Crystalline representations and Wach modules in the relative case II

Abhinandan

Abstract. We study the notion of Wach modules in relative setting and show a categorical equivalence between lattices inside relative crystalline representations and relative Wach modules. Along the way, we also show an equivalence between lattices inside crystalline representations and Wach modules in the imperfect residue field case. Finally, we deduce a purity result for relative crystalline representations.

1. Introduction

In [Fon90] Fontaine introduced the notion of *représentations de cr-hauteur finie* of the absolute Galois group of a finite unramified extension of $\mathbb{Q}_p$. This notion was further developed by Wach [Wac96; Wac97], Colmez [Col99] and Berger [Ber04]. In [Abh21] we studied a similar notion in relative case, i.e. over certain étale algebras over a formal torus (see §1.3 for precise setup). In loc. cit. we defined a notion of relative Wach modules and showed that these give rise to crystalline representations of the fundamental group of the generic fiber of the torus. In this article, we further generalize the definition of relative Wach modules to characterize all crystalline representations of the fundamental group.

1.1. The arithmetic case. Let p be a fixed prime number and let κ denote a finite field of characteristic p; set $O_F = W(\kappa)$ to be the ring of p-typical Witt vectors with coefficients in κ and $F = \mathrm{Frac}(O_F)$. Let $\overline{F}$ denote a fixed algebraic closure of F, $\mathbb{C}_p := \widehat{\overline{F}}$ the p-adic completion, and $G_F = \mathrm{Gal}(\overline{F}/F)$ the absolute Galois group of F. Moreover, let $F_\infty = \cup_n F(\mu_{p^n})$ with $\Gamma_F := \mathrm{Gal}(F_\infty/F)$ and $H_F := \mathrm{Gal}(\overline{F}/F_\infty)$. Furthermore, let $F_\infty^\flat$ denote the tilt of F_∞ and fix $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in O_{F_\infty}^\flat$, $\mu = [\varepsilon] - 1$, $[p]_q = \tilde{\xi} = \varphi(\mu)/\mu \in \mathbf{A}_{\mathrm{inf}}(O_{F_\infty})$.

In [Fon90] Fontaine estalished an equivalence of categories between $\mathbb{Z}_p$-representations of G_F and étale (φ, Γ_F)-modules over a certain period ring $\mathbf{A}_F = O_F[\![\mu]\!][1/\mu]^\wedge \subset W(\mathbb{C}_p^\flat)$, where $^\wedge$ denotes the p-adic completion. For a fixed finite free $\mathbb{Z}_p$-representation T of G_F the associated finite free étale (φ, Γ_F)-module over $\mathbf{A}_F$ is given as $\mathbf{D}_F(T) = (\mathbf{A} \otimes_{\mathbb{Z}_p} T)^{H_F}$, where $\mathbf{A} \subset W(\mathbb{C}_p^\flat)$ is maximal unramified extension of $\mathbf{A}_F$ inside $W(\mathbb{C}_p^\flat)$. In loc. cit. Fontaine conjectured that if $V = T[1/p]$ is crystalline then there exist lattices inside $\mathbf{D}_F(V) = \mathbf{D}_F(T)[1/p]$ over which the action of Γ_F admits a simpler form. Let $\mathbf{A}_F^+ = O_F[\![\mu]\!]$ and we note the following

Definition 1.1. Let $a, b \in \mathbb{Z}$ with $b \geq a$. A *Wach module* over $\mathbf{A}_F^+$ with weights in the interval $[a, b]$ is a finite free $\mathbf{A}_F^+$-module N equipped with a continuous and semilinear action of Γ_F such that Γ_F acts trivially on $N/\mu N$, and there exists a Frobenius-semilinear operator $\varphi : N[1/\mu] \to N[1/\varphi(\mu)]$ commuting with the action of Γ_F such that $\varphi(\mu^b N) \subset \mu^b N$ and the map $(1 \otimes \varphi) : \varphi^*(\mu^b N) \to \mu^b N$ is injective and its cokernel is killed by $\tilde{\xi}^{b-a} = [p]_q^{b-a}$.

Say N is *effective* if we have $b = 0$ and $a \leq 0$. Denote the category of Wach modules over $\mathbf{A}_F^+$ as $(\varphi, \Gamma_F)\text{-}\mathrm{Mod}_{\mathbf{A}_F^+}^{[p]_q}$ with morphisms between objects being $\mathbf{A}_F^+$-linear, Γ_F-equivariant and φ-equivariant (after inverting μ) morphisms.

Let $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_F)$ denote the category of $\mathbb{Z}_p$-lattices inside p-adic crystalline representations of G_F. To any T in $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_F)$ Berger functorially attaches the Wach module $\mathbf{N}_F(T)$ over $\mathbf{A}_F^+$ in [Ber04]. The main result in the arithmetic case is as follows:

Theorem 1.2 ([Wac96, Wach], [Col99, Colmez], [Ber04, Berger]). *The Wach module functor induces an equivalence of $\otimes$-catgeories*

$$\begin{aligned} \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_F) &\xrightarrow{\sim} (\varphi, \Gamma_F)\text{-}\mathrm{Mod}_{\mathbf{A}_F^+}^{[p]_q} \\ T &\longmapsto \mathbf{N}_F(T), \end{aligned}$$

with a quasi-inverse given as $N \mapsto (W(\mathbb{C}_p^\flat) \otimes_{\mathbf{A}_R^+} N)^{\varphi=1}$.

1.2. The relative case. Let $d \in \mathbb{N}$ and $X_1, X_2, \ldots, X_d$ be some indeterminates. Set $O_F\langle X_1^{\pm 1}, \ldots, X_d^{\pm 1}\rangle$ to be the p-adic completion of a d-dimensional torus over O_F and let R denote the p-adic completion of an étale algebra over $O_F\langle X, X^{-1}\rangle$ with non-empty and geometrically integral special fiber. Let G_R denote the étale fundamental group of $R[1/p]$ and Γ_R the Galois group of $R_\infty[1/p]$ over $R[1/p]$ where R_∞ is obtained by adjoining to R all p-power roots of unity and all p-power roots of X_i for all $1 \leq i \leq d$. We have $\Gamma_R \xrightarrow{\sim} \mathbb{Z}_p(1)^d \rtimes \mathbb{Z}_p^\times$ and set $H_R = \mathrm{Ker}\,(G_R \to \Gamma_R)$ (see §3 for precise definitions). Let $O_L = (R_{(p)})^\wedge$ be complete discrete valuation ring with fraction field L and residue field an étale extension of $\kappa(X_1, \ldots, X_d)$. Let G_L denote the absolute Galois group of L such that we have a continuous homomorphism $G_L \to G_R$; let Γ_L denote the Galois group of the L_∞ over L where L_∞ is obtained by adjoining to L all p-power roots of unity and all p-power roots of X_i for all $1 \leq i \leq d$. The homomorphism $G_L \to G_R$ induces a continuous isomorphism $\Gamma_L \xrightarrow{\sim} \Gamma_R$. In the following, working over base R will be referred to as the relative case and working over base O_L will be referred to as the imperfect residue field case. In both these settings, we have respective theories of crystalline representations of G_R [Bri08] and G_L [Bri06] as well as étale (φ, Γ)-modules [And06; AB08].

1.2.1. Relative Wach modules. For $1 \leq i \leq d$, fix $X_i^\flat = (X_i, X_i^{1/p}, \ldots) \in R_\infty^\flat$ and $[X_i^\flat] \in \mathbf{A}_{\mathrm{inf}}(R_\infty)$. Let $\mathbf{A}_R^+$ denote the (p,μ)-adic completion of the unique extension of (p,μ)-adic completion of $O_F[\mu, [X_1^\flat]^{\pm 1}, \ldots, [X_d^\flat]^{\pm 1}]$ along p-adically completed étale map $O_F\langle X_1^{\pm 1}, \ldots, X_d^{\pm 1}\rangle \to R$ (see §1.3 and §3.1). The ring $\mathbf{A}_R^+$ is equipped with a Frobenius endomorphism φ and a continuous action of Γ_R, set $\mathbf{A}_L^+$ to be the (p,μ)-adic completion of $(\mathbf{A}_R^+)_{(p,\mu)}$ equipped with a Frobenius endomorphism φ and a continuous action of Γ_L. We define the following:

Definition 1.3. Let $a, b \in \mathbb{Z}$ with $b \geq a$. A *Wach module* over $\mathbf{A}_R^+$ with weights in the interval $[a, b]$ is a finitely generated $\mathbf{A}_R^+$-module N equipped with a continuous and semilinear action of Γ_R satisfying the following:

(1) The sequences $\{p, \mu\}$ and $\{\mu, p\}$ are regular on N.

(2) The action of Γ_R is trivial on $N/\mu N$.

(3) There is a Frobenius-semilinear operator $\varphi : N[1/\mu] \to N[1/\varphi(\mu)]$ commuting with the action of Γ_R such that $\varphi(\mu^b N) \subset \mu^b N$ and the map $(1 \otimes \varphi) : \varphi^*(\mu^b N) \to \mu^b N$ is injective and its cokernel is killed by $\tilde{\xi}^{b-a} = [p]_q^{b-a}$.

Define the *height* of N to be the smallest value of $b - a$, where $a, b \in \mathbb{Z}$ as above. Say N is *effective* if one can take $b = 0$ and $a \leq 0$. Denote the category of Wach modules over $\mathbf{A}_R^+$ as $(\varphi, \Gamma)\text{-}\mathrm{Mod}_{\mathbf{A}_R^+}^{[p]_q}$ with morphisms between objects being $\mathbf{A}_R^+$-linear, Γ_R-equivariant and φ-equivariant (after inverting μ) morphisms.

Remark 1.4. The condition (1) in Definition 1.3 is rephrased as $\{p, \mu\}$ being strictly N-regular in Definition 3.1. This condition is equivalent to the vanishing of local cohomology of N with respect to the ideal $(p, \mu) \subset \mathbf{A}_R^+$ in degrees 0 and 1 (see Lemma 3.3 and Remark 3.4). Furthermore, one can also show that $N[1/p]$ is finite projective over $\mathbf{A}_R^+[1/p]$ (see Proposition A.1), $N[1/\mu]$ is finite projective over $\mathbf{A}_R^+[1/\mu]$ (see Proposition 3.7) and $N = N[1/p] \cap N[1/\mu] \subset N[1/p, 1/\mu]$ (see Lemma 3.5).

Remark 1.5. Definition 1.3 may be adapted to the imperfect residue field case, i.e. over the ring $\mathbf{A}_L^+$. Then the preceding definition coincides with Definition 2.15, in particular, a Wach module N over $\mathbf{A}_L^+$ is necessarily finite free. Indeed, if N is a Wach module over $\mathbf{A}_L^+$ in the sense of Definition 1.3 then one first observes that N is in fact torsion-free since $N \subset N[1/p]$ and the latter is finite free over $\mathbf{A}_L^+[1/p]$ by [BMS18, Proposition 4.3] (the proof of loc. cit. does not depend on the prefectness of the residue field, in particular, it applies to L, also see Lemma A.2 for handling the difference in Frobenius endomorphism on $\mathbf{A}_L^+$). Then using [Fon90, §B.1.2.4 Proposition] it follows that N is finite free (see Lemma 3.5 and Remark 2.8).

Set $\mathbf{A}_R = \mathbf{A}_R^+[1/\mu]^\wedge$ as the p-adic completion, equipped with a Frobenius endomorphism φ and a continuous action of Γ_R, similarly set $\mathbf{A}_L = \mathbf{A}_L^+[1/\mu]^\wedge$ equipped with a Frobenius endomorphism φ and a continuous action of Γ_L.

Remark 1.6. The category of Wach modules over $\mathbf{A}_R^+$ (resp. $\mathbf{A}_L^+$) can be realized as a full subcategory of étale (φ, Γ)-modules over $\mathbf{A}_R$, see Proposition 2.17 (resp. $\mathbf{A}_L$, see Proposition 3.10).

Let T be a finite free $\mathbb{Z}_p$-module equipped with a continuous action of G_R, then one can functorially attach to T a projective and étale (φ, Γ_R)-module $\mathbf{D}_R(T)$ over $\mathbf{A}_R$ of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$ equipped with a Frobenius-semilinear operator φ and a semilinear and continuous action of Γ_R. In fact, the preceding functor induces an equialence between finite free $\mathbb{Z}_p$-representations of G_R and projective and étale (φ, Γ_R)-modules over $\mathbf{A}_R$. If T is a lattice inside a p-adic crystalline representation of G_R, then we construct Wach module $\mathbf{N}_R(T)$ over $\mathbf{A}_R^+$ functorial in T and contained in $\mathbf{D}_R(T)$, see Theorem 3.21. Similar statements are also true over L, see §2.2 and Theorem 2.24.

Let $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_F)$ denote the category of $\mathbb{Z}_p$-lattices inside p-adic crystalline representations of G_R. The main result of this article is as follows:

Theorem 1.7 (Corollaries 3.22 & 2.25)**.** *The Wach module functor induces an equivalence of $\otimes$-categories*

$$\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_R) \xrightarrow{\sim} (\varphi, \Gamma)\text{-}\mathrm{Mod}_{\mathbf{A}_R^+}^{[p]_q}$$
$$T \longmapsto \mathbf{N}_R(T),$$

with a quasi-inverse given as $N \mapsto \big(W(\overline{R}^\flat[1/p^\flat]) \otimes_{\mathbf{A}_R^+} N\big)^{\varphi=1}$.

Remark 1.8. In Theorem 1.7 note that the functor $\mathbf{N}_R$ is not exact. But, it becomes exact after passing to corresponding isogeny categories, i.e. the induced functor from p-adic crystalline representations of G_R to Wach modules over $\mathbf{B}_R^+$ is exact. We leave the details to the reader.

Remark 1.9. With recent developments in the theory of prismatic F-crystals in [BS21], [DLMS22] and [GR22] it can be seen that the above categorical equivalence implies an equivalence between Wach modules over $\mathbf{A}_R^+$ and analytic/completed prismatic F-crystals on the absolute prismatic site $(\mathrm{Spf}\ R)_{\Delta}$. In fact, it is possible to directly construct the aforementioned equivalence, i.e. without passing through the category $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_R)$. We will report on this in a future work.

As an application of Theorem 1.7, we obtain the following purity statement:

Theorem 1.10 (Theorem 3.24)**.** *Let V be a p-adic representation of G_R. Then V is crystalline as a representation of G_R if and only if it is crystalline as a representation of G_L.*

Remark 1.11. A similar purity statement has been obtained in [Moo22, Theorem 1.4] using the results of [DLMS22].

1.2.2. Strategy for the proof of Theorem 1.7. Proof of the relative case depends on results obtained in the imperfect residue field case, which can be shown independently. In the imperfect residue field case, starting with a Wach module N over $\mathbf{A}_L^+$, we use ideas developed in [Abh21] to show that N gives rise to a $\mathbb{Z}_p$-lattice $\mathbf{T}_L(N)$ inside a p-adic crystalline representation of G_L (see Proposition 2.20). In the opposite direction, starting with a $\mathbb{Z}_p$-lattice T inside a p-adic crystalline representation of G_L, we use Kisin's ideas from [Kis06] and the results of Wach [Wac96] and Berger [Ber04] in the perfect residue field case to construct a finite free $\mathbf{A}_L^+$-module $\mathbf{N}_L(T)$ of finite $[p]_q$-height. However, the action of Γ_L is not immediate from these constructions. So we use the Galois action on $T[1/p]$ and its crystalline property together with (φ, Γ)-module theory to construct a natural action of Γ_L on $\mathbf{N}_L(T)$ (see Theorem 2.24).

In the relative case, starting with a Wach module N over $\mathbf{A}_R^+$, we observe that $\mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} N$ is a Wach module over $\mathbf{A}_L^+$ and use the corresponding statement from the imperfect residue field case to prove non-trivial statements for N which help in establising that $\mathbf{T}_R(N)$ is a $\mathbb{Z}_p$-lattice inside a p-adic crystalline representation of G_R (see Theorem 3.17). For the converse, starting with a $\mathbb{Z}_p$-lattice T inside a p-adic crystalline representation of G_R, first we observe that $T[1/p]$ is a p-adic crystalline representation of G_L and use the equivalence in imperfect residue field case to obtain the Wach module $\mathbf{N}_L(T)$ over $\mathbf{A}_L^+$. Then from the theory of (φ, Γ)-modules we have an étale (φ, Γ_R)-module $\mathbf{D}_R(T)$ over $\mathbf{A}_R$. We define an $\mathbf{A}_R^+$-module as $\mathbf{N}_R(T) = \mathbf{N}_L(T) \cap \mathbf{D}_R(T) \subset \mathbf{D}_L(T)$ which is naturally equipped with a Frobenius-semilinear endomorphism φ and a continuous action of Γ_R. To show that $\mathbf{N}_R(T)$ is a Wach module, we use some ideas from [DLMS22] to deduce some properties of $\mathbf{N}_R(T)$. Using these properties we show that $\mathbf{N}_R(T)$ satisfies all the desired axioms, thus proving Theorem 1.7 (see Theorem 3.21).

1.3. Setup and notations.

In this section we describe our setup and fix some notations which are essentially the same as in [Abh21, §1.4].

Convention. We will work under the convention that $0 \in \mathbb{N}$, the set of natural numbers.

Let p be a fixed prime number, κ a finite field of characteristic p, $W := W(\kappa)$ the ring of p-typical Witt vectors with coefficients in κ and $F := W[1/p]$, the fraction field of W. In particular, F is an unramified extension of $\mathbb{Q}_p$ with ring of integers $O_F = W$. Let $\overline{F}$ be a fixed algebraic closure of F so that its residue field, denoted as $\overline{\kappa}$, is an algebraic closure of κ. Further, we denote by $G_F = \mathrm{Gal}(\overline{F}/F)$, the absolute Galois group of F.

Let $Z = (Z_1, \ldots, Z_s)$ denote a set of indeterminates and $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}^s$ be a multi-index, then we write $Z^{\mathbf{k}} := Z_1^{k_1} \cdots Z_s^{k_s}$. For $\mathbf{k} \to +\infty$ we will mean that $\sum k_i \to +\infty$. Now for a topological algebra Λ we define

$$\Lambda\langle Z\rangle := \Big\{ \sum_{\mathbf{k} \in \mathbb{N}^s} a_{\mathbf{k}} Z^{\mathbf{k}}, \text{ where } a_{\mathbf{k}} \in \Lambda \text{ and } a_{\mathbf{k}} \to 0 \text{ as } \mathbf{k} \to +\infty \Big\}.$$

We fix $d \in \mathbb{N}$ and let $X = (X_1, X_2, \ldots, X_d)$ be some indeterminates. Let R be the p-adic completion of an étale algebra over $O_F\langle X, X^{-1}\rangle$ with non-empty geometrically integral special fiber. In particular, we have a presentation

$$R = O_F\langle X, X^{-1}\rangle\langle Z_1, \ldots, Z_s\rangle/(Q_1, \ldots, Q_s), \tag{1.1}$$

where $Q_i(Z_1, \ldots, Z_s) \in O_F\langle X, X^{-1}\rangle[Z_1, \ldots, Z_s]$ for $1 \leq i \leq s$ are multivariate polynomials such that $\det\big(\frac{\partial Q_i}{\partial Z_j}\big)_{1 \leq i,j \leq s}$ is invertible in R.

We fix an algebraic closure $\overline{\mathrm{Frac}(R)}$ of $\mathrm{Frac}(R)$ containing $\overline{F}$. Let $\overline{R}$ denote the union of finite R-subalgebras $S \subset \overline{\mathrm{Frac}(R)}$, such that $S[1/p]$ is étale over $R[1/p]$. Let $\overline{\eta}$ denote the fixed geometric point of the generic fiber $\mathrm{Spec}\, R[1/p]$ (defined by $\overline{\mathrm{Frac}(R)}$) and let $G_R := \pi_1^{\text{ét}}(\mathrm{Spec}\, R[1/p], \overline{\eta})$ denote the étale fundamental group. We can write this étale fundamental group as the Galois group (of the fraction field of $\overline{R}[1/p]$ over the fraction field of $R[1/p]$)

$$G_R = \pi_1^{\text{ét}}(\mathrm{Spec}\, R[\tfrac{1}{p}], \overline{\eta}) = \mathrm{Gal}\big(\overline{R}[\tfrac{1}{p}]/R[\tfrac{1}{p}]\big).$$

For $k \in \mathbb{N}$, let Ω^k_R denote the p-adic completion of module of k-differentials of R relative to $\mathbb{Z}$. Then, we have

$$\Omega^1_R = \oplus_{i=1}^d R\, d\log X_i, \text{ and } \Omega^k_R = \wedge^k_R \Omega^1_R.$$

Let $\varphi : O_F\langle X, X^{-1}\rangle \to O_F\langle X, X^{-1}\rangle$ denote a morphism extending the natural Frobenius on O_F by setting $\varphi(X_i) = X_i^p$ for all $1 \leq i \leq d$. The endomorphism φ of $O_F\langle X, X^{-1}\rangle$ is flat by [Bri08, Lemma 7.1.5] and faithfully flat since $\varphi(\mathfrak{m}) \subset \mathfrak{m}$ for any maximal ideal $\mathfrak{m} \subset O_F\langle X, X^{-1}\rangle$. Moreover, it is finite of degree p^d using Nakayama Lemma and the fact that φ modulo p is evidently of degree p^d. Recall that the O_F-algebra R is given as p-adic completion of an étale algebra $O_F\langle X, X^{-1}\rangle$, therefore the Frobenius endomorphism φ on $O_F\langle X, X^{-1}\rangle$ admits a unique extension $\varphi : R \to R$ such that the induced map $\varphi : R/p \to R/p$ is the absolute Frobenius $x \mapsto x^p$ (see [CN17, Proposition 2.1]). Similar to above, again note that the endomorphism $\varphi : R \to R$ is finite and faithfully flat of degree p^d.

Let $O_L = (R_{(p)})^\wedge$ where $^\wedge$ denotes the p-adic completion. The R-algebra O_L is a complete discrete valuation ring with uniformizer p and let $L = O_L[1/p]$ denote its fraction field. Fix an algebraic closure $\overline{L}/L$, with ring of integers $O_{\overline{L}}$ such that we have an injective map $\overline{R} \to O_{\overline{L}}$. Let $G_L = \mathrm{Gal}(\overline{L}/L)$ denote the absolute Galois group of L and we have an induced continuous morphism $G_L \to G_R$. The Frobenius on R extends to a unique finite and faithfuly flat of degree p^d Frobenius endomorphism $\varphi : O_L \to O_L$ lifting the absolute Frobenius on O_L/pO_L.

2. Imperfect residue field case

We will use the setup and notations from §1.3. Recall that R is the p-adic completion of an étale algebra over $O_F\langle X_1^{\pm 1}, \ldots, X_d^{\pm 1}\rangle$ and $O_L = (R_{(p)})^\wedge$. Set $L_\infty = \cup_{i=1}^d L(\mu_{p^\infty}, X_i^{1/p^\infty})$ and for $1 \leq i \leq d$, we fix $X_i^\flat = (X_i, X_i^{1/p}, X_i^{1/p^2}, \ldots) \in O^\flat_{L_\infty}$. Then we have the following Galois groups (see [Hyo86, §1.1] for details)

$$\begin{aligned}&G_L = \mathrm{Gal}(\overline{L}/L),\ H_L = \mathrm{Gal}(\overline{L}/L_\infty),\ \Gamma_L = G_L/H_L = \mathrm{Gal}(L_\infty/L) \xrightarrow{\sim} \mathbb{Z}_p(1)^d \rtimes \mathbb{Z}_p^\times,\\ &\Gamma'_L = \mathrm{Gal}(L_\infty/L(\mu_{p^\infty})) \xrightarrow{\sim} \mathbb{Z}_p(1)^d,\ \mathrm{Gal}(L(\mu_{p^\infty})/L) = \Gamma_L/\Gamma'_L \xrightarrow{\sim} \mathbb{Z}_p^\times.\end{aligned}$$

Let $O_{\breve{L}} = (\cup_{i=1}^d O_L[X_i^{1/p^\infty}])^\wedge$, where $^\wedge$ denotes the p-adic completion. The O_L-algebra $O_{\breve{L}}$ is a complete discrete valuation ring with perfect residue field, uniformizer p and fraction field $\breve{L} = O_{\breve{L}}[1/p]$. The Witt vector Frobenius on $O_{\breve{L}}$ is given by the Frobenius on O_L described in §1.3 and setting $\varphi(X_i^{1/p^n}) = X_i^{1/p^{n-1}}$ for all $1 \leq i \leq d$ and $n \in \mathbb{N}$. Let $\breve{L}_\infty = \breve{L}(\mu_{p^\infty})$ and let $\overline{\breve{L}} \supset \overline{L}$ denote a fixed algebraic closure of $\breve{L}$. We have

$$\begin{aligned}&G_{\breve{L}} = \mathrm{Gal}(\overline{\breve{L}}/\breve{L}) \xrightarrow{\sim} \mathrm{Gal}(\overline{L}/\cup_{i=1}^d L(X_i^{1/p^\infty})),\ H_{\breve{L}} = \mathrm{Gal}(\overline{\breve{L}}/\breve{L}_\infty) = \mathrm{Gal}(\overline{L}/L_\infty),\\ &\Gamma_{\breve{L}} = G_{\breve{L}}/H_{\breve{L}} = \mathrm{Gal}(\breve{L}_\infty/\breve{L}) \xrightarrow{\sim} \mathrm{Gal}(L_\infty/\cup_{i=1}^d L(X_i^{1/p^\infty})) \xrightarrow{\sim} \mathrm{Gal}(L(\mu_{p^\infty})/L) \xrightarrow{\sim} \mathbb{Z}_p^\times.\end{aligned}$$

From the description above note that $G_{\breve{L}}$ can be identified with a subgroup of G_L, $H_{\breve{L}} \xrightarrow{\sim} H_L$ and $\Gamma_{\breve{L}}$ can be identified with a quotient of Γ_L.

2.1. Period rings.

In this section we will quickly recall and fix notations for the period rings to be used in the rest of this section. For details refer to [And06], [Bri06] and [Ohk13].

2.1.1. Crystalline period rings.

Let $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty}) := W(O^\flat_{L_\infty})$ and $\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}}) := W(O^\flat_{\overline{L}})$ admitting the Frobenius on Witt vectors and continuous G_L-action (for the weak topology). Moreover, we have $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty}) = \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})^{H_L}$. We fix $\overline{\mu} := \varepsilon - 1$, where $\varepsilon := (1, \zeta_p, \zeta_{p^2}, \ldots) \in O^\flat_{F_\infty}$ and let $\mu := [\varepsilon] - 1, \xi := \mu/\varphi^{-1}(\mu) \in \mathbf{A}_{\mathrm{inf}}(O_{F_\infty})$. Recall that for $1 \leq i \leq d$, we fixed $X_i^\flat = (X_i, X_i^{1/p}, X_i^{1/p^2}, \ldots) \in O_{L_\infty}$ and let us fix Teichmüller lifts $[X_i^\flat] \in \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$. We set $\mathbf{A}_{\mathrm{cris}}(O_{L_\infty}) := \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})\langle \xi^k/k!, k \in \mathbb{N}\rangle$. Let $t := \log(1+\mu) \in \mathbf{A}_{\mathrm{cris}}(O_{F_\infty})$ and set $\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) := \mathbf{A}_{\mathrm{cris}}(O_{L_\infty})[1/p]$ and $\mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) := \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty})[1/t]$. Furthermore, one can define period rings $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$, $\mathcal{O}\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty})$ and $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{L_\infty})$. These rings are equipped with a Frobenius endomorphism φ and continuous Γ_L-action. Rings with a subscript "cris" are equipped with a decreasing filtration and rings with a prefix "$\mathcal{O}$" are further equipped with an integrable connection satisfying Griffiths transversality with respect to the filtration (see [Abh21, §2.2] for definitions over R with similar notations). One can define variations of these rings over $\overline{L}$ as well.

We have two O_L-algebra structures on $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{\overline{L}})$: a canonical structure coming from the definition of $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{\overline{L}})$; a non-canonical structure $O_L \to \mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{\overline{L}})$ given by the map $x \mapsto \sum_{\mathbf{k}\in\mathbb{N}^d} \prod_{i=1}^d \partial_i^{k_i}(x) \prod_{i=1}^d ([X_i^\flat] - X_i)^{[k_i]}$, in particular $X_i \mapsto [X_i^\flat]$.

2.1.2. Rings of (φ, Γ)-modules. For details see [And06]. Let $O_\square = (O_F[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]_{(p)})^\wedge$ where $^\wedge$ denotes the p-adic completion. We have an injective map $O_\square \to O_L$ given as p-adic completion of an étale algebra over $O_\square$, in particular, O_L admits a presentation over $O_\square$ using the same multivariate polynomials appearing in (1.1). Let $A_\square^+$ denote the (p,μ)-adic completion of the localization $O_F[\mu, [X_1^\flat]^{\pm 1}, \ldots, [X_d^\flat]^{\pm 1}]_{(p,\mu)}$. We have a natural embedding $A_\square^+ \subset \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ and $A_\square^+$ is stable under the induced Frobenius and Γ_L-action. Moreover, we have an embedding $\iota : O_\square \to A_\square^+$ via the map $X_i \mapsto [X_i^\flat]$ and it extends to an isomorphism of rings $O_\square[\![\mu]\!] \xrightarrow{\sim} A_\square^+$.

Let $\mathbf{A}_L^+$ denote the (p,μ)-adic completion of the unique extension of the embedding $A_\square^+ \to \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ along the p-adically completed étale map $O_\square \to O_L$ (see [CN17, Proposition 2.1]). In particular, the embedding $\iota : O_\square \to A_\square^+$ and the isomorphism $O_\square[\![\mu]\!] \xrightarrow{\sim} A_\square^+$ extend to a unique embedding $\iota : O_L \to \mathbf{A}_L^+$ and an isomorphism of rings $O_L[\![\mu]\!] \xrightarrow{\sim} \mathbf{A}_L^+$. We have a natural embedding $\mathbf{A}_L^+ \subset \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ and $\mathbf{A}_L^+$ is stable under the induced Frobenius and Γ_L-action. Equip $O_L[\![\mu]\!]$ with a finite and faithfully flat of degree p^{d+1} Frobenius endomorphism using the Frobenius on O_L and setting $\varphi(\mu) = (1+\mu)^p - 1$. Then the embedding ι and the isomorphism $O_L[\![\mu]\!] \xrightarrow{\sim} \mathbf{A}_L^+$ are Frobenius-equivariant. In particular, the Frobenius $\varphi : \mathbf{A}_L^+ \to \mathbf{A}_L^+$ is finite and faithfully flat of degree p^{d+1}. Let $u_\alpha = (1+\mu)^{\alpha_0}[X_1^\flat]^{\alpha_1} \cdots [X_d^\flat]^{\alpha_d}$ where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \{0, 1, \ldots, p-1\}^{[0,d]}$ then we have $\varphi^*(\mathbf{A}_L^+) = \mathbf{A}_L^+ \otimes_{\varphi, \mathbf{A}_L^+} \mathbf{A}_L^+ = \oplus_\alpha \varphi(\mathbf{A}_L^+) u_\alpha$.

Let $\mathbb{C}_L := \widehat{\overline{L}}$, $\tilde{\mathbf{A}} := W(\mathbb{C}_L^\flat)$ and $\tilde{\mathbf{B}} := \tilde{\mathbf{A}}[1/p]$ admitting the Frobenius on Witt vectors and continuous G_L-action (for the weak topology). Set $\mathbf{A}_L := \mathbf{A}_L^+[1/\mu]^\wedge$ equipped with induced Frobenius endomorphism and continuous Γ_L-action. Note that $\mathbf{A}_L$ is a two dimensional regular local ring and $\mathbf{B}_L = \mathbf{A}_L[1/p]$ is its fraction field. Similar to above, $\varphi : \mathbf{A}_L \to \mathbf{A}_L$ is finite and faithfully flat of degree p^{d+1} and we can write $\varphi^*(\mathbf{A}_L) = \mathbf{A}_L \otimes_{\varphi, \mathbf{A}_L} \mathbf{A}_L = \oplus_\alpha \varphi(\mathbf{A}_L) u_\alpha = (\oplus_\alpha \varphi(\mathbf{A}_L^+) u_\alpha) \otimes_{\varphi(\mathbf{A}_L^+)} \varphi(\mathbf{A}_L) = \mathbf{A}_L^+ \otimes_{\varphi, \mathbf{A}_L^+} \mathbf{A}_L$. Furthermore, we have a natural Frobenius and Γ_L-equivariant embedding $\mathbf{A}_L \subset \tilde{\mathbf{A}}^{H_L}$. Let $\mathbf{A}$ denote the p-adic completion of the maximal unramified extension of $\mathbf{A}_L$ inside $\tilde{\mathbf{A}}$ and set $\mathbf{B} := \mathbf{A}[1/p] \subset \tilde{\mathbf{B}}$, i.e. $\mathbf{A}$ is the ring of integers of $\mathbf{B}$. The rings $\mathbf{A}$ and $\mathbf{B}$ are stable under induced Frobenius and G_L-action and we have $\mathbf{A}_L = \mathbf{A}^{H_L}$ and $\mathbf{B}_L = \mathbf{B}^{H_L}$ stable under induced Frobenius and residual Γ_L-action.

2.1.3. Overconvergent rings. We begin by definining the ring of overconvergent coefficients stable under Frobenius and G_L-action (see [CC98] and [AB08]) Denote the natural valuation on $O_{\overline{L}}^\flat$ by $v^\flat$ extending the valuation on $O_{\overline{F}}^\flat$. Let $r > 0$ and let $\alpha \in O_{\overline{F}}^\flat$ such that $v^\flat(\alpha) = pr/(p-1)$. Set

$$\tilde{\mathbf{A}}^{\dagger,r} := \Big\{ \sum_{k\in\mathbb{N}} p^k[x_k] \in \tilde{\mathbf{A}},\ v^\flat(x_k) + \tfrac{pr}{p-1}k \to +\infty \text{ as } k \to +\infty \Big\}.$$

The G_L-action and Frobenius φ on $\tilde{\mathbf{A}}$ induce commuting actions of G_L and φ on $\tilde{\mathbf{A}}^{\dagger,r}$ such that $\varphi(\tilde{\mathbf{A}}^{\dagger,r}) = \tilde{\mathbf{A}}^{\dagger,pr}$. Define the ring of *overconvergent coefficients* as $\tilde{\mathbf{A}}^\dagger := \cup_{r\in\mathbb{Q}_{>0}} \tilde{\mathbf{A}}^{\dagger,r} \subset \tilde{\mathbf{A}}$ equipped with induced Frobenius and G_L-action. Moreover, inside $\tilde{\mathbf{A}}$ we take $\mathbf{A}_L^{\dagger,r} := \mathbf{A}_L \cap \tilde{\mathbf{A}}^{\dagger,r}$ and $\mathbf{A}^{\dagger,r} := \mathbf{A} \cap \tilde{\mathbf{A}}^{\dagger,r}$. Define $\mathbf{A}_L^\dagger := \mathbf{A}_L \cap \mathbf{A}_L^\dagger = \cup_{r\in\mathbb{Q}_{>0}} \mathbf{A}_L^{\dagger,r}$ and $\mathbf{A}^\dagger := \mathbf{A} \cap \tilde{\mathbf{A}}^\dagger = \cup_{r\in\mathbb{Q}_{>0}} \mathbf{A}^{\dagger,r}$ equipped with induced Frobenius endomorphism and G_L-action from respective actions on $\tilde{\mathbf{A}}$;

we have $\mathbf{A}_L^\dagger = (\mathbf{A}^\dagger)^{H_L}$. Upon inverting p in the definitions above one obtains $\mathbb{Q}_p$-algebras inside $\tilde{\mathbf{B}}$, i.e. set $\tilde{\mathbf{B}}^{\dagger,r} = \tilde{\mathbf{A}}^{\dagger,r}[1/p]$, $\tilde{\mathbf{B}}^\dagger = \tilde{\mathbf{A}}^\dagger[1/p]$, $\mathbf{B}^{\dagger,r} = \mathbf{A}^{\dagger,r}[1/p]$, $\mathbf{B}^\dagger = \mathbf{A}^\dagger[1/p]$, equipped with induced Frobenius and G_L-action. Moreover, set $\tilde{\mathbf{B}}_L^{\dagger,r} = (\tilde{\mathbf{B}}^{\dagger,r})^{H_L}$, $\tilde{\mathbf{B}}_L^\dagger = (\tilde{\mathbf{B}}^\dagger)^{H_L}$, $\mathbf{B}_L^{\dagger,r} = (\mathbf{B}^{\dagger,r})^{H_L} = \mathbf{A}_L^{\dagger,r}[1/p]$ and $\mathbf{B}_L^\dagger = (\mathbf{B}^{\dagger,r})^{H_L} = \mathbf{A}_L^\dagger[1/p]$ equipped with induced Frobenius and residual Γ_L-action.

2.1.4. Analytic rings. In this section, we will define the Robba ring over L following [Ked05, §2] and [Ohk15, §1]. However, we will use the notations of [Ber02, §2] in the perfect residue field case (see [Ohk15, §1.10] for compatibility between different notations). Define

$$\tilde{\mathbf{B}}_{\mathrm{rig}}^\dagger := \cup_{r\geq 0} \cap_{s\geq r} \left(\mathbf{A}_{\inf}(O_{\overline{L}})\langle \tfrac{p}{[\overline{\mu}]^r}, \tfrac{[\overline{\mu}]^s}{p}\rangle\left[\tfrac{1}{p}\right]\right).$$

The ring $\tilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$ can also be defined as $\cup_{r\in\mathbb{Q}_{>0}}\tilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$ where $\tilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$ denotes the Fréchet completion of $\tilde{\mathbf{B}}^{\dagger,r} = \tilde{\mathbf{A}}^{\dagger,r}[1/p]$ for a certain family of valuations (see [Ked05, §2] and [Ohk15, §1.6]). The Frobenius and G_L-action on $\tilde{\mathbf{B}}^{\dagger,r}$ respectively induce Frobenius and G_L-action on $\tilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$ which extend to respective actions on $\tilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$. In particular, we have a Frobenius and G_L-equivariant inclusion $\tilde{\mathbf{B}}^\dagger \subset \tilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$ (see [Ohk15, §1.6 & §1.10]). Set

$$\tilde{\mathbf{B}}_{\mathrm{rig}}^+ = \cap_{n\in\mathbb{N}}\varphi^n(\mathbf{B}_{\mathrm{cris}}^+(O_{\overline{L}}))$$

equipped with an induced Frobenius endomorphism and G_L-action from the respective actions on $\mathbf{B}_{\mathrm{cris}}^+(O_{\overline{L}})$. Description of rings in [Ber02, Lemme 2.5, Exemple 2.8 & §2.3] directly extend to our situation as the aforementioned results do not depend on structure of the residue field of base ring O_L. Therefore, from loc. cit. it follows that $\tilde{\mathbf{B}}_{\mathrm{rig}}^+ \subset \tilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$ compatible with Frobenius and G_L-action. Moreover, we set $\tilde{\mathbf{B}}_{\mathrm{rig},L}^{\dagger,r} = (\tilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r})^{H_L}$, $\tilde{\mathbf{B}}_{\mathrm{rig},L}^\dagger = (\tilde{\mathbf{B}}_{\mathrm{rig}}^\dagger)^{H_L}$ and $\tilde{\mathbf{B}}_{\mathrm{rig},L}^+ = (\tilde{\mathbf{B}}_{\mathrm{rig}}^+)^{H_L} \subset \tilde{\mathbf{B}}_{\mathrm{rig},L}^\dagger$ equipped with induced Frobenius endomorphism and residual Γ_L-action. Furthermore, [Ber02, Lemma 2.18, Corollaire 2.28] are independent of structure of residue field of O_L. So we have the following description of $\tilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$:

Lemma 2.1. *Let $\mathbf{B}_{\inf}(O_{\overline{L}}) = \mathbf{A}_{\inf}(O_{\overline{L}})[1/p]$, then we have an exact sequence*

$$0 \longrightarrow \mathbf{B}_{\inf}(O_{\overline{L}}) \longrightarrow \tilde{\mathbf{B}}^{\dagger,r} \oplus \tilde{\mathbf{B}}_{\mathrm{rig}}^+ \longrightarrow \tilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r} \longrightarrow 0.$$

Similar statement also holds over L_∞, i.e. we have an exact sequence

$$0 \longrightarrow \mathbf{B}_{\inf}(O_{L_\infty}) \longrightarrow \tilde{\mathbf{B}}_L^{\dagger,r} \oplus \tilde{\mathbf{B}}_{\mathrm{rig},L}^+ \longrightarrow \tilde{\mathbf{B}}_{\mathrm{rig},L}^{\dagger,r} \longrightarrow 0.$$

Recall that from §2.1.2 we have a Frobenius-equivariant embedding $\iota : O_L \to \mathbf{A}_L^+$. From [Ohk15, §1.6] the ring $\mathbf{A}_L^{\dagger,r}$ has the following description

$$\mathbf{A}_L^{\dagger,r} \xrightarrow{\sim} \Big\{\sum_{k\in\mathbb{Z}} \iota(a_k)\mu^k \text{ such that } a_k \in O_L \text{ and for any } p^{-1/r} \leq \rho < 1, \lim_{k\to-\infty} |a_k|\rho^k = 0\Big\}.$$

We have $\mathbf{B}_L^{\dagger,r} = \mathbf{A}_L^{\dagger,r}[1/p]$ and we set

$$\mathbf{B}_{\mathrm{rig},L}^{\dagger,r} := \Big\{\sum_{k\in\mathbb{Z}} \iota(a_k)\mu^k \text{ such that } a_k \in L \text{ and for any } p^{-1/r} \leq \rho < 1, \lim_{k\to\pm\infty} |a_k|\rho^k = 0\Big\}.$$

The ring $\mathbf{B}_{\mathrm{rig},L}^{\dagger,r}$ can also be defined as Fréchet completion of $\mathbf{B}_L^{\dagger,r}$ for a family of valuations induced by the inclusion $\mathbf{B}_L^{\dagger,r} \subset \tilde{\mathbf{B}}^{\dagger,r}$ (see [Ked05, §2] and [Ohk15, §1.6]). Define the *Robba ring* over L as $\mathbf{B}_{\mathrm{rig},L}^\dagger := \cup_{r\geq 0}\mathbf{B}_{\mathrm{rig},L}^{\dagger,r}$. The Frobenius and G_L-action on $\mathbf{B}_L^{\dagger,r}$ induce respective Frobenius and G_L-action on $\mathbf{B}_{\mathrm{rig},L}^{\dagger,r}$ which extend to respective actions on $\mathbf{B}_{\mathrm{rig},L}^\dagger$ (also see [Ohk15, §4.3] where

Ohkubo constructs the differential action of Lie Γ_L; one may also obtain the action of Γ_L by exponentiating the action of Lie Γ_L). From the preceding discussion, we have a Frobenius and Γ_L-equivariant injection $\mathbf{B}_L^{\dagger} \subset \mathbf{B}_{\mathrm{rig},L}^{\dagger}$ and the former ring $\mathbf{B}_L^{\dagger}$ is also known as the *bounded Robba ring*. Furthermore, note that $\mathbf{B}_L^{\dagger,r} \subset \tilde{\mathbf{B}}_L^{\dagger,r} = (\tilde{\mathbf{B}}^{\dagger,r})^{H_L} \subset \tilde{\mathbf{B}}_{\mathrm{rig},L}^{\dagger,r}$, where the last term can also be described as the Fréchet completion of the middle term for a family of valuations induced by the inclusion $\tilde{\mathbf{B}}_L^{\dagger,r} \subset \tilde{\mathbf{B}}^{\dagger,r}$ (see [Ked05, §2] and [Ohk15, §1.6]).

To summarize, for $r \in \mathbb{Q}_{>0}$ we have the following commutative diagram with injective arrows

$$\begin{tikzcd}
\mathbf{B}^{\dagger,r} \arrow[r] & \tilde{\mathbf{B}}^{\dagger,r} \arrow[dr] & \\
\mathbf{B}_L^{\dagger,r} \arrow[u] \arrow[r] \arrow[d] & \tilde{\mathbf{B}}_L^{\dagger,r} \arrow[u] \arrow[r] \arrow[d] & \tilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r} \\
\mathbf{B}_{\mathrm{rig},L}^{\dagger,r} \arrow[r] & \tilde{\mathbf{B}}_{\mathrm{rig},L}^{\dagger,r}, \arrow[ur] &
\end{tikzcd}$$

where in the second row, two rings on the left row are obtained from the rings in first row by taking H_L-invariants and the rightmost ring is obtained as Fréchet completion of the rightmost ring in first row. The bottom row is obtained as Fréchet completion of two rings on the left in the second row. These inclusions are compatible with Frobenius and Γ_L-action and these compatibilities are preserved after passing to respective Fréchet completions. In particular, we get a Frobenius and Γ_L-equivariant embedding $\mathbf{B}_{\mathrm{rig},L}^{\dagger} \subset \tilde{\mathbf{B}}_{\mathrm{rig},L}^{\dagger}$.

Definition 2.2. Define $\mathbf{B}_{\mathrm{rig},L}^{+} := \mathbf{B}_{\mathrm{rig},L}^{\dagger} \cap \tilde{\mathbf{B}}_{\mathrm{rig},L}^{+} \subset \tilde{\mathbf{B}}_{\mathrm{rig},L}^{\dagger}$, equipped with induced Frobenius endomorphism and Γ_L-action.

Lemma 2.3. *The ring $\mathbf{B}_{\mathrm{rig},L}^{+}$ can be identified with the ring of convergent power series over the open unit disk in one variable over L, i.e.*

$$\mathbf{B}_{\mathrm{rig},L}^{+} \xrightarrow{\sim} \Big\{ \sum_{k \in \mathbb{N}} \iota(a_k)\mu^k \textit{ such that } a_k \in L \textit{ and for any } 0 \leq \rho < 1, \lim_{k \to +\infty} |a_k|\rho^k = 0 \Big\},$$

Proof. Let $x \in \mathbf{B}_{\mathrm{rig},L}^{\dagger}$. Using the explicit description of $\mathbf{B}_{\mathrm{rig},L}^{\dagger,r}$ and $\mathbf{B}_L^{\dagger,r}$ for $r \in \mathbb{Q}_{>0}$, we can write $x = x^+ + x^-$ with x^+ convergent on the open unit disk over L and $x^- \in \mathbf{B}_L^{\dagger}$, in particular, $x^+ \in (\tilde{\mathbf{B}}_{\mathrm{rig}}^{+})^{H_L} = \tilde{\mathbf{B}}_{\mathrm{rig},L}^{+}$. From Lemma 2.1, $x \in \mathbf{B}_{\mathrm{rig},L}^{+} \subset \tilde{\mathbf{B}}_{\mathrm{rig},L}^{+}$ if and only if $x^- \in \mathbf{B}_{\mathrm{inf}}(O_{L_\infty}) \cap \mathbf{B}_L^{\dagger} = \mathbf{B}_L^{+}$, i.e. x converges on the open unit disk over L. The other inclusion is obvious. ∎

Remark 2.4. From §1.3 recall that $\varphi : L \to L$ is finite of degree p^d and we also have $\varphi(\mu) = (1+\mu)^p - 1$. Therefore, from the explicit description of $\mathbf{B}_{\mathrm{rig},L}^{+}$ in Lemma 2.3 it follows that the Frobenius endomorphism $\varphi : \mathbf{B}_{\mathrm{rig},L}^{+} \to \mathbf{B}_{\mathrm{rig},L}^{+}$ is finite and faithfully flat of degree p^{d+1}.

2.1.5. Period rings for $\breve{L}$. Definitions above may be adopted almost verbatim to define corresponding period rings for $\breve{L}$, in particular, one recovers definitions of period rings in [Fon90], [CC98] and [Ber02], in particular, one obtain period rings $\mathbf{A}_{\breve{L}}^{+}$, $\mathbf{A}_{\breve{L}}$, $\mathbf{A}_{\breve{L}}^{\dagger}$, $\mathbf{B}_{\mathrm{rig},\breve{L}}^{+}$ and $\mathbf{B}_{\mathrm{rig},\breve{L}}^{\dagger}$ equipped with a Frobenius endomorphism φ and $\Gamma_{\breve{L}}$-action. Note that we have a natural identification $\mathbf{A}_{\breve{L}}^{+} \xrightarrow{\sim} O_{\breve{L}}[\![\mu]\!]$ where the right hand side is equipped with a finite and faithfully flat of degree p Frobenius endomorphism using the natural Frobenius on $O_{\breve{L}}$ and setting $\varphi(\mu) = (1+\mu)^p - 1$ and a $\Gamma_{\breve{L}}$-action given as $g(\mu) = (1+\mu)^{\chi(g)} - 1$ for $g \in \Gamma_{\breve{L}}$. Moreover, the preceding isomorphism naturally extends to a Frobenius and $\Gamma_{\breve{L}}$-equivariant isomorphism $\mathbf{A}_{\breve{L}}^{+} \xrightarrow{\sim} O_{\breve{L}}[\![\mu]\!][1/\mu]^\wedge$, where $^\wedge$ denotes the p-adic completion.

Recall that we have a Frobenius-equivariant faithfully flat embedding $O_L \to O_{\breve{L}}$ and it naturally extends to a Frobenius and $\Gamma_{\breve{L}}$-equivariant faithfully flat embedding $O_L[\![\mu]\!] \to O_{\breve{L}}[\![\mu]\!]$.

Using Frobenius and $\Gamma_{\breve{L}}$-equivariant isomorphisms $\mathbf{A}_L^+ \xrightarrow{\sim} O_L[\![\mu]\!]$ and $\mathbf{A}_{\breve{L}} \xrightarrow{\sim} O_{\breve{L}}[\![\mu]\!]$ we obtain a Frobenius and $\Gamma_{\breve{L}}$-equivariant faithfully flat embedding $\mathbf{A}_L^+ \to \mathbf{A}_{\breve{L}}^+$ sending $[X_i^\flat] \mapsto X_i$. Inverting μ and completing p-adically we obtain a Frobenius and $\Gamma_{\breve{L}}$-equivariant faithfully flat embedding $\mathbf{A}_L \to \mathbf{A}_{\breve{L}}$.

We can equip $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ with a non-canonical O_L-algebra structure by first defining an injection $O_\square \to \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ via the map $X_i \mapsto [X_i^\flat]$ and extending it uniquely along the p-adically complete étale map $O_\square \to O_L$, to an injection $O_L \to \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$. Note that the preceding maps are Frobenius-equivariant but not Γ_L-equivariant. The O_L-algebra structure on $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ naturally extends to a Frobenius-equivariant $O_{\breve{L}}$-algebra structure by sending $X_i^{1/p^n} \mapsto [(X_i^{1/p^n})^\flat]$ for all $1 \leq i \leq d$ and $n \in \mathbb{N}$. We can further extend this to a Frobenius and $\Gamma_{\breve{L}}$-equivariant embedding $\mathbf{A}_{\breve{L}}^+ = O_{\breve{L}}[\![\mu]\!] \to \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$.

Using the embeddings described above and following the definitions of various period rings discussed so far, we obtain a commutative diagram with injective arrows

$$\begin{tikzcd}
\tilde{\mathbf{B}}^+_{\mathrm{rig},L} & \mathbf{B}^+_{\mathrm{rig},L} \arrow[l] \arrow[r] \arrow[d] & \mathbf{B}^\dagger_{\mathrm{rig},L} \arrow[r] \arrow[d] & \tilde{\mathbf{B}}^\dagger_{\mathrm{rig},L} \\
& \mathbf{B}^+_{\mathrm{rig},\breve{L}} \arrow[ul] \arrow[r] & \mathbf{B}^\dagger_{\mathrm{rig},\breve{L}}, \arrow[ur] &
\end{tikzcd}$$

where the top horizontal arrows are Frobenius and Γ_L-equivariant and the rest are Frobenius and $\Gamma_{\breve{L}}$-equivariant.

Remark 2.5. Similar to Lemma 2.3 we have

$$\mathbf{B}^+_{\mathrm{rig},\breve{L}} \xrightarrow{\sim} \Big\{ \sum_{k \in \mathbb{N}} a_k \mu^k \text{ such that } a_k \in \breve{L} \text{ and for any } 0 \leq \rho < 1, \lim_{k \to +\infty} |a_k|\rho^k = 0 \Big\}.$$

Moreover, since $\varphi : \breve{L} \xrightarrow{\sim} \breve{L}$ and $\varphi(\mu) = (1+\mu)^p - 1$, the Frobenius endomorphism on $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$ is finite and faithfully flat of degree p.

Lemma 2.6. *The morphism $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ is faithfully flat.*

Proof. Let $f : \mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},L}$ denote the map acting as Frobenius on L and $f(\mu) = \mu$. Explicitly, using Lemma 2.3 write $x = \sum_{k \in \mathbb{N}} \iota(a_k)\mu^k \in \mathbf{B}^+_{\mathrm{rig},L}$ where $a_k \in L$ for all $k \in \mathbb{N}$. Then $f(x) = \sum_{k \in \mathbb{N}} \iota(\varphi(a_k))\mu^k$. Using Remark 2.4 we note that the map f is finite and faithfully flat of degree p^d. Recall that $\breve{L} = \cup_{i=1}^d L(X_i^{1/p^\infty}) = \varphi^{-\infty}(L)$, therefore using Remark 2.5 we get $\mathbf{B}^+_{\mathrm{rig},\breve{L}} = f^{-\infty}(\mathbf{B}^+_{\mathrm{rig},L})$. Hence, $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ is faithfully flat. ∎

2.1.6. Modules over certain period rings. Let $\varphi\text{-Mod}_{\mathbf{B}^\dagger_{\mathrm{rig},L}}$ denote the category of finite free $\mathbf{B}^\dagger_{\mathrm{rig},L}$-modules equipped with an isomorphism $1 \otimes \varphi : \varphi^*M \xrightarrow{\sim} M$ and morphisms between objects are $\mathbf{B}^\dagger_{\mathrm{rig},L}$-linear maps compatible with $1 \otimes \varphi$ on both sides; denote by $\varphi\text{-Mod}^0_{\mathbf{B}^\dagger_{\mathrm{rig},L}}$ the full subcategory of objects that are pure of slope 0 in the sense of [Ked04, §6.3]. Similarly, one can define the category $\varphi\text{-Mod}_{\mathbf{B}^\dagger_L}$ and denote by $\varphi\text{-Mod}^0_{\mathbf{B}^\dagger_L}$ the full subcategory of objects that are pure of slope 0 (as φ-modules over a discretely valued field).

Let $\text{Eff-}\varphi\text{-Mod}^{[p]_q}_{\mathbf{A}_L^+}$ denote the category of effective and finite $[p]_q$-height $\mathbf{A}_L^+$-modules, i.e. finite free $\mathbf{A}_L^+$-module N equipped with a Frobenius-semilinear endomorphism $\varphi : N \to N$ such that the map $1 \otimes \varphi : \varphi^*(N) \to N$ is injective and its cokernel is killed by $[p]_q^s = \tilde{\xi}^s$ for some $s \in \mathbb{N}$; denote by $\text{Eff-}\varphi\text{-Mod}^{[p]_q}_{\mathbf{A}_L^+} \otimes \mathbb{Q}_p$ the associated isogeny category. Similarly, define $\text{Eff-}\varphi\text{-Mod}^{[p]_q}_{\mathbf{B}^+_{\mathrm{rig},L}}$ as the category of effective and finite $[p]_q$-height $\mathbf{B}^+_{\mathrm{rig},L}$-modules and let $\text{Eff-}\varphi\text{-Mod}^{[p]_q,0}_{\mathbf{B}^+_{\mathrm{rig},L}}$ denote the full subcategory of objects that are pure of slope 0, i.e. M such $\mathbf{B}^{\dagger}_{\mathrm{rig},L} \otimes_{\mathbf{B}^{+}_{\mathrm{rig},L}} M$ is pure of slope 0.

Lemma 2.7. (1) *There is a natural equivalence of categories* $\varphi\text{-Mod}^0_{\mathbf{B}^{\dagger}_L} \xrightarrow{\sim} \varphi\text{-Mod}^0_{\mathbf{B}^{\dagger}_{\mathrm{rig},L}}$ *induced by the functor* $M \mapsto M \otimes_{\mathbf{B}^{\dagger}_L} \mathbf{B}^{\dagger}_{\mathrm{rig},L}$.

(2) *There is a natural equivalence of categories* $\text{Eff-}\varphi\text{-Mod}^{[p]_q}_{\mathbf{A}^{+}_L} \otimes \mathbb{Q}_p \xrightarrow{\sim} \text{Eff-}\varphi\text{-Mod}^{[p]_q,0}_{\mathbf{B}^{+}_{\mathrm{rig},L}}$ *induced by the functor* $N \mapsto N \otimes_{\mathbf{A}^{+}_L} \mathbf{B}^{+}_{\mathrm{rig},L}$.

Proof. The claim in (1) follows from [Ked05, Theorem 6.3.3]. The claim in (2) follows from part (1), [Kis06, Lemma 1.3.13] and [Ked04, Proposition 6.5]. Note that in [Kis06] Kisin assumes the residue field of the discrete valuation base field (L in our case) to be perfect. However, the proof of [Kis06, Lemma 1.3.13] depends only on [Ked04, Proposition 6.5] and [Ked05, Theorem 6.3.3] which are independent of the structure of the residue field of L. In particular, the proof of [Kis06, Lemma 1.3.13] applies almost verbatim to our case. We will recall the construction of the quasi-inverse functor from loc. cit., this will be useful in the sequel (see §2.5.4).

Let M be finite height effective $\mathbf{B}^{+}_{\mathrm{rig},L}$-module pure of slope 0, then $M^{\dagger}_{\mathrm{rig}} = \mathbf{B}^{\dagger}_{\mathrm{rig},L} \otimes_{\mathbf{B}^{+}_{\mathrm{rig},L}} M$ is pure of slope 0 and (1) implies that there exists a finite free $\mathbf{B}^{\dagger}_L$-module $M^{\dagger}$ pure of slope 0 such that $\mathbf{B}^{\dagger}_{\mathrm{rig},L} \otimes_{\mathbf{B}^{\dagger}_L} M^{\dagger} \xrightarrow{\sim} M^{\dagger}_{\mathrm{rig}} \xleftarrow{\sim} \mathbf{B}^{\dagger}_{\mathrm{rig},L} \otimes_{\mathbf{B}^{+}_{\mathrm{rig},L}} M$. Take $N = M \cap M^{\dagger} \subset M^{\dagger}_{\mathrm{rig}}$, it is a finitely generated φ-stable $\mathbf{B}^{+}_L$-module. Since $M^{\dagger}$ is of slope 0, there exists an $\mathbf{A}^{\dagger}_L$-lattice $M^{\dagger}_0 \subset M^{\dagger}$. Let $N'_0 = N \cap M^{\dagger}_0 \subset M^{\dagger}$ and set $N_0 = (\mathbf{A}^{\dagger}_L \otimes_{\mathbf{A}^{+}_L} N'_0) \cap N'_0[1/p] \subset M^{\dagger}$. Using [Kis06, Lemma 1.3.13] and the discussion above, we see that $N_0 \subset N$ is a finite free φ-stable $\mathbf{A}^{+}_L$-submodule such that cokernel of the injective map $1 \otimes \varphi : \varphi^*(N_0) \to N_0$ is killed by some finite power of $[p]_q$. ∎

We note two useful facts about structure of certain finitely generated modules over $\mathbf{A}^{+}_L$ and $\mathbf{B}^{+}_{\mathrm{rig},L}$ respectively.

Remark 2.8. Let M be a finitely generated torsion-free $\mathbf{A}^{+}_L$-module. Then $D = \mathbf{A}_L \otimes_{\mathbf{A}^{+}_L} N$ is a finite free $\mathbf{A}_L$-module and $M \subset D$ and $\mathbf{A}^{+}_L$-submodule. Moreover, the $\mathbf{A}^{+}_L$-module $N = M[1/p] \cap D$ is finite free. The claim essentially follows from [Fon90, Proposition B.1.2.4]. Note that Fontaine assumes the residue field of the discrete valuation base field (L in our case) to be perfect. However, the proof of [Fon90, Proposition B.1.2.4] only depends on [Lan90, Chapter 5 Theorem 3] which is independent of the structure of residue field of L. Therefore, using Fontaine's proof we conclude that N is finite free.

Remark 2.9. Let M be a finite free $\mathbf{B}^{+}_{\mathrm{rig},L}$-module and $N \subset M$ a $\mathbf{B}^{+}_{\mathrm{rig},L}$-submodule. Then N is finite free if and only if it is finitely generated if and only if it is a closed submodule of M. Equivalences in the preceding statement essentially follow from [Kis06, Lemma 1.1.4]. Note that Kisin assumes the residue field of the discrete valuation base field (L in our case) to be perfect. However, the proof of loc. cit. depends on results of [Laz62, §7-§8], [Ked04, Lemma 2.4] and [Ber02, Proposition 4.12 & Lemme 4.13], where the proof of latter depends on [Laz62] and [Hel43]. The relevant results of [Laz62], [Ked04] and [Hel43] are independent of the structure of the residue field of the discrete valuation field L. Hence, we get the equivalence by using the proof of [Kis06, Lemma 1.1.4] almost verbatim.

2.2. p-adic representations and (φ, Γ)-modules.

Let T be a finite free $\mathbb{Z}_p$-representation of G_L. By the theory of (φ, Γ_L)-modules (see [Fon90] and [And06]) one can functorially associate to T a finite free étale (φ, Γ_L)-module $\mathbf{D}_L(T) = (\mathbf{A} \otimes_{\mathbb{Z}_p} T)^{H_L}$ over $\mathbf{A}_L$ of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$, i.e. a finite free $\mathbf{A}_L$-module equipped with a Forbenius-semilinear endomorphism φ and a semilinear and continuous action of Γ_L such that the natural map $1 \otimes \varphi : \varphi^*(\mathbf{D}_L(T)) \to \mathbf{D}_L(T)$ is an isomorphism. Moreover, we have $\tilde{\mathbf{D}}_L(T) := (\tilde{\mathbf{A}} \otimes_{\mathbb{Z}_p} T)^{H_L} \xrightarrow{\sim} \tilde{\mathbf{A}}^{H_L} \otimes_{\mathbf{A}_L} \mathbf{D}_L(T)$. Furthermore, by the theory overconvergence of p-adic and $\mathbb{Z}_p$-representations (see [CC98] and [AB08]) one can functorially associate to T a finite free étale (φ, Γ_L)-module $\mathbf{D}_L^\dagger(T) = (\mathbf{A}^\dagger \otimes_{\mathbb{Z}_p} T)^{H_L}$ over $\mathbf{A}_L^\dagger$ of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$ and such that $\mathbf{A}_L \otimes_{\mathbf{A}_L^\dagger} \mathbf{D}_L^\dagger(T) \xrightarrow{\sim} \mathbf{D}_L(T)$. We have natural (φ, Γ_L)-equivariant isomorphisms

$$\mathbf{A} \otimes_{\mathbf{A}_L} \mathbf{D}_L(T) \xrightarrow{\sim} \mathbf{A} \otimes_{\mathbb{Z}_p} T, \quad \mathbf{A}^\dagger \otimes_{\mathbf{A}_L^\dagger} \mathbf{D}_L^\dagger(T) \xrightarrow{\sim} \mathbf{A}^\dagger \otimes_{\mathbb{Z}_p} T. \tag{2.1}$$

More generally, the constructions described above are functorial and induce equivalence of categories

$$\mathrm{Rep}_{\mathbb{Z}_p}(G_L) \xrightarrow{\sim} (\varphi, \Gamma_L)\text{-}\mathrm{Mod}_{\mathbf{A}_L}^{\text{ét}} \xleftarrow{\sim} (\varphi, \Gamma_L)\text{-}\mathrm{Mod}_{\mathbf{A}_L^\dagger}^{\text{ét}}. \tag{2.2}$$

Similar statements are also true for p-adic representations of G_L. For a p-adic representation V of G_L, extending scalars of $\mathbf{D}_L^\dagger(V)$ along $\mathbf{B}_L^\dagger \to \mathbf{B}_{\mathrm{rig},L}^\dagger$ we obtain a finite free $\mathbf{B}_{\mathrm{rig},L}^\dagger$-module $\mathbf{D}_{\mathrm{rig},L}^\dagger(V) := \mathbf{B}_{\mathrm{rig},L}^\dagger \otimes_{\mathbf{B}_L^\dagger} \mathbf{D}_L^\dagger(V)$ which is pure of slope 0 (see [Ber02], [Ked05] and [Ohk15]). Moreover, we have natural (φ, Γ_L)-equivariant isomorphism

$$\tilde{\mathbf{B}}_{\mathrm{rig}}^\dagger \otimes_{\mathbf{B}_{\mathrm{rig},L}^\dagger} \mathbf{D}_{\mathrm{rig},L}^\dagger(V) \xrightarrow{\sim} \tilde{\mathbf{B}}_{\mathrm{rig}}^\dagger \otimes_{\mathbb{Q}_p} V. \tag{2.3}$$

Remark 2.10. We have variations of the results mentioned above for p-adic (resp. $\mathbb{Z}_p$-representations) of $G_{\breve{L}}$ as well (see [Fon90], [CC98] and [Ber02] for details).

Let N be a finite free $\mathbf{A}_{\breve{L}}^+$-module. Say that N is *effective* and of *finite* $[p]_q$*-height* if N is equipped with a Frobenius-semilinear endomorphism φ such that the natural map $1 \otimes \varphi : \varphi^*(N) \to N$ is injective and its cokernel is killed by some finite power of $[p]_q$.

Let $D_{\breve{L}}$ be a finite free étale φ-module over $\mathbf{A}_{\breve{L}}$. Let $\mathcal{S}(D_{\breve{L}})$ denote the set of finitely generated $\mathbf{A}_{\breve{L}}^+$-submodules $M \subset D_{\breve{L}}$ such that M is stable under induced φ from $D_{\breve{L}}$ and cokernel of the injective map $1 \otimes \varphi : \varphi^*(M) \to M$ is killed by some finite power of $[p]_q$. In [Fon90, §B.1.5.5], Fontaine functorially attached to $D_{\breve{L}}$ an $\mathbf{A}_{\breve{L}}^+$-submodule $\tilde{j}_*(D_{\breve{L}}) = \cup_{M \in \mathcal{S}(D_{\breve{L}})} M \subset D_{\breve{L}}$ (Fontaine uses the notation j_*^q to denote the functor $\tilde{j}_*$; we change notations to avoid obvious confusions).

Lemma 2.11. *The $\mathbf{A}_{\breve{L}}^+$-module $\tilde{j}_*(D_{\breve{L}})$ is free of rank $\leq \mathrm{rk}_{\mathbf{A}_{\breve{L}}} D_{\breve{L}}$. Moreover, if N is an effective $\mathbf{A}_{\breve{L}}^+$-module of finite $[p]_q$-height, then cokernel of the injective map $N \to \tilde{j}_*(\mathbf{A}_{\breve{L}} \otimes_{\mathbf{A}_{\breve{L}}^+} N)$ is killed by some finite power of μ.*

Proof. The first claim is shown in [Fon90, §B.1.5.5]. For the second claim note that N is finite free over $\mathbf{A}_{\breve{L}}^+$ and of finite $[p]_q$-height, therefore it is p-étale in the sense of [Fon90, §B.1.3.1] by the equivalence shown in [Fon90, Proposition B.1.3.3]. In particular, we get that $D_{\breve{L}} = \mathbf{A}_{\breve{L}} \otimes_{\mathbf{A}_{\breve{L}}^+} N$ is an étale φ-module and $N \in \mathcal{S}(D_{\breve{L}})$. Now from [Fon90, Proposition B.1.5.6] it follows that cokernel of the injective map $N \to \tilde{j}_*(D_{\breve{L}})$ is killed by some finite power of μ. ∎

Finally, let V be a p-adic representation of G_L and $T \subset V$ a G_L-stable $\mathbb{Z}_p$-lattice. Since $G_{\breve{L}}$ is a subgroup of G_L, therefore by restriction V is a p-adic representation of $G_{\breve{L}}$ and $T \subset V$ a $G_{\breve{L}}$-stable $\mathbb{Z}_p$-lattice. Furthermore, we have a $\Gamma_{\breve{L}}$-equivariant embedding $\mathbf{A}_L \subset \mathbf{A}_{\breve{L}}$ (via the map $[X_i^\flat] \mapsto X_i$) and thus we have isomorphisms of étale $(\varphi, \Gamma_{\breve{L}})$-modules $\mathbf{D}_{\breve{L}}(T) \xrightarrow{\sim} \mathbf{A}_{\breve{L}} \otimes_{\mathbf{A}_L} \mathbf{D}_L(T)$ and $\tilde{\mathbf{D}}_{\breve{L}}(T) = (\tilde{\mathbf{A}} \otimes_{\mathbb{Z}_p} T)^{H_{\breve{L}}} \xrightarrow{\sim} \tilde{\mathbf{A}}^{H_{\breve{L}}} \otimes_{\mathbf{A}_{\breve{L}}} \mathbf{D}_{\breve{L}}(T)$. Similar statements are also true for V.

2.3. Crystalline representations. Let V be a p-adic crystalline representation of G_L (see [Bri06] or [Abh21, §2.4]). Using the theory described in loc. cit. one can attach to V a rank $= \dim_{\mathbb{Q}_p} V$ filtered (φ, ∂)-module over L, denoted as $\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. Similarly, as a representation of $G_{\breve{L}}$ one can attach to V a rank $= \dim_{\mathbb{Q}_p} V$ filtered φ-module over $\breve{L}$, denoted as $\mathbf{D}_{\mathrm{cris},\breve{L}}(V)$. From [BT08, Proposition 4.14], we have that $\mathbf{D}_{\mathrm{cris},\breve{L}}(V) \xrightarrow{\sim} \breve{L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ compatible with

Frobenius and filtration (via the map $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \twoheadrightarrow \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$, $X_i \mapsto [X_i^\flat]$). We equip $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$ with an L-algebra structure via the composition $L \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \twoheadrightarrow \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$ where the first map is the non-canonical L-algebra structure on $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$ (see §2.1.1).

Lemma 2.12. *There exists a natural $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$-linear isomorphism*

$$\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V \xrightarrow{\sim} (\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V))^{\partial=0} \xrightarrow{\sim} \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V),$$

induced by the surjective map $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \twoheadrightarrow \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$ given by $X_i \mapsto [X_i^\flat]$ for $1 \leq i \leq d$.

Proof. Let $J = ([X_1^\flat] - X_1, \ldots, [X_d^\flat] - X_d)\mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{\overline{L}})$. We have a projection,

$$\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \longrightarrow \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V),$$

via the map $X_i \mapsto [X_i^\flat]$ with kernel given as $J\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$. Moreover, using the non-canonical L-algebra structure on $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$, we have an L-linear map

$$\begin{aligned}\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) &\longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)\\ d &\longmapsto \sum_{\mathbf{k}\in\mathbb{N}^d} \prod_{i=1}^d \partial_i^{k_i}(d) \prod_{i=1}^d ([X_i^\flat] - X_i)^{[k_i]}.\end{aligned}$$

The map above extends $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$-linearly to a map

$$\begin{aligned}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) &\longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V),\\ a \otimes d &\longmapsto a \otimes \sum_{\mathbf{k}\in\mathbb{N}^d} \prod_{i=1}^d \partial_i^{k_i}(d) \prod_{i=1}^d ([X_i^\flat] - X_i)^{[k_i]}.\end{aligned} \tag{2.4}$$

and it provides a section to the projection described above. In particular, we obtain a $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$-linear direct sum decomposition

$$\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) = (J\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)) \oplus (\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)).$$

Note that the image of the section (2.4) lies in $(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V))^{\partial=0}$. Moreover, since V is crystalline we have $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L V$ and then one can easily show that $(J\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V))^{\partial=0} = 0$. Therefore, from the direct sum decomposition we conclude that $(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V))^{\partial=0} \xrightarrow{\sim} \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. ∎

Remark 2.13. Using the $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$-linear map in (2.4) we can describe the action of G_L on $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ explitcitly. The action can be given by the formula $g(a \otimes d) = g(a) \otimes \sum_{\mathbf{k}\in\mathbb{N}^d} \prod_{i=1}^d \partial_i^{k_i}(d) \prod_{i=1}^d (g([X_i^\flat]) - [X_i^\flat])^{[k_i]}$, for $g \in G_L$.

Remark 2.14. Using the description in Remark 2.13 we have that $\mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \subset \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is stable under the action of G_L as well. Moreover, we note that the H_L-action on $\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ in the tensor product $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is trivial. Therefore, we get that

$$(\mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V))^{H_L} = \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V). \tag{2.5}$$

2.4. Wach modules.

In this section we will describe Wach modules in the imperfect residue field case and finite $[p]_q$-height representations of G_L and relate them to crystalline representations. We start with the notion of Wach modules.

2.4.1. Wach modules over $\mathbf{A}_L^+$. Inside $\mathbf{A}_{\mathrm{inf}}(O_{F_\infty})$ fix $q = [\varepsilon]$, $\mu = [\varepsilon] - 1 = q - 1$ and $\tilde{\xi} = \varphi(\pi)/\pi = [p]_q$.

Definition 2.15. Let $a, b \in \mathbb{Z}$ with $b \geq a$. A *Wach module* over $\mathbf{A}_L^+$ with weights in the interval $[a, b]$ is a finite free $\mathbf{A}_L^+$-module N equipped with a continuous and semilinear action of Γ_L satisfying the following:

(1) Γ_L acts trivially on $N/\mu N$.

(2) There is a Frobenius-semilinear operator $\varphi : N[1/\mu] \to N[1/\varphi(\mu)]$ commuting with the action of Γ_L such that $\varphi(\mu^b N) \subset \mu^b N$ and the map $(1 \otimes \varphi) : \varphi^*(\mu^b N) \to \mu^b N$ is injective and its cokernel is killed by $\tilde{\xi}^{b-a} = [p]_q^{b-a}$.

Define the *height* of N to be the smallest value of $b - a$, where $a, b \in \mathbb{Z}$ as above. Say N is *effective* if one can take $b = 0$ and $a \leq 0$. A Wach module over $\mathbf{B}_L^+$ is defined to be a finitely generated module M admitting an $\mathbf{A}_L^+$-submodule $N \subset M$ with N as above and $N[1/p] = M$.

Remark 2.16. The definition of Wach modules in the imperfect residue field case is a direct and natural generalization of Wach modules in the perfect residue field case (see [Ber04, Définition III.4.1]).

Denote the category of Wach modules over $\mathbf{A}_L^+$ as $(\varphi, \Gamma)\text{-}\mathrm{Mod}_{\mathbf{A}_L^+}^{[p]_q}$ with morphisms between objects being $\mathbf{A}_L^+$-linear Γ_L-equivariant and φ-equivariant morphisms (after inverting μ). Extending scalars along $\mathbf{A}_L^+ \to \mathbf{A}_L$ induces a functor $(\varphi, \Gamma)\text{-}\mathrm{Mod}_{\mathbf{A}_L^+}^{[p]_q} \to (\varphi, \Gamma)\text{-}\mathrm{Mod}_{\mathbf{A}_L}^{\text{ét}}$ and we make the following claim:

Proposition 2.17. *The following natural functor is fully faithful*

$$\begin{aligned} (\varphi, \Gamma)\text{-}\mathrm{Mod}_{\mathbf{A}_L^+}^{[p]_q} &\longrightarrow (\varphi, \Gamma)\text{-}\mathrm{Mod}_{\mathbf{A}_L}^{\text{ét}} \\ N &\longmapsto \mathbf{A}_L \otimes_{\mathbf{A}_L^+} N. \end{aligned}$$

Proof. We need to show that for Wach modules N and N', we have a bijection

$$\mathrm{Hom}_{(\varphi,\Gamma)\text{-}\mathrm{Mod}_{\mathbf{A}_L^+}^{[p]_q}}(N, N') \xrightarrow{\sim} \mathrm{Hom}_{(\varphi,\Gamma)\text{-}\mathrm{Mod}_{\mathbf{A}_L}^{\text{ét}}}(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N, \mathbf{A}_L \otimes_{\mathbf{A}_L^+} N') \tag{2.6}$$

Note that $\mathbf{A}_L^+ \to \mathbf{A}_L = \mathbf{A}_L^+[1/\mu]^\wedge$ is injective, in particular, the map in (2.6) is injective. To check that (2.6) is surjective let $D_L = \mathbf{A}_L \otimes_{\mathbf{A}_L^+} N$, $D_L' = \mathbf{A}_L \otimes_{\mathbf{A}_L^+} N'$ and take an $\mathbf{A}_L$-linear and (φ, Γ_L)-equivariant map $f : D_L \to D_L'$. Base changing f along the embedding $\mathbf{A}_L \to \mathbf{A}_{\breve{L}}$ (see §2.1.5) we obtain an $\mathbf{A}_{\breve{L}}$-linear and $(\varphi, \Gamma_{\breve{L}})$-equivariant map $f_{\breve{L}} : D_{\breve{L}} \to D_{\breve{L}}'$. Using the definition and notation preceding Lemma 2.11 we further obtain an $\mathbf{A}_{\breve{L}}^+$-linear and $(\varphi, \Gamma_{\breve{L}})$-equivariant map $f_{\breve{L}} : \tilde{j}_*(D_{\breve{L}}) \to \tilde{j}_*(D_{\breve{L}}')$ where we abuse notations by writing $f_{\breve{L}}$ instead of $\tilde{j}_*(f_{\breve{L}})$. From Lemma 2.11 note that for some $s \in \mathbb{N}$ and $N_{\breve{L}} = \mathbf{A}_{\breve{L}}^+ \otimes_{\mathbf{A}_L^+} N$, we have $\mu^s N_{\breve{L}} \subset \tilde{j}_*(D_{\breve{L}})$ and its cokernel is killed by some finite power of μ. Hence, $N_{\breve{L}}[1/\mu] \xrightarrow{\sim} \tilde{j}_*(D_{\breve{L}})[1/\mu]$. Similarly, one can also show that $N_{\breve{L}}'[1/\mu] \xrightarrow{\sim} \tilde{j}_*(D_{\breve{L}}')[1/\mu]$.

Now from the map $f_{\breve{L}} : \tilde{j}_*(D_{\breve{L}}) \to \tilde{j}_*(D_{\breve{L}}')$ we obtain an induced $\Gamma_{\breve{L}}$-equivariant map $f_{\breve{L}} : N_{\breve{L}}[1/\mu] = \tilde{j}_*(D_{\breve{L}}')[1/\mu] \to \tilde{j}_*(D_{\breve{L}}')[1/\mu] = N_{\breve{L}}'[1/\mu]$ and from Lemma 2.18 we get that $f_{\breve{L}}(N_{\breve{L}}) \subset N_{\breve{L}}'$. It is easy to see that $N = N_{\breve{L}} \cap D_L \subset D_{\breve{L}}$ and $N' = N_{\breve{L}}' \cap D_L' \subset D_{\breve{L}}'$, so we conclude that $f(N) = f_{\breve{L}}(N_{\breve{L}}) \cap f(D_L) \subset N_{\breve{L}}' \cap D_L' = N'$. This proves the surjectivity of (2.6). ∎

Lemma 2.18. *Let N and N' be Wach modules over $\mathbf{A}_{\breve{L}}^+$ and let $f : N[1/\mu] \to N'[1/\mu]$ be an* $\mathbf{A}_{\breve{L}}^+$-linear *and $\Gamma_{\breve{L}}$-equivariant map. Then $f(N) \subset N'$.*

Proof. The proof is similar to the proof of [Abh21, Lemma 5.31]. Assume $f(N) \subset \mu^{-k}N'$ for some $k \in \mathbb{N}$ and consider the reduction of f modulo μ, which is again $\Gamma_{\breve{L}}$-equivariant. By definition we have that $\Gamma_{\breve{L}}$ acts trivially over $N/\mu N$, whereas $\mu^{-k}N'/\mu^{-k+1}N' \xrightarrow{\sim} N'/\mu N'(-k)$, i.e. the action of $\Gamma_{\breve{L}}$ on $\mu^{-k}N'/\mu^{-k+1}N'$ is given by χ^{-k} where χ is the p-adic cyclotomic character, in particular, $(\mu^{-k}N'/\mu^{-k+1}N')^{\Gamma_{\breve{L}}} = 0$. Since f is $\Gamma_{\breve{L}}$-equivariant, we must have $k = 0$, i.e. $f(N) \subset N'$. ∎

Composing the functor in Proposition 2.17 with the equivalence in (2.2), we obtain a fully faithful functor

$$\begin{aligned} \mathbf{T}_L : (\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_L^+}^{[p]_q} &\longrightarrow \mathrm{Rep}_{\mathbb{Z}_p}(G_L) \\ N &\longmapsto (\mathbf{A} \otimes_{\mathbf{A}_L^+} N)^{\varphi=1} \xrightarrow{\sim} (W(\mathbb{C}_L^\flat) \otimes_{\mathbf{A}_L^+} N)^{\varphi=1}. \end{aligned} \tag{2.7}$$

Lemma 2.19. *Let N be Wach module of height s and let $T = \mathbf{T}_L(N)$. Then we have a G_L-equivariant isomorphism $\mathbf{A}^+[1/\mu] \otimes_{\mathbf{A}_L^+} N \xrightarrow{\sim} \mathbf{A}^+[1/\mu] \otimes_{\mathbb{Z}_p} T$. Moreover, if N is effective, then we have G_L-equivariant inclusions $\mu^s(\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T) \subset \mathbf{A}^+ \otimes_{\mathbf{A}_L^+} N \subset \mathbf{A}^+ \otimes_{\mathbb{Z}_p} T$.*

Proof. For $r \in \mathbb{N}$ large enough, the Wach module $\mu^r N(-r)$ is always effective and we have $\mathbf{T}_L(\mu^r N(-r)) = T(-r)$ (the twist $(-r)$ denotes the Tate twist on which Γ_L acts via the cyclotomic character). Therefore, it is enough to show both the claims for effective Wach modules. Assume N is effective. Since N is finite free over $\mathbf{A}_L^+$, using Definition 2.15 (2) and tensor product Frobenius we obtain an isomorphism $\varphi : \mathbf{A}_{\inf}(O_{\overline{L}})[1/\xi] \otimes_{\mathbf{A}_L^+} N \xrightarrow{\sim} \mathbf{A}_{\inf}(O_{\overline{L}})[1/\tilde{\xi}] \otimes_{\mathbf{A}_L^+} N$. So from [MT20, Proposition 6.15] we get G_L-equivariant inclusions

$$\mu^s(\mathbf{A}_{\inf}(O_{\overline{L}}) \otimes_{\mathbb{Z}_p} T) \subset \mathbf{A}_{\inf}(O_{\overline{L}}) \otimes_{\mathbf{A}_L^+} N \subset \mathbf{A}_{\inf}(O_{\overline{L}}) \otimes_{\mathbb{Z}_p} T \subset \tilde{\mathbf{A}} \otimes_{\mathbf{A}_L^+} N.$$

Moreover, from (2.1) we have $\mathbf{A} \otimes_{\mathbf{A}_L^+} N \xrightarrow{\sim} \mathbf{A} \otimes_{\mathbb{Z}_p} T$. Therefore, taking intersection inside $\tilde{\mathbf{A}} \otimes_{\mathbf{A}_L^+} N \xrightarrow{\sim} \tilde{\mathbf{A}} \otimes_{\mathbb{Z}_p} T$ we obtain G_L-equivariant inclusions

$$\mu^s(\mathbf{A}_{\inf}(O_{\overline{L}}) \cap \mathbf{A}) \otimes_{\mathbb{Z}_p} T \subset (\mathbf{A}_{\inf}(O_{\overline{L}}) \cap \mathbf{A}) \otimes_{\mathbf{A}_L^+} N \subset (\mathbf{A}_{\inf}(O_{\overline{L}}) \cap \mathbf{A}) \otimes_{\mathbb{Z}_p} T.$$

Since $\mathbf{A}^+ = \mathbf{A}_{\inf}(O_{\overline{L}}) \cap \mathbf{A}$ we get the desired statements. ∎

Proposition 2.20. *Let N be a Wach module over $\mathbf{A}_L^+$ then $V = \mathbf{T}_L(N)[1/p]$ is a p-adic crystalline representation of G_L.*

Proof. Note that N is free over $\mathbf{A}_L^+$ and $\mathbf{T}_L(N)$ is a finite $[p]_q$-height $\mathbb{Z}_p$-representation of G_L in the sense of Definition 2.21 (see Remark 2.23). Then the results of [Abh21, §4.3-§4.5] can be adapted to the case of base ring O_L almost verbatim since all objects appearing in loc. cit. admit a natural variation for O_L. In particular, techniques used in the proof of [Abh21, Theorem 4.25, Corollary 4.27] can be adapted to our case to obtain that $V = \mathbf{T}_L(N)[1/p]$ is a crystalline representation of G_L. ∎

2.4.2. Finite $[p]_q$-height representations. In this section we generalize the definition of finite $[p]_q$-height representations from [Abh21, Definition 4.9] in the imperfect residue field case. Let $\mathbf{D}_L^+(T) = (\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T)^{H_L}$ be the (φ, Γ_L)-module over $\mathbf{A}_L^+$ associated to T and let $\mathbf{D}_L^+(V) = \mathbf{D}_L^+(T)[1/p]$ be the (φ, Γ_L)-module over $\mathbf{B}_L^+$ associated to V.

Definition 2.21. A *finite $[p]_q$-height* $\mathbb{Z}_p$-representation of G_L is a finite free $\mathbb{Z}_p$-module T admitting a linear and continuous action of G_L such that there exists a finite free $\mathbf{A}_L^+$-submodule $\mathbf{N}_L(T) \subset \mathbf{D}_L(T)$ satisfying the following:

(1) $\mathbf{N}_L(T)$ is a Wach module in the sense of Definition 2.15.

(2) We have $\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T) \xrightarrow{\sim} \mathbf{D}_L(T)$.

The height of T is defined to be the height of $\mathbf{N}_L(T)$. Say T is *positive* if $\mathbf{N}_L(T)$ is effective.

A finite $[p]_q$-height p-adic representation of G_L is a finite dimensional $\mathbb{Q}_p$-vector space admitting a linear and continuous action of G_L such that there exists a G_L-stable $\mathbb{Z}_p$-lattice $T \subset V$ with T of finite $[p]_q$-height. We set $\mathbf{N}_L(V) = \mathbf{N}_L(T)[\frac{1}{p}]$ satisfying analogous properties. The height of V is defined to be the height of T. Say V is positive if $\mathbf{N}_L(V)$ is effective.

Lemma 2.22. *Let T be a finite $[p]_q$-height $\mathbb{Z}_p$-representation of G_R. Then,*

(1) *If T is positive then $\mu^s \mathbf{D}_L^+(T) \subset \mathbf{N}_L(T) \subset \mathbf{D}_L^+(T)$.*

(2) *The $\mathbf{A}_L^+$-module $\mathbf{N}_L(T)$ is unique.*

Similar statements hold for $V = T[1/p]$.

Proof. Since $\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T) \xrightarrow{\sim} \mathbf{D}_L(T)$ and this scalar extension is fully faithful, we obtain that $\mathbf{T}_L(\mathbf{N}_L(T)) \xrightarrow{\sim} T$ as representations of G_L (here $\mathbf{T}_L$ is the functor defined in (2.7)). This also implies that Lemma 2.19 holds for $\mathbf{N}_L(T)$, so taking H_L-invariants there we obtain $\mu^s \mathbf{D}_L^+(T) \subset \mathbf{N}_L(T) \subset \mathbf{D}_L^+(T)$ which shows (1). The claim in (2) follows from Proposition 2.17, or using an argument similar to [Abh21, Proposition 4.13]. ∎

Remark 2.23. From the definition of finite $[p]_q$-height representations, Lemma 2.22 and the fully faithful functor in (2.7) it follows that the data of a finite $[p]_q$-height representation is equivalent to the data of a Wach module.

2.5. Crystalline implies finite height. The aim of rest of this section is to prove the following claim:

Theorem 2.24. *Let T be a finite free $\mathbb{Z}_p$-representation of G_R such that $V = T[1/p]$ is a p-adic crystalline representation of G_R. Then there exists a unique Wach module $\mathbf{N}_L(T)$ over $\mathbf{A}_L^+$. In other words, T is of finite $[p]_q$-height.*

Before carrying out the proof of Theorem 2.24, we note the following corollary: let $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_L)$ denote the category of $\mathbb{Z}_p$-lattices inside p-adic crystalline representations of G_L. Then combining Proposition 2.20 and Theorem 2.24 we obtain:

Corollary 2.25. *The Wach module functor induces an equivalence of $\otimes$-catgeories*

$$\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_L) \xrightarrow{\sim} (\varphi, \Gamma)\text{-}\mathrm{Mod}_{\mathbf{A}_L^+}^{[p]_q}$$
$$T \longmapsto \mathbf{N}_L(T),$$

with a quasi-inverse given as $N \mapsto (W(\mathbb{C}_L^\flat) \otimes_{\mathbf{A}_R^+} N)^{\varphi=1}$.

Remark 2.26. In Corollary 2.25 compatibility with tensor products follows from [Abh21, Proposition 4.14]. However, note that the functor $\mathbf{N}_L$ is not exact. But, it becomes exact after passing to corresponding isogeny categories, i.e. the induced functor from p-adic crystalline representations of G_L to Wach modules over $\mathbf{B}_L^+$ is exact. We leave details of these claims to the reader.

In the rest of this section we will carry out the proof of Theorem 2.24 by constructing $\mathbf{N}_L(T)$. Note that the property of being crystalline and of finite $[p]_q$-height is invariant under twisting the representation by χ^r for $r \in \mathbb{N}$. So from now onwards we will assume that V is a p-adic positive crystalline representation of G_L, i.e. all its Hodge-Tate weights are ≤ 0. We have $T \subset V$ a G_L-stable $\mathbb{Z}_p$-lattice.

Recall that $G_{\breve{L}}$ is a subgroup of G_L, so from [BT08, Proposition 4.14] it follows that V is a p-adic positive crystalline representation of $G_{\breve{L}}$ and $T \subset V$ a $G_{\breve{L}}$-stable $\mathbb{Z}_p$-lattice. Note that $\breve{L}$ is an unramified extension of $\mathbb{Q}_p$ with perfect residue field, therefore the $G_{\breve{L}}$-representation V is of finite $[p]_q$-height (see [Col99] and [Ber04]). Let the $[p]_q$-height of V be $s \in \mathbb{N}$. One associates to V a finite free $(\varphi, \Gamma_{\breve{L}})$-module over $\mathbf{B}^+_{\breve{L}}$ of rank $= \dim_{\mathbb{Q}_p} V$ called the Wach module $\mathbf{N}_{\breve{L}}(V)$ and to T a finite free $(\varphi, \Gamma_{\breve{L}})$-module over $\mathbf{A}^+_{\breve{L}}$ of rank $= \dim_{\mathbb{Q}_p} V$ called the Wach module $\mathbf{N}_{\breve{L}}(T)$ (see [Wac96; Wac97; Ber04] and [Abh21, §4.1] for a recollection). Let $\tilde{\mathbf{D}}^+_L(T) = (\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}}) \otimes_{\mathbb{Z}_p} T)^{H_L}$ be the (φ, Γ_L)-module over $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ associated to T and let $\tilde{\mathbf{D}}^+_L(V) = \tilde{\mathbf{D}}^+_L(T)[1/p]$ be the (φ, Γ_L)-module over $\mathbf{B}_{\mathrm{inf}}(O_{L_\infty})$ associated to V.

Lemma 2.27 ([Ber04]). (1) $\mathbf{N}_{\breve{L}}(T) = \mathbf{N}_{\breve{L}}(V) \cap \mathbf{D}_{\breve{L}}(T) \subset \mathbf{D}_{\breve{L}}(V)$.

(2) $\mu^s \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}}) \otimes_{\mathbb{Z}_p} T \subset \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}}) \otimes_{\mathbf{A}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(T) \subset \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}}) \otimes_{\mathbb{Z}_p} T$ *and taking H_L-invariants gives* $\mu^s \tilde{\mathbf{D}}^+_L(T) \subset \mathbf{A}_{\mathrm{inf}}(O_{L_\infty}) \otimes_{\mathbf{A}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(T) \subset \tilde{\mathbf{D}}^+_L(T)$. *Similar claims are also true for V.*

By properties of Wach modules, we have funtorial isomorphisms of étale (φ, Γ_L)-modules

$$\begin{gathered}\mathbf{A}_{\breve{L}} \otimes_{\mathbf{A}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(T) \xrightarrow{\sim} \mathbf{D}_{\breve{L}}(T) \ \text{ and } \ \mathbf{A}^\dagger_{\breve{L}} \otimes_{\mathbf{A}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(T) \xrightarrow{\sim} \mathbf{D}^\dagger_{\breve{L}}(T),\\ \mathbf{B}_{\breve{L}} \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \xrightarrow{\sim} \mathbf{D}_{\breve{L}}(V) \ \text{ and } \ \mathbf{B}^\dagger_{\breve{L}} \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \xrightarrow{\sim} \mathbf{D}^\dagger_{\breve{L}}(V),\\ \mathbf{B}^\dagger_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \xrightarrow{\sim} \mathbf{D}^\dagger_{\mathrm{rig},\breve{L}}(V).\end{gathered} \tag{2.8}$$

Let $N_{\mathrm{rig},\breve{L}}(V) := \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)$ equipped with (diagonally) induced Frobenius-semilinear operator φ and $\Gamma_{\breve{L}}$-action. From [Ber04, Théorème III.4.4], we have a natural inclusion $\mathbf{D}_{\mathrm{cris},\breve{L}}(V) \subset N_{\mathrm{rig},\breve{L}}(V)$ which extends $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$-linearly to a Frobenius and $\Gamma_{\breve{L}}$-equivariant inclusion $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \subset N_{\mathrm{rig},\breve{L}}(V)$ such that its cokernel is killed by $(t/\mu)^s \in \mathbf{B}^+_{\mathrm{rig},\breve{L}}$. In particular, we obtain a $(\varphi, \Gamma_{\breve{L}})$-equivariant isomorphism

$$\mathbf{B}^+_{\mathrm{rig},\breve{L}}\left[\tfrac{\mu}{t}\right] \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},\breve{L}}\left[\tfrac{\mu}{t}\right] \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V). \tag{2.9}$$

Moreover, from loc. cit. we have a natural isomorphism of filtered φ-modules $\mathbf{D}_{\mathrm{cris},\breve{L}}(V) \xrightarrow{\sim} N_{\mathrm{rig},\breve{L}}(V)/\mu N_{\mathrm{rig},\breve{L}}(V) = \mathbf{N}_{\breve{L}}(V)/\mu \mathbf{N}_{\breve{L}}(V)$ such that the largest Hodge-Tate weight of V equals s, i.e. the $[p]_q$-height of V. Since t/μ is a unit in $\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty})$ and $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \subset \tilde{\mathbf{B}}^+_{\mathrm{rig},L} \subset \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty})$, by extending scalars in (2.9) we obtain a $\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty})$-linear isomorphism compatible with the action of φ and $\Gamma_{\breve{L}}$

$$\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V). \tag{2.10}$$

2.5.1. Kisin's construction. Our goal is to show the existence of Wach module $\mathbf{N}_L(T)$. To this end, we will adapt techniques from [BT08] and [KR09] generalizing the results of Kisin in [Kis06].

Let $E(X) = \frac{(1+X)^p-1}{X} \in \mathbb{Z}_p[\![X]\!]$ denote the cyclotomic polynomial. We equip $\mathbb{Z}_p[\![X]\!]$ with the cyclotomic Frobenius $X \mapsto (1+X)^p - 1$ and for $n \in \mathbb{N}$ we set $E_n(X) = \varphi^n(E(X))$. In particular, $\zeta_{p^{n+1}} - 1$ is a simple zero of $E_n(X)$. For $X = \mu$, we will write $E_n(X) = \tilde{\xi}_n$ for $n \in \mathbb{N}$ and $\varphi(\mu)/\mu = \tilde{\xi} = \tilde{\xi}_0 = E(\mu) = [p]_q$.

Remark 2.28. Let $\varphi_{\breve{L}} : \mathbf{B}^+_{\mathrm{rig},\breve{L}} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ denote the map given by Frobenius on $\breve{L}$ and $\varphi_{\breve{L}}(\mu) = \mu$. Similarly, let $\varphi_L : \mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},L}$ denote the map $\sum_{k \in \mathbb{N}} \iota(a_k)\mu^k \mapsto \sum_{k \in \mathbb{N}} \iota(\varphi(a_k))\mu^k$. Then $\mathbf{B}^+_{\mathrm{rig},L}$ is finite free of rank p^d over $\mathbf{B}^+_{\mathrm{rig},L}$ via the map φ_L, in particular, flat. Also from §2.1.4 note that the injection $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ is compatible with φ_L on left and $\varphi_{\breve{L}}$ on right.

Remark 2.29. We have $t/\mu \in \mathbf{B}^+_{\mathrm{rig},L}$ and we can write $t/\mu = \prod_{n\in\mathbb{N}}(\tilde{\xi}_n/p)$. The zeros of t/μ are $\zeta_{p^{n+1}} - 1$ for $n \in \mathbb{N}$. Moreover, we see that $\varphi_L^{-n}(t/\mu) = t/\mu$, therefore the zeros of $\varphi_L^{-n}(t/\mu)$ are $\zeta_{p^{n+1}} - 1$ as well.

Now let $\widehat{\mathbf{B}}_{\breve{L},n}$ denote the completion of $\breve{L}(\zeta_{p^n}) \otimes_{O_{\breve{L}}} \mathbf{B}^+_{\breve{L}}$ with respect to the maximal ideal generated by $\mu - (\zeta_{p^{n+1}} - 1)$. Moreover, since $\zeta_{p^{n+1}} - 1$ is a simple root of $\tilde{\xi}_n$ we obtain that $(\mu-(\zeta_{p^{n+1}}-1)) = (\tilde{\xi}_n) \subset \widehat{\mathbf{B}}_{\breve{L},n}$. The local ring $\widehat{\mathbf{B}}_{\breve{L},n}$ naturally admits an action of $\Gamma_{\breve{L}}$ obtained by the diagonal action of $\Gamma_{\breve{L}}$ on the tensor product $\breve{L}(\zeta_{p^n})\otimes_{O_{\breve{L}}} \mathbf{B}^+_{\breve{L}}$. We put a filtration on $\widehat{\mathbf{B}}_{\breve{L},n}[1/\tilde{\xi}_n]$ as $\mathrm{Fil}^r\widehat{\mathbf{B}}_{\breve{L},n}[1/\tilde{\xi}_n] = \tilde{\xi}_n^r\widehat{\mathbf{B}}_{\breve{L},n}$ for $r \in \mathbb{Z}$. We have inclusions $\mathbf{B}^+_{\breve{L}} \subset \mathbf{B}^+_{\mathrm{rig},\breve{L}} \subset \widehat{\mathbf{B}}_{\breve{L},n}[1/\tilde{\xi}_n]$.

Let $D_L := \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ and $D_{\breve{L}} := \mathbf{D}_{\mathrm{cris},\breve{L}}(V)$ then using the φ-equivariant injection $L \to \breve{L}$, we obtain an isomorphism of filtered φ-modules $\breve{L} \otimes_L D_L \xrightarrow{\sim} D_{\breve{L}}$ from [BT08, Proposition 4.14]. Note that D_L (resp. $D_{\breve{L}}$) is an effective filtered φ-module over L (resp. over $\breve{L}$), i.e. $\mathrm{Fil}^0 D_L = D_L$ (resp. $\mathrm{Fil}^0 D_{\breve{L}} = D_{\breve{L}}$) and we have a φ-equivariant inclusion $D_L \subset D_{\breve{L}}$. Consider a map

$$i_n : \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{O_{\breve{L}}} D_{\breve{L}} \xrightarrow{\varphi_{\breve{L}}^{-n}\otimes\varphi_{D_{\breve{L}}}^{-n}} \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}} \longrightarrow \widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\breve{L}} D_{\breve{L}},$$

where $\varphi_{\breve{L}} : \mathbf{B}^+_{\mathrm{rig},\breve{L}} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ is defined in Remark 2.28 and $\varphi_{D_{\breve{L}}}$ is the Frobenius-semilinear operator on $D_{\breve{L}}$. Since the residue field of $\breve{L}$ is perfect, the map i_n is well-defined and it extends to a map

$$i_n : \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\tfrac{\mu}{t}] \otimes_{\breve{L}} D_{\breve{L}} \longrightarrow \widehat{\mathbf{B}}_{\breve{L},n}[\tfrac{1}{\tilde{\xi}_n}] \otimes_{\breve{L}} D_{\breve{L}}.$$

We define

$$\mathcal{M}_{\breve{L}}(D_{\breve{L}}) := \{x \in \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\tfrac{\mu}{t}] \otimes_{O_{\breve{L}}} D_{\breve{L}}, \text{ such that } \forall n \in \mathbb{N}, i_n(x) \in \mathrm{Fil}^0(\widehat{\mathbf{B}}_{\breve{L},n}[\tfrac{1}{\tilde{\xi}_n}] \otimes_{\breve{L}} D_{\breve{L}})\},$$

where $\widehat{\mathbf{B}}_{\breve{L},n}[1/\tilde{\xi}_n] \otimes_{\breve{L}} D_{\breve{L}}$ is equipped with tensor product Frobenius and filtration. By [Kis06, Lemma 1.2.2] and [KR09, Lemma 2.2.1], the $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$-module $\mathcal{M}_{\breve{L}}(D_{\breve{L}})$ is finite free of rank $= \dim_{\breve{L}} D_{\breve{L}}$ stable under φ and $\Gamma_{\breve{L}}$ such that cokernel of the injective map $1 \otimes \varphi : \varphi^*(\mathcal{M}_{\breve{L}}(D_{\breve{L}})) \to \mathcal{M}_{\breve{L}}(D_{\breve{L}})$ is killed by $\tilde{\xi}^s$ and the action of $\Gamma_{\breve{L}}$ is trivial modulo μ. Moreover, from [KR09, Proposition 2.2.6] reduction modulo μ induces an isomorphism $D_{\breve{L}} \xrightarrow{\sim} \mathcal{M}_{\breve{L}}(D_{\breve{L}})/\mu\mathcal{M}_{\breve{L}}(D_{\breve{L}})$ compatible with Frobenius and filtration.

By [Kis06, Theorem 1.3.8] and [KR09, Proposition 2.3.3], $\mathbf{B}^\dagger_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} \mathcal{M}_{\breve{L}}(D_{\breve{L}})$ is pure of slope 0. Then from [KR09, Corollay 2.4.2] one obtains an $\mathbf{A}^+_{\breve{L}}$-module $N_{\breve{L}}$ finite free of rank $= \dim_{\breve{L}} D_{\breve{L}}$ equipped with a Frobenius-seimilinear endomorphism φ and semilinear and conitnuous action of $\Gamma_{\breve{L}}$ such that cokernel of the injective map $1 \otimes \varphi : \varphi^*(N_{\breve{L}}) \to N_{\breve{L}}$ is killed by $\tilde{\xi}^s$, the action of $\Gamma_{\breve{L}}$ is trivial modulo μ and $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{A}^+_{\breve{L}}} N_{\breve{L}} \xrightarrow{\sim} \mathcal{M}_{\breve{L}}(D_{\breve{L}})$ compatible with $(\varphi, \Gamma_{\breve{L}})$-action.

Lemma 2.30. *There is a natural $(\varphi, \Gamma_{\breve{L}})$-equivariant isomorphism $\mathcal{M}_{\breve{L}}(D_{\breve{L}}) \xrightarrow{\sim} N_{\mathrm{rig},\breve{L}}(V)$. Moreover, it restricts to a $(\varphi, \Gamma_{\breve{L}})$-equivariant isomorphism $N_{\breve{L}}[1/p] \xrightarrow{\sim} \mathbf{N}_{\breve{L}}(V)$.*

Proof. Note that $N_{\mathrm{rig},\breve{L}}(V) = \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)$ is a finite free $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$-module of rank $= \dim_{\mathbb{Q}_p} V$ equipped with a Frobenius-semilinear endomorphism φ and a semilinear and continuous action of $\Gamma_{\breve{L}}$ such that cokernel of the injective map $1 \otimes \varphi : \varphi^*(N_{\mathrm{rig},\breve{L}}(V)) \to N_{\mathrm{rig},\breve{L}}(V)$ is killed by $\tilde{\xi}^s$ and the action of $\Gamma_{\breve{L}}$ is trivial on $N_{\mathrm{rig},\breve{L}}(V)/\mu N_{\mathrm{rig},\breve{L}}(V) \xrightarrow{\sim} D_{\breve{L}}$. From [KR09, Lemma 2.1.2] the action of $\Gamma_{\breve{L}}$ on $N_{\mathrm{rig},\breve{L}}$ is "$\mathbb{Z}_p$-analytic" in the sense of [KR09, §2.1.3]. Therefore, from [KR09, Proposition 2.2.6] and its proof we note that $\mathcal{M}_{\breve{L}}(D_{\breve{L}}) \xrightarrow{\sim} \mathcal{M}_{\breve{L}}(N_{\mathrm{rig},\breve{L}}(V)/\mu N_{\mathrm{rig},\breve{L}}(V)) \xrightarrow{\sim} N_{\mathrm{rig},\breve{L}}(V)$ compatible with $(\varphi, \Gamma_{\breve{L}})$-action. Finally, since $\mathbf{B}^\dagger_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} \mathcal{M}_{\breve{L}}(D_{\breve{L}}) \xrightarrow{\sim} \mathbf{B}^\dagger_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} N_{\mathrm{rig},\breve{L}}(V)$ is pure of slope 0, from [KR09, Corollary 2.4.2] we conclude that $N_{\breve{L}}[1/p] \xrightarrow{\sim} \mathbf{N}_{\breve{L}}(V)$ compatible with $(\varphi, \Gamma_{\breve{L}})$-action. ∎

Definition 2.31. Define

$$\begin{aligned}\mathcal{M}_L(D_L) &:= \{x \in \mathbf{B}^+_{\mathrm{rig},L}[\tfrac{\mu}{t}] \otimes_L D_L, \text{ such that } \forall n \in \mathbb{N}, i_n(x) \in \mathrm{Fil}^0(\widehat{\mathbf{B}}_{\breve{L},n}[\tfrac{1}{\tilde{\xi}_n}] \otimes_{\breve{L}} D_{\breve{L}})\}\\ &= (\mathbf{B}^+_{\mathrm{rig},L}[\tfrac{\mu}{t}] \otimes_L D_L) \cap \mathcal{M}_{\breve{L}}(D_{\breve{L}}) \subset \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\tfrac{\mu}{t}] \otimes_{O_{\breve{L}}} D_{\breve{L}}.\end{aligned}$$

From §2.1.5 recall that we have a φ-equivariant injection $\mathbf{B}^+_{\mathrm{rig},L} \rightarrow \mathbf{B}^+_{\mathrm{rig},\breve{L}}$, therefore by definition $\mathcal{M}_L(D_L)$ is stable under the induced tensor product Frobenius semilinear-operator φ on $\mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{O_{\breve{L}}} D_{\breve{L}}$. Since we have φ-equivariant inclusions $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}} \subset \mathcal{M}_{\breve{L}}(D_{\breve{L}}) \subset (\mu/t)^s\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}}$, it follows that we have φ-equivariant inclusions

$$\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L \subset \mathcal{M}_L(D_L) \subset (\tfrac{\mu}{t})^s\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L. \tag{2.11}$$

Therefore, similar to (2.9), we obtain a φ-equivariant isomorphism

$$\mathbf{B}^+_{\mathrm{rig},L}[\tfrac{\mu}{t}] \otimes_L D_L \xrightarrow{\sim} \mathcal{M}_L(D_L)[\tfrac{\mu}{t}], \tag{2.12}$$

Note that the natural map $(\breve{L}(\zeta_{p^n}) \otimes_L \mathbf{B}^+_{\mathrm{rig},L})_{(\tilde{\xi}_n)} \rightarrow \widehat{\mathbf{B}}_{\breve{L},n}$ is obtained as completion of a discrete valuation ring, we get the following:

Lemma 2.32. *For all $n \in \mathbb{N}$ the composition of maps $\mathbf{B}^+_{\mathrm{rig},L} \rightarrow \breve{L}(\zeta_{p^n}) \otimes_L \mathbf{B}^+_{\mathrm{rig},L} \rightarrow (\breve{L}(\zeta_{p^n}) \otimes_L \mathbf{B}^+_{\mathrm{rig},L})_{(\tilde{\xi}_n)} \rightarrow \widehat{\mathbf{B}}_{\breve{L},n}$ is flat.*

Lemma 2.33. *Consider $\widehat{\mathbf{B}}_{\breve{L},n}$ as a $\mathbf{B}^+_{\mathrm{rig},L}$-algebra via the composition $\mathbf{B}^+_{\mathrm{rig},L} \rightarrow \mathbf{B}^+_{\mathrm{rig},\breve{L}} \xrightarrow{\varphi_{\breve{L}}^{-n}} \widehat{\mathbf{B}}_{\breve{L},n}$. Then*

(1) *The homomorphism*

$$\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} (\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L) \longrightarrow \widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\breve{L}} D_{\breve{L}} \xleftarrow{\sim} \widehat{\mathbf{B}}_{\breve{L},n} \otimes_L D_L,$$

induced by i_n is an isomorphism.

(2) *The isomorphism in (1) induces an isomorphism*

$$\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L) \xrightarrow{\sim} \sum_{i \in \mathbb{N}} \tilde{\xi}_n^{-i}\widehat{\mathbf{B}}_{\breve{L},n} \otimes_L \mathrm{Fil}^i D_L$$

(3) *The φ-equivariant homomorphism $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L) \rightarrow \mathcal{M}_{\breve{L}}(D_{\breve{L}})$ obtained by extending $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$-linearly the φ-equivariant inclusion $\mathcal{M}_L(D_L) \subset \mathcal{M}_{\breve{L}}(D_{\breve{L}})$ is an isomorphism.*

Proof. The proof follows in a manner similar to [Kis06, Lemma 1.2.1]. The claim in (1) can be checked after reducing modulo $\tilde{\xi}_n$. To show (2), let us write for $k \in \mathbb{N}$

$$\mathcal{M}_{L,k}(D_L) = \{x \in \mathbf{B}^+_{\mathrm{rig},L}[\tfrac{\mu}{t}] \otimes_L D_L \text{ such that } i_k(x) \in \mathrm{Fil}^0(\widehat{\mathbf{B}}_{\breve{L},k}[\tfrac{1}{E_k(\mu)}] \otimes_{\breve{L}} D_{\breve{L}})\}.$$

Then we have $\mathcal{M}_L(D_L) = \cap_{k\in\mathbb{N}}\mathcal{M}_{L,k}(D_L)$. By flatness of $\widehat{\mathbf{B}}_{\breve{L},n}$ over $\mathbf{B}^+_{\mathrm{rig},L}$ (see Lemma 2.32) and of $\varphi_L : \mathbf{B}^+_{\mathrm{rig},L} \rightarrow \mathbf{B}^+_{\mathrm{rig},L}$ (see Remark 2.28), we get that $\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L) = \cap_{k\in\mathbb{N}}(\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_{L,k}(D_L))$. To prove our claim, it suffices to show the following two equalities:

$$\begin{aligned}\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_{L,n}(D_L) &= \sum_{r\in\mathbb{N}} \tilde{\xi}_n^{-r}\widehat{\mathbf{B}}_{\breve{L},n} \otimes_L \mathrm{Fil}^r D_L,\\ \widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_{L,k}(D_L) &= \widehat{\mathbf{B}}_{\breve{L},n}[\tfrac{1}{\tilde{\xi}_n}] \otimes_L D_L, \quad \text{for } k \neq n.\end{aligned}$$

For the first equality note that we have $\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_{L,n}(D_L) \subset \sum_{r\in\mathbb{N}} \tilde{\xi}_n^{-r}\widehat{\mathbf{B}}_{\breve{L},n} \otimes_L \mathrm{Fil}^r D_L$ by definition. For the converse, note that we have $\tilde{\xi}_n^{-1} = \frac{1}{p}\varphi^n(\mu/t)\varphi^{n+1}(t/\mu) \in \mathbf{B}_{\mathrm{rig},L}[\mu/t]$ and $\varphi_L^{-n}(\tilde{\xi}_n^{-1}) = \tilde{\xi}_n^{-1}$. So for any $r \in \mathbb{N}$ and $\tilde{\xi}_n^{-r}a \otimes d \in \tilde{\xi}_n^{-r}\widehat{\mathbf{B}}_{\breve{L},n} \otimes_L \mathrm{Fil}^r D_L$, we have $\tilde{\xi}_n^{-r} \otimes \varphi^n(d) \in \mathcal{M}_{L,n}(D_L)$ since $i_n(\tilde{\xi}_n^{-r} \otimes \varphi^n(d)) = \tilde{\xi}_n^{-r} \otimes d \in \mathrm{Fil}^0(\widehat{\mathbf{B}}_{\breve{L},n}[1/\tilde{\xi}_n] \otimes_{\breve{L}} D_{\breve{L}})$. Therefore, $\tilde{\xi}_n^{-r}a \otimes d = a \otimes i_n(\tilde{\xi}_n^{-r} \otimes \varphi^n(d)) \in \widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_{L,n}(D_L)$. For the second equality again note that by definition we have $\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_{L,k}(D_L) \subset \widehat{\mathbf{B}}_{\breve{L},n}[1/\tilde{\xi}_n] \otimes_L D_L$. For the converse, note that $E_k(\mu)$ is a unit in $\widehat{\mathbf{B}}_{\breve{L},n}$ since $\zeta_{p^{k+1}} - 1$ is not a root of $E_n(X)$. So for any $j, r \in \mathbb{N}$ and $\tilde{\xi}_n^{-r}E_k(\mu)^{-j}a \otimes d \in \widehat{\mathbf{B}}_{\breve{L},n}[1/\tilde{\xi}_n] \otimes_L \mathrm{Fil}^j D_L$, we have $\tilde{\xi}_n^{-r}E_k(\mu)^{-j} \otimes \varphi^k(d) \in \mathcal{M}_{L,k}(D_L)$ since $i_k(\tilde{\xi}_n^{-r}E_k(\mu)^{-j} \otimes \varphi^k(d)) = \tilde{\xi}_n^{-r}E_k(\mu)^{-j} \otimes d \in \mathrm{Fil}^0(\widehat{\mathbf{B}}_{\breve{L},k}[1/E_k(\mu)] \otimes_{\breve{L}} D_{\breve{L}})$. Therefore, $\tilde{\xi}_n^{-r}E_k(\mu)^{-j}a \otimes d = a \otimes i_k(\tilde{\xi}_n^{-r}E_k(\mu)^{-j} \otimes \varphi^k(d)) \in \widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_{L,k}(D_L)$.

To show (3), note that we have inclusions $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_L D_L \subset \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L) \subset \mathcal{M}_{\breve{L}}(D_{\breve{L}}) \subset (\mu/t)^s\mathbf{B}_{\mathrm{rig},\breve{L}} \otimes_L D_L$. So we get that some finite power of t/μ kills the cokernel of the injective map $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L) \to \mathcal{M}_{\breve{L}}(D_{\breve{L}})$. Since the cokernel is a finitely generated $\mathbf{B}^+_{\mathrm{rig},L}$-module, it is supported at zeros of the annihilator (see Remark 2.29 for zeros of t/μ). Then it suffices to check that the cokernel vanishes at zeros of t/μ. This follows from (2). ∎

Lemma 2.34. *We have following properties for the $\mathbf{B}^+_{\mathrm{rig},L}$-module $\mathcal{M}_L(D_L)$:*

1. *$\mathcal{M}_L(D_L)$ is a finite free $\mathbf{B}^+_{\mathrm{rig},L}$-module of rank $= \dim_L D_L$.*
2. *Cokernel of the injective map $1 \otimes \varphi : \varphi^*(\mathcal{M}_L(D_L)) \to \mathcal{M}_L(D_L)$ is killed by $\tilde{\xi}^s = [p]_q^s$.*
3. *$\mathcal{M}_L(D_L)$ is pure of slope 0, i.e. the $\mathbf{B}^\dagger_{\mathrm{rig},L}$-module $\mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L)$ is pure of slope 0.*

Proof. Note that $\mathcal{M}_{\breve{L}}(D_{\breve{L}}) \subset (\mu/t)^s\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}}$ is a closed submodule by [Kis06, Lemma 1.1.5, Lemma 1.2.2]. Moreover, since $\mathbf{B}^+_{\mathrm{rig},L} \subset \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ is a closed subring, we get that $\mathcal{M}_L(D_L) \subset (\mu/t)^s\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L$ is closed and hence finite free by Remark 2.9 and of rank $= \dim_L D_L$ by Lemma 2.33 (3). This shows (1).

For (2), let us first note the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M}_L(D_L) & \longrightarrow & (\frac{\mu}{t})^s\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L & \longrightarrow & Q_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{M}_{\breve{L}}(D_{\breve{L}}) & \longrightarrow & (\frac{\mu}{t})^s\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}} & \longrightarrow & Q_2 & \longrightarrow & 0. \end{array}$$

All maps are φ-equivariant and vertical maps are injective (see Lemma 2.30, Defnition 2.31, (2.11) and Lemma 2.33 (3)). From Remark 2.4, Remark 2.5 and Lemma 2.6 recall that the maps $\varphi_L : \mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},L}$, $\varphi_{\breve{L}} : \mathbf{B}^+_{\mathrm{rig},\breve{L}} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ and $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ are faithfully flat (we write φ with subscripts to avoid confusion). Using Lemma 2.33 (3) and $D_{\breve{L}} = \breve{L} \otimes_L D_L$, we get that $\varphi^*_{\breve{L}}(\mathcal{M}_{\breve{L}}(D_{\breve{L}})) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \varphi^*_L(\mathcal{M}_L(D_L))$ and $\varphi^*_{\breve{L}}((\mu/t)^s\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}}) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \varphi^*_L((\mu/t)^s\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L)$. From faithful flatness of aforementioned maps and exactness of both rows in the diagram above, it follows that $\varphi^*_L(\mathcal{M}_L(D_L)) = \varphi^*_L((\mu/t)^s\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L) \cap \varphi^*_{\breve{L}}(\mathcal{M}_{\breve{L}}(D_{\breve{L}})) \subset \varphi^*_{\breve{L}}((\mu/t)^s\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}})$. Now let $x \in \mathcal{M}_L(D_L) \subset \mathcal{M}_{\breve{L}}(D_{\breve{L}})$, then there exists $y \in \varphi^*(\mathcal{M}_{\breve{L}}(D_{\breve{L}}))$ such that $(1 \otimes \varphi)y = \tilde{\xi}^s x$. Recall that $1 \otimes \varphi : \varphi^*(D_L) \xrightarrow{\sim} D_L$ and $\varphi(\mu/t) = (\tilde{\xi}\mu)/(pt)$, therefore the cokernel of $1 \otimes \varphi : \varphi^*((\mu/t)^s\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L) \to (\mu/t)^s\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L$ is killed by $\tilde{\xi}^s$, in particular, $\tilde{\xi}^s x \in (1 \otimes \varphi)\varphi^*((\mu/t)^s\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L)$. Since $1 \otimes \varphi$ is injective on

$\varphi^*((\mu/t)^s\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}})$, therefore we get that $y \in \varphi^*((\mu/t)^s\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L) \cap \varphi^*(\mathcal{M}_{\breve{L}}(D_{\breve{L}})) = \varphi^*(\mathcal{M}_L(D_L))$. In particular, the cokernel of $1 \otimes \varphi : \varphi^*(\mathcal{M}_L(D_L)) \to \mathcal{M}_L(D_L)$ is killed by $\tilde{\xi}^s$.

For (3), note that from Lemma 2.33 (3) $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L) \xrightarrow{\sim} \mathcal{M}_{\breve{L}}(D_{\breve{L}})$. Moreover, from [Ked04, Theorem 6.10] we obtain a slope filtration on $\mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L)$ such that base changing this slope filtration along $\mathbf{B}^\dagger_{\mathrm{rig},L} \to \mathbf{B}^\dagger_{\mathrm{rig},\breve{L}}$ gives a slope filtration on $\mathbf{B}^\dagger_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} \mathcal{M}_{\breve{L}}(D_{\breve{L}})$. However, from [Kis06, Theorem 1.3.8] and [KR09, Proposition 2.3.3] $\mathbf{B}^\dagger_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} \mathcal{M}_{\breve{L}}(D_{\breve{L}})$ is pure of slope 0. Therefore, we must have that $\mathcal{M}_L(D_L)$ is pure of slope 0. ∎

2.5.2. Stability under Galois action. In this section we will equip the $\mathbf{B}^+_{\mathrm{rig},L}$-module $\mathcal{M}_L(D_L)$ with a natural action of Γ_L.

From §2.1.4 recall that $\tilde{\mathbf{B}}^+_{\mathrm{rig},L} = (\tilde{\mathbf{B}}^+_{\mathrm{rig}})^{H_L} = \cap_{n\in\mathbb{N}}\varphi^n(\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}))$, then from Remark 2.13 and (2.5) it follows that $\tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is stable under the action of Γ_L and for $a \otimes d \in \tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ this action can be explicitly described by the formula

$$g(a \otimes d) = g(a) \otimes \sum_{\mathbf{k}\in\mathbb{N}^d} \prod_{i=1}^{d} \partial_i^{k_i}(d) \prod_{i=1}^{d}(g([X_i^\flat]) - [X_i^\flat])^{[k_i]}, \ \text{ for } g \in \Gamma_L.$$

Using that $\mathbf{D}_{\mathrm{cris},\breve{L}}(V) = \breve{L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ and extending the isomorphism in (2.9) along the map $\mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \to \tilde{\mathbf{B}}^+_{\mathrm{rig},L}[1/t]$ (see §2.1.5), we obtain an isomorphism $\tilde{\mathbf{B}}^+_{\mathrm{rig},L}[1/t] \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \xrightarrow{\sim} \tilde{\mathbf{B}}^+_{\mathrm{rig},L}[1/t] \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)$. Recall that for $g \in \Gamma_L$ we have $g(t) = \chi(g)t$ where χ is the p-adic cyclotomic character. So the preceding discussion induces an action of Γ_L over $\tilde{\mathbf{B}}^+_{\mathrm{rig},L}[1/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} N_{\mathrm{rig},\breve{L}}(V) = \tilde{\mathbf{B}}^+_{\mathrm{rig},L}[1/t] \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)$. Our first objective is to show that $\tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} N_{\mathrm{rig},\breve{L}}(V) \subset \tilde{\mathbf{B}}^+_{\mathrm{rig},L}[1/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} N_{\mathrm{rig},\breve{L}}(V)$ is stable under the action of Γ_L. We will do this by embedding everything into $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V$.

Let us fix some elements in $\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$. For $n \in \mathbb{N}$, let $n = (p-1)f(n) + r(n)$ with $r(n), f(n) \in \mathbb{N}$ and $0 \leq r(n) < p-1$. Set $t^{\{n\}} = \frac{t^n}{f(n)!p^{f(n)}}$ for $p \geq 3$ (resp. $t^{\{n\}} = \frac{t^n}{n!2^n}$ for $p = 2$). Additionally, set

$$\Lambda := \Big\{ \sum_{n\in\mathbb{N}} a_n t^{\{n\}} \text{ with } a_n \in O_F \text{ such that } a_n = 0 \text{ if } p-1 \nmid n \text{ for } p \geq 3$$
$$\text{(resp. } 2 \nmid n) \text{ for } p = 2\Big\}.$$

Let $z = \sum_{a\in\mathbb{F}_p}[\varepsilon]^{[a]}$ for $p \geq 3$ (resp. $z = [\varepsilon] + [\varepsilon]^{-1}$ for $p = 2$) and set $\mu_0 = z - p$. Furthermore, for $r \in \mathbb{N}$ and $A = \mathbf{A}_{\mathrm{inf}}(O_{L_\infty}), \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}}), \mathbf{A}_{\mathrm{cris}}(O_{L_\infty}), \mathbf{A}_{\mathrm{cris}}(O_{\overline{L}})$ set

$$I^{(r)}A = \{a \in A \text{ such that } \varphi^n(a) \in \mathrm{Fil}^r A \text{ for all } n \in \mathbb{N}\}. \tag{2.13}$$

Lemma 2.35. *We note the following facts from [Fon94, §5.2]:*

(1) $\mu_0 = (p-1)\sum_{n\geq 1, p-1|n} \frac{t^n}{n!} \in \Lambda$ *for* $p \geq 3$ *(resp.* $\mu_0 = 2\sum_{n\geq 1, 2|n} \frac{t^n}{n!} \in \Lambda$ *for* $p = 2$*).*

(2) $\mu_0 \in p\Lambda$ *for* $p \geq 3$ *(resp.* $\mu_0 \in 8\Lambda$ *for* $p = 2$*) and there exists* $v \in \Lambda^\times$ *such that* $\mu_0/p = vt^{p-1}/p$ *for* $p \geq 3$ *(resp.* $\mu_0/8 = vt^2/8$ *for* $p = 2$*).*

(3) $t^{p-1} \in p\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$, $t^n \in \mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$ *and* t/μ *is a unit in* $\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$*.*

(4) *For* $r \in \mathbb{N}$ *we have* $I^{(r)}\mathbf{A}_{\mathrm{inf}}(O_{L_\infty}) = \mu^r\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ *and* $I^{(p-1)}\mathbf{A}_{\mathrm{inf}}(O_{L_\infty}) = \mu_0\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ *for* $p \geq 3$ *(resp.* $I^{(2)}\mathbf{A}_{\mathrm{inf}}(O_{L_\infty}) = \mu_0\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ *for* $p = 2$*).*

(5) *Let $S = O_F[[\mu]]$, then the natural map*

$$\begin{aligned}\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})\widehat{\otimes}_S\Lambda &\longrightarrow \mathbf{A}_{\mathrm{cris}}(O_{L_\infty})\\ \sum_{k\in\mathbb{N}} a_k \otimes \big(\tfrac{\mu_0}{p}\big)^{[k]} &\longmapsto \sum_{k\in\mathbb{N}} a_k\big(\tfrac{\mu_0}{p}\big)^{[k]},\end{aligned}$$

is continuous for the p-adic topology and an isomorphism of $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$-algebras.

(6) *The ideal $I^{(r)}\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$ is generated by $t^{\{s\}}$ for $s \geq r$.*

(7) *The natural map $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})/I^{(r)} \to \mathbf{A}_{\mathrm{cris}}(O_{L_\infty})/I^{(r)}$ is injective and the cokernel is killed by $m!p^m$ where $m = \lfloor \frac{r}{p-1}\rfloor$.*

Similar statements are true for $\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})$ and $\mathbf{A}_{\mathrm{cris}}(O_{\overline{L}})$.

Remark 2.36. For rings $\mathbf{B}_{\mathrm{inf}}(O_{L_\infty}) = \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})[1/p]$, $\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) = \mathbf{A}_{\mathrm{cris}}(O_{L_\infty})[1/p]$ and $\tilde{\mathbf{B}}^+_{\mathrm{rig},L}$ (equipped with induced filtration), one can define ideals similar to (2.13). Then from Lemma 2.35 (7) we obtain an isomorphism $\mathbf{B}_{\mathrm{inf}}(O_{L_\infty})/I^{(r)} \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty})/I^{(r)}$. Similarly, we have $\mathbf{B}_{\mathrm{inf}}(O_{L_\infty})/I^{(r)} \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty})/I^{(r)}$.

Proposition 2.37. *The $\mathbf{B}_{\mathrm{inf}}(O_{L_\infty})$-module*

$$N_{\breve{L},\infty}(V) := \mathbf{B}_{\mathrm{inf}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \subset (\mathbf{B}_{\mathrm{inf}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V)^{H_L} = \tilde{\mathbf{D}}^+_L(V),$$

is stable under the induced action of Γ_L and we have a natural Γ_L-equivariant embedding $N_{\breve{L},\infty}(V) \subset \mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbb{Q}_p} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ (see Remark 2.13 and (2.5) for Γ_L-action on the latter).

Recall that $N^+_{\mathrm{rig},\breve{L}} = \mathbf{B}_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)$ and we note the following:

Corollary 2.38. *Extending the scalars of $N_{\breve{L},\infty}(V)$ along the (φ,Γ_L)-compatible embedding $\mathbf{B}_{\mathrm{inf}}(O_{L_\infty}) \subset \tilde{\mathbf{B}}^+_{\mathrm{rig},L}$ gives an identification of $\tilde{\mathbf{B}}^+_{\mathrm{rig},L}$-submodules of $\mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbb{Q}_p} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$*

$$\tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}_{\mathrm{inf}}(O_{L_\infty})} N_{\breve{L},\infty}(V) = \tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) = \tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} N_{\mathrm{rig},\breve{L}}(V),$$

stable under the action of Γ_L.

Proof of Proposition 2.37. From Lemma 2.27 (2) consider the exact sequence

$$0 \longrightarrow \mu^s\tilde{\mathbf{D}}^+_L(V) \longrightarrow N_{\breve{L},\infty}(V) \longrightarrow N_{\breve{L},\infty}(V)/\mu^s\tilde{\mathbf{D}}^+_L(V) \longrightarrow 0, \tag{2.14}$$

where we know that $\mu^s\tilde{\mathbf{D}}^+_L(V) \subset \tilde{\mathbf{D}}^+_L(V)$ is stable under the action of Γ_L. Therefore, to show that the middle term above is stable under the action of Γ_L, it is enough to show that the image of the inclusion $N_{\breve{L},\infty}(V)/\mu^s\tilde{\mathbf{D}}^+_L(V) \subset \mathbf{B}_{\mathrm{inf}}(O_{\overline{L}})/\mu^s \otimes_{\mathbb{Q}_p} V$ is stable under the action of G_L.

In view of Remark 2.36, let us consider the following intersection inside $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V$

$$M := (I^{(s)}\mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V) \cap (\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)).$$

Then we obtain a diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mu^s\tilde{\mathbf{D}}^+_L(V) \arrow[r] \arrow[d] & N_{\breve{L},\infty}(V) \arrow[r] \arrow[d] & N_{\breve{L},\infty}(V)/\mu^s\tilde{\mathbf{D}}^+_L(V) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \arrow[r] & (\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V))/M \arrow[r] & 0.
\end{tikzcd}$$

The left vertical arrow is injective by Lemma 2.27 (2) and the middle arrow is obviously injective.

Lemma 2.39. *The inclusion* $N_{\breve{L},\infty}(V) \subset \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)$ *induces a* $\Gamma_{\breve{L}}$*-equivariant isomorphism* $N_{\breve{L},\infty}(V)/\mu^s\tilde{\mathbf{D}}^+_{\breve{L}}(V) \xrightarrow{\sim} (\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V))/M$.

Proof. First, we observe that by Lemma 2.27 (2) we have

$$\begin{aligned} M \cap N_{\breve{L},\infty}(V) &= (I^{(s)}\mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V) \cap N_{\breve{L},\infty}(V) \\ &\subset (I^{(s)}\mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V) \cap \tilde{\mathbf{D}}^+_{\breve{L}}(V) \subset \mu^s\tilde{\mathbf{D}}^+_{\breve{L}}(V). \end{aligned}$$

Therefore, we get that the rightmost vertical map in the diagram is injective. Next, we need to show that $N_{\breve{L},\infty}(V) + M = \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)$. The left expression is clearly contained in the right. To show the other direction, let $x \in \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)$. Then for $m \in \mathbb{N}$ large enough $p^m x \in \mathbf{A}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{A}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(T)$. By the isomorphism in Lemma 2.35 (5), for $r = \lceil \frac{s}{p-1} \rceil$, $k \in \mathbb{N}$ and $x_k \in \mathbf{N}_{\breve{L}}(T)$ we can write

$$p^m x = \sum_{k \in \mathbb{N}} x_k \big(\tfrac{\mu_0}{p}\big)^{[k]} = \sum_{k \leq r-1} x_k \big(\tfrac{\mu_0}{p}\big)^{[k]} + \sum_{k \geq r} x_k \big(\tfrac{\mu_0}{p}\big)^{[k]}.$$

Clearly, the first summand in the rightmost expression is in $N_{\breve{L},\infty}(V)$. Moreover, from Lemma 2.35 (2) there exists $v \in \Lambda^\times$ such that $\mu_0/p = vt^{p-1}/p$ for $p \geq 3$ (resp. $\mu_0/8 = vt^2/8$ for $p = 2$). Therefore, we obtain that the second summand is in $(I^{(s)}\mathbf{A}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Z}_p} T) \cap (\mathbf{A}_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{A}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(T)) \subset M$. Hence, $x \in N_{\breve{L},\infty}(V) + M$. ∎

From (2.10) we have a natural isomorphism

$$\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V),$$

compatible with Frobenius, filtration and $\Gamma_{\breve{L}}$-action. Now consider the following diagram

$$\begin{tikzcd} \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \arrow[r, "\sim"] \arrow[d, hook] & \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \arrow[d, hook] \\ \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V \arrow[d, "\wr"] \\ \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \arrow[r, "\sim"] & \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V \end{tikzcd}$$

In the diagram, the top horizontal map and vertical maps between first and second row are $\Gamma_{\breve{L}}$-equivariant. The middle horizontal map and vertical maps between second and third row are $G_{\breve{L}}$-equivariant. The bottom horizontal map is G_L-equivariant by the isomorphism in Remark 2.13. Recall that we have $\breve{L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \mathbf{D}_{\mathrm{cris},\breve{L}}(V)$, so we get that

$$\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V).$$

From (2.5) note that the image of right term in $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is stable under the action of Γ_L. Therefore, in the diagram, the image of top row in the bottom row is stable under the action of G_L. In particular, we obtain a G_L-equivariant isomorphism

$$\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \subset \mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V.$$

Using the stability of $\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)$ under the action of G_L, we also obtain that its image in $\mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}})/I^{(s)} \otimes_{\mathbb{Q}_p} V \xrightarrow{\sim} \mathbf{B}_{\mathrm{inf}}(O_{\overline{L}})/\mu^s \otimes_{\mathbb{Q}_p} V$ is stable under the action of G_L. In other words, $(\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V))/M$ is stable under the action of G_L, where $M = (I^{(s)}\mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V) \cap (\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V))$. Therefore, from the preceding lemma we obtain that the image of $N_{\breve{L},\infty}(V)/\mu^s\tilde{\mathbf{D}}^+_{\breve{L}}(V) \subset \mathbf{B}_{\mathrm{inf}}(O_{\overline{L}})/\mu^s \otimes_{\mathbb{Q}_p} V$ is stable under the action of G_L. Hence, from (2.14) we conclude that $N_{\breve{L},\infty}(V)$ is stable under the action of Γ_L and the natural composition

$$\mathbf{B}_{\mathrm{inf}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \subset \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V),$$

is compatible with the action of Γ_L. ∎

From Definition 2.31 and Lemma 2.30 we have that

$$N_{\mathrm{rig},L}(V) := \mathcal{M}_L(D_L) = (\mathbf{B}^+_{\mathrm{rig},L}[\tfrac{\mu}{t}] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)) \cap N_{\mathrm{rig},\breve{L}}(V) \subset \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\tfrac{\mu}{t}] \otimes_{O_{\breve{L}}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V),$$

such that $N_{\mathrm{rig},L}(V)$ is a finite free $\mathbf{B}^+_{\mathrm{rig},L}$-module of rank $= \dim_{\mathbb{Q}_p} V$ (see Lemma 2.34 (1)) and $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} N_{\mathrm{rig},L}(V) \xrightarrow{\sim} N_{\mathrm{rig},\breve{L}}(V)$ compatible with the Frobenius-semilinear operator φ (see Lemma 2.33 (3)). Moreover, from (2.12) we have a φ-equivariant inclusion

$$N_{\mathrm{rig},L}(V) \subset \mathbf{B}^+_{\mathrm{rig},L}[\tfrac{1}{t}] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V),$$

where the right hand side is equipped with a Γ_L-action induced by the isomorphism in Remark 2.13 and given by the formula $g(a \otimes d) = g(a) \otimes \sum_{\mathbf{k}\in\mathbb{N}^d} \prod_{i=1}^d \partial_i^{k_i}(d) \prod_{i=1}^d (g([X_i^\flat]) - [X_i^\flat])^{[k_i]}$, for $g \in \Gamma_L$.

Proposition 2.40. *The $\mathbf{B}^+_{\mathrm{rig},L}$-module $N_{\mathrm{rig},L}(V) \subset \mathbf{B}^+_{\mathrm{rig},L}[\tfrac{1}{t}] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is stable under the action of Γ_L.*

Proof. From Corollary 2.38, inside $\mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbb{Q}_p} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$, we have that

$$\tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} N_{\mathrm{rig},\breve{L}}(V) \xrightarrow{\sim} \tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} N_{\mathrm{rig},L}(V),$$

is stable under the action of Γ_L. Moreover, we have a Γ_L-equivariant embedding $\mathbf{B}^+_{\mathrm{rig},L}[\tfrac{1}{t}] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \subset \mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbb{Q}_p} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. Therefore, inside $\mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbb{Q}_p} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$, the following intersection is stable under Γ_L-action

$$\begin{aligned}
&(\tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} N_{\mathrm{rig},L}(V)) \cap (\mathbf{B}^+_{\mathrm{rig},L}[\tfrac{1}{t}] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)) \\
&\qquad = (\tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} N_{\mathrm{rig},L}(V)) \cap (\mathbf{B}^+_{\mathrm{rig},L}[\tfrac{1}{t}] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} N_{\mathrm{rig},L}(V)) \\
&\qquad = (\tilde{\mathbf{B}}^+_{\mathrm{rig},L} \cap \mathbf{B}^+_{\mathrm{rig},L}[\tfrac{1}{t}]) \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} N_{\mathrm{rig},L}(V) = N_{\mathrm{rig},L}(V).
\end{aligned}$$

The first equality follows from (2.12) and the second equality follows because $N_{\mathrm{rig},L}(V)$ is finite free over $\mathbf{B}^+_{\mathrm{rig},L}$. This proves the claim. ∎

Corollary 2.41. *The action of Γ_L on $N_{\mathrm{rig},L}(V)$ is trvial modulo μ.*

Proof. From (2.12) and Proposition 2.40 we have (φ, Γ_L)-equivariant inclusion

$$N_{\mathrm{rig},L}(V) \subset \mathbf{B}^+_{\mathrm{rig},L}[\tfrac{\mu}{t}] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} N_{\mathrm{rig},L}(V) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},L}[\tfrac{\mu}{t}] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V),$$

where the rightmost term is equipped with a Γ_L-action induced by the isomorphism in Remark 2.13 and given by the formula $g(a \otimes d) = g(a) \otimes \sum_{\mathbf{k}\in\mathbb{N}^d} \prod_{i=1}^d \partial_i^{k_i}(d) \prod_{i=1}^d (g([X_i^\flat]) - [X_i^\flat])^{[k_i]}$, for

$g \in \Gamma_L$. Since $(g-1)\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \subset \mu\mathbf{B}^+_{\mathrm{rig},L}[\mu/t]$ and $(g-1)[X_i^\flat] \in \mu\mathbf{B}^+_L$, we obtain that Γ_L-action is trivial modulo μ on $\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} N_{\mathrm{rig},L}(V)$.

Now let $x \in N_{\mathrm{rig},L}(V)$, then for $g \in \Gamma_L$ we have that $(g-1)x \in N_{\mathrm{rig},L}(V) \subset N_{\mathrm{rig},\breve{L}}(V)$ and $(g-1)x \in \mu\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} N_{\mathrm{rig},L}(V)$. Inside $N_{\mathrm{rig},\breve{L}}(V)[\mu/t]$, we have

$$\begin{aligned} N_{\mathrm{rig},\breve{L}}(V) &\cap (\mu\mathbf{B}^+_{\mathrm{rig},L}[\tfrac{\mu}{t}] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} N_{\mathrm{rig},L}(V)) \\ &= (\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} N_{\mathrm{rig},L}(V)) \cap (\mu\mathbf{B}^+_{\mathrm{rig},L}[\tfrac{\mu}{t}] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} N_{\mathrm{rig},L}(V)) \\ &= (\mathbf{B}^+_{\mathrm{rig},\breve{L}} \cap \mu\mathbf{B}^+_{\mathrm{rig},L}[\tfrac{\mu}{t}]) \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} N_{\mathrm{rig},L}(V) = \mu N_{\mathrm{rig},L}(V), \end{aligned}$$

where the first equality follows from Lemma 2.33 (3), the second equality follows since $N_{\mathrm{rig},L}(V)$ is finite free over $\mathbf{B}^+_{\mathrm{rig},L}$ and the last equality follows since $\mu\mathbf{B}^+_{\mathrm{rig},L} = (\mathbf{B}^+_{\mathrm{rig},\breve{L}} \cap \mu\mathbf{B}^+_{\mathrm{rig},L}[\mu/t]) \subset \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t]$. Hence, we get that $(g-1)N_{\mathrm{rig},L}(V) \subset \mu N_{\mathrm{rig},L}(V)$ for $g \in \Gamma_L$. ∎

2.5.3. Compatibility with (φ, Γ_L)-modules. From §2.2 recall that $\mathbf{D}^\dagger_{\mathrm{rig},L}(V)$ is a pure of slope 0 finite free (φ, Γ_L)-module over $\mathbf{B}^\dagger_{\mathrm{rig},L}$.

Proposition 2.42. *There is a natural (φ, G_L)-equivariant isomorphism*

$$\tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[\tfrac{1}{t}] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[\tfrac{1}{t}] \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},L}} \mathbf{D}^\dagger_{\mathrm{rig},L}(V).$$

Proof. From Remark 2.13, there is a natural $\tilde{\mathbf{B}}^+_{\mathrm{rig}}$-linear and (φ, G_L)-equivariant map

$$\tilde{\mathbf{B}}^+_{\mathrm{rig}} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \longrightarrow \mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V.$$

Furthermore, $\mathbf{B}_{\mathrm{inf}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)$ is stable under the action of Γ_L (see Proposition 2.37) and we have a G_L-equivariant isomorphism

$$\tilde{\mathbf{B}}^+_{\mathrm{rig}}[\tfrac{1}{t}] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \tilde{\mathbf{B}}^+_{\mathrm{rig}}[\tfrac{1}{t}] \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \xrightarrow{\sim} \tilde{\mathbf{B}}^+_{\mathrm{rig}}[\tfrac{1}{t}] \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} N_{\mathrm{rig},\breve{L}}(V).$$

where the second isomorphism follows from (2.9). Extending the inclusion in Lemma 2.27 (2) along $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty}) \subset \tilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t]$, we obtain a (φ, G_L)-equivariant commutative diagram

$$\begin{tikzcd} \tilde{\mathbf{B}}^+_{\mathrm{rig}}[\tfrac{1}{t}] \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} N_{\mathrm{rig},\breve{L}}(V) \arrow[r, "\sim"] & \tilde{\mathbf{B}}^+_{\mathrm{rig}}[\tfrac{1}{t}] \otimes_{\mathbb{Q}_p} V \arrow[d] \\ \tilde{\mathbf{B}}^+_{\mathrm{rig}}[\tfrac{1}{t}] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \arrow[u, "\wr"] \arrow[r] & \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V. \end{tikzcd}$$

The image of bottom left and top right coincide inside $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V$. Hence, we obtain a (φ, G_L)-equivariant isomorphism

$$\tilde{\mathbf{B}}^+_{\mathrm{rig}}[\tfrac{1}{t}] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \tilde{\mathbf{B}}^+_{\mathrm{rig}}[\tfrac{1}{t}] \otimes_{\mathbb{Q}_p} V.$$

In the commutative diagram above, extending the left vertical map and top horizontal map $\tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[\frac{1}{t}]$-linearly and using (2.3) and Lemma 2.33 (3), we obtain a string of (φ, G_L)-equivariant isomorphisms

$$\begin{aligned} \tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[\tfrac{1}{t}] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) &\xrightarrow{\sim} \tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[\tfrac{1}{t}] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} N_{\mathrm{rig},L}(V) \\ &\xrightarrow{\sim} \tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[\tfrac{1}{t}] \otimes_{\mathbb{Q}_p} V \xrightarrow{\sim} \tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[\tfrac{1}{t}] \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},L}} \mathbf{D}^\dagger_{\mathrm{rig},L}(V). \end{aligned}$$

This proves the claim. ∎

Extending scalars along the injective map $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^\dagger_{\mathrm{rig},L}$, we obtain that $\mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} N_{\mathrm{rig},L}(V)$ is a pure of slope 0 finite free (φ, Γ_L)-module over $\mathbf{B}^\dagger_{\mathrm{rig},L}$ of rank $= \dim_{\mathbb{Q}_p} V$ (see Lemma 2.34 and Proposition 2.40). Moreover, from Propositon 2.42 we have a (φ, G_L)-equivariant isomorphism

$$\tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[\tfrac{1}{t}] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} N_{\mathrm{rig},L}(V) \xrightarrow{\sim} \tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[\tfrac{1}{t}] \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},L}} \mathbf{D}^\dagger_{\mathrm{rig},L}(V).$$

Therefore, by functoriality of $\mathbf{D}^\dagger_{\mathrm{rig},L}(V)$, we get a natural $\mathbf{B}^\dagger_{\mathrm{rig},L}$-linear and (φ, Γ_L)-equivariant isomorphism

$$\mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} N_{\mathrm{rig},L}(V) \xrightarrow{\sim} \mathbf{D}^\dagger_{\mathrm{rig},L}(V).$$

By definition $\mathbf{D}^\dagger_{\mathrm{rig},L}(V) = \mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^\dagger_L} \mathbf{D}^\dagger_L(V)$, so we get that

Corollary 2.43. *There exists a natural $\mathbf{B}^\dagger_{\mathrm{rig},L}$-linear and (φ, Γ_L)-equivariant isomorphism*

$$\mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} N_{\mathrm{rig},L}(V) \xrightarrow{\sim} \mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^\dagger_L} \mathbf{D}^\dagger_L(V).$$

2.5.4. Obtaining Wach module. The finite free $\mathbf{B}^+_{\mathrm{rig},L}$-module $N_{\mathrm{rig},L}(V)$ is of finite $[p]_q$-height s and pure of slope 0 (see Lemma 2.34), therefore from Lemma 2.7 there exists a unique finite free $\mathbf{B}^+_L$-module of rank $= \dim_{\mathbb{Q}_p} V$ and finite height s (see the proof of Lemma 2.7).

Definition 2.44. Define $\mathbf{N}_L(V) := N_{\mathrm{rig},L}(V) \cap \mathbf{D}^\dagger_L(V) \subset \mathbf{D}^\dagger_{\mathrm{rig},L}(V)$.

The module $\mathbf{N}_L(V)$ is equipped with induced Frobenius-semilinear endomorphism φ such that cokernel of the induced injective map $(1 \otimes \varphi) : \varphi^*(\mathbf{N}_L(V)) \to \mathbf{N}_L(V)$ is killed by $[p]_q^s$ since the same is true for $N_{\mathrm{rig},L}(V)$ and $1 \otimes \varphi : \varphi^*(\mathbf{D}^\dagger_L(V)) \xrightarrow{\sim} \mathbf{D}^\dagger_L(V)$. Moreover, we have $\mathbf{N}_L(V) \subset \mathbf{D}^+_L(V)$ because inside $\mathbf{D}^\dagger_{\mathrm{rig},L}(V)$ we have

$$\begin{aligned}
\mathbf{N}_L(V) = N_{\mathrm{rig},L}(V) \cap \mathbf{D}^\dagger_L(V) &\subset (\tilde{\mathbf{B}}^+_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V)^{H_L} \cap (\mathbf{B}^\dagger \otimes_{\mathbb{Q}_p} V)^{H_L} \\
&\subset ((\tilde{\mathbf{B}}^+_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V) \cap (\mathbf{B}^\dagger \otimes_{\mathbb{Q}_p} V))^{H_L} \\
&\subset ((\tilde{\mathbf{B}}^+_{\mathrm{rig}} \cap \mathbf{B}^\dagger) \otimes_{\mathbb{Q}_p} V)^{H_L} = (\mathbf{B}^+ \otimes_{\mathbb{Q}_p} V)^{H_L} = \mathbf{D}^+_L(V).
\end{aligned}$$

Furthemore, since $N_{\mathrm{rig},L}(V)$ and $\mathbf{D}^\dagger_L(V)$ are stable under compatible action of Γ_L (see Proposition 2.40 and Corollary 2.43), we conclude that $\mathbf{N}_L(V)$ is stable under Γ_L-action. In particular, from the preceding discussion and Lemma 2.7 we have (φ, Γ_L)-equivariant isomorphisms

$$\mathbf{B}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_L} \mathbf{N}_L(V) \xrightarrow{\sim} N_{\mathrm{rig},L}(V) \text{ and } \mathbf{B}^\dagger_L \otimes_{\mathbf{B}^+_L} \mathbf{N}_L(V) \xrightarrow{\sim} \mathbf{D}^\dagger_L(V). \tag{2.15}$$

Lemma 2.45. *The action of Γ_L on $\mathbf{N}_L(V)$ is trvial modulo μ.*

Proof. Let $g \in \Gamma_L$ and $x \in \mathbf{N}_L(V)$. Then, $(g-1)x \in \mathbf{N}_L(V) \subset \mathbf{D}^\dagger_L(V)$. Moreover, from Corollary 2.41 we have $(g-1)x \in \mu N_{\mathrm{rig},L}(V)$. Therefore, using (2.15) inside $\mathbf{D}^\dagger_{\mathrm{rig},L}(V)$ we get

$$(g-1)x \in \mathbf{D}^\dagger_L(V) \cap \mu N_{\mathrm{rig},L}(V) = (\mathbf{B}^\dagger_L \cap \mu \mathbf{B}^+_{\mathrm{rig},L}) \otimes_{\mathbf{B}^+_L} \mathbf{N}_L(V) = \mu \mathbf{N}_L(V). \qquad \blacksquare$$

Definition 2.46. Define the Wach module over $\mathbf{A}^+_L = \mathbf{B}^+_L \cap \mathbf{A}_L \subset \mathbf{B}_L$ as

$$\mathbf{N}_L(T) := \mathbf{N}_L(V) \cap \mathbf{D}_L(T) \subset \mathbf{D}_L(V).$$

Proof of Theorem 2.24. We will show that $\mathbf{N}_L(T)$ from Definition 2.46 satisfies all the axioms of Definition 2.21. First, using Remark 2.8 we observe that $\mathbf{N}_L(T)$ is finite free of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$ over $\mathbf{A}_L^+$. Alternatively, one can also use [Ber04, Lemme II.1.3] (the proof of loc. cit. does not require the residue field of O_L to be perfect).

Next, $\mathbf{N}_L(T)$ is equipped with an induced Frobenius-semilinear endomorphism φ such that cokernel of the injective map $1 \otimes \varphi : \varphi^*(\mathbf{N}_L(T)) \to \mathbf{N}_L(T)$ is killed by $[p]_q^s$. Indeed, we have $\varphi : \mathbf{A}_L^+ \to \mathbf{A}_L^+$ is finite and faithfully flat of degree p^{d+1} and we can write $\varphi^*(\mathbf{A}_L) = \mathbf{A}_L^+ \otimes_{\varphi, \mathbf{A}_L^+} \mathbf{A}_L$ and $\varphi^*(\mathbf{B}_L^+) = \mathbf{A}_L^+ \otimes_{\varphi, \mathbf{A}_L^+} \mathbf{B}_L^+$ (see §2.1.2). Therefore, we get that $\varphi^*(\mathbf{N}_L(V)) = \mathbf{B}_L^+ \otimes_{\varphi, \mathbf{B}_L^+} \mathbf{N}_L(V) = \mathbf{A}_L^+ \otimes_{\varphi, \mathbf{A}_L^+} \mathbf{N}_L(V)$ and $\varphi^*(\mathbf{D}_L(T)) = \mathbf{A}_L \otimes_{\varphi, \mathbf{A}_L} \mathbf{D}_L(T) = \mathbf{A}_L^+ \otimes_{\varphi, \mathbf{A}_L^+} \mathbf{D}_L(T)$. It follows that $\varphi^*(\mathbf{N}_L(T)) = \varphi^*(\mathbf{N}_L(V)) \cap \varphi^*(\mathbf{D}_L(T)) \subset \varphi^*(\mathbf{D}_L(V))$. Since $1 \otimes \varphi$ is injective on $\varphi^*(\mathbf{D}_L(V))$, $1 \otimes \varphi : \varphi^*(\mathbf{D}_L(T)) \xrightarrow{\sim} \mathbf{D}_L(T)$ and cokernel of $1 \otimes \varphi : \varphi^*(\mathbf{N}_L(V)) \to \mathbf{N}_L(V)$ is killed by $[p]_q^s$, we get that cokernel of $1 \otimes \varphi : \varphi^*(\mathbf{N}_L(T)) \to \mathbf{N}_L(T)$ is killed by $[p]_q^s$.

Furthermore, note that $\mathbf{N}_L(T)$ is equipped with an induced Γ_L-action such that Γ_L acts trivially on $\mathbf{N}_L(T)/\mu\mathbf{N}_L(T)$ (follows easily from Lemma 2.45). Finally, $[p]_q$ is invertible in $\mathbf{A}_L$, therefore $\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T)$ is a finite free of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$ étale (φ, Γ_L)-module. Since we have a (φ, Γ_L)-equivariant injection $\mathbf{N}_L(T) \subset \mathbf{D}_L(T)$, by functoriality of étale (φ, Γ_L)-modules it follows that $\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T) \xrightarrow{\sim} \mathbf{D}_L(T)$. It is easy to see that $\mathbf{B}_L^+ \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T) \xrightarrow{\sim} \mathbf{N}_L(V)$. This concludes the proof. ∎

3. Relative case

We will use the setup and notations from §1.3. Recall that R is the p-adic completion of an étale algebra over $O_F\langle X_1^{\pm 1}, \ldots, X_d^{\pm 1}\rangle$ and $O_L = (R_{(p)})^\wedge$. Set $R_\infty = \cup_{i=1}^d R[\mu_{p^\infty}, X_i^{1/p^\infty}]$ and $\overline{R}$ is the union of finite R-subalgebras S in a fixed algebraic closure $\overline{\mathrm{Frac}(R)} \supset \overline{F}$, such that $S[1/p]$ is étale over $R[1/p]$. We have (see [Abh21, §2 & §3])

$$
\begin{aligned}
&G_R = \mathrm{Gal}(\overline{R}[\tfrac{1}{p}]/R[\tfrac{1}{p}]),\ H_R = \mathrm{Gal}(\overline{R}[\tfrac{1}{p}]/R_\infty[\tfrac{1}{p}]),\\
&\Gamma_R = G_R/H_R = \mathrm{Gal}(R_\infty[\tfrac{1}{p}]/R[\tfrac{1}{p}]) \xrightarrow{\sim} \mathbb{Z}_p(1)^d \rtimes \mathbb{Z}_p^\times,\\
&\Gamma_R' = \mathrm{Gal}(R_\infty[\tfrac{1}{p}]/R(\mu_{p^\infty})[\tfrac{1}{p}]) \xrightarrow{\sim} \mathbb{Z}_p(1)^d,\ \mathrm{Gal}(R(\mu_{p^\infty})[\tfrac{1}{p}]/R[\tfrac{1}{p}]) = \Gamma_R/\Gamma_R' \xrightarrow{\sim} \mathbb{Z}_p^\times.
\end{aligned}
$$

3.1. Period rings. In this section we briefly recall the period rings to be used in the sequel. Please refer to [Abh21, §2 and §3] for more details. Let $\mathbf{A}_{\mathrm{inf}}(R_\infty) := W(R_\infty^\flat)$ and $\mathbf{A}_{\mathrm{inf}}(\overline{R}) := W(\overline{R}^\flat)$ admitting the Frobenius on Witt vectors and continuous G_R-action (for the weak topology). Moreover, we have $\mathbf{A}_{\mathrm{inf}}(R_\infty) = \mathbf{A}_{\mathrm{inf}}(\overline{R})^{H_R}$. We fix $\overline{\mu} := \varepsilon - 1$, where $\varepsilon := (1, \zeta_p, \zeta_{p^2}, \ldots) \in O_{F_\infty}^\flat$ and let $\mu := [\varepsilon] - 1, \xi := \mu/\varphi^{-1}(\mu) \in \mathbf{A}_{\mathrm{inf}}(O_{F_\infty})$. Moreover, for $1 \leq i \leq d$, we fix $X_i^\flat = (X_i, X_i^{1/p}, X_i^{1/p^2}, \ldots) \in R_\infty$ and its Teichmüller lift $[X_i^\flat] \in \mathbf{A}_{\mathrm{inf}}(R_\infty)$. We set $\mathbf{A}_{\mathrm{cris}}(R_\infty) := \mathbf{A}_{\mathrm{inf}}(R_\infty)\langle \xi^k/k!, k \in \mathbb{N}\rangle$. Let $t := \log(1+\mu) \in \mathbf{A}_{\mathrm{cris}}(O_{F_\infty})$ and set $\mathbf{B}_{\mathrm{cris}}^+(R_\infty) := \mathbf{A}_{\mathrm{cris}}(R_\infty)[1/p]$ and $\mathbf{B}_{\mathrm{cris}}(R_\infty) := \mathbf{B}_{\mathrm{cris}}^+(R_\infty)[1/t]$. These rings are stable under induced Frobenius and Γ_R-action. Furthermore, one can define period rings $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_\infty)$, $\mathcal{O}\mathbf{B}_{\mathrm{cris}}^+(R_\infty)$ and $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(R_\infty)$. These rings are equipped with a Frobenius endomorphism φ and continuous Γ_R-action. Rings with a subscript "cris" are equipped with a decreasing filtration and rings with a prefix "$\mathcal{O}$" are further equipped with an integrable connection satisfying Griffiths transversality with respect to the the filtration (see [Abh21, §2.2]). One can define variations of these rings over $\overline{R}$ as well.

Let $A_\square^+$ denote the (p, μ)-adic completion of $O_F[\mu, [X_1^\flat]^{\pm 1}, \ldots, [X_d^\flat]^{\pm 1}]$. Note that $A_\square^+$ naturally embeds into $\mathbf{A}_{\mathrm{inf}}(R_\infty)$ and it is stable under the induced Frobenius endomorphism φ and the action of Γ_R (see [Abh21, §3]). Similar to §1.3 and §2.1.2, it follows that the induced Frobenius endomorphism φ on $A_\square^+$ is finite and faithfully flat of degree p^{d+1}. Let $\mathbf{A}_R^+$

denote the (p, μ)-adic completion of the unique extension of $A_\square^+ \subset \mathbf{A}_{\mathrm{inf}}(R_\infty)$ along the p-adically completed étale map $O_F\langle X_1^{\pm 1}, \ldots, X_d^{\pm 1}\rangle \to R$ (see [Abh21, §3.3.2]). In particular, $\mathbf{A}_R^+$ can be described as the (p, μ)-adic completion of an étale algebra over $A_\square^+$ defined by the same multivariate polynomials as in (1.1). The algebra $\mathbf{A}_R^+ \subset \mathbf{A}_{\mathrm{inf}}(R_\infty)$ is stable under the induced Frobenius endomorphism φ and the action of Γ_R. Moreover, similar to §2.1.2 the induced Frobenius endomorphism φ on $\mathbf{A}_R^+$ is finite and faithfully flat of degree p^{d+1} and $\varphi^*(\mathbf{A}_R^+) = \oplus_\alpha \varphi(\mathbf{A}_R^+)u_\alpha$ where $u_\alpha = (1+\mu)^{\alpha_0}[X_1^\flat]^{\alpha_1} \cdots [X_d^\flat]^{\alpha_d}$ for $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \{0, 1, \ldots, p-1\}^{[0,d]}$. We set $\mathbf{A}_R := \mathbf{A}_R^+[\frac{1}{\mu}]^\wedge$ and the Frobenius endomorphism φ and continuous Γ_R-action on $\mathbf{A}_R^+$ naturally extend to $\mathbf{A}_R$. Similar to above, the induced Frobenius endomorphism φ on $\mathbf{A}_R$ is finite and faithfully flat of degree p^{d+1} and $\varphi^*(\mathbf{A}_R) = \mathbf{A}_R \otimes_{\varphi, \mathbf{A}_R} \mathbf{A}_R = \oplus_\alpha \varphi(\mathbf{A}_R)u_\alpha = (\oplus_\alpha \varphi(\mathbf{A}_R^+)u_\alpha) \otimes_{\varphi(\mathbf{A}_R^+)} \varphi(\mathbf{A}_R) = \mathbf{A}_R^+ \otimes_{\varphi, \mathbf{A}_R^+} \mathbf{A}_R$.

Let $\mathbb{C}(\overline{R}) := \widehat{\overline{R}}$, $\tilde{\mathbf{A}} := W(\mathbb{C}(\overline{R})^\flat)$ and $\tilde{\mathbf{B}} := \tilde{\mathbf{A}}[1/p]$ admitting the Frobenius on Witt vectors and continuous G_R-action (for the weak topology). We have natural Frobenius and Γ_R-equivariant embeddings $\mathbf{A}_R^+ \subset \mathbf{A}_{\mathrm{inf}}(R_\infty)$ and $\mathbf{A}_R \subset \tilde{\mathbf{A}}^{H_R}$ and we set $\mathbf{B}_R = \mathbf{A}_R[1/p]$. Let $\mathbf{A}$ denote the p-adic completion of the maximal unramified extension of $\mathbf{A}_R$ inside $\tilde{\mathbf{A}}$ and set $\mathbf{B} := \mathbf{A}[1/p] \subset \tilde{\mathbf{B}}$. The rings $\mathbf{A}$ and $\mathbf{B}$ are stable under induced Frobenius and Γ_R-action and we have $\mathbf{A}_R = \mathbf{A}^{H_R}$ and $\mathbf{B}_R = \mathbf{B}^{H_R}$. Moreover, set $\mathbf{A}^+ = \mathbf{A}_{\mathrm{inf}}(\overline{R}) \cap \mathbf{A} \subset \tilde{\mathbf{A}}$ and $\mathbf{B}^+ = \mathbf{A}^+[1/p]$, then we have $\mathbf{A}_R^+ = (\mathbf{A}^+)^{H_R}$ and $\mathbf{B}_R^+ = (\mathbf{B}^+)^{H_R}$. Via the identification $\Gamma_L \xrightarrow{\sim} \Gamma_R$, we have a (φ, Γ_L)-equivariant isomorphism $\mathbf{A}_L^+ \xrightarrow{\sim} ((\mathbf{A}_R^+)_{(p,\mu)})^\wedge$ where $^\wedge$ denotes the (p, μ)-adic completion. Moreover, $\mathbf{A}_R^+ = \mathbf{A}_L^+ \cap \mathbf{A}_R \subset \mathbf{A}_L$ and $\mathbf{B}_R^+ = \mathbf{A}_R^+[1/p] = \mathbf{B}_L^+ \cap \mathbf{B}_R \subset \mathbf{B}_L$.

3.2. p-adic representations. Let T be a finite free $\mathbb{Z}_p$-representation of G_R. By the theory of (φ, Γ_R)-modules (see [Fon90] and [And06]) we have a finite projective étale (φ, Γ_R)-module $\mathbf{D}_R(T) = (\mathbf{A} \otimes_{\mathbb{Z}_p} T)^{H_R}$ over $\mathbf{A}_R$ of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$. Moreover, $\tilde{\mathbf{D}}_R(T) = (\tilde{\mathbf{A}} \otimes_{\mathbb{Z}_p} T)^{H_R} \xrightarrow{\sim} \tilde{\mathbf{A}}^{H_R} \otimes_{\mathbf{A}_R} \mathbf{D}_R(T)$. We also have natural (φ, Γ_R)-equivariant isomorphisms

$$\mathbf{A} \otimes_{\mathbf{A}_R} \mathbf{D}_R(T) \xrightarrow{\sim} \mathbf{A} \otimes_{\mathbb{Z}_p} T. \tag{3.1}$$

These constructions are functorial and induce equivalence of categories

$$\mathrm{Rep}_{\mathbb{Z}_p}(G_R) \xrightarrow{\sim} (\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{\mathbf{A}_R}^{\text{ét}}. \tag{3.2}$$

Similar statements are also true for p-adic representations of G_R. Furthermore, let $\mathbf{D}_R^+(T) := (\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T)^{H_R}$ be the (φ, Γ_R)-module over $\mathbf{A}_R^+$ associated to T and for $V = T[1/p]$ let $\mathbf{D}_R^+(V) = \mathbf{D}_R^+(T)[1/p]$ be the (φ, Γ_R)-module over $\mathbf{B}_R^+$ associated to V.

Let V be a p-adic crystalline representation of G_R, $T \subset V$ a G_R-stable $\mathbb{Z}_p$-lattice. From p-adic Hodge theory of G_R (see [Bri08]), one can attach to V a rank $= \dim_{\mathbb{Q}_p} V$ filtered (φ, ∂)-module over $R[1/p]$, denoted as $\mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$. Moreover, we have a continuous homomorphism $G_L \to G_R$, in particular, V is a p-adic representation of G_L. Base changing the isomorphism $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V$ along $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$, one obtains that V is a crystalline representation of G_L. By functoriality and comparing dimensions over L, we obtain a natural isomorphism $L \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ compatible with Frobenius, filtration and connection.

3.3. Relative Wach modules. In this section we will describe Wach modules in the relative case and finite $[p]_q$-height representations of G_R and relate them to crystalline representations. We start with the notion of Wach modules.

3.3.1. Wach modules over $\mathbf{A}_R^+$. In $\mathbf{A}_{\mathrm{inf}}(O_{F_\infty})$ we fix $q = [\varepsilon]$, $\mu = [\varepsilon] - 1 = q - 1$ and $\tilde{\xi} = \varphi(\pi)/\pi = [p]_q$. Let N be a finitely generated $\mathbf{A}_R^+$-module.

Definition 3.1. The sequence $\{p, \mu\}$ in $\mathbf{A}_R^+$ is said to be N*-regular* if N is p-torsion free and N/pN is μ-torsion free. The sequence $\{p, \mu\}$ in $\mathbf{A}_R^+$ is said to be *strictly* N*-regular* if both $\{p, \mu\}$ and $\{\mu, p\}$ are N-regular.

Remark 3.2. In Definition 3.1 it is easy to see that $\{p, \mu\}$ is strictly N-regular if and only if N is (p, μ)-torsion free and $N/\mu N$ is p-torsion free.

Lemma 3.3. *Let N be a finitely generated $\mathbf{A}_R^+$-module and set $\mathscr{C}^\bullet : N \xrightarrow{(p,\mu)} N \oplus N \xrightarrow{(\mu,-p)} N$ where the first map is given by $x \mapsto (px, \mu x)$ and the second map is given by $(x, y) \mapsto \mu x - py$. Then $\{p, \mu\}$ is strictly N-regular if and only if $H^0(\mathscr{C}^\bullet) = H^1(\mathscr{C}^\bullet) = 0$.*

Proof. If $\{p, \mu\}$ is strictly N-regular then $(N/p)[\mu] = (N/\mu)[p] = 0$ by a simple diagram chase. This implies $H^0(\mathscr{C}^\bullet) = H^1(\mathscr{C}^\bullet) = 0$. The converse follows by a direct computation. ∎

Remark 3.4. The complex $\mathscr{C}^\bullet$ in Lemma 3.3 computes local cohomology of N with respect to the ideal $(p, \mu) \subset \mathbf{A}_R^+$ (see [Wei94, Theorem 4.6.8]). So if we set $Z = V(p, \mu) \subset \mathrm{Spec}\,(\mathbf{A}_R^+) = X$ as a closed subset, then one also says that $\mathscr{C}^\bullet$ computes cohomology with compact support along Z, written as $H_Z^i(X, N)$ (see [Wei94, Generalization 4.6.2] for details).

Lemma 3.5. *Let N be a finitely generated $\mathbf{A}_R^+$-module such that $\{p, \mu\}$ is strictly N-regular. Then we have $N = N[1/p] \cap N[1/\mu] \subset N[1/p, 1/\mu]$ as $\mathbf{A}_R^+$-modules. Moreover, $N = N[1/p] \cap N[1/\mu]^\wedge \subset N[1/\mu]^\wedge[1/p]$, where $^\wedge$ denotes the p-adic completion.*

Proof. A simple diagram chase shows that $(N/p)[\mu] = (N/\mu)[p] = 0$ and $(N[1/\mu])/p = (N/p)[1/\mu]$. So we have $N/p^nN \subset (N/p^n)[1/\mu]$ for all $n \in \mathbb{N}$ and therefore $N \cap p^nN[1/\mu] = p^nN$. Hence, $N[1/p] \cap N[1/\mu] = N$. Furthermore, since $(N[1/\mu]^\wedge)/p^n = (N[1/\mu])/p^n = (N/p^n)[1/\mu]$, similar to above we also get that $N \cap p^nN[1/\mu]^\wedge = p^nN$ for all $n \in N$. Hence, $N[1/p] \cap N[1/\mu]^\wedge = N$. ∎

Definition 3.6. Let $a, b \in \mathbb{Z}$ with $b \geq a$. A *Wach module* over $\mathbf{A}_R^+$ with weights in the interval $[a, b]$ is a finitely generated $\mathbf{A}_R^+$-module N equipped with a continuous and semilinear action of Γ_R satisfying the following:

(1) The sequence $\{p, \mu\}$ is strictly N-regular.

(2) The action of Γ_R on $N/\mu N$ is trivial.

(3) There is a Frobenius-semilinear operator $\varphi : N[1/\mu] \to N[1/\varphi(\mu)]$ commuting with the action of Γ_R such that $\varphi(\mu^bN) \subset \mu^bN$ and the map $(1 \otimes \varphi) : \varphi^*(\mu^bN) \to \mu^bN$ is injective and its cokernel is killed by $\tilde{\xi}^{b-a} = [p]_q^{b-a}$.

Define the *height* of N to be the smallest value of $b - a$, where $a, b \in \mathbb{Z}$ as above. Say N is *effective* if we have $b = 0$ and $a \leq 0$. A Wach module over $\mathbf{B}_R^+$ is defined to be a finitely generated module M admitting an $\mathbf{A}_R^+$-submodule $N \subset M$ with N as above and $N[1/p] = M$.

Denote the category of Wach modules over $\mathbf{A}_R^+$ as $(\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_R^+}^{[p]_q}$ with morphisms between objects being $\mathbf{A}_R^+$-linear Γ_R-equivariant and φ-equivariant (after inverting μ) morphisms.

Proposition 3.7. *Let N be a Wach module over $\mathbf{A}_R^+$. Then $N[1/p]$ is finite projective over $\mathbf{A}_R^+[1/p]$ and $N[1/\mu]$ is finite projective over $\mathbf{A}_R^+[1/\mu]$.*

Proof. Without loss of generality we may assume that N is effective. Then the first claim follows from Proposition A.1. For the second claim, note that N is p-torsion free, so $\mathbf{A}_R \otimes_{\mathbf{A}_R^+} N$ is a p-torsion free étale (φ, Γ_R)-module over $\mathbf{A}_R$ and therefore finite projective by Lemma A.4. Since $\mathbf{A}_R^+[1/\mu]$ is noetherian we have $N[1/\mu]^\wedge \xrightarrow{\sim} \mathbf{A}_R \otimes_{\mathbf{A}_R^+[1/\mu]} N[1/\mu] = \mathbf{A}_R \otimes_{\mathbf{A}_R^+} N$, where $^\wedge$ denotes the p-adic completion. So we obtain an isomorphism $f : N[1/\mu]^\wedge[1/p] \xrightarrow{\sim} \mathbf{A}_R \otimes_{\mathbf{A}_R^+[1/\mu]} N[1/\mu, 1/p]$. Note that the natural map $\mathrm{Spec}\,(\mathbf{A}_R^+[1/\mu]^\wedge) \cup \mathrm{Spec}\,(\mathbf{A}_R^+[1/\mu, 1/p]) \to \mathrm{Spec}\,(\mathbf{A}_R^+[1/\mu])$ is a flat cover. Therefore, from the data $(N[1/p], N[1/\mu]^\wedge, f)$ and faithfully flat descent we obtain that $N[1/\mu]$ is finite projective over $\mathbf{A}_R^+[1/\mu]$. ∎

Remark 3.8. Note that for a Wach module N over $\mathbf{A}_R^+$, we have that N is p-torsion free, in particular, $N \subset N[1/p]$. Since $N[1/p]$ is projective over $\mathbf{A}_R^+[1/p]$ by Proposition 3.7, we obtain that N is a torsion free $\mathbf{A}_R^+$-module.

Lemma 3.9. *Let N be a Wach module over $\mathbf{A}_R^+$, then we have $N = (\mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} N) \cap (\mathbf{A}_R \otimes_{\mathbf{A}_R^+} N) \subset \mathbf{A}_L \otimes_{\mathbf{A}_R^+} N$ as $\mathbf{A}_R^+$-modules.*

Proof. Let $N_R = N$, $N_L = \mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} N$ and $D_R = \mathbf{A}_R \otimes_{\mathbf{A}_R^+} N$. Note that $N_R[1/p]$ is finite projective over $\mathbf{B}_R^+$ with $N_L[1/p] = \mathbf{B}_L^+ \otimes_{\mathbf{B}_R^+} N_R[1/p]$ and $D_R[1/p] = \mathbf{B}_R \otimes_{\mathbf{B}_R^+} N_R[1/p]$, therefore $N_L[1/p] \cap D_R[1/p] = (\mathbf{B}_L^+ \cap \mathbf{B}_R) \otimes_{\mathbf{B}_R^+} N_R[1/p] = N_R[1/p]$. Moreover, we have $N_L \cap D_R \subset N_L[1/p] \cap D_R[1/p] = N_R[1/p]$ and using Lemma 3.5 we see that $N_L \cap D_R = N_L \cap D_R \cap N_R[1/p] = N_R$. ∎

From the proof of Proposition 3.7, it is clear that extending scalars along $\mathbf{A}_R^+ \to \mathbf{A}_R$ induces a functor $(\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_R^+}^{[p]_q} \to (\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_R}^{\text{ét}}$ and we make the following claim:

Proposition 3.10. *The following natural functor is fully faithful*

$$\begin{aligned} (\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_R^+}^{[p]_q} &\longrightarrow (\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_R}^{\text{ét}} \\ N &\longmapsto \mathbf{A}_R \otimes_{\mathbf{A}_R^+} N. \end{aligned}$$

Proof. Let N, N' be two Wach modules over $\mathbf{A}_R^+$. Write $N_R = N$, $N_L = \mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} N$, $D_L = \mathbf{A}_R \otimes_{\mathbf{A}_R^+} N$ and similarly for N'. We need to show that for Wach modules N and N', we have

$$\operatorname{Hom}_{(\varphi,\Gamma)\text{-Mod}_{\mathbf{A}_R^+}^{[p]_q}}(N_R, N'_R) \xrightarrow{\sim} \operatorname{Hom}_{(\varphi,\Gamma)\text{-Mod}_{\mathbf{A}_R}^{\text{ét}}}(D_R, D'_R) \tag{3.3}$$

Note that $\mathbf{A}_R^+ \to \mathbf{A}_R = \mathbf{A}_R^+[1/\mu]^\wedge$ is injective, in particular, the map in (3.3) is injective. To check that (3.3) is surjective take an $\mathbf{A}_R$-linear and (φ, Γ_R)-equivariant map $f : D_R \to D'_R$. We need to show that $f(N_R) \subset N'_R$. Base changing f along $\mathbf{A}_R \to \mathbf{A}_L$ and using the isomorphism $\Gamma_L \xrightarrow{\sim} \Gamma_R$ induces an $\mathbf{A}_L$-linear and (φ, Γ_L)-equivariant map $f : D_L \to D'_L$. Then by Proposition 2.17 we have $f(N_L) \subset N'_L$. Using Lemma 3.9 we get that $f(N_R) = f(N_L \cap D_R) = f(N_L) \cap f(D_R) \subset N'_L \cap D'_R = N'_R \subset D'_L$. ∎

Similar to the imperfect residue field case, composing the functor in Proposition 3.10 with the equivalence in (3.2), we obtain a fully faithful functor

$$\begin{aligned} \mathbf{T}_R : (\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_R^+}^{[p]_q} &\longrightarrow \operatorname{Rep}_{\mathbb{Z}_p}(G_R) \\ N &\longmapsto (\mathbf{A} \otimes_{\mathbf{A}_R^+} N)^{\varphi=1} \xrightarrow{\sim} (W(\mathbb{C}(\overline{R})^\flat) \otimes_{\mathbf{A}_R^+} N)^{\varphi=1}. \end{aligned} \tag{3.4}$$

Proposition 3.11. *Let N be a Wach module over $\mathbf{A}_R^+$ and $T = \mathbf{T}_R(N)$ the associated finite free $\mathbb{Z}_p$-representation of G_R. Then we have a natural G_R-equivariant comparison isomorphism*

$$\mathbf{A}_{\text{inf}}(\overline{R})[\tfrac{1}{\mu}] \otimes_{\mathbf{A}_R^+} N \xrightarrow{\sim} \mathbf{A}_{\text{inf}}(\overline{R})[\tfrac{1}{\mu}] \otimes_{\mathbb{Z}_p} T.$$

Proof. For $r \in \mathbb{N}$ large enough, the Wach module $\mu^r N(-r)$ is always effective and we have $\mathbf{T}_R(\mu^r N(-r)) = T(-r)$ (the twist $(-r)$ denotes the Tate twist on which Γ_L acts via the cyclotomic character). Therefore, it is enough to show the claim for effective Wach modules. Assume N is effective and to avoid confusion let us write $N_R = N$ and $N_L = \mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} N$. Under the identification $\Gamma_L \xrightarrow{\sim} \Gamma_R$ we obtain that N_L is a Wach module over $\mathbf{A}_L^+$, in particular, finite free of rank $= \operatorname{rk}_{\mathbb{Z}_p} T$. From Lemma 2.19 we have G_L-equivariant inclusions

$\mu^s \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}}) \otimes_{\mathbb{Z}_p} T \subset \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}}) \otimes_{\mathbf{A}_L^+} N_L = \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}}) \otimes_{\mathbf{A}_R^+} N_R \subset \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}}) \otimes_{\mathbb{Z}_p} T$. In particular, we have a G_L-equivariant isomorphism

$$\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})[\tfrac{1}{\mu}] \otimes_{\mathbf{A}_R^+} N_R \xrightarrow{\sim} \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})[\tfrac{1}{\mu}] \otimes_{\mathbb{Z}_p} T.$$

Now let $\mathcal{S}$ denote the set of minimal primes of $\overline{R}$ above $pR \subset R$ and for each prime $\mathfrak{p} \in \mathcal{S}$ let $\mathbb{C}(\mathfrak{p})$ denote the p-adic completion of a fixed algebraic closure of $(\overline{R})_{\mathfrak{p}}[1/p]$ with $\mathbb{C}^+(\mathfrak{p})$ as its ring of integers. Let $\widehat{G}_R(\mathfrak{p})$ denote the Galois group of an algebraic closure over L such that the algebraically closed field is contained in $\mathbb{C}(\mathfrak{p})$ and it contains $(\overline{R})_{\mathfrak{p}}$, in particular, we have a natural homomorphism of groups $\widehat{G}_R(\mathfrak{p}) \to G_R$ (see [Bri08, §3.3]). Therefore, similar to the case of G_L, we obtain $\widehat{G}_R(\mathfrak{p})$-equivariant inclusions $\mu^s \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Z}_p} T \subset \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbf{A}_R^+} N_R \subset \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Z}_p} T$ for each $\mathfrak{p} \in \mathcal{S}$. Taking the product over all $\mathfrak{p} \in \mathcal{S}$, we obtain

$$\mu^s \prod_{\mathfrak{p}\in\mathcal{S}} (\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Z}_p} T) \subset \prod_{\mathfrak{p}\in\mathcal{S}} (\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbf{A}_R^+} N_R) \subset \prod_{\mathfrak{p}\in\mathcal{S}} (\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Z}_p} T). \tag{3.5}$$

Inverting μ in the equation above we get

$$\Big(\prod_{\mathfrak{p}\in\mathcal{S}} (\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbf{A}_R^+} N_R)\Big)[\tfrac{1}{\mu}] \xrightarrow{\sim} \Big(\prod_{\mathfrak{p}\in\mathcal{S}} (\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Z}_p} T)\Big)[\tfrac{1}{\mu}]. \tag{3.6}$$

The right hand term of (3.6) can be written as $\Big(\prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))\Big)[\tfrac{1}{\mu}] \otimes_{\mathbb{Z}_p} T$ and the left hand term can be written as

$$\begin{aligned}\Big(\prod_{\mathfrak{p}\in\mathcal{S}} (\mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbf{A}_R^+} N_R)\Big)[\tfrac{1}{\mu}] &= \Big(\Big(\prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))\Big) \otimes_{\mathbf{A}_R^+} N_R\Big)[\tfrac{1}{\mu}] \\ &= \Big(\prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))\Big)[\tfrac{1}{\mu}] \otimes_{\mathbf{A}_R^+[1/\mu]} N_R[\tfrac{1}{\mu}],\end{aligned} \tag{3.7}$$

where the first equality follows from the fact that product is an exact functor on the category of $\mathbf{A}_R^+$-modules and N_R is finitely generated over the noetherian ring $\mathbf{A}_R^+$ (see [Sta23, Tag 059K]). Furthermore, by theory of (φ, Γ)-modules, we have a G_R-equivariant comparison isomorphism

$$W(\mathbb{C}(\overline{R})^\flat) \otimes_{\mathbf{A}_R^+[1/\mu]} N_R[\tfrac{1}{\mu}] \xrightarrow{\sim} W(\mathbb{C}(\overline{R})^\flat) \otimes_{\mathbb{Z}_p} T. \tag{3.8}$$

Using Lemma 3.12, we embed all the terms above inside $\big(\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)\big) \otimes_{\mathbb{Z}_p} T$.

Note that $N_R[1/\mu]$ is finite projective over $\mathbf{A}_R^+[1/\mu]$, so taking the intersection of left hand terms in (3.6) (also see the last term in (3.7)) and (3.8), inside $\big(\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)\big) \otimes_{\mathbb{Z}_p} T$, we obtain

$$\begin{aligned}&\Big(W(\mathbb{C}(\overline{R})^\flat) \otimes_{\mathbf{A}_R^+[1/\mu]} N_R[\tfrac{1}{\mu}]\Big) \cap \Big(\Big(\prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))\Big)[\tfrac{1}{\mu}] \otimes_{\mathbf{A}_R^+[1/\mu]} N_R[\tfrac{1}{\mu}]\Big) \\ &\qquad = \mathbf{A}_{\mathrm{inf}}(\overline{R})[\tfrac{1}{\mu}] \otimes_{\mathbf{A}_R^+[1/\mu]} N_R[\tfrac{1}{\mu}] = \mathbf{A}_{\mathrm{inf}}(\overline{R})[\tfrac{1}{\mu}] \otimes_{\mathbf{A}_R^+} N_R,\end{aligned}$$

where the first equality follows from Lemma 3.12. Similarly, taking the intersection of right hand terms in (3.6) and (3.8), inside $\big(\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)\big) \otimes_{\mathbb{Z}_p} T$, we obtain

$$\Big(W(\mathbb{C}(\overline{R})^\flat) \otimes_{\mathbb{Z}_p} T\Big) \cap \Big(\Big(\prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))\Big)[\tfrac{1}{\mu}] \otimes_{\mathbb{Z}_p} T\Big) = \mathbf{A}_{\mathrm{inf}}(\overline{R})[\tfrac{1}{\mu}] \otimes_{\mathbb{Z}_p} T,$$

where the equality again follows from Lemma 3.12. Since (3.6) and (3.8) are isomorphisms, we obtain a natural commutative diagram

$$\begin{tikzcd}
\mathbf{A}_{\inf}(\overline{R})[\frac{1}{\mu}] \otimes_{\mathbf{A}_R^+} N_R \arrow[r, "\simeq"] \arrow[d] & \mathbf{A}_{\inf}(\overline{R})[\frac{1}{\mu}] \otimes_{\mathbb{Z}_p} T \arrow[d] \\
W(\mathbb{C}(\overline{R})^\flat) \otimes_{\mathbf{A}_R^+} N_R \arrow[r, "\simeq"] & W(\mathbb{C}(\overline{R})^\flat) \otimes_{\mathbb{Z}_p} T,
\end{tikzcd}$$

where the horizontal arrows are bijective and vertical arrows are injective. Finally, as the bottom horizontal isomorphism is G_R-equivariant, we get that the top horizontal isomorphism is G_R-equivariant as well. ∎

Following result was used above:

Lemma 3.12. *Let $\mathcal{S}$ denote the set of minimal primes of $\overline{R}$ above $pR \subset R$ and for each prime $\mathfrak{p} \in \mathcal{S}$ let $\mathbb{C}(\mathfrak{p})$ denote the p-adic completion of a fixed algebraic closure of $(\overline{R})_{\mathfrak{p}}[1/p]$. Set $\mathbb{C}^+(\mathfrak{p})$ as the ring of integers of $\mathbb{C}(\mathfrak{p})$, then we have*

$$\mathbf{A}_{\inf}(\overline{R}) = W(\mathbb{C}(\overline{R})^\flat) \cap \prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\inf}(\mathbb{C}^+(\mathfrak{p})) \subset \prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat).$$

Proof. Recall that $\overline{R}$ is a direct limit of finite and normal $\overline{R}$-algebras, therefore the natural map $\overline{R}/p^n \to \oplus_{\mathfrak{p}\in\mathcal{S}}(\overline{R})_{\mathfrak{p}}/p^n$ is injective (see also [Bri08, p. 24]). Passing to the limit over n, we obtain injective maps $\mathbb{C}^+(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}}(\overline{R})_{\mathfrak{p}}^{\wedge} \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}^+(\mathfrak{p})$. Applying the tilting functor we further obtain a commutative diagram

$$\begin{tikzcd}
\mathbb{C}^+(\overline{R})^\flat \arrow[r] \arrow[d] & \prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}^+(\mathfrak{p})^\flat \arrow[d] & \\
\mathbb{C}(\overline{R})^\flat \arrow[r] & \left(\prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}^+(\mathfrak{p})^\flat\right)[\frac{1}{p^\flat}] \arrow[r] & \prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}(\mathfrak{p})^\flat,
\end{tikzcd}$$

where the bottom right horizontal arrow and vertical arrows are injective. From the injectivity of $\overline{R}/p \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}^+(\mathfrak{p})/p$ and left exactness of $\lim_{\varphi}$, we obtain that the top horizontal arrow and hence the bottom left horizontal arrow are injective. In particular,

$$\mathbb{C}^+(\overline{R})^\flat = \mathbb{C}(\overline{R})^\flat \cap \prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}^+(\mathfrak{p})^\flat \subset \prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}(\mathfrak{p})^\flat. \tag{3.9}$$

Furthermore, recall that the p-typical Witt vector functor is left exact since it is right adjoint to the forgetful functor from the category of rings to the category of δ-rings. Therefore, all maps in the following natural commutative diagram are injective

$$\begin{tikzcd}
\mathbf{A}_{\inf}(\overline{R}) \arrow[r] \arrow[d] & \prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\inf}(\mathbb{C}^+(\mathfrak{p})) \arrow[d] \\
W(\mathbb{C}(\overline{R})^\flat) \arrow[r] & \prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat).
\end{tikzcd}$$

Then from (3.9) we obtain

$$\mathbf{A}_{\inf}(\overline{R}) = W(\mathbb{C}(\overline{R})^\flat) \cap \prod_{\mathfrak{p}\in\mathcal{S}} \mathbf{A}_{\inf}(\mathbb{C}^+(\mathfrak{p})) \subset \prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat).$$ ∎

Corollary 3.13. *Let N be a Wach module over $\mathbf{A}_R^+$ and $T = \mathbf{T}_R(N)$ the associated finite free $\mathbb{Z}_p$-representation of G_R. Then we have a natural G_R-equivariant comparison isomorphism*

$$\mathbf{A}^+[\tfrac{1}{\mu}] \otimes_{\mathbf{A}_R^+} N \xrightarrow{\sim} \mathbf{A}^+[\tfrac{1}{\mu}] \otimes_{\mathbb{Z}_p} T.$$

Proof. Since $N[1/\mu]$ is finite projective over $\mathbf{A}_R^+[1/\mu]$, taking the intersection of the isomorphism in Proposition 3.11 with the isomorphism in (3.1) inside $\tilde{\mathbf{A}} \otimes_{\mathbb{Z}_p} T$ we obtain $\mathbf{A}^+[1/\mu] \otimes_{\mathbf{A}_R^+[1/\mu]} N[1/\mu] \xrightarrow{\sim} \mathbf{A}^+[1/\mu] \otimes_{\mathbb{Z}_p} T$. ∎

Proposition 3.14. *Let N be an effective Wach module over $\mathbf{A}_R^+$ and $T = \mathbf{T}_R(N)$ the associated finite free $\mathbb{Z}_p$-representation of G_R. Then we have Γ_R-equivariant inclusions $\mu^s \mathbf{D}_R^+(T) \subset N \subset \mathbf{D}_R^+(T)$ (see §3.2 for notations).*

Proof. The proof follows in a manner similar to the proof of Proposition 3.11, so we will freely use the notation of that proof. Inverting p in (3.5) we get

$$\begin{aligned}\mu^s\Big(\prod_{\mathfrak{p}\in\mathcal{S}}(\mathbf{A}_{\inf}(\mathbb{C}^+(\mathfrak{p}))\otimes_{\mathbb{Z}_p}T)\Big)[\tfrac{1}{p}] &\subset \Big(\prod_{\mathfrak{p}\in\mathcal{S}}(\mathbf{A}_{\inf}(\mathbb{C}^+(\mathfrak{p}))\otimes_{\mathbf{A}_R^+}N_R)\Big)[\tfrac{1}{p}]\\ &\subset \Big(\prod_{\mathfrak{p}\in\mathcal{S}}(\mathbf{A}_{\inf}(\mathbb{C}^+(\mathfrak{p}))\otimes_{\mathbb{Z}_p}T)\Big)[\tfrac{1}{p}].\end{aligned} \tag{3.10}$$

The last term of (3.10) can be written as $\Big(\prod_{\mathfrak{p}\in\mathcal{S}}\mathbf{A}_{\inf}(\mathbb{C}^+(\mathfrak{p}))\Big)[1/p]\otimes_{\mathbb{Q}_p}V$ and similarly for the first term. Using an argument similar to the one given for (3.7), the middle term of (3.10) can be written as $\Big(\prod_{\mathfrak{p}\in\mathcal{S}}\mathbf{A}_{\inf}(\mathbb{C}^+(\mathfrak{p}))\Big)[1/p]\otimes_{\mathbf{B}_R^+}N_R[1/p]$. Furthermore, by the theory of (φ,Γ)-modules, we have a G_R-equivariant comparison isomorphism

$$W(\mathbb{C}(\overline{R})^\flat)[\tfrac{1}{p}]\otimes_{\mathbf{B}_R^+}N_R[\tfrac{1}{p}] \xrightarrow{\sim} W(\mathbb{C}(\overline{R})^\flat)[\tfrac{1}{p}]\otimes_{\mathbb{Q}_p}V. \tag{3.11}$$

Using Lemma 3.12, we embed all the terms above inside $(\prod_{\mathfrak{p}\in\mathcal{S}}W(\mathbb{C}(\mathfrak{p})^\flat))[1/p]\otimes_{\mathbb{Q}_p}V$.

Now since $N_R[1/p]$ is finite projective over $\mathbf{B}_R^+$, taking the intersection of middle term in (3.10) and left hand term in (3.11), inside $(\prod_{\mathfrak{p}\in\mathcal{S}}W(\mathbb{C}(\mathfrak{p})^\flat))[1/p]\otimes_{\mathbb{Q}_p}V$, we obtain

$$\Big(W(\mathbb{C}(\overline{R})^\flat)[\tfrac{1}{p}]\otimes_{\mathbf{B}_R^+}N_R[\tfrac{1}{p}]\Big)\cap\Big(\Big(\prod_{\mathfrak{p}\in\mathcal{S}}\mathbf{A}_{\inf}(\mathbb{C}^+(\mathfrak{p}))\Big)[\tfrac{1}{p}]\otimes_{\mathbf{B}_R^+}N_R[\tfrac{1}{p}]\Big) = \mathbf{A}_{\inf}(\overline{R})[\tfrac{1}{p}]\otimes_{\mathbf{B}_R^+}N_R[\tfrac{1}{p}].$$

where the equality follows from Lemma 3.12. Similarly, taking the intersection of right hand terms in (3.6) and (3.11), inside $(\prod_{\mathfrak{p}\in\mathcal{S}}W(\mathbb{C}(\mathfrak{p})^\flat))[1/p]\otimes_{\mathbb{Q}_p}V$, we obtain

$$\Big(W(\mathbb{C}(\overline{R})^\flat)[\tfrac{1}{p}]\otimes_{\mathbb{Q}_p}V\Big)\cap\Big(\Big(\prod_{\mathfrak{p}\in\mathcal{S}}\mathbf{A}_{\inf}(\mathbb{C}^+(\mathfrak{p}))\Big)[\tfrac{1}{p}]\otimes_{\mathbb{Q}_p}V\Big) = \mathbf{A}_{\inf}(\overline{R})[\tfrac{1}{p}]\otimes_{\mathbb{Q}_p}V,$$

where the equality again follows from Lemma 3.12. Therefore, from (3.10) and G_R-equivariance of (3.11), we obtain G_R-equivariant inclusions

$$\mu^s(\mathbf{A}_{\inf}(\overline{R})[\tfrac{1}{p}]\otimes_{\mathbb{Q}_p}V)\subset\mathbf{A}_{\inf}(\overline{R})[\tfrac{1}{p}]\otimes_{\mathbf{A}_R^+}N_R[\tfrac{1}{p}]\subset\mathbf{A}_{\inf}(\overline{R})[\tfrac{1}{p}]\otimes_{\mathbb{Q}_p}V. \tag{3.12}$$

Inverting p in the isomorphism of Corollary 3.13 and taking its intersection with the equation above, inside $W(\mathbb{C}(\overline{R})^\flat)[1/p]\otimes_{\mathbb{Q}_p}V$, we obtain G_R-equivariant inclusions

$$\mu^s(\mathbf{B}^+\otimes_{\mathbb{Q}_p}V)\subset\mathbf{B}^+\otimes_{\mathbf{B}_R^+}N_R[\tfrac{1}{p}]\subset\mathbf{B}^+\otimes_{\mathbb{Q}_p}V.$$

Taking the H_R-invariants and taking the intersection with $\mathbf{D}_R(T) = N[1/\mu]^\wedge$ in the preceding equation, inside $\mathbf{D}_R(V)$ we obtain $\mu^s\mathbf{D}_R^+(T)\subset N\subset\mathbf{D}_R^+(T)$ since $N = N[1/p]\cap N[1/\mu]^\wedge$ from Lemma 3.5 and $\mathbf{D}_R^+(T) = \mathbf{D}_R(T)\cap\mathbf{D}_R^+(V)\subset\mathbf{D}_R(V)$ by definitions. ∎

Proposition 3.15. *Let N be an effective Wach module over $\mathbf{A}_R^+$ and $T = \mathbf{T}_R(N)$ the associated finite free $\mathbb{Z}_p$-representation of G_R. Set $M = (\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_\infty)\otimes_{\mathbf{A}_R^+}N[1/p])^{\Gamma_R}$, then we have a natural comparison isomorphism*

$$\mathcal{O}\mathbf{B}_{\mathrm{cris}}^+(R_\infty)\otimes_{R[1/p]}M\xrightarrow{\sim}\mathcal{O}\mathbf{B}_{\mathrm{cris}}^+(R_\infty)\otimes_{\mathbf{A}_R^+}N.$$

Proof. We will adapt the proof of [Abh21, Proposition 4.28]. Recall the following rings from [Abh21, §4.4.1]. For $n \in \mathbb{N}$, define a p-adically complete ring $S_n^{\mathrm{PD}} := \mathbf{A}_R^+\{\frac{\pi}{p^n}, \frac{\pi^2}{2!p^{2n}}, \ldots, \frac{\pi^k}{k!p^{kn}}, \ldots\}$. Let $I_n^{[i]}$ denote the ideal of S_n^{PD} generated by $\frac{\pi^k}{k!p^{kn}}$ for $k \geq i$ and set $\widehat{S}_n^{\mathrm{PD}} := \lim_i S_n^{\mathrm{PD}}/I_n^{[i]}$. Now consider the O_F-linear homomorphism of rings $\iota : R \to \widehat{S}_n^{\mathrm{PD}}$ sending $X_j \mapsto [X_j^\flat]$ for $1 \leq j \leq d$. Using ι define an O_F-linear morphism of rings $f : R \otimes_{O_F} \widehat{S}_n^{\mathrm{PD}} \to \widehat{S}_n^{\mathrm{PD}}$ via $a \otimes b \mapsto \iota(a)b$. Let $\mathcal{O}\widehat{S}_n^{\mathrm{PD}}$ denote the p-adic completion of the divided power envelope of $R \otimes_{O_F} \widehat{S}_n^{\mathrm{PD}}$ with respect to Ker f. The divided power ring $\mathcal{O}\widehat{S}_n^{\mathrm{PD}}$ is equipped with a continuous action of Γ_R, an integrable connection and a Frobenius $\varphi : \mathcal{O}\widehat{S}_n^{\mathrm{PD}} \to \mathcal{O}\widehat{S}_{n-1}^{\mathrm{PD}}$ such that $\varphi^n(\mathcal{O}\widehat{S}_n^{\mathrm{PD}}) \subset \mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_\infty)$. Moreover, we have $R = (\mathcal{O}\widehat{S}_n^{\mathrm{PD}})^{\Gamma_R}$ and divided power ideals of $\mathcal{O}\widehat{S}_n^{\mathrm{PD}}$

$$J^{[i]}\mathcal{O}\widehat{S}_n^{\mathrm{PD}} := \Big\langle \tfrac{\pi^{[k_0]}}{p^{nk_0}} \prod_{j=1}^{d} (1 - V_j)^{[k_j]},\ \mathbf{k} = (k_0, k_1, \ldots, k_d) \in \mathbb{N}^{d+1} \text{ such that } \sum_{j=0}^{d} k_j \geq i \Big\rangle.$$

Let $M_n = \big(\mathcal{O}\widehat{S}_n^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N[1/p]\big)^{\Gamma_R}$ an $R[1/p]$-module equipped with an integrable connection. Equip M_n with the tensor product Frobenius $\varphi : M_n \to M_{n-1}$, in particular, $\varphi^n(M_n) \subset M = \big(\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_\infty) \otimes_{\mathbf{A}_R^+} N[1/p]\big)^{\Gamma_R}$. Let $V = T[1/p]$, then using (3.12) we note that $\big(\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_\infty) \otimes_{\mathbf{A}_R^+} N[1/p]\big)^{\Gamma_R} = \big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}^+(\overline{R}) \otimes_{\mathbf{B}_R^+} N[1/p]\big)^{G_R} \subset \big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}^+(\overline{R}) \otimes_{\mathbb{Q}_p} V\big)^{G_R} \subset \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$. Since φ is faithfully flat over $R[1/p]$, we get that M_n is a finitely generated $R[1/p]$-module equipped with an integrable connection, in particular, it is finite projective over $R[1/p]$ by [Bri08, Proposition 7.1.2]. Furthermore, for $n \geq 1$ similar to the proof of [Abh21, Lemmas 4.32 & 4.36], one can show that $\nabla_i = \log \gamma_i$ converge as a series of operators on M_n, where $\{\gamma_0, \gamma_1, \ldots, \gamma_d\}$ are topological generators of Γ_R (see [Abh21, §3.1] for a description of these generators).

Lemma 3.16. *Let $m \geq 1$, then we have $\mathcal{O}\widehat{S}_m^{\mathrm{PD}} \otimes_R M_m \xrightarrow{\sim} \mathcal{O}\widehat{S}_m^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N[1/p]$.*

Proof. Define the following differential operators on $\mathcal{O}N_m^{\mathrm{PD}} = \mathcal{O}\widehat{S}_m^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N[1/p]$ (see [Abh21, §4.4.2])

$$\partial_i = \begin{cases} -t^{-1}\nabla_0 & \text{for } i = 0, \\ t^{-1}V_i^{-1}\nabla_i & \text{for } 1 \leq i \leq d, \end{cases}$$

where $V_i = \frac{X_i \otimes 1}{1 \otimes [X_i^\flat]}$ for $1 \leq i \leq d$. Note that these operators commute with each other by [Abh21, Lemma 4.38]. Then for $x \in N[1/p]$, similar to the proof of [Abh21, Lemma 4.39 & Lemma 4.41], it follows that the following sum

$$y = \sum_{\mathbf{k} \in \mathbb{N}^{d+1}} \partial_0^{k_0} \circ \partial_1^{k_1} \circ \cdots \circ \partial_d^{k_d}(x) \tfrac{t^{[k_0]}}{p^{mk_0}} (1 - V_1)^{[k_1]} \cdots (1 - V_d)^{[k_d]}, \tag{3.13}$$

converges in M_m. Using the construction above we can define a transformation α on the finite projective $\mathcal{O}\widehat{S}_m^{\mathrm{PD}}[1/p]$-module $\mathcal{O}N_m^{\mathrm{PD}}$. We claim that α is an automorphism of $\mathcal{O}N_m^{\mathrm{PD}}$. Indeed, first choose a presentation $\mathcal{O}N_m^{\mathrm{PD}} \oplus N' = (\mathcal{O}\widehat{S}_m^{\mathrm{PD}})^r$ for some $r \in \mathbb{N}$. Then on $(\mathcal{O}\widehat{S}_m^{\mathrm{PD}})^r$ we can define a transformation β by (3.13) over $\mathcal{O}N_m^{\mathrm{PD}}$ and identity on N'. Note that this transformation preserves $\mathcal{O}N_m^{\mathrm{PD}}$ and we set $\det \alpha = \det \beta$ which is independent of the chosen presentation (see [Gol61, Proposition 1.2]). Now similar to the proof of [Abh21, Lemma 4.43] it easily follows that for some large enough $N \in \mathbb{N}$ we can write $p^N \det \alpha = p^N \det \beta \in 1 + J^{[1]}\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$, i.e. $\det \alpha$ is a unit in $\mathcal{O}\widehat{S}_m^{\mathrm{PD}}[1/p]$ and α defines an automorphism of $\mathcal{O}N_m^{\mathrm{PD}}$ (see [Gol61, Proposition 1.3]). Finally, since the formula in (3.13) converges in M_m, we obtain that $\mathcal{O}\widehat{S}_m^{\mathrm{PD}} \otimes_R M_m \xrightarrow{\sim} \mathcal{O}\widehat{S}_m^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} N[1/p]$. ∎

We also have that M is an $R[1/p]$-module equipped with an integrable connection and $M \subset \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$, in particular, it is finite projective over $R[1/p]$ by [Bri08, Proposition 7.1.2]. Recall that we have $\varphi^m : \mathcal{O}\widehat{S}_m^{\mathrm{PD}} \to \mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_\infty)$, so extending scalars of the isomorphism above along this map, we obtain

$$\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_\infty) \otimes_{R,\varphi^m} M_m \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_\infty) \otimes_{\mathbf{A}_R^+,\varphi^m} N[1/p] \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_\infty) \otimes_{\mathbf{A}_R^+} N[1/p],$$

where the last isomorphism follows from [Abh21, Lemma 4.45]. Since we have $\varphi^m(M_m) \subset M$, we get the desired isomorphism

$$\mathcal{O}\mathbf{B}_{\mathrm{cris}}^+(R_\infty) \otimes_{R[1/p]} M \xrightarrow{\sim} \mathcal{O}\mathbf{B}_{\mathrm{cris}}^+(R_\infty) \otimes_{\mathbf{B}_R^+} N[1/p]. \qquad \blacksquare$$

Theorem 3.17. *Let N be a Wach module over $\mathbf{A}_R^+$ and let $T = \mathbf{T}_R(N)$ the associated finite free $\mathbb{Z}_p$-representation of G_R. Then $V = T[1/p]$ is a p-adic crystalline representation of G_R.*

Proof. Note that the crystalline property is invariant under Tate twists, so without loss of generality we can assume that N is effective. Let $M = \big(\mathcal{O}\mathbf{A}_{\mathrm{cris}}(R_\infty) \otimes_{\mathbf{A}_R^+} N[1/p]\big)^{\Gamma_R}$ as in Proposition 3.15, then we have a commutative diagram

$$\begin{tikzcd}
\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{R[1/p]} M \arrow[r, "\simeq"] \arrow[d, tail] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbf{B}_R^+} N[1/p] \arrow[d, "\simeq"] \\
\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \arrow[r, tail] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V,
\end{tikzcd}$$

where the top horizontal arrow is extension of scalars along $\mathcal{O}\mathbf{B}_{\mathrm{cris}}^+(R_\infty) \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ of the isomorphism in Proposition 3.15, right vertical arrow is extension of scalars along $\mathbf{A}_{\mathrm{inf}}(\overline{R})[1/\mu] \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ of the isomorphism in Proposition 3.11 and left vertical arrow is injective since $R[1/p] \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ is faithfully flat by [Bri08, Thèoréme 6.3.8]. It follows that the bottom arrow is bijective and hence V is a crystalline representation of G_R. $\blacksquare$

3.3.2. Finite $[p]_q$-height representations.

In this section we generalize the definition of finite $[p]_q$-height representations from [Abh21, Definition 4.9] in the relative case.

Definition 3.18. A finite $[p]_q$-height $\mathbb{Z}_p$-representation of G_R is a finite free $\mathbb{Z}_p$-module T admitting a linear and continuous action of G_R such that there exists a finitely generated torsion-free $\mathbf{A}_R^+$-submodule $\mathbf{N}_R(T) \subset \mathbf{D}_R(T)$ satisfying the following:

(1) $\mathbf{N}_R(T)$ is a Wach module in the sense of Defintion 3.6.

(2) $\mathbf{A}_R \otimes_{\mathbf{A}_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{D}_R(T)$.

The height of T is defined to be the height of $\mathbf{N}_R(T)$. Say T is *positive* if $\mathbf{N}_R(T)$ is effective.

A finite $[p]_q$-height p-adic representation of G_R is a finite dimensional $\mathbb{Q}_p$-vector space admitting a linear and continuous action of G_R such that there exists a G_R-stable $\mathbb{Z}_p$-lattice $T \subset V$ with T of positive finite $[p]_q$-height. We set $\mathbf{N}_R(V) = \mathbf{N}_R(T)[1/p]$ satisfying analogous properties. The height of V is defined to be the height of T. Say V is positive if $\mathbf{N}_R(V)$ is effective.

Lemma 3.19. *Let T be a finite $[p]_q$-height $\mathbb{Z}_p$-representation of G_R then the $\mathbf{A}_R^+$-module $\mathbf{N}_R(T)$ is unique. Similar statement holds for $V = T[1/p]$.*

Proof. By definition $\mathbf{A}_R \otimes_{\mathbf{A}_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{D}_R(T)$ and this scalar extension induces a fullly faithful functor in Proposition 3.10. So from (3.2) we obtain the uniqueness of $\mathbf{N}(T)$. Alternatively, the uniqess can also be deduced using Proposition 3.14 and [Abh21, Proposition 4.13]. $\blacksquare$

Remark 3.20. By the definition of finite $[p]_q$-height representations, Lemma 3.19 and the fully faithful functor in (3.4) it follows that the data of a finite height representation is equivalent to the data of a Wach module.

3.4. Crystalline implies finite height. The aim of this section is to prove the following claim:

Theorem 3.21. *Let T be a finite free $\mathbb{Z}_p$-representation of G_R such that $V = T[1/p]$ is a p-adic crystalline representation of G_R. Then there exists a unique Wach module $\mathbf{N}_R(T)$ over $\mathbf{A}_R^+$. In other words, T is of finite $[p]_q$-height.*

Proof. The property of being crystalline and of finite $[p]_q$-height is invariant under twisting the representation by χ^r for $r \in \mathbb{N}$. Therefore, we will assume that V is positive crystalline. Note that V is also a positive crystalline representation of G_L and hence positive and of finite $[p]_q$-height as a p-adic representation of G_L (see Definition 2.21). This implies that T is positive and of finite $[p]_q$-height as a $\mathbb{Z}_p$-representation of G_L. In particular, we have the Wach module $\mathbf{N}_L(T)$ over $\mathbf{A}_L^+$ and we set $\mathbf{N}_R(T) := \mathbf{N}_L(T) \cap \mathbf{D}_R(T) \subset \mathbf{D}_L(T)$ as an $\mathbf{A}_R^+$-module. The module $\mathbf{N}_R(T)$ satisfies axioms of Definitions 3.6 & 3.18 by Proposition 3.25. Hence, it follows that $\mathbf{N}_R(T)$ is a unique Wach module, or equivalently, T is of finite $[p]_q$-height. ∎

Let $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_R)$ denote the category of $\mathbb{Z}_p$-lattices inside p-adic crystalline representations of G_R. Then combining Theorem 3.17 and Theorem 3.21 we obtain:

Corollary 3.22. *The Wach module functor induces an equivalence of $\otimes$-categories*

$$\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_R) \xrightarrow{\sim} (\varphi, \Gamma)\text{-}\mathrm{Mod}_{\mathbf{A}_R^+}^{[p]_q}$$
$$T \longmapsto \mathbf{N}_R(T),$$

with a quasi-inverse given as $N \mapsto \big(W(\overline{R}^\flat[1/p^\flat]) \otimes_{\mathbf{A}_R^+} N\big)^{\varphi=1}$.

Remark 3.23. In Corollary 3.22 compatibility with tensor products follows from [Abh21, Proposition 4.14]. However, note that the functor $\mathbf{N}_R$ is not exact. But, it becomes exact after passing to corresponding isogeny categories, i.e. the induced functor from p-adic crystalline representations of G_R to Wach modules over $\mathbf{B}_R^+$ is exact. We leave details of these claims to the reader.

We obtain an application of Theorem 3.21 as a purity statement:

Theorem 3.24. *Let V be a p-adic representation of G_R. Then V is crystalline as a representation of G_R if and only if it is crystalline as a representation of G_L.*

Proof. If $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_R)$ then it is obvious that $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_L)$. Conversely, if $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_L)$ then for a choice of G_R-stable $\mathbb{Z}_p$-lattice $T \subset V$ we have that T is of finite $[p]_q$-height as a representation of G_L. Then from Proposition 3.25 it follows that T is of finite $[p]_q$-height as a representation of G_R. Hence, $V = T[1/p]$ is a crystalline representation of G_R by Theorem 3.17. ∎

In the rest of the section we will carry out steps in the proof of the following claim:

Proposition 3.25. *The $\mathbf{A}_R^+$-module $\mathbf{N}_R(T) = \mathbf{N}_L(T) \cap \mathbf{D}_R(T)$ satisfies all the axioms of Definition 3.18.*

Proof. It is immediate that $\mathbf{N}_R(T)$ is p-torsion free and μ-torsion free. From Lemma 3.26 note that $\mathbf{N}_R(T)$ is finitely generated over $\mathbf{A}_R^+$ and from its proof we have $\mathbf{N}_R(T)/p \subset (\mathbf{N}_L(T)/p) \cap (\mathbf{D}_R(T)/p) \subset \mathbf{D}_L(T)/p$, in particular, $\mathbf{N}_R(T)/p$ is μ-torsion free. Next, from Lemma 3.27 we know that $\mathbf{N}_R(T)$ is of finite $[p]_q$-height, i.e. cokernel of the injective map $1 \otimes \varphi : \varphi^*(\mathbf{N}_R(T)) \to \mathbf{N}_R(T)$ is killed by $[p]_q^s$ where s is the height of $\mathbf{N}_L(T)$. Furthermore, from Proposition 3.30 we have $\mathbf{A}_R \otimes_{\mathbf{A}_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{D}_R(T)$. Finally, since $\mathbf{N}_L(T) \xrightarrow{\sim} \mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}_R(T)$ (see Proposition 3.30), the action of Γ_L is trivial on $\mathbf{N}_L(T)/\mu\mathbf{N}_L(T)$ and $\Gamma_L \xrightarrow{\sim} \Gamma_R$, therefore it easily follows that Γ_R acts trivially on $\mathbf{N}_R(T)/\mu\mathbf{N}_R(T)$. This concludes the proof. ∎

Lemma 3.26. *The $\mathbf{A}_R^+$-module $\mathbf{N}_R(T) = \mathbf{N}_L(T) \cap \mathbf{D}_R(T)$ is finitely generated.*

Proof. It is enough to show that $\mathbf{N}_R(T)/p \to (\mathbf{N}_L(T)/p) \cap (\mathbf{D}_R(T)/p) \subset \mathbf{D}_L(T)/p$ is injective and $\mathbf{N}_L(T)/p \cap \mathbf{D}_R(T)/p$ is finite over $\mathbf{A}_R^+/p = \mathbf{E}_R^+$. Note that we have $p\mathbf{N}_L(T) \cap \mathbf{N}_R(T) \subset p\mathbf{D}_L(T) \cap \mathbf{D}_R(T) = p\mathbf{D}_R(T)$, so

$$p\mathbf{N}_R(T) \subset p\mathbf{N}_L(T) \cap \mathbf{N}_R(T) \subset p\mathbf{N}_L(T) \cap p\mathbf{D}_R(T) = p\mathbf{N}_R(T),$$

therefore $\mathbf{N}_R(T)/p \subset \mathbf{N}_L(T)/p$. Moreover, $p\mathbf{D}_R(T) \cap \mathbf{N}_R(T) \subset p\mathbf{D}_L(T) \cap \mathbf{N}_L(T) = p\mathbf{N}_L(T)$, so

$$p\mathbf{N}_R(T) \subset p\mathbf{D}_R(T) \cap \mathbf{N}_R(T) \subset p\mathbf{D}_R(T) \cap p\mathbf{N}_L(T) = p\mathbf{N}_R(T),$$

therefore $\mathbf{N}_R(T)/p \subset \mathbf{D}_R(T)/p$.

Assume that $\overline{D}_R = \mathbf{D}_R(T)/p$ is finite free (a priori it is finite projective) of rank h over $\mathbf{E}_R = \mathbf{A}_R/p$. Let $\mathbf{e} = \{e_1, \ldots, e_h\}$ be a basis of $\overline{N}_L = \mathbf{N}_L(T)/p$ over $\mathbf{E}_L^+ = \mathbf{A}_L^+/p$ and $\mathbf{f} = \{f_1, \ldots, f_h\}$ a basis of $\overline{D}_R = \mathbf{D}_R(T)/p$ over $\mathbf{E}_R$. Then we have $\mathbf{f} = A\mathbf{e}$ for $A = (a_{ij}) \in \mathrm{GL}(h, \mathbf{E}_L)$ and write $A^{-1} = (b_{ij}) \in \mathrm{GL}(h, \mathbf{E}_L)$. Set $M = \oplus_{i=1}^h \mathbf{E}_R^+ f_i$, so that $M[1/\mu] = \overline{D}_R$. Let $x \in M[1/\mu] \cap \overline{N}_L$ and write $x = \sum_{i=1}^h c_i e_i = \sum_{i=1}^h d_i f_i$ with $c_i \in \mathbf{E}_L^+$ and $d_i \in \mathbf{E}_R$ for all $1 \leq i \leq h$. So we obtain that $d_i = \sum_{j=1}^h b_{ji} c_j$ for all $1 \leq i \leq h$. Therefore, we have $d_i \in \frac{1}{\mu^k}\mathbf{E}_L^+$ for all $1 \leq i \leq h$ for k large enough. Note that we have $\frac{1}{\mu^k}\mathbf{E}_L^+ \cap \mathbf{E}_R = \frac{1}{\mu^k}\mathbf{E}_R^+$, so we obtain that $d_i \in \frac{1}{\mu^k}\mathbf{E}_R^+$. Hence, $M[1/\mu] \cap \overline{N}_L \subset \frac{1}{\mu^k}M$, in particular $M[1/\mu] \cap \overline{N}_L = \overline{D}_R \cap \overline{N}_L$ is finitely generated over $\mathbf{E}_R^+$.

In the general case when $\overline{D}_R$ is finite projective, we can take a nonzerodivisior $a \in \mathbf{E}_R^+$ to obtain $\overline{D}_R[1/a]$ as a finite free module over $\mathbf{E}_R[1/a]$ of rank h. Similar to above one can choose $\mathbf{e} = \{e_1, \ldots, e_h\}$ to be a basis of $\overline{N}_L$ over $\mathbf{E}_L^+$ and $\mathbf{f} = \{f_1, \ldots, f_h\}$ a basis of $\overline{D}_R[1/a]$ over $\mathbf{E}_R[1/a]$ such that $\mathbf{f} = A\mathbf{e}$ with $A \in \mathrm{GL}(h, \mathbf{E}_L)$ (since $\mathbf{E}_L$ is a field). Set $M = \oplus_{i=1}^h \mathbf{E}_R^+ f_i$ as an $\mathbf{E}_R^+$-submodule of $\overline{D}_R[1/a]$ and we have that $\overline{D}_R \subset M[1/\mu]$. Proceeding as above, we obtain that $M[1/\mu] \cap \overline{N}_L \subset \frac{1}{\mu^k}M$. Hence, in the general case $\overline{D}_R \cap \overline{N}_L \subset M[1/\mu] \cap \overline{N}_L$ is finitely generated over $\mathbf{E}_R^+$. ∎

Lemma 3.27. *The $\mathbf{A}_R^+$-module $\mathbf{N}_R(T)$ is of finite $[p]_q$-height, i.e. cokernel of the injective map $1 \otimes \varphi : \varphi^*(\mathbf{N}_R(T)) \to \mathbf{N}_R(T)$ is killed by $[p]_q^s$ for some $s \in \mathbb{N}$.*

Proof. Note that $\varphi : \mathbf{A}_R^+ \to \mathbf{A}_R^+$ is finite and faithfully flat of degree p^{d+1} (see §3.1). Moreover, from §3.1 and §3.1 we have $\varphi^*(\mathbf{A}_R) = \mathbf{A}_R^+ \otimes_{\varphi, \mathbf{A}_R^+} \mathbf{A}_R$ and $\varphi^*(\mathbf{A}_L^+) = \oplus_\alpha \varphi(\mathbf{A}_L^+) u_\alpha = (\oplus_\alpha \varphi(\mathbf{A}_R^+) u_\alpha) \otimes_{\varphi(\mathbf{A}_R^+)} \varphi(\mathbf{A}_L^+) = \mathbf{A}_R^+ \otimes_{\varphi, \mathbf{A}_R^+} \mathbf{A}_L^+$. Therefore, we also obtain that $\varphi^*(\mathbf{N}_L(T)) = \mathbf{A}_L^+ \otimes_{\varphi, \mathbf{A}_L^+} \mathbf{N}_L(T) = \mathbf{A}_R^+ \otimes_{\varphi, \mathbf{A}_R^+} \mathbf{N}_L(T)$ and $\varphi^*(\mathbf{D}_R(T)) = \mathbf{A}_R \otimes_{\varphi, \mathbf{A}_R} \mathbf{D}_R(T) = \mathbf{A}_R^+ \otimes_{\varphi, \mathbf{A}_R^+} \mathbf{D}_R(T)$. Hence, inside $\varphi^*(\mathbf{D}_L(T))$ we conclude that

$$\begin{aligned}
\varphi^*(\mathbf{N}_R(T)) = \mathbf{A}_R^+ \otimes_{\varphi, \mathbf{A}_R^+} \mathbf{N}_R(T) &= \mathbf{A}_R^+ \otimes_{\varphi, \mathbf{A}_R^+} (\mathbf{N}_L(T) \cap \mathbf{D}_R(T)) \\
&= (\mathbf{A}_R^+ \otimes_{\varphi, \mathbf{A}_R^+} \mathbf{N}_L(T)) \cap (\mathbf{A}_R^+ \otimes_{\varphi, \mathbf{A}_R^+} \mathbf{D}_R(T)) = \varphi^*(\mathbf{N}_L(T)) \cap \varphi^*(\mathbf{D}_R(T)).
\end{aligned}$$

Since cokernel of the injective map $(1 \otimes \varphi) : \varphi^*(\mathbf{N}_L(T)) \to \mathbf{N}_L(T)$ is killed by $[p]_q^s$ for some $s \in \mathbb{N}$ and $(1 \otimes \varphi) : \varphi^*(\mathbf{D}_R(T)) \xrightarrow{\sim} \mathbf{D}_R(T)$, it easily follows that the cokernel of $(1 \otimes \varphi) : \varphi^*(\mathbf{N}_R(T)) \to \mathbf{N}_R(T)$ is killed by $[p]_q^s$ as well. ∎

Finally, we will show that $\mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{N}_L(T)$ and $\mathbf{A}_R \otimes_{\mathbf{A}_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{D}_R(T)$. Our approach is motivated by some techniques in [DLMS22] in proving some auxiliary lemmas below.

For $n \in \mathbb{N}_{\geq 1}$ let $N_{R,n} = \mathbf{N}_R(T)/p^n$, $D_{R,n} = \mathbf{D}_R(T)/p^n$, $N_{L,n} = \mathbf{N}_L(T)/p^n$, $D_{L,n} = \mathbf{D}_L(T)/p^n$ and $M_n := N_{L,n} \cap D_{R,n} \subset D_{L,n}$. We have a commutative diagram

$$\begin{array}{ccc}
M_n & \xrightarrow{f_n} & M_1 \\
\downarrow & & \downarrow \\
D_{R,n} & \xrightarrow{f_n} & D_{R,1},
\end{array}$$

where the vertical arrows are natural inclusions, the bottom horizontal arrow f_n is the natural projection map and the top arrow is the induced map. We have a similar diagram with the bottom row replaced by $N_{L,n} \twoheadrightarrow N_{L,1}$.

Lemma 3.28. *We have*

(1) $\mathbf{N}_R(T) \xrightarrow{\sim} \lim_n M_n$

(2) *M_n is a finitely generated $\mathbf{A}_R^+/p^n$-module.*

(3) *M_n is of finite height s.*

(4) *$M_n[1/\mu] = \mathbf{A}_R \otimes_{\mathbf{A}_R^+} M_n \xrightarrow{\sim} D_{R,n}$ and $\mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} M_n \xrightarrow{\sim} N_{L,n}$.*

Proof. Similar to the mod p case it is easy to see that $\mathbf{N}_R(T)/p^n \subset D_{R,n} \cap N_{L,n} = M_n \subset D_{L,n}$ for each $n \geq 1$. Therefore the natural map $\mathbf{N}_R(T) = \lim_n \mathbf{N}_R(T)/p^n \to \lim_n M_n$ is injective. Now let $x = (x_n) \in \lim_n M_n$, then $x_n \in M_n = D_{R,n} \cap N_{L,n}$ for each $n \geq 1$. In particular, $x \in (\lim_n D_{R,n}) \cap (\lim_n N_{L,n}) = \mathbf{D}_R(T) \cap \mathbf{N}_L(T) = \mathbf{N}_R(T)$, i.e. the preceding map is surjective as well. This shows (1). The claim in (2) follows in a manner similar to Lemma 3.26 and the claim in (3) follows similar to Lemma 3.27. As the maps $\mathbf{A}_R^+ \to \mathbf{A}_R$ and $\mathbf{A}_R^+ \to \mathbf{A}_L^+$ are flat, the last claim follows from the equalities below:

$$\mathbf{A}_R \otimes_{\mathbf{A}_R^+} M_n = (\mathbf{A}_R \otimes_{\mathbf{A}_R^+} D_{R,n}) \cap (\mathbf{A}_R \otimes_{\mathbf{A}_R^+} N_{L,n}) = (\mathbf{A}_R \otimes_{\mathbf{A}_R^+} D_{R,n}) \cap (\mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} D_{R,n}) = D_{R,n},$$
$$\mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} M_n = (\mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} D_{R,n}) \cap (\mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} N_{L,n}) = (\mathbf{A}_R \otimes_{\mathbf{A}_R^+} N_{L,n}) \cap (\mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} N_{L,n}) = N_{L,n}.$$

Hence we conclude. ∎

Let $\mathcal{S}$ denote the set of $\mathbf{A}_R^+$-submodules $M' \subset M_1$ stable under the action of φ, is of finite height s and satisfies $M'[1/\mu] = M_1[1/\mu] = D_{R,1} = \mathbf{D}_R(T)/p$. Set $M^\circ := \cap_{M' \in \mathcal{S}} M' \subset M_1$.

Lemma 3.29. *We have $M^\circ \in \mathcal{S}$ and $f_n(M_n) \in S$ for all $n \in \mathbb{N}_{\geq 1}$.*

Proof. The idea of the proof is motivated by [DLMS22, Lemma 4.25]. Let $M' \in \mathcal{S}$. For the first claim we need to show that there exists $r \in \mathbb{N}$ such that $\mu^r M_1 \subset M' \subset M_1$. Let $M'' = M_1/M'$ such that $M'' \neq 0$ and let $k = p(p-1)s \in \mathbb{N}$. Since M_1 and M' are of finite height k (since $s < k$), we define $\mathbf{A}_R^+/p$-linear maps $\psi_M : M_1 \xrightarrow{\mu^k} \mu^k M_1 \to \varphi^*(M_1)$ and $\psi_{M'} : M' \xrightarrow{\mu^k} \mu^k M' \to \varphi^*(M')$ such that $\psi_M \circ (1 \otimes \varphi_M) = \mu^k \mathrm{Id}_{\varphi_M^*}$ and $\psi_{M'} \circ (1 \otimes \varphi_{M'}) = \mu^k \mathrm{Id}_{\varphi_{M'}^*}$. So we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \varphi^*(M') \arrow[r] \arrow[d, "1\otimes\varphi_{M'}"] & \varphi^*(M_1) \arrow[r] \arrow[d, "1\otimes\varphi_M"] & \varphi^*(M'') \arrow[r] \arrow[d, "1\otimes\varphi_{M''}"] & 0 \\
0 \arrow[r] & M' \arrow[r] \arrow[d, "1\otimes\psi_{M'}"] & M_1 \arrow[r] \arrow[d, "1\otimes\psi_M"] & M'' \arrow[r] \arrow[d, "1\otimes\psi_{M''}"] & 0 \\
0 \arrow[r] & \varphi^*(M') \arrow[r] & \varphi^*(M_1) \arrow[r] & \varphi^*(M'') \arrow[r] & 0.
\end{tikzcd}$$

Now note that $[p]_q = \mu^{p-1} \mod p$, $\varphi(\mu) = \mu^p \mod p$ and $\varphi([p]_q) = \mu^{p(p-1)} \mod p$. Since $M_1[1/\mu] = M'[1/\mu]$, let $i \in \mathbb{N}_{\geq 1}$ such that $\mu^{pi} M'' = 0$ and $\mu^{pi-1} M'' \neq 0$. Let $x \in M''$ such that $\mu^{pi} x \neq 0$ and set $y = 1 \otimes x \in \varphi^*(M'')$. Then $\varphi(\mu^{pi}) y = 1 \otimes \mu^{pi} x = 0$ but $\mu^{p^2 i - 1} y = \varphi(\mu^{pi-1}) y = 1 \otimes \mu^{pi-1} x \neq 0$. Let $z = (1 \otimes \varphi_{M''}) y \in M''$, then $\mu^{pi} z = 0$. So we have $0 = \psi_{M''}(\mu^{pi} z) = \mu^{pi}(\psi_{M''} \circ (1 \otimes \varphi_{M''}) y) = \mu^{pi+k} y$. Therefore, we get that $pi + k = pi + p(p-1)s > p^2 i - 1 > p^2(i-1)$, i.e. $i < s + \frac{p}{p-1}$. Hence, $\mu^{s+1} M'' = 0$. Since the constant obtained is independent of M' we also get that $\mu^{s+1} M_1 \subset M^\circ \subset M_1$ and $M^\circ[1/\mu] = M_1[1/\mu]$.

Next, we will show that M° is of finite height s. Let $x \in M^\circ$ then $x \in M'$ for each $M' \in \mathcal{S}$ and there exists some $y \in \varphi^*(M') \subset \varphi^*(M_1)$ such that $(1 \otimes \varphi)y = [p]_q^s x$. Note that y is unique in $\varphi^*(M_1)$ and since $\varphi : \mathbf{A}_R^+ \to \mathbf{A}_R^+$ is flat, we get that $y \in \cap_{M' \in \mathcal{S}}(\mathbf{A}_R^+ \otimes_{\varphi, \mathbf{A}_R^+} M') = \mathbf{A}_R^+ \otimes_{\varphi, \mathbf{A}_R^+} (\cap_{M' \in \mathcal{S}} M') = \varphi^*(M^\circ)$. Therefore, we get that $M^\circ \in \mathcal{S}$.

For the second part of the claim note that $M_n[1/\mu] = D_{R,n}$ and $f_n(D_{R,n}) = D_{R,n}/p = \mathbf{D}_R(T)/p$ (see Lemma 3.28). So we get that $f_n(M_n[1/\mu]) = \mathbf{D}_R(T)/p$ and we are left to show that $f_n(M_n)$ is of finite height s. Note that we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \varphi^*(\mathrm{kernel}) \arrow[r] \arrow[d, "1\otimes\varphi"] & \varphi^*(M_n) \arrow[r] \arrow[d, "1\otimes\varphi"] & \varphi^*(f_n(M_n)) \arrow[r] \arrow[d, "1\otimes\varphi"] & 0 \\
0 \arrow[r] & \mathrm{kernel} \arrow[r] & M_n \arrow[r, "f_n"] & f_n(M_n) \arrow[r] & 0.
\end{tikzcd}$$

The rightmost vertical arrow is injective since $f_n(M_n) \subset D_{R,n}$ and the cokernel of the middle vertical arrow is killed by $[p]_q^s$ by Lemma 3.28. Hence, the cokernel of the rightmost vertical arrow is killed by $[p]_q^s$. ■

Proposition 3.30. *We have $\mathbf{A}_R \otimes_{\mathbf{A}_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{D}_R(T)$ and $\mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{N}_L(T)$.*

Proof. Since everything is p-adically complete and $\mathbf{D}_R(T)$ and $\mathbf{N}_L(T)$ are p-torsion free, it is enough to show the claim modulo p. Recall that we have $\mathbf{N}_R(T)/p \subset M_1 = \mathbf{D}_R(T)/p \cap \mathbf{N}_L(T)/p \subset \mathbf{D}_L(T)/p$ and from Lemma 3.29 we have $M^\circ \subset \mathbf{N}_R(T)/p$. Therefore, we get that $\mathbf{D}_R(T)/p = M^\circ[1/\mu] \subset \mathbf{A}_R/p \otimes_{\mathbf{A}_R^+/p} \mathbf{N}_R(T)/p \subset M_1[1/\mu] = \mathbf{D}_R(T)/p$. Finally, the isomorphism $\mathbf{A}_L^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{N}_L(T)$ follows by functoriality of Wach modules. ■

A. Structure of φ-modules

We will use setup and notations from §1.3 and the rings defined in §3.1. Let q be an indeterminate, then we have a Frobenius-equivariant isomorphism $R[\![q-1]\!] \xrightarrow{\sim} \mathbf{A}_R^+$ via the map $X_i \mapsto [X_i^\flat]$ and $q \mapsto 1 + \mu$. We will show the following structural result:

Proposition A.1. *Let N be a finitely generated $\mathbf{A}_R^+$-module and suppose that N is equipped with a Frobenius-semilinear endomorphism $\varphi : N \to N$ such that $1 \otimes \varphi : \varphi^*(N)[1/[p]_q] \xrightarrow{\sim} N[1/[p_q]]$. Then $N[1/p]$ is finite projective over $\mathbf{B}_R^+$.*

Proof. The proof is essentially the same as [DLMS22, Proposition 4.13]. Compared to loc. cit. the Frobenius endomorphism on $\mathbf{A}_R^+$ and finite height assumption on N are different and we do not assume N to be torsion free. However, one observes that torsion freeness of N is not used in the proof and one can use Lemmas A.2 and A.3 instead of [BMS18, Proposition 4.3] and [DLMS22, Lemma 4.12]. ■

Lemma A.2. *Let k be a perfect field of characteristic p and let $A = W(k)[\![q-1]\!]$ equipped with a Frobenius endorphism extending the Witt vector Frobenius on $W(k)$ by $\varphi(q) = q^p$. Let N be a finitely generated A-module equipped with a Frobenius-semilinear endomorphism such that $1 \otimes \varphi : \varphi^*(N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$. Then $N[1/p]$ is finite free over $A[1/p]$.*

Proof. The proof is essentially the same as [BMS18, Proposition 4.3]. Let J denote the smallest non-zero Fitting ideal of N over A. Set $K = W(k)[1/p]$ and $T = q-1$, in particular, $A = W(k)[\![T]\!]$ with $\varphi(T) = (1+T)^p - 1$ and set $\overline{A} = A/J$. From loc. cit. the claim can be reduced to checking that $\overline{A}[1/p] = 0$. Note that the Frobenius endomorphism on A and finite height condition on N are different from loc. cit. Therefore, we need some modifications in the arguments of loc. cit.; we us point out the differences in terms of their notations. Fix an algebraic closure $\overline{K}$ of K and consider the finite set $Z = \mathrm{Spec}\,(\overline{A}[1/p])(\overline{K})$ of $\overline{K}$-valued points of $\overline{A}[1/p]$. Let

$Z' = \{x \in \mathfrak{m}$ such that $(1+x)^p - 1 \in Z\}$, where $\mathfrak{m} \subset O_{\overline{K}}$ is the maximal ideal. Then from the equality $(A/J)[1/[p]_q] = (A/\varphi(J))[1/[p]_q]$ we get that $Z \cap U = Z' \cap U$ where $U = \mathfrak{m} - \{\zeta_p - 1, \ldots, \zeta_p^{p-1} - 1\}$. All the arguments from loc. cit. then easily adapt to give an isomorphism $K[T]/(T^r) \xrightarrow{\sim} K[T]/(\varphi(T)^r)$ where $K = W(k)[1/p]$ and $T = q - 1$. But then we get that $(\varphi(T)/T)^r$ is a unit in $K[T]$, whereas $\varphi(T)/T \in K[T]$ is an irreducible polynomial. Hence, we must have $r = 0$ and thus $(A/J)[1/p] = 0$, proving the claim. ∎

Lemma A.3. *Let k be a perfect field of characteristic p and $S = W(k)[\![u_1, \ldots, u_m]\!]$ equipped with a Frobenius endomorphism φ extending the Witt vector Frobenius on $W(k)$ such that $\varphi(u_i) \in S$ has zero constant term for each $1 \leq i \leq m$. Let $A = S[\![q-1]\!]$ equipped with a Frobenius endomorphism extending the one on S by $\varphi(q) = q^p$ and let N be a finitely generated A-module equipped with a Frobenius-semilinear endomorphism such that $1 \otimes \varphi : \varphi^*(N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$. Then $N[1/p]$ is finite projective over $A[1/p]$.*

Proof. The proof is essentially the same as [DLMS22, Lemma 4.12], except for a few changes. One proceeds by induction on m. The case $m = 0$ follows from Lemma A.2, so let $m \geq 1$. Take J to be the smallest non-zero Fitting ideal of N over A. It suffices to show that $JA[1/p] = A[1/p]$. Compatibility of Fitting ideals under base change implies that $JA[1/[p]_q] = \varphi(J)A[1/[p]_q]$ as ideals of $A[1/[p]_q]$, therefore $(A/J)[1/[p]_q] = (A/\varphi(J))[1/[p]_q]$. Let us assume $JA[1/p] \neq A[1/p]$ and we will reach a contradiction.

In our setting, the Frobenius endomorphism on A and the finite height condition are different from loc. cit. Therefore, we need some modifications in the arguments of loc. cit.; let us point out the differences in terms of their notations. Let $K = W(k)[1/p]$, fix $\overline{K}$ as an algebraic closure of K. Consider the $\overline{K}$-valued points of Spec $(A[1/p]/J)$ and let $Z = \{(|u_1|, \ldots, |u_m|, |q-1|) \in \mathbb{R}^{m+1}\}$ be the corresponding set of $(m+1)$-tuple norms. Define the set $Z' = \{(|u_1|, \ldots, |u_m|, |q-1|) \in \mathbb{R}^{m+1}$ such that $(|\varphi(u_1), \ldots |\varphi(u_m)|, |q^p - 1|) \in Z\}$ and take $\zeta_p - 1$ as the chosen uniformizer. Then one proceeds as in loc. cit. to show that $JA[1/p] \subset (u_1, \ldots, u_m, q-1)A[1/p]$ and $JA[1/p] \not\subset IA[1/p]$ where $I = (u_1, \ldots, u_m) \subset A[1/p]$.

Finally, consider the Frobenius-equivariant projection $A \to \overline{A} = A/I = W(k)[\![q-1]\!]$ and let $\overline{J} \subset \overline{A}$ denote the image of J. Since $JA[1/p] \not\subset IA[1/p]$, we get that $\overline{J} \neq 0$. Moreover, $\overline{JA}[1/p] \neq \overline{A}[1/p]$ since $JA[1/p] \subset (u_1, \ldots, u_m, q-1)A[1/p]$. However, the equality $(A/J)[1/[p]_q] = (A/\varphi(J))[1/[p]_q]$ implies that $(\overline{A}/\overline{J})[1/[p]_q] = (\overline{A}/\varphi(\overline{J}))[1/[p]_q]$, i.e. $\overline{JA}[1/p] = \overline{A}[1/p]$ by inductive hypothesis (see Lemma A.2). This gives a contradiction. Hence, we must have $JA[1/p] = A[1/p]$, thus proving the lemma. ∎

Finally, we will show a structural result for étale φ-modules over $\mathbf{A}_R$.

Lemma A.4. *Let D be a finitely generated p-torsion free étale φ-module over $\mathbf{A}_R$. Then D is finite projective.*

Proof. Note that $\mathbf{A}_R$ is a noetherian ring, so it is enough to show that D is flat over $\mathbf{A}_R$. Moreover, since $\mathbf{A}_R$ and D are p-adically complete and p-torsion free, therefore using [Sta23, Tag 0912, Tag 051C] it is enough to show that D/pD is flat over $\mathbf{A}_R/p\mathbf{A}_R$. Now from [And06, Lemma 7.10] it follows that D/pD is flat over $\mathbf{A}_R/p\mathbf{A}_R$. Note that the proof of loc. cit. only uses finiteness and étaleness of D over $\mathbf{A}_R$. In particular, one does not need to add the data of Γ_R-action on D. This proves the lemma. ∎

References

[Abh21] Abhinandan. “Crystalline representations and Wach modules in the relative case”. In: arXiv:2103.17097 (Mar. 2021). To appear in Annales de l’Institut Fourier.

[And06] Fabrizio Andreatta. “Generalized ring of norms and generalized (ϕ, Γ)-modules”. In: *Ann. Sci. École Norm. Sup. (4)* 39.4 (2006), pp. 599–647. ISSN: 0012-9593.

[AB08] Fabrizio Andreatta and Olivier Brinon. “Surconvergence des représentations p-adiques: le cas relatif”. In: 319. Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (ϕ, Γ)-modules. 2008, pp. 39–116. ISBN: 978-2-85629-256-3.

[Ber02] Laurent Berger. “Représentations p-adiques et équations différentielles”. In: *Invent. Math.* 148.2 (2002), pp. 219–284. ISSN: 0020-9910.

[Ber04] Laurent Berger. “Limites de représentations cristallines”. In: *Compos. Math.* 140.6 (2004), pp. 1473–1498. ISSN: 0010-437X.

[BMS18] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. “Integral p-adic Hodge theory”. In: *Publ. Math. Inst. Hautes Études Sci.* 128 (2018), pp. 219–397. ISSN: 0073-8301.

[BS21] Bhargav Bhatt and Peter Scholze. “Prismatic F-crystals and crystalline Galois representations”. In: arXiv:2106.14735 (June 2021).

[Bri06] Olivier Brinon. “Représentations cristallines dans le cas d’un corps résiduel imparfait”. In: *Ann. Inst. Fourier (Grenoble)* 56.4 (2006), pp. 919–999. ISSN: 0373-0956.

[Bri08] Olivier Brinon. “Représentations p-adiques cristallines et de de Rham dans le cas relatif”. In: *Mém. Soc. Math. Fr. (N.S.)* 112 (2008), pp. vi+159. ISSN: 0249-633X.

[BT08] Olivier Brinon and Fabien Trihan. “Représentations cristallines et F-cristaux: le cas d’un corps résiduel imparfait”. In: *Rend. Semin. Mat. Univ. Padova* 119 (2008), pp. 141–171. ISSN: 0041-8994.

[CC98] F. Cherbonnier and P. Colmez. “Représentations p-adiques surconvergentes”. In: *Invent. Math.* 133.3 (1998), pp. 581–611. ISSN: 0020-9910.

[Col99] Pierre Colmez. “Représentations cristallines et représentations de hauteur finie”. In: *J. Reine Angew. Math.* 514 (1999), pp. 119–143. ISSN: 0075-4102.

[CN17] Pierre Colmez and Wiesława Nizioł. “Syntomic complexes and p-adic nearby cycles”. In: *Invent. Math.* 208.1 (2017), pp. 1–108. ISSN: 0020-9910.

[DLMS22] Heng Du, Tong Liu, Yong Suk Moon, and Koji Shimizu. “Completed prismatic F-crystals and crystalline $\mathbf{Z}_p$-local systems”. In: arXiv:2203.03444 (Mar. 2022).

[Fon90] Jean-Marc Fontaine. “Représentations p-adiques des corps locaux. I”. In: *The Grothendieck Festschrift, Vol. II*. Vol. 87. Progr. Math. Birkhäuser Boston, Boston, MA, 1990, pp. 249–309.

[Fon94] Jean-Marc Fontaine. “Le corps des périodes p-adiques”. In: *Astérisque* 223 (1994). With an appendix by Pierre Colmez, Périodes p-adiques (Bures-sur-Yvette, 1988), pp. 59–111. ISSN: 0303-1179.

[Gol61] Oscar Goldman. “Determinants in projective modules”. In: *Nagoya Math. J.* 18 (1961), pp. 27–36. ISSN: 0027-7630.

[GR22] Haoyang Guo and Emanuel Reinecke. “A prismatic approach to crystalline local systems”. In: arXiv:2203.09490 (Mar. 2022).

[Hel43] Olaf Helmer. “The elementary divisor theorem for certain rings without chain condition”. In: *Bull. Amer. Math. Soc.* 49 (1943), pp. 225–236. ISSN: 0002-9904.

[Hyo86] Osamu Hyodo. “On the Hodge-Tate decomposition in the imperfect residue field case”. In: *J. Reine Angew. Math.* 365 (1986), pp. 97–113. ISSN: 0075-4102.

[Jon95] A. J. de Jong. “Crystalline Dieudonné module theory via formal and rigid geometry”. In: *Inst. Hautes Études Sci. Publ. Math.* 82 (1995), 5–96 (1996). ISSN: 0073-8301.

[Ked04] Kiran S. Kedlaya. “A p-adic local monodromy theorem”. In: *Ann. of Math. (2)* 160.1 (2004), pp. 93–184. ISSN: 0003-486X.

[Ked05] Kiran S. Kedlaya. “Slope filtrations revisited”. In: *Doc. Math.* 10 (2005), pp. 447–525. ISSN: 1431-0635.

[Kis06] Mark Kisin. “Crystalline representations and F-crystals”. In: *Algebraic geometry and number theory.* Vol. 253. Progr. Math. Birkhäuser Boston, Boston, MA, 2006, pp. 459–496.

[KR09] Mark Kisin and Wei Ren. “Galois representations and Lubin-Tate groups”. In: *Doc. Math.* 14 (2009), pp. 441–461. ISSN: 1431-0635.

[Lan90] Serge Lang. *Cyclotomic fields I and II.* second. Vol. 121. Graduate Texts in Mathematics. With an appendix by Karl Rubin. Springer-Verlag, New York, 1990, pp. xviii+433. ISBN: 0-387-96671-4.

[Laz62] Michel Lazard. “Les zéros des fonctions analytiques d’une variable sur un corps valué complet”. In: *Inst. Hautes Études Sci. Publ. Math.* 14 (1962), pp. 47–75. ISSN: 0073-8301.

[Mar17] Florent Martin. “Analytic functions on tubes of nonarchimedean analytic spaces”. In: *Algebra Number Theory* 11.3 (2017). With an appendix by Christian Kappen and Martin, pp. 657–683. ISSN: 1937-0652.

[Moo22] Yong Suk Moon. “Note on purity of crystalline local systems”. In: arXiv:2210.07368 (Oct. 2022).

[MT20] Matthew Morrow and Takeshi Tsuji. “Generalised representations as q-connections in integral p-adic Hodge theory”. In: arXiv:2010.04059 (Oct. 2020).

[Ohk13] Shun Ohkubo. “The p-adic monodromy theorem in the imperfect residue field case”. In: *Algebra Number Theory* 7.8 (2013), pp. 1977–2037. ISSN: 1937-0652.

[Ohk15] Shun Ohkubo. “On differential modules associated to de Rham representations in the imperfect residue field case”. In: *Algebra Number Theory* 9.8 (2015), pp. 1881–1954. ISSN: 1937-0652.

[Sta23] The Stacks project authors. *The Stacks project.* `https://stacks.math.columbia.edu`. 2023.

[Wac96] Nathalie Wach. “Représentations p-adiques potentiellement cristallines”. In: *Bull. Soc. Math. France* 124.3 (1996), pp. 375–400. ISSN: 0037-9484.

[Wac97] Nathalie Wach. “Représentations cristallines de torsion”. In: *Compositio Math.* 108.2 (1997), pp. 185–240. ISSN: 0010-437X.

[Wei94] Charles A. Weibel. *An introduction to homological algebra.* Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994, pp. xiv+450. ISBN: 0-521-43500-5; 0-521-55987-1.

ABHINANDAN
UNIVERSITY OF TOKYO, 3 CHOME-8-1, KOMABA, MEGURO CITY, TOKYO, JAPAN
E-mail: abhi@ms.u-tokyo.ac.jp, *Web*: https://abhinandan1729.github.io/