this module given as

$$\mathrm{Fil}^k N^{[u,v]}(T(r)) := \sum_{i+j=k} \mathrm{Fil}^i \mathbf{A}_R^{[u,v]} \widehat{\otimes}_{\mathbf{A}_{R_0}^+} \mathrm{Fil}^j \mathbf{N}(T(r)), \quad \text{for } k \in \mathbb{Z}.$$

These submodules are stable under the action of Γ_R. Let ϵ^{-r} denote a $\mathbb{Z}_p$-basis of $\mathbb{Z}_p(-r)$, then we have

$$\begin{aligned} (t^r \otimes \epsilon^{-r})\, \mathrm{Fil}^k N^{[u,v]}(T(r)) &= (t^r \otimes \epsilon^{-r}) \sum_{i+j=k} \mathrm{Fil}^i \mathbf{A}_R^{[u,v]} \widehat{\otimes}_{\mathbf{A}_{R_0}^+} \mathrm{Fil}^j \mathbf{N}(T(r)) \\ &= \tfrac{t^r}{\pi^r} \sum_{i+j=k} \mathrm{Fil}^i \mathbf{A}_R^{[u,v]} \widehat{\otimes}_{\mathbf{A}_{R_0}^+} \mathrm{Fil}^{j+r} \mathbf{N}(T) = \mathrm{Fil}^{r+k} M^{[u,v]}, \end{aligned} \tag{5.14}$$

where the second equality is the result of observation made in Lemma 3.11, and the third equality comes from the fact that $\frac{t}{\pi}$ is a unit in $\mathbf{A}_R^{[u,v]}$ (see Lemma 2.43). Moreover, we also have that $(t^r \otimes \epsilon^{-r}) N^{[u,v/p]}(T(r)) = t^r \pi^{-r} M^{[u,v/p]} = M^{[u,v/p]}$.

From Remark 5.18, we have that ∇_i is well-defined over $N^{[u,v]}(T(r))$, for $0 \leq i \leq d$. Now using Definition 4.19 we have the complex $\mathrm{Kos}\big(\mathrm{Lie}\,\Gamma'_R, \mathrm{Fil}^0 N^{[u,v]}(T(r))\big)$, and we consider the subcomplex

$$\begin{aligned} \mathcal{K}\big(\mathrm{Lie}\,\Gamma'_R, \mathrm{Fil}^0 N^{[u,v]}(T(r))\big) : \mathrm{Fil}^0 N^{[u,v]}(T(r)) \xrightarrow{(\nabla_i)} \big(t\mathrm{Fil}^{-1} N^{[u,v]}(T(r))\big)^{I'_1} \longrightarrow \cdots \\ \cdots \longrightarrow \big(t^q \mathrm{Fil}^{-q} N^{[u,v]}(T(r))\big)^{I'_q} \longrightarrow \cdots. \end{aligned}$$

Similar to above, we can define the complex $\mathcal{K}\big(\mathrm{Lie}\,\Gamma'_R, t\mathrm{Fil}^{-1} N^{[u,v]}(T(r))\big)$ as a subcomplex of $\mathrm{Kos}\big(\mathrm{Lie}\,\Gamma'_R, \mathrm{Fil}^0 N^{[u,v]}(T(r))\big)$, and a map

$$\nabla_0 : \mathcal{K}\big(\mathrm{Lie}\,\Gamma'_R, \mathrm{Fil}^0 N^{[u,v]}(T(r))\big) \longrightarrow \mathcal{K}\big(\mathrm{Lie}\,\Gamma'_R, t\mathrm{Fil}^{-1} N^{[u,v]}(T(r))\big).$$

The total complex of the latter map, written as $\mathcal{K}\big(\mathrm{Lie}\,\Gamma_R, \mathrm{Fil}^r M^{[u,v]}\big)$, is a subcomplex of $\mathrm{Kos}\big(\mathrm{Lie}\,\Gamma_R, \mathrm{Fil}^0 N^{[u,v]}(T(r))\big)$. Again, in a similar manner, we can define complexes $\mathcal{K}\big(\mathrm{Lie}\,\Gamma'_R, N^{[u,v/p]}(T(r))\big)$ and $\mathcal{K}\big(\mathrm{Lie}\,\Gamma'_R, tN^{[u,v/p]}(T(r))\big)$, and a map ∇_0 from the former to the latter complex.

Consider the morphism $\mathcal{K}\big(\mathrm{Lie}\,\Gamma'_R, \mathrm{Fil}^0 N^{[u,v]}(T(r))\big) \longrightarrow \mathcal{K}\big(\mathrm{Lie}\,\Gamma'_R, \mathrm{Fil}^r M^{[u,v]}\big)$ given by the diagram

$$\begin{tikzcd}
\mathrm{Fil}^0 N^{[u,v]}(T(r)) \arrow[r, "(\nabla_i)"] \arrow[d, "t^r\otimes\epsilon^{-r}"] & \big(t\mathrm{Fil}^{-1}N^{[u,v]}(T(r))\big)^{I_1'} \arrow[r] \arrow[d, "t^r\otimes\epsilon^{-r}"] & \cdots \arrow[r] & \big(t^q\mathrm{Fil}^{-q}N^{[u,v]}(T(r))\big)^{I_q'} \arrow[r] \arrow[d, "t^r\otimes\epsilon^{-r}"] & \cdots \\
\mathrm{Fil}^r M^{[u,v]} \arrow[r, "(\nabla_i)"] & \big(t\mathrm{Fil}^{r-1}M^{[u,v]}\big)^{I_1'} \arrow[r] & \cdots \arrow[r] & \big(t^q\mathrm{Fil}^{r-q}M^{[u,v]}\big)^{I_q'} \arrow[r] & \cdots,
\end{tikzcd}$$

which is is bijective in each term and therefore an isomorphism. Considering similar maps between complexes considered above, we obtain a morphism (multiplication by $t^r\otimes\epsilon^{-r}$ on each term)

$$\left[\begin{tikzcd}
\mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', \mathrm{Fil}^0N^{[u,v]}(T(r))\big) \arrow[r, "p^r(1-\varphi)"] \arrow[d, "\nabla_0"] & \mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', N^{[u,v/p]}(T(r))\big) \arrow[d, "\nabla_0"] \\
\mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', t\mathrm{Fil}^{-1}N^{[u,v]}(T(r))\big) \arrow[r, "p^r(1-\varphi)"] & \mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', tN^{[u,v/p]}(T(r))\big)
\end{tikzcd}\right] \longrightarrow$$
$$\left[\begin{tikzcd}
\mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', \mathrm{Fil}^rM^{[u,v]}\big) \arrow[r, "p^r-\varphi"] \arrow[d, "\nabla_0"] & \mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', M^{[u,v/p]}\big) \arrow[d, "\nabla_0"] \\
\mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', t\mathrm{Fil}^{r-1}M^{[u,v]}\big) \arrow[r, "p^r-\varphi"] & \mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', tM^{[u,v/p]}\big)
\end{tikzcd}\right]. \tag{5.15}$$

Again, it is immediate that

Lemma 5.34. *The map constructed in (5.15) is a quasi-isomorphism of complexes.*

In order to proceed from "Lie Γ_R-Koszul complexes" discussed above to "Γ_R-Koszul complexes", we modify the source complex in the map of Lemma 5.34 as follows:

$$\mathcal{K}\big(\varphi, \mathrm{Lie}\,\Gamma_R, N^{[u,v]}(T(r))\big) := \left[\begin{tikzcd}
\mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', \mathrm{Fil}^0N^{[u,v]}(T(r))\big) \arrow[r, "1-\varphi"] \arrow[d, "\nabla_0"'] & \mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', N^{[u,v/p]}(T(r))\big) \arrow[d, "\nabla_0"] \\
\mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', t\mathrm{Fil}^{-1}N^{[u,v]}(T(r))\big) \arrow[r, "1-\varphi"] & \mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', tN^{[u,v/p]}(T(r))\big)
\end{tikzcd}\right].$$

Remark 5.35. The complex $\mathcal{K}\big(\varphi, \mathrm{Lie}\,\Gamma_R, N^{[u,v]}(T(r))\big)$ is p^{4r}-isomorphic to the source complex in the map of Lemma 5.34.

Combining Lemmas 5.33 & 5.34, and Remark 5.35, we get

Proposition 5.36. *There exists a p^{4r}-quasi-isomorphism of complexes*

$$\mathrm{Kos}\big(\varphi, \partial_A, \mathrm{Fil}^rM^{[u,v]}\big) \simeq \mathcal{K}\big(\varphi, \mathrm{Lie}\,\Gamma_R, N^{[u,v]}(T(r))\big).$$

5.2.3. From infinitesimal action of Γ_R to continuous action of Γ_R

In the previous section, we changed from complexes involving the operators ∂_i to complexes invoving the operators ∇_i. In this section, we will further replace these complexes with complexes involving operators $\gamma_i - 1$. Note that we are working under the assumption that $\frac{p-1}{p} \leq u \leq \frac{v}{p} < 1 < v$, for example, one can take $u = \frac{p-1}{p}$ and $v = p-1$.

Next, we want to construct similar complexes for the action of Γ_R. Note that we have

$$(\gamma_i - 1)\mathrm{Fil}^kN^{[u,v]}(T(r)) \subset \mathrm{Fil}^kN^{[u,v]}(T(r)) \cap \pi N^{[u,v]}(T(r)) = \pi\mathrm{Fil}^{k-1}N^{[u,v]}(T(r))$$

where the last equality follows from Lemma 3.17. We can define a subcomplex of $\mathrm{Kos}\big(\Gamma'_R, \mathrm{Fil}^0 N^{[u,v]}(T(r))\big)$ as

$$\mathcal{K}\big(\Gamma'_R, \mathrm{Fil}^0 N^{[u,v]}(T(r))\big) \,:\, \mathrm{Fil}^0 N^{[u,v]}(T(r)) \xrightarrow{(\tau_i)} \big(\pi\mathrm{Fil}^{-1} N^{[u,v]}(T(r))\big)^{I'_1} \longrightarrow \big(\pi^2\mathrm{Fil}^{-2} N^{[u,v]}(T(r))\big)^{I'_2} \longrightarrow \cdots. \tag{5.16}$$

Similarly, we can define the complex $\mathcal{K}^c\big(\Gamma'_R, \pi\mathrm{Fil}^{-1} N^{[u,v]}(T(r))\big)$ as a subcomplex of $\mathrm{Kos}^c\big(\Gamma'_R, \mathrm{Fil}^0 N^{[u,v]}(T(r))\big)$ (see Definition 4.10). Now, consider the map

$$\tau_0 \,:\, \mathcal{K}\big(\Gamma'_R, \mathrm{Fil}^0 N^{[u,v]}(T(r))\big) \longrightarrow \mathcal{K}^c\big(\Gamma'_R, t\mathrm{Fil}^{-1} N^{[u,v]}(T(r))\big), \tag{5.17}$$

defined by the commutative diagram

$$\begin{tikzcd}
\mathrm{Fil}^0 N^{[u,v]}(T(r)) \arrow[r, "(\tau_i)"] \arrow[d, "\tau_0^0"] & \big(\pi\mathrm{Fil}^{-1} N^{[u,v]}(T(r))\big)^{I'_1} \arrow[r] \arrow[d, "\tau_0^1"] & \big(\pi^2\mathrm{Fil}^{-2} N^{[u,v]}(T(r))\big)^{I'_2} \arrow[r] \arrow[d, "\tau_0^2"] & \cdots \\
\pi\mathrm{Fil}^{-1} N^{[u,v]}(T(r)) \arrow[r, "(\tau_i)"] & \big(\pi^2\mathrm{Fil}^{-2} N^{[u,v]}(T(r))\big)^{I'_1} \arrow[r] & \big(\pi^3\mathrm{Fil}^{-3} N^{[u,v]}(T(r))\big)^{I'_2} \arrow[r] & \cdots,
\end{tikzcd}$$

where the vertical maps are as in Definitions 4.9 & 4.12. We write the total complex of the diagram above as $\mathcal{K}\big(\Gamma_R, \mathrm{Fil}^0 N^{[u,v]}(T(r))\big)$, which is a subcomplex of $\mathrm{Kos}\big(\Gamma_R, \mathrm{Fil}^0 N^{[u,v]}(T(r))\big)$. In a similar manner, we can define complexes $\mathcal{K}\big(\Gamma'_R, N^{[u,v/p]}(T(r))\big)$ and $\mathcal{K}^c\big(\Gamma'_R, \pi N^{[u,v/p]}(T(r))\big)$ and a map τ_0 from the former to the latter complex.

Next, we consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Fil}^0 N^{[u,v]}(T(r)) \arrow[r, "(\tau_i)"] \arrow[d, "id"] & \big(t\mathrm{Fil}^{-1} N^{[u,v]}(T(r))\big)^{I'_1} \arrow[r] \arrow[d, "\beta_1"] & \big(t^2\mathrm{Fil}^{-2} N^{[u,v]}(T(r))\big)^{I'_2} \arrow[r] \arrow[d, "\beta_2"] & \cdots \\
\mathrm{Fil}^0 N^{[u,v]}(T(r)) \arrow[r, "(\nabla_i)"] & \big(t\mathrm{Fil}^{-1} N^{[u,v]}(T(r))\big)^{I'_1} \arrow[r] & \big(t^2\mathrm{Fil}^{-2} N^{[u,v]}(T(r))\big)^{I'_2} \arrow[r] & \cdots,
\end{tikzcd}$$

where $\beta_q \,:\, (a_{i_1\cdots i_q}) \mapsto \big(\nabla_{i_q}\cdots\nabla_{i_1}\tau_{i_1}^{-1}\cdots\tau_{i_q}^{-1}(a_{i_1\cdots i_q})\big)$ for $1 \leq q \leq d$. Notice that since $\frac{t}{\pi}$ is a unit in $\mathrm{A}_R^{[u,v]}$ (see Lemma 2.43), the top complex in the diagram above is exactly the complex $\mathcal{K}\big(\Gamma'_R, \mathrm{Fil}^0 N^{[u,v]}(T(r))\big)$ from (5.16). This defines a map

$$\beta \,:\, \mathcal{K}\big(\Gamma'_R, \mathrm{Fil}^0 N^{[u,v]}(T(r))\big) \longrightarrow \mathcal{K}\big(\mathrm{Lie}\,\Gamma'_R, \mathrm{Fil}^0 N^{[u,v]}(T(r))\big),$$

Similarly, we can consider the commutative diagram

$$\begin{tikzcd}
t\mathrm{Fil}^{-1} N^{[u,v]}(T(r)) \arrow[r, "(\tau_i^c)"] \arrow[d, "\beta_0^c"] & \big(t^2\mathrm{Fil}^{-2} N^{[u,v]}(T(r))\big)^{I'_1} \arrow[r] \arrow[d, "\beta_1^c"] & \big(t^3\mathrm{Fil}^{-3} N^{[u,v]}(T(r))\big)^{I'_2} \arrow[r] \arrow[d, "\beta_2^c"] & \cdots \\
t\mathrm{Fil}^{-1} N^{[u,v]}(T(r)) \arrow[r, "(\nabla_i)"] & \big(t^2\mathrm{Fil}^{-2} N^{[u,v]}(T(r))\big)^{I'_1} \arrow[r] & \big(t^3\mathrm{Fil}^{-3} N^{[u,v]}(T(r))\big)^{I'_2} \arrow[r] & \cdots,
\end{tikzcd}$$

with $\beta_0^c = \nabla_0\tau_0^{-1}$ and

$$\beta_q^c \,:\, (a_{i_1\cdots i_q}) \longmapsto \big(\nabla_{i_q}\cdots\nabla_{i_1}\nabla_0\tau_0^{-1}\tau_{i_1}^{c,-1}\cdots\tau_{i_q}^{c,-1}(a_{i_1\cdots i_q})\big) \;\text{ for } 1 \leq q \leq d.$$

Recall that $c = \chi(\gamma_0) = \exp(p^m)$. Again, this defines a map

$$\beta^c \,:\, \mathcal{K}^c\big(\Gamma'_R, t\mathrm{Fil}^{-1} N^{[u,v]}(T(r))\big) \longrightarrow \mathcal{K}^c\big(\mathrm{Lie}\,\Gamma'_R, t\mathrm{Fil}^{-1} N^{[u,v]}(T(r))\big).$$

Remark 5.37. The definition of maps β and β^c continue to hold after base changing each term of the complexes to the ring $\mathbf{A}_R^{[u,v/p]}$.

Next, for $j \in \mathbb{N}$, we have $t^j\mathrm{Fil}^{-j}N^{[u,v]}(T(r)) \subset N^{[u,v]}(T(r))$ and the induced Frobenius gives

$$\varphi(t^j\mathrm{Fil}^{-j}N^{[u,v]}(T(r))) = \varphi(\pi^{j-r}\mathrm{Fil}^{r-j}M^{[u,v]}(r)) \subset \pi^{j-r}M^{[u,v/p]}(r) = t^jN^{[u,v/p]}(T(r)),$$

where we have used the fact that $\frac{t}{\pi} \in \mathbf{A}_R^{[u,v]}$ is a unit (see Lemma 2.43). Using the Frobenius morphism and the map between complexes discussed above, we obtain an induced morphism

$$\left[\begin{array}{ccc} \mathcal{K}\big(\Gamma_R', \mathrm{Fil}^0N^{[u,v]}(T(r))\big) & \xrightarrow{1-\varphi} & \mathcal{K}\big(\Gamma_R', N^{[u,v/p]}(T(r))\big) \\ \big\downarrow{\scriptstyle \tau_0} & & \big\downarrow{\scriptstyle \tau_0} \\ \mathcal{K}^c\big(\Gamma_R', t\mathrm{Fil}^{-1}N^{[u,v]}(T(r))\big) & \xrightarrow{1-\varphi} & \mathcal{K}^c\big(\Gamma_R', tN^{[u,v/p]}(T(r))\big) \end{array}\right] \xrightarrow{(\beta,\beta^c)}$$

$$\left[\begin{array}{ccc} \mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', \mathrm{Fil}^0N^{[u,v]}(T(r))\big) & \xrightarrow{1-\varphi} & \mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', N^{[u,v/p]}(T(r))\big) \\ \big\downarrow{\scriptstyle \nabla_0} & & \big\downarrow{\scriptstyle \nabla_0} \\ \mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', t\mathrm{Fil}^{-1}N^{[u,v]}(T(r))\big) & \xrightarrow{1-\varphi} & \mathcal{K}\big(\mathrm{Lie}\,\Gamma_R', tN^{[u,v/p]}(T(r))\big) \end{array}\right].$$

We denote the complex on left as $\mathcal{K}\big(\varphi, \Gamma_R, N^{[u,v]}(T(r))\big)$ and write the map as

$$\mathcal{L} = (\beta, \beta^c) : \mathcal{K}\big(\varphi, \Gamma_R, N^{[u,v]}(T(r))\big) \longrightarrow \mathcal{K}\big(\varphi, \mathrm{Lie}\,\Gamma_R, N^{[u,v]}(T(r))\big),$$

Proposition 5.38. *The morphism of complexes $\mathcal{L}$ from the construction above is an isomorphism.*

Proof. The proof follows in a manner similar to [CN17, Lemma 4.6]. From the fact that $\nabla_i\tau_i^{-1}$, for $0 \leq i \leq d$, is invertible (see Corollary 5.17) and $[\nabla_i, \nabla_j] = 0$, for $1 \leq i, j \leq d$, we get that the map β above is an isomorphism.

Next, we will show that the map β_q^c, for $1 \leq q \leq d$, is a well-defined isomorphism. For this, we need to show that $\nabla_{i_q}\cdots\nabla_{i_1}\nabla_0\tau_0^{-1}\tau_{i_1}^{c,-1}\cdots\tau_{i_q}^{c,-1}$ are well-defined isomorphisms, for $1 \leq i_1 < \cdots < i_q \leq d$. We can reduce the map to

$$(\nabla_{i_q}/\tau_{i_q})\cdots(\nabla_{i_1}/\tau_{i_1})\tau_{i_q}\cdots\tau_{i_1}\nabla_0\tau_0^{-1}\tau_{i_1}^{c,-1}\cdots\tau_{i_q}^{c,-1},$$

and since ∇_i/τ_i is invertible for $0 \leq i \leq d$, we only need to show that $\tau_{i_q}\cdots\tau_{i_1}\nabla_0\tau_0^{-1}\tau_{i_1}^{c,-1}\cdots\tau_{i_q}^{c,-1}$ is a well-defined isomorphism. Using the proof of Lemma 4.17, we can write

$$\tau_{i_q}\cdots\tau_{i_1}\nabla_0\tau_0^{-1}\tau_{i_1}^{c,-1}\cdots\tau_{i_q}^{c,-1} = \sum_{k\geq 0} a_k\tau_{i_q}\cdots\tau_{i_1}(\gamma_0-1)^k\tau_{i_1}^{c,-1}\cdots\tau_{i_q}^{c,-1},$$

where $a_k \in O_F$. Using the fact that $\gamma_0\gamma_i^{a/c} = \gamma_i^a\gamma_0$, we get that

$$(\gamma_i^a - 1)(\gamma_0 - x) = (\gamma_0 - x\delta(\gamma_i^a))(\gamma_i^{a/c} - 1), \quad \text{where} \quad \delta(\gamma_i^a) := \tfrac{\gamma_i^a - 1}{\gamma_i^{a/c} - 1},$$

which yields

$$(\gamma_i^a - 1)(\gamma_0 - 1)^k = (\gamma_0 - \delta(\gamma_i^a))(\gamma_0 - \delta(\gamma_i^{a/c}))\cdots\big(\gamma_0 - \delta\big(\gamma_i^{a/c^{k-1}}\big)\big)\big(\gamma_i^{a/c^k} - 1\big).$$

So we can write

$$\tau_{i_q} \cdots \tau_{i_1}(\gamma_0 - 1)^k \tau_{i_1}^{c,-1} \cdots \tau_{i_q}^{c,-1} = (\gamma_0 - \delta_k) \cdots (\gamma_0 - \delta_1) \frac{\gamma_{i_q}^{1/c^k} - 1}{\gamma_{i_q}^c - 1} \cdots \frac{\gamma_{i_1}^{1/c^k} - 1}{\gamma_{i_1}^c - 1}$$
$$= (\gamma_0 - \delta_k) \cdots (\gamma_0 - \delta_1)\delta_0. \tag{5.18}$$

Observe that for $0 \leq i \leq d$ and $j \in \mathbb{Z}$, we have

$$\frac{\gamma_i^{1/c^j} - 1}{\gamma_i^{1/c^{j+1}} - 1} = \frac{\gamma_i^{1/c^j} - 1}{\gamma_i - 1} \cdot \frac{\gamma_i - 1}{\gamma_i^{1/c^{j+1}} - 1} \quad \text{and} \quad \frac{\gamma_i^{1/c^k} - 1}{\gamma_i^c - 1} = \frac{\gamma_i^{1/c^k} - 1}{\gamma_i - 1} \cdot \frac{\gamma_i - 1}{\gamma_i^c - 1} \in 1 + (p^m, \gamma_i - 1)\mathbb{Z}_p[[\Gamma_R]].$$

Therefore, in (5.18) we have that $\delta_j \in 1 + (p^m, (\gamma_1 - 1), \ldots, (\gamma_d - 1))$. Writing $(\gamma_0 - \delta_j) = (\gamma_0 - 1) + (1 - \delta_j)$, we conclude that

$$\tau_{i_q} \cdots \tau_{i_1}(\gamma_0 - 1)^k \tau_{i_1}^{c,-1} \cdots \tau_{i_q}^{c,-1} \in (p^m, \gamma_0 - 1, \ldots, \gamma_d - 1)^k.$$

Now from Lemma 2.45, it follows that the series of operators

$$\sum_{k \geq 0} a_k \tau_{i_q} \cdots \tau_{i_1}(\gamma_0 - 1)^k \tau_{i_1}^{c,-1} \cdots \tau_{i_q}^{c,-1}$$

converge and therefore $\nabla_{i_q} \cdots \nabla_{i_1} \nabla_0 \tau_0^{-1} \tau_{i_1}^{c,-1} \cdots \tau_{i_q}^{c,-1}$ is well-defined. The same arguments show that the series of operators $\sum_{k \geq 0} b_k \tau_{i_q}^c \cdots \tau_{i_1}^c (\gamma_0 - 1)^k \tau_{i_1}^{-1} \cdots \tau_{i_q}^{-1}$ converge as an inverse to the previous operator. This establishes the claim. ∎

5.2.4. Change of annulus of convergence : Part 1

Now that we have changed our original complex to a complex involving operators $\gamma_i - 1$, in this section, we will pass from the ring $\mathbf{A}_R^{[u,v]}$ to the overconvergent ring $\mathbf{A}_R^{(0,v]+}$ and also twist our module by r. Note that we are working under the assumption that $\frac{p-1}{p} \leq u \leq \frac{v}{p} < 1 < v$, for example, one can take $u = \frac{p-1}{p}$ and $v = p - 1$.

Let us set $N^{(0,v]+}(T(r)) := \mathbf{A}_R^{(0,v]+} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T(r))$. We can equip this module with a filtration given as

$$\mathrm{Fil}^k N^{(0,v]+}(T(r)) := \sum_{i+j=k} \mathrm{Fil}^i \mathbf{A}_R^{(0,v]+} \widehat{\otimes}_{\mathbf{A}_{R_0}^+} \mathrm{Fil}^j \mathbf{N}(T(r)), \quad \text{for } k \in \mathbb{Z},$$

where we put the filtration on $\mathbf{A}_R^{(0,v]+}$ by identifying it with the ring $R_\varpi^{(0,v]+}$ via the map ι_{cycl} (see §2.4), and the latter ring has a filtration described in Definition 2.27. These submodules are stable under the action of Γ_R.

Next, we define a subcomplex of $\mathrm{Kos}\big(\Gamma_R', \mathrm{Fil}^0 N^{(0,v]+}(T(r))\big)$ as

$$\mathcal{K}\big(\Gamma_R', \mathrm{Fil}^0 N^{(0,v]+}(T(r))\big) \,:\, \mathrm{Fil}^0 N^{(0,v]+}(T(r)) \xrightarrow{(\tau_i)} \big(\pi \mathrm{Fil}^{-1} N^{(0,v]+}(T(r))\big)^{I_1'} \longrightarrow \big(\pi^2 \mathrm{Fil}^{-2} N^{(0,v]+}(T(r))\big)^{I_2'} \longrightarrow \cdots.$$

Similarly, we can define the complex $\mathcal{K}^c\big(\Gamma_R', \pi \mathrm{Fil}^{-1} N^{(0,v]+}(T(r))\big)$ as a subcomplex of $\mathrm{Kos}^c\big(\Gamma_R', \mathrm{Fil}^0 N^{(0,v]+}(T(r))\big)$ (see Definition 4.10). Now, consider the map

$$\tau_0 \,:\, \mathcal{K}\big(\Gamma_R', \mathrm{Fil}^0 N^{(0,v]+}(T(r))\big) \longrightarrow \mathcal{K}^c\big(\Gamma_R', \pi \mathrm{Fil}^{-1} N^{(0,v]+}(T(r))\big),$$

defined by a commutative diagram similar to (5.17) (see also Definitions 4.9 & 4.12)

$$\begin{tikzcd}
\mathrm{Fil}^0 N^{(0,v]+}(T(r)) \arrow[r, "(\tau_i)"] \arrow[d, "\tau_0^0"] & \left(\pi \mathrm{Fil}^{-1} N^{(0,v]+}(T(r))\right)^{I_1'} \arrow[r] \arrow[d, "\tau_0^1"] & \left(\pi^2 \mathrm{Fil}^{-2} N^{(0,v]+}(T(r))\right)^{I_2'} \arrow[r] \arrow[d, "\tau_0^2"] & \cdots \\
\pi \mathrm{Fil}^{-1} N^{(0,v]+}(T(r)) \arrow[r, "(\tau_i)"] & \left(\pi^2 \mathrm{Fil}^{-2} N^{(0,v]+}(T(r))\right)^{I_1'} \arrow[r] & \left(\pi^3 \mathrm{Fil}^{-3} N^{(0,v]+}(T(r))\right)^{I_2'} \arrow[r] & \cdots.
\end{tikzcd}$$

We write the total complex of the diagram as $\mathcal{K}\big(\Gamma_R, \mathrm{Fil}^0 N^{(0,v]+}(T(r))\big)$, which is a subcomplex of $\mathrm{Kos}\big(\Gamma_R, \mathrm{Fil}^0 N^{(0,v]+}(T(r))\big)$. In a similar manner, we can define complexes $\mathcal{K}\big(\Gamma_R', N^{(0,v/p]+}(T(r))\big)$ and $\mathcal{K}^c\big(\Gamma_R', \pi N^{(0,v/p]+}(T(r))\big)$ and a map τ_0 from former to the latter complex.

Next, for $j \in \mathbb{N}$, we have $\pi^j \mathrm{Fil}^{-j} N^{(0,v]+}(T(r)) \subset N^{(0,v]+}(T(r))$ and the induced Frobenius gives

$$\varphi(\pi^j \mathrm{Fil}^{-j} N^{(0,v]+}(T(r))) = \varphi(\pi^{j-r} \mathrm{Fil}^{r-j} N^{(0,v]+}(T)(r)) \subset \pi^{j-r} N^{(0,v/p]+}(T)(r) = \pi^j N^{(0,v/p]+}(T(r)).$$

Using the Forbenius morphism and the map between complexes discussed above, we define the complex

$$\mathcal{K}\big(\varphi, \Gamma_R, N^{(0,v]+}(T(r))\big) := \left[\begin{tikzcd} \mathcal{K}\big(\Gamma_R', \mathrm{Fil}^0 N^{(0,v]+}(T(r))\big) \arrow[r, "1-\varphi"] \arrow[d, "\tau_0"] & \mathcal{K}\big(\Gamma_R', N^{(0,v/p]+}(T(r))\big) \arrow[d, "\tau_0"] \\ \mathcal{K}^c\big(\Gamma_R', \pi \mathrm{Fil}^{-1} N^{(0,v]+}(r)\big) \arrow[r, "1-\varphi"] & \mathcal{K}^c\big(\Gamma_R', \pi N^{(0,v/p]+}(T(r))\big) \end{tikzcd}\right].$$

It is obvious that we can compare this to the complex defined in the previous section.

Proposition 5.39. *The natural map*

$$\mathcal{K}\big(\varphi, \Gamma_R, N^{(0,v]+}(T(r))\big) \longrightarrow \mathcal{K}\big(\varphi, \Gamma_R, N^{[u,v]}(T(r))\big)$$

induced by the inclusion $N^{(0,v]+}(T(r)) \subset N^{[u,v]}(T(r))$ is a p^{3r}-quasi-isomorphism.

Proof. The map in the claim is injective, so we only need to show that the cokernel complex is killed by p^{3r}. In the cokernel complex, we have maps

$$1-\varphi : \pi^k \mathrm{Fil}^{-k} N^{[u,v]}(T(r))/\pi^k \mathrm{Fil}^{-k} N^{(0,v]+}(T(r)) \longrightarrow \pi^k N^{[u,v/p]}(T(r))/\pi^k N^{(0,v/p]+}(T(r)) \ \text{ for } \ k \in \mathbb{Z}, \tag{5.19}$$

and it is enough to show that these maps are p^{4r}-bijective. Let us define the modules

$$M^{(0,v]+}(r) := \mathbf{A}_R^{(0,v]+} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)(r) \quad \text{and} \quad M^{[u,v]}(r) := \mathbf{A}_R^{[u,v]} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)(r),$$

equipped with filtrations given by the usual filtration on tensor products. It is also immediately clear that $\pi^k \mathrm{Fil}^{-k} N^{(0,v]+}(T(r)) = \pi^{k-r} \mathrm{Fil}^{r-k} M^{(0,v]+}(r)$ and $\pi^k \mathrm{Fil}^{-k} N^{[u,v]}(T(r)) = \pi^{k-r} \mathrm{Fil}^{r-k} M^{[u,v]}(r)$, for $k \in \mathbb{Z}$ (see (5.14) for a similar conclusion).

Let $n = r - k$ and we rewrite (5.19) as

$$1-\varphi : \pi^{-n} \mathrm{Fil}^n M^{[u,v]}(r)/\pi^{-n} \mathrm{Fil}^n M^{(0,v]+}(r) \longrightarrow \pi^{-n} M^{[u,v/p]}(r)/\pi^{-n} M^{(0,v/p]+}(r), \tag{5.20}$$

For $n \leq 0$, the claim follows from Lemma 5.40. For $n > 0$, we begin by showing that the natural map

$$\pi_1^{-n} M^{[u,v]}(r)/\pi_1^{-n} M^{(0,v]+}(r) \longrightarrow \pi^{-n} \mathrm{Fil}^n M^{[u,v]}(r)/\pi^{-n} \mathrm{Fil}^n M^{(0,v]+}(r), \tag{5.21}$$

is p^n-bijective. Recall that $\xi = \frac{\pi}{\pi_1}$, so we have

$$\pi_1^{-n} M^{[u,v]}(r) = \pi^{-n} \xi^n M^{[u,v]}(r) \subset \pi^{-n} \mathrm{Fil}^n M^{[u,v]}(r), \ \text{ and } \ \pi_1^{-n} M^{[u,v]}(r) \cap \pi^{-n} \mathrm{Fil}^n M^{(0,v]+}(r) = \pi_1^{-n} M^{(0,v]+}(r).$$

Therefore, we get that (5.21) is injective. Next, we note that from the definitions we can write $\mathbf{A}_R^{[u,v]} = \mathbf{A}_R^{[u]} + \mathbf{A}_R^{(0,v]+}$. So we take $M^{[u]} := \mathbf{A}_R^{[u]} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)$ and $M^+ := \mathbf{A}_R^+ \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)$ and we endow these modules with filtrations by considering the tensor product of filtrations on each component (note that for simplicity in notation we consider modules without the twist - this is harmless). This reduces (5.21) to the map

$$\pi_1^{-n} M^{[u]}/\pi_1^{-n} M^+ \longrightarrow \pi^{-n}\mathrm{Fil}^n M^{[u]}/\pi^{-n}\mathrm{Fil}^n M^+,$$

and we need to show that for any $x \in \pi^{-n}\mathrm{Fil}^n M^{[u]}$, there exists $y \in \pi_1^{-n} M^{[u]}$ such that under the natural map above, y maps to the image of $p^n x$. Let

$$x = \pi^{-n} \sum_{i+j=n} a_i \otimes x_j \in \pi^{-n}\mathrm{Fil}^n M^{[u]} = \pi^{-n} \sum_{i+j=n} \mathrm{Fil}^i \mathbf{A}_R^{[u]} \widehat{\otimes} \mathrm{Fil}^j \mathbf{N}(T).$$

From Lemma 2.28, for $i < n$, we can write $a_i = a_{i1} + a_{i2}$, with $a_{i1} \in \mathrm{Fil}^n \mathbf{A}_R^{[u]}$ and $a_{i2} \in \frac{1}{p^{[nu]}}\mathbf{A}_R^+$. However, note that $a_{i2} = a_i - a_{i1} \in \mathrm{Fil}^i \mathbf{A}_R^{[u]} \cap \frac{1}{p^{[nu]}}\mathbf{A}_R^+$, therefore we get that $a_{i2} \in \frac{1}{p^{[nu]}}\mathrm{Fil}^i \mathbf{A}_R^+$. Now we set

$$y = \frac{p^n}{\pi^n} \sum_{\substack{i+j=n \\ i<n}} a_{i1} \otimes x_j + \frac{p^n}{\pi^n} \sum_{\substack{i+j=n \\ i \geq n}} a_i \otimes x_j \in \frac{p^n}{\pi^n}\mathrm{Fil}^n \mathbf{A}_R^{[u]} \otimes \mathbf{N}(T) \subset \pi_1^{-n}\mathbf{A}_R^{[u]} \otimes \mathbf{N}(T).$$

and we get that $p^n x - y = \pi^{-n} p^n (\sum a_{i2} \otimes x_j) \in \pi^{-n} M^+$ (since $u = \frac{p-1}{p} < 1$). So (5.20) is p^n-isomorphic to the equation

$$1 - \varphi \,:\, \pi_1^{-n} M^{[u,v]}(r)/\pi_1^{-n} M^{(0,v]+}(r) \longrightarrow \pi^{-n} M^{[u,v/p]}(r)/\pi^{-n} M^{(0,v/p]+}(r),$$

Next, recall that we have $v = p - 1$, so it follows from Lemma 2.47 (v) that π divides p in $\mathbf{A}_R^{(0,v/p]+}$, whereas π_1 divides p in $\mathbf{A}_R^{(0,v]+}$, therefore (5.20) is p^{2n}-isomorphic to the equation

$$1 - \varphi \,:\, M^{[u,v]}(r)/M^{(0,v]+}(r) \longrightarrow M^{[u,v/p]}(r)/M^{(0,v/p]+}(r).$$

But from Lemma 5.40, we have that this map is bijective (note that Frobenius has no effect on twist). Therefore, we conclude that (5.19) is p^{3n}-bijective. As $n = r - k \leq r$, the cokernel complex of the map in the claim is killed by p^{3r}. This proves the claim. ■

Following observation was used above,

Lemma 5.40. *The natural map*

$$1 - \varphi \,:\, \mathbf{A}_R^{[u,v]} \otimes \mathbf{N}(T)/\mathbf{A}_R^{(0,v]+} \otimes \mathbf{N}(T) \longrightarrow \mathbf{A}_R^{[u,v/p]} \otimes \mathbf{N}(T)/\mathbf{A}_R^{(0,v/p]+} \otimes \mathbf{N}(T),$$

is bijective.

Proof. We will follow the strategy of the proof of [CN17, Lemma 4.8]. Let us note that the natural map

$$\mathbf{A}_R^{[u,v]} \otimes \mathbf{N}(T)/\mathbf{A}_R^{(0,v]+} \otimes \mathbf{N}(T) \longrightarrow \mathbf{A}_R^{[u,v/p]} \otimes \mathbf{N}(T)/\mathbf{A}_R^{(0,v/p]+} \otimes \mathbf{N}(T)$$

induced by the inclusion $\mathbf{A}_R^{[u,v]} \rightarrowtail \mathbf{A}_R^{[u,v/p]}$ is an isomorphism. Indeed, the map above is injective because the kernel consists of analytic functions that take values in $\mathbf{N}(T)$ and are integral on the annulus $\frac{u}{e} \leq v_p(X_0) \leq \frac{v}{e}$ and which extend to analytic functions taking values in $\mathbf{N}(T)$ and integral on the annulus $0 < v_p(X_0) \leq \frac{v}{pe}$, hence belong to $\mathbf{A}_R^{(0,v]+} \otimes \mathbf{N}(T)$. It is surjective because we can write $\mathbf{A}_R^{[u,v/p]} = \mathbf{A}_R^{[u]} + \mathbf{A}_R^{(0,v/p]+}$ (clear from the definitions). So, we can consider $(1 - \varphi)$ as an endomorphism of the module $M = \mathbf{A}_R^{[u,v]} \otimes \mathbf{N}(T)/\mathbf{A}_R^{(0,v]+} \otimes \mathbf{N}(T)$.

An element $x \in \mathbf{A}_R^{[u,v]}$ can be written as $x = \sum_{k\in\mathbb{N}} \frac{\pi_m^k}{p^{\lfloor ku/e \rfloor}} x_k$, with $x_k \in \mathbf{A}_R^{(0,v]+}$ going to 0, p-adically. So,

$$\varphi(x) = \sum_{k\in\mathbb{N}} p^{\lfloor pku/e \rfloor - \lfloor ku/e \rfloor} \left(\tfrac{\varphi(\pi_m)}{\pi_m^p}\right)^k \tfrac{\pi_m^{pk}}{p^{\lfloor pku/e \rfloor}} \varphi(x_k),$$

and since $\lfloor pku/e \rfloor - \lfloor ku/e \rfloor \geq 1$ if $\lfloor ku/e \rfloor \neq 0$, we see that $\varphi(x) \in \mathbf{A}_R^{(0,v/p]+} + p\mathbf{A}_R^{[u,v/p]}$. As $\varphi(\mathbf{N}(T)) \subset \mathbf{N}(T)$, we get $\varphi(M) \subset pM$. To show the bijectivity of $1-\varphi$, it remains to check that M does not contain p-divisible elements, which would then imply that $1+\varphi+\varphi^2+\cdots$ converges on M. Let $(f_j)_{j\in J}$ be a collection of elements of $\mathbf{A}_R^+$ whose images form a basis of $\mathbf{A}_R^+/(p, \pi_m)$ over $\kappa = \mathbf{A}_K^+/(p, \pi_m)$. Then $(f_j)_{j\in J}$ is a topological basis of $\mathbf{A}_R^{[u,v]}$ over $\mathbf{A}_K^{[u,v]}$ and of $\mathbf{A}_R^{(0,v]+}$ over $\mathbf{A}_K^{(0,v]+}$. Writing everything in the basis $\{f_j \otimes e_i,\ \text{for } 1 \leq i \leq h, j \in J\}$, where $\{e_i,\ 1 \leq i \leq h\}$ is a basis of $\mathbf{N}(T)$, reduces the question to proving that $\mathbf{A}_K^{[u,v]}/\mathbf{A}_K^{(0,v]+}$ has no p-divisible element. Since all such elements can be written as a power series in $\mathbf{A}_K^{[u]}/\mathbf{A}_K^+$, we conclude that there can be no p-divisible elements in this quotient. Hence, we get the desired conclusion. ■

5.2.5. Change of annulus of convergence : Part 2

In this section, we will change the ring of coefficients from $\mathbf{A}_R^{(0,v]+}$ to $\mathbf{A}_R^{(0,v/p]+}$ by replacing the action of φ with its left inverse ψ in the complexes discussed so far : these steps are required in order to obtain a complex comparable to Koszul complexes computing the Galois cohomology of $T(r)$. Note that we are working under the assumption that $\frac{p-1}{p} \leq u \leq \frac{v}{p} < 1 < v$, for example, one can take $u = \frac{p-1}{p}$ and $v = p-1$.

Recall from Proposition 2.13 that we have a left inverse ψ of the Frobenius such that $\psi(\mathbf{A}) \subset \mathbf{A}$, which induces the operator $\psi : \mathbf{A}^+ \longrightarrow \mathbf{A}^+$. For the overconvergent rings we can consider the induced operator over $\mathbf{A}^\dagger$ and we have that $\psi(\mathbf{A}^\dagger) \subset \mathbf{A}^\dagger$. This gives us an operator $\psi : \mathbf{A}_R^{(0,v/p]+} \longrightarrow \mathbf{A}_R^{(0,v]+}$. Note that we can also define ψ by identifying $\mathbf{A}_R^{(0,v/p]+} \simeq R_\varpi^{(0,v/p]+}$ via the isomorphism ι_{cycl} in §2.4, and considering the left inverse of the cyclotomic Frobenius over $R_\varpi^{(0,v/p]+}$ (see §2.3.2). Both these definitions coincide since ι_{cycl} commutes with the Frobenius on each side.

Next, let $\ell = p^{m-1}$, then from Proposition 2.40 (i) we have inclusions

$$\psi\left(\pi_m^{-\ell}\mathbf{A}_R^{(0,v]+}\right) \subset \psi\left(\pi_m^{-\ell}\mathbf{A}_R^{(0,v/p]+}\right) \subset \pi_m^{-p^{m-2}}\mathbf{A}_R^{(0,v]+} \subset \pi_m^{-\ell}\mathbf{A}_R^{(0,v]+} \subset \pi_m^{-\ell}\mathbf{A}_R^{(0,v/p]+}. \tag{5.22}$$

Using this, we deduce that $\pi_m^{-\ell}\mathbf{A}_R^{(0,v]+}$ is stable under ψ. Define

$$D^{(0,v]+}(r) := \mathbf{A}_R^{(0,v]+} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{D}^+(T(r)).$$

Note that this module is stable under the action of Γ_R.

Notation. We write $D^{(0,v]+}(r)$ instead of $D^{(0,v]+}(T(r))$ as we have $\mathbf{D}^+(T(r)) = \mathbf{D}^+(T)(r)$. We hope this change in notation is not too confusing for the reader.

Recall from Lemma 2.37 that we have $\psi\left(\mathbf{A}_R^{(0,v/p]+}\right) \subset \mathbf{A}_R^{(0,v]+}$. Further, for $v = p-1$, by Lemma 2.47 (v) we have that $\pi_m^{-p\ell}\pi$ is a unit in $\mathbf{A}_R^{(0,v/p]+}$. So by combining Lemma 2.39 and Proposition 2.40 (i), we see that $\psi\left(\pi_m^{-p\ell r}\mathbf{A}_R^{(0,v/p]+}\right) \subset \pi_m^{-\ell r}\mathbf{A}_R^{(0,v]+}$. Now, we know that ψ commutes with the action of G_R, so by linearity we can extend this map to get $\psi(\mathbf{D}^+(T)) \subset \mathbf{D}^+(T)$, and therefore we have that $\psi\left(D^{(0,v/p]+}(r)\right) \subset D^{(0,v]+}(r)$. Coupling this with the observation above, we note that $\psi\left(\pi_m^{-p\ell r}D^{(0,v/p]+}(r)\right) \subset \pi_m^{-\ell r}D^{(0,v]+}(r)$. Now since $\psi(\mathbf{N}(T)) \subset \psi(\mathbf{D}^+(T)) \subset \mathbf{D}^+(T)$, therefore from the inclusion $\pi_m^{-\ell}\mathbf{A}_R^{(0,v]+} \subset \pi_m^{-\ell}\mathbf{A}_R^{(0,v/p]+}$ and (5.22), we deduce that

$$\begin{aligned}\psi\left(N^{(0,v]+}(T(r))\right) &\subset \psi\left(N^{(0,v/p]+}(T(r))\right) \subset \psi\left(\pi^{-r}D^{(0,v/p]+}(r)\right) \\ &= \psi\left(\pi_m^{-p\ell r}D^{(0,v/p]+}(r)\right) \subset \pi_m^{-\ell r}D^{(0,v]+}(r) \subset \pi_m^{-\ell r}D^{(0,v/p]+}(r).\end{aligned} \tag{5.23}$$

Next, for the filtration on $\mathbf{A}_R^{(0,v]+}$ and $k \in \mathbb{N}$ such that $k \leq r$, we observe that $\varphi\big(\pi^k \mathrm{Fil}^{-k} N^{(0,v]+}(T(r))\big) \subset \pi^k N^{(0,v/p]+}(T(r)) \subset \pi_m^{-p\ell(r-k)} D^{(0,v/p]+}(r)$, therefore

$$\begin{aligned}\pi^k \mathrm{Fil}^{-k} N^{(0,v]+}(T(r)) = \psi\big(\varphi\big(\pi^k \mathrm{Fil}^{-k} N^{(0,v]+}(T(r))\big)\big) &\subset \psi\big(\pi^k N^{(0,v/p]+}(T(r))\big) \\ &\subset \psi\big(\pi_m^{-p\ell(r-k)} D^{(0,v/p]+}(r)\big) \subset \pi_m^{-\ell(r-k)} D^{(0,v]+}(r).\end{aligned} \tag{5.24}$$

Equally obvious is the inclusion

$$\psi\big(\pi^k \mathrm{Fil}^{-k} N^{(0,v]+}(T(r))\big) \subset \psi\big(\pi^k N^{(0,v]+}(T(r))\big) \subset \psi\big(\pi^k N^{(0,v/p]+}(T(r))\big) \subset \pi_m^{-\ell(r-k)} D^{(0,v]+}(r).$$

In conclusion, we obtain that

$$(\psi - 1)\big(\pi^k \mathrm{Fil}^{-k} N^{(0,v]+}(T(r))\big) \subset \pi_m^{-\ell(r-k)} D^{(0,v]+}(r) \subset \pi_m^{-\ell r} D^{(0,v]+}(r). \tag{5.25}$$

We now turn to complexes. Recall that we have,

$$\mathrm{Kos}\big(\Gamma'_R, \pi_m^{-\ell r} D^{(0,v]+}(r)\big) \ : \ \pi_m^{-\ell r} D^{(0,v]+}(r) \xrightarrow{(\tau_i)} \big(\pi_m^{-\ell r} D^{(0,v]+}(r)\big)^{I'_1} \longrightarrow \big(\pi_m^{-\ell r} D^{(0,v]+}(r)\big)^{I'_2} \longrightarrow \cdots,$$

and similarly $\mathrm{Kos}^c\big(\Gamma'_R, \pi_m^{-\ell r} D^{(0,v]+}(r)\big)$. In the previous section, we already defined the complexes $\mathcal{K}\big(\Gamma'_R, \mathrm{Fil}^0 N^{(0,v]+}(T(r))\big)$, $\mathcal{K}^c\big(\Gamma'_R, \pi \mathrm{Fil}^{-1} N^{(0,v]+}(T(r))\big)$ and a map τ_0 from the former complex to the latter. Therefore, similar to the complex $\mathcal{K}\big(\varphi, \Gamma_R, N^{(0,v]+}(T(r))\big)$ from the previous section and using (5.25) define the complex

$$\mathcal{K}\big(\psi, \Gamma_R, N^{(0,v]+}(T(r))\big) \ := \left[\begin{array}{ccc} \mathcal{K}\big(\Gamma'_R, \mathrm{Fil}^0 N^{(0,v]+}(T(r))\big) & \xrightarrow{\psi-1} & \mathrm{Kos}\big(\Gamma'_R, \pi_m^{-\ell r} D^{(0,v]+}(r)\big) \\ \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau_0} \\ \mathcal{K}^c\big(\Gamma'_R, \pi \mathrm{Fil}^{-1} N^{(0,v]+}(T(r))\big) & \xrightarrow{\psi-1} & \mathrm{Kos}^c\big(\Gamma'_R, \pi_m^{-\ell r} D^{(0,v]+}(r)\big) \end{array}\right].$$

Proposition 5.41. *With notations as above, the natural map*

$$\tau_{\leq r} \mathcal{K}\big(\varphi, \Gamma_R, N^{(0,v]+}(T(r))\big) \longrightarrow \tau_{\leq r} \mathcal{K}\big(\psi, \Gamma_R, N^{(0,v]+}(T(r))\big),$$

induced by identity in the first column and ψ in the second column is a p^{5r+s+2}-quasi-isomorphism, where s is the maximum among the absolute values of Hodge-Tate weights of V (see Definition 3.8).

Proof. We will show that the kernel and cokernel complex are killed by some power of p.

First, let us look at the cokernel complex, which is made up of modules $\pi_m^{-\ell r} D^{(0,v]+}(r)/\psi\big(\pi^k N^{(0,v/p]+}(T(r))\big)$ for $0 \leq k \leq r$. We want to show that these modules are killed by p^{4r+s}. Now, note that $\varphi\big(D^{(0,v]+}(r)\big) \subset D^{(0,v/p]+}(r)$, therefore $D^{(0,v]+}(r) \subset \psi\big(D^{(0,v/p]+}(r)\big)$. Moreover, from (5.22) we get that

$$\psi\big(D^{(0,v/p]+}(r)\big) \subset \psi\big(\pi_m^{-\ell r} D^{(0,v/p]+}(r)\big) \subset \pi_m^{-\ell r} D^{(0,v]+}(r).$$

Therefore, $\pi_m^{-\ell r} D^{(0,v]+}(r)/\psi\big(D^{(0,v/p]+}(r)\big)$ is killed by $\pi_m^{\ell r}$. But, from Lemma 2.47 we have that π_m^{ℓ} divides p in $\mathbf{A}_R^{(0,v]+}$ (for $v = p-1$), therefore $\pi_m^{-\ell r} D^{(0,v]+}(r)/\psi\big(D^{(0,v/p]+}(r)\big)$ is killed by p^r.

Further, from Definition 3.8 we have $\pi^s \mathbf{D}^+(T) \subset \mathbf{N}(T) \subset \mathbf{D}^+(T)$. So we obtain

$$\pi^{k+s-r} D^{(0,v/p]+}(r) \subset \pi^k N^{(0,v/p]+}(T(r)) \subset \pi^{k-r} D^{0,v/p]+}(r).$$

Since π divides p in $\mathbf{A}_R^{(0,v/p]+}$ (see Lemma 2.47 (v) for $v = p-1$), we obtain that $\pi^k N^{(0,v/p]+}(T(r))$ is p^{k+s}-isomorphic to $\pi^{-r} D^{(0,v/p]+}(r)$. Similarly, we see that the natural inclusion $D^{(0,v/p]+}(r) \subset \pi^{-r} D^{(0,v/p]+}(r)$ is a p^r-isomorphism. Combining both these statements we get that $D^{(0,v/p]+}(r)$ is p^{k+r+s}-isomorphic to $N^{(0,v/p]+}(T(r))$. Therefore, the natural map

$$\pi_m^{-\ell r} D^{(0,v]+}(r)/\psi\big(N^{(0,v/p]+}(T(r))\big) \longrightarrow \pi_m^{-\ell r} D^{(0,v]+}(r)/\psi\big(D^{(0,v/p]+}\big)(r)$$

is a p^{k+r+s}-isomorphism. Since the latter module is killed by p^r, we conclude that the module $\pi_m^{-\ell r} D^{(0,v]+}(r)/\psi\big(N^{(0,v/p]+}(T(r))\big)$ is killed by p^{k+3r+s}. As this value grows with the degree of the complex, we see that after truncating in degree $\leq r$, we obtain that the cokernel complex of the map in the claim is p^{4r+s}-acyclic.

Next, we look at the kernel complex. Our strategy is to replace the kernel complex with a simpler complex, up to some power of p, and show that the latter complex is p^2-acyclic.

Note that the map is identity on the first column, so the kernel complex can be written as

$$\tau_{\leq r}\big[\mathcal{K}\big(\Gamma_R', \big(N^{(0,v/p]+}(T(r))\big)^{\psi=0}\big) \xrightarrow{\tau_0} \mathcal{K}^c\big(\Gamma_R', \big(\pi N^{(0,v/p]+}(T(r))\big)^{\psi=0}\big)\big].$$

Since π divides p in $\mathbf{A}_R^{(0,v/p]+}$ (see Lemma 2.47 (v)), we obtain that $\pi^k N^{(0,v/p]+}(T(r))$ is p^{r-k}-isomorphic to $N^{(0,v/p]+}(T)(r)$, for $k \leq r$. Using this we see that the kernel complex is p^r-quasi-isomorphic to the complex

$$\tau_{\leq r}\big[\mathrm{Kos}\big(\Gamma_R', \big(N^{(0,v/p]+}(T)(r)\big)^{\psi=0}\big) \xrightarrow{\tau_0} \mathrm{Kos}^c\big(\Gamma_R', \big(N^{(0,v/p]+}(T)(r)\big)^{\psi=0}\big)\big].$$

Now, we will analyze the module $\big(N^{(0,v/p]+}(T)\big)^{\psi=0}$. Let us write $\mathbf{N}(T) = \sum_{j=1}^h \mathbf{A}_{R_0}^+ e_j$, for a choice of basis. Since the attached (φ, Γ_R)-module $\mathbf{D}_R(T)$ over $\mathbf{A}_R$ is étale, we obtain that $\mathbf{D}_R(T) = \sum_{j=1}^h \mathbf{A}_R \varphi(e_j)$. Now note that $z = \sum_{j=1}^h z_j \varphi(e_j) \in (\mathbf{D}_R(T))^{\psi=0} = \big(\oplus_{j=1}^h \mathbf{A}_R \varphi(e_j)\big)^{\psi=0}$, if and only if $z_j \in \big(\mathbf{A}_R\big)^{\psi=0}$, for each $1 \leq j \leq h$. Indeed, $\psi(z) = 0$ if and only if $\sum_{j=1}^h \psi(z_j \varphi(e_j)) = \sum_{j=1}^h \psi(z_j) e_j = 0$. As e_j are linearly independent over $\mathbf{A}_R$, we get the desired conclusion.

Next, using Lemma 2.37 (ii), we have a decomposition

$$\mathbf{A}_R^{\psi=0} \simeq \bigoplus_{\alpha \in \{0,\ldots,p-1\}^{[0,d]}, \alpha \neq 0} \varphi\big(\mathbf{A}_R\big)[X^\flat]^\alpha, \quad \text{where } [X^\flat]^\alpha = (1+\pi_m)^{\alpha_0}[X_1^\flat]^{\alpha_0} \cdots [X_d^\flat]^{\alpha_d}.$$

Therefore, we obtain that

$$\begin{aligned} \big(\mathbf{D}_R(T)\big)^{\psi=0} &\simeq \big(\mathbf{D}_R(T)\big)^{\psi=0} \simeq \Big(\sum_{j=1}^h \mathbf{A}_R \varphi(e_j)\Big)^{\psi=0} \\ &\simeq \bigoplus_{\substack{\alpha \in \{0,\ldots,p-1\}^{[0,d]} \\ \alpha \neq 0}} \sum_{j=1}^h \varphi\big(\mathbf{A}_R e_j\big)[X^\flat]^\alpha = \bigoplus_{\substack{\alpha \in \{0,\ldots,p-1\}^{[0,d]} \\ \alpha \neq 0}} \varphi\big(\mathbf{D}_R(T)\big)[X^\flat]^\alpha. \end{aligned}$$

Now observe that $\big(N^{(0,v/p]+}(T)\big)^{\psi=0} = \big(\mathbf{D}_R(T)\big)^{\psi=0} \cap N^{(0,v/p]+}(T)$. Using the decomposition above, we set

$$M[X^\flat]^\alpha := \varphi\big(\mathbf{D}_R(T)\big)[X^\flat]^\alpha \cap N^{(0,v/p]+}(T), \ \text{ for } \ \alpha \in \{0,\ldots,p-1\} \ \text{ and } \ \alpha \neq 0,$$

where we take the intersection inside $\big(\mathbf{D}_R(T)\big)^{\psi=0}$. Note that the module M is an $\mathbf{A}_R^{(0,v/p]+}$-module contained in $N^{(0,v/p]+}(T)$, stable under the action of Γ_R and independent of α. Indeed, for $\alpha \neq \alpha'$, if we have $\sum_{i=1}^h x_i e_i [X^\flat]^\alpha \in M[X^\flat]^\alpha$ then $\sum_{i=1}^h x_i e_i [X^\flat]^{\alpha'} \in M[X^\flat]^{\alpha'}$, and vice versa.

From the discussion above, we see that the kernel complex of the map in the claim is p^r-isomorphic to the complex

$$\tau_{\leq r} \bigoplus_{\alpha \in \{0,\ldots,p-1\}^{[0,d]}, \alpha \neq 0} \Big[\ \mathrm{Kos}\big(\Gamma_R', M(r)[X^\flat]^\alpha\big) \xrightarrow{\tau_0} \mathrm{Kos}^c\big(\Gamma_R', M(r)[X^\flat]^\alpha\big)\ \Big]. \tag{5.26}$$

Lemma 5.42. *The complex described in (5.26) above is p^2-acyclic.*

Proof. The proof will follow the technique used in the proof of [CN17, Lemma 4.10]. We will treat terms corresponding to each α separately. First, let us assume that $\alpha_k \neq 0$ for some $k \neq 0$. We want to show that both $\mathrm{Kos}\big(\Gamma_R', M[X^\flat]^\alpha\big)$ and $\mathrm{Kos}^c\big(\Gamma_R', M[X^\flat]^\alpha\big)$ complexes are p-acyclic (the twist has disappeared because the cyclotomic character is trivial on Γ_R'). As the proof is same in both the cases, we only treat the first case. We can write the complex as a double complex

$$\begin{array}{ccccccc}
M[X^\flat]^\alpha & \xrightarrow{(\gamma_i - 1)} & M^{I_1''}[X^\flat]^\alpha & \longrightarrow & M^{I_2''}[X^\flat]^\alpha & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \gamma_k - 1} & & \downarrow{\scriptstyle \gamma_k - 1} & & \downarrow{\scriptstyle \gamma_k - 1} & & \\
M[X^\flat]^\alpha & \xrightarrow{(\gamma_i - 1)} & M^{I_1''}[X^\flat]^\alpha & \longrightarrow & M^{I_2''}[X^\flat]^\alpha & \longrightarrow & \cdots,
\end{array}$$

where the horizontal maps involve γ_i's with $i \neq k$, $1 \leq i \leq d$. Now, we have

$$(\gamma_k - 1) \cdot \big(y[X^\flat]^\alpha\big) = \pi_1 G(y)[X^\flat]^\alpha,$$

where

$$G(y) = \tfrac{(1+\pi)^{\alpha_k}(\gamma_k - 1)y}{\pi} + \tfrac{((1+\pi)^{\alpha_k} - 1)y}{\pi}, \quad \text{for} \quad y \in M,$$

and we have used the fact that

$$\gamma_k\big([X^\flat]^\alpha\big) = [\varepsilon]^{\alpha_k}[X^\flat]^\alpha = (1+\pi)^{\alpha_k}[X^\flat]^\alpha.$$

Now, G is π_m-linear and $\gamma_k - 1$ is trivial modulo π on $\mathbf{A}_R^{(0,v]+}$ and $\mathbf{N}(T)$ (see Lemma 2.46 and Definition 3.8). Since π divides p in $\mathbf{A}_R^{(0,v/p]+}$ (see Lemma 2.47 for $v = p-1$), therefore it follows that modulo π, G is just multiplication by α_k on M. This shows that G is invertible over M, therefore $\gamma_k - 1$ is injective on $M[X^\flat]^\alpha$. Finally, since we have that $\frac{p}{\pi} \in \mathbf{A}_R^{(0,v/p]+}$, the cokernel of $\gamma_k - 1$ is killed by p.

Next, let $\alpha_k = 0$ for all $k \neq 0$ and $\alpha_0 \neq 0$. To prove that the kernel complex is p-acyclic, we will show that $\tau_0 : \mathrm{Kos} \longrightarrow \mathrm{Kos}^c$ is injective and the cokernel complex is killed by p. This amounts to showing the same statement for

$$\gamma_0 - \delta_{i_1} \cdots \delta_{i_q} : M[X^\flat]^\alpha(r) \longrightarrow M[X^\flat]^\alpha(r), \quad \delta_{i_j} = \tfrac{\gamma_{i_j}^c - 1}{\gamma_{i_j} - 1}.$$

We have

$$(\gamma_0 - \delta_{i_1} \cdots \delta_{i_q})\big(y[X^\flat]^\alpha(r)\big) = \big(c^r \gamma_0(y)(1+\pi)^{p^{-m}(c-1)\alpha_0}[X^\flat]^\alpha\big)(r) - \big(\delta_{i_1} \cdots \delta_{i_q}(y)[X^\flat]^\alpha\big)(r).$$

So we are lead to study the map F defined by

$$F = c^r(1+\pi)^z\gamma_0 - \delta_{i_1} \cdots \delta_{i_q}, \quad z = p^{-m}(c-1)\alpha_0 \in \mathbb{Z}_p^*.$$

Now $c^r - 1$ is divisible by p^m, $(1+\pi)^z = 1 + z\pi \bmod \pi^2$ and $\delta_{i_j} - 1 \in (p^m, \gamma_{i_j} - 1)\mathbb{Z}_p[[\gamma_{i_j} - 1]]$. Therefore, we can write $\pi^{-1}F$ in the form $\pi^{-1}F = z + \pi^{-1}F'$, with $F' \in \big(p^m, \pi^2, \gamma_0 - 1, \ldots, \gamma_d - 1\big)\mathbb{Z}_p[[\pi, \Gamma_R]]$. It follows from Lemma 2.46, Lemma 2.47 and Definition 3.8, that for $N = 2p^{m-1} > 0$ we have that $\pi^{-1}F' = 0$ on $\pi_m^a M/\pi_m^{a+N}M$, for all $a \in \mathbb{N}$. Hence, $\pi^{-1}F$ induces multiplication by z on $\pi_m^a M/\pi_m^{a+N}M$ for all $a \in \mathbb{N}$, which implies that it is an isomorphism of M. This shows what we want since π divides p in $\mathbf{A}_R^{(0,v/p]+}$ by Lemma 2.47 (for $v = p-1$). ■

Combining the analysis for the kernel and cokernel complex, we conclude that the map in the claim of Proposition 5.41 is a p^{5r+s+2}-quasi-isomorphism. ■

By replacing v by v/p in §5.2.4, define the complex

$$\mathcal{K}\big(\Gamma_R', N^{(0,v/p]+}(T(r))\big) : N^{(0,v/p]+}(T(r)) \xrightarrow{(\tau_i)} \big(\pi N^{(0,v/p]+}(T(r))\big)^{I_1'} \longrightarrow \big(\pi^2 N^{(0,v/p]+}(T(r))\big)^{I_2'} \longrightarrow \cdots.$$

Similarly, we can define the complex $\mathcal{K}^c\big(\Gamma_R', N^{(0,v/p]+}(T(r))\big)$ and a map τ_0 from former to the latter complex. Moreover, from (5.23) and the natural inclusion $N^{(0,v/p]+}(T(r)) \subset \pi^{-r}D^{(0,v/p]+}(r) = \pi_m^{-\ell r}D^{(0,v/p]+}(r)$ (since $\pi_m^{-p^m}\pi$ is a unit in $\mathbf{A}_R^{(0,v/p]+}$), we deduce that

$$(\psi-1)\big(\pi^k N^{(0,v/p]+}(T(r))\big) \subset \pi_m^{-p\ell(r-k)}D^{(0,v/p]+}(r) \subset \pi_m^{-p\ell r}D^{(0,v/p]+}(r).$$

Therefore, similar to $\mathcal{K}\big(\psi, \Gamma_R, N^{(0,v]+}(T(r))\big)$, define the complex

$$\mathcal{K}\big(\psi, \Gamma_R, N^{(0,v/p]+}(T(r))\big) := \left[\begin{array}{ccc} \mathcal{K}\big(\Gamma_R', N^{(0,v/p]+}(T(r))\big) & \xrightarrow{\psi-1} & \mathrm{Kos}\big(\Gamma_R', \pi_m^{-p\ell r}D^{(0,v/p]+}(r)\big) \\ \downarrow{\scriptstyle\tau_0} & & \downarrow{\scriptstyle\tau_0} \\ \mathcal{K}^c\big(\Gamma_R', \pi N^{(0,v/p]+}(T(r))\big) & \xrightarrow{\psi-1} & \mathrm{Kos}^c\big(\Gamma_R', \pi_m^{-p\ell r}D^{(0,v/p]+}(r)\big) \end{array}\right].$$

We can compare it to the complex defined before Proposition 5.41:

Lemma 5.43. *The natural map*

$$\tau_{\le r}\mathcal{K}\big(\psi, \Gamma_R, N^{(0,v]+}(T(r))\big) \longrightarrow \tau_{\le r}\mathcal{K}\big(\psi, \Gamma_R, N^{(0,v/p]+}(T(r))\big),$$

induced by inclusions $N^{(0,v]+}(T(r)) \subset N^{(0,v/p]+}(T(r))$ and $\pi_m^{-\ell r}D^{(0,v]+}(r) \subset \pi_m^{-p\ell r}D^{(0,v/p]+}(r)$ is a p^{r+s}-quasi-isomorphism.

Proof. As the map is injective it is enough to show that the cokernel complex is killed by p^{r+s}. For $k \in \mathbb{N}$ and $k \le r$, in the cokernel complex, we have maps

$$\psi - 1 \,:\, \pi^k N^{(0,v/p]+}(T(r))/\pi^k \mathrm{Fil}^{-k} N^{(0,v]+}(T(r)) \longrightarrow \pi_m^{-p\ell r}D^{(0,v/p]+}(r)/\pi_m^{-\ell r}D^{(0,v]+}(r), \tag{5.27}$$

and it is enough to show that these are p^{r+s}-bijective. Let us show the p^{r+s}-surjectivity first. Note that from 5.24 we have $\psi\big(\pi^k N^{(0,v/p]+}(T(r))\big) \subset \pi_m^{-\ell r}D^{(0,v]+}(r)$, therefore the cokernel of (5.27) is given as $\pi_m^{-p\ell r}D^{(0,v/p]+}(r)/\pi^k N^{(0,v/p]+}(T(r))$. Recall from Definition 3.8 that we have

$$\pi^s \mathbf{D}^+(T)(r) \subset \mathbf{N}(T)(r) = \pi^r \mathbf{N}(T(r)) \subset \mathbf{D}^+(T(r)) = \mathbf{D}^+(T)(r).$$

Extending scalars to $\mathbf{A}_R^{(0,v/p]+}$ in the equation above and dividing by π^r, we obtain a natural inclusion $\pi^{s-r}D^{(0,v/p]+}(r) \subset N^{(0,v/p]+}(T(r))$. Therefore, we see that $\pi_m^{-p\ell r}D^{(0,v/p]+}(r)/\pi^k N^{(0,v/p]+}(T(r)) = \pi^{-r}D^{(0,v/p]+}(r)/\pi^k N^{(0,v/p]+}(T(r))$ is killed by π^{k+s}. But π divides p in $\mathbf{A}_R^{(0,v/p]+}$ (see Lemma 2.47 for $v = p-1$), therefore (5.27) is p^{k+s}-surjective (this also shows that truncation in degree $\le r$ is necessary in order to bound the power of p).

For injectivity, let $x \in \pi^k N^{(0,v/p]+}(T(r))$ such that $(\psi-1)x \in \pi_m^{-\ell r}D^{(0,v]+}(r)$. We want to show that $x \in \pi^k \mathrm{Fil}^{-k} N^{(0,v]+}(T(r))$. Note that from 5.23, we have

$$\psi\big(\pi^k N^{(0,v/p]+}(T(r))\big) \subset \psi\big(N^{(0,v/p]+}(T(r))\big) \subset \psi\big(\pi_m^{-p\ell r}D^{(0,v/p]+}(r)\big) \subset \pi_m^{-\ell r}D^{(0,v]+}(r).$$

So we get that $x \in \pi_m^{-\ell r}D^{(0,v]+}$. We write $x = \pi^{k-r}a \otimes e$, for $a \in \mathbf{A}_R^{(0,v/p]+}$ and $e \in \mathbf{N}(T)(r)$. As $\pi_m^{-\ell}\pi_1$ is a unit in $\mathbf{A}_R^{(0,v]+}$, we also get that

$$x = \tfrac{1}{\pi^{r-k}}a \otimes e \in \tfrac{1}{\pi_1^r}\mathbf{A}_R^{(0,v]+} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{D}^+(T)(r).$$

But then we must have $a \in \pi_1^k \xi^{r-k}\mathbf{A}_R^{(0,v]+} \subset \mathrm{Fil}^{r-k}\mathbf{A}_R^{(0,v]+}$, which implies that $x = \pi^{k-r}a \otimes e \in \pi^{k-r}\mathrm{Fil}^{r-k}\mathbf{A}_R^{(0,v]+} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)(r) \subset \pi^k \mathrm{Fil}^{-k} N^{(0,v]+}(T(r))$. This shows that (5.27) is injective.

Finally, putting everything together for $k \le r$, we conclude that the map in the claim is a p^{r+s}-quasi-isomorphism. ∎

Recall from (5.23) that we have $\psi\big(\pi_m^{-p\ell}D^{(0,v/p]+}(r)\big) \subset \pi_m^{-\ell}D^{(0,v/p]+}(r) \subset \pi_m^{-p\ell}D^{(0,v/p]+}(r)$. So, by the general formalism of Koszul complexes in §4.2, let us define

$$\mathrm{Kos}\big(\psi,\Gamma_R,D^{(0,v/p]+}(r)\big) := \left[\begin{array}{ccc} \mathrm{Kos}\big(\Gamma_R',\pi_m^{-p\ell r}D^{(0,v/p]+}(r)\big) & \xrightarrow{\psi-1} & \mathrm{Kos}\big(\Gamma_R',\pi_m^{-p\ell r}D^{(0,v/p]+}(r)\big) \\ \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau_0} \\ \mathrm{Kos}^c\big(\Gamma_R',\pi_m^{-p\ell r}D^{(0,v/p]+}(r)\big) & \xrightarrow{\psi-1} & \mathrm{Kos}^c\big(\Gamma_R',\pi_m^{-p\ell r}D^{(0,v/p]+}(r)\big)\end{array}\right].$$

Lemma 5.44. *The natural map*

$$\tau_{\le r}\mathcal{K}\big(\psi,\Gamma_R,N^{(0,v/p]+}(T(r))\big) \longrightarrow \tau_{\le r}\mathrm{Kos}\big(\psi,\Gamma_R,D^{(0,v/p]+}(r)\big),$$

induced by the inclusion $N^{(0,v/p]+}(T(r)) \subset \pi_m^{-p\ell r}D^{(0,v/p]+}(r)$, is a p^{r+s}-quasi-isomorphism.

Proof. Since the map is injective it is enough to show that the cokernel complex is killed by p^{r+s}. Note that the cokernel is a complex made up of $\mathbf{A}_R^{(0,v/p]+}$-modules $\pi_m^{-p\ell r}D^{(0,v/p]+}(r)/\pi^kN^{(0,v/p]+}(T(r))$, for $k \in \mathbb{N}$ such that $k \le r$. Recall from Definition 3.8 that we have $\pi^s\mathbf{D}^+(T)(r) \subset \mathbf{N}(T)(r) = \pi^r\mathbf{N}(T(r)) \subset \mathbf{D}^+(T(r))$. Extending scalars to $\mathbf{A}_R^{(0,v/p]+}$ in the equation above and dividing by π^r, we obtain natural inclusions

$$\pi^{s-r}D^{(0,v/p]+}(r) \subset N^{(0,v/p]+}(T(r)) \subset \pi^{-r}D^{(0,v/p]+}(r).$$

As $v = p-1$, from Lemma 2.47 (v) we have that π divides p in $\mathbf{A}_R^{(0,v/p]+}$. Therefore, the module $\pi_m^{-p\ell r}D^{(0,v/p]+}(r)/\pi^kN^{(0,v/p]+}(T(r)) = \pi^{-r}D^{(0,v/p]+}(r)/\pi^kN^{(0,v/p]+}(T(r))$ is killed by p^{k+s}. Hence, the cokernel complex (for the truncated complex) is p^{r+s}-acyclic, which proves the claim. ■

5.2.6. Change of disk of convergence

Finally, we are in a position to relate our complexes to the Koszul complex computing continuous G_R-cohomology of $T(r)$. Recall that in §2.1, we defined an operator $\psi : \mathbf{D}_R(T(r)) \longrightarrow \mathbf{D}_R(T(r))$, as the left inverse of φ. Using this operator, we can define the complex

$$\mathrm{Kos}\big(\psi,\Gamma_R,\mathbf{D}_R(T(r))\big) := \left[\begin{array}{ccc} \mathrm{Kos}\big(\Gamma_R',\mathbf{D}_R(T(r))\big) & \xrightarrow{\psi-1} & \mathrm{Kos}\big(\Gamma_R',\mathbf{D}_R(T(r))\big) \\ \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau_0} \\ \mathrm{Kos}^c\big(\Gamma_R',\mathbf{D}_R(T(r))\big) & \xrightarrow{\psi-1} & \mathrm{Kos}^c\big(\Gamma_R',\mathbf{D}_R(T(r))\big)\end{array}\right].$$

This complex is related to the one from the previous section:

Lemma 5.45. *The natural map*

$$\mathrm{Kos}\big(\psi,\Gamma_R,D^{(0,v/p]+}(r)\big) \longrightarrow \mathrm{Kos}\big(\psi,\Gamma_R,\mathbf{D}_R(T(r))\big),$$

induced by the inclusion $\pi_m^{-p\ell r}D^{(0,v/p]+}(r) \subset \mathbf{D}_R(T(r))$, is a quasi-isomorphism.

Proof. The proof is similar to [CN17, Lemma 4.12]. First we note that the map on complexes is induced by inclusion, so the kernel complex is 0. Next, to examine the cokernel complex we write

$$\mathbf{D}_R(T(r)) = D^{(0,v/p]+}(r)\big[\tfrac{1}{\pi_m}\big]^\wedge,$$

where ^ denotes the p-adic completion.

Let $\ell = p^{m-1}$, and recall from Lemma 2.37 that we have $\psi\big(\mathbf{A}_R^{(0,v/p]+}\big) \subset \mathbf{A}_R^{(0,v]+} \subset \mathbf{A}_R^{(0,v/p]+}$. Further, for $v = p-1$, by Lemma 2.47 (v) we have that $\pi_m^{-p\ell}\pi$ is a unit in $\mathbf{A}_R^{(0,v/p]+}$. So by combining Lemma

2.39 and Proposition 2.40 (i), we see that $\psi\big(\pi_m^{-p^k\ell r}\mathbf{A}_R^{(0,v/p]+}\big) \subset \pi_m^{-p^{k-1}\ell r}\mathbf{A}_R^{(0,v/p]+}$ for $k \geq 1$. Moreover, we have that $\psi(\mathbf{D}^+(T)) \subset \mathbf{D}^+(T)$, and therefore $\psi\big(D^{(0,v/p]+}(r)\big) \subset D^{(0,v/p]+}(r)$. Coupling this with the observation above, we note that $\psi\big(\pi_m^{-p^k\ell r}D^{(0,v/p]+}(r)\big) \subset \pi_m^{-p^{k-1}\ell r}D^{(0,v/p]+}(r)$.

From this dicsussion, we note that the map

$$\psi\ :\ \mathbf{D}_R(T(r))/\pi_m^{-p\ell r}D^{(0,v/p]+}(r) \longrightarrow \mathbf{D}_R(T(r))/\pi_m^{-p\ell r}D^{(0,v/p]+}(r)$$

is (pointwise) topologically nilpotent, therefore $1-\psi$ is bijective over this quotient of modules. But, this also means that the complexes

$$\big[\mathrm{Kos}\big(\Gamma_R', \mathbf{D}_R(T(r))/\pi_m^{-p\ell r}D^{(0,v/p]+}(r)\big) \xrightarrow{\psi-1} \mathrm{Kos}\big(\Gamma_R', \mathbf{D}_R(T(r))/\pi_m^{-p\ell r}D^{(0,v/p]+}(r)\big)\big], \quad \text{and}$$
$$\big[\mathrm{Kos}^c\big(\Gamma_R', \mathbf{D}_R(T(r))/\pi_m^{-p\ell r}D^{(0,v/p]+}(r)\big) \xrightarrow{\psi-1} \mathrm{Kos}^c\big(\Gamma_R', \mathbf{D}_R(T(r))/\pi_m^{-p\ell r}D^{(0,v/p]+}(r)\big)\big],$$

are acyclic. Hence the cokernel complex is acyclic. ∎

Next, recall that we have the complex

$$\mathrm{Kos}\big(\varphi, \Gamma_R, \mathbf{D}_R(T(r))\big) = \left[\begin{array}{ccc} \mathrm{Kos}\big(\Gamma_R', \mathbf{D}_R(T(r))\big) & \xrightarrow{1-\varphi} & \mathrm{Kos}\big(\Gamma_R', \mathbf{D}_R(T(r))\big) \\ \big\downarrow{\scriptstyle \tau_0} & & \big\downarrow{\scriptstyle \tau_0} \\ \mathrm{Kos}^c\big(\Gamma_R', \mathbf{D}_R(T(r))\big) & \xrightarrow{1-\varphi} & \mathrm{Kos}^c\big(\Gamma_R', \mathbf{D}_R(T(r))\big) \end{array}\right].$$

Proposition 5.46. *With notations as above, the natural map*

$$\mathrm{Kos}\big(\varphi, \Gamma_R, \mathbf{D}_R(T(r))\big) \longrightarrow \mathrm{Kos}\big(\psi, \Gamma_R, \mathbf{D}_R(T(r))\big),$$

induced by identity on the first column and ψ on the second column is a quasi-isomorphism.

Proof. We will examine the kernel and cokernel of the map above. Notice that the map ψ is surjective on $\mathbf{D}_R(T(r))$, so the cokernel complex is 0. For the kernel complex, we need to show that the complex

$$\big[\mathrm{Kos}\big(\Gamma_R', \mathbf{D}_R(T(r))^{\psi=0}\big) \xrightarrow{\tau_0} \mathrm{Kos}\big(\Gamma_R', \mathbf{D}_R(T(r))^{\psi=0}\big)\big],$$

is acyclic. For this, we will analyze the module $(\mathbf{D}_R(T(r)))^{\psi=0}$. Let us write $\mathbf{N}(T) = \oplus_{j=1}^h \mathbf{A}_{R_0}^+ e_j$ for a choice of $\mathbf{A}_{R_0}^+$-basis. Since $\mathbf{D}(T(r)) \simeq \mathbf{D}(T)(r) \simeq \mathbf{A}_{R_0} \otimes_{\mathbf{A}_{R_0}^+} \mathbf{N}(T)(r)$, we obtain that $\{e_1 \otimes \epsilon^{\otimes r}, \ldots, e_h \otimes \epsilon^{\otimes r}\}$ is an $\mathbf{A}_{R_0}$-basis of $\mathbf{D}(T(r))$, where $\epsilon^{\otimes r}$ is a basis of $\mathbb{Z}_p(r)$. Further, since $\mathbf{D}(T(r))$ is étale and $\mathbf{D}_R(T(r)) = \mathbf{A}_R \otimes_{\mathbf{A}_{R_0}} \mathbf{D}(T(r))$, we obtain a decomposition

$$\mathbf{D}_R(T(r)) \simeq \bigoplus_{j=1}^h \mathbf{A}_R \varphi(e_j) \otimes \epsilon^{\otimes r}.$$

Using this decomposition, note that we can write

$$z = \sum_{j=1}^h z_j \varphi(e_j) \in \Big(\bigoplus_{j=1}^h \mathbf{A}_R \varphi(e_j)\Big)^{\psi=0} = \big(\mathbf{D}_R(T)\big)^{\psi=0}$$

if and only if $z_j \in \mathbf{A}_R^{\psi=0}$ for each $1 \leq j \leq h$. Indeed, $\psi(z) = 0$ if and only if $\sum_{j=1}^h \psi(z_j\varphi(e_j)) = \sum_{j=1}^h \psi(z_j)e_j = 0$. As e_j are linearly independent over $\mathbf{A}_R$, we get the desired conclusion.

Next, according to Proposition 2.40, we have a decomposition

$$\mathbf{A}_R^{\psi=0} \simeq \bigoplus_{\alpha \in \{0,\ldots,p-1\}^{[0,d]}, \alpha \neq 0} \varphi\big(\mathbf{A}_R\big)[X^\flat]^\alpha, \quad \text{where } [X^\flat]^\alpha = (1+\pi_m)^{\alpha_0}[X_1^\flat]^{\alpha_0} \cdots [X_d^\flat]^{\alpha_d}.$$

Therefore, we obtain that

$$\big(\mathbf{D}_R(T(r))\big)^{\psi=0} \simeq \big(\mathbf{D}_R(T)\big)^{\psi=0}(r) \simeq \Big(\bigoplus_{i=1}^{h} \mathbf{A}_R e_j\Big)^{\psi=0}(r) \simeq \bigoplus_{\substack{\alpha\in\{0,\ldots,p-1\}^{[0,d]},\alpha\neq 0\\ j\in\{1,\ldots,h\}}} \varphi\big(\mathbf{A}_R e_j\big)(r)[X^\flat]^\alpha,$$

We have $\mathbf{D}_R(T) = \oplus_{j=1}^h \mathbf{A}_R e_j$ and we see that the kernel complex of the map in the claim is isomorphic to the complex

$$\bigoplus_{\alpha\in\{0,\ldots,p-1\}^{[0,d]},\alpha\neq 0} \Big[\,\mathrm{Kos}\big(\Gamma_R', \varphi\big(\mathbf{D}_R(T)\big)(r)[X^\flat]^\alpha\big) \xrightarrow{\tau_0} \mathrm{Kos}^c\big(\Gamma_R', \varphi\big(\mathbf{D}_R(T)\big)(r)[X^\flat]^\alpha\big)\,\Big]. \tag{5.28}$$

Lemma 5.47. *The complex described in (5.28) is acyclic.*

Proof. The proof will follow the technique used in the proof of [CN17, Lemma 4.10, Remark 4.11] and will be essentially similar to Lemma 5.42. We will treat terms corresponding to each α separately. First, let us assume that $\alpha_k \neq 0$ for some $k \neq 0$. We want to show that both $\mathrm{Kos}\big(\Gamma_R', \varphi\big(\mathbf{D}_R(T)\big)[X^\flat]^\alpha\big)$ and $\mathrm{Kos}^c\big(\Gamma_R', \varphi\big(\mathbf{D}_R(T)\big)[X^\flat]^\alpha\big)$ complexes are acyclic (the twist has disappeared because the cyclotomic character is trivial on Γ_R'). As the proof is same in both the cases, we only treat the first case. We can write the complex as a double complex

$$\begin{tikzcd}
\varphi\big(\mathbf{D}_R(T)\big)[X^\flat]^\alpha \arrow[r, "(\gamma_i-1)"] \arrow[d, "\gamma_k-1"] & \varphi\big(\mathbf{D}_R(T)\big)^{I_1''}[X^\flat]^\alpha \arrow[r] \arrow[d, "\gamma_k-1"] & \varphi\big(\mathbf{D}_R(T)\big)^{I_2''}[X^\flat]^\alpha \arrow[r] \arrow[d, "\gamma_k-1"] & \cdots \\
\varphi\big(\mathbf{D}_R(T)\big)[X^\flat]^\alpha \arrow[r, "(\gamma_i-1)"] & \varphi\big(\mathbf{D}_R(T)\big)^{I_1''}[X^\flat]^\alpha \arrow[r] & \varphi\big(\mathbf{D}_R(T)\big)^{I_2''}[X^\flat]^\alpha \arrow[r] & \cdots,
\end{tikzcd}$$

where the first horizontal maps involve γ_i's with $i \neq k$, $1 \leq i \leq d$. Since $\mathbf{D}_R(T)$ is p-adically complete, it enough to show that $\gamma_k - 1$ is bijective on $\varphi\big(\mathbf{D}_R(T)\big)[X^\flat]^\alpha$ modulo p. Indeed, this follows from inductively applying five lemma to following exact sequences, for $k \in \mathbb{N}$,

$$\begin{tikzcd}
0 \arrow[r] & p^k\varphi\big(\mathbf{D}_R(T)\big)[X^\flat]^\alpha/p^{k+1} \arrow[r] \arrow[d, "\gamma_k-1"] & \varphi\big(\mathbf{D}_R(T)\big)[X^\flat]^\alpha/p^{k+1} \arrow[r] \arrow[d, "\gamma_k-1"] & \varphi\big(\mathbf{D}_R(T)\big)[X^\flat]^\alpha/p^k \arrow[r] \arrow[d, "\gamma_k-1"] & 0 \\
0 \arrow[r] & p^k\varphi\big(\mathbf{D}_R(T)\big)[X^\flat]^\alpha/p^{k+1} \arrow[r] & \varphi\big(\mathbf{D}_R(T)\big)[X^\flat]^\alpha/p^{k+1} \arrow[r] & \varphi\big(\mathbf{D}_R(T)\big)[X^\flat]^\alpha/p^k \arrow[r] & 0.
\end{tikzcd}$$

So below, we will work modulo p, however with slight abuse, we will hide this from the notation.

Note that we have

$$(\gamma_k - 1)\cdot\big(\varphi(y)[X^\flat]^\alpha\big) = \varphi(\pi_1(G(y)))[X^\flat]^\alpha,$$

where

$$G(y) = \tfrac{(1+\pi_1)^{\alpha_k}(\gamma_k-1)y}{\pi_1} + \tfrac{((1+\pi_1)^{\alpha_k}-1)y}{\pi_1}, \quad \text{for} \quad y \in \mathbf{D}_R(T).$$

Note that we have $\mathbf{E}_R = \mathbf{E}_R^+\big[\frac{1}{\pi_m}\big]$, and setting $\overline{N}_R = \oplus_{i=1}^h \mathbf{E}_R^+ e_i$, we obtain that $\mathbf{D}_R(T)/p = \overline{N}_R\big[\frac{1}{\pi_m}\big]$. Now, G is π_m-linear, $\gamma_k - 1$ is trivial modulo π on $\mathbf{N}(T)$ (see Definition 3.8), and γ_k fixes π_m. Therefore, G is just multiplication by α_k on $\pi_m^a\overline{N}_R/\pi_m^{a+N}\overline{N}_R$ for $a \in \mathbb{Z}$ and $N = p^m$. Looking at the following diagram and applying five lemma for $a \in \mathbb{Z}$,

$$\begin{tikzcd}
0 \arrow[r] & \pi_m^{a+N}\overline{N}_R/\pi_m^{a+2N}\overline{N}_R \arrow[r] \arrow[d, "G"] & \pi_m^a\overline{N}_R/\pi_m^{a+2N}\overline{N}_R \arrow[r] \arrow[d, "G"] & \pi_m^a\overline{N}_R/\pi_m^{a+N}\overline{N}_R \arrow[r] \arrow[d, "G"] & 0 \\
0 \arrow[r] & \pi_m^{a+N}\overline{N}_R/\pi_m^{a+2N}\overline{N}_R \arrow[r] & \pi_m^a\overline{N}_R/\pi_m^{a+2N}\overline{N}_R \arrow[r] & \pi_m^a\overline{N}_R/\pi_m^{a+N}\overline{N}_R \arrow[r] & 0,
\end{tikzcd}$$

we obtain that, G is bijective over $\mathbf{D}_R(T)/p$. Finally, since π_1 is invertible in $\mathbf{E}_R$, we obtain that $\gamma_k - 1$ is bijective over $\varphi\big(\mathbf{D}_R(T)\big)[X^\flat]^\alpha$ modulo p, as desired.

Next, let $\alpha_k = 0$ for all $k \neq 0$ and $\alpha_0 \neq 0$. To prove that the kernel complex is acyclic, we will show that the map $\tau_0 : \mathrm{Kos} \longrightarrow \mathrm{Kos}^c$ is bijective. This amounts to showing the same statement for

$$\gamma_0 - \delta_{i_1} \cdots \delta_{i_q} : \varphi\big(\mathbf{D}_R(T)\big)[X^\flat]^\alpha(r) \longrightarrow \varphi\big(\mathbf{D}_R(T)\big)[X^\flat]^\alpha(r), \quad \delta_{i_j} = \tfrac{\gamma_{i_j}^c - 1}{\gamma_{i_j} - 1}.$$

Again, arguing as in the previous part, we see that it is enough to show this statement modulo p. We have

$$(\gamma_0 - \delta_{i_1} \cdots \delta_{i_q})\big(\varphi(y)[X^\flat]^\alpha(r)\big) = \big(c^r \varphi(\gamma_0(y))(1+\pi)^{p^{-m}(c-1)\alpha_0}[X^\flat]^\alpha\big)(r) - \big(\varphi(\delta_{i_1} \cdots \delta_{i_q}(y))[X^\flat]^\alpha\big)(r).$$

So we are lead to study the map F defined by

$$F = c^r(1+\pi_1)^z \gamma_0 - \delta_{i_1} \cdots \delta_{i_q}, \quad z = p^{-m}(c-1)\alpha_0 \in \mathbb{Z}_p^*.$$

Now $c^r - 1$ is divisible by p^m, $(1+\pi_1)^z = 1 + z\pi_1 \bmod \pi_1^2$ and $\delta_{i_j} - 1 \in (p^m, \gamma_{i_j} - 1)\mathbb{Z}_p[[\gamma_{i_j} - 1]]$. Therefore, we can write $\pi_1^{-1}F$ in the form $\pi_1^{-1}F = z + \pi_1^{-1}F'$, with $F' \in \big(p^m, \pi_1^2, \gamma_0 - 1, \ldots, \gamma_d - 1\big)\mathbb{Z}_p[[\pi_1, \Gamma_R]]$. Working modulo p, it follows from Lemma 2.46, Lemma 2.47 and Definition 3.8, that for $N = 2p^{m-1} > 0$ we have that $\pi_1^{-1}F' = 0$ on $\pi_m^a \overline{N}_R / \pi_m^{a+N} \overline{N}_R$, for all $a \in \mathbb{Z}$. Hence, $\pi_1^{-1}F$ induces multiplication by z on $\pi_m^a \overline{N}_R / \pi_m^{a+N} \overline{N}_R$ for all $a \in \mathbb{Z}$, which implies that it is an isomorphism of $\mathbf{D}_R(T)$ modulo p. This shows what we want since π_1 is invertible in $\mathbf{A}_R$. ∎

Combining the analysis for the kernel and cokernel complex, we conclude that the map in the claim of Proposition 5.46 is a quasi-isomorphism.

∎

Proof of Proposition 5.31. Recall that s is the maximum among the absolute values of Hodge-Tate length of V (see Definition 3.8). From Lemmas 5.33 & 5.34 and Remark 5.35, we have a p^{4r}-quasi-isomorphism

$$\mathrm{Kos}\big(\varphi, \partial_A, \mathrm{Fil}^r M^{[u,v]}\big) \simeq \mathcal{K}\big(\varphi, \mathrm{Lie}\,\Gamma_R, N^{[u,v]}(T(r))\big).$$

Changing from infinitesimal action of Γ_R to the continuous action of Γ_R is an isomorphism of complexes by Proposition 5.38,

$$\mathcal{K}\big(\varphi, \mathrm{Lie}\,\Gamma_R, N^{[u,v]}(T(r))\big) \simeq \mathcal{K}\big(\varphi, \Gamma_R, N^{[u,v]}(T(r))\big).$$

Further, from Proposition 5.39 we have a p^{3r}-quasi-isomorphism

$$\mathcal{K}\big(\varphi, \Gamma_R, N^{[u,v]}(T(r))\big) \simeq \mathcal{K}\big(\varphi, \Gamma_R, N^{(0,v]+}(T(r))\big).$$

Next, from Proposition 5.41 and Lemmas 5.43 & 5.44, we have $p^{7r+3s+2}$-quasi-isomorphisms

$$\begin{aligned} \tau_{\leq r}\mathcal{K}\big(\varphi, \Gamma_R, N^{(0,v]+}(T(r))\big) &\simeq \tau_{\leq r}\mathcal{K}\big(\psi, \Gamma_R, N^{(0,v]+}(T(r))\big) \\ &\simeq \tau_{\leq r}\mathcal{K}\big(\psi, \Gamma_R, N^{(0,v/p]+}(T(r))\big) \simeq \tau_{\leq r}\mathrm{Kos}\big(\varphi, \Gamma_R, D^{(0,v/p]+}(r)\big). \end{aligned}$$

Finally, From Lemma 5.45 and Proposition 5.46 we obtain quasi-isomorphisms

$$\mathrm{Kos}\big(\psi, \Gamma_R, D^{(0,v/p]+}(r)\big) \simeq \mathrm{Kos}\big(\psi, \Gamma_R, \mathbf{D}_R(T(r))\big) \simeq \mathrm{Kos}\big(\varphi, \Gamma_R, \mathbf{D}_R(T(r))\big).$$

Combining these statements we get the claim with $N = 14r + 3s + 2$. ∎

APPENDIX A

Galois cohomology and classical Wach modules

Let F be a finite unramified extension of $\mathbb{Q}_p$ and V a crystalline p-adic representation of $G_F = \mathrm{Gal}(\overline{F}/F)$. The aim of this chapter is to emphasize the importance of Wach modules from the point of view of Galois cohomology. In [Her98], Herr obtained a three term complex in terms of the attached (φ, Γ_F)-module computing continuous G_F-cohomology of V. Since the Wach module of V is an "integral" lattice inside the (φ, Γ_F)-module, it is interesting to explore whether some part of Galois cohomology groups of V could be captured in terms of a complex written down completely in terms of the Wach module. This could be answered positively via some concrete statements, for example, see Proposition A.4. In order to establish these claims, we will need to introduce some more background from (classical) p-adic Hodge theory. After recalling these facts, we will describe a complex and carry out some concrete computations involving Wach modules.

A.1. Crystalline extension classes

We fix a compatible system of p-power roots of unity $(\zeta_{p^n})_{n\in\mathbb{N}}$ such that $\zeta_{p^0} = 1$, $\zeta_p \neq 1$ and $\zeta_{p^{n+1}}^p = \zeta_{p^n}$. Moreover, we set $F_n = F(\zeta_{p^n})$, $F_\infty = \cup_n F_n$, $\Gamma_F = \mathrm{Gal}(F_\infty/F)$ and $\gamma \in \Gamma_F$ a topological generator.

Let V be an h-dimensional p-adic crystalline representation of G_F with Hodge-Tate weights $-r_1 \leq -r_2 \leq \cdots \leq -r_d \leq 0$. Let T be a free $\mathbb{Z}_p$-lattice of rank h inside V stable under the action of G_F. Set $V(r) := V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(r)$ and $T(r) := T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(r)$, then the Hodge-Tate weights of $V(r)$ are $r - r_1 \leq r - r_2 \leq \cdots \leq r - r_d$. From §3.1, we have Wach modules $\mathbf{N}(T)$ and $\mathbf{N}(V)$, such that

$$\mathbf{N}(T(r)) = \pi^{-r}\mathbf{N}(T)(r) \quad \text{and} \quad \mathbf{N}(V(r)) = \mathbf{B}_F^+ \otimes_{\mathbf{A}_F^+} \mathbf{N}(T(r)) = \pi^{-r}\mathbf{N}(V)(r).$$

From Theorem 4.4 we have that the Fontaine-Herr complex

$$\mathcal{C}^\bullet(V(r)) : \mathbf{D}(V(r)) \xrightarrow{(1-\varphi, \gamma-1)} \mathbf{D}(V(r)) \oplus \mathbf{D}(V(r)) \xrightarrow{\binom{\gamma-1}{1-\varphi}} \mathbf{D}(V(r)),$$

computes the Galois cohomology of $V(r)$ i.e., for all $k \in \mathbb{N}$, we have natural isomorphisms

$$H^k(\mathcal{C}^\bullet(V(r))) \xrightarrow{\simeq} H^k(G_F, V(r)).$$

In particular, any extension class in $H^1(G_F, V(r))$ can be represented by a pair (x, y) with $x, y \in \mathbf{D}(V(r))$ and satisfying the relation $(1 - \varphi)x = (\gamma - 1)y$. We want to look at extension classes in

$H^1_f(G_F, V(r))$ which come from crystalline extensions of $\mathbb{Q}_p$ by $V(r)$.

Let V be a positive crystalline representation of G_F as above. Let X be an extension of $\mathbb{Q}_p(-r)$ by V such that it is crystalline as a representation of G_F

$$0 \longrightarrow V \longrightarrow X \longrightarrow \mathbb{Q}_p(-r) \longrightarrow 0.$$

Equivalently, we have that $X(r) := X \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(r)$ is a crystalline extension of $\mathbb{Q}_p$ by $V(r)$

$$0 \longrightarrow V(r) \longrightarrow X(r) \longrightarrow \mathbb{Q}_p \longrightarrow 0. \tag{A.1}$$

From Proposition 3.6 we have that the Wach functor $\mathbf{N}$ is exact. Therefore, we have an exact sequence

$$0 \longrightarrow \mathbf{N}(V(r)) \longrightarrow \mathbf{N}(X(r)) \longrightarrow \mathbf{N}(\mathbb{Q}_p) \longrightarrow 0. \tag{A.2}$$

Lemma A.1. *The sequence*

$$0 \longrightarrow \mathrm{Fil}^0\mathbf{N}(V(r)) \longrightarrow \mathrm{Fil}^0\mathbf{N}(X(r)) \longrightarrow \mathrm{Fil}^0\mathbf{N}(\mathbb{Q}_p) \longrightarrow 0, \tag{A.3}$$

is exact.

Proof. First, we want to show exactness of (A.3) on the right. Let $e \in \mathbf{N}(X(r))$ be a lift of $1 \in \mathbf{B}_F^+ = \mathbf{N}(\mathbb{Q}_p) = \mathrm{Fil}^0\mathbf{N}(\mathbb{Q}_p)$, and we want to show that $e \in \mathrm{Fil}^0\mathbf{N}(X(r))$. Recall that we also have the exact sequence

$$0 \longrightarrow V \longrightarrow X \longrightarrow \mathbb{Q}_p(-r) \longrightarrow 0.$$

Applying the exact Wach functor to it we obtain an exact sequence

$$0 \longrightarrow \mathbf{N}(V) \longrightarrow \mathbf{N}(X) \longrightarrow \mathbf{N}(\mathbb{Q}_p(-r)) \longrightarrow 0, \tag{A.4}$$

such that $\pi^r e \otimes \epsilon^{\otimes(-r)} \in \mathbf{N}(X)$ and its image in $\mathbf{N}(\mathbb{Q}_p(-r)) = (\pi^r \otimes \epsilon^{\otimes(-r)})\mathbf{B}_F^+$ is a basis. Here ϵ denotes a basis of $\mathbb{Q}_p(1)$.

Let $\{e_1, \ldots, e_h\}$ denote a $\mathbf{B}_F^+$-basis of $\mathbf{N}(V)$, then we have that $\{e_1, \ldots, e_h, \pi^r e \otimes \epsilon^{\otimes(-r)}\}$ is a $\mathbf{B}_F^+$-basis of $\mathbf{N}(X)$. Since each module in (A.4) is stable under the action of Frobenius, we obtain that

$$\varphi\big(\pi^r e \otimes \epsilon^{\otimes(-r)}\big) = \sum_{i=1}^{h} a_i e_i + a_{h+1}\pi^r e \otimes \epsilon^{\otimes(-r)},$$

where $a_i \in \mathbf{B}_F^+$ for $1 \leq i \leq h+1$. But from the exact sequence (A.4), we have that $q^{-r}\varphi\big(\pi^r e \otimes \epsilon^{\otimes(-r)}\big)$ and $\pi^r e \otimes \epsilon^{\otimes(-r)}$ must have the same image in $\mathbf{N}(\mathbb{Q}_p(-r))$. Therefore,

$$q^{-r}\varphi\big(\pi^r e \otimes \epsilon^{\otimes(-r)}\big) - \pi^r e \otimes \epsilon^{\otimes(-r)} = \sum_{i=1}^{h} q^{-r} a_i e_i + (q^{-r}a_{h+1} - 1)\pi^r e \otimes \epsilon^{\otimes(-r)} \in \mathbf{N}(V).$$

This means that we must have $a_{h+1} = q^r$ and $a_i \in q^r\mathbf{B}_F^+$ for $1 \leq i \leq h$. Therefore, $\pi^r e \otimes \epsilon^{\otimes(-r)} \in \mathrm{Fil}^r\mathbf{N}(X)$, or equivalently $e \in \pi^{-r}\mathrm{Fil}^r\mathbf{N}(X)(r) = \mathrm{Fil}^0\mathbf{N}(X(r))$.

Next, to show exactness in the middle, let $e, e' \in \mathrm{Fil}^0\mathbf{N}(X(r))$ be two such lifts. Then arguing as above, we obtain that $\pi^r(e - e') \otimes \epsilon^{\otimes(-r)} \in \mathrm{Fil}^r\mathbf{N}(V)$, or equivalently $e - e' \in \pi^{-r}\mathrm{Fil}^r\mathbf{N}(V)(r) = \mathrm{Fil}^0\mathbf{N}(V(r))$. Hence, the sequence (A.3) is exact. ∎

Lemma A.2. *The class of the extension (A.1) in $H^1(\mathcal{C}^\bullet(V(r)))$ is represented by a pair (x, y) for some $x \in \mathbf{N}(V)(r)$ and $y \in \mathbf{N}(V(r))$ satisfying the relation $(1 - \varphi)x = (\gamma - 1)y$.*

Proof. Consider the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Fil}^0\, \mathbf{N}(V(r)) \arrow[r] \arrow[d, two heads] & \mathrm{Fil}^0\, \mathbf{N}(X(r)) \arrow[r] \arrow[d, two heads] & \mathrm{Fil}^0\, \mathbf{B}_F^+ \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \mathrm{Fil}^0\, \mathbf{D}_{\mathrm{cris}}(V(r)) \arrow[r] & \mathrm{Fil}^0\, \mathbf{D}_{\mathrm{cris}}(X(r)) \arrow[r] & F \arrow[r] & 0,
\end{tikzcd}$$

where the top row is a consequence of Lemma A.1 and the vertical maps are reduction modulo π. Let $e \in \mathrm{Fil}^0\, \mathbf{N}(X(r))$ be a lift of $1 \in \mathbf{B}_F^+$ in a manner compatible with the above diagram. Then we have that

$$y_r = (1-\varphi)e \in \mathbf{N}(V(r)),$$

since y_r maps to $0 \in \mathbf{B}_F^+$ in (A.2). Now, since the action of Γ_F on $\mathbf{N}(X(r))/\pi\mathbf{N}(X(r))$ is trivial and the filtration on Wach modules is stable under the action of Γ_F, we get that

$$x_r = (\gamma-1)e \in \mathrm{Fil}^0\mathbf{N}(V(r)) \cap \pi\mathbf{N}(V(r)) = \tfrac{1}{\pi^r}\mathrm{Fil}^r\mathbf{N}(V)(r) \cap \tfrac{1}{\pi^{r-1}}\mathbf{N}(V)(r) = \tfrac{1}{\pi^{r-1}}\mathrm{Fil}^{r-1}\mathbf{N}(V)(r),$$

where the last equality is a consequence of Lemma 3.17. The pair (x_r, y_r) represents the class of this extension in $H^1(\mathcal{C}^\bullet)$.

Next, we want to modify the pair (x_r, y_r) by adding coboundaries to get a pair (x, y) with $x \in \mathbf{N}(V)(r)$ and $y \in \mathbf{N}(V(r))$ cohomologous to (x_r, y_r). We do this iteratively by clearing out negative powers of π in the expression of x_r. It is easy to observe that for any $z \in \mathrm{Fil}^r\, \mathbf{N}(V(r))$, the pair $(x_r + (1-\gamma)z, y_r + (\varphi-1)z)$ is cohomologous to the pair (x_r, y_r) in $H^1(\mathcal{C}^\bullet(V(r)))$. Let us represent

$$x_r = \tfrac{a_r}{\pi^{r-1}} \otimes \epsilon^{\otimes r} \in \tfrac{1}{\pi^{r-1}}\mathrm{Fil}^{r-1}\mathbf{N}(V)(r),$$

and take

$$z_{r-1} = \tfrac{b_{r-1}}{\pi^{r-1}} \otimes \epsilon^{\otimes r} \in \tfrac{1}{\pi^{r-1}}\mathrm{Fil}^{r-1}\mathbf{N}(V)(r),$$

where

$$b_{r-1} = \tfrac{a_r}{\chi(\gamma)-1}.$$

Clearly, $b_{r-1} \in \mathrm{Fil}^{r-1}\mathbf{N}(V)$. Now observe that,

$$\begin{aligned} x_r + (1-\gamma)z_{r-1} &= \tfrac{a_r}{\pi^{r-1}} \otimes \epsilon^{\otimes r} + (1-\gamma)\big(\tfrac{b_{r-1}}{\pi^{r-1}} \otimes \epsilon^{\otimes r}\big) \\ &= \frac{a_r + b_{r-1} - \chi(\gamma)u^{r-1}\gamma(b_{r-1})}{\pi^{r-1}} \otimes \epsilon^{\otimes r}. \end{aligned}$$

where we have used the expression $\gamma(\pi) = \chi(\gamma)\pi u^{-1}$, for a unit $u \in 1 + \mathbf{B}_F^+$. By a small computation we can write

$$a_r + b_{r-1} - \chi(\gamma)u^{r-1}\gamma(b_{r-1}) = -\frac{\chi(\gamma)(\gamma-1)a_r}{\chi(\gamma)-1} + \frac{\chi(\gamma)(1-u^{r-1})\gamma(a_r)}{\chi(\gamma)-1}.$$

Since π divides $(\gamma-1)a_r$ and $(1-u^{r-1})$, from Lemma 3.17 we have that

$$\tfrac{1}{\pi}\big(a_r + b_{r-1} - \chi(\gamma)u^{r-1}\gamma(b_{r-1})\big) \in \mathrm{Fil}^{r-2}\mathbf{N}(V),$$

and therefore,

$$x_r + (1-\gamma)z_{r-1} \in \tfrac{1}{\pi^{r-2}}\mathrm{Fil}^{r-2}\mathbf{N}(V)(r).$$

So, we can write

$$\begin{aligned} (\varphi-1)z_{r-1} &= \tfrac{\varphi(b_{r-1})}{\varphi(\pi^{r-1})} \otimes \epsilon^{\otimes r} - \tfrac{b_{r-1}}{\pi^{r-1}} \otimes \epsilon^{\otimes r} \\ &= \tfrac{q^{-r+1}\varphi(b_{r-1}) - b_{r-1}}{\pi^{r-1}} \otimes \epsilon^{\otimes r} \\ &= \tfrac{q^{-r}\varphi(\pi b_{r-1}) - \pi b_{r-1}}{\pi^{r}} \otimes \epsilon^{\otimes r}. \end{aligned}$$

Since $b_{r-1} \in \mathrm{Fil}^{r-1}\mathbf{N}(V)$, we get that

$$q^{-r}\varphi(\pi b_{r-1}) - \pi b_{r-1} \in \mathbf{N}(V),$$

i.e. $(1-\varphi)z_{r-1} \in \mathbf{N}(V(r))$ and therefore,

$$y_r + (\varphi - 1)z_{r-1} \in \mathbf{N}(V(r)).$$

Next, let $a_{r-1} = \frac{1}{\pi}(a_r + b_{r-1} - \chi(\gamma)u^{r-1}\gamma(b_{r-1}))$ and $b_{r-2} = \frac{a_{r-1}}{(\chi(\gamma)^2-1)}$. So we set

$$\begin{aligned} x_{r-1} &:= x_r + (1-\gamma)z_{r-1} = \tfrac{a_{r-1}}{\pi^{r-2}} \otimes \epsilon^{\otimes r} \in \tfrac{1}{\pi^{r-2}}\mathrm{Fil}^{r-2}\mathbf{N}(V)(r), \\ y_{r-1} &:= y_r + (\varphi - 1)z_{r-1} \in \mathbf{N}(V(r)), \end{aligned}$$

as well as

$$z_{r-2} := \tfrac{b_{r-2}}{\pi^{r-2}} \otimes \epsilon^{\otimes r} \in \tfrac{1}{\pi^{r-2}}\mathrm{Fil}^{r-2}\mathbf{N}(V)(r).$$

Now, we can repeat the argument above with r replaced by $r-1$ and iterate this process until $r = 1$ and get

$$\begin{aligned} x_1 &= x_2 + (1-\gamma)z_2 \in \mathrm{Fil}^0\mathbf{N}(V)(r) = \mathbf{N}(V)(r), \\ y_1 &= y_2 + (\varphi - 1)z_2 \in \mathbf{N}(V(r)), \end{aligned}$$

where x_2, y_2 and z_2 come from the step $r = 2$. We set $(x, y) = (x_1, y_1)$, where we have

$$x \in \mathbf{N}(V)(r), \quad \text{and} \quad y \in \mathbf{N}(V(r)),$$

satisfying the relation $(1-\varphi)x = (\gamma - 1)y$ and which is cohomologous to (x_r, y_r) in $H^1(\mathcal{C}^\bullet(V(r)))$. This shows the claim. ∎

Let $V(r)$ be a crystalline representation of G_F as above. For the associated Wach module over $\mathbf{B}_F^+$, define

$$\mathcal{K}^\bullet(V(r)) : \mathrm{Fil}^0\mathbf{N}(V(r)) \xrightarrow{(1-\varphi,\gamma-1)} \mathrm{Fil}^0\mathbf{N}(V(r)) \oplus \mathbf{N}(V(r)) \xrightarrow{\left(\begin{smallmatrix}\gamma-1\\ 1-\varphi\end{smallmatrix}\right)} \mathbf{N}(V(r)).$$

Lemma A.3. *For a crystalline representation $V(r)$ as above and $r \geq r_1$, we have*

$$H^0(\mathcal{K}^\bullet) = (\mathrm{Fil}^0\mathbf{N}(V(r)))^{\varphi=1,\gamma=1} \simeq V(r)^{G_F}.$$

Proof. First, note that we have $(\mathrm{Fil}^0\mathbf{N}(V(r)))^{\varphi=1,\gamma=1} \subset \mathbf{D}(V(r))^{\varphi=1,\gamma=1} = V(r)^{G_F} = (\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V(r)))^{\varphi=1}$. On the other hand, from Proposition 3.2 we have $\pi^{r_1}(\mathbf{D}^+(V)) \subset \mathbf{N}(V)$, therefore $\mathbf{D}^+(V(r)) \subset \pi^{r_1-r}\mathbf{D}^+(V(r)) \subset \mathbf{N}(V(r))$. Since $\mathbf{D}^+(V(r))^{\varphi=1,\gamma=1} = V(r)^{G_F}$, we get the claim. ∎

Proposition A.4. *For a crystalline representation $V(r)$ as above, we have $H^1_{\mathrm{f}}(G_F, V(r)) \simeq H^1(\mathcal{K}^\bullet(V(r)))$.*

Proof. Since we know that any cohomology class in $H^1_{\mathrm{f}}(G_F, V(r))$ corresponds to a crystalline extension of $\mathbb{Q}_p$ by $V(r)$, it will be enough to construct a bijection between such extensions and cohomology classes in $H^1(\mathcal{K}^\bullet(V(r)))$. Let $X(r)$ denote a crystalline representation of G_F given as an extension of $\mathbb{Q}_p$ by $V(r)$, i.e. we have an exact sequence of G_F-modules

$$0 \longrightarrow V(r) \longrightarrow X(r) \longrightarrow \mathbb{Q}_p \longrightarrow 0.$$

Since $\mathbf{N}$ is an exact functor, we get an exact sequence of Wach modules over $\mathbf{B}_F^+$

$$0 \longrightarrow \mathbf{N}(V(r)) \longrightarrow \mathbf{N}(X(r)) \longrightarrow \mathbf{B}_F^+ \longrightarrow 0.$$

We can write $\mathbf{N}(X(r)) = \mathbf{N}(V(r)) + \mathbf{B}_F^+ \cdot e$ with

$$(\gamma - 1)e = x,$$
$$(1 - \varphi)e = y,$$

for some $x, y \in \mathbf{N}(V(r))$. Recall from Lemma A.1 that we have $e \in \mathrm{Fil}^0\mathbf{N}(X(r))$, therefore $x = (\gamma-1)e \in \mathrm{Fil}^0\mathbf{N}(V(r))$. By the commutativity of φ and γ, we get that

$$(1 - \varphi)(\gamma - 1)e = (\gamma - 1)(1 - \varphi)e,$$

or equivalently, we have

$$(1 - \varphi)x = (\gamma - 1)y,$$

which implies that (x, y) represents a cohomological class in $H^1(\mathcal{K}^\bullet(V(r)))$.

Conversely, let $w \in \mathrm{Fil}^0\,\mathbf{N}(V(r))$ and $z \in \mathbf{N}(V(r))$ such that

$$(1 - \varphi)w = (\gamma - 1)z.$$

Then we have that the pair (w, z) represents a cohomological class in $H^1(\mathcal{K}^\bullet(V(r)))$. Set $E = \mathbf{N}(V(r)) + \mathbf{B}_F^+ \cdot e$ with

$$\gamma(e) = w + e,$$
$$\varphi(e) = z + e.$$

Clearly, E is an extension of $\mathbf{B}_F^+$ by $\mathbf{N}(V(r))$, i.e. by sending e to $1 \in \mathbf{B}_F^+$ we have an exact sequence

$$0 \longrightarrow \mathbf{N}(V(r)) \longrightarrow E \longrightarrow \mathbf{B}_F^+ \longrightarrow 0, \tag{A.5}$$

of Wach modules over $\mathbf{B}_F^+$. From Proposition 3.6, applying the quasi-inverse exact functor of $\mathbf{N}$ to (A.5), we get a crystalline extension of $\mathbb{Q}_p$ by $V(r)$

$$0 \longrightarrow V(r) \longrightarrow Y \longrightarrow \mathbb{Q}_p \longrightarrow 0,$$

where we set $Y = (\mathbf{B} \otimes_{\mathbf{B}_F^+} E)^{\varphi=1}$. This extension represents a cohomology class in $H^1_{\mathrm{f}}(G_F, V(r))$.

It is clear that these constructions are inverse to each other. Therefore, we conclude that

$$H^1(\mathcal{K}^\bullet(V(r))) \simeq H^1_{\mathrm{f}}(G_F, V(r)).$$

■

Bibliography

[AB08] Fabrizio Andreatta and Olivier Brinon. Surconvergence des représentations p-adiques: le cas relatif. *Astérisque*, (319):39–116, 2008. Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (ϕ, Γ)-modules.

[Abh21] Abhinandan. Crystalline representations and Wach modules in the relative case. *arXiv e-prints*, page arXiv:2103.17097, March 2021.

[Abr07] Victor Abrashkin. An analogue of the field-of-norms functor and of the Grothendieck conjecture. *J. Algebraic Geom.*, 16(4):671–730, 2007.

[AGV71] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier. *Theorie de Topos et Cohomologie Etale des Schemas I, II, III*, volume 269, 270, 305 of *Lecture Notes in Mathematics*. Springer, 1971.

[AI08] Fabrizio Andreatta and Adrian Iovita. Global applications of relative (φ, Γ)-modules. I. *Astérisque*, (319):339–420, 2008. Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (φ, Γ)-modules.

[AI12] Fabrizio Andreatta and Adrian Iovita. Semistable sheaves and comparison isomorphisms in the semistable case. *Rend. Semin. Mat. Univ. Padova*, 128:131–285 (2013), 2012.

[AI13] Fabrizio Andreatta and Adrian Iovita. Comparison isomorphisms for smooth formal schemes. *J. Inst. Math. Jussieu*, 12(1):77–151, 2013.

[And06] Fabrizio Andreatta. Generalized ring of norms and generalized (ϕ, Γ)-modules. *Ann. Sci. École Norm. Sup. (4)*, 39(4):599–647, 2006.

[BB08] Denis Benois and Laurent Berger. Théorie d'Iwasawa des représentations cristallines. II. *Comment. Math. Helv.*, 83(3):603–677, 2008.

[BB10] Laurent Berger and Christophe Breuil. Sur quelques représentations potentiellement cristallines de $\mathrm{GL}_2(\mathbb{Q}_p)$. *Astérisque*, (330):155–211, 2010.

[Bei12] Alexander Beilinson. p-adic periods and derived de Rham cohomology. *J. Amer. Math. Soc.*, 25(3):715–738, 2012.

[Bei13] Alexander Beilinson. On the crystalline period map. *Camb. J. Math.*, 1(1):1–51, 2013.

[Ben00] Denis Benois. On Iwasawa theory of crystalline representations. *Duke Math. J.*, 104(2):211–267, 2000.

[Ben11] Denis Benois. A generalization of Greenberg's $\mathcal{L}$-invariant. *Amer. J. Math.*, 133(6):1573–1632, 2011.

[Ber02] Laurent Berger. Représentations p-adiques et équations différentielles. *Invent. Math.*, 148(2):219–284, 2002.

[Ber03] Laurent Berger. Bloch and Kato's exponential map: three explicit formulas. *Doc. Math.*, (Extra Vol.):99–129, 2003. Kazuya Kato's fiftieth birthday.

[Ber04] Laurent Berger. Limites de représentations cristallines. *Compos. Math.*, 140(6):1473–1498, 2004.

[Ber08] Laurent Berger. Équations différentielles p-adiques et (ϕ, N)-modules filtrés. *Astérisque*, (319):13–38, 2008. Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (ϕ, Γ)-modules.

[Bha17] Bhargav Bhatt. Lecture notes for a class on perfectoid spaces. *Lecture notes*, 2017.

[BM90] Pierre Berthelot and William Messing. Théorie de Dieudonné cristalline. III. Théorèmes d'équivalence et de pleine fidélité. In *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, pages 173–247. Birkhäuser Boston, Boston, MA, 1990.

[BMS18] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Integral p-adic Hodge theory. *Publ. Math. Inst. Hautes Études Sci.*, 128:219–397, 2018.

[BO78] Pierre Berthelot and Arthur Ogus. *Notes on crystalline cohomology.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978.

[Bou98] Nicolas Bourbaki. *Algebra I: Chapters 1-3.* Actualités scientifiques et industrielles. Springer, 1998.

[Bre99] Christophe Breuil. Une application de corps des normes. *Compositio Math.*, 117(2):189–203, 1999.

[Bre02] Christophe Breuil. Integral p-adic Hodge theory. In *Algebraic geometry 2000, Azumino (Hotaka)*, volume 36 of *Adv. Stud. Pure Math.*, pages 51–80. Math. Soc. Japan, Tokyo, 2002.

[Bri06] Olivier Brinon. Représentations cristallines dans le cas d'un corps résiduel imparfait. *Ann. Inst. Fourier (Grenoble)*, 56(4):919–999, 2006.

[Bri08] Olivier Brinon. Représentations p-adiques cristallines et de de Rham dans le cas relatif. *Mém. Soc. Math. Fr. (N.S.)*, (112):vi+159, 2008.

[BS19] Bhargav Bhatt and Peter Scholze. Prisms and Prismatic Cohomology. *arXiv e-prints*, page arXiv:1905.08229, May 2019.

[BS21] Bhargav Bhatt and Peter Scholze. Prismatic F-crystals and crystalline Galois representations. *arXiv e-prints*, page arXiv:2106.14735, June 2021.

[Car13] Xavier Caruso. Représentations galoisiennes p-adiques et (φ, τ)-modules. *Duke Math. J.*, 162(13):2525–2607, 2013.

[CC98] Frédéric Cherbonnier and Pierre Colmez. Représentations p-adiques surconvergentes. *Invent. Math.*, 133(3):581–611, 1998.

[CC99] Frédéric Cherbonnier and Pierre Colmez. Théorie d'Iwasawa des représentations p-adiques d'un corps local. *J. Amer. Math. Soc.*, 12(1):241–268, 1999.

[ČK19] Kęstutis Česnavičius and Teruhisa Koshikawa. The A_{inf}-cohomology in the semistable case. *Compos. Math.*, 155(11):2039–2128, 2019.

[CN17] Pierre Colmez and Wiesława Nizioł. Syntomic complexes and p-adic nearby cycles. *Invent. Math.*, 208(1):1–108, 2017.

[Col98] Pierre Colmez. Théorie d'Iwasawa des représentations de de Rham d'un corps local. *Ann. of Math. (2)*, 148(2):485–571, 1998.

[Col99] Pierre Colmez. Représentations cristallines et représentations de hauteur finie. *J. Reine Angew. Math.*, 514:119–143, 1999.

[Col02] Pierre Colmez. Espaces de Banach de dimension finie. *J. Inst. Math. Jussieu*, 1(3):331–439, 2002.

[DM82] Pierre Deligne and James Milne. *Tannakian Categories*, pages 101–228. Springer Berlin Heidelberg, Berlin, Heidelberg, 1982.

[DR31] Georges De Rham. *Sur l'analysis situs des variétés à n dimensions*. NUMDAM, [place of publication not identified], 1931.

[Fal88] Gerd Faltings. p-adic Hodge theory. *J. Amer. Math. Soc.*, 1(1):255–299, 1988.

[Fal89] Gerd Faltings. Crystalline cohomology and p-adic Galois-representations. In *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, pages 25–80. Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[Fal02] Gerd Faltings. Almost étale extensions. *Astérisque*, (279):185–270, 2002. Cohomologies p-adiques et applications arithmétiques, II.

[FF18] Laurent Fargues and Jean-Marc Fontaine. Courbes et fibrés vectoriels en théorie de Hodge p-adique. *Astérisque*, (406):xiii+382, 2018. With a preface by Pierre Colmez.

[FL82] Jean-Marc Fontaine and Guy Laffaille. Construction de représentations p-adiques. *Ann. Sci. École Norm. Sup. (4)*, 15(4):547–608 (1983), 1982.

[FM87] Jean-Marc Fontaine and William Messing. p-adic periods and p-adic étale cohomology. In *Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985)*, volume 67 of *Contemp. Math.*, pages 179–207. Amer. Math. Soc., Providence, RI, 1987.

[FO08] Jean-Marc Fontaine and Yi Ouyang. Theory of p-adic Galois representations. *preprint*, 2008.

[Fon79] Jean-Marc Fontaine. Modules galoisiens, modules filtrés et anneaux de Barsotti-Tate. In *Journées de Géométrie Algébrique de Rennes. (Rennes, 1978), Vol. III*, volume 65 of *Astérisque*, pages 3–80. Soc. Math. France, Paris, 1979.

[Fon82] Jean-Marc Fontaine. Sur certains types de représentations p-adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti-Tate. *Ann. of Math. (2)*, 115(3):529–577, 1982.

[Fon83] Jean-Marc Fontaine. Cohomologie de de Rham, cohomologie cristalline et représentations p-adiques. In *Algebraic geometry (Tokyo/Kyoto, 1982)*, volume 1016 of *Lecture Notes in Math.*, pages 86–108. Springer, Berlin, 1983.

[Fon90] Jean-Marc Fontaine. Représentations p-adiques des corps locaux. I. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progr. Math.*, pages 249–309. Birkhäuser Boston, Boston, MA, 1990.

[Fon94a] Jean-Marc Fontaine. Le corps des périodes p-adiques. *Astérisque*, (223):59–111, 1994. With an appendix by Pierre Colmez, Périodes p-adiques (Bures-sur-Yvette, 1988).

[Fon94b] Jean-Marc Fontaine. Représentations p-adiques semi-stables. *Astérisque*, (223):113–184, 1994. With an appendix by Pierre Colmez, Périodes p-adiques (Bures-sur-Yvette, 1988).

[Fon04] Jean-Marc Fontaine. Arithmétique des représentations galoisiennes p-adiques. *Astérisque*, (295):xi, 1–115, 2004. Cohomologies p-adiques et applications arithmétiques. III.

[Fon82] Jean-Marc Fontaine. Formes différentielles et modules de Tate des variétés abéliennes sur les corps locaux. *Invent. Math.*, 65(3):379–409, 1981/82.

[FW79a] Jean-Marc Fontaine and Jean-Pierre Wintenberger. Extensions algébrique et corps des normes des extensions APF des corps locaux. *C. R. Acad. Sci. Paris Sér. A-B*, 288(8):A441–A444, 1979.

[FW79b] Jean-Marc Fontaine and Jean-Pierre Wintenberger. Le "corps des normes" de certaines extensions algébriques de corps locaux. *C. R. Acad. Sci. Paris Sér. A-B*, 288(6):A367–A370, 1979.

[Gol61] Oscar Goldman. Determinants in projective modules. *Nagoya Math. J.*, 18:27–36, 1961.

[Gro63] Alexander Grothendieck. *Revêtements étales et groupe fondamental. Fasc. I: Exposés 1 à 5.* Institut des Hautes Études Scientifiques, Paris, 1963. Troisième édition, corrigée, Séminaire de Géométrie Algébrique, 1960/61.

[Gro66] Alexander Grothendieck. On the de Rham cohomology of algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (29):95–103, 1966.

[Gro74] Alexandre Grothendieck. *Groupes de Barsotti-Tate et cristaux de Dieudonné.* Les Presses de l'Université de Montréal, Montreal, Que., 1974. Séminaire de Mathématiques Supérieures, No. 45 (Été, 1970).

[Her98] Laurent Herr. Sur la cohomologie galoisienne des corps p-adiques. *Bull. Soc. Math. France*, 126(4):563–600, 1998.

[HK94] Osamu Hyodo and Kazuya Kato. Semi-stable reduction and crystalline cohomology with logarithmic poles. *Astérisque*, (223):221–268, 1994. Périodes p-adiques (Bures-sur-Yvette, 1988).

[Kat79] Nicholas Katz. Slope filtration of F-crystals. In *Journées de Géométrie Algébrique de Rennes (Rennes, 1978), Vol. I*, volume 63 of *Astérisque*, pages 113–163. Soc. Math. France, Paris, 1979.

[Kat87] Kazuya Kato. On p-adic vanishing cycles (application of ideas of Fontaine-Messing). In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 207–251. North-Holland, Amsterdam, 1987.

[Kat89] Kazuya Kato. Logarithmic structures of Fontaine-Illusie. In *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, pages 191–224. Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[Kat94] Kazuya Kato. Semi-stable reduction and p-adic étale cohomology. *Astérisque*, (223):269–293, 1994. Périodes p-adiques (Bures-sur-Yvette, 1988).

[Ked04] Kiran Kedlaya. A p-adic local monodromy theorem. *Ann. of Math. (2)*, 160(1):93–184, 2004.

[Kim15] Wansu Kim. The relative Breuil-Kisin classification of p-divisible groups and finite flat group schemes. *Int. Math. Res. Not. IMRN*, (17):8152–8232, 2015.

[Kis06] Mark Kisin. Crystalline representations and F-crystals. In *Algebraic geometry and number theory*, volume 253 of *Progr. Math.*, pages 459–496. Birkhäuser Boston, Boston, MA, 2006.

[KM92] Kazuya Kato and William Messing. Syntomic cohomology and p-adic étale cohomology. *Tohoku Math. J. (2)*, 44(1):1–9, 1992.

[Kur87] Masato Kurihara. A note on p-adic étale cohomology. *Proc. Japan Acad. Ser. A Math. Sci.*, 63(7):275–278, 1987.

[Laz65] Michel Lazard. Groupes analytiques p-adiques. *Inst. Hautes Études Sci. Publ. Math.*, (26):389–603, 1965.

[Mor08] Kazuma Morita. Galois cohomology of a p-adic field via (Φ, Γ)-modules in the imperfect residue field case. *J. Math. Sci. Univ. Tokyo*, 15(2):219–241, 2008.

[MT20] Matthew Morrow and Takeshi Tsuji. Generalised representations as q-connections in integral p-adic Hodge theory. *arXiv e-prints*, page arXiv:2010.04059, October 2020.

[Niz98] Wiesława Nizioł. Crystalline conjecture via K-theory. *Ann. Sci. École Norm. Sup. (4)*, 31(5):659–681, 1998.

[PR94] Bernadette Perrin-Riou. Théorie d'Iwasawa des représentations p-adiques sur un corps local. *Invent. Math.*, 115(1):81–161, 1994. With an appendix by Jean-Marc Fontaine.

[Sam01] Hans Samelson. Differential forms, the early days; or the stories of Deahna's theorem and of Volterra's theorem. *Amer. Math. Monthly*, 108(6):522–530, 2001.

[Sch06] Anthony J. Scholl. Higher fields of norms and (ϕ, Γ)-modules. *Doc. Math.*, (Extra Vol.):685–709, 2006.

[Sch12] Peter Scholze. Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.*, 116:245–313, 2012.

[Sch13] Peter Scholze. p-adic Hodge theory for rigid-analytic varieties. *Forum Math. Pi*, 1:e1, 77, 2013.

[Sch17] Peter Scholze. Canonical q-deformations in arithmetic geometry. *Ann. Fac. Sci. Toulouse Math. (6)*, 26(5):1163–1192, 2017.

[Sta20] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2020.

[Tat67] John Tate. p-divisible groups. In *Proc. Conf. Local Fields (Driebergen, 1966)*, pages 158–183. Springer, Berlin, 1967.

[Tsu96] Takeshi Tsuji. Syntomic complexes and p-adic vanishing cycles. *J. Reine Angew. Math.*, 472:69–138, 1996.

[Tsu99] Takeshi Tsuji. p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case. *Invent. Math.*, 137(2):233–411, 1999.

[Tsu20] Takeshi Tsuji. Crystalline $\mathbb{Z}_p$-representations and A_{inf}-Representations with Frobenius. *Proceeedings in Simons Symposium: p-adic Hodge theory*, pages 161–319, 2020.

[Tyc88] Andrzej Tyc. Differential basis, p-basis, and smoothness in characteristic $p > 0$. *Proc. Amer. Math. Soc.*, 103(2):389–394, 1988.

[Wac96] Nathalie Wach. Représentations p-adiques potentiellement cristallines. *Bull. Soc. Math. France*, 124(3):375–400, 1996.

[Wac97] Nathalie Wach. Représentations cristallines de torsion. *Compositio Math.*, 108(2):185–240, 1997.

[Win83] Jean-Pierre Wintenberger. Le corps des normes de certaines extensions infinies de corps locaux; applications. *Ann. Sci. École Norm. Sup. (4)*, 16(1):59–89, 1983.

[Yam11] Go Yamashita. p-adic Hodge theory for open varieties. *C. R. Math. Acad. Sci. Paris*, 349(21-22):1127–1130, 2011.