Syntomic complex and p-adic nearby cycles

Abhinandan

Abstract. In local relative p-adic Hodge theory, we show that Galois cohomology of a finite crystalline height representation (upto a twist), is essentially computed by (Fontaine-Messing) syntomic complex with coefficient in the associated F-isocrystal. In global applications, for smooth (p-adic formal) schemes, we establish a comparison between syntomic complex with coefficient in a locally free Fontaine-Laffaille module and p-adic nearby cycles of the associated étale local system on the (rigid) generic fiber.

Contents

1. Introduction

Let p denote a fixed prime, κ a perfect field of characteristic p, K a discrete valuation field of mixed characteristic with ring of integers O_K and residue field κ and $F = W(\kappa)[\frac{1}{p}]$ the fraction field of the ring of p-typical Witt vectors with coefficients in κ. Fontaine's *crystalline comparison theorem* for an O_K-scheme $\mathfrak{X}$ examines the relationship between p-adic étale cohomology of its generic fiber and crystalline cohomology of its special fiber. More precisely,

Theorem 1.1. *Let $\mathfrak{X}$ be a proper and smooth scheme defined over O_K, with $X = \mathfrak{X} \otimes_{O_K} K$ its generic fiber $\mathfrak{X}_\kappa = \mathfrak{X} \otimes_{O_K} \kappa$ its special fiber. Then for each $k \in \mathbb{N}$ there exists a natural isomorphism*

$$H^k_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\text{cris}}(O_{\overline{K}}) \xrightarrow{\sim} H^k_{\text{cris}}(\mathfrak{X}_\kappa/W(\kappa)) \otimes_{W(\kappa)} \mathbf{B}_{\text{cris}}(O_{\overline{K}}),$$

compatible with filtration, Frobenius and action of G_K on each side.

Here $\mathbf{B}_{\text{cris}}(O_{\overline{K}})$ denotes the crystalline period ring constructed by Fontaine (see [Fon94a]), and it is equipped with a filtration, Frobenius and continuous action of G_K.

In [FM87] Fontaine and Messing initiated a program for proving the statement via *syntomic* methods. By subsequent works of [KM92, Kato-Messing], [Kat94, Kato] and the remarkable work of [Tsu99, Tsuji] this program was concluded with a proof of the crystalline comparison theorem (more generally, the semistable comparison theorem). There have been several other proofs as well as generalizations of crystalline comparison theorem: [Fal89; Fal02, Faltings], [Niz98, Nizioł], [Bei12; Bei13, Beilinson], [Sch13, Scholze], [YY14, Yamashita-Yasuda], [CN17, Colmez-Nizioł], [BMS18, Bhatt-Morrow-Scholze] among others.

Theorem 1.1 also holds for proper and smooth p-adic formal schemes. This was shown by Andreatta and Iovita in [AI13] using Faltings approach of almost étale extensions. The natural variation of Theorem 1.1 for proper semistable p-adic formal schemes was obtained by Colmez and Nizioł in [CN17].

1.1. p-adic nearby cycles.

Let $\mathfrak{X}$ be a smooth (p-adic formal) scheme over O_K with X as its (rigid) generic fiber and $\mathfrak{X}_\kappa$ as its special fiber. Let $j : X_{\text{ét}} \to \mathfrak{X}_{\text{ét}}$ and $i : \mathfrak{X}_{\kappa,\text{ét}} \to \mathfrak{X}_{\text{ét}}$ denote natural morphisms between sites. For $r \geq 0$, let $\mathscr{S}_n(r)_{\mathfrak{X}}$ denote the syntomic sheaf modulo p^n on $\mathfrak{X}_{\kappa,\text{ét}}$. It can be thought of as a derived Frobenius and filtration eigenspace of crystalline cohomology. In [FM87], Fontaine and Messing constructed a period morphism

$$\alpha^{\text{FM}}_{r,n} : \mathscr{S}_n(r)_{\mathfrak{X}} \longrightarrow i^*\mathrm{R}j_*\mathbb{Z}/p^n(r)'_X, \tag{1.1}$$

from the syntomic complex to the complex of p-adic nearby cycles, where $\mathbb{Z}_p(r)' := \frac{1}{a(r)!p^{a(r)}}\mathbb{Z}_p(r)$, for $r = (p-1)a(r) + b(r)$ with $0 \leq b(r) < p-1$. In the case of schemes, for $0 \leq r \leq p-1$ and after truncating the complexes in (1.1) in degrees $\leq r$ the map $\alpha^{\text{FM}}_{r,n}$ was shown to be a quasi-isomorphism in the work of Kato [Kat87; Kat94], Kurihara [Kur87], and Tsuji [Tsu99]. In [Tsu96], Tsuji generalized the result for schemes to some non-trivial étale local systems arising from Fontaine-Laffaille modules over O_F (see [FL82]).

Colmez and Nizioł have shown that the Fontaine-Messing period map $\alpha^{\text{FM}}_{r,n}$, after a suitable truncation, is essentially a quasi-isomorphism. More precisely,

Theorem 1.2 ([CN17, Theorem 1.1])**.** *For $0 \leq k \leq r$, the map*

$$\alpha^{\text{FM}}_{r,n} : \mathcal{H}^k(\mathscr{S}_n(r)_{\mathfrak{X}}) \longrightarrow i^*\mathrm{R}^k j_*\mathbb{Z}/p^n(r)'_X,$$

is a p^N-isomorphism, i.e. the kernel and cokernel of this map is killed by p^N, where $N = N(e,p,r) \in \mathbb{N}$ depends on the absolute ramification index e of K, the prime p and the twist r but not on X or n.

Theorem 1.2 also holds for base change of proper and smooth (p-adic formal) schemes. In particular, after passing to the limit and inverting p, for $0 \leq k \leq r$ we obtain isomorphisms (see [Tsu99, Theorem 3.3.4])

$$\alpha_r^{\mathrm{FM}} : H^k_{\mathrm{syn}}(\mathfrak{X}_{O_{\overline{K}}}, r)_{\mathbb{Q}} \xrightarrow{\sim} H^k_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r)). \tag{1.2}$$

The isomorphism in (1.2) is one of the most important step in proving Theorem 1.1 via syntomic methods. These ideas have been used in [FM87], [KM92], [Kat87], [Kat94], [Tsu99] and [YY14].

The proof of Colmez and Nizioł is different from earlier approaches. They prove Theorem 1.2 first and deduce the comparison in (1.2) via base change in proper and smooth case. To prove their claim, they reduce the problem to local setting and construct another local period map $\alpha_r^{\mathcal{L}\mathrm{az}}$, employing techniques from the theory of (φ, Γ)-modules and a version of integral Lazard isomorphism between Lie algebra cohomology and continuous group cohomology. They show that $\alpha_r^{\mathcal{L}\mathrm{az}}$ is a quasi-isomorphism and coincides with local Fontaine-Messing period map up to some fixed power of p.

Remark 1.3. The results of [CN17] have been worked out in the setting of semistable (p-adic formal) schemes. So to obtain the claim for $0 \leq k \leq r$ as in Theorem 1.2, one should work with log-crystalline cohomology. Working without log structures, one would obtain the p-power isomorphism in Theorem 1.2 for $0 \leq k \leq r-1$ (also see Remark 1.12 (i) below).

1.1.1. Local result of Colmez and Nizioł. Most of the work done for the proof of Theorem 1.1 in [CN17] involves computations in the local setting, i.e. over an étale algebra over a (formal) torus. More precisely, a smooth (p-adic formal) scheme $\mathfrak{X}$ defined over O_K can be covered by affine schemes given as (formal) spectrum of (p-adic completion of an) étale algebra over $O_K[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ for some indeterminates $X_1, \ldots, X_d$. In the local setting, Colmez and Nizioł also show that it is enough to work with p-adic completions, i.e. formal schemes and deduce results for schemes by invoking Elkik's approximation theorem and a form of rigid GAGA (see [CN17, §5.1]).

For simplicity in the introduction, we will state their results over the algebra R taken as the p-adic completion of $O_F[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ and let $S := O_K \otimes_{O_F} R$. Let $G_S = \pi_1^{\text{ét}}(S[\frac{1}{p}], \overline{\eta})$ for a fixed geometric generic point of $\mathrm{Sp}\,(S[\frac{1}{p}])$. Let R_ϖ^+ denote the (p, X_0)-adic completion of $W(\kappa)[X_0, X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$, and let R_ϖ^{PD} denote the p-adic completion of the divided power envelope with respect to the kernel of the map $R_\varpi^+ \twoheadrightarrow S$ sending X_0 to ϖ (a uniformizer of K). Further, let $\Omega^1_{R_\varpi^{\mathrm{PD}}}$ denote the p-adic completion of the module of differentials of R_ϖ^{PD} relative to $\mathbb{Z}$. The syntomic cohomology of S can be computed by the complex

$$\mathrm{Syn}(S, r) := \mathrm{Cone}\big(F^r \Omega^\bullet_{R_\varpi^{\mathrm{PD}}} \xrightarrow{p^r - p^\bullet \varphi} \Omega^\bullet_{R_\varpi^{\mathrm{PD}}}\big)[-1].$$

Theorem 1.4 ([CN17, Theorem 1.6])**.** *If K contains enough roots of unity, then the maps*

$$\alpha_r^{\mathcal{L}az} : \tau_{\leq r}\mathrm{Syn}(S, r) \longrightarrow \tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{cont}}(G_S, \mathbb{Z}_p(r)),$$
$$\alpha_{r,n}^{\mathcal{L}az} : \tau_{\leq r}\mathrm{Syn}(S, r)_n \longrightarrow \tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{cont}}(G_S, \mathbb{Z}/p^n(r)) \longrightarrow \tau_{\leq r}\mathrm{R}\Gamma\big((\mathrm{Sp}\ S[\tfrac{1}{p}])_{\text{ét}}, \mathbb{Z}/p^n(r)\big),$$

are p^{Nr}-quasi-isomorphisms for a universal constant N.

Note that the truncation here denotes canonical truncation in literature. Having enough roots of unity in K is a technical condition (see [CN17, §2.2.1]). In general, if K does not contain enough roots of unity (for example $K = F$ or), then one passes to an extension $K(\zeta_{p^m})$ for m large enough and then using Galois descent we obtain the result over K with the constant N depending on the ramification index $e = [K : F]$, p and r (see [CN17, Theorem 5.4]). The proof of Colmez and Nizioł relies on comparing the syntomic complex with the relative version of Fontaine-Herr complex in terms of (φ, Γ)-modules computing the continuous G_S-cohomology of $\mathbb{Z}_p(r)$ (see [Her98] and [AI08]).

Remark 1.5. Similar to Remark 1.3 let us note that in Theorem 1.4 Colmez and Nizioł work with semistable affinoids and log-syntomic complex. Without log structures one should truncate in degree $\leq r-1$ (see Remark 1.12 (i) below).

Our goal is to generalize Theorem 1.2 to non-trivial coefficients. Clearly, one needs to restrict themselves to a "friendly" category of coefficients, i.e. on which local computations similar to [CN17] could be carried out. In the local setting, by techniques employed in the proof of Theorem 1.4 (and applying $K(\pi,1)$-Lemma of Scholze for p-coefficients, see [Sch13, Theorem 4.9]), the problem could be formulated as: can one obtain a statement similar to Theorem 1.4 for more general $\mathbb{Z}_p$-representations of G_R? A natural object to consider for such a local result is a G_R-stable $\mathbb{Z}_p$-lattice T inside a crystalline representation V of G_R (in the sense of [Bri08, Chapitre 8]). However, as local computations involve complexes of (φ,Γ)-modules, we should further restrict ourselves to a representation whose corresponding étale (φ,Γ)-module is "crystalline". Representations capturing these ideas are referred to as *finite crystalline height representations.*

1.2. Finite height representations. In the classical case, i.e. for a mixed characteristic local field K, in [Fon90] Fontaine established an equivalence of catgeories between $\mathbb{Z}_p$-representations (resp. p-adic representations) of G_K and étale (φ,Γ)-modules over a certain period ring $\mathbf{A}_K$ (resp. $\mathbf{B}_K$). Moreover, in [Fon79; Fon82; Fon94a; Fon94b] Fontaine described crystalline representations of G_K in terms of certain filtered φ-modules over F. For $K=F$, by the works of [Wac96; Wac97, Wach], [Col99, Colmez] and [Ber04, Berger] it is known that crystalline representations can be described in terms of finite height (φ,Γ)-modules (closely related to the étale (φ,Γ)-module of Fontaine).

In the relative case, let us now fix $p\geq 3$, an absolutely unramified extension F over $\mathbb{Q}_p$, $K=F(\zeta_{p^m})$ for a fixed $m\geq 1$ and let $\varpi=\zeta_{p^m}-1$ (see Remark 1.12 on rationale behind our assumptions). For simplicity, let R denote the p-adic completion of $O_F[X_1^{\pm 1},\ldots,X_d^{\pm 1}]$ and let $S:=O_K\otimes_{O_F}R$.

Remark 1.6. Note that all of the following results in this section are also true for p-adic completion of an étale algebra over $O_F[X_1^{\pm 1},\ldots,X_d^{\pm 1}]$ with non-empty geometrically integral special fiber (see Assumption 2.1).

1.2.1. (φ,Γ)-modules. Let us fix an algebraically closed field $\overline{\mathrm{Fr}\ (R)}$ containing $\overline{F}$. Let $\overline{R}$ denote the union of finite R-subalgebras $R'\subset\overline{\mathrm{Fr}\ (R)}$ such that $R'[\frac{1}{p}]$ is étale over $R[\frac{1}{p}]$. We write $\mathbb{C}^+(\overline{R})=\widehat{\overline{R}}$ as the p-adic completion, $\mathbb{C}(\overline{R})=\mathbb{C}^+(\overline{R})[\frac{1}{p}]$ and $G_R=\mathrm{Gal}(\overline{R}[\frac{1}{p}]/R[\frac{1}{p}])$. Now for $n\in\mathbb{N}$, let $F_n=F(\zeta_{p^n})$ and let R_n denote the integral closure of $R\otimes O_{F_n}[X_1^{1/p^n},\ldots X_d^{1/p^n}]$ inside $\overline{R}[\frac{1}{p}]$ and let $R_\infty:=\cup_n R_n$. We set $\Gamma_R:=\mathrm{Gal}(R_\infty[\frac{1}{p}]/R[\frac{1}{p}])$, $H_R:=\mathrm{Ker}\ (G_R\rightarrow\Gamma_R)$ and we have an exact sequence

$$1\longrightarrow\Gamma'_R\longrightarrow\Gamma_R\longrightarrow\Gamma_F\longrightarrow 1,$$

where we have $\Gamma'_R=\mathrm{Gal}(R_\infty[\frac{1}{p}]/F_\infty R[\frac{1}{p}])\simeq\mathbb{Z}_p^d$, and $\Gamma_F=\mathrm{Gal}(F_\infty/F)\simeq\mathbb{Z}_p^\times$.

Using a certain period ring $\mathbf{A}\subset W(\mathbb{C}(\overline{R})^\flat)$, stable under Frobenius on Witt vectors and G_R-action, in [And06] Andreatta generalized Fontaine's results to $\mathbb{Z}_p$-representations (resp. p-adic representations) of G_R. To any $\mathbb{Z}_p$-representation T of G_R, Andreatta functorially attaches an étale (φ,Γ_R)-module $\mathbf{D}(T)=(\mathbf{A}\otimes_{\mathbb{Z}_p}T)^{H_R}$ over the period ring $\mathbf{A}_R=\mathbf{A}^{H_R}$. This induces an equivalence of categories between $\mathbb{Z}_p$-representations and étale (φ,Γ_R)-modules over $\mathbf{A}_R$. Similarly, to any p-adic representation V of G_R, using the period ring $\mathbf{B}=\mathbf{A}[\frac{1}{p}]$, one can attach an étale (φ,Γ_R)-module $\mathbf{D}(T)=(\mathbf{B}\otimes_{\mathbb{Q}_p}V)^{H_R}$ over $\mathbf{B}_R=\mathbf{B}^{H_R}$. Again, this induces an equivalence of categories between p-adic representations and étale (φ,Γ_R)-modules over $\mathbf{B}_R$.

Next, let $\mathbf{A}_{\inf}(\overline{R}) = W(\mathbb{C}^+(\overline{R}))$, $\mathbf{A}^+ = \mathbf{A} \cap \mathbf{A}_{\inf}(\overline{R}) \subset W(\mathbb{C}(\overline{R}))$ and set $\mathbf{D}^+(T) = (\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T)^{H_R}$, a finitely generated (φ, Γ_R)-module over $\mathbf{A}_R^+ = (\mathbf{A}^+)^{H_R}$. Let $q = \frac{\varphi(\pi)}{\pi}$, where π is the usual element in Fontaine's constructions (see §2.2 for notations). In [Abh21], we studied the notion of a finite q-height representation, i.e. a representation which (up to twisting by the p-adic cyclotomic character) admits a unique projective $\mathbf{A}_R^+$-submodule $\mathbf{N}(T) \subset \mathbf{D}^+(T)$ of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$ with actions of φ and Γ_R satisfying certain conditions (see Definition 3.2). Such representations are motivated by the classical definition of finite height crystalline representations [Wac96; Wac97; Col99; Ber04] (see [Abh21, Remark 1.4]). Moreover, finite q-height representations are closely related to crystalline representations of G_R (see below).

1.2.2. Crystalline representations. Akin to Fontaine's formalism in [Fon82], Brinon studied p-adic representations of G_R in [Bri08]. To classify crystalline representations, Brinon constructs the (big) crystalline period ring $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$, a p-adically complete $R[\frac{1}{p}]$-algebra equipped with a G_R-action, a Frobenius endomorphism, a filtration and a $\mathbf{B}_{\mathrm{cris}}(\overline{R})$-linear connection satisfying Griffiths transversality (see §2.2 for details). For V a p-adic representation of G_R let

$$\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) := (\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V)^{G_R}.$$

This construction is functorial in V and takes values in the category of filtered (φ, ∂)-modules over $R[\frac{1}{p}]$. The representation V is said to be *crystalline* if and only if it is $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$-admissible (see §2.3). The restriction of the functor $\mathcal{O}\mathbf{D}_{\mathrm{cris}}$ to the subcategory of crystalline representations of G_R establishes an equivalence with the essential image of the restriction.

Let us recall the following result relating finite q-height representations of G_R to crystalline representations using the period ring $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$ compatible with filtration, Frobenius G_R-action and connection constructed in [Abh21, §4.3].

Theorem 1.7 ([Abh21, Theorem 4.24, Proposition 4.27])**.** *Let V be a positive finite q-height representation of G_R, then*

(i) *V is a positive crystalline representation.*

(ii) *We have an isomorphism of $R[\frac{1}{p}]$-modules*

$$\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xleftarrow{\sim} \left(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\right)^{\Gamma_R}[\tfrac{1}{p}],$$

compatible with Frobenius, filtration, and connection on each side.

(iii) *After extension of scalars to $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, we obtain a natural isomorphism*

$$\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xleftarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V),$$

compatible with Frobenius, filtration, connection and the action of Γ_R on each side.

The preceding result helps us in constructing an R-submodule inside $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ satisfying certain key properties helpful in establishing our main local result (see Theorem 1.9).

1.3. Syntomic coefficients and (φ, Γ)-modules. In this section, let us consider the following class of representations: Let V be a positive finite q-height representation of G_R with $T \subset V$ a G_R-stable $\mathbb{Z}_p$-lattice as in Definition 3.2 such that the $\mathbf{A}_R^+$-module is free of rank $= \dim \mathbb{Q}_p V$. We consider $M \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ a finite free R-submodule of rank $= \dim_{\mathbb{Q}_p} V$ such that $M[\frac{1}{p}] \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ satisfying certain conditions (see Assumption 5.4). Also see Example 5.5 for a discussion on obtaining M from $\mathbf{N}(T)$ such that M satisfies Assumption 5.4.

Our objective is to relate the relative Fontaine-Herr complex computing continuous G_R-cohomology of $T(r)$ to syntomic complex with coefficients in the R-lattice $M \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$.

Let us first consider the case of cyclotomic extension $S = R[\varpi]$. From §2.5 we have the divided power ring $R_{\varpi}^{\mathrm{PD}} \twoheadrightarrow S$ and let $M_{\varpi}^{\mathrm{PD}} := R_{\varpi}^{\mathrm{PD}} \otimes_R M$ equipped with a Frobenius-semilinear endomorphism φ, a filtration and a connection satistfying Griffiths transversality with respect to the filtration. In particular, we have a filtered de Rham complex

$$\mathrm{Fil}^r \mathscr{D}^{\bullet}_{S,M} : \mathrm{Fil}^r M_{\varpi}^{\mathrm{PD}} \longrightarrow \mathrm{Fil}^{r-1} M_{\varpi}^{\mathrm{PD}} \otimes_{R_{\varpi}^{\mathrm{PD}}} \Omega^1_{R_{\varpi}^{\mathrm{PD}}} \longrightarrow \mathrm{Fil}^{r-2} M_{\varpi}^{\mathrm{PD}} \otimes_{R_{\varpi}^{\mathrm{PD}}} \Omega^2_{R_{\varpi}^{\mathrm{PD}}} \longrightarrow \cdots.$$

Definition 1.8. Let $r \in \mathbb{N}$ and consider the complex $\mathrm{Fil}^r \mathscr{D}^{\bullet}_{S,M}$ as above. Define the *syntomic complex* of S with coefficients in M as

$$\begin{aligned} \mathrm{Syn}(S, M, r) &:= \big[\, \mathrm{Fil}^r \mathscr{D}^{\bullet}_{S,M} \xrightarrow{p^r - p^{\bullet}\varphi} \mathscr{D}^{\bullet}_{S,M} \,\big]; \\ \mathrm{Syn}(S, M, r)_n &:= \mathrm{Syn}(S, M, r) \otimes \mathbb{Z}/p^n. \end{aligned}$$

Our main local result is as follows:

Theorem 1.9 (see Theorem 5.8)**.** *Let V be a p-adic finite q-height representation of G_R of height s with $T \subset V$ a G_R-stable $\mathbb{Z}_p$-lattice as above and let $r \in \mathbb{N}$ such that $r \geq s+1$. Then there exists p^N-quasi-isomorphisms*

$$\begin{aligned} \alpha_r^{\mathcal{L}az} &: \tau_{\leq r-s-1}\mathrm{Syn}(S, M, r) \simeq \tau_{\leq r-s-1}\mathrm{R}\Gamma_{\mathrm{cont}}(G_S, T(r)), \\ \alpha_{r,n}^{\mathcal{L}az} &: \tau_{\leq r-s-1}\mathrm{Syn}(S, M, r)_n \simeq \tau_{\leq r-s-1}\mathrm{R}\Gamma_{\mathrm{cont}}(G_S, T/p^n(r)), \end{aligned}$$

where $N = N(T, e, r) \in \mathbb{N}$ depending on the representation T, the absolute ramification index e of K and the twist r.

The proof of Theorem 5.8 proceeds in two main steps: First, we modify the syntomic complex with coefficients in M to relate it to a "differential" Koszul complex with coefficients in $\mathbf{N}(T)$ (see Proposition 5.35). Next, in the second step we modify Koszul complex from the first step to obtain Koszul complex computing continuous G_S-cohomology of $T(r)$ (see Definition 5.8 and Proposition 6.1). The key to the connection between these two steps is provided by the comparison isomorphism in Theorem 1.7 and a version of Poincaré Lemma (see §5.6). The idea for the proof is inspired by the work of Colmez and Nizioł [CN17], however our setting demands several non-trivial technical refinements.

We can descend the quasi-isomorphism in Theorem 1.9 to R. Note that we have a filtered de Rham complex over R with coeffcients in M as

$$\mathrm{Fil}^r \mathscr{D}^{\bullet}_{R,M} : \mathrm{Fil}^r M \longrightarrow \mathrm{Fil}^{r-1} M \otimes_R \Omega^1_R \longrightarrow \mathrm{Fil}^{r-2} M \otimes_R \Omega^2_R \longrightarrow \cdots.$$

Definition 1.10. Let $r \in \mathbb{N}$ and define the *syntomic complex* of R with coefficients in M as

$$\begin{aligned} \mathrm{Syn}(R, M, r) &:= \big[\, \mathrm{Fil}^r \mathscr{D}^{\bullet}_{R,M} \xrightarrow{p^r - p^{\bullet}\varphi} \mathscr{D}^{\bullet}_{R,M} \,\big]; \\ \mathrm{Syn}(R, M, r)_n &:= \mathrm{Syn}(R, M, r) \otimes \mathbb{Z}/p^n. \end{aligned}$$

Using Theorem 1.9 for $\varpi = \zeta_{p^2} - 1$ and Galois descent (see Lemma 6.26), we obtain

Corollary 1.11 (see Corollary 5.8)**.** *Let V be a finite q-height representation of G_R of height s with $T \subset V$ a G_R-stable $\mathbb{Z}_p$-lattice as above and let $r \in \mathbb{N}$ such that $r \geq s+1$. Then there exists p^N-quasi-isomorphisms*

$$\begin{aligned} \tau_{\leq r-s-1}\mathrm{Syn}(R, M, r) &\simeq \tau_{\leq r-s-1}\mathrm{R}\Gamma_{\mathrm{cont}}(G_R, T(r)), \\ \tau_{\leq r-s-1}\mathrm{Syn}(R, M, r)_n &\simeq \tau_{\leq r-s-1}\mathrm{R}\Gamma_{\mathrm{cont}}(G_R, T/p^n(r)), \end{aligned}$$

where $N = N(p, r, s) \in \mathbb{N}$ depending on the prime p, the twist r and the height s of the representation V.

Remark 1.12. (i) Taking $T = \mathbb{Z}_p$ in Theorem 1.9 we obtain a statement similar to Theorem 1.2. However, note that we have to truncate in degree $\leq r-1$. This is due to the fact that we do not work with log-structures unlike [CN17]. Working with log-syntomic complex, where we consider log-structure over R_ϖ^+ with respect to the arithmetic variable X_0 and Kummer Frobenius as explained below, would enable us to show a p-power quasi-isomorphism also in degree r.

(ii) Note that Theorem 1.2 is shown for all finite extensions K/F, whereas in Theorem 1.9, we restrict ourselves to the cyclotomic case. This is due to the fact that we use cyclotomic Frobenius ($X_0 \mapsto (1+X_0)^p - 1$) in Definition 1.8, whereas Colmez and Nizioł used Kummer Frobenius ($X_0 \mapsto X_0^p$). Note that for general K, the definition of cyclotomic Frobenius for X_0 is different from the formula displayed above (see [CN17, §2.3]).

(iii) For a finite extension K/F, one should use log-structure over R_ϖ^+ with respect to the arithmetic variable X_0 and Kummer Frobenius instead of the cyclotomic Frobenius to define a log-syntomic complex. Then using [CN17, §3.5] (an application of Poincaré Lemma), it is possible to obtain an analogue of Theorem 1.9 for all finite extensions K/F (with truncation in degree $\leq r-s$).

(iv) To obtain the statement over $\overline{F}$ one could proceed as in (iii) and pass to the limit over all finite extensions K/F. Alternatively, one could directly work over $\mathbb{C}_p = \widehat{\overline{F}}$ as in [Gil21] to avoid complications arising from Frobenius on arithmetic variable X_0. In that case, our proofs can be adapted for syntomic complex (without log-structure with respect to X_0) to obtain a statement analogous to Theorem 5.8 for $S = R \otimes_{O_F} O_{\mathbb{C}_p}$ (with truncation in degree $\leq r-s-1$).

(v) The case $p = 2$ is slightly different than the case of $p \geq 3$. But similar to [CN17], the proofs could be appropriately modified to include $p = 2$ as well.

To conclude this section, let us note that for S as in Theorem 1.9, using the fundamental exact sequence in p-adic Hodge theory (2.2), one can define the local version of Fontaine-Messing period map (see §6.7) for T as in Theorem 1.9. Then we are able to show that

Theorem 1.13. *The period map $\tilde{\alpha}_{r,n,S}^{\mathrm{FM}}$ is $p^{N(T,e,r)}$-equal to $\alpha_{r,n}^{\mathcal{L}az}$ from Theorem 1.9.*

1.4. Fontaine-Laffaille modules and p-adic nearby cycles.

We finally come to global applications of results described in the previous section. In this section we will consider locally free Fontaine-Laffaille modules introduced by Faltings in [Fal89, §II]. These objects are obtained by gluing together local data.

Let R denote the p-adic completion of an étale algebra over $O_F[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ for some $d \in \mathbb{N}$ and such that R has non-empty geometrically integral special fiber (see §2.1 for details). In Definition 3.16, we consider the category $\mathrm{MF}_{[0,s],\,\mathrm{free}}(R, \Phi, \partial)$ of free relative Fontaine-Laffaille modules of level $[0,s]$, a full subcategory of the abelian category $\mathfrak{MF}_{[0,s]}^{\nabla}(R)$ of [Fal89, §II]. One can functorially attach to such a module, a free $\mathbb{Z}_p$-module $T_{\mathrm{cris}}(M)$ equipped with a continuous G_R-action such that $V_{\mathrm{cris}}(M)$ is crystalline and s equals the maximum among the absolute value of Hodge-Tate weights of $V_{\mathrm{cris}}(M)$. Moreover, in [Abh21, Theorem 5.4] it has been shown that $V_{\mathrm{cris}}(M)$ is a finite q-height representation of height s. Furthermore, $V_{\mathrm{cris}}(M)$ satisfies assumptions of Theorem 1.7 and Theorem 1.9 (with very precise bounds on the constant $N(p,r,s)$, see Remark 3.20 and Example 5.5 (iii)).

The category of free relative Fontaine-Laffaille modules globalizes well. Let $\mathfrak{X}$ be a smooth (p-adic formal) scheme defined over O_F with X as its (rigid) generic fiber and $\mathfrak{X}_\kappa$ as its special fiber. Cover $\mathfrak{X}$ by affine (formal) schemes $\{\mathfrak{U}_i\}_{i \in I}$ where $\mathfrak{U}_i = \operatorname{Spec} A_i$ (resp. $\mathfrak{U}_i = \operatorname{Spf} A_i$) such that p-adic completions $\widehat{A}_i$ satisfy Assumption 2.1 and fix Frobenius lifts $\varphi_i : \widehat{A}_i \to \widehat{A}_i$.

Definition 1.14. Define $\mathrm{MF}_{[0,s],\,\mathrm{free}}(\mathfrak{X}, \Phi, \partial)$ as the category of finite locally free filtered $\mathcal{O}_{\mathfrak{X}}$-modules $\mathcal{M}$ equipped with a p-adically quasi-nilpotent integrable connection satisfying Griffiths transversality with respect to the filtration and such that there exists a covering $\{\mathfrak{U}_i\}_{i \in I}$ of $\mathfrak{X}$ as above with $\mathcal{M}_{\mathfrak{U}_i} \in \mathrm{MF}_{[0,s],\,\mathrm{free}}(\widehat{A}_i, \Phi, \partial)$ for all $i \in I$ and on $\mathfrak{U}_{ij}$ the two structures glue well for different Frobenii (see Remark 8.2).

By [Fal89, Theorem 2.6*], the functor T_{cris} associates to any object of $\mathrm{MF}_{[0,s],\,\mathrm{free}}(\mathfrak{X}, \Phi, \partial)$ a compatible system of étale sheaves on $\mathrm{Spec}\,(\widehat{A}_i[\frac{1}{p}])$. Again, these sheaves glue well to give us an étale sheaf on the (rigid) generic fiber X of $\mathfrak{X}$. The étale $\mathbb{Z}_p$-local system on the generic fiber associated to $\mathcal{M}$ will be denoted as $\mathbb{L}$. Our global result is as follows:

Theorem 1.15 (see Theorem 8.8)**.** *Let $\mathfrak{X}$ be a smooth (p-adic formal) scheme over O_F, $\mathcal{M} \in \mathrm{MF}_{[0,s],\,\mathrm{free}}(\mathfrak{X}, \Phi, \partial)$ a Fontaine-Laffaille module of level $[0, s]$ for $0 \leq s \leq p-2$ and let $\mathbb{L}$ be the associated $\mathbb{Z}_p$-local system on the (rigid) generic fiber X of $\mathfrak{X}$. Then for $0 \leq k \leq r-s-1$ the Fontaine-Messing period map*

$$\alpha^{\mathrm{FM}}_{r,n,\mathfrak{X}} : \mathcal{H}^k(\mathscr{S}_n(\mathcal{M}, r)_{\mathfrak{X}}) \longrightarrow i^* \mathrm{R}^k j_* \mathbb{L}/p^n(r)'_X,$$

is a p^N-isomorphism for an integer $N = N(p, r, s)$, which depends on p, r and s but not on $\mathfrak{X}$ or n.

The theorem is proved by reducing it to the local setting, where we can directly apply Theorem 1.9. Note that for schemes we require a non-trivial argument to in order to use Theorem 1.9 to deduce the local result.

Remark 1.16. (i) In light of Remark 1.12 (iii), it should be possible to base change the isomorphism of Theorem 1.15 to $\overline{F}$.

(ii) In personal communications with Takeshi Tsuji, I learnt that in some unpublished work he obtained similar results over $\overline{F}$ and large enough p. However, our respective approach are completely different and this paper includes more general local results as well as the arithmetic case.

Remark 1.17. In their work [BMS19, §10] Bhatt, Morrow and Scholze have refined the definition of syntomic complex (using prismatic cohomology) and showed that it computes p-adic nearby cycles for trivial coefficients. By the work of Morrow and Tsuji on coefficients in integral p-adic Hodge theory and prismatic cohomology [MT20], we should be able to refine our results and obtain an integral result for coefficients (in the geometric case). Furthermore, by recent introduction of completed/analytic prismatic F-crystals on the absolute prismatic site [DLMS22; GR22], we should be able to further refine these results, thus including the arithmetic case. We will report on these ideas in future.

Outline of the paper. Sections 2-6 comprise the local part of the paper, while sections 7-8 consist of the global applications. In §2.1 we describe our local setup, notations and some conventions. We recall the relative de Rham and crystalline representations studied by Brinon [Bri08] and the fundamental exact sequence in §2.2 and §2.3. Next, we recall the theory of relative (φ, Γ)-modules developed by Andreatta [And06], the overconvergent theory developed by [AB08] and a variation of fundamental exact sequence in §2.4. Section 2.5 introduces "good" crystalline coordinates using which we define several rings and describe their properties. In §2.6, we equip these rings with a Frobenius endomorphism and in §2.7 we consider their Frobenius-equivariant embedding into period rings described in previous sections. Finally, in §2.8 we consider certain fat period rings and prove a version of filtered Poincaré Lemma.

Section 3 recounts the theory of finite height representations in relative p-adic Hodge theory from [Abh21] and we prove some technical lemmas to be used in §6. We also recall the theory of free Fontaine-Laffaille modules and its relation with finite height representations from [Abh21].

In §4 we recollect the theory of Fontaine-Herr complex [Her98] and its relative version from [AI08]. Then in §4.2 we study Koszul complexes computing Γ_S-cohomology of a $\mathbb{Z}_p[[\Gamma_S]]$-module, where $\Gamma_S = \mathrm{Gal}(R_\infty[\frac{1}{p}]/S[\frac{1}{p}])$. In §4.3 we define Koszul complexes computing Lie Γ_S-cohomology of modules defined over certain period rings studied in §2.7.

We formulate our main local result Theorem 5.8 in §5 and carry out local syntomic computations for its proof. In §5.2, we define several syntomic complexes with coefficients in M over rings introduced in 2.5. Then in §5.3 and §5.4 we show that the aforementioned syntomic complexes are p-power quasi-isomorphic. Section 5.5 interprets syntomic complex in terms of differential Koszul complex with coefficients in M and in §5.6 we relate the latter complex to differential Koszul complex with coefficients in the Wach module $\mathbf{N}(T)$ using filtered Poincaré Lemma.

The aim of §6 is to carry out (φ, Γ)-module side computations for the proof of Theorem 5.8. In §6.2 we modify differential Koszul complex to obtain a subcomplex of the Koszul complex computing Lie Γ_S-cohomology over an analytic ring. The latter complex is then modified in §6.3 to obtain a subcomplex of the Koszul complex computing Γ_S-cohomology over an analytic ring. Then in §6.4, §6.5 and §6.6 a careful analysis of the complex from preceding section is carried out to show that it is p-power quasi-isomorphic to relative Fontaine-Herr complex concluding the proof of Theorem 5.8. In §6.7 we define the local version of Fontaine-Messing period map using the fundamental exact sequence and show that it coincides with the Lazard period map in Theorem 5.8 up to some power of p. Finally, we conclude the local part with a techincal lemma on Galois descent for syntomic complex helpful in concluding Corollary 5.12 over base ring R.

In §7 we give a recount of locally free filtered crystals equipped with Frobenius structure over a (p-adic formal) scheme. Moreover, in §7.2 we define syntomic complex with coefficients globally. An expert reader could skip this section entirely.

Lastly, in §8 we give a global application. In this section, we define global Fontaine-Laffaille modules and give a global construction of Fontaine-Messing period map following [Tsu99, §3.1]. Finally, in §8.3 we state and prove Theorem 8.8.

Acknowledgements. Local results of the paper were part of my PhD thesis at Université de Bordeaux. I would like to thank my advisor Denis Benois for several discussions related to the project as well as guidance and motivation during my time in Bordeaux. I would also like to thank Nicola Mazzari for discussions concerning crystalline cohomology and syntomic coefficients. Ideas employed in this paper have been heavily influenced by the article [CN17] of Colmez and Nizioł and I would like to thank them for their work. Finally, I would also like to thank Takeshi Tsuji for discussions concerning relative Fontaine-Laffaille modules.

The last part of the project was carried out while I was supported by ANR project GALF (ANR-18-CE40-0029) and I-SITE ULNE project PAFAGEO (ANR-16-IDEX-0004) at Université de Lille.

2. Relative p-adic Hodge theory

In this section we will recall some constructions and results in local relative p-adic Hodge theory developed in [And06; Bri08; AB08].

2.1. Setup and notations. We begin by describing the setup of §2 to §2 and fix some notations.

Convention. We will work under the convention that $0 \in \mathbb{N}$, the set of natural numbers.

Let $p \geq 3$ be a fixed prime number, κ a perfect field of characteristic p, $W := W(\kappa)$ the ring of p-typical Witt vectors with coefficients in κ and $F := W[\frac{1}{p}]$, the fraction field of W. In particular, F is an unramified extension of $\mathbb{Q}_p$ with ring of integers $O_F = W$. Let $\overline{F}$ be a fixed algebraic closure of F so that its residue field, denoted as $\overline{\kappa}$, is an algebraic closure of κ. Further, we denote by $G_F = \mathrm{Gal}(\overline{F}/F)$, the absolute Galois group of F.

Let $Z = (Z_1, \ldots, Z_s)$ denote a set of indeterminates and $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}^s$ be a multi-index, then we write $Z^{\mathbf{k}} := Z_1^{k_1} \cdots Z_s^{k_s}$. For $\mathbf{k} \to +\infty$ we will mean that $\sum k_i \to +\infty$. Now for a topological algebra Λ we define

$$\Lambda\{Z\} := \Big\{ \sum_{\mathbf{k} \in \mathbb{N}^s} a_{\mathbf{k}} Z^{\mathbf{k}}, \text{ where } a_{\mathbf{k}} \in \Lambda \text{ and } a_{\mathbf{k}} \to 0 \text{ as } \mathbf{k} \to +\infty \Big\}.$$

We fix $d \in \mathbb{N}$ and let $X = (X_1, X_2, \ldots, X_d)$ be some indeterminates.

Assumption 2.1. Let R be the p-adic completion of an étale algebra over $W\{X, X^{-1}\}$ with non-empty geometrically integral special fiber. In particular, we have a presentation

$$R = W\{X, X^{-1}\}\{Z_1, \ldots, Z_s\}/(Q_1, \ldots, Q_s),$$

where $Q_i(Z_1, \ldots, Z_s) \in W[X, X^{-1}][Z_1, \ldots, Z_s]$ for $1 \leq i \leq s$ are multivariate polynomials such that $\det\big(\frac{\partial Q_i}{\partial Z_j}\big)_{1 \leq i,j \leq s}$ is invertible in R. The algebra $R[\frac{1}{p}]$ is the relative analogue of "finite unramified extension of $\mathbb{Q}_p$" (indeed, by removing the geometric coordinates we will obtain $R[\frac{1}{p}]$ = finite unramified extension of F).

The p-adic Hodge theory over R is the study of p-adic representations of the étale fundamental group of $R[\frac{1}{p}]$, which we introduce next. We fix an algebraic closure $\overline{\mathrm{Fr}\,(R)}$ of $\mathrm{Fr}\,(R)$ containing $\overline{F}$. Let $\overline{R}$ denote the union of finite R-subalgebras $S \subset \overline{\mathrm{Fr}\,R}$, such that $S[\frac{1}{p}]$ is étale over $R[\frac{1}{p}]$. Let $\overline{\eta}$ denote the geometric point of the generic fiber $\mathrm{Sp}\,(R[\frac{1}{p}])$ (corresponding to $\overline{\mathrm{Fr}\,(R)}$) and let $G_R := \pi_1^{\text{ét}}(\mathrm{Sp}\,R[\frac{1}{p}], \overline{\eta})$ denote the étale fundamental group. By [Gro63, Exposé V, §8], we can write this étale fundamental group as the Galois group (of the fraction field of $\overline{R}[\frac{1}{p}]$ over the fraction field of $R[\frac{1}{p}]$)

$$G_R = \pi_1^{\text{ét}}(\mathrm{Sp}\,(R[\tfrac{1}{p}]), \overline{\eta}) = \mathrm{Gal}(\overline{R}[\tfrac{1}{p}]/R[\tfrac{1}{p}]).$$

For $n \in \mathbb{N}$, let $F_n := F(\mu_{p^n})$. From now onwards, we will fix some $m \in \mathbb{N}_{\geq 1}$ and set $K := F_m$, with its ring of integers O_K. The element $\varpi = \zeta_{p^m} - 1 \in O_K$ is a uniformizer of K, and its minimal polynomial $P_\varpi(X) = \frac{(1+X)^{p^m}-1}{(1+X)^{p^{m-1}}-1}$ is an Eisenstein polynomial in $W[X]$ of degree $e := [K : F] = p^{m-1}(p-1)$. Finally, for $S = R[\varpi] = O_K \otimes_{O_F} R$ we have that it is totally ramified at the prime ideal $(p) \subset R[\varpi]$. And similar to above, we obtain Galois groups $G_K \triangleleft G_F$ and $G_S \triangleleft G_R$ respectively, such that $G_R/G_S = G_F/G_K = \mathrm{Gal}(K/F)$. Finally, we have that R and $R[\varpi]$ are *small* algebras in the sense of Faltings (see [Fal88, §II 1(a)]).

For $k \in \mathbb{N}$, let Ω_R^k denote the p-adic completion of the module of k-differentials of R relative to $\mathbb{Z}$. Then, we have

$$\Omega_R^1 = \oplus_{i=1}^d R\, d\log X_i, \quad \text{and} \quad \Omega_R^k = \bigwedge_R^k \Omega_R^1.$$

For $S = R[\varpi]$, the natural map $\Omega^k_R \otimes_R S \to \Omega^k_S$ is bijective. In particular, we get that

$$\Omega^k_S = \bigwedge_R^k \left(\oplus_{i=1}^d S\, d\log X_i \right).$$

We also have that $R/pR \xrightarrow{\sim} S/\varpi S$ and for any $n \in \mathbb{N}$, R/p^nR is a smooth $\mathbb{Z}/p^n\mathbb{Z}$-algebra. Finally, we fix a lift $\varphi : R \to R$ of the absolute Frobenius $x \mapsto x^p$ over R/pR such that $\varphi(X_i) = X_i^p$, for $1 \leq i \leq d$.

Let us remark that to carry out some computations in later sections, we will need to extend our base field (hence the base ring) by adjoining some p-power roots of unity (see the field K and the ring $S = R[\varpi]$ above). As a consequence, we will also require the corresponding period rings defined for such rings. However, in §2.2 & §2.3 we will only recall results by fixing our base as R. As we shall see the period rings will only depend on $\overline{R}$ and we have $\overline{S} = \overline{R} \subset \overline{\mathrm{Fr}\ (R)} = \overline{\mathrm{Fr}\ (S)}$, therefore fixing our base as R is sufficient (see [And06; Bri08; AB08] for general constructions).

Convention. Let A be a ring and $I \subsetneq A$ an ideal. We say that an A-module M is I-adically complete if and only if $M \xrightarrow{\sim} \lim_n M/I^nM$.

Notation. Let S be a $\mathbb{Z}_p$-algebra. A homomorphism $f : M \to N$ between two S-modules is said to be a *p^n-isomorphism*, for some $n \in \mathbb{N}$ if the kernel and cokernel of f are killed by p^n.

2.2. Period rings. We will recall definitions and properties of the relative version of Fontaine's period ring $\mathbf{B}_{\mathrm{dR}}$ and $\mathbf{B}_{\mathrm{cris}}$ (see [Fon94a] for classical case).

Let $\mathbb{C}_p$ denote the p-adic completion of $\overline{F}$. Recall that $\overline{R}$ is the union of finite R-subalgebras $S \subset \overline{\mathrm{Fr}\ (R)} = \overline{\mathrm{Fr}\ (R[\varpi])}$, such that $S[\frac{1}{p}]$ is étale over $R[\frac{1}{p}]$. Let $\mathbb{C}^+(\overline{R})$ denote the p-adic completion of $\overline{R}$ and $\mathbb{C}(\overline{R}) = \mathbb{C}^+(\overline{R})[\frac{1}{p}]$. We define the tilt $\mathbb{C}^+(\overline{R})^\flat := \lim_{x \mapsto x^p} \mathbb{C}^+(\overline{R})/p = \lim_{x \mapsto x^p} \overline{R}/p$ and equip it with the inverse limit topology (where we equip $\overline{R}/p$ with the discrete topology) and let $\mathbb{C}(\overline{R})^\flat := \mathbb{C}^+(\overline{R})^\flat[\frac{1}{p^\flat}]$ for $p^\flat := (p, p^{1/p}, p^{1/p^2}, \ldots) \in \mathbb{C}^+(\overline{R})^\flat$ equipped with the coarsest ring topology such that $\mathbb{C}^+(\overline{R})^\flat$ is an open subring. These rings admit a continuous G_R-action for the topology described.

Let us fix $\varepsilon := (1, \zeta_p, \zeta_{p^2}, \ldots) \in \mathbb{C}_p^\flat$, $X_i^\flat := (X_i, X_i^{1/p}, X_i^{1/p^2}, \ldots) \in \mathbb{C}(\overline{R})^\flat$ for $1 \leq i \leq d$. Set $\mathbf{A}_{\mathrm{inf}}(\overline{R}) := W(\mathbb{C}^+(\overline{R})^\flat)$, the ring of p-typical Witt vectors with coefficients in $\mathbb{C}^+(\overline{R})^\flat$. The absolute Frobenius on $\mathbb{C}^+(\overline{R})^\flat$ lifts to an endomorphism $\varphi : \mathbf{A}_{\mathrm{inf}}(\overline{R}) \to \mathbf{A}_{\mathrm{inf}}(\overline{R})$ and the G_R-action extends to $\mathbf{A}_{\mathrm{inf}}(\overline{R})$ such that the action is continuous for the weak topology. For $x \in \mathbb{C}^+(\overline{R})^\flat$, let $[x] = (x, 0, 0, \ldots) \in \mathbf{A}_{\mathrm{inf}}(\overline{R})$ denote its Teichmüller representative. So we have $[\varepsilon] \in \mathbf{A}_{\mathrm{inf}}(\overline{R})$ with $g[\varepsilon] = [\varepsilon]^{\chi(g)}$ for $g \in G_R$ and $\chi : G_R \to \mathbb{Z}_p^\times$ the p-adic cyclotomic character and $\varphi([\varepsilon]) = [\varepsilon]^p$. Furthermore, let $\pi := [\varepsilon] - 1, \pi_1 := \varphi^{-1}(\pi) = [\varepsilon^{1/p}] - 1$, and $\xi := \frac{\pi}{\pi_1}$. Clearly we have $g(\pi) = (1+\pi)^{\chi(g)} - 1$ for $g \in G_R$ and $\varphi(\pi) = (1+\pi)^p - 1$.

2.2.1. The de Rham period ring. We have Fontaine's θ-map defined as $\theta : \mathbf{A}_{\mathrm{inf}}(\overline{R}) \to \mathbb{C}^+(\overline{R})$ sending $\sum_{k \in \mathbb{N}} p^k[x_k] \mapsto \sum_{k \in \mathbb{N}} p^k x_k^\sharp$, it is a G_R-equivariant surjective ring homomorphism whose kernel is principal and generated by, for example, $p - [p^\flat]$ or ξ (see [Fon82, Proposition 2.4 (ii)]). Define

$$\mathbf{B}^+_{\mathrm{dR}}(\overline{R}) := \lim_n \mathbf{A}_{\mathrm{inf}}(\overline{R})[\tfrac{1}{p}]/(\mathrm{Ker}\ \theta)^n.$$

The ring $\mathbf{B}^+_{\mathrm{dR}}(\overline{R})$ is an F-algebra. Let $t := \log[\varepsilon] = \log(1+\pi) = \sum_{k \in \mathbb{N}} (-1)^k \frac{\pi^{k+1}}{k+1} \in \mathbf{B}^+_{\mathrm{dR}}(\overline{R})$ on which $g \in G_R$ acts by $g(t) = \chi(g)t$. We set $\mathbf{B}_{\mathrm{dR}}(\overline{R}) := \mathbf{B}^+_{\mathrm{dR}}(\overline{R})[\frac{1}{t}]$. The homomorphism θ and the action of G_R extend to $\mathbf{B}^+_{\mathrm{dR}}(\overline{R})$ and $\mathbf{B}_{\mathrm{dR}}(\overline{R})$. The ring $\mathbf{B}_{\mathrm{dR}}(\overline{R})$ admits a natural G_R-stable filtration and we equip $\mathbf{B}^+_{\mathrm{dR}}(\overline{R})$ with the induced filtration (see [Bri08, §5.1] for details).

We can extend the map θ by R-linearity to obtain a ring homomorphism $\theta_R : R \otimes_{\mathbb{Z}} \mathbf{A}_{\mathrm{inf}}(\overline{R}) \to \mathbb{C}^+(\overline{R})$. Let $\mathcal{O}\mathbf{A}_{\mathrm{inf}}(\overline{R})$ denote the $\theta_R^{-1}(p\mathbb{C}^+(\overline{R}))$-adic completion of $R \otimes_{\mathbb{Z}} \mathbf{A}_{\mathrm{inf}}(\overline{R})$, then θ_R

extends to a surjective homomorphism $\theta_R : \mathcal{O}\mathbf{A}_{\mathrm{inf}}(\overline{R})[\frac{1}{p}] \to \mathbb{C}(\overline{R})$. Define

$$\mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(\overline{R}) := \lim_{n} \mathcal{O}\mathbf{A}_{\mathrm{inf}}(\overline{R})[\tfrac{1}{p}]/(\mathrm{Ker}\ \theta_R)^n.$$

The ring $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(\overline{R})$ is an $R[\frac{1}{p}]$-algebra. We set $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) := \mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(\overline{R})[\frac{1}{t}]$. Moreover, the homomorphism θ_R and the action of G_R extends to $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(\overline{R})$ and $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$.

A more explicit description of the ring $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(\overline{R})$ can also be given (see [Bri08, §5.2] or [Abh21, §2.1.3]) and one can identify $\mathbf{B}_{\mathrm{dR}}^{+}(\overline{R})$ with a subring of $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(\overline{R})$. There are natural G_R-stable filtrations on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(\overline{R})$ and $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$. Moreover, the induced filtrations on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(\overline{R})$, $\mathbf{B}_{\mathrm{dR}}^{+}(\overline{R})$ and $\mathbf{B}_{\mathrm{dR}}(\overline{R})$ match with the ones defined before. Furthermore, these rings are equipped with a G_R-equivariant connection,

$$\partial : \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) \longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) \otimes_{R[\frac{1}{p}]} \Omega^1_R[\tfrac{1}{p}],$$

satisfying Griffiths transversality with respect to the filtration, i.e. $\partial(\mathrm{Fil}^r \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})) \subset \mathrm{Fil}^{r-1}\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) \otimes_{R[\frac{1}{p}]} \Omega^1_R[\frac{1}{p}]$. Its restriction to $R[\frac{1}{p}]$ is the canonical differential operator. Moreover, we have $\big(\mathcal{O}\mathbf{B}_{\mathrm{dR}}^{+}(\overline{R})\big)^{\partial=0} = \mathbf{B}_{\mathrm{dR}}^{+}(\overline{R})$ and $\big(\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})\big)^{\partial=0} = \mathbf{B}_{\mathrm{dR}}(\overline{R})$ (see [Bri08, §5.3] for details). Finally, we have $\big(\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})\big)^{G_R} = R[\frac{1}{p}]$ (see [Bri08, §5.2] for details).

2.2.2. The crystalline period ring. Let us consider the map $\theta : \mathbf{A}_{\mathrm{inf}}(\overline{R}) \to \mathbb{C}^{+}(\overline{R})$ whose kernel is a principal ideal generated by ξ or $p - [p^\flat]$. Let us denote $x^{[k]} := \frac{x^k}{k!}$ for $x \in \mathrm{Ker}\ \theta \subset \mathbf{A}_{\mathrm{inf}}(\overline{R})$ and $k \in \mathbb{N}$. The divided power envelope of $\mathbf{A}_{\mathrm{inf}}(\overline{R})$ with respect to Ker θ is given as $\mathbf{A}_{\mathrm{inf}}(\overline{R})[x^{[k]},\ x \in \mathrm{Ker}\ \theta]_{k\in\mathbb{N}} = \mathbf{A}_{\mathrm{inf}}(\overline{R})[\xi^{[k]}]_{k\in\mathbb{N}}$. We define a p-torsion free $W(\kappa)$-algebra

$$\mathbf{A}_{\mathrm{cris}}(\overline{R}) := p\text{-adic completion of } \mathbf{A}_{\mathrm{inf}}(\overline{R})[\xi^{[k]}]_{k\in\mathbb{N}}.$$

The Witt vector Frobenius on $\mathbf{A}_{\mathrm{inf}}(\overline{R})$ and the map θ extend to $\mathbf{A}_{\mathrm{cris}}(\overline{R})$. Also, we have $t = \log(1+\pi) \in \mathrm{Ker}\ \theta \subset \mathbf{A}_{\mathrm{cris}}(\overline{R})$ with $\varphi(t) = pt$. The ring $\mathbf{A}_{\mathrm{cris}}(\overline{R})$ is t-torsion free, so we set $\varphi(\frac{1}{t}) = \frac{1}{pt}$ and define $\mathbf{B}_{\mathrm{cris}}^{+}(\overline{R}) := \mathbf{A}_{\mathrm{cris}}(\overline{R})[\frac{1}{p}]$ and $\mathbf{B}_{\mathrm{cris}}(\overline{R}) := \mathbf{B}_{\mathrm{cris}}^{+}(\overline{R})[\frac{1}{t}]$. These are F-algebras, equipped with a Frobenius endormorphim φ and a continuous action of G_R (see [Bri08, §6.1 and §6.2]).

Next, let us consider the map $\theta_R : R \otimes_{\mathbb{Z}} \mathbf{A}_{\mathrm{inf}}(\overline{R}) \to \mathbb{C}^{+}(\overline{R})$. The kernel of this map is an ideal generated by $\{1 \otimes \xi, X_1 \otimes 1 - 1 \otimes [X_1^\flat], \ldots, X_d \otimes 1 - 1 \otimes [X_d^\flat]\}$. The divided power envelope of $R \otimes_{\mathbb{Z}} \mathbf{A}_{\mathrm{inf}}(\overline{R})$ with respect to Ker θ_R is given as $R \otimes_{\mathbb{Z}} \mathbf{A}_{\mathrm{inf}}(\overline{R})[x^{[k]},\ x \in \mathrm{Ker}\ \theta_R]_{k\in\mathbb{N}}$. We define a p-torsion free R-algebra

$$\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}) := p\text{-adic completion of } R \otimes_{\mathbb{Z}} \mathbf{A}_{\mathrm{inf}}(\overline{R})[x^{[k]},\ x \in \mathrm{Ker}\ \theta_R]_{k\in\mathbb{N}}.$$

Taking the diagonal action of the Frobenius on $R \otimes_{\mathbb{Z}} \mathbf{A}_{\mathrm{inf}}(\overline{R})$ it follows that the Frobenius extends to $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$ and we denote this extension again by φ. The homomorphism θ_R and the G_R-action extend to $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$ (see [Bri08, p. 64]).

Let $T = (T_1, \ldots, T_d)$ be some indeterminates as above and let $\mathbf{A}_{\mathrm{cris}}(\overline{R})\langle T\rangle^\wedge$ denote the p-adic completion of the divided power polynomial algebra in indeterminates T and coefficients in $\mathbf{A}_{\mathrm{cris}}(\overline{R})$. Then we have an isomorphism of $\mathbf{A}_{\mathrm{cris}}(\overline{R})$-algebras (see [Bri08, Proposition 6.1.5])

$$\begin{aligned} f_{\mathrm{cris}} : \mathbf{A}_{\mathrm{cris}}(\overline{R})\langle T\rangle^\wedge &\longrightarrow \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}) \\ T_i &\longmapsto z_i \quad \text{for } 1 \leq i \leq d. \end{aligned}$$

We set $\mathcal{O}\mathbf{B}_{\mathrm{cris}}^{+}(\overline{R}) := \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})[\frac{1}{p}]$ and $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) := \mathcal{O}\mathbf{B}_{\mathrm{cris}}^{+}(\overline{R})[\frac{1}{t}]$. These $R[\frac{1}{p}]$-algebras are equipped with a continuous action of G_R, the action of Frobenius extends to these rings and we denote these extensions again by φ (see [Bri08, §6.1 and §6.2] for details).

Note that there exist natural G_R-equivariant injective morphisms of rings $\mathbf{B}^+_{\mathrm{cris}}(\overline{R}) \to \mathbf{B}^+_{\mathrm{dR}}(\overline{R})$, $\mathcal{O}\mathbf{B}^+_{\mathrm{cris}}(\overline{R}) \to \mathcal{O}\mathbf{B}^+_{\mathrm{dR}}(\overline{R})$, $\mathbf{B}_{\mathrm{cris}}(\overline{R}) \to \mathbf{B}_{\mathrm{dR}}(\overline{R})$ and $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \to \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$. Using this, we get induced filtrations on crystalline period rings as $\mathrm{Fil}^r\mathbf{B}_{\mathrm{cris}}(\overline{R}) := \mathbf{B}_{\mathrm{cris}}(\overline{R}) \cap \mathrm{Fil}^r\mathbf{B}_{\mathrm{dR}}(\overline{R})$ and $\mathrm{Fil}^r\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) := \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \cap \mathrm{Fil}^r\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$ for $r \in \mathbb{Z}$ (see [Bri08, §6.2]). Moreover, from the connection on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$, we obtain a G_R-equivariant induced connection

$$\partial : \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{R[\frac{1}{p}]} \Omega^1_R[\tfrac{1}{p}].$$

The connection ∂ satisfies Griffiths transversality with respect to the filtration, since the same is true over $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$. Its restriction to $R[\frac{1}{p}]$ is the canonical differential operator. Moreover, $\big(\mathcal{O}\mathbf{A}^+_{\mathrm{cris}}(\overline{R})\big)^{\partial=0} = \mathbf{A}_{\mathrm{cris}}(\overline{R})$, $\big(\mathcal{O}\mathbf{B}^+_{\mathrm{cris}}(\overline{R})\big)^{\partial=0} = \mathbf{B}^+_{\mathrm{cris}}(\overline{R})$ and $\big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})\big)^{\partial=0} = \mathbf{B}_{\mathrm{cris}}(\overline{R})$. Over $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$, the Frobenius operator commutes with the connection, i.e. $\varphi\partial = \partial\varphi$. Furthermore, we have $\big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})\big)^{G_R} = R[\frac{1}{p}]$ (see [Bri08, §6.2 and §6.3]).

2.2.3. Fundamental exact sequence. Let us recall the statement of fundamental exact sequence of p-adic Hodge theory over $\mathbf{A}_{\mathrm{cris}}(\overline{R})$. From Artin-Schrier theory in [AI08, §8.1.1], we have an exact sequence

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbf{A}_{\mathrm{inf}}(\overline{R}) \xrightarrow{1-\varphi} \mathbf{A}_{\mathrm{inf}}(\overline{R}) \longrightarrow 0. \tag{2.1}$$

Let $r \in \mathbb{N}$ and write $r = (p-1)a(r) + b(r)$ with $0 \leq b(r) < p-1$ and set $\mathbb{Z}_p(r)' = \frac{1}{p^{a(r)}}\mathbb{Z}_p(r)$. From [Tsu99, Theorem A3.26] and [CN17, Lemma 2.23], we have a p^r-exact sequence

$$0 \longrightarrow \mathbb{Z}_p(r)' \longrightarrow \mathrm{Fil}^r\mathbf{A}_{\mathrm{cris}}(\overline{R}) \xrightarrow{p^r-\varphi} \mathbf{A}_{\mathrm{cris}}(\overline{R}) \longrightarrow 0. \tag{2.2}$$

2.3. p-adic Galois representations.

In this section we will recall results on linear algebra data associated to p-adic de Rham and crystalline representations of the Galois group G_R. We will use the G_R-regularity of a topological $\mathbb{Q}_p$-algebra B in the sense of [Bri08, p. 106]. If V is a p-adic representation of G_R, we set

$$\mathbf{D}_B(V) := (B \otimes_{\mathbb{Q}_p} V)^{G_R}.$$

This is a B^{G_R}-module and we have a natural morphism of B-modules, functorial in V

$$\begin{aligned} \alpha_B(V) : B \otimes_{B^{G_R}} \mathbf{D}_B(V) &\longrightarrow B \otimes_{\mathbb{Q}_p} V \\ b \otimes d &\longmapsto bd. \end{aligned}$$

The representation V is said to be *B-admissible* if α_B is an isomorphism.

We begin with de Rham representations of G_R. Note that $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$ is a G_R-regular $R[\frac{1}{p}]$-algebra. We set

$$\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V) := \big(\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) \otimes_{\mathbb{Q}_p} V\big)^{G_R}.$$

The representation V is de Rham if it is $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$-admissible. The $R[\frac{1}{p}]$-module $\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ is equipped with a decreasing, separated and exhaustive filtration induced from the filtration on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) \otimes_{\mathbb{Q}_p} V$ where we consider the G_R-stable filtration on $\mathcal{O}\mathbf{D}_{\mathrm{dR}}(\overline{R})$ from §2.2. Moreover, the module $\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ is equipped with an integrable connection, induced from the G_R-equivariant integrable connection

$$\begin{aligned} \partial : V \otimes_{\mathbb{Q}_p} \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) &\longrightarrow V \otimes_{\mathbb{Q}_p} \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R}) \otimes_{R[\frac{1}{p}]} \Omega^1_R[\tfrac{1}{p}] \\ v \otimes b &\longmapsto v \otimes \partial(b). \end{aligned}$$

We denote the induced connection on $\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ again by ∂. Since the connection ∂ on $\mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$ satisfies Griffiths transversality, the same is true for $\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$, i.e. $\partial(\mathrm{Fil}^r\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)) \subset$

$\mathrm{Fil}^{r-1}\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)\otimes_{R[\frac{1}{p}]}\Omega^1_R[\frac{1}{p}]$. Further, $\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ is projective over $R[\frac{1}{p}]$ of rank $\leq \dim(V)$. If V is de Rham then for all $r \in \mathbb{Z}$, the $R[\frac{1}{p}]$-modules $\mathrm{Fil}^r\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ and $\mathrm{gr}^r\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ are projective of finite type and for such a representation the collection of integers r_i for $1 \leq i \leq \dim_{\mathbb{Q}_p}(V)$ such that $\mathrm{gr}^{-r_i}\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V) \neq 0$ are called *Hodge-Tate weights* of V. Moreover, we say that V is positive if and only if $r_i \leq 0$ for all $1 \leq i \leq \dim_{\mathbb{Q}_p}(V)$ (see [Bri08, §8.3] for details).

Finally, let us describe crystalline representations of G_R. Note that $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$ is a G_R-regular $R[\frac{1}{p}]$-algebra. We set

$$\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) := \big(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V\big)^{G_R}.$$

We will denote the category of crystalline representations (i.e. $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$-admissible) as $\mathrm{Rep}^{\mathcal{O}\mathrm{cris}}_{\mathbb{Q}_p}(G_R)$. The $R[\frac{1}{p}]$-module $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is equipped with a Frobenius-semilinear operator φ induced from the Frobenius on $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V$, where we consider the G_R-equivariant Frobenius on $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R})$. Moreover, $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is an $R[\frac{1}{p}]$-submodule of $\mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$, and we equip the former with induced filtration and connection which satisfies Griffiths transversality with respect to the filtration. Additionally, we have $\partial\varphi = \varphi\partial$ over $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ (see [Bri08, §8.3] for details).

The module $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is projective over $R[\frac{1}{p}]$ of rank $\leq \dim(V)$. If V is crystalline, then the $R[\frac{1}{p}]$-linear homomorphism $1 \otimes \varphi : R[\frac{1}{p}] \otimes_{R[\frac{1}{p}],\varphi} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \to \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is an isomorphism and $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is called a filtered (φ, ∂)-module. The inclusion $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \rightarrowtail \mathcal{O}\mathbf{B}_{\mathrm{dR}}(\overline{R})$ induces the inclusion $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \rightarrowtail \mathcal{O}\mathbf{D}_{\mathrm{dR}}(V)$ (see [Bri08, §8.2 and §8.3] for details).

In conclusion, we have a functor

$$\mathcal{O}\mathbf{D}_{\mathrm{cris}} : \mathrm{Rep}^{\mathcal{O}\mathrm{cris}}_{\mathbb{Q}_p}(G_R) \longrightarrow \text{filtered } (\varphi, \partial)\text{-modules over } R[\tfrac{1}{p}].$$

The objects in the essential image are called *admissible* filtered (φ, ∂)-modules and the functor induces an equivalence of categories with the essential image (see [Bri08, Théorèmes 8.4.2, 8.5.1]).

2.4. (φ, Γ)-modules. In this section, we briefly recall the theory of relative (φ, Γ)-modules from [And06; AB08; AI08].

2.4.1. The Galois group Γ_R. Let $F_n = F(\mu_{p^n})$ for $n \in \mathbb{N}$ and $F_\infty = \cup_n F_n$. We take R_n to be the integral closure of $R \otimes_{O_F[X^{\pm 1}]} O_{F_n}[X_1^{p^{-n}}, \ldots X_d^{p^{-n}}]$ inside $\overline{R}[\frac{1}{p}]$ and set $R_\infty := \cup_{n \geq m} R_n$ noting that $F_\infty \subset R_\infty[\frac{1}{p}]$. From §2.2 recall that $\mathbb{C}(\overline{R}) = \mathbb{C}^+(\overline{R})[\frac{1}{p}]$ and $\mathbb{C}(\overline{R})^\flat$ denotes its tilt. The ring $\mathbb{C}(\overline{R})^\flat$ is perfect of characteristic p and we set $\mathbf{A}_{\overline{R}} := W(\mathbb{C}(\overline{R})^\flat)$, the ring of p-typical Witt vectors with coefficients in $\mathbb{C}(\overline{R})^\flat$ and endow it with the weak topology (see [AI08, §2.10]). The absolute Frobenius over $\mathbb{C}(\overline{R})^\flat$ lifts to an endomorphism $\varphi : \mathbf{A}_{\overline{R}} \to \mathbf{A}_{\overline{R}}$, which we again call the Frobenius. The action of G_R on $\mathbb{C}(\overline{R})^\flat$ extends to a continuous action on $\mathbf{A}_{\overline{R}}$ which commutes with the Frobenius. The inclusion $\overline{F} \subset \overline{R}[\frac{1}{p}]$ induces inclusions $\mathbb{C}_p^\flat \subset \mathbb{C}(\overline{R})^\flat$ and $\mathbf{A}_{\overline{F}} \subset \mathbf{A}_{\overline{R}}$. Recall that we set $\mathbf{A}_{\mathrm{inf}}(\overline{R}) := W(\mathbb{C}^+(\overline{R})^\flat)$. The inclusion $O_{\overline{F}} \subset \overline{R}$ induces inclusions $O_{\mathbb{C}_p}^\flat \subset \mathbb{C}^+(\overline{R})^\flat$ and $\mathbf{A}_{\mathrm{inf}}(O_{\overline{F}}) \subset \mathbf{A}_{\mathrm{inf}}(\overline{R})$.

The ring $R_\infty[\frac{1}{p}]$ is a Galois extension of $R[\frac{1}{p}]$ with Galois group $\Gamma_R := \mathrm{Gal}(R_\infty[\frac{1}{p}]/R[\frac{1}{p}])$ isomorphic to the semidirect product of Γ_F and Γ'_R, where $\Gamma_F = \mathrm{Gal}(F_\infty/F)$ and $\Gamma'_R = \mathrm{Gal}(R_\infty[\frac{1}{p}]/F_\infty R[\frac{1}{p}])$. In particular, we have an exact sequence

$$1 \longrightarrow \Gamma'_R \longrightarrow \Gamma_R \longrightarrow \Gamma_F \longrightarrow 1, \tag{2.3}$$

where (see [Bri08, p. 9] and [And06, §2.4])

$$\begin{aligned} \Gamma'_R &= \mathrm{Gal}(R_\infty[\tfrac{1}{p}]/F_\infty R[\tfrac{1}{p}]) \simeq \mathbb{Z}_p^d, \\ \chi : \Gamma_F &= \mathrm{Gal}(F_\infty/F) \simeq \mathbb{Z}_p^\times. \end{aligned}$$

The group Γ_F can be viewed as a subgroup of Γ_R, i.e. we can take a section of the projection map in (2.3) such that for $\gamma \in \Gamma_F$ and $g \in \Gamma'_R$, we have $\gamma g \gamma^{-1} = g^{\chi(\gamma)}$. So we can choose topological generators $\{\gamma, \gamma_1, \ldots, \gamma_d\}$ of Γ_R such that

$$\begin{aligned} \gamma(\varepsilon) = \varepsilon^{\chi(\gamma)}, \;\; \gamma_i(\varepsilon) = \varepsilon \qquad & \text{for } 1 \leq i \leq d, \\ \gamma_i(X_i^\flat) = \varepsilon X_i^\flat, \;\; \gamma_i(X_j^\flat) = X_j^\flat \qquad & \text{for } i \neq j \text{ and } 1 \leq j \leq d, \end{aligned}$$

and that $\gamma_0 = \gamma^e$ with $\chi(\gamma_0) = \exp(p^m)$, is a topological generator of $\Gamma_K = \mathrm{Gal}(K_\infty/K)$, where $K_\infty = F_\infty$ and $e = [K : F]$. It follows that $\{\gamma_1, \ldots, \gamma_d\}$ are topological generators of Γ'_R, γ is a topological generator of Γ_F, and γ_0 is a topological generator of Γ_K. In particular,

$$\chi : \Gamma_K = \mathrm{Gal}(F_\infty/K) \simeq 1 + p^m\mathbb{Z}_p.$$

2.4.2. Étale (φ, Γ_R)-modules. Let $S \subset \overline{R}$ be an R_n-algebra which is finite as an R_n-module and $S[\frac{1}{p}]$ is étale over $R_n[\frac{1}{p}]$. For $k \geq n$ denote by S_k the integral closure of $S \otimes_{R_n} R_k$ in $\overline{R}[\frac{1}{p}]$ and set $S_\infty := \cup_{k \geq n} S_k$. We have that S_∞ is a normal R_∞-algebra and an integral domain as a subring of $\overline{R}$. As in the case of R, for S we define $G_S := \mathrm{Gal}(\overline{R}[\frac{1}{p}]/S[\frac{1}{p}])$, $\Gamma_S := \mathrm{Gal}(S_\infty[\frac{1}{p}]/S[\frac{1}{p}])$ and $H_S := \mathrm{Ker}\ (G_S \to \Gamma_S)$. Again, Γ_S is isomorphic to the semidirect product of Γ_{F_n} and Γ'_S, where $\Gamma'_S = \mathrm{Gal}(S_\infty[\frac{1}{p}]/F_\infty S[\frac{1}{p}])$ is a finite index subgroup of $\Gamma'_R \simeq \mathbb{Z}_p^d$.

Generalizing [FW79b; FW79a; Win83] to the relative setting, in [And06] Andreatta functorially associated a ring $\mathbf{E}_S \subset \mathrm{Fr}\ \widehat{S}_\infty^\flat$ to any S as above. Furthermore, he defined a subring $\mathbf{E}_R^+ \subset \mathbf{E}_R$ and for S as above a $\mathbf{E}_R^+$-subalgebra $\mathbf{E}_S^+ \subset \mathbf{E}_S$ (see [And06, Definition 4.2]). The ring $\mathbf{E}_S^+$ is finite and torsion free as an $\mathbf{E}_R^+$-module. It is a reduced Noetherian ring which is $\overline{\pi}$-adically complete, where $\overline{\pi}$ denotes the reduction of $\pi \in W(\widehat{F}_\infty^\flat)$ modulo p. By construction, $\mathbf{E}_S^+$ is endowed with a $\overline{\pi}$-adically continuous action of Γ_S and a Frobenius endomorphism φ commuting with each other and compatible with respective structures on $\widehat{S}_\infty^\flat$ (see [And06, Proposition 4.5, Corollaries 5.3 & 5.4] for more details). Finally, set $\mathbf{E}_S := \mathbf{E}_S^+[\frac{1}{\overline{\pi}}]$.

Definition 2.2. Define $\mathbf{E}^+ := \cup_S \mathbf{E}_S^+$, where the union runs over R_n-subalgebras $S \subset \overline{R}$ for some $n \in \mathbb{N}$ such that S is normal and finite as an R_n-module and $S[\frac{1}{p}]$ is étale over $R_n[\frac{1}{p}]$. Also, we set $\mathbf{E} := \mathbf{E}^+[\frac{1}{\overline{\pi}}]$. These rings are $\overline{\pi}$-adically complete and equipped with a Frobenius and a continuous G_R-action.

Remark 2.3. From [AI08, Proposition 2.9], we have $(\mathbb{C}^+(\overline{R}))^{H_R} = \widehat{R}_\infty$, $(\mathbb{C}^+(\overline{R})^\flat)^{H_R} = \widehat{R}_\infty^\flat$, $(\mathbb{C}(\overline{R})^\flat)^{H_R} = \widehat{R}_\infty^\flat[\frac{1}{\overline{\pi}}]$, $(\mathbf{E}^+)^{H_R} = \mathbf{E}_R^+$ and $\mathbf{E}^{H_R} = \mathbf{E}_R$.

Next, we have liftings of these rings to characteristic 0. In other words, there exists a Noetherian regular domain $\mathbf{A}_R \subset W(\widehat{R}_\infty^\flat[\frac{1}{\overline{\pi}}])$, complete for the weak topology and endowed with a continuous action of Γ_R and a Frobenius such that $\mathbf{A}_R/p\mathbf{A}_R = \mathbf{E}_R$. Moreover, $\mathbf{A}_R$ contains a Frobenius and Γ_R-stable subring $\mathbf{A}_R^+$ lifting $\mathbf{E}_R^+$ and it is complete for the weak topology with $\pi, [X_1^\flat], \ldots, [X_d^\flat] \in \mathbf{A}_R^+$ (see [And06, Appendix C]). Furthermore, for S as in Definition 2.2 let $\mathbf{A}_S$ denote the unique finite étale $\mathbf{A}_R$-algebra lifting the finite étale extension $\mathbf{E}_R \subset \mathbf{E}_S$. It is a Noetherian regular domain, complete for the weak topology and endowed with a continuous action of Γ_S and a Frobenius, lifting the ones defined on $\mathbf{E}_S$. Furthermore, it contains a Frobenius and Γ_S-stable $\mathbf{A}_R$-subalgebra $\mathbf{A}_S^+$ lifting $\mathbf{E}_S^+$ such that the former is complete for the weak topology. Finally, we set $\mathbf{B}_{\overline{R}} := \mathbf{A}_{\overline{R}}[\frac{1}{p}] = \cup_{j \in \mathbb{N}} p^{-j}\mathbf{A}_{\overline{R}}$ equipped with the direct limit topology (see [And06, §7] for details).

Definition 2.4. Define $\mathbf{A}$:= completion of $\cup_S \mathbf{A}_S \subset \mathbf{A}_{\overline{R}}$ for the p-adic topology, where the union runs over all R_n-subalgebras $S \subset \overline{R}$ as in Definition 2.2. Equip $\mathbf{A}$ with the weak topology induced by the inclusion $\mathbf{A} \subset \mathbf{A}_{\overline{R}}$. Moreover, we set $\mathbf{A}^+ := \mathbf{A} \cap \mathbf{A}_{\mathrm{inf}}(\overline{R})$, $\mathbf{B}^+ := \mathbf{A}^+[\frac{1}{p}]$ and $\mathbf{B} := \mathbf{A}[\frac{1}{p}]$ equipped with induced weak topology. These rings are stable under φ and admit a continuous G_R-action.

Remark 2.5. From [AI08, Lemma 2.11] we have $\mathbf{A}^{H_R} = \mathbf{A}_R$ and $(\mathbf{A}^+)^{H_R} = \mathbf{A}_R^+$ and from [Abh21, Remark 3.7] we have $\mathbf{A}^+/p\mathbf{A}^+ = \mathbf{E}^+$.

Having introduced all the necessary rings, we finally come to (φ, Γ_R)-modules.

Definition 2.6. A (φ, Γ_R)-module D over $\mathbf{A}_R$ is a finitely generated module equipped with

(i) A semilinear action of Γ_R, continuous for the weak topology,

(ii) A Frobenius-semilinear homomorphism φ which is Γ_R-equivariant.

We say that D is *étale* if the natural map $1 \otimes \varphi : \mathbf{A}_R \otimes_{\mathbf{A}_R,\varphi} D \to D$ is an isomorphism of $\mathbf{A}_R$-modules.

Denote by $(\varphi, \Gamma_R)\text{-Mod}_{\mathbf{A}_R}^{\text{ét}}$ the category of étale (φ, Γ_R)-modules over $\mathbf{A}_R$ with morphisms between objects being continuous, φ-equivariant and Γ_R-equivariant morphisms of $\mathbf{A}_R$-modules. Next, denote by $\text{Rep}_{\mathbb{Z}_p}(G_R)$ the category of finitely generated $\mathbb{Z}_p$-modules equipped with a linear and continuous action of G_R, with morphisms between objects being continuous and G_R-equivariant morphisms of $\mathbb{Z}_p$-modules.

Let T be a $\mathbb{Z}_p$-representation of G_R. The $\mathbf{A}_R$-module $\mathbf{D}(T) := (\mathbf{A} \otimes_{\mathbb{Z}_p} T)^{H_R}$ is equipped with a semilinear operator φ and a continuous (for the weak topology) and semilinear action of Γ_R, which commute with each other. Moreover, $\mathbf{D}(T)$ is an étale (φ, Γ_R)-module. Furthermore, if T is free of finite rank, then $\mathbf{D}(T)$ is a projective module of rank $= \text{rk}_{\mathbb{Z}_p} T$ (see [And06, Theorem 7.11]). The functor

$$\mathbf{D} : \text{Rep}_{\mathbb{Z}_p}(G_R) \longrightarrow (\varphi, \Gamma_R)\text{-Mod}_{\mathbf{A}_R}^{\text{ét}}, \tag{2.4}$$

induces an equivalence of categories (see [And06, Theorem 7.11]), and the natural map $\mathbf{A} \otimes_{\mathbf{A}_R} \mathbf{D}(T) \xrightarrow{\sim} \mathbf{A} \otimes_{\mathbb{Z}_p} T$ is an isomorphism of $\mathbf{A}$-modules compatible with Frobenius and the action of G_R on each side.

2.4.3. Overconvergence. In [CC98], Cherbonnier and Colmez showed that all $\mathbb{Z}_p$-representations (resp. p-adic representations) of G_F are overconvergent. Generalizing this to the relative case, Andreatta and Brinon in [AB08] have shown that all $\mathbb{Z}_p$-representations (resp. p-adic representations) of G_R are overconvergent. In this section we will recall some of these results.

Let us denote the natural valuation on $O_{\mathbb{C}_p}^\flat$ by $v^\flat$. We extend it to a map $v^\flat : \mathbb{C}^+(\overline{R})^\flat \to \mathbb{R} \cup \{+\infty\}$ by setting $v^\flat(x) = \frac{p}{p-1} \max\{n \in \mathbb{Q}, x \in \overline{\pi}^{-n}\mathbb{C}^+(\overline{R})^\flat\}$. Let $v > 0$ and let $\alpha \in O_{\mathbb{C}_p}^\flat$ such that $v^\flat(\alpha) = 1/v$. Set

$$\begin{aligned}
\mathbf{A}_R^{(0,v]} &:= \Big\{ \sum_{k \in \mathbb{N}} p^k [x_k],\ v v^\flat(x_k) + k \to +\infty \text{ when } k \to +\infty \Big\} \\
\mathbf{A}_R^{(0,v]+} &:= \Big\{ \sum_{k \in \mathbb{N}} p^k [x_k] \in \mathbf{A}_R^{(0,v]} \text{ with } v v^\flat(x_k) + k \geq 0 \Big\} \\
&= p\text{-adic completion of } \mathbf{A}_{\inf}(R)\big[\tfrac{p}{[\alpha]}\big].
\end{aligned}$$

Note that we have $\mathbf{A}_R^{(0,v]} = \mathbf{A}_R^{(0,v]+}\big[\frac{1}{[p^\flat]}\big]$. The action of G_R on $\mathbf{A}_{\inf}(R)$ extends to these rings and it commutes with the induced Frobenius φ. For the homomorphism φ, we have

$$\varphi(\mathbf{A}_R^{(0,v]+}) = \mathbf{A}_R^{(0,v/p]+} \quad \text{and} \quad \varphi(\mathbf{A}_R^{(0,v]}) = \mathbf{A}_R^{(0,v/p]}.$$

Moreover, we have injections (see [CN17, §2.4.2])

$$\mathbf{A}_R^{(0,v]+} \rightarrowtail \mathbf{B}_{\text{dR}}^+(\overline{R}) \quad \text{and} \quad \mathbf{A}_R^{(0,v]} \rightarrowtail \mathbf{B}_{\text{dR}}^+(\overline{R}) \quad \text{if } v \geq 1.$$

Definition 2.7. Define the ring of *overconvergent coefficients* as

$$\mathbf{A}^{\dagger}_{\overline{R}} := \bigcup_{v \in \mathbb{Q}_{>0}} \mathbf{A}^{(0,v]}_{\overline{R}} \text{ and } \mathbf{B}^{\dagger}_{\overline{R}} := \bigcup_{v \in \mathbb{Q}_{>0}} \mathbf{B}^{(0,v]}_{\overline{R}} = \bigcup_{v \in \mathbb{Q}_{>0}} \mathbf{A}^{(0,v]}_{\overline{R}}[\tfrac{1}{p}].$$

Next, set

$$\mathbf{A}^{(0,v]}_{R} := \mathbf{A}_R \cap \mathbf{A}^{(0,v]}_{\overline{R}} \text{ and } \mathbf{A}^{(0,v]} := \mathbf{A} \cap \mathbf{A}^{(0,v]}_{\overline{R}},$$

and define

$$\mathbf{A}^{\dagger}_{R} := \mathbf{A}_R \cap \mathbf{A}^{\dagger}_{\overline{R}} = \bigcup_{v \in \mathbb{Q}_{>0}} \mathbf{A}^{(0,v]}_{R} \text{ and } \mathbf{A}^{\dagger} := \mathbf{A} \cap \mathbf{A}^{\dagger}_{\overline{R}} = \bigcup_{v \in \mathbb{Q}_{>0}} \mathbf{A}^{(0,v]}.$$

Now, let us describe the topology on the rings defined above. For $x = \sum_{k \in \mathbb{Z}} p^k [x_k] \in \mathbf{B}^{(0,v]+}_{\overline{R}}$, we set

$$w_v(z) := \inf_{k \in \mathbb{Z}} \; (v v^{\flat}(x_k) + k).$$

This induces a valuation on $\mathbf{A}^{(0,v]+}_{\overline{R}}$ and it is complete for the topology induced by the valuation (see [AB08, Proposition 4.2]). We will equip $\mathbf{A}^{\dagger}_{\overline{R}}$ with the topology induced by the inductive limit of the topology described above. Further, $\mathbf{A}^{\dagger}$ is also endowed with a Frobenius endomorphism φ and a continuous action of G_R which commutes with φ (see [And06, Proposition 7.2]). These actions are induced from the inclusion $\mathbf{A}^{\dagger}_{\overline{R}} \subset \mathbf{A}_{\overline{R}}$. Further, all subrings of $\mathbf{A}^{\dagger}_{\overline{R}}$ appearing above are equipped with the induced structures as well.

Remark 2.8. From [AI08, Lemma 2.11], we have $\big(\mathbf{A}^{(0,v]}\big)^{H_R} = \mathbf{A}^{(0,v]}_{R}$, $(\mathbf{A}^{\dagger})^{H_R} = \mathbf{A}^{\dagger}_{R}$ and $\mathbf{A}^{\dagger}_{R}/p\mathbf{A}^{\dagger}_{R} = \mathbf{E}_R$.

Now we come to overconvergent (φ, Γ_R)-modules.

Definition 2.9. A (φ, Γ_R)-module D over $\mathbf{A}^{\dagger}_{R}$ is a finitely generated module equipped with

(i) A semilinear action of Γ_R, continuous for the weak topology;

(ii) A Frobenius-semilinear homomorphism φ commuting with Γ_R.

These modules are called *étale* if the natural map,

$$1 \otimes \varphi : \mathbf{A}^{\dagger}_{R} \otimes_{\mathbf{A}^{\dagger}_{R}, \varphi} D \longrightarrow D,$$

is an isomorphism of $\mathbf{A}^{\dagger}_{R}$-modules. Let $(\varphi, \Gamma_R)\text{-Mod}^{\text{ét}}_{\mathbf{A}^{\dagger}_{R}}$ denote the category of such modules.

Denote by $(\varphi, \Gamma_R)\text{-Mod}^{\text{ét}}_{\mathbf{A}^{\dagger}_{R}}$ the category of étale (φ, Γ_R)-modules over $\mathbf{A}^{\dagger}_{R}$ with morphisms between objects being continuous, φ-equivariant and Γ_R-equivariant morphisms of $\mathbf{A}^{\dagger}_{R}$-modules. Recall that $\text{Rep}_{\mathbb{Z}_p}(G_R)$ is the category of finitely generated $\mathbb{Z}_p$-modules equipped with a linear and continuous action of G_R, with morphisms between objects being continuous and G_R-equivariant morphisms of $\mathbb{Z}_p$-modules.

Let $T \in \text{Rep}_{\mathbb{Z}_p}(G_R)$ then the module

$$\mathbf{D}^{\dagger}(T) := (\mathbf{A}^{\dagger} \otimes_{\mathbb{Z}_p} T)^{H_R},$$

is equipped with a semilinear action of φ and a continuous and semilinear action of Γ_R commuting with each other. The functor $\mathbf{D}^{\dagger}$ takes values in the category $(\varphi, \Gamma_R)\text{-Mod}^{\text{ét}}_{\mathbf{A}^{\dagger}_{R}}$, i.e. $\mathbf{D}^{\dagger}(T)$ is an étale (φ, Γ_R)-module over $\mathbf{A}^{\dagger}_{R}$. Furthermore, if T is free of finite rank, then $\mathbf{D}^{\dagger}(T)$ is projective of rank $= \text{rk}_{\mathbb{Z}_p} T$. The functor

$$\mathbf{D}^{\dagger} : \text{Rep}_{\mathbb{Z}_p}(G_R) \longrightarrow (\varphi, \Gamma_R)\text{-Mod}^{\text{ét}}_{\mathbf{A}^{\dagger}_{R}},$$

induces an equivalence of categories (see [AB08, Théorème 4.35]). Moreover, the natural map

$$\mathbf{A}^{\dagger} \otimes_{\mathbf{A}_R} \mathbf{D}^{\dagger}(T) \xrightarrow{\sim} \mathbf{A}^{\dagger} \otimes_{\mathbb{Z}_p} T$$

is an isomorphism of $\mathbf{A}^{\dagger}$-modules compatible with Frobenius and the action of G_R on each side. Furthermore, the scalar extension along $\mathbf{A}_R^{\dagger} \rightarrowtail \mathbf{A}_R$ gives an isomorphism of (φ, Γ_R)-modules over $\mathbf{A}_R$,

$$\mathbf{A}_R \otimes_{\mathbf{A}_R^{\dagger}} \mathbf{D}^{\dagger}(T) \xrightarrow{\sim} \mathbf{D}(T).$$

Finally, if T is free of rank h, then there exists an R-algebra S such that S is normal and finite over R, $S[\frac{1}{p}]$ is Galois over $R[\frac{1}{p}]$ and $\mathbf{A}_S^{\dagger} \otimes_{\mathbf{A}_R^{\dagger}} \mathbf{D}^{\dagger}(T)$ is a free $\mathbf{A}_S^{\dagger}$-module of rank h.

We will end this section by introducing certain analytic rings which will be useful in §5. Let $0 < u \leq v$ and let $\alpha, \beta \in O_{\mathbb{C}_p}^{\flat}$ such that $v^{\flat}(\alpha) = 1/v$ and $v^{\flat}(\beta) = 1/u$. Set

$$\mathbf{A}_{\overline{R}}^{[u]} := p\text{-adic completion of } \mathbf{A}_{\inf}(\overline{R})\left[\tfrac{[\beta]}{p}\right],$$
$$\mathbf{A}_{\overline{R}}^{[u,v]} := p\text{-adic completion of } \mathbf{A}_{\inf}(\overline{R})\left[\tfrac{p}{[\alpha]}, \tfrac{[\beta]}{p}\right].$$

The action of G_R on $\mathbf{A}_{\inf}(\overline{R})$ extends to a continuous action of G_R on these rings and this action commutes with the induced Frobenius φ. For the homomorphism φ, we have

$$\varphi(\mathbf{A}_{\overline{R}}^{[u]}) = \mathbf{A}_{\overline{R}}^{[u/p]} \text{ and } \varphi(\mathbf{A}_{\overline{R}}^{[u,v]}) = \mathbf{A}_{\overline{R}}^{[u/p,v/p]}.$$

Moreover, we have injections (see [CN17, §2.4.2])

$$\mathbf{A}_{\overline{R}}^{[u]} \rightarrowtail \mathbf{B}_{\mathrm{dR}}^{+}(\overline{R}) \text{ if } u \leq 1 \text{ and } \mathbf{A}_{\overline{R}}^{[u,v]} \rightarrowtail \mathbf{B}_{\mathrm{dR}}^{+}(\overline{R}) \text{ if } u \leq 1 \leq v.$$

2.4.4. Fundamental exact sequences. The Artin-Schreier exact sequence in (2.1) can be ugraded to following exact sequences (see [AI08, §8.1] and [CN17, Lemma 2.23])

$$\begin{aligned} &0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbf{A}_{\overline{R}} \xrightarrow{1-\varphi} \mathbf{A}_{\overline{R}} \longrightarrow 0, \\ &0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbf{A}_{\overline{R}}^{(0,v]+} \xrightarrow{1-\varphi} \mathbf{A}_{\overline{R}}^{(0,v/p]+} \longrightarrow 0, \text{ for } v > 0. \end{aligned} \tag{2.5}$$

Furthermore, for $0 < u \leq 1 \leq v$ the exact sequence in (2.2) can be upgraded to a p^{4r}-exact sequence (see [CN17, Lemma 2.23])

$$0 \longrightarrow \mathbb{Z}_p(r) \longrightarrow \mathrm{Fil}^r \mathbf{A}_{\overline{R}}^{[u,v]} \xrightarrow{p^r - \varphi} \mathbf{A}_{\overline{R}}^{[u,v/p]} \longrightarrow 0. \tag{2.6}$$

2.4.5. The operator ψ. In this section, we will define a left inverse ψ of the Frobenius operator φ on the ring $\mathbf{A}$. Let S be an R-algebra as in Definition 2.2. Then, from [AB08, Corollaire 4.10] we note that the $\mathbf{A}_S$-module $\varphi^{-1}(\mathbf{A}_S)$ is free with a basis given as

$$u_{\alpha/p} = (1+\pi)^{\alpha_0/p}[X_1^{\flat}]^{\alpha_1/p} \cdots [X_d^{\flat}]^{\alpha_d/p} \quad \text{for } \alpha = (\alpha_0, \ldots, \alpha_d) \in \{0, 1, \ldots, p-1\}^{[0,d]}.$$

Considering the union over all such S we get that $\varphi^{-1}(\mathbf{A})$ is a free $\mathbf{A}$-module with a basis given as above (slight caveat is that we should replace $\varphi^{-1}(\mathbf{A}_S)$ by $\mathbf{A}_S$ and take p-th root of all the basis elements in loc. cit.).

Define the operator

$$\begin{aligned} \psi : \mathbf{A} &\longrightarrow \mathbf{A} \\ x &\longmapsto \tfrac{1}{p^{d+1}} \circ \mathrm{Tr}_{\varphi^{-1}(\mathbf{A})/\mathbf{A}} \circ \varphi^{-1}(x). \end{aligned}$$

Proposition 2.10 ([AB08, §4.8])**.** *The operator ψ satisfies the following properties:*

(i) $\psi \circ \varphi = id$; *let* $x \in \mathbf{A}$ *and write* $\varphi^{-1}(x) = \sum_{\alpha} x_{\alpha} u_{\alpha/p}$, *then we have* $\psi(x) = x_0$;

(ii) ψ *commutes with the action of* G_R;

(iii) $\psi(\mathbf{A}^{+}) \subset \mathbf{A}^{+}$ *and* $\psi(\mathbf{A}^{\dagger}) \subset \mathbf{A}^{\dagger}$.

2.5. Crystalline coordinates. In this section we will introduce certain "coordinate" rings. As we shall see in the next section, these rings are related to period rings appearing in §2 & §2.4.

Let r_ϖ^+ and r_ϖ denote the algebras $O_F[[X_0]]$ and $O_F[[X_0]]\{X_0^{-1}\}$ respectively. Sending X_0 to ϖ (the uniformizer of K) induces a surjective homomorphism $r_\varpi^+ \twoheadrightarrow O_K$, whose kernel is generated by a degree $e = [K : F] = p^{m-1}(p-1)$ Eisenstein polynomial $P_\varpi = P_\varpi(X_0)$. Let $R_{\varpi,\square}^+$ denote the completion of $O_F[X_0, X, X^{-1}]$ for the (p, X_0)-adic topology. Sending X_0 to ϖ induces a surjective homomorphism $R_{\varpi,\square}^+ \twoheadrightarrow O_K\{X, X^{-1}\}$, whose kernel is again generated by P_ϖ. This provides a closed embedding of $\operatorname{Spf} O_K\{X, X^{-1}\}$ into a formal scheme $\operatorname{Spf} R_{\varpi,\square}^+$, which is smooth over O_F. Recall that R is étale over $O_F\{X, X^{-1}\}$ and we have multivariate polynomials $Q_i(Z_1, \ldots, Z_s) \in O_F[X, X^{-1}][Z_1, \ldots, Z_s]$ for $1 \leq i \leq s$ such that $\det\big(\frac{\partial Q_i}{\partial Z_j}\big)$ is invertible in R. So we can set R_ϖ^+ to be the quotient of (p, X_0)-adic completion of $O_F[X_0, X, X^{-1}][Z_1, \ldots, Z_s]$ by the ideal $(Q_1, \ldots, Q_s)$. Again, we have that $\det\big(\frac{\partial Q_i}{\partial Z_j}\big)$ is invertible in R_ϖ^+ (since $R \rightarrowtail R_\varpi^+$). Hence, R_ϖ^+ is étale over $R_{\varpi,\square}^+$ and smooth over O_F. Sending X_0 to ϖ induces a surjective homomorphism $R_\varpi^+ \twoheadrightarrow R[\varpi]$ whose kernel is generated by P_ϖ. This can be summarized by the commutative diagram

$$\begin{tikzcd}
\operatorname{Spf} R[\varpi] \arrow[rr, tail] \arrow[dr] \arrow[ddd] & & \operatorname{Spf} R_\varpi^+ \arrow[dl] \arrow[ddd] \\
& \operatorname{Spf} R \arrow[d] & \\
& \operatorname{Spf} W(\kappa)\{X, X^{-1}\} & \\
\operatorname{Spf} O_K\{X, X^{-1}\} \arrow[rr, tail] \arrow[ur] & & \operatorname{Spf} R_{\varpi,\square}^+, \arrow[ul]
\end{tikzcd}$$

where the vertical arrows are étale extensions and the horizontal maps are obtained by sending $X_0 \mapsto \varpi$, and the rest are natural maps. Finally, we set $R_\varpi = R_\varpi^+\big[\frac{1}{X_0}\big]$.

Next, since $P_\varpi \equiv X_0^e \mod p$, we have $R_\varpi^+\big[\frac{P_\varpi^k}{k!}\big]_{k\in\mathbb{N}} = R_\varpi^+\big[\frac{X_0^k}{[k/e]!}\big]_{k\in\mathbb{N}}$. So, we set $R_\varpi^{\mathrm{PD}} :=$ p-adic completion of $R_\varpi^+\big[\frac{P_\varpi^k}{k!}\big]_{k\in\mathbb{N}}$. In summary, we have a diagram of formal schemes where the horizontal arrows are closed embeddings into formal schemes smooth over O_F, obtained by sending $X_0 \mapsto \varpi$ on the level of algebras,

$$\begin{tikzcd}
& \operatorname{Spf} R_\varpi^{\mathrm{PD}} \arrow[dr] & \\
\operatorname{Spf} R[\varpi] \arrow[ur, tail] \arrow[rr, tail] \arrow[d] & & \operatorname{Spf} R_\varpi^+ \arrow[d] \\
\operatorname{Spf} O_K\{X, X^{-1}\} \arrow[rr, tail] \arrow[d] & & \operatorname{Spf} R_{\varpi,\square}^+ \arrow[d] \\
\operatorname{Spf} O_K \arrow[rr, tail] \arrow[d] & & \operatorname{Spf} r_\varpi^+ \arrow[dll] \\
\operatorname{Spf} O_F. & &
\end{tikzcd} \tag{2.7}$$

Let Ω_R^q denote the p-adic completion of the modules of differential of R relative to $\mathbb{Z}$, so we have

$$\Omega^1_R = \oplus_{i=1}^d R\, d\log X_i \ \text{ and } \ \Omega^k_R = \bigwedge_R^k \Omega^1_R,$$

Moreover, since R^+_ϖ is étale over $R^+_{\varpi,\Box}$, for $S = R^+_\varpi, R^+_{\varpi,\Box}$ we have that

$$\Omega^1_S = S\tfrac{dX_0}{1+X_0} \oplus \big(\oplus_{i=1}^d S\, d\log X_i\big).$$

Definition 2.11. For $0 < u \le v$ define the rings,

$$\begin{aligned}
R^{(0,v]+}_\varpi &:= p\text{-adic completion of } R^+_\varpi\big[\tfrac{p^{\lceil vk/e\rceil}}{X_0^k}\big]_{k\in\mathbb{N}}, \quad R^{(0,v]}_\varpi := R^{(0,v]+}_\varpi\big[\tfrac{1}{X_0}\big],\\
R^{[u]}_\varpi &:= p\text{-adic completion of } R^+_\varpi\big[\tfrac{X_0^k}{p^{\lfloor uk/e\rfloor}}\big]_{k\in\mathbb{N}},\\
R^{[u,v]}_\varpi &:= p\text{-adic completion of } R^+_\varpi\big[\tfrac{X_0^k}{p^{\lfloor uk/e\rfloor}}, \tfrac{p^{\lceil vk/e\rceil}}{X_0^k}\big]_{k\in\mathbb{N}},\\
R_\varpi &:= p\text{-adic completion of } R^+_\varpi\big[\tfrac{1}{X_0}\big].
\end{aligned}$$

We will write $R^\star_\varpi$ for $\star \in \{\ , +, \mathrm{PD}, [u], (0,v]+, [u,v]\}$ and for the arithmetic case $R = O_F$, we will write $r^\star_\varpi$ instead. Going from R^+_ϖ to $R^\star_\varpi$ involves only the arithmetic variable X_0, so we have isomorphisms

$$R^\star_\varpi = r^\star_\varpi \widehat{\otimes}_{r^+_\varpi} R^+_\varpi,$$

where $\widehat{\otimes}$ is the completion of tensor product for the p-adic topology.

Remark 2.12. Unless otherwise stated, we will assume $\frac{p-1}{p} \le u \le \frac{v}{p} < 1 < v$, for example, we can take $u = \frac{p-1}{p}$ and $v = p-1$.

Definition 2.13. We define a filtration on the rings in Definition 2.11 by order of vanishing at $X_0 = \varpi = \zeta_{p^m} - 1$.

(a) Let $S = R^{(0,v]+}_\varpi$ $(v < 1)$, $R^{(0,v]}_\varpi$ $(v < 1)$, $R^{[u,v]}_\varpi$ $(1 \notin [u,v])$ or R_ϖ. As P_ϖ is invertible in $S[\frac{1}{p}]$, we put the trivial filtration on S.

(b) Let S be the placeholder for all other rings occuring in Definition 2.11, such that P_ϖ is not invertible in $S[\frac{1}{p}]$. Then there is a natural embedding $S \to R[\frac{1}{p}][[P_\varpi]]$ by completing $S[\frac{1}{p}]$ for the P_ϖ-adic topology. We use this embedding to endow S with the natural filtration $\mathrm{Fil}^k S = S \cap P^k_\varpi R[\frac{1}{p}][[P_\varpi]]$ for $k \in \mathbb{Z}$.

Next, we note a lemma that will be useful in §5.

Lemma 2.14 ([CN17, Lemma 2.6])**.** *Let $r \in \mathbb{N}$.*

(i) *For $f \in R^{\mathrm{PD}}_\varpi$ we can write $f = f_1 + f_2$ with $f_1 \in \mathrm{Fil}^r R^{\mathrm{PD}}_\varpi$ and $f_2 \in \frac{1}{(r-1)!} R^+_\varpi$.*

(ii) *For $f \in R^{[u]}_\varpi$ we can write $f = f_1 + f_2$ with $f_1 \in \mathrm{Fil}^r R^{[u]}_\varpi$ and $f_2 \in \frac{1}{p^{\lfloor ru\rfloor}} R^+_\varpi$.*

Proof. First we note that from the definitions an element $f \in r^{\mathrm{PD}}_\varpi$ (resp. $f \in r^{[u]}_\varpi$) can be written (uniquely) in the form $f = f^+ + f^-$ with $f^+ \in \mathrm{Fil}^r r^{\mathrm{PD}}_\varpi$ and $f^- \in \frac{1}{(r-1)!} O_F[X_0]$ (resp. $f^- \in \frac{1}{p^{\lfloor ru\rfloor}} O_F[X_0]$) of degree $\le re - 1$. Next, from the equality $R^{\mathrm{PD}}_\varpi = r^{\mathrm{PD}}_\varpi \widehat{\otimes}_{r^+_\varpi} R^+_\varpi$ (resp. $R^{[u]}_\varpi = r^{[u]}_\varpi \widehat{\otimes}_{r^+_\varpi} R^+_\varpi$), it follows that we can write any $f \in R^{\mathrm{PD}}_\varpi$ as $f_1 + f_2$ with $f_1 \in \mathrm{Fil}^r R^{\mathrm{PD}}_\varpi$ and $f_2 \in \frac{1}{(r-1)!} R^+_\varpi$ (resp. any $f \in R^{[u]}_\varpi$ as $f_1 \in \mathrm{Fil}^r R^{[u]}_\varpi$ and $f_2 \in \frac{1}{p^{\lfloor ru\rfloor}} R^+_\varpi$). ∎

Lemma 2.15 ([CN17, Lemma 2.11])**.** *Let $t := p^m \log(1 + X_0)$. If $\frac{p-1}{p} \le u \le \frac{v}{p} < 1 < v$, then*

(i) *t belongs to $pr^{[u,v]}_\varpi$ and to $pr^{[u,v/p]}_\varpi$;*

(ii) $\frac{t}{P_\varpi} \in p^{-1} r_\varpi^{[u,v]}$ *and* $t \in p^{-2} r_\varpi^{[u,v/p]}$;

(iii) $x \mapsto t^r x$ *induces a* p^r*-isomorphism* $r_\varpi^{[u,v]} \simeq \mathrm{Fil}^r r_\varpi^{[u,v]}$ *and a* p^{2r}*-isomorphism* $r_\varpi^{[u,v/p]} \simeq r_\varpi^{[u,v/p]}$.

We note an important fact from [CN17], the *implicit function theorem*, which would enable us to lift certain maps over étale extensions. Let $\lambda : R^+_{\varpi,\square} \to \Lambda$ be a continuous morphism of topological rings. Recall that we have $R^+_\varpi = R^+_{\varpi,\square}\{Z\}/(Q)$, where $Q = (Q_1, \ldots, Q_s)$ are multivariate polynomials in indetereminates $Z = (Z_1, \ldots, Z_s)$. We would like to extend the morphism λ to R^+_ϖ which amounts to solving the equation $Q^\lambda(Y) = 0$ in Λ, where if $F \in R^+_{\varpi,\square}\{Z\}$, we note $F^\lambda \in \Lambda\{Z\}$ the series obtained by applying λ to the coeffficients of F. Then,

Proposition 2.16 ([CN17, Proposition 2.1 & Remark 2.2])**.** *The equation* $Q^\lambda(Y)$ *has a unique solution in* $Z_\lambda + I^s$.

Proof. For the sake of completeness, we recall the proof in our special case. Let $J = \left(\frac{\partial Q_i}{\partial Z_j}\right)_{1 \leq i,j \leq s} \in \mathrm{Mat}(s, R^+_{\varpi,\square}\{Z_1, \ldots, Z_s\})$. Suppose that there exists an ideal $I \subset \Lambda$ such that Λ is complete with respect to the I-adic topology, $Z_\lambda = (Z_{1,\lambda}, \ldots, Z_{s,\lambda}) \in \Lambda^s$ and $H_\lambda \in \mathrm{Mat}(s, \Lambda)$, such that the entries of $Q^\lambda(Z_\lambda)$ belong to I. Now, since R^+_ϖ is étale over Λ, so $\det J$ is invertible in $R^+_{\varpi,\square}$ and therefore there exists $H \in \mathrm{Mat}(s, R^+_{\varpi,\square}\{Z_1, \ldots, Z_s\})$ such that $HJ - 1$ has its entries in $(Q_1, \ldots, Q_s)$. But $Q^\lambda(Z_\lambda)$ has coordinates in the ideal I, therefore $H^\lambda J^\lambda - 1$ has entries in I. Thus, we can apply [CN17, Proposition 2.1], by taking (in the notation of loc. cit.) $z = 1$ and $H_\lambda = H^\lambda(Z_\lambda)$. Hence, the equation $Q^\lambda(Y)$ has a unique solution in $Z_\lambda + I^s$. ∎

2.6. Cyclotomic Frobenius.

In this section, we will define (cyclotomic) Frobenius endomorphism on the rings studied in the previous section. Furthermore, we will introduce a left inverse to the Frobenius operator which will be helpful in our study of syntomic complexes later.

Definition 2.17. Over $R^+_{\varpi,\square}$ we define a lift of the absolute Frobenius modulo p as

$$\begin{aligned} \varphi : R^+_{\varpi,\square} &\longrightarrow R^+_{\varpi,\square} \\ X_0 &\longmapsto (1 + X_0)^p - 1 \\ X_i &\longmapsto X_i^p \ \text{ for } \ i \leq i \leq d, \end{aligned}$$

which we will call the (cyclotomic) Frobenius. Clearly, $\varphi(x) - x^p \in pR^+_{\varpi,\square}$ for $x \in R^+_{\varpi,\square}$. Using Proposition 2.16 with $\Lambda_1 = R^+_{\varpi,\square}$, $\Lambda_1' = \Lambda_2 = R^+_\varpi$, $\lambda = \varphi$, $I = (p)$ and $Z_\lambda = Z^p$, we can extend the Frobenius homomorphism to $\varphi : R^+_\varpi \to R^+_\varpi$. By continuity, the Frobenius endomorphism φ admits unique extensions

$$R_\varpi^{\mathrm{PD}} \longrightarrow R_\varpi^{\mathrm{PD}}, \quad R_\varpi^{[u]} \longrightarrow R_\varpi^{[u]}, \quad R_\varpi^{(0,v]+} \longrightarrow R_\varpi^{(0,v/p]+}, \quad R_\varpi^{[u,v]} \longrightarrow R_\varpi^{[u,v/p]} \ \text{ and } \ R_\varpi \longrightarrow R_\varpi.$$

We mention an important fact which will be useful in §5. Recall that we have explicit description of rings,

$$r_\varpi^{\mathrm{PD}} = \Big\{f = \sum_{k \in \mathbb{N}} a_k \frac{X_0^k}{[k/e]!}, \text{ such that } a_k \in O_F \text{ goes to } 0 \text{ as } i \to \infty\Big\},$$
$$r_\varpi^{[u]} = \Big\{f = \sum_{k \in \mathbb{N}} a_k \frac{X_0^k}{p^{\lfloor \frac{ku}{e} \rfloor}}, \text{ such that } a_k \in O_F \text{ goes to } 0 \text{ as } i \to \infty\Big\}.$$

Let $S = r_\varpi^{\mathrm{PD}}$ or $r_\varpi^{[u]}$. Denote by $\upsilon_{X_0} : S \to \mathbb{N} \cup \{+\infty\}$ the valuation relative to X_0, i.e. if $f = \sum a_k X_0^k$, then $\upsilon_{X_0}(f) = \inf\{i \in \mathbb{N}, a_i \neq 0\}$. For $N \in \mathbb{N}$, we define $S_N = \{f \in S,\ \upsilon_{X_0}(f) \geq N\}$. Define $R^{\mathrm{PD}}_{\varpi,N}$ and $R^{[u]}_{\varpi,N}$ as the topological closures of $r^{\mathrm{PD}}_{\varpi,N} \otimes_{r^+_\varpi} R^+_\varpi \subset R^{\mathrm{PD}}_\varpi$ and $r^{[u]}_{\varpi,N} \otimes_{r^+_\varpi} R^+_\varpi \subset R^{[u]}_\varpi$, respectively.

Lemma 2.18 ([CN17, Proposition 3.1])**.** *Let $N \in \mathbb{N}_{>0}$, $s \in \mathbb{Z}$ and $N \geq se$ (resp. $N \geq se/u(p-1)$), then $1 - p^{-s}\varphi$ is bijective on $R^{\mathrm{PD}}_{\varpi,N}$ (resp. $R^{[u]}_{\varpi,N}$).*

Next, we will define a left inverse of the cyclotomic Frobenius φ, which we will denote by ψ. This operator is closely related to the operator defined in Proposition 2.10 (this will become clear in §2.7). However, we prefer to give an explicit definition here. Let

$$u_\alpha = (1+X_0)^{\alpha_0} X_1^{\alpha_1} \cdots X_d^{\alpha_d} \quad \text{for } \alpha = (\alpha_0, \ldots, \alpha_d) \in \{0, 1, \ldots, p-1\}^{[0,d]}.$$

We set

$$\partial_0 = (1+X_0)\frac{d}{dX_0}, \qquad \partial_i = X_i \frac{d}{dX_i} \quad \text{for } 1 \leq i \leq d.$$

Therefore, for $0 \leq i \leq d$ we have

$$\partial_i u_\alpha = \alpha_i u_\alpha \text{ and } \varphi(u_\alpha) = u_\alpha^p.$$

Remark 2.19. Note that X_0 is in the Jacobson radical of R_ϖ^+ therefore $1 + X_0$ is invertible in it. Moreover, by definition $X_1, \ldots, X_d$ are invertible in R_ϖ^+, therefore u_α is invertible in R_ϖ^+ for $\alpha = (\alpha_0, \ldots, \alpha_d) \in \{0, 1, \ldots, p-1\}^{[0,d]}$.

Lemma 2.20 ([CN17, Proposition 2.15])**.**

(i) *Any $x \in R_\varpi/p$ can be written uniquely as $x = \sum_\alpha c_\alpha(x)$, with $\partial_i \circ c_\alpha(x) = \alpha_i c_\alpha(x)$ for $0 \leq i \leq d$.*

(ii) *There exists a unique $x_\alpha \in R_\varpi/p$ such that $c_\alpha(x) = x_\alpha^p u_\alpha$.*

(iii) *If $x \in R_\varpi^+/p$, then $c_\alpha(x) \in R_\varpi^+/p$.*

Proof. Let $S = R_\varpi/p$, $S^+ = R_\varpi^+/p$. Then $\partial_i(\partial_i - 1)\cdots(\partial_i - (p-1))$ is identically 0 on $R_{\varpi,\Box}/p$, hence also on S by étaleness. It follows that ∂_i is diagonalizable for all i and since these operators commute pairwise, we can decompose S and S^+ into the direct sum of common eigenspaces. This shows (i) and (iii). Now, differentials of the elements in the set $\{1 + X_0, X_1, \ldots, X_d\}$ form a basis of the module of differentials of $R_{\varpi,\Box}/p$, hence also of S, since it is obtained as the completion of an étale algebra over $R_{\varpi,\Box}/p$. From [Tyc88, §III, Theorem 1], it follows that $\{1 + X_0, X_1, \ldots, X_d\}$ is a p-basis of S which can be rephrased by saying that any element x of S can be written uniquely as $x = \sum_\alpha x_\alpha^p u_\alpha$. Since $\partial_i(x_\alpha^p u_\alpha) = \alpha_i x_\alpha^p u_\alpha$ for $1 \leq i \leq d$, this proves (ii). ∎

Proposition 2.21.

(i) *Any $x \in R_\varpi$ can be written uniquely as $x = \sum_\alpha c_\alpha(x)$, with $c_\alpha(x) \in \varphi(R_\varpi)u_\alpha$.*

(ii) *If $x \in R_\varpi^+$ and if $c_\alpha(x) = \varphi(x_\alpha)u_\alpha$, then $c_\alpha(x) \in R_\varpi^+$ for all α and*

$$\partial_i c_\alpha(x) - \alpha_i c_\alpha(x) \in pR_\varpi^+ \quad \text{for } 0 \leq i \leq d.$$

(iii) *For $x \in R_\varpi^{(0,v]+}$, we have $c_\alpha(x) \in R_\varpi^{(0,v]+}$ for all α.*

Proof. (i) and (ii) follow from the lemma above. (iii) follows from [CN17, Proposition 2.15]. ∎

Definition 2.22. Define the left inverse ψ of the Frobenius φ on $S = R_\varpi^+$ or $S = R_\varpi$, by the formula

$$\psi(x) = \varphi^{-1}(c_0(x)).$$

Since R_ϖ is an extension of degree p^{d+1} of $\varphi(R_\varpi)$ with basis the u_α's and since $\varphi(u_\alpha) = u_\alpha^p$ for all α, we have

$$\mathrm{Tr}_{R_\varpi/\varphi(R_\varpi)}(u_\alpha) = 0 \text{ if } \alpha \neq 0,$$

and we can define ψ intrinsically, by the formula

$$\psi(x) := \tfrac{1}{p^{d+1}}\varphi^{-1} \circ \mathrm{Tr}_{R_\varpi/\varphi(R_\varpi)}(x).$$

Note that ψ is not a ring morphism; it is a left inverse to φ and more generally, we have $\psi(\varphi(x)y) = x\psi(y)$. Also,

$$\partial_i \circ \varphi = p\varphi \circ \partial_i \ \text{ and } \ \partial_i \circ \psi = p^{-1}\psi \circ \partial_i \ \text{ for } \ i = 0, 1, \ldots, d.$$

The first equality can be obtained by checking on the basis elements u_α. For the second equality, note that for $x \in R_\varpi$ and in the notation of Proposition 2.21 we have

$$\partial_i(\varphi(x_\alpha)u_\alpha) = \partial_i \circ \varphi(x_\alpha)u_\alpha + \varphi(x_\alpha)\partial_i(u_\alpha) = (p\varphi \circ \partial_i(x_\alpha) + \alpha_i\varphi(x_\alpha))u_\alpha = \varphi(p\partial_i(x_\alpha) + \alpha_i x_\alpha)u_\alpha.$$

Applying ψ to the latter expression we note that it is nonzero only if $\alpha = 0$, in which case we get that $\psi \circ \partial_i \in pR_\varpi$ for all $0 \leq i \leq d$, the equality follows from this.

For any $k \in \mathbb{N}$, we can write $X_0^k = \sum_{j=0}^{p-1} \varphi(a_{j,k})(1 + X_0)^j$ for $a_{j,k} \in R_\varpi^+$. Therefore, by continuity

Lemma 2.23. (i) *The explicit formula for ψ extends to surjective maps $R_\varpi^{(0,v]+} \to R_\varpi^{(0,pv]+}$, $R_\varpi^{[u]} \to R_\varpi^{[pu]}$ and $R_\varpi^{[u,v]} \to R_\varpi^{[pu,pv]}$.*

(ii) *For the same reasons, the maps $x \mapsto c_\alpha(x)$ also extend and lead to decompositions $S = \oplus_\alpha S_\alpha$, where $S_\alpha = S \cap \varphi(R_\varpi)u_\alpha$ for $S = R_\varpi^\star$ with $\star \in \{, +, [u], (0, v]+, [u, v]\}$. Since $\psi(x) = \varphi^{-1}(c_0(x))$, we have*

$$S^{\psi=0} = \oplus_{\alpha\neq 0} S_\alpha.$$

Lemma 2.24. *If $S = R_\varpi^\star$ for $\star \in \{\ , +, [u], (0, v]+, [u, v]\}$, then for $0 \leq i \leq d$ the operator ∂_i on $S_\alpha^\star/pS_\alpha^\star$ is given by multiplication by α_i, where α_i is the i-th entry in $\alpha = (\alpha_0, \ldots, \alpha_d)$.*

Proof. If $\star \in \{\ , +\}$, this is part of Proposition 2.21. For $\star \in \{[u], (0, v]+, [u, v]\}$, elements of $S_\alpha^\star$ are those of the form $\sum_{k\in\mathbb{Z}} p^{r_k} X_0^k x_k$, where $x_k \in S^+$ goes to 0 when $k \to +\infty$ and r_k is determined by "$\star$". Let $x = \sum_{k\in\mathbb{Z}} p^{r_k} X_0^k x_k$. For $1 \leq i \leq d$, we have

$$\partial_i(X_0^k a_k) - \alpha_i X_0^k a_k = X_0^k(\partial_i(a_k) - \alpha_i a_k) \in pS^+,$$

by Proposition 2.21.

For $i = 0$, first we look at $S^{[u]}$ and write

$$x = \sum_{k\in\mathbb{N}} p^{r_k} x_k \sum_{j=0}^{p-1} \varphi(a_{j,k})(1 + X_0)^j \quad \text{for } \ a_{j,k} \in S^+.$$

Then

$$c_\alpha(x) = \sum_{j=0}^{p-1} \sum_{k\in\mathbb{N}} p^{r_k} \varphi(a_{j,k}) c_{(\alpha_0 - j, \alpha_1, \cdots, \alpha_d)}(x_k)(1 + X_0)^j,$$

where $\alpha_0 - j$ is to be understood as its representative modulo p between 0 and $p - 1$. Since $\partial_0(c_{(\alpha_0-j,\alpha_1,\cdots,\alpha_d)}(x_k)) - (\alpha_0 - j)c_{(\alpha_0-j,\alpha_1,\cdots,\alpha_d)}(x_k) \in pS^+$ and $\partial_0 \circ \varphi = p\varphi \circ \partial_0$, we get the desired conclusion for $S^{[u]}$. Next, for $S^{(0,v]+}$, using the result for S we get that $\partial_0(x) - \alpha_0 x \in pS \cap S^{(0,v]+} = pS^{(0,v]+}$. Finally, combining the results for $S^{[u]}$ and $S^{(0,v]+}$ we get the conclusion for $S^{[u,v]}$. ∎

Next, we note a lemma which will be useful in the proof of Propositions 2.26 & 6.11.

Lemma 2.25. *Let $x \in R_\varpi^{\psi=0}$, then $X_0^k \psi(x) = \psi(\varphi(X_0)^k x)$ for $k \in \mathbb{Z}$.*

Proof. Note that it is enough to prove the statement for $k = 1$. Indeed, $k \geq 2$ case immediately follows from this, whereas for $k = -1$ we observe that since X_0 is invertible in R_ϖ, we have $X_0\psi(\varphi(X_0^{-1})x) = \psi(\varphi(X_0)\varphi(X_0^{-1})x) = \psi(x)$.

Now, to show the case $k = 1$, we recall that $\varphi(X_0) = (1 + X_0)^p - 1$. Next, from Proposition 2.21 let us write $x = \sum_\alpha c_\alpha$, then we have $\psi(x) = \varphi^{-1}(c_0)$. It follows that,

$$\psi(\varphi(X_0)x) = \psi(((1+X_0)^p - 1)x) = \psi((1+X_0)^p x) - \psi(x) = (1+X_0)\varphi^{-1}(c_0) - \varphi^{-1}(c_0) = X_0\psi(x),$$

as desired. ■

Proposition 2.26 ([CN17, Proposition 2.16])**.** *Let $v < p$.*

(i) $\psi(X_0^{-pN} R_\varpi^{(0,v/p]+}) \subset X_0^{-N} R_\varpi^{(0,v]+}$*;*

(ii) *If $\ell = p^m$, then $X_0^{-\ell} R_\varpi^{(0,v]+}$ is stable under ψ;*

(iii) *The natural map*

$$\oplus_{\alpha \neq 0}\varphi(R_\varpi^{(0,v]+})u_\alpha \longrightarrow (R_\varpi^{(0,v/p]+})^{\psi=0}$$

is an isomorphism.

Proof. (i) follows from Proposition 2.21 (ii) and (iii), and taking into account the facts that $\psi(\varphi(X_0)^{-N}x) = X_0^{-N}\psi(x)$ and $\frac{\varphi(X_0)}{X_0^p}$ is a unit in $R_\varpi^{(0,v/p]+}$. (ii) is an immediate consequence of (i) and the inclusion $R_\varpi^{(0,v]+} \subset R_\varpi^{(0,v/p]+}$. Finally, if $x \in (R_\varpi^{(0,v/p]+})^{\psi=0}$, using Proposition 2.21 (ii), we can write $x = \sum_{\alpha\neq 0}\varphi(x_\alpha)u_\alpha$ with $\varphi(x_\alpha)u_\alpha \in R_\varpi^{(0,v/p]+}$. But, u_α is invertible in $R_\varpi^{(0,v/p]+}$ (see Remark 2.19), hence $\varphi(x_\alpha) \in R_\varpi^{(0,v/p]+}$. From [CN17, Lemma 2.14], we have that if $x_\alpha \in R_\varpi$ such that $\varphi(x_\alpha) \in R_\varpi^{(0,v/p]+}$, then $x_\alpha \in R_\varpi^{(0,v]+}$. This gives us (iii). ■

2.7. Cyclotomic embedding. In this section, we will describe the relationship between $R_\varpi^\star$ for $\star \in \{\ , +, \mathrm{PD}\}$ and the period rings discussed in §2 & §2.4. We begin by defining an embedding

$$\begin{aligned} \iota_{\mathrm{cycl}} : R_{\varpi,\square}^+ &\longrightarrow \mathbf{A}_{\inf}(\overline{R}) \\ X_0 &\longmapsto \pi_m = \varphi^{-m}(\pi), \\ X_i &\longmapsto [X_i^\flat], \ \text{ for } \ 1 \leq i \leq d. \end{aligned}$$

Lemma 2.27. *The map ι_{cycl} has a unique extension to an embedding $R_\varpi^+ \to \mathbf{A}_{\inf}(\overline{R})$ such that $\theta \circ \iota_{\mathrm{cycl}}$ is the projection $R_\varpi^+ \to R[\varpi]$.*

Proof. We can apply Proposition 2.16 with $\Lambda_1 = R_{\varpi,\square}^+$, $\Lambda_2 = \mathbf{A}_{\inf}(\overline{R})$, $\Lambda_1' = R_\varpi^+$, $\lambda = \iota_{\mathrm{cycl}}$, $I = (\xi)$ and $Z_\lambda = ([Z_1^\flat], \ldots, [Z_s^\flat])$. Next, from defintions we already have that $\theta \circ \iota_{\mathrm{cycl}} : R_{\varpi,\square}^+ \twoheadrightarrow O_K\{X, X^{-1}\}$ coincides with the canonical projection and R_ϖ^+ is étale over $R_{\varpi,\square}^+$, hence the second claim follows. ■

This embedding commutes with Frobenius on either side, i.e. $\iota_{\mathrm{cycl}} \circ \varphi_{\mathrm{cycl}} = \varphi \circ \iota_{\mathrm{cycl}}$. By continuity, the morphism ι_{cycl} extends to embeddings

$$R_\varpi^{\mathrm{PD}} \rightarrowtail \mathbf{A}_{\mathrm{cris}}(\overline{R}), \quad R_\varpi^{[u]} \rightarrowtail \mathbf{A}_{\overline{R}}^{[u]}, \quad R_\varpi^{(0,v]+} \rightarrowtail \mathbf{A}_{\overline{R}}, \quad R_\varpi^{[u,v]} \rightarrowtail \mathbf{A}_{\overline{R}}^{[u,v]} \ \text{ and } \ R_\varpi \rightarrowtail \mathbf{A}_{\overline{R}}.$$

Denote by $\mathbf{A}_{R,\varpi}^\star$ the image of $R_\varpi^\star$ under ι_{cycl}. These rings are stable under the action of G_R. Moreover, this embedding induces a filtration on $\mathbf{A}_{R,\varpi}^\star$ for $\star \in \{+, \mathrm{PD}, [u], [u,v], (0,v]+\}$ and $r \in \mathbb{Z}$ (use Definition 2.13).

Remark 2.28. From [CN17, §2.4.2], we have an inclusion of rings $\mathbf{A}_{R,\varpi}^{[u']} \subset \mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathbf{A}_{R,\varpi}^{[u]}$ for $u \geq \frac{1}{p-1}$ and $u' \leq \frac{1}{p}$.

Remark 2.29. Note that we write $\mathbf{A}_{R,\varpi}^{+}$ and so on instead of slightly cumbersome notation $\mathbf{A}_{R[\varpi]}^{+}$ or simpler notation $\mathbf{A}_{S}^{+}$ for $S = R[\varpi]$, in order to emphasize the choice of root of unity in the definition.

Note that the preceding discussion works well for $R[\varpi]$ where $\varpi = \zeta_{p^m} - 1$ with $m \geq 1$. For R one can repeat the construction above to obtain the period ring $\mathbf{A}_R^+ \subset \mathbf{A}_{R,\varpi}^+$. Then restriction of the map θ gives us a surjective map $\theta : \mathbf{A}_R^+ \twoheadrightarrow R$ whose kernel is principal and generated by π (since $\theta \circ \iota_{\mathrm{cycl}} = id$ on R). Recall that over $\mathbf{A}_{R,\varpi}^+$ the filtration is given as $\mathrm{Fil}^k \mathbf{A}_{R,\varpi}^+ = \xi^k \mathbf{A}_{R,\varpi}^+$, where $\xi = \frac{\pi}{\pi_1}$. However, $\xi \notin \mathbf{A}_R^+$. Therefore, we equip $\mathbf{A}_R^+$ with the induced filtration $\mathrm{Fil}^k \mathbf{A}_R^+ = \mathbf{A}_R^+ \cap \mathrm{Fil}^k \mathbf{A}_{R,\varpi}^+ = \pi^k \mathbf{A}_R^+$ (see [Abh21, Lemma 3.17]).

We note the following result from [Abh21, Lemma 3.14]:

Lemma 2.30. *$\frac{t}{\pi}$ is a unit in $\mathbf{A}_{F,\varpi}^{\mathrm{PD}} \subset \mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathbf{A}_{R,\varpi}^{[u]} \subset \mathbf{A}_{R,\varpi}^{[u,v]}$.*

Next, we prove some claims for the action of Γ_R. These results will be used in the study of Koszul complexes computing Lie Γ_R-cohomology in §4.3.

Lemma 2.31. *Let $k \in \mathbb{N}$ and $i \in \{0, 1, \ldots, d\}$. Then $(\gamma_i - 1)(p^m, \pi_m^{p^m})^k \mathbf{A}_{R,\varpi}^{\star} \subset (p^m, \pi_m^{p^m})^{k+1} \mathbf{A}_{R,\varpi}^{\star}$ for $\star \in \{+, \mathrm{PD}, [u]\}$;*

Proof. First, let $i = 0$. Then we have

$$\begin{aligned}(\gamma_0 - 1)\pi_m &= (1+\pi_m)\big((1+\pi_m)^{\chi(\gamma_0)-1} - 1\big) = (1+\pi_m)\big((1+\pi_m)^{p^m a} - 1\big)\\ &= (1+\pi_m)\big((1+\pi)^a - 1\big) = (1+\pi_m)\big(a\pi + \tfrac{a(a-1)}{2!}\pi^2 + \tfrac{a(a-1)(a-2)}{3!}\pi^3 + \cdots\big) = \pi x,\end{aligned}$$

for some $x \in \mathbf{A}_{R,\varpi}^+$. Since $\pi = (1+\pi_m)^{p^m} - 1 = \pi_m^{p^m} + p^m \pi_m^{p^m - 1} + \cdots + p^m \pi_m$, we get that $\pi \in (p^m, \pi_m^{p^m})\mathbf{A}_{R,\varpi}^+$, therefore $(\gamma_0 - 1)\pi_m \in (p^m, \pi_m^{p^m})\mathbf{A}_{R,\varpi}^+$. Next, we observe that

$$\begin{aligned}(\gamma_0 - 1)\pi_m^{p^m} &= \gamma_0(\pi_m)^{p^m} - \pi_m^{p^m} = (\pi x + \pi_m)^{p^m} - \pi_m^{p^m}\\ &= \pi^{p^m} x^{p^m} + \cdots + p^m \pi x \pi_m^{p^m - 1} \in (p^m, \pi_m^{p^m})^2 \mathbf{A}_{R,\varpi}^+.\end{aligned}$$

Proceeding by induction on $k \geq 1$ and using the fact that $\gamma_0 - 1$ acts as a twisted derivation (i.e. for $x, y \in \mathbf{A}_{R,\varpi}^+$ we have $(\gamma_0 - 1)xy = (\gamma_0 - 1)x \cdot y + \gamma_0(x)(\gamma_0 - 1)y$), we conclude that

$$(\gamma_0 - 1)(p^m, \pi_m^{p^m})^k \mathbf{A}_{R,\varpi}^+ \subset (p^m, \pi_m^{p^m})^{k+1} \mathbf{A}_{R,\varpi}^+.$$

Now any $f \in \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ can be written as $f = \sum_{n \in \mathbb{N}} f_n \frac{\pi_m^n}{\lfloor n/e \rfloor !}$ such that $f_n \in \mathbf{A}_{R,\varpi}^+$ goes to 0 as $n \to +\infty$. For notational convenience, we take $n = je$ for some $j \in \mathbb{N}$ and see that

$$\begin{aligned}\frac{(\gamma_0 - 1)\pi_m^{je}}{j!} &= \frac{\gamma_0(\pi_m)^{je} - \pi_m^{je}}{j!} = \frac{(\pi x + \pi_m)^{je} - \pi_m^{je}}{j!} = \frac{(\pi x)^{je} + je(\pi x)^{je-1}\pi_m + \cdots + je(\pi x)\pi_m^{je-1}}{j!}\\ &= \frac{(\pi x)^{je}}{j!} + \pi \frac{\pi_m^{je-1} z}{(j-1)!} \in \frac{1}{j!}(p^m, \pi_m^{p^m})^{je} \mathbf{A}_{R,\varpi}^{\mathrm{PD}} + (p^m, \pi_m^{p^m})\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset (p^m, \pi_m^{p^m})\mathbf{A}_{R,\varpi}^{\mathrm{PD}}.\end{aligned}$$

Proceeding by induction on $k \geq 1$ and using the fact that $\gamma_0 - 1$ acts as a twisted derivation, we conclude that

$$(\gamma_0 - 1)(p^m, \pi_m^{p^m})^k \mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset (p^m, \pi_m^{p^m})^{k+1} \mathbf{A}_{R,\varpi}^{\mathrm{PD}}.$$

Next, for $i \in \{1, \ldots, d\}$ we have $(\gamma_i - 1)[X_i^\flat] = \pi[X_i^\flat] \in \pi \mathbf{A}_{R,\varpi}^+ \subset (p^m, \pi_m^{p^m})\mathbf{A}_{R,\varpi}^+$ and $(\gamma_i - 1)\big([X_i^\flat]^{-1}\big) = -\pi(1+\pi)^{-1}[X_i^\flat]^{-1} \in \pi \mathbf{A}_{R,\varpi}^+ \subset (p^m, \pi_m^{p^m})\mathbf{A}_{R,\varpi}^+$. Proceeding by induction on $k \geq 0$ and using the fact that $\gamma_i - 1$ acts as a twisted derivation, we conclude that

$$(\gamma_i - 1)(p^m, \pi_m^{p^m})^k \mathbf{A}_{R,\varpi}^+ \subset (p^m, \pi_m^{p^m})^{k+1} \mathbf{A}_{R,\varpi}^+.$$

Since any $f \in \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ can be written as $f = \sum_{j\in\mathbb{N}} f_j \frac{\pi_m^j}{\lfloor j/e \rfloor !}$ such that $f_j \in \mathbf{A}_{R,\varpi}^{+}$ goes to 0 as $j \to +\infty$, from the discussion for $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ and $\mathbf{A}_{R,\varpi}^{+}$ and using the fact that $\gamma_i - 1$ acts as a twisted derivation, we conclude that

$$(\gamma_i - 1)(p^m, \pi_m^{p^m})^k \mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset (p^m, \pi_m^{p^m})^{k+1} \mathbf{A}_{R,\varpi}^{\mathrm{PD}}.$$

■

The next claim will be useful in analyzing Koszul complexes for Γ_R-cohomology in Propositions 6.11 & 6.18.

Lemma 2.32. *Let $k \in \mathbb{N}$.*

(i) *We have $(\gamma_0 - 1)\mathbf{A}_{R,\varpi}^{(0,v]+} \subset (p^m\pi_m, \pi_m^{p^m})\mathbf{A}_{R,\varpi}^{(0,v]+}$ and $(\gamma_i - 1)\mathbf{A}_{R,\varpi}^{(0,v]+} \subset \pi\mathbf{A}_{R,\varpi}^{(0,v]+}$ for $i \in \{1, \ldots, d\}$.*

(ii) *We have $(\gamma_i - 1)(p^m, \pi_m^{p^m})^k \mathbf{A}_{R,\varpi}^{[u,v]} \subset (p^m, \pi_m^{p^m})^{k+1} \mathbf{A}_{R,\varpi}^{[u,v]}$ for $i \in \{0, 1, \ldots, d\}$..*

Proof. First, let $i = 0$. Then from Lemma 2.31 we have $(\gamma_0 - 1)\pi_m = \pi x$ for some $x \in \mathbf{A}_{R,\varpi}^{+}$. Since $\pi = (1 + \pi_m)^{p^m} - 1 = \pi_m^{p^m} + p^m\pi_m^{p^m-1} + \cdots + p^m\pi_m$, we get that $\pi \in (p^m\pi_m, \pi_m^{p^m})\mathbf{A}_{R,\varpi}^{+}$, therefore $(\gamma_0 - 1)\pi_m \in (p^m\pi_m, \pi_m^{p^m})\mathbf{A}_{R,\varpi}^{+}$. We observe that

$$\gamma_0(\pi_m) = (1 + \pi_m)^{\chi(\gamma_0)} - 1 = \chi(\gamma_0)\pi_m(1 + \tfrac{\chi(\gamma_0)-1}{2}\pi_m + \cdots) = \chi(\gamma_0)\pi_m f,$$

where $\chi(\gamma_0) = \exp(p^m) \in \mathbb{Z}_p^*$ and f is a unit in $\mathbf{A}_{R,\varpi}^{+}$. From the expression above we also have that $1 - \chi(\gamma_0)f = p^m z$ for some $z \in \mathbf{A}_{R,\varpi}^{+}$. So we can write

$$(\gamma_0 - 1)\pi_m^{-1} = \gamma_0(\pi_m)^{-1} - \pi_m^{-1} = (\chi(\gamma_0)f\pi_m)^{-1} - \pi_m^{-1} = \frac{1 - \chi(\gamma_0)f}{\chi(\gamma_0)f\pi_m} = \frac{p^m z}{\chi(\gamma_0)f\pi_m}$$

Now from the definitions we know that $\frac{p}{\pi_m} \in \mathbf{A}_{R,\varpi}^{(0,v]+}$, therefore $(\gamma_0 - 1)\frac{p}{\pi_m} \in (p^m\pi_m, \pi_m^{p^m})\mathbf{A}_{R,\varpi}^{(0,v]+}$. From Lemma 2.31 we already have that $(\gamma_0 - 1)\mathbf{A}_{R,\varpi}^{+} \in (p^m, \pi_m^{p^m})\mathbf{A}_{R,\varpi}^{+}$. Combining this with the discussion above and using the fact that $\gamma_0 - 1$ acts as a twisted derivation, we conclude that

$$(\gamma_0 - 1)\mathbf{A}_{R,\varpi}^{(0,v]+} \subset (p^m\pi_m, \pi_m^{p^m})^{k+1}\mathbf{A}_{R,\varpi}^{(0,v]+}.$$

For $1 \leq i \leq d$ from the analysis for $\mathbf{A}_{R,\varpi}^{+}$ in Lemma 2.31 we already have that $(\gamma_i - 1)\mathbf{A}_{R,\varpi}^{+} \subset \pi\mathbf{A}_{R,\varpi}^{+}$. Since passing from $\mathbf{A}_{R,\varpi}^{+}$ to $\mathbf{A}_{R}^{(0,v]+}$ involves only the arithmetic variable π_m on which γ_i acts trivially. So using the fact that $\gamma_i - 1$ acts as a twisted derivation we conclude that

$$(\gamma_i - 1)\mathbf{A}_{R,\varpi}^{(0,v]+} \subset \pi\mathbf{A}_{R,\varpi}^{(0,v]+}.$$

This shows (i). Finally, The claim for $\mathbf{A}_{R,\varpi}^{[u,v]}$ follows by combining (i) with the discussion in Lemma 2.31 for $\mathbf{A}_{R,\varpi}^{[u]}$. ■

Finally, we show a claim which will be useful for changing the annulus of convergence in §6.

Lemma 2.33 ([CN17, Lemma 2.35])**.** *If $v \leq p$, then*

(i) *$\pi_m^{-p^{m-1}}\pi_1$ is a unit in $\mathbf{A}_{R,\varpi}^{(0,v]+}$;*

(ii) *p is divisible by $\pi_m^{\lfloor (p-1)p^{m-1}/v \rfloor}$, hence also by $\pi_m^{(p-1)p^{m-2}}$;*

(iii) *$\frac{p^2}{\pi_1} \in \mathbf{A}_{R,\varpi}^{(0,v]+}$ and is divisible by $\pi_m^{(2(p-1)-v)p^{m-2}}$;*

(iv) $\frac{\pi}{\pi_1} \in (p, \pi_m^{(p-1)p^{m-1}})\mathbf{A}_{R,\varpi}^{(0,v]+}$ *and is divisible by* $\pi_m^{(p-1)p^{m-2}}$;

(v) *Let* $v = p-1$, *then* $\pi_m^{-p^m}\pi$ *is a unit in* $\mathbf{A}_{R,\varpi}^{(0,v/p]+}$ *and* $\frac{p}{\pi} \in \mathbf{A}_{R,\varpi}^{(0,v/p]+}$.

Proof. We can work in $r_\varpi^{(0,v]+}$, in which case π_m becomes X_0 and π_1 becomes $(1+X_0)^{p^{m-1}}-1$ and we are looking at the annulus $0 < v_p(T) \leq \frac{v}{p^{m-1}(p-1)}$ on which $(1+X_0)^{p^{m-1}}-1$ has no zero and $v_p((1+X_0)^{p^{m-1}}-1) = p^{m-1}v_p(X_0)$ since $v < p$. This shows (i). The claim in (ii) comes from the definition of $R_\varpi^{(0,v]+}$. (iii) follows from (i) and (ii) since $2\lfloor\frac{(p-1)p^{m-1}}{v}\rfloor - p^{m-1} \geq (2(p-1)-v)p^{m-2}$. The claim in (iv) follows from (i), (ii) and the identity

$$\frac{\pi}{\pi_1} = \pi_1^{p-1} + p\pi_1^{p-2} + \cdots + p.$$

For (v), replacing π by $(1+X_0)^{p^m}-1$, we see that $v_p((1+X_0)^{p^m}-1) = p^m v_p(X_0)$. Using arguments similar to (i) gives us first part of (v). The second half of (v) follows from the first part and (ii) since $\lfloor\frac{(p-1)p^{m-1}}{(p-1)/p}\rfloor = p^m$. ■

2.8. Fat period rings.

In this section we will give an alternative construction of fat period rings and a version of PD-Poincaré lemma. The Poinaré lemma will be useful for relating complexes computing Galois cohomology and syntomic complex with coefficients in §5.

2.8.1. Structural properties.

Let Σ and Λ be p-adically complete filtered O_F-algebras. Let $\iota : \Sigma \to \Lambda$ be a continuous injective morphism of filtered O_F-algebras and let $f : \Sigma \otimes \Lambda \to \Lambda$ be the morphism sending $x \otimes y \mapsto \iota(x)y$.

Definition 2.34. Define $\Sigma\Lambda$ to be the p-adic completion of the divided power envelope of $\Sigma \otimes \Lambda$ with respect to Ker f.

Now, let $\Sigma = R$ or $R_\varpi^\star$ for $\star \in \{\mathrm{PD}, [u], [u,v]\}$, where over R we consider the trivial filtration, whereas over R_ϖ^{PD} we consider the filtration described in Definition 2.13. Then we have,

Remark 2.35. (i) The ring $\Sigma\Lambda$ is the p-adic completion of $\Sigma \otimes \Lambda$ adjoined $(x \otimes 1 - 1 \otimes \iota(x))^{[k]}$, for $x \in \Sigma$ and $n \in \mathbb{N}$ and $(V_i - 1)^{[k]}$ for $1 \leq i \leq d$ and $k \in \mathbb{N}$, where $V_i = \frac{X_i \otimes 1}{1 \otimes \iota(X_i)}$ for $1 \leq i \leq d$.

(ii) The morphism $f : \Sigma \otimes \Lambda \to \Lambda$ extends uniquely to a continuous morphism $f : \Sigma\Lambda \to \Lambda$.

(iii) There is a natural filtration over $\Sigma\Lambda$ where we define $\mathrm{Fil}^r\Sigma\Lambda$ to be the topological closure of the ideal generated by the products of the form $x_1x_2\prod(V_i-1)^{[k_i]}$, with $x_1 \in \mathrm{Fil}^{r_1}\Sigma$, $x_2 \in \mathrm{Fil}^{r_2}\Lambda$ and $r_1 + r_2 + \sum k_i \geq r$.

Lemma 2.36 ([CN17, Lemma 2.36]). *Any element* $x \in \Sigma\Lambda$ *can be uniquely written as* $x = \sum_{\mathbf{k}\in\mathbb{N}^{d+1}} x_{\mathbf{k}}(1-V_1)^{[k_1]}\cdots(1-V_d)^{[k_d]}$ *with* $x_{\mathbf{k}} \in \Lambda$ *for all* $\mathbf{k} = (k_0, \ldots, k_d) \in \mathbb{N}^{d+1}$ *and* $x_{\mathbf{k}} \to 0$ *as* $k \to +\infty$. *Moreover, an element* $x \in \mathrm{Fil}^r\Sigma\Lambda$ *if and only if* $x_{\mathbf{k}} \in \mathrm{Fil}^{r-|\mathbf{k}|}\Lambda$ *for all* $\mathbf{k} \in \mathbb{N}^{d+1}$.

2.8.2. Filtered Poincaré Lemma.

Let $\Omega^1 := \mathbb{Z}\frac{dX_0}{1+X_0} \oplus \left(\oplus_{i=1}^d \mathbb{Z}\frac{dX_i}{X_i}\right)$ and $\Omega^k := \wedge^k\Omega^1$. Therefore, we have $\Omega^k_{\Sigma\Lambda/\Lambda} = \Sigma\Lambda \otimes_{\mathbb{Z}} \Omega^k$. For $r \in \mathbb{Z}$, we have the filtered de Rham complex of $\Sigma\Lambda$:

$$\mathrm{Fil}^r\Omega^\bullet_{\Sigma\Lambda/\Lambda} : \mathrm{Fil}^r\Sigma\Lambda \longrightarrow \mathrm{Fil}^{r-1}\Sigma\Lambda \otimes_{\mathbb{Z}} \Omega^1 \longrightarrow \mathrm{Fil}^{r-2}\Sigma\Lambda \otimes_{\mathbb{Z}} \Omega^2 \longrightarrow \cdots.$$

Now, let D be a finitely generated filtered Λ-module. We set $\Delta := \Sigma\Lambda \otimes_\Lambda D$ and define a filtration on Δ by $\mathrm{Fil}^r\Delta := \sum_{a+b=r}\mathrm{Fil}^a\Sigma\Lambda\widehat{\otimes}_\Lambda\mathrm{Fil}^bD$. Then Δ is a finitely generated filtered $\Sigma\Lambda$-module equipped with an integrable connection $\partial : \Delta \to \Delta \otimes_{\Sigma\Lambda} \Omega^1_{\Sigma\Lambda/\Lambda}$. For the differential operator on $S\Lambda$ we have $\partial(\mathrm{Fil}^k\Sigma\Lambda) \subset \mathrm{Fil}^{k-1}\Sigma\Lambda$, therefore the connection on Δ satisfies Griffiths transversality with respect to the filtration on it. For $r \in \mathbb{Z}$, we have the filtered de Rham complex with coefficients in Δ as

$$\begin{aligned}\mathrm{Fil}^r\Delta \otimes \Omega^\bullet_{\Sigma\Lambda/\Lambda} : \mathrm{Fil}^r\Delta &\longrightarrow \mathrm{Fil}^{r-1}\Delta \otimes_{\Sigma\Lambda} \Omega^1_{\Sigma\Lambda/\Lambda} \longrightarrow \mathrm{Fil}^{r-2}\Delta \otimes_{\Sigma\Lambda} \Omega^2_{\Sigma\Lambda/\Lambda} \longrightarrow \cdots \\ = \mathrm{Fil}^r\Delta &\longrightarrow \mathrm{Fil}^{r-1}\Delta \otimes_{\mathbb{Z}} \Omega^1 \longrightarrow \mathrm{Fil}^{r-2}\Delta \otimes_{\mathbb{Z}} \Omega^2 \longrightarrow \cdots .\end{aligned}$$

Since $\mathrm{Fil}^r D = (\mathrm{Fil}^r\Delta)^{\partial=0}$, we get a filtered Poincaré Lemma:

Lemma 2.37. *The natural map*

$$\mathrm{Fil}^r D \longrightarrow \mathrm{Fil}^r\Delta \otimes \Omega^\bullet_{\Sigma\Lambda/\Lambda}$$

is a quasi-isomorphism.

Proof. We have a natural injection $\epsilon : \mathrm{Fil}^r D \to \mathrm{Fil}^r\Delta$, so we give a contracting (Λ-linear) homotopy. Define

$$\begin{aligned} h^0 : \mathrm{Fil}^r\Delta &\longrightarrow \mathrm{Fil}^r D \\ \sum_{j+k=r} x \otimes a &\longmapsto \sum_{j+k=r} x_0 \otimes a, \end{aligned}$$

where $x \in \mathrm{Fil}^j\Sigma\Lambda$, $a \in \mathrm{Fil}^k D$ and x_0 is the projection to the 0-th component (see Lemma 2.36). Clearly, $h^0\epsilon = id$. For $q > 0$, define the map

$$h^q : \mathrm{Fil}^{j-q}\Delta \otimes \Omega^q \longrightarrow \mathrm{Fil}^{j-q+1}\Delta \otimes \Omega^{q-1}$$

by the formula

$$\begin{aligned} &x \otimes a \prod_{i=0}^{d} (V_i - 1)^{[k_i]} V_{i_1} \frac{dX_{i_1}}{X_{i_1}} \wedge \cdots \wedge V_{i_q} \frac{dX_{i_q}}{X_{i_q}} \\ &\longmapsto \begin{cases} x \otimes a \prod_{i=0}^{d} (V_i - 1)^{[k_i + \delta_{ji_1}]} V_{i_2} \frac{dX_{i_2}}{X_{i_2}} \wedge \cdots \wedge V_{i_q} \frac{dX_{i_q}}{X_{i_q}} & \text{if } k_j = 0 \text{ for } 0 \leq j \leq i_1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We have $\epsilon h^0 + h^1 d = id$ and $dh^q + h^{q+1}d = id$, as desired. ■

Next, let $R_1 = \Sigma = R^\star_\varpi$, $R_2 = \Lambda = \mathbf{A}^\star_{R,\varpi}$ for $\star \in \{\mathrm{PD}, [u], [u,v]\}$, such that $\iota = \iota_{\mathrm{cycl}}$ is an isomorphism of filtered W-algebras, and $R_3 = \Sigma\Lambda$. We set $X_{0,1} = X_0$, $X_{0,2} = \pi_m$ and for $1 \leq i \leq d$, we set $X_{i,1} = X_i$ and $X_{i,2} = [X_i^\flat]$. Now for $j = 1, 2$, we set

$$\Omega^1_j := \mathbb{Z}\tfrac{dX_{0,j}}{1+X_{0,j}} \oplus_{i=1}^{d} \mathbb{Z}\tfrac{dX_{i,j}}{X_{i,j}},$$

and $\Omega^1_3 := \Omega^1_1 \oplus \Omega^1_2$. For $j = 1, 2, 3$, let $\Omega^k_i = \wedge^k \Omega_j$. Therefore, $\Omega^k_{R_j} = R_j \otimes \Omega^k_j$.

Let Δ be a finitely generated filtered R_3-module equipped with a quasi-nilpotent integrable connection satisfying Griffiths transversality with respect to the filtration. In other words, for each $k \in \mathbb{N}$, we have a complex

$$\mathrm{Fil}^k\Delta \otimes \Omega^\bullet_3 : \mathrm{Fil}^k\Delta \xrightarrow{\partial_{R_3}} \mathrm{Fil}^{k-1}\Delta \otimes \Omega^1_3 \xrightarrow{\partial_{R_3}} \mathrm{Fil}^{k-2}\Delta \otimes \Omega^2_3 \xrightarrow{\partial_{R_3}} \cdots .$$

Now, let $D_1 = \Delta^{\partial_2=0}$ be a finitely generated R_1-module equipped with a filtration $\mathrm{Fil}^k D_1 = (\mathrm{Fil}^k\Delta)^{\partial_2=0}$, and a quasi-nilpotent integrable connection satisfying Griffiths transversality with respect to the filtration, i.e. for $k \in \mathbb{Z}$, we have

$$\partial_{R_1} : \mathrm{Fil}^k D_1 \longrightarrow \mathrm{Fil}^{k-1} D_1 \otimes_{\mathbb{Z}} \Omega^1_1,$$

In other words, we obtain a filtered de Rham complex

$$\mathrm{Fil}^k D_1 \otimes \Omega_1^\bullet : \mathrm{Fil}^k D_1 \xrightarrow{\partial_{R_1}} \mathrm{Fil}^{k-1} D_1 \otimes \Omega_1^1 \xrightarrow{\partial_{R_1}} \mathrm{Fil}^{k-2} D_1 \otimes \Omega_1^2 \xrightarrow{\partial_{R_1}} \cdots,$$

Similarly, let $D_2 = \Delta^{\partial_1 = 0}$ be a finitely generated R_2-module equipped with a filtration $\mathrm{Fil}^k D_2 = (\mathrm{Fil}^k \Delta)^{\partial_1 = 0}$, and a quasi-nilpotent integrable connection satisfying Griffiths transversality with respect to the filtration, i.e. for $k \in \mathbb{Z}$, we have

$$\partial_{R_2} : \mathrm{Fil}^k D_2 \longrightarrow \mathrm{Fil}^{k-1} D_2 \otimes_{\mathbb{Z}} \Omega_2^1,$$

In other words, we obtain a filtered de Rham complex

$$\mathrm{Fil}^k D_2 \otimes \Omega_2^\bullet : \mathrm{Fil}^k D_2 \xrightarrow{\partial_{R_2}} \mathrm{Fil}^{k-1} D_2 \otimes \Omega_2^1 \xrightarrow{\partial_{R_2}} \mathrm{Fil}^{k-2} D_2 \otimes \Omega_2^2 \xrightarrow{\partial_{R_2}} \cdots,$$

Proposition 2.38. *The natural maps*

$$\mathrm{Fil}^k D_1 \otimes \Omega_1^\bullet \longrightarrow \mathrm{Fil}^k \Delta \otimes \Omega_3^\bullet \longleftarrow \mathrm{Fil}^k D_2 \otimes \Omega_2^\bullet$$

are quasi-isomorphism of complexes.

Proof. Note that the claim is symmetric in R_1 and R_2, so we only prove the quasi-isomorphism for the map on the left. Since we have $\mathrm{Fil}^k D_1 = (\mathrm{Fil}^k \Delta)^{\partial_{R_2} = 0}$, from Lemma 2.37 we obtain that the sequence

$$0 \longrightarrow \mathrm{Fil}^k D_1 \longrightarrow \mathrm{Fil}^k \Delta \xrightarrow{\partial_{R_2}} \mathrm{Fil}^{k-1} \Delta \otimes \Omega_2^1 \xrightarrow{\partial_{R_2}} \cdots,$$

is exact. We can extend the sequence above to a sequence of maps of de Rham complexes

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Fil}^k D_1 \arrow[r] \arrow[d, "\partial_{R_1}"] & \mathrm{Fil}^k \Delta \arrow[r, "\partial_{R_2}"] \arrow[d, "\partial_{R_1}"] & \mathrm{Fil}^{k-1} \Delta \otimes \Omega_2^1 \arrow[r, "\partial_{R_2}"] \arrow[d, "\partial_{R_1}"] & \cdots \\
0 \arrow[r] & \mathrm{Fil}^k D_1 \otimes \Omega_1^1 \arrow[r] \arrow[d, "\partial_{R_1}"] & \mathrm{Fil}^k \Delta \otimes \Omega_1^1 \arrow[r, "\partial_{R_2}"] \arrow[d, "\partial_{R_1}"] & \mathrm{Fil}^{k-1} \Delta \otimes (\Omega_2^1 \wedge \Omega_1^1) \arrow[r, "\partial_{R_2}"] \arrow[d, "\partial_{R_1}"] & \cdots \\
& \vdots & \vdots & \vdots & .
\end{tikzcd}$$

The contracting homotopy in the proof of Lemma 2.37 is R_1-linear, so it extends as well, which shows that the rows of the double complex above are exact. The total complex of the double complex

$$\mathrm{Fil}^k \Delta \otimes \Omega_1^\bullet \xrightarrow{\partial_{R_2}} \mathrm{Fil}^{k-1} \Delta \otimes (\Omega_2^1 \wedge \Omega_1^\bullet) \xrightarrow{\partial_{R_2}} \cdots,$$

is equal to the de Rham complex $\mathrm{Fil}^k \Delta \otimes \Omega_3^\bullet$. This allows us to conclude. ■

Lemma 2.37 and Proposition 2.38 will play a key role in connecting syntomic complex with coefficients to "Koszul (φ, ∂)-complexes" (see Lemmas 5.31 & 5.32 and Proposition 5.35).

3. Finite height representations

In this section we will recall the notion of a relative Wach module and its relationship with crystalline representations. This notion was studied by the author in [Abh21] following the arithmetic case developed in [Fon90; Wac96; Col99; Ber02].

Recall that we fixed $m \in \mathbb{N}_{\geq 1}$ and we have $K = F_m = F(\zeta_{p^m})$. The element $\varpi = \zeta_{p^m} - 1$ is a uniformizer of K. We have $X = (X_1, \ldots, X_d)$ as a set of indeterminates and we defined R to be the p-adic completion of an étale algebra over $O_F[X, X^{-1}]$ having non-empty and geometrically integral special fiber and $R[\varpi] = O_K \otimes_{O_F} R$. For R and $R[\varpi]$, we can use the (φ, Γ)-module theory discussed in §2.4, as well as the constructions in §2.5 and §2.7.

Notation. For an algebra S admitting an action of the Frobenius and an S-module M admitting a Frobenius-semilinear endomorphism $\varphi : M \to M$, we denote by $\varphi^*(M) \subset M$ the S-submodule generated by the image of φ.

3.1. Relative Wach modules.

Set $q = \frac{\varphi(\pi)}{\pi} \in \mathbf{A}_R^+$ and define relative Wach modules as follows:

Definition 3.1. Let a, $b \in \mathbb{Z}$ with $b \geq a$. A *Wach module* over $\mathbf{A}_R^+$ (resp. $\mathbf{B}_R^+$) with weights in the interval $[a, b]$ is a finite projective $\mathbf{A}_R^+$-module (resp. $\mathbf{B}_R^+$-module) N, equipped with a continuous and semilinear action of Γ_R such that the action of Γ_R is trivial on $N/\pi N$. Further, there is a Frobenius-semilinear operator $\varphi : N[\frac{1}{\pi}] \to N[\frac{1}{\varphi(\pi)}]$ which commutes with the action of Γ_R such that $\varphi(\pi^b N) \subset \pi^b N$ and $\pi^b N/\varphi^*(\pi^b N)$ is killed by q^{b-a}.

Let V be a p-adic representation of the Galois group G_R admitting a $\mathbb{Z}_p$-lattice $T \subset V$ stable under the action of G_R. Then we have a finitely generated $\mathbf{A}_R^+$-submodule $\mathbf{D}^+(T) := (\mathbf{A}^+ \otimes_{\mathbb{Q}_p} T)^{H_R} \subset \mathbf{D}(T)$. We introduce the following definition:

Definition 3.2. A *positive finite q-height* $\mathbb{Z}_p$-representation of G_R is a finite free $\mathbb{Z}_p$-module T admitting a linear and continuous action of G_R such that there exists a finite projective $\mathbf{A}_R^+$-submodule $\mathbf{N}(T) \subset \mathbf{D}^+(T)$ of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$ satisfying the following conditions:

(i) $\mathbf{N}(T)$ is stable under the action of φ and Γ_R, and $\mathbf{A}_R \otimes_{\mathbf{A}_R^+} \mathbf{N}(T) \simeq \mathbf{D}(T)$;

(ii) The $\mathbf{A}_R^+$-module $\mathbf{N}(T)/\varphi^*(\mathbf{N}(T))$ is killed by q^s for some $s \in \mathbb{N}$;

(iii) The action of Γ_R is trivial on $\mathbf{N}(T)/\pi\mathbf{N}(T)$;

(iv) There exists $R' \subset \overline{R}$ finite étale over R such that the $\mathbf{A}_{R'}^+$-module $\mathbf{A}_{R'}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ is free.

The module $\mathbf{N}(T)$ is a Wach module associated to T with weights in the interval $[-s, 0]$. The *height* of T is defined to be the smallest $s \in \mathbb{N}$ satisfying (ii) above.

Furthermore, a positive finite q-height p-adic representation of G_R is a representation admitting a finite q-height $\mathbb{Z}_p$-lattice $T \subset V$ and we set $\mathbf{N}(V) := \mathbf{N}(T)[\frac{1}{p}]$ satisfying properties analogous to (i)-(iv) above. The height of V is defined to be the height of T.

For $r \in \mathbb{Z}$, we set $V(r) := V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(r)$ and $T(r) := T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(r)$. We will call these twists as representations of *finite q-height* and define

$$\mathbf{N}(T(r)) := \tfrac{1}{\pi^r}\mathbf{N}(T)(r) \text{ and } \mathbf{N}(V(r)) := \tfrac{1}{\pi^r}\mathbf{N}(V)(r).$$

Since $\mathbf{N}(V)$ and $\mathbf{N}(T)$ are Wach modules with weights in the interval $[-s, 0]$, twisting by r gives us Wach modules in the sense of Definition 3.1 with weights in the interval $[r - s, r]$. We will say that *height* of $V(r)$ = height of V. For general properties of Wach modules we refer the reader to [Abh21, §4.2].

The operator ψ defined in §2.4 commutes with the action of G_R, so by linearity we extend it to a map $\psi : \mathbf{D}(T) \to \mathbf{D}(T)$ and from Proposition 2.10 we get that $\psi(\mathbf{D}^+(T)) \subset \mathbf{D}^+(T)$.

Lemma 3.3. *Let T be positive finite q-height $\mathbb{Z}_p$-representation of G_R of height s. Then for $r \geq s$, we have $\psi(\mathbf{N}(T(r))) \subset \mathbf{N}(T(r))$.*

Proof. Note that we have $q^s\mathbf{N}(T) \subset \varphi^*(\mathbf{N}(T))$. So for $r \geq s$ and $x \in \mathbf{N}(T(r))$, we must have $\varphi(\pi^r)x = q^r\pi^r x \in \varphi^*(\mathbf{N}(T)(r))$. Therefore, $\psi(x) \in \frac{1}{\pi^r}\mathbf{N}(T)(r) = \mathbf{N}(T(r))$. ■

There is a natural filtration on Wach modules attached to finite q-height representations. We will introduce this filtration next and prove some properties concerning this filtration.

Definition 3.4. Let V be a positive finite q-height represenation of G_R and $r \in \mathbb{N}$. Then there is a natural filtration on the associated Wach modules given as

$$\mathrm{Fil}^k\mathbf{N}(V(r)) := \{x \in \mathbf{N}(V(r)), \text{ such that } \varphi(x) \in q^k\mathbf{N}(V(r))\} \text{ for } k \in \mathbb{Z},$$

and we set $\mathrm{Fil}^k\mathbf{N}(T(r)) := \mathrm{Fil}^k\mathbf{N}(V(r)) \cap \mathbf{N}(T(r)) \subset \mathbf{N}(V(r))$.

Lemma 3.5 ([Abh21, Lemma 4.16])**.** *With notations as above, we have*

(i) $\mathrm{Fil}^k\mathbf{N}(T(r)) = \{x \in \mathbf{N}(T(r)),$ *such that* $\varphi(x) \in q^k\mathbf{N}(T(r))\}$.

(ii) $\mathrm{Fil}^k\mathbf{N}(V(r)) = \mathrm{Fil}^k\pi^{-r}\mathbf{N}(V)(r) = \pi^{-r}\mathrm{Fil}^{k+r}\mathbf{N}(V)(r)$ *and similarly for* $\mathrm{Fil}^k\mathbf{N}(T(r))$.

Lemma 3.6. *Let T be a finite q-height $\mathbb{Z}_p$-representation of G_R such that the $\mathbf{A}_R^+$-module $\mathbf{N}(T)$ is free. Then for $k \in \mathbb{Z}$, we have*

$$\mathrm{Fil}^k\mathbf{N}(T) \cap \pi\mathbf{N}(T) = \pi\mathrm{Fil}^{k-1}\mathbf{N}(T),$$

as submodules of $\mathbf{N}(T)$. Iterating this $j \in \mathbb{N}$ times, we obtain $\mathrm{Fil}^k\mathbf{N}(T) \cap \pi^j\mathbf{N}(T) = \pi^j\mathrm{Fil}^{k-j}\mathbf{N}(T)$. For $V = T[1/p]$, similar statement is true for the $\mathbf{B}_R^+$-module $\mathbf{N}(V)$.

Proof. Using Lemma 3.5, one can reduce to the case of positive finite q-height representations. The claim is obvious if $\mathrm{Fil}^{k-1}\mathbf{N}(T) = \mathbf{N}(T)$. So we assume that $\mathrm{Fil}^{k-1}\mathbf{N}(T) \subsetneq \mathbf{N}(T)$, i.e. $k \geq 2$. Let $x \in \mathrm{Fil}^k\mathbf{N}(T)$ then $x \in \mathrm{Fil}^k\mathbf{N}(T) \cap \pi\mathbf{N}(T)$ if and only if $x = \pi y$ for some $y \in \mathbf{N}(T)$. So $\varphi(x) \in q^k\mathbf{N}(V) \cap \mathbf{N}(T) = q^k\mathbf{N}(T)$ (see Lemma 3.5), where $q = \frac{\varphi(\pi)}{\pi} = p + \pi w$ for some $w \in \mathbf{A}_F^+$. Therefore, $\pi\varphi(y) \in q^{k-1}\mathbf{N}(T)$, i.e. $\pi\varphi(y) = q^{k-1}z$ for some $z \in \mathbf{N}(T)$. So $q^{k-1}z \equiv p^{k-1}z \equiv 0 \mod \pi\mathbf{N}(T)$. However, $\mathbf{N}(T)/\pi\mathbf{N}(T)$ is p-torsion free since $\mathbf{A}_R^+/\pi\mathbf{A}_R^+ \xrightarrow{\sim} R$ and $\mathbf{N}(T)$ is projective over $\mathbf{A}_R^+$. Therefore, π divides z, i.e. $y \in \mathrm{Fil}^{k-1}\mathbf{N}(T)$. The other inclusion is obvious, since $\pi\mathrm{Fil}^{k-1}\mathbf{N}(T) \subset \mathrm{Fil}^k\mathbf{N}(T)$. ■

3.2. Wach modules and crystalline representations. Recall that we have $\mathbf{A}_{R,\varpi}^+$ defined in §2.7, equipped with an action of the Frobenius φ and a continuous action of Γ_R. Since we have a natural injection $\mathbf{A}_{R,\varpi}^+ \rightarrowtail \mathbf{A}_{\mathrm{inf}}(\overline{R})$, we obtain a G_R-equivariant commutative diagram

$$\begin{tikzcd}
\mathbf{A}_{R,\varpi}^+ \arrow[r, two heads, "\theta"] \arrow[d, tail] & R[\varpi] \arrow[d, tail] \\
\mathbf{A}_{\mathrm{inf}}(\overline{R}) \arrow[r, two heads, "\theta"] & \mathbb{C}^+(\overline{R}).
\end{tikzcd}$$

By R-linearlity, extending scalars for the map θ above, we obtain a ring homomorphism

$$\theta_R : R \otimes_{\mathbb{Z}} \mathbf{A}_{R,\varpi}^+ \longrightarrow R[\varpi],$$

sending $X_i \otimes 1 \mapsto X_i$, $1 \otimes [X_i^\flat] \mapsto X_i$ for $1 \leq i \leq d$ and $1 \otimes \pi_m \mapsto \zeta_{p^m} - 1$. Note that we have inclusion of ideals $(\xi, X_i \otimes 1 - 1 \otimes [X_i^\flat], \text{ for } 1 \leq i \leq d) \subset \mathrm{Ker}\ \theta_R \subset R \otimes_{\mathbb{Z}} \mathbf{A}_{R,\varpi}^+$, where $\xi = \frac{\pi}{\pi_1}$.

Definition 3.7. Let $x^{[n]} := x^n/n!$ for $x \in \mathrm{Ker}\ \theta_R$. Define $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ to be the p-adic completion of the divided power envelope of $R \otimes_{O_F} \mathbf{A}_{R,\varpi}^{+}$ with respect to $\mathrm{Ker}\ \theta_R$.

From [Abh21, §4.3], we note that $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is equipped with a Frobenius endomorphism φ, a continuous action of Γ_R and $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$ compatible with all structures. Now let $T = (T_1, \ldots, T_d)$ denote a set of indeterminates and let $\mathbf{A}_{\mathrm{cris}}(\overline{R})\langle T\rangle^{\wedge}$ denote the p-adic completion of the divided power polynomial algebra $\mathbf{A}_{\mathrm{cris}}(\overline{R})\langle T\rangle = \mathbf{A}_{\mathrm{cris}}(\overline{R})[T_i^{[n]},\ n \in \mathbb{N},\ 1 \leq i \leq d]$. Recall from §2.2 that we have an isomorphism of rings

$$\begin{aligned} f_{\mathrm{cris}} : \mathbf{A}_{\mathrm{cris}}(\overline{R})\langle T\rangle^{\wedge} &\xrightarrow{\sim} \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}) \\ T_i &\longmapsto X_i \otimes 1 - 1 \otimes [X_i^{\flat}], \text{ for } 1 \leq i \leq d. \end{aligned}$$

Next, let $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\langle T\rangle^{\wedge}$ denote the p-adic completion of the divided power polynomial algebra $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\langle T\rangle = \mathbf{A}_{R,\varpi}^{\mathrm{PD}}[T_i^{[n]},\ n \in \mathbb{N},\ 1 \leq i \leq d]$. Then the preimage of $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, under f_{cris} is exactly $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\langle T\rangle^{\wedge}$. In other words, we have an isomorphism (see [Abh21, Lemma 4.19])

$$\begin{aligned} f^{\mathrm{PD}} : \mathbf{A}_{R,\varpi}^{\mathrm{PD}}\langle T\rangle^{\wedge} &\xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \\ T_i &\longmapsto X_i \otimes 1 - 1 \otimes [X_i^{\flat}], \text{ for } 1 \leq i \leq d. \end{aligned}$$

There is a natural filtration over the ring $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ by Γ_R-stable submodules:

Definition 3.8. Let $U_i := \frac{1\otimes[X_i^{\flat}]}{X_i\otimes 1}$ for $1 \leq i \leq d$ and $r \in \mathbb{Z}$, define the filtration over $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ as

$$\mathrm{Fil}^r \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} := \Big\langle (a\otimes b) \prod_{i=1}^{d} (U_i - 1)^{[k_i]} \in \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}, \text{ such that } a \in R, b \in \mathrm{Fil}^j \mathbf{A}_{R,\varpi}^{\mathrm{PD}}, \text{ and } j + \sum_i k_i \geq r \Big\rangle.$$

Finally, we have a connection over $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ induced by the connection on $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$,

$$\partial : \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \longrightarrow \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes \Omega_R^1,$$

where we have $\partial\big(X_i \otimes 1 - 1 \otimes [X_i^{\flat}]\big)^{[n]} = \big(X_i \otimes 1 - 1 \otimes [X_i^{\flat}]\big)^{[n-1]} dX_i$. This connection over $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ satisfies Griffiths transversality with respect to the filtration since it does so over $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$.

The main result concerning finite q-height representations is as follows:

Theorem 3.9 ([Abh21, Theorem 4.24, Proposition 4.27]). *Let V be a positive finite q-height representation of G_R, then*

(i) *V is a positive crystalline representation.*

(ii) *We have an isomorphism of $R[\frac{1}{p}]$-modules*

$$\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xleftarrow{\sim} \big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\big)^{\Gamma_R}[\tfrac{1}{p}],$$

compatible with Frobenius, filtration, and connection on each side.

(iii) *After extension of scalars to $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, we obtain a natural isomorphism*

$$\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xleftarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V),$$

compatible with Frobenius, filtration, connection and the action of Γ_R on each side.

Recall that from Definition 3.2 any finite q-height representation is a twist of a positive finite q-height representation by $\mathbb{Q}_p(r)$, for $r \in \mathbb{N}$. Since twist of crystalline representations by $\mathbb{Q}_p(r)$ are again crystalline, we obtain that all finite q-height representations of G_R are crystalline. In [Abh21], the proof of Theorem 3.9 depends on the following important observation:

Lemma 3.10 ([Abh21, Proposition 4.27])**.** *Let V be a positive finite q-height representation of G_R such that $\mathbf{N}(T)$ is free over $\mathbf{A}_R^+$. Then there exists a free R-module $M_0 \subset M = (\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T))^{\Gamma_R}$ such that $M_0[\frac{1}{p}] = M[\frac{1}{p}] \simeq \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ are free modules of rank $= \dim_{\mathbb{Q}_p} V$ over $R[\frac{1}{p}]$.*

Finally, we make an observation which will be useful in §5.

Proposition 3.11. *Let V be a positive finite q-height representation of G_R of height s such that $\mathbf{N}(T)$ is a free over $\mathbf{A}_R^+$. Let $M_0 \subset M = (\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T))^{\Gamma_R}$ be the free R-module obtained in Lemma 3.10. Then, the R-module $M_0/\varphi^*(M_0)$ is killed by p^{ms}.*

Proof. In order to prove the claim, we will use without recalling constructions and notations from proof of [Abh21, Proposition 4.28]. Let $\mathbf{f} = \{f_1, \ldots, f_h\}$ be an $\mathbf{A}_R^+$-basis of $\mathbf{N}(T)$. Then from Lemma 3.10 and proof of [Abh21, Proposition 4.28] M_0 is a free R-module with basis given as $\mathbf{g} = \{g_1, \ldots, g_h\}$, where $\mathbf{g} = \varphi^m(\mathbf{f})\varphi^m(A)$ for $A \in \mathrm{GL}(h, \mathcal{O}\widehat{S}_m^{\mathrm{PD}})$. It is easy to see that M_0 is independent of the choice of an $\mathbf{A}_R^+$-basis of $\mathbf{N}(T)$. Note that $q = \frac{\varphi(\pi)}{\pi} = p\varphi(\frac{\pi}{t})\frac{t}{\pi}$ and since $\frac{\pi}{t}$ is a unit in $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ (see Lemma 2.30) we obtain that q and p are associates in $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. Furthermore, $\mathbf{N}(T)/\varphi^*(\mathbf{N}(T))$ is killed by q^s, where s is the height of V. So $(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T))/\varphi^{m,*}(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T))$ is killed by p^{ms}, where we write $\varphi^{m,*}(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)) = \oplus_{i=1}^h \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}\varphi^m(f_i)$. Recall that $\det A$ is a unit in $\mathcal{O}\widehat{S}_m^{\mathrm{PD}}$ (see [Abh21, Lemma 4.43]), therefore $\varphi^m(\det A)$ is a unit in $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ and $\varphi(A)$ is invertible over $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, therefore $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M_0 \xrightarrow{\sim} \varphi^{m,*}(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T))$. Thus, cokernel of the natural inclusion $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M_0 \subset \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ is killed by p^{ms}. It also implies that cokernel of the natural inclusion $\varphi^{m,*}(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M_0) \subset \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M_0 \xrightarrow{\sim} \varphi^{m,*}(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T))$ is killed by p^{ms}. In other words, we have $p^{ms}(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M_0) \subset \varphi^{m,*}(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M_0) \subset \varphi^*(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M_0)$. Finally, we note that the action of Frobenius commutes with the action of Γ_R, therefore taking Γ_R-invariants, we obtain that $p^{ms}M_0 \subset \varphi^*(M_0)$, i.e. $M_0/\varphi^*(M_0)$ is killed by p^{ms}. ■

Remark 3.12. From the proof of Proposition 3.11, we have $p^s(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)) \subset \varphi^*(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T))$. Taking Γ_R invariants, we get that $p^sM \subset \varphi^*(M)$. Furthermore, putting Lemma 3.10 and Proposition 3.11 together we obtain that the cokernel of the natural injection $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M \to \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ is killed by p^{ms}.

Remark 3.13. Using Theorem 3.9 (ii), we equip $M \subset M[\frac{1}{p}]$ with a p-adically quasi-nilpotent integrable connection $\partial : M \to M \otimes_R \Omega_R^1$. Moreover, M is equipped with an induced filtration compatible with the tensor product filtration (see [Abh21, §4.5.1]) and the connection satisfies Griffiths transversality with respect to the filtration. Furthermore, using the explicit description of M_0 in Proposition 3.11 it follows that M_0 is stable under the induced connection (since the connection is trivial over $\mathbf{N}(T)$). In particular, we obtain a p-adically quasi-nilpotent integrable connection $\partial : M_0 \to M_0 \otimes_R \Omega_R^1$. Finally, we equip $M_0 \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ with the induced filtration and the connection ∂ satisfies Griffiths transversality with respect to the filtration.

Remark 3.14. Note that we fixed a choice of $m \in \mathbb{N}_{\geq 1}$ in the beginning. The R-modules that we have obtained above depend on this choice. In particular, let $1 \leq m \leq m'$ with $\varpi = \zeta_{p^m} - 1$ and $\varpi' = \zeta_{p^{m'}} - 1$. Then we have that $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathcal{O}\mathbf{A}_{R,\varpi'}^{\mathrm{PD}}$ and $M = (\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T))^{\Gamma_R}$ and $M' = (\mathcal{O}\mathbf{A}_{R,\varpi'}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T))^{\Gamma_R}$. Furthermore, let M_0 and M_0' be the R-modules obtained for m and m' respectively in Lemma 3.10. We note that $\varphi^{m'-m}(M') \subset M$ and $\varphi^{m'-m}(M_0') \subset M_0$ (this esentially follows from the fact that $\varphi^{m'-m}(\mathcal{O}\widehat{S}_{m'}^{\mathrm{PD}}) \subset \mathcal{O}\widehat{S}_m^{\mathrm{PD}}$ in the notation of the proof of [Abh21, Proposition 4.27]).

Remark 3.15. In the case when $\mathbf{N}(T)$ is a free $\mathbf{A}_R^+$- module of rank h, from Lemma 3.10 we obtain that $M_0[\frac{1}{p}] = M[\frac{1}{p}] \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is a free $R[\frac{1}{p}]$-module of rank h. In particular, for finite q-height representations there exists a finite étale extension R' over R such that $R'[\frac{1}{p}] \otimes_{R[\frac{1}{p}]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ is a free module of rank h.

3.3. Relative Fontaine-Laffaille modules. In this section we will recall from [Abh21, §5] the fact that finite free relative Fontaine-Laffaille modules give rise to finite q-height representations in a natural way. Explicitly, we consider the category $\mathrm{MF}_{[0,s],\,\mathrm{free}}(R, \Phi, \partial)$ defined by [Tsu20, §4] as a full subcategory of the abelian category $\mathfrak{MF}^{\nabla}_{[0,s]}(R)$ introduced by Faltings in [Fal89, §II]. Let $s \in \mathbb{N}$ such that $s \leq p-2$.

Definition 3.16. Define the category of *free relative Fontaine-Laffaille* modules of level $[0, s]$, denoted by $\mathrm{MF}_{[0,s],\,\mathrm{free}}(R, \Phi, \partial)$, as follows:
An object with weights in the interval $[0, s]$ is a quadruple $(M, \mathrm{Fil}^\bullet M, \partial, \Phi)$ such that,

(i) M is a free R-module of finite rank.

(ii) M is equipped with a decreasing filtration $\{\mathrm{Fil}^k M\}_{k\in\mathbb{Z}}$ by finite R-submodules with $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{s+1} M = 0$ such that $\mathrm{gr}^k_{\mathrm{Fil}} M$ is a finite free R-module for every $k \in \mathbb{Z}$.

(iii) The connection $\partial : M \to M \otimes_R \Omega^1_R$ is quasi-nilpotent and integrable, and satisfies Griffiths transversality with respect to the filtration, i.e. $\partial(\mathrm{Fil}^k M) \subset \mathrm{Fil}^{k-1} M \otimes_R \Omega^1_R$ for $k \in \mathbb{Z}$.

(iv) Let $(\varphi^*(M), \varphi^*(\partial))$ denote the pullback of (M, ∂) by $\varphi : R \to R$, and equip it with a decreasing filtration $\mathrm{Fil}^k_p(\varphi^*(M)) = \sum_{i\in\mathbb{N}} p^{[i]} \varphi^*(\mathrm{Fil}^{k-i} M)$ for $k \in \mathbb{Z}$. We suppose that there is an R-linear morphism $\Phi : \varphi^*(M) \to M$ such that Φ is compatible with connections, $\Phi(\mathrm{Fil}^k_p(\varphi^*(M))) \subset p^k M$ for $0 \leq k \leq s$, and $\sum_{k=0}^{s} p^{-k} \Phi(\mathrm{Fil}^k_p(\varphi^*(M))) = M$. We denote the composition $M \to \varphi^*(M) \xrightarrow{\Phi} M$ by φ.

A morphism between two objects of the category $\mathrm{MF}_{[0,s],\,\mathrm{free}}(R, \Phi, \partial)$ is a continuous R-linear map compatible with the homomorphism Φ and the connection ∂ on each side.

Notation. By a slight abuse of notations, we will denote $(M, \mathrm{Fil}^k M, \partial, \Phi) \in \mathrm{MF}_{[0,s],\,\mathrm{free}}(R, \Phi, \partial)$ by M and say that it is of level $[0, s]$.

To an object $M \in \mathrm{MF}_{[0,s],\,\mathrm{free}}(R, \varphi, \mathrm{Fil})$, let us associate a $\mathbb{Z}_p$-module as

$$T^*_{\mathrm{cris}}(M) := \mathrm{Hom}_{R,\,\mathrm{Fil},\,\varphi,\,\partial}(M, \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})), \tag{3.1}$$

i.e. R-linear maps from M to $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$ compatible with filtration, Frobenius and connection.

Proposition 3.17 ([Fal89], [Tsu20])**.** (i) *For a free Fontaine-Laffaille module M of level $[0, s]$, the $\mathbb{Z}_p$-module $T^*_{\mathrm{cris}}(M)$ is a free module of rank $= \mathrm{rk}_R M$ equipped with a continuous action of G_R. Further, the p-adic representation $V^*_{\mathrm{cris}}(M) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T^*_{\mathrm{cris}}(M)$ is a crystalline representation of G_R with Hodge-Tate weights in the interval $[0, s]$.*

(ii) *The contravariant $\mathbb{Z}_p$-linear functor*

$$T^*_{\mathrm{cris}} : \mathrm{MF}_{[0,s],\,\mathrm{free}}(R, \Phi, \partial) \longrightarrow \mathrm{Rep}_{\mathbb{Z}_p,\,\mathrm{free}}(G_R),$$

is fully faithful. Here $\mathrm{Rep}_{\mathbb{Z}_p,\,\mathrm{free}}(G_R)$ denotes the category of finite free $\mathbb{Z}_p$-modules equipped with a continuous action of G_R.

Definition 3.18. Let M be a free relative Fontaine-Laffaille module of level $[0, s]$, and set

$$T_{\mathrm{cris}}(M) := \mathrm{Hom}_{\mathbb{Z}_p}(T^*_{\mathrm{cris}}(M), \mathbb{Z}_p),$$

which is a free $\mathbb{Z}_p$-module of rank $= \mathrm{rk}_R M$, admitting a continuous action of G_R.

The main result connecting Fontaine-Laffaille modules and finite q-height representations is as follows:

Theorem 3.19 ([Abh21, Theorem 5.4])**.** *For a free relative Fontaine-Laffaille module M over R of level $[0, s]$, the associated p-adic representation $V_{\mathrm{cris}}(M) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_{\mathrm{cris}}(M)$ of G_R is a positive finite q-height representation (in the sense of Definition 3.2).*

Remark 3.20. (i) The results of [Abh21] are shown for the case $s = p - 2$. However, all the arguments can be adapted almost verbatim (by replacing $p - 2$ everywhere by any $0 \leq s \leq p - 2$).

(ii) For a free relative Fontaine-Laffaille module M over R of level $[0, s]$ and the associated $\mathbb{Z}_p$-representation $T = T_{\mathrm{cris}}(M)$ of G_R, from Theorem 3.19 we obtain a free relative Wach module $\mathbf{N}(T)$ over $\mathbf{A}_R^+$. Moreover, combining [Abh21, Propositions 5.23 & 5.27] and the proof of [Abh21, Theorem 5.4], we obtain a natural isomorphism

$$\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T) \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M,$$

compatible with filtration, Frobenius and the action of Γ_R on each side. In low Hodge-Tate weights $0 \leq s \leq p - 2$, this statement is a strictly stronger integral version of the comparison obtained in Theorem 3.9.

(iii) From the proof of [Abh21, Theorem 5.4], one can observe that $M/\varphi^*(M)$ is p^s-torsion and s equals the maximum among the absolute value of Hodge-Tate weights of $V_{\mathrm{cris}}(M)$.

Remark 3.21. In Defintion 3.16, we considered finite free modules over R. For the R/p^n-module M/p^n the associated $\mathbb{Z}/p^n$-representation of G_R is given as $T_{\mathrm{cris}}(M/p^n) = T_{\mathrm{cris}}(M)/p^n$. Moreover, we will associate a Wach module to $T/p^n = T_{\mathrm{cris}}(M)/p^n$ as $\mathbf{N}(T/p^n) := \mathbf{N}(T)/p^n$. In this case, we again have a natural isomorphism

$$\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}/p^n \otimes_{\mathbf{A}_R^+/p^n} \mathbf{N}(T/p^n) \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}/p^n \otimes_{R/p^n} M/p^n,$$

compatible with filtration, Frobenius and the action of Γ_R on each side (see [Abh21, §5.3]).

4. Galois cohomology complexes

By the equivalence between the category of $\mathbb{Z}_p$-representations of G_F and étale (φ, Γ_F)-modules over $\mathbf{A}_F$ (see §2.4), it is natural to expect that the continuous cohomology groups of a $\mathbb{Z}_p$-representation T could be computed using a complex written in terms of (φ, Γ_K)-module $\mathbf{D}(T)$. This question was first answered in the article of Herr (see [Her98]) where we have a three term complex which computes the continuous cohomology of the representation in each cohomological degree. More precisely,

Theorem 4.1 (Fontaine-Herr). *Let T be a $\mathbb{Z}_p$-representation of G_F, and let $\mathbf{D}(T)$ denote the associated étale (φ, Γ_F)-module over $\mathbf{A}_F$. Then we have a complex*

$$\mathscr{C}^\bullet : \mathbf{D}(T) \xrightarrow{(1-\varphi, \gamma-1)} \mathbf{D}(V) \oplus \mathbf{D}(T) \xrightarrow{\binom{\gamma-1}{1-\varphi}} \mathbf{D}(T),$$

where the second map is $(x, y) \mapsto (\gamma-1)x - (1-\varphi)y$. The complex $\mathscr{C}^\bullet$ computes the continuous cohomology of T in each cohomological degree, i.e. for $k \in \mathbb{N}$, we have natural isomorphims

$$H^k(\mathscr{C}^\bullet) \xrightarrow{\sim} H^k_{\mathrm{cont}}(G_F, V).$$

Before discussing the relative case, let us introduce some shorthand notation for writing certain complexes.

Notation. Let $f : C_1 \to C_2$ be a morphism of complexes. The *mapping cone* of f is the complex $\mathrm{Cone}(f)$ whose degree n part is given as $C_1^{n+1} \oplus C_2^n$ and the differential is given by $d(c_1, c_2) = (-d(c_1), d(c_2) - f(c_1))$. Next, we denote the *mapping fiber* of f by

$$[C_1 \xrightarrow{f} C_2] := \mathrm{Cone}(f)[-1].$$

We also set

$$\begin{bmatrix} C_1 & \xrightarrow{f} & C_2 \\ \downarrow & & \downarrow \\ C_3 & \xrightarrow{g} & C_4 \end{bmatrix} := [[C_1 \xrightarrow{f} C_2] \longrightarrow [C_3 \xrightarrow{g} C_4]].$$

In other words, this amounts to taking the total complex of the associated double complex.

Using the notation introduced above, we can also write the quasi-isomorphism of complexes in Theorem 4.1 as

$$[\mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_F, \mathbf{D}(V)) \xrightarrow{1-\varphi} \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_F, \mathbf{D}(V))] \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cont}}(G_F, V).$$

4.1. Relative Fontaine-Herr complex. Now we turn our attention towards the relative case, where we have R as the p-adic completion of an étale algebra over a torus and G_R as its absolute Galois group. Similar to Theorem 4.1, we have results in the relative case where a complex of (φ, Γ)-modules computes the continuous $G_{R,\varpi}$-cohomology of a p-adic representation of $G_{R,\varpi}$. For this reason, we consider the continuous cohomology (for the weak topology) of (φ, Γ_R)-modules over $\mathbf{A}_R$ and $\mathbf{A}_R^\dagger$ (see §2.4).

Definition 4.2. Let D be an étale (φ, Γ_R)-module over $\mathbf{A}_R$ or $\mathbf{A}_R^\dagger$. Define $\mathcal{C}^\bullet(\Gamma_R, D)$ to be the complex of continuous cochains with values in D and let $\mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, D)$ denote this complex in the derived category of abelian groups.

Let T be a $\mathbb{Z}_p$-module, equipped with a continuous and linear action of G_R. Let $\mathbf{D}(T)$ and $\mathbf{D}^\dagger(T)$ denote the associated (φ, Γ_R)-module over $\mathbf{A}_R$ and $\mathbf{A}_R^\dagger$, respectively. Then we have,

Theorem 4.3 ([AI08, Theorem 7.10.6])**.** *The natural maps*

$$\mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{D}(T)) \longrightarrow \mathrm{R}\Gamma_{\mathrm{cont}}(G_R, T \otimes_{\mathbb{Z}_p} \mathbf{A}_{\overline{R}}),$$
$$\mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{D}^\dagger(T)) \longrightarrow \mathrm{R}\Gamma_{\mathrm{cont}}(G_R, T \otimes_{\mathbb{Z}_p} \mathbf{A}_{\overline{R}}^\dagger),$$

are isomorphisms.

Moreover, from [AI08, Proposition 8.1] we have that the sequence

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbf{A}_{\overline{R}} \xrightarrow{1-\varphi} \mathbf{A}_{\overline{R}} \longrightarrow 0$$

is exact and remains exact if we replace $\mathbf{A}_{\overline{R}}$ above with $\mathbf{A}_{\overline{R}}^\dagger$, $\mathbf{A}$ or $\mathbf{A}^\dagger$. Combining the short exact sequence above with Theorem 4.3 and by explicit computations, Andreatta and Iovita have shown that

Theorem 4.4 ([AI08, Theorem 3.3])**.** *There are isomorphisms of δ-functors from the category* $\mathrm{Rep}_{\mathbb{Z}_p}(G_R)$ *to the category of abelian groups*

$$\beta : [\mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{D}(-)) \xrightarrow{1-\varphi} \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{D}(-))] \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cont}}(G_R, -),$$
$$\beta^\dagger : [\mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{D}^\dagger(-)) \xrightarrow{1-\varphi} \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{D}^\dagger(-))] \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cont}}(G_R, -).$$

Furthermore, for $T \in \mathrm{Rep}_{\mathbb{Z}_p}(G_R)$, the natural inclusion of (φ, Γ_R)-modules $\mathbf{D}^\dagger(T) \subset \mathbf{D}(T)$ induces a natural isomorphism

$$[\mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{D}^\dagger(-)) \xrightarrow{1-\varphi} \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{D}^\dagger(-))] \xrightarrow{\sim} [\mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{D}(-)) \xrightarrow{1-\varphi} \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{D}(-))],$$

compatible with β and $\beta^\dagger$.

Remark 4.5. The discussion above remains valid if we replace R by $S = R[\varpi]$ for $\varpi = \zeta_{p^m} - 1$, G_R by G_S, Γ_R by $\Gamma_S = \Gamma'_R \rtimes \Gamma_K$ and considering complexes in terms of étale (φ, Γ_S)-modules over the period rings $\mathbf{A}_S$ and $\mathbf{A}_S^\dagger$ respectively.

4.2. Koszul complexes. Recall that we have $K = F(\zeta_{p^m})$ for some $m \in \mathbb{N}_{\geq 1}$. Let $S = R[\varpi]$ for $\varpi = \zeta_{p^m} - 1$ a uniformizer of K. From §2.4 we know that the ring $S_\infty[\frac{1}{p}] = R_\infty[\frac{1}{p}]$ is a Galois extension of $S[\frac{1}{p}]$, with Galois group Γ_S fitting into an exact sequence

$$1 \longrightarrow \Gamma'_S \longrightarrow \Gamma_S \longrightarrow \Gamma_K \longrightarrow 1,$$

and we have topological generators $\{\gamma_0, \gamma_1, \ldots, \gamma_d\}$ of Γ_S such that $\{\gamma_1, \ldots, \gamma_d\}$ are topological generators of $\Gamma'_S = \Gamma'_R$ and γ_0 is a lift of a topological generator of Γ_K. Further, let χ denote the p-adic cyclotomic character and recall the convention that $c = \chi(\gamma_0) = \exp(p^m)$.

In this section, we will introduce Koszul complexes which will be used to compute continuous Γ_S-cohomology of topological modules admitting a continuous action of Γ_R, in particular, (φ, Γ_R)-modules (see Remark 4.5). Koszul complexes have the advantage of being explicit and therefore easier to manipulate. We will follow the exposition in [CN17, §4.2].

Let us set $\tau_i = \gamma_i - 1$ for $1 \leq i \leq d$. We consider the case of an Iwasawa algebra $A = \mathbb{Z}_p[[\tau_1, \ldots, \tau_d]]$.

Definition 4.6. The *Koszul complex* associated to $(\tau_1, \ldots, \tau_d)$ is the complex

$$K(\tau_1, \ldots, \tau_d) = K(\tau_1)\widehat{\otimes}_{\mathbb{Z}_p} K(\tau_2)\widehat{\otimes}_{\mathbb{Z}_p} \cdots \widehat{\otimes}_{\mathbb{Z}_p} K(\tau_d),$$

where $K(\tau_i)$ is the complex

$$0 \longrightarrow \mathbb{Z}_p[[\tau_i]] \xrightarrow{\tau_i} \mathbb{Z}_p[[\tau_i]] \longrightarrow 0,$$

where the non-trivial map is multiplication by τ_i and the right-hand term is placed in degree 0.

Remark 4.7. The Koszul complex defined above, in degree q, equals the exterior power $\wedge^q A^d$. In the standard basis $\{e_{i_1\cdots i_q}\}$ of $\wedge^q A^d$ for $1 \leq i_1 < \cdots < i_q \leq d$, the differential $d_q^1 : \wedge^q A^d \to \wedge^{q-1} A^d$ is given by the formula

$$d_q^1(a_{i_1\cdots i_q}) = \sum_{k=1}^{q} (-1)^{k+1} a_{i_1\cdots \widehat{i_k} \cdots i_q} \tau_{i_k}. \tag{4.1}$$

The augmentation map $A \to \mathbb{Z}_p$ makes $K(\tau_1, \ldots, \tau_d)$ into a resolution of $\mathbb{Z}_p$ in the category of topological A-modules.

We can use this to define Koszul complex for modules equipped with an action of Γ'_S or Γ_S. Let $\mathbb{Z}_p[[\Gamma'_S]]$ denote the Iwasawa algebra of Γ'_S, i.e. the completed group ring

$$\mathbb{Z}_p[[\Gamma'_S]] := \lim_{H \trianglelefteq \Gamma'_S} \mathbb{Z}_p[\Gamma'_S/H],$$

where the (projective) limit runs over all open normal subgroups H of Γ'_S and every group ring $\mathbb{Z}_p[\Gamma'_S/H]$ is equipped with the p-adic topology. We have $\mathbb{Z}_p[[\Gamma'_S]] \simeq \mathbb{Z}_p[[\tau_1, \ldots, \tau_d]]$, $\tau_i = \gamma_i - 1$ for $i \in \{1, \ldots, d\}$.

Definition 4.8. The Koszul complex $K(\tau_1, \ldots, \tau_d)$ is given as

$$0 \longrightarrow \mathbb{Z}_p[[\Gamma'_S]]^{I'_d} \xrightarrow{d_{d-1}^1} \cdots \xrightarrow{d_1^1} \mathbb{Z}_p[[\Gamma'_S]]^{I'_1} \xrightarrow{d_0^1} \mathbb{Z}_p[[\Gamma'_S]] \longrightarrow 0,$$

where $I'_j = \{(i_1, \ldots, i_j),\ 1 \leq i_1 < \cdots < i_j \leq d\}$ and differentials as in (4.1). Similarly, for $c = \chi(\gamma_0) = \exp(p^m)$ we can define the Koszul complex $K(\tau_1^c, \ldots, \tau_d^c)$ (with differentials d_q^c), where $\tau_i^c := \gamma_i^c - 1$.

Both $K(\tau_1, \ldots, \tau_d)$ and $K(\tau_1^c, \ldots, \tau_d^c)$ are resolutions of $\mathbb{Z}_p$ in the category of $\mathbb{Z}_p[[\Gamma'_S]]$-modules.

Definition 4.9. Let $\Lambda := \mathbb{Z}_p[[\Gamma'_S]]$, and define the complex

$$K(\Lambda) : 0 \longrightarrow \Lambda^{I'_d} \xrightarrow{d_{d-1}^1} \cdots \xrightarrow{d_1^1} \Lambda^{I'_1} \xrightarrow{d_0^1} \Lambda \longrightarrow 0.$$

We have an isomorphism

$$\lim_m \mathbb{Z}_p\left[\Gamma_K/\left(\Gamma_K\right)^{p^m}\right] \otimes_{\mathbb{Z}_p} K(\tau_1, \ldots, \tau_d) \simeq K(\Lambda),$$

of left Λ- and right $\mathbb{Z}_p[[\tau_1, \ldots, \tau_d]]$-modules (see [Mor08, Lemma 4.3]). Therefore, the complex $K(\Lambda)$ is a resolution of $\mathbb{Z}_p[[\Gamma_K]]$ in the category of topological left Λ-modules. Similarly, we have the complex $K^c(\Lambda)$, obtained from $K(\tau_1^c, \ldots, \tau_d^c)$, which is again a resolution of $\mathbb{Z}_p[[\Gamma_K]]$.

Definition 4.10. Define a map

$$\tau_0 : K^c(\Lambda) \longrightarrow K(\Lambda),$$

by the commutative diagram of topological left Λ-modules

$$\begin{tikzcd}
0 \arrow[r] & \Lambda^{I'_d} \arrow[r, "d_{d-1}^c"] \arrow[d, "\tau_0^d"] & \cdots \arrow[r, "d_1^c"] & \Lambda^{I'_1} \arrow[r, "d_0^c"] \arrow[d, "\tau_0^1"] & \Lambda \arrow[r] \arrow[d, "\tau_0^0"] & \mathbb{Z}_p[[\Gamma_K]] \arrow[r] \arrow[d, "\gamma_0 - 1"] & 0 \\
0 \arrow[r] & \Lambda^{I'_d} \arrow[r, "d_{d-1}^1"] & \cdots \arrow[r, "d_1^1"] & \Lambda^{I'_1} \arrow[r, "d_0^1"] & \Lambda \arrow[r] & \mathbb{Z}_p[[\Gamma_K]] \arrow[r] & 0,
\end{tikzcd}$$

where the vertical maps are defined as

$$\tau_0^0 = \gamma_0 - 1$$
$$\tau_0^q : (a_{i_1\cdots i_q}) \mapsto (a_{i_1\cdots i_q}(\gamma_0 - \delta_{i_1\cdots i_q})),$$

for $1 \leq q \leq d$, $1 \leq i_1 < \cdots < i_q \leq d$, and $\delta_{i_1\cdots i_q} = \delta_{i_q}\cdots\delta_{i_1}$, where $\delta_{i_j} = (\gamma_{i_j}^c - 1)(\gamma_{i_j} - 1)^{-1}$.

Let M be a topological $\mathbb{Z}_p$-module admitting a continuous action of Γ_S.

Definition 4.11. Define the complexes

$$\mathrm{Kos}(\Gamma_S', M) := \mathrm{Hom}_{\Lambda,\mathrm{cont}}(K(\Lambda), M) = \mathrm{Hom}_\Lambda(K(\Lambda), M),$$
$$\mathrm{Kos}^c(\Gamma_S', M) := \mathrm{Hom}_{\Lambda,\mathrm{cont}}(K^c(\Lambda), M) = \mathrm{Hom}_\Lambda(K^c(\Lambda), M).$$

Remark 4.12. Using Definition 4.9, we can write the complexes in Definition 4.11 as

$$\mathrm{Kos}(\Gamma_S', M) : M \xrightarrow{(\tau_i)} M^{I_1'} \longrightarrow \cdots \longrightarrow M^{I_d'},$$

$$\mathrm{Kos}^c(\Gamma_S', M) : M \xrightarrow{(\tau_i^c)} M^{I_1'} \longrightarrow \cdots \longrightarrow M^{I_d'}.$$

The map $\tau_0 : K^c(\Lambda) \longrightarrow K(\Lambda)$ induces a map of complexes

$$\tau_0 : \mathrm{Kos}(\Gamma_S', M) \longrightarrow \mathrm{Kos}^c(\Gamma_S', M),$$

which can be represented by the commutative diagram

$$\begin{tikzcd}
M \arrow[r, "(\tau_i)"] \arrow[d, "\tau_0^0"] & M^{I_1'} \arrow[r] \arrow[d, "\tau_0^1"] & \cdots \arrow[r] & M^{I_d'} \arrow[d, "\tau_0^d"] \\
M \arrow[r, "(\tau_i^c)"] & M^{I_1'} \arrow[r] & \cdots \arrow[r] & M^{I_d'}.
\end{tikzcd}$$

Let $K(\Lambda, \tau) := [K^c(\Lambda) \xrightarrow{\tau_0} K(\Lambda)]$. This complex is a resolution of $\mathbb{Z}_p$ in the category of topological left Λ-modules.

Definition 4.13. Define the Γ_S-*Koszul complex* with values in M as

$$\mathrm{Kos}(\Gamma_S, M) := \mathrm{Hom}_{\Lambda,\mathrm{cont}}(K(\Lambda, \tau), M) = [\mathrm{Kos}(\Gamma_S', M) \xrightarrow{\tau_0} \mathrm{Kos}^c(\Gamma_S', M)].$$

By the general theory of continuous group cohomology for p-adic Lie groups, we have the following conclusion:

Proposition 4.14 ([Laz65, Lazard])**.** *There exists a natural quasi-isomorphism*

$$\mathrm{Kos}(\Gamma_S, M) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_S, M).$$

Definition 4.15. Let D be an étale (φ, Γ_S)-module over $\mathbf{A}_S = \mathbf{A}_{R,\varpi}$ from Definition 2.6. Define the complex

$$\mathrm{Kos}(\varphi, \Gamma_S, D) := \left[\begin{tikzcd}
\mathrm{Kos}(\Gamma_S', D) \arrow[r, "1-\varphi"] \arrow[d, "\tau_0"] & \mathrm{Kos}(\Gamma_S', D) \arrow[d, "\tau_0"] \\
\mathrm{Kos}^c(\Gamma_S', D) \arrow[r, "1-\varphi"] & \mathrm{Kos}^c(\Gamma_S', D)
\end{tikzcd}\right].$$

Therefore, from Proposition 4.14 we have a natural quasi-isomorphism

$$\mathrm{Kos}(\varphi, \Gamma_S, D) \xrightarrow{\sim} [\mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_S, D) \xrightarrow{1-\varphi} \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_S, D)].$$

Using the definition above, we have the following conclusion for $\mathbb{Z}_p$-representations of G_S:

Proposition 4.16. *Let T be a $\mathbb{Z}_p$-representation of G_S and $\mathbf{D}(T)$ the associated (φ, Γ_S)-module over $\mathbf{A}_S = \mathbf{A}_{R,\varpi}$ (see §2.4). Then from the discussion above, Theorem 4.4 and Remark 4.5 we obtain a natural quasi-isomorphism*

$$\mathrm{Kos}(\varphi, \Gamma_S, \mathbf{D}(T)) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cont}}(G_S, T).$$

4.3. Lie algebra cohomology. In this section we will study the infinitesimal action of Γ_S on some of the rings constructed in previous sections (recall that $S = R[\varpi]$ for $\varpi = \zeta_{p^m} - 1$ with $m \in \mathbb{N}_{\geq 1}$). This will help us in computing continuous Lie algebra cohomology of certain $\mathbb{Z}_p[[\mathrm{Lie}\ \Gamma_S]]$-modules, which is roughly the same as continuous Lie group cohomology of these modules. Recall from the previous section that we have topological generators $\{\gamma_0, \gamma_1, \ldots, \gamma_d\}$ of Γ_S such that $\{\gamma_1, \ldots, \gamma_d\}$ are topological generators of Γ'_S and γ_0 is a lift of a topological generator of Γ_K.

In the rest of this section we will fix constants $u, v \in \mathbb{R}$ such that $\frac{p-1}{p} \leq u \leq \frac{v}{p} < 1 < v$, for example, one can fix $u = \frac{p-1}{p}$ and $v = p-1$.

4.3.1. Convergence of operators. Recall from §2.7 that we have rings $\mathbf{A}_S^{\mathrm{PD}} = \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, $\mathbf{A}_S^{[u]} = \mathbf{A}_{R,\varpi}^{[u]}$ and $\mathbf{A}_S^{[u,v]} = \mathbf{A}_{R,\varpi}^{[u,v]}$ equipped with a continuous action of Γ_S. For the sake of consistency with §2.7, we will continue to use the latter notation.

Lemma 4.17. *For $i \in \{0, 1, \ldots, d\}$ the operators*

$$\nabla_i := \log \gamma_i = \sum_{k \in \mathbb{N}} (-1)^k \tfrac{(\gamma_i - 1)^{k+1}}{k+1},$$

converge as series of operators on $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, $\mathbf{A}_{R,\varpi}^{[u]}$ and $\mathbf{A}_{R,\varpi}^{[u,v]}$.

Proof. From Lemma 2.31, we have that for $k \geq 0$,

$$(\gamma_0 - 1)(p^m, \pi_m^{p^m})^k \mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset (p^m, \pi_m^{p^m})^{k+1} \mathbf{A}_{R,\varpi}^{\mathrm{PD}}.$$

Using the fact that $\gamma_0 - 1$ acts as a twisted derivation (i.e. for $x, y \in \mathbf{A}_{R,\varpi}^+$ we have $(\gamma_0 - 1)xy = (\gamma_0 - 1)x \cdot y + \gamma_0(x)(\gamma_0 - 1)y)$, we conclude that for $x \in \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$,

$$(\gamma_0 - 1)^k x \in (p^m, \pi_m^{p^m})^k \mathbf{A}_{R,\varpi}^{\mathrm{PD}}. \tag{4.2}$$

To check that the series

$$\nabla_0(x) = \sum_{k \in \mathbb{N}} (-1)^k \tfrac{(\gamma_0 - 1)^{k+1}(x)}{k+1}$$

converges in $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, it is enough to show that for $k \in \mathbb{N}$ and $0 \leq j \leq k$, the p-adic valuation of $\frac{p^{m(k-j)}}{k}(\lfloor \frac{p^m j}{e} \rfloor !)$ goes to $+\infty$ as $k \to +\infty$. The p-adic valuation of this term is

$$m(k-j) + \upsilon_p\Big(\tfrac{\lfloor p^m j/e \rfloor !}{k}\Big) \geq m(k-j) - \tfrac{k}{p-1} + \upsilon_p(\lfloor \tfrac{pj}{p-1} \rfloor !) \geq \tfrac{pm-m-1}{p-1}(k-j) + \upsilon_p(\lfloor \tfrac{j}{p-1} \rfloor !) - 1,$$

where the last inequality follows from an easy computation following Remark 4.19. Clearly we have that the sum above goes to $+\infty$ as $k \to +\infty$. Therefore, $\nabla_0(x)$ converges in $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$.

Next, consider γ_i for $i \in \{1, \ldots, d\}$. Again from Lemma 2.31 we have that for $k \geq 0$,

$$(\gamma_i - 1)(p^m, \pi_m^{p^m})^k \mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset (p^m, \pi_m^{p^m})^{k+1} \mathbf{A}_{R,\varpi}^{\mathrm{PD}}.$$

Using the fact that $\gamma_i - 1$ acts as a twisted derivation, we conclude that for $x \in \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$,

$$(\gamma_i - 1)^k x \in (p^m, \pi_m^{p^m})^k \mathbf{A}_{R,\varpi}^{\mathrm{PD}}. \tag{4.3}$$

Therefore, using a similar estimate as in the case of γ_0 we conclude that the following series converges

$$\nabla_i(x) = \sum_{k \in \mathbb{N}} (-1)^k \tfrac{(\gamma_i - 1)^{k+1}(x)}{k+1} \in \mathbf{A}_{R,\varpi}^{\mathrm{PD}}.$$

The arguments in the case of $\mathbf{A}_{R,\varpi}^{[u]}$ and $\mathbf{A}_{R,\varpi}^{[u,v]}$ follow similarly (use Lemma 2.32 for $\mathbf{A}_{R,\varpi}^{[u,v]}$). ∎

Remark 4.18. One can explicitly check that the series $\nabla_0(\pi_m)$ converges in $\mathbf{A}_{F,\varpi}^{\mathrm{PD}}$. Similar to above, we have

$$(\gamma_0 - 1)^k \pi_m \subset (p^m, \pi_m^{p^m})^k \mathbf{A}_{F,\varpi}^{\mathrm{PD}}.$$

So to check that the series $\nabla_0(\pi_m)$ converges over $\mathbf{A}_{F,\varpi}^{\mathrm{PD}}$ we write it as $\sum_j c_j \pi_m^j$ and we collect the coefficients of $\pi_m^{p^m k}$ for $k \geq 1$, having the smallest p-adic valuation, which will also have the least p-adic valuation among the coefficients of π_m^j for $p^m k \leq j \leq p^m(k+1)$. We write the collection of these terms as

$$\sum_{k \geq 1} (-1)^{k+1} \frac{\pi_m^{p^m k}}{k} = \sum_{k \geq 1} (-1)^{k+1} \frac{\lfloor p^m k/e \rfloor !}{k} \frac{\pi_m^{p^m k}}{\lfloor p^m k/e \rfloor !},$$

and it is enough to show that these coefficients go to 0 as $k \to +\infty$. Let $k = (p-1)a + b$ with $0 \leq b < p-1$, then by Remark 4.19 we have

$$\upsilon_p\Big(\tfrac{\lfloor p^m k/e \rfloor !}{k}\Big) = \upsilon_p\Big(\lfloor \tfrac{pk}{p-1} \rfloor !\Big) - \upsilon_p(k) \geq \upsilon_p((pa+b)!) - \upsilon_p(k) \geq \tfrac{pa - s_p(pa)}{p-1} - \tfrac{k}{p-1} \geq \upsilon_p(a!) + \tfrac{(p-2)a}{p-1} - 1,$$

which goes to $+\infty$ as $k \to +\infty$.

The following elementary observation was used above,

Remark 4.19. Let $n \in \mathbb{N}$, so we can write $n = \sum_{i=0}^k n_i p^i$ for some $k \in \mathbb{N}$, where $0 \leq n_i \leq p-1$ for $0 \leq i \leq k$. Let us set $s_p(n) = \sum_{i=0}^k n_i$. Then we have

$$\begin{aligned} \upsilon_p(n!) &= \sum_{j \geq 1} \lfloor \tfrac{n}{p^j} \rfloor = \sum_{j \geq 0} \lfloor \tfrac{\sum_{i=0}^k n_i p^i}{p^j} \rfloor = \sum_{j=1}^k \sum_{i=j}^k n_i p^{i-j} \\ &= \sum_{i=1}^k n_i \sum_{j=1}^i p^j = \sum_{i=1}^k n_i \tfrac{p^i - 1}{p-1} = \tfrac{n - s_p(n)}{p-1}. \end{aligned}$$

Also, note that we have $s_p(pn) = s_p(n)$ for any $n \in \mathbb{N}$.

Note that formally we can write

$$\begin{aligned} \frac{\log(1+X)}{X} &= 1 + a_1 X + a_2 X^2 + a_3 X^3 + \cdots, \\ \frac{X}{\log(1+X)} &= 1 + b_1 X + b_2 X^2 + b_3 X^3 + \cdots, \end{aligned}$$

where $\upsilon_p(a_k) \geq -\frac{k}{p-1}$ for all $k \geq 1$ and therefore, $\upsilon_p(b_k) \geq -\frac{k}{p-1}$ for all $k \geq 1$. Setting $X = \gamma_i - 1$ for $i \in \{0, 1, \ldots, d\}$, we make the following claim:

Lemma 4.20. *For $i \in \{0, 1, \ldots, d\}$, the operators*

$$\frac{\nabla_i}{\gamma_i - 1} = \frac{\log \gamma_i}{\gamma_i - 1} \quad \textit{and} \quad \frac{\gamma_i - 1}{\nabla_i} = \frac{\gamma_i - 1}{\log \gamma_i}$$

converge as series of operators on $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, $\mathbf{A}_{R,\varpi}^{[u]}$ and $\mathbf{A}_{R,\varpi}^{[u,v]}$.

Proof. We will only show that these series converge on $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, the case of $\mathbf{A}_{R,\varpi}^{[u]}$ and $\mathbf{A}_{R,\varpi}^{[u,v]}$ follow similarly (use Lemma 2.32 for $\mathbf{A}_{R,\varpi}^{[u,v]}$). Moreover, we have $\upsilon_p(a_k) \geq -\frac{k}{p-1}$ and $\upsilon_p(b_k) \geq -\frac{k}{p-1}$ for all $k \geq 1$, so it is enough to show the convergence of $\frac{\gamma_i - 1}{\log \gamma_i}$.

From Lemma 2.31, we have that for $k \geq 1$,

$$(\gamma_i - 1)(p^m, \pi_m^{p^m})^k \mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset (p^m, \pi_m^{p^m})^{k+1} \mathbf{A}_{R,\varpi}^{\mathrm{PD}}.$$

Using the fact that $\gamma_i - 1$ acts as a twisted derivation and (4.2) and (4.3), we have

$$(\gamma_i - 1)^k x \in \left(p^m, \pi_m^{p^m}\right)^k \mathbf{A}_{R,\varpi}^{\mathrm{PD}}.$$

To check that the series

$$\sum_{k\in\mathbb{N}} (-1)^k b_k (\gamma_i - 1)^k x$$

converges in $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, it is enough to show that for $0 \leq j \leq k$, the p-adic valuation of $b_k p^{m(k-j)}(\lfloor \frac{p^m j}{e} \rfloor !)$ goes to $+\infty$ as $k \to +\infty$. The p-adic valuation of this term is

$$m(k-j) + \upsilon_p\Big(b_k \lfloor \tfrac{p^m j}{e} \rfloor !\Big) \geq m(k-j) - \tfrac{k}{p-1} + \upsilon_p(\lfloor \tfrac{pj}{p-1} \rfloor !) \geq \tfrac{pm-m-1}{p-1}(k-j) + \upsilon_p(\lfloor \tfrac{j}{p-1} \rfloor !) - 1,$$

where the last inequality follows from an easy computation following Remark 4.19. Clearly we have that the sum above goes to $+\infty$ as $k \to +\infty$. Therefore, $\frac{\gamma_i - 1}{\log \gamma_i}(x)$ converges in $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. ■

4.3.2. Koszul Complexes for Lie Γ_S. In this section, we turn our attention to the computation of Lie algebra cohomology using Koszul complexes. The Lie algebra Lie Γ'_S of the p-adic Lie group Γ'_S is a free $\mathbb{Z}_p$-module of rank d, i.e. Lie $\Gamma'_S = \mathbb{Z}_p[\nabla_i]_{1\leq i\leq d}$ with

$$\nabla_i := \log \gamma_i = \sum_{k\in\mathbb{N}} (-1)^k \tfrac{(\gamma_i - 1)^{k+1}}{k+1} : M \longrightarrow M,$$

for any Lie Γ'_S-module M. Moreover, Lie Γ'_S is commutative. Similarly, the Lie algebra Lie Γ_S of the p-adic Lie group Γ_S is a free $\mathbb{Z}_p$-module of rank $d+1$, i.e. Lie $\Gamma_S = \mathbb{Z}_p[\nabla_i]_{0\leq i\leq d}$ (∇_i defined as above for $0 \leq i \leq d$). We have

$$\begin{aligned} [\nabla_i, \nabla_j] &= 0, &&\text{for } \ 1 \leq i, j \leq d, \\ [\nabla_0, \nabla_i] &= p^m \nabla_i, &&\text{for } \ 1 \leq i \leq d. \end{aligned} \tag{4.4}$$

It follows that Lie Γ_S is not commutative.

Let M be a topological $\mathbb{Z}_p$-module admitting a continuous action of the Lie algebra Lie Γ_S. Similar to the definition of Koszul complexes in the case of Γ_S (see §4.2), we define Koszul complexes for Lie Γ_S.

Definition 4.21. Define the complex

$$\mathrm{Kos}(\mathrm{Lie}\ \Gamma'_S, M) : M \longrightarrow M^{I'_1} \longrightarrow \cdots \longrightarrow M^{I'_d},$$

with differentials dual to those in (4.1) (with τ_i replaced by ∇_i).

Now, consider the map

$$\nabla_0 : \mathrm{Kos}(\mathrm{Lie}\ \Gamma'_S, M) \longrightarrow \mathrm{Kos}(\mathrm{Lie}\ \Gamma'_S, M),$$

defined by the diagram

$$\begin{tikzcd} M \arrow[r, "(\nabla_i)"] \arrow[d, "\nabla_0"] & M^{I'_1} \arrow[r] \arrow[d, "\nabla_0 - p^m"] & \cdots \arrow[r] & M^{I'_r} \arrow[r] \arrow[d, "\nabla_0 - rp^m"] & \cdots \\ M \arrow[r, "(\nabla_i)"] & M^{I'_1} \arrow[r] & \cdots \arrow[r] & M^{I'_r} \arrow[r] & \cdots, \end{tikzcd}$$

which commutes since $\nabla_0\nabla_i - \nabla_i\nabla_0 = p^m\nabla_i$ for $1 \leq i \leq d$ (see (4.4)). Note that the k-th vertical arrow is $\nabla_0 - kp^m$ since the $(k-1)$-th vertical arrow is $\nabla_0 - (k-1)p^m$ and using (4.4) we have $(\nabla_0 - kp^m)\nabla_i = \nabla_i(\nabla_0 - (k-1)p^m)$.

Definition 4.22. Define the Lie Γ_S-*Koszul complex* for M as

$$\mathrm{Kos}(\mathrm{Lie}\ \Gamma_S, M) := \left[\mathrm{Kos}(\mathrm{Lie}\ \Gamma'_S, M) \xrightarrow{\nabla_0} \mathrm{Kos}(\mathrm{Lie}\ \Gamma'_S, M)\right].$$

Proposition 4.23 ([Laz65, Lazard])**.** *The Koszul complexes in Definitions 4.21 and 4.22 compute Lie algebra cohomology of* Lie Γ'_S *and* Lie Γ_S *respectively, with values in M. In other words, we have natural quasi-isomorphisms*

$$\mathrm{R}\Gamma_{\mathrm{cont}}(\mathrm{Lie}\ \Gamma'_S, M) \simeq \mathrm{Kos}(\mathrm{Lie}\ \Gamma'_S, M),$$
$$\mathrm{R}\Gamma_{\mathrm{cont}}(\mathrm{Lie}\ \Gamma_S, M) \simeq \mathrm{Kos}(\mathrm{Lie}\ \Gamma_S, M).$$

5. Syntomic complex and finite height representations

The goal of current and next section is to compare syntomic complexes with coefficient and relative Fontaine-Herr complex computing continuous Galois cohomology of a finite height representation. For the coefficient of syntomic complex, we will use relative Wach modules and its comparison with the associated F-isocrystal as stated in Theorem 3.9. Our result and methods are greatly inspired by the computations done by Colmez and Nizioł in [CN17]. So before introducing our main result, let us recall the result of Colmez and Nizioł.

We will assume the setup of §2. Recall that we fixed $p \geq 3$, F to be a finite unramified extension of $\mathbb{Q}_p$ and $K = F(\zeta_{p^m})$, where $m \in \mathbb{N}_{\geq 1}$ and we let $\varpi = \zeta_{p^m} - 1$ be a uniformizer of K. Further, we take R to be the p-adic completion of an étale algebra over d-dimensional torus and $S = R[\varpi]$. From §2.5, we also have rings $r_\varpi^\star$ and $R_\varpi^\star$ for $\star \in \{\ , +, \mathrm{PD}, [u], (0, v]+, [u, v]\}$. Throughout this section, we will assume $u = \frac{p-1}{p}$ and $v = p - 1$. The p-adic completion of the module of differentials of R relative to $\mathbb{Z}$ is given as

$$\Omega_R^1 = \oplus_{i=1}^d R\, d\log X_i \ \text{ and } \ \Omega_R^k = \bigwedge_R^k \Omega_R^1, \ \text{ for } \ k \in \mathbb{N}.$$

Moreover, we have a natural isomorphism $\Omega_R^k \otimes_R S \to \Omega_S^k$, i.e.

$$\Omega_S^k = \bigwedge_S^k \big(\oplus_{i=1}^d S\, d\log X_i \big).$$

Also, for $R_\varpi^\star$ where $\star \in \{+, \mathrm{PD}, [u], [u, v]\}$, we have

$$\Omega_{R_\varpi^\star}^1 = R_\varpi^\star \tfrac{dX_0}{1+X_0} \oplus \big(\oplus_{i=1}^d R_\varpi^\star\, d\log X_i \big).$$

The syntomic cohomology of S can be computed by the complex

$$\mathrm{Syn}(S, r) := \mathrm{Cone}\big(F^r \Omega_{R_\varpi^{\mathrm{PD}}}^\bullet \xrightarrow{p^r - p^\bullet \varphi} \Omega_{R_\varpi^{\mathrm{PD}}}^\bullet\big)[-1],$$

such that we have $H_{\mathrm{syn}}^i(S, r) = H^i(\mathrm{Syn}(S, r))$.

Remark 5.1. Note that R is formally smooth over O_F, so the syntomic complex for R can be defined using $\Omega_R^\bullet$. However, one can also define syntomic complex for R as above: one needs to replace the ring R_ϖ^{PD} by the divided power envelope of the surjective map $R_\varpi^+ \twoheadrightarrow R$ sending $X_0 \to 0$ (note that this map does not depend on ϖ and therefore neither does the divided power envelope). In the statement of Theorem 5.2, by abuse of notations, we use the latter definition to include the case of R.

Theorem 5.2 ([CN17, Theorems 1.1 & 1.6])**.** *Consider the natural maps*

$$\begin{aligned} &\alpha_r^{\mathcal{L}az} : \tau_{\leq r}\mathrm{Syn}(S, r) \longrightarrow \tau_{\leq r}\mathrm{R\Gamma}_{\mathrm{cont}}(G_S, \mathbb{Z}_p(r)), \\ &\alpha_{r,n}^{\mathcal{L}az} : \tau_{\leq r}\mathrm{Syn}(S, r)_n \longrightarrow \tau_{\leq r}\mathrm{R\Gamma}_{\mathrm{cont}}(G_S, \mathbb{Z}/p^n(r)) \longrightarrow \tau_{\leq r}\mathrm{R\Gamma}\big((\mathrm{Sp}\ S[\tfrac{1}{p}])_{\text{ét}}, \mathbb{Z}/p^n(r)\big). \end{aligned} \tag{5.1}$$

(i) *If K contains enough roots of unity, i.e. for m large enough, the maps in (5.1) are p^{Nr+c_p}-quasi-isomorphisms for a universal constant $N \in \mathbb{N}$ (not depending on p, R, K, n, r) and a constant c_p depending only on p.*

(ii) *In general, the kernel and cokernel of the maps (5.1) are annihilated by p^N for $N = N(K, p, r) \in \mathbb{N}$ but not depending on R (and not on n for mod p^n complexes).*

Note that the truncation here denotes the canonical truncation in literature.

Remark 5.3. (i) To be very precise, in Theorem 5.2 either one should use log versions of the syntomic complex following [CN17] or one should truncate in degrees $\leq r - 1$ following Theorem 5.8.

(ii) In Theorem 5.2, the statement in (ii) is a consequence of (i). More precisely, if K does not contain enough roots of unity then one passes to a larger extension containing enough roots of unity and uses Galois descent to obtain the statement for K (see [CN17, §5.1.6]). In particular, Theorem 5.2 can be obtained for base field F and base ring R.

5.1. Formulation of the main result. Considering Theorem 5.2 for R, in the first map $\alpha_r^{\mathcal{L}\mathrm{az}}$ of (5.1) we would like to insert a $\mathbb{Z}_p$-representation T of G_R on the right hand side (resp. T/p^n in the second map $\alpha_{r,n}^{\mathcal{L}\mathrm{az}}$) and an appropriate syntomic object (resp. mod p^n syntomic object) on the left. To realize this goal, let us consider the following class of representations:

Assumption 5.4. Let V be a positive finite q-height representation of G_R (see Definition 3.2). Assume that the Wach module $\mathbf{N}(T)$ is free of rank $= \dim_{\mathbb{Q}_p} V$ over $\mathbf{A}_R^+$ and let $M \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ be a free R-submodule of rank $= \dim_{\mathbb{Q}_p} V$ such that $M[\frac{1}{p}] = \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ and the induced connection over M is p-adically quasi-nilpotent, integrable and satisfies Griffiths transversality with respect to the induced filtration. Furthermore, assume that $p^s M \subset \varphi^*(M)$ and there exists a p^N-isomorphism $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M \simeq \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ with $N = n(T,e) \in \mathbb{N}$ for $e = [K:F] = p^{m-1}(p-1)$ and compatible with Frobenius, filtration, connection and Γ_R-action.

Example 5.5. (i) Assuming that $\mathbf{N}(T)$ is a free $\mathbf{A}_R^+$-module, from Proposition 3.11 and Remark 3.13 we have that the R-module $M := M_0$ (in the notation of the proposition) satisfies Assumption 5.4 with $n(T,e) = ms$ where $m = v_p(e) + 1$.

(ii) Let $M = (\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T))^{\Gamma_R}$ with an additional assumption that it is free over R of rank $= \dim_{\mathbb{Q}_p} V$. Then, the module M depends on T and $m \in \mathbb{N}_{\geq 1}$ (see Remark 3.14) and satisfies the Assumption 5.4 with $n(T,e) = ms$ (see Remarks 3.12 & 3.13).

(iii) For our intended global applications to relative Fontaine-Laffaille modules, we note that for representations arising from finite free relative Fontaine-Laffaille modules of level $[0,s]$ with $s \leq p-2$ as in §3.3, the Assumption 5.4 is automatically satisfied, with M being the relative Fontaine-Laffaille module (see Remark 3.20) and $n(T,e) = 0$.

Our objective is to relate the (φ,Γ)-module complex computing the continuous G_R-cohomology of $T(r)$ (see Theorem 4.4), to syntomic complex with coefficients in the R-lattice $M \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$.

Let us first consider the case of cyclotomic extension $S = R[\varpi]$. From §2.5 we have the divided power ring $R_\varpi^{\mathrm{PD}} \twoheadrightarrow S$ and consider the following finite free module

$$M_\varpi^{\mathrm{PD}} := R_\varpi^{\mathrm{PD}} \otimes_R M.$$

The R_ϖ^{PD}-module M_ϖ^{PD} is equipped with a Frobenius-semilinear endomorphism φ given by the diagonal action of the Frobenius on each component of the tensor product, and a filtration given as

$$\mathrm{Fil}^k M_\varpi^{\mathrm{PD}} = \text{closure of } \sum_{i+j=k} \mathrm{Fil}^i R_\varpi^{\mathrm{PD}} \otimes_R \mathrm{Fil}^j M \subset M_\varpi^{\mathrm{PD}}, \quad \text{for } k \in \mathbb{Z}. \tag{5.2}$$

Furthermore, M_ϖ^{PD} is equipped with a connection

$$\begin{aligned} \partial : M_\varpi^{\mathrm{PD}} &\to M_\varpi^{\mathrm{PD}} \otimes_{R_\varpi^{\mathrm{PD}}} \Omega^1_{R_\varpi^{\mathrm{PD}}}, \\ a \otimes x &\longmapsto a \otimes \partial_M(x) + x da, \end{aligned}$$

arising from the connection on M and the differential operator on R_ϖ^{PD}. Moreover, the connection on M_ϖ^{PD} satistfies Griffiths transversality with respect to the filtration. In particular, for $r \in \mathbb{Z}$, we have a filtered de Rham complex

$$\mathrm{Fil}^r \mathcal{D}_{S,M}^\bullet : \mathrm{Fil}^r M_\varpi^{\mathrm{PD}} \longrightarrow \mathrm{Fil}^{r-1} M_\varpi^{\mathrm{PD}} \otimes_{R_\varpi^{\mathrm{PD}}} \Omega^1_{R_\varpi^{\mathrm{PD}}} \longrightarrow \mathrm{Fil}^{r-2} M_\varpi^{\mathrm{PD}} \otimes_{R_\varpi^{\mathrm{PD}}} \Omega^2_{R_\varpi^{\mathrm{PD}}} \longrightarrow \cdots. \tag{5.3}$$

Next, we describe the action of Frobenius on $\Omega^1_{R_\varpi^{\mathrm{PD}}}$. We fix a basis of $\Omega^1_{R_\varpi^{\mathrm{PD}}}$ as $\{\frac{dX_0}{1+X_0}, \frac{dX_1}{X_1}, \ldots, \frac{dX_d}{X_d}\}$. For $j \in \mathbb{N}$, let $I_j = \{0 \leq i_1 < \cdots < i_j \leq d\}$ and for $\mathbf{i} = (i_1, \ldots, i_j) \in I_j$, let

$$\omega_{\mathbf{i}} = \begin{cases} \frac{dX_0}{1+X_0} \wedge \frac{dX_{i_2}}{X_{i_2}} \wedge \cdots \wedge \frac{dX_{i_j}}{X_{i_j}} & \text{if } i_1 = 0, \\ \frac{dX_{i_1}}{X_{i_1}} \wedge \cdots \wedge \frac{dX_{i_j}}{X_{i_j}} & \text{otherwise.} \end{cases}$$

We define operators φ and ψ on $\Omega^j_{R_\varpi^{\mathrm{PD}}}$ by

$$\varphi\big(\sum_{\mathbf{i} \in I_j} x_{\mathbf{i}}\omega_{\mathbf{i}}\big) = \sum_{\mathbf{i} \in I_j} \varphi(x_{\mathbf{i}})\omega_{\mathbf{i}} \text{ and } \psi\big(\sum_{\mathbf{i} \in I_j} x_{\mathbf{i}}\omega_{\mathbf{i}}\big) = \sum_{\mathbf{i} \in I_j} \psi(x_{\mathbf{i}})\omega_{\mathbf{i}}. \tag{5.4}$$

Remark 5.6. Note that this is not the natural definition of Frobenius, as we have $d(\varphi(x)) = p\varphi(dx)$ by the definition above. But in order to define ψ integrally, we need to divide the usual Frobenius on $\Omega^1_{R_\varpi^\star}$ by powers of p. Furthermore, recall that from §2.3 with the usual definition of Frobenius we have $\varphi\partial = \partial\varphi$ over $M \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$. However, using (5.4) for Ω^1_R as well, we note that for $f \in M$, we now have $\partial_M(\varphi(f)) = \sum_{i=1}^d \partial_i(\varphi(f))\omega_i = \sum p\varphi(\partial_i(f))\omega_i = p\varphi(\partial_M(f))$.

Definition 5.7. Let $r \in \mathbb{N}$ and consider the complex $\mathrm{Fil}^r\mathscr{D}^\bullet_{S,M}$ as above. Define the *syntomic complex* $\mathrm{Syn}(S, M, r)$ and the *syntomic cohomology* of S with coefficients in M as

$$\begin{aligned} \mathrm{Syn}(S, M, r) &:= \big[\, \mathrm{Fil}^r\mathscr{D}^\bullet_{S,M} \xrightarrow{p^r - p^\bullet\varphi} \mathscr{D}^\bullet_{S,M} \,\big]; \\ H^*_{\mathrm{syn}}(S, M, r) &:= H^*(\mathrm{Syn}(S, M, r)). \end{aligned}$$

For $n \in \mathbb{N}$, let $S_n = S \otimes \mathbb{Z}/p^n$ and $M_n = M \otimes \mathbb{Z}/p^n$. Define the modulo p^n *syntomic complex* and *syntomic cohomology* of S with coefficients in M as

$$\begin{aligned} \mathrm{Syn}(S, M, r)_n &:= \mathrm{Syn}(S, M, r) \otimes \mathbb{Z}/p^n; \\ H^*_{\mathrm{syn}}(S_n, M_n, r) &:= H^*(\mathrm{Syn}(S, M, r)_n). \end{aligned}$$

Our objective is to relate the syntomic complex with coefficients in Definition 5.7 to the relative Fontaine-Herr complex computing the continuous G_S-cohomology of $T(r)$ (see §4.1). The key idea is to interpret both the complexes in terms of Koszul complexes, and by applying a version of Poincaré lemma, we can further relate the "differential Koszul complexes" to "(φ, Γ)-module Koszul complexes". The main local result is as follows:

Theorem 5.8. *Let V be a p-adic finite q-height representation of G_R of height s, $T \subset V$ a G_R-stable $\mathbb{Z}_p$-lattice and satisfying Assumption 5.4, and let $r \in \mathbb{Z}$ such that $r \geq s+1$. Then there exists p^N-quasi-isomorphisms*

$$\begin{aligned} \alpha_r^{\mathcal{L}az} &: \tau_{\leq r-s-1}\mathrm{Syn}(S, M, r) \simeq \tau_{\leq r-s-1}\mathrm{R}\Gamma_{\mathrm{cont}}(G_S, T(r)), \\ \alpha_{r,n}^{\mathcal{L}az} &: \tau_{\leq r-s-1}\mathrm{Syn}(S, M, r)_n \simeq \tau_{\leq r-s-1}\mathrm{R}\Gamma_{\mathrm{cont}}(G_S, T/p^n(r)), \end{aligned}$$

where $N = N(T, e, r) \in \mathbb{N}$ depending on the representation T, the absolute ramification index e of K and the twist r.

Remark 5.9. Sections §5 and §6 are devoted to the proof of Theorem 5.8. Almost all of the statements and proofs in these two sections are valid for $m \geq 1$. However, in Lemmas 6.19 and 6.14, one needs to assume $m \geq 2$. But to conclude Theorem 5.8, one can pass to $R[\zeta_{p^2} - 1]$ to obtain the claim and then apply Galois descent twice as in Lemma 6.26 (also see Corollary 5.12).

Before proceeding with the proof of Theorem 5.8, let us recall that we are interested in obtaining a similar statement over R. This will be achieved in essentially the same way as (ii) is obtained from (i) in Theorem 5.2 (using Galois descent, see Remark 5.3). To state the result in a more precise manner, let us first introduce the syntomic complex over R with coefficients in M.

Recall that R is the p-adic completion of an étale algebra over $O_F[X, X^{-1}]$, in particular, it is smooth over O_F. Furthermore, the finite free R-module M is equipped with a Frobenius-semilinear endomorphism φ an induced filtration and an induced integrable connection satisfying Griffiths transversality with respect to the filtration. In particular, for $r \in \mathbb{Z}$, we have a filtered de Rham complex

$$\mathrm{Fil}^r \mathscr{D}^\bullet_{R,M} : \mathrm{Fil}^r M \longrightarrow \mathrm{Fil}^{r-1} M \otimes_R \Omega^1_R \longrightarrow \mathrm{Fil}^{r-2} M \otimes_R \Omega^2_R \longrightarrow \cdots . \tag{5.5}$$

Remark 5.10. One can also consider the formulation of filtered de Rham complex for R as in (5.3). In that case one considers a surjection $R^+_\varpi \twoheadrightarrow R$ via the map $X_0 \mapsto 0$. Let R_0^{PD} denote the p-adic completion of the divided power envelope and set $M_0^{\mathrm{PD}} = R_0^{\mathrm{PD}} \otimes_R M$ equipped with tensor product filtration. Then we have the filtered de Rham complex

$$\mathrm{Fil}^r \mathscr{E}^\bullet_{R,M} : \mathrm{Fil}^r M_0^{\mathrm{PD}} \longrightarrow \mathrm{Fil}^{r-1} M_0^{\mathrm{PD}} \otimes_{R_0^{\mathrm{PD}}} \Omega^1_{R_0^{\mathrm{PD}}} \longrightarrow \mathrm{Fil}^{r-2} M_0^{\mathrm{PD}} \otimes_{R_0^{\mathrm{PD}}} \Omega^2_{R_0^{\mathrm{PD}}} \longrightarrow \cdots ,$$

and a quasi-isomorphism $\mathrm{Fil}^r \mathscr{E}^\bullet_{R,M} \simeq \mathrm{Fil}^r \mathscr{D}^\bullet_{R,M}$.

Definition 5.11. Let $r \in \mathbb{N}$ and consider the complex $\mathrm{Fil}^r \mathscr{D}^\bullet_{R,M}$ as above. Define the *syntomic complex* $\mathrm{Syn}(R, M, r)$ and the *syntomic cohomology* of R with coefficients in M as

$$\begin{aligned} \mathrm{Syn}(R, M, r) &:= \big[\, \mathrm{Fil}^r \mathscr{D}^\bullet_{R,M} \xrightarrow{p^r - p^\bullet \varphi} \mathscr{D}^\bullet_{R,M} \,\big]; \\ H^*_{\mathrm{syn}}(R, M, r) &:= H^*(\mathrm{Syn}(R, M, r)). \end{aligned}$$

For $n \in \mathbb{N}$, let $R_n = R \otimes \mathbb{Z}/p^n$ and $M_n = M \otimes \mathbb{Z}/p^n$. Define the modulo p^n *syntomic complex* and *syntomic cohomology* of R with coefficients in M as

$$\begin{aligned} \mathrm{Syn}(R, M, r)_n &:= \mathrm{Syn}(R, M, r) \otimes \mathbb{Z}/p^n; \\ H^*_{\mathrm{syn}}(R_n, M_n, r) &:= H^*(\mathrm{Syn}(R, M, r)_n). \end{aligned}$$

Using Theorem 5.8 for $\varpi = \zeta_{p^2} - 1$ (in particular, Example 5.5(ii) for $m = 2$) and Corollary 6.25 (by applying Galois descent in Lemma 6.26 for $e = p(p-1)$), we conclude that

Corollary 5.12. *Let V be a finite q-height representation of G_R of height s, $T \subset V$ a G_R-stable $\mathbb{Z}_p$-lattice and satisfying Assumption 5.4, and let $r \in \mathbb{Z}$ such that $r \geq s+1$. Then there exists p^N-quasi-isomorphisms*

$$\begin{aligned} \tau_{\leq r-s-1} \mathrm{Syn}(R, M, r) &\simeq \tau_{\leq r-s-1} \mathrm{R}\Gamma_{\mathrm{cont}}(G_R, T(r)), \\ \tau_{\leq r-s-1} \mathrm{Syn}(R, M, r)_n &\simeq \tau_{\leq r-s-1} \mathrm{R}\Gamma_{\mathrm{cont}}(G_R, T/p^n(r)), \end{aligned}$$

where $N = N(p, r, s) \in \mathbb{N}$ depending on the prime p, the twist r and the height s of the representation V.

Now let us turn to the proof of Theorem 5.8. We will mainly proceed by proving the first p-power-quasi-isomorphism, i.e. the p-adic case. The modulo p^n case follows in a similar manner and we will point out the main differences (wherever they may occur). The proof of Theorem 5.8 will proceed in two main steps: First, we will modify the syntomic complex with coefficients in M to relate it to a “differential” Koszul complex with coefficients in $\mathbf{N}(T)$ (see Proposition 5.35). Next, in the second step we will modify the Koszul complex from the first step to obtain Koszul complex computing continuous G_S-cohomology of $T(r)$ (see Definition 5.8 and Proposition 6.1). The key to the connection between these two steps will be provided by the comparison isomorphism in Theorem 3.9.

5.2. Syntomic complex with coefficients. In this section we will carry out computations involving syntomic complexes in order to prove Theorem 5.8. More precisely, we will define syntomic complexes with coefficients in M, over various rings introduced in §2.5. Moreover, we will relate these complexes to differential Koszul complex with coefficients in $\mathbf{N}(T)$. Further computations clarifying relations between differential Koszul complex and relative Fontaine-Herr will be worked out in §6.

We begin by fixing some notations for the rest of this section. For $\star \in \{[u], [u,v], [u, v/p]\}$, we define a finite free module over $R_\varpi^\star$

$$M_\varpi^\star := R_\varpi^\star \otimes_R M.$$

By considering the diagonal action of the Frobenius on each component of the tensor product, we can define Frobenius-semilinear operators $\varphi : M_\varpi^{[u]} \to M_\varpi^{[u]}$ and $\varphi : M_\varpi^{[u,v]} \to M_\varpi^{[u,v/p]}$. We equip $M_\varpi^\star$ with a filtration

$$\mathrm{Fil}^k M_\varpi^\star = \text{closure of } \sum_{i+j=k} \mathrm{Fil}^i R_\varpi^\star \otimes_R \mathrm{Fil}^j M \subset M_\varpi^\star, \quad \text{for } k \in \mathbb{Z}. \tag{5.6}$$

Further, if ∂_M denotes the connection on M then we can equip M_ϖ with a connection

$$\begin{aligned} \partial : M_\varpi^\star &\longrightarrow M_\varpi^\star \otimes \Omega^1_{R_\varpi^\star} \\ a \otimes x &\longmapsto a \otimes \partial_M(x) + x da, \end{aligned}$$

satisfying Griffiths transversality with respect to the filtration, since the differential operator on $R_\varpi^\star$ as well as ∂_M satisfy this condition. In particular, for $r \in \mathbb{Z}$, we have a filtered de Rham complex,

$$\mathrm{Fil}^r \mathscr{D}^\bullet_{R_\varpi^\star, M} := \mathrm{Fil}^r M_\varpi^\star \longrightarrow \mathrm{Fil}^{r-1} M_\varpi^\star \otimes \Omega^1_{R_\varpi^\star} \longrightarrow \mathrm{Fil}^{r-2} M_\varpi^\star \otimes \Omega^2_{R_\varpi^\star} \longrightarrow \cdots. \tag{5.7}$$

Moreover, for $\star \in \{[u], [u,v], [u, v/p]\}$, we define operators φ and ψ on $\Omega^j_{R_\varpi^\star}$ as in (5.4).

Now we are ready to define syntomic cohomology with coefficients. From (5.7), let $\mathscr{D}^\bullet_{R_\varpi^\star, M}$ denote the de Rham complex with $\star \in \{[u], [u,v]\}$ and $\mathscr{E}^\bullet_{R_\varpi^\star, M}$ denote the de Rham complex with coefficients in the module which are target under the Frobenius-semilinear operator φ, i.e. $\star \in \{[u], [u, v/p]\}$.

Definition 5.13. Define the *syntomic complex* $\mathrm{Syn}(M_\varpi^\star, r)$ and the *syntomic cohomology* of with coefficients in $M_\varpi^\star$ as

$$\begin{aligned} \mathrm{Syn}(M_\varpi^\star, r) &:= [\, \mathrm{Fil}^r \mathscr{D}^\bullet_{R_\varpi^\star, M} \xrightarrow{p^r - p^\bullet \varphi} \mathscr{E}^\bullet_{R_\varpi^\star, M} \,]; \\ H^*_{\mathrm{syn}}(M_\varpi^\star, r) &:= H^*(\mathrm{Syn}(M_\varpi^\star, r)). \end{aligned}$$

Remark 5.14. Note that for $\star = [u]$, we have $\mathscr{D}^\bullet_{R_\varpi^{[u]}, M} = \mathscr{E}^\bullet_{R_\varpi^{[u]}, M}$.

5.3. Change of disk of convergence. In this section, we will write the syntomic complex $\mathrm{Syn}(S, M, r)$ in Definition 5.7 as $\mathrm{Syn}(M_\varpi^{\mathrm{PD}}, r)$.

In order to relate $\mathrm{Syn}(M_\varpi^{\mathrm{PD}}, r)$ to Koszul complexes, we will first pass to the analytic ring $R_\varpi^{[u]}$ and then to $R_\varpi^{[u,v]}$. Recall that we have $M_\varpi^{\mathrm{PD}} = R_\varpi^{\mathrm{PD}} \otimes_R M$ and $M_\varpi^{[u]} = R_\varpi^{[u]} \otimes_R M$.

Proposition 5.15. (i) *For $\frac{1}{p-1} \leq u \leq 1$, the morphism of complexes*

$$\mathrm{Syn}(M_\varpi^{\mathrm{PD}}, r) \longrightarrow \mathrm{Syn}(M_\varpi^{[u]}, r)$$

induced by the inclusion $M_\varpi^{\mathrm{PD}} \subset M_\varpi^{[u]}$ is a p^{2r}-isomorphism.

(ii) *For $u' \leq u \leq pu'$, the morphism of complexes*

$$\mathrm{Syn}(M_{\varpi}^{[u']}, r) \longrightarrow \mathrm{Syn}(M_{\varpi}^{[u]}, r)$$

induced by the inclusion $M_{\varpi}^{[u']} \subset M_{\varpi}^{[u]}$ is a p^{2r}-isomorphism.

The proposition follows from the following lemma by setting $k = r$.

Lemma 5.16. *Let $k \in \mathbb{N}$.*

(i) *If $\frac{1}{p-1} \leq u \leq 1$, the map*

$$p^k - p^j\varphi : \mathrm{Fil}^r M_{\varpi}^{[u]} \otimes \Omega^j_{R_{\varpi}^{[u]}} / \mathrm{Fil}^r M_{\varpi}^{\mathrm{PD}} \otimes \Omega^j_{R_{\varpi}^{\mathrm{PD}}} \longrightarrow M_{\varpi}^{[u]} \otimes \Omega^j_{R_{\varpi}^{[u]}} / M_{\varpi}^{\mathrm{PD}} \otimes \Omega^j_{R_{\varpi}^{\mathrm{PD}}},$$

is a p^{k+r}-isomorphism.

(ii) *If $u' \leq u \leq pu'$, the map*

$$p^k - p^j\varphi : \mathrm{Fil}^r M_{\varpi}^{[u]} \otimes \Omega^j_{R_{\varpi}^{[u]}} / \mathrm{Fil}^r M_{\varpi}^{[u']} \otimes \Omega^j_{R_{\varpi}^{[u']}} \longrightarrow M_{\varpi}^{[u]} \otimes \Omega^j_{R_{\varpi}^{[u]}} / M_{\varpi}^{[u']} \otimes \Omega^j_{R_{\varpi}^{[u']}},$$

is a p^{k+r}-isomorphism.

Proof. The proof follows in a manner similar to [CN17, Lemma 3.2].

(i) Note that we can decompose everything in the basis of the $\omega_{\mathbf{i}}$'s, where $\mathbf{i} \in I_j$. By the definition of Frobenius on $\omega_{\mathbf{i}}$ we are reduced to showing that

$$p^k - p^j\varphi : \mathrm{Fil}^r M_{\varpi}^{[u]} / \mathrm{Fil}^r M_{\varpi}^{\mathrm{PD}} \longrightarrow M_{\varpi}^{[u]} / M_{\varpi}^{\mathrm{PD}},$$

is a p^{k+r}-isomorphism. We have $M_{\varpi}^{\mathrm{PD}} \subset M_{\varpi}^{[u]}$ and $\varphi(M_{\varpi}^{[u]}) \subset M_{\varpi}^{\mathrm{PD}}$ since $\varphi(R_{\varpi}^{[u]}) \subset R_{\varpi}^{[u/p]} \subset R_{\varpi}^{\mathrm{PD}}$, for $\frac{1}{p-1} \leq u \leq 1$.

For p^k-injectivity, we note that we have $\mathrm{Fil}^r M_{\varpi}^{[u]} = M_{\varpi}^{[u]} \cap \mathrm{Fil}^r M_{\varpi}^{\mathrm{PD}}$, so it suffices to show that if $(p^k - p^j\varphi)x \in M_{\varpi}^{\mathrm{PD}}$ then $p^k x \in M_{\varpi}^{\mathrm{PD}}$. But since we can write $p^k x = (p^k - p^j\varphi)x + p^j\varphi(x)$ and $\varphi(M_{\varpi}^{[u]}) \subset M_{\varpi}^{\mathrm{PD}}$, we get that $p^k x \in M_{\varpi}^{\mathrm{PD}}$.

Now, let $\{f_1, \ldots, f_h\}$ be an R-basis of M. Then, to show p^{k+r}-surjectivity we write $x = \sum_{i=1}^h a_i \otimes f_i \in R_{\varpi}^{[u]} \otimes_R M = M_{\varpi}^{[u]}$. We will write $p^{k+r}x$ as a sum of elements in $(p^k - p^j\varphi)\mathrm{Fil}^r M_{\varpi}^{[u]}$ and M_{ϖ}^{PD}. Let $N = \frac{ke}{u(p-1)}$, then from the definition of $R_{\varpi}^{[u]}$ we can write

$$a_i = a_{i1} + a_{i2}, \ \text{ with } \ a_{i2} \in R_{\varpi,N}^{[u]} \ \text{ and } \ a_{i1} \in p^{-\lfloor Nu/e \rfloor} R_{\varpi}^{+} \subset p^{-k} R_{\varpi}^{\mathrm{PD}},$$

where we write $R_{\varpi,N}^{[u]}$ as in the notation of Lemma 2.18 (it consists of power series in X_0 involving terms X_0^s for $s \geq N$). Now let $x_1 = \sum_{i=1}^h a_{i1} \otimes f_i$ and $x_2 = \sum_{i=1}^h a_{i2} \otimes f_i$, so that $x = x_1 + x_2$. By Lemma 2.18, we can write

$$x_2 = (1 - p^{j-k}\varphi)z, \ \text{ for some } \ z = \sum_{i=1}^h b_i \otimes f_i \in R_{\varpi}^{[u]} \otimes M = M_{\varpi}^{[u]}.$$

Also, by Lemma 2.14 we can write $b_i = b_{i1} + b_{i2}$ with $b_{i1} \in \mathrm{Fil}^r R_{\varpi}^{[u]}$ and $b_{i2} \in p^{-\lfloor ru \rfloor} R_{\varpi}^{+}$. Let $z_1 = \sum_{i=1}^h b_{i1} \otimes f_i \in \mathrm{Fil}^r M_{\varpi}^{[u]}$ and $z_2 = \sum_{i=1}^h b_{i2} \otimes f_i \in p^{-r} M_{\varpi}^{\mathrm{PD}}$, then

$$(1 - p^{j-k}\varphi)z_2 = p^{-k}(p^k - p^j\varphi)z_2 \in p^{-k-r} M_{\varpi}^{\mathrm{PD}},$$

and

$$\begin{aligned} x - (1-p^{j-k}\varphi)z_1 &= x_1 + x_2 - (1-p^{j-k}\varphi)z_1 \\ &= x_1 + (1-p^{j-k}\varphi)z_2 \in p^{-k}M_\varpi^{\mathrm{PD}} + p^{-k-r}M_\varpi^{\mathrm{PD}} \subset p^{-k-r}M_\varpi^{\mathrm{PD}}. \end{aligned}$$

Therefore, we obtain that

$$x \in p^{-k-r}M_\varpi^{\mathrm{PD}} + p^{-k}(p^k - p^j\varphi)\mathrm{Fil}^r M_\varpi^{[u]},$$

which allows us to conclude.

(ii) We can repeat the arguments in (i) by replacing M_ϖ^{PD} with $M_\varpi^{[u']}$, since $R_\varpi^{[u']} \subset R_\varpi^{[u]}$ and $\varphi(R_\varpi^{[u]}) \subset R_\varpi^{[u/p]} \subset R_\varpi^{[u']}$, for $u' \leq u \leq pu'$.

∎

5.4. Change of annulus of convergence. Recall that our objective is to relate the syntomic complexes discussed in the last section to differential Koszul complexes. To realize this goal, we further base change our complex to the ring $R_\varpi^{[u,v]}$. Recall that we have $M_\varpi^{[u]} = R_\varpi^{[u]} \otimes_R M$, and $M_\varpi^{[u,v]} = R_\varpi^{[u,v]} \otimes_R M = R_\varpi^{[u,v]} \otimes_{R_\varpi^{[u]}} M_\varpi^{[u]}$.

Proposition 5.17. *For $pu \leq v$, there exists a p^{2r+4s}-quasi-isomorphism*

$$\tau_{\leq r-s-1}\mathrm{Syn}(M_\varpi^{[u]}, r) \simeq \tau_{\leq r-s-1}\mathrm{Syn}(M_\varpi^{[u,v]}, r),$$

i.e. we have p^{2r+4s}-isomorphisms

$$H^k_{\mathrm{syn}}(M_\varpi^{[u]}, r) \simeq H^k_{\mathrm{syn}}(M_\varpi^{[u,v]}, r),$$

for $0 \leq k \leq r-s-1$.

Proof. Combining the results from Lemmas 5.18, 5.21 & 5.19, we get the claim. ∎

From the definition of complexes displayed in the claim above, it is not at all immediate that we should expect them (before and after scalar extension) to be quasi-isomorphic. Adapting a technique used in the theory of (φ, Γ)-modules of passing to the corresponding (quasi-isomorphic) ψ-complex, we will establish a p-power quasi-isomorphism, between the complexes of interest. This motivates our next definition for an operator ψ over $R_\varpi^{[u]} \otimes_R M$, which would act as a left inverse to φ.

First of all, we know that $\varphi^*(\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)) \simeq \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$, or equivalently $\varphi(\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V))$ generates $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ as an $R[\frac{1}{p}]$-module. Let $\mathbf{f} = \{f_1, \ldots, f_h\}$ denote an R-basis of M, i.e. $M = \oplus_{i=0}^h Rf_i$. Then $\mathbf{f}$ is also a basis of $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ over $R[\frac{1}{p}]$. Hence, $\varphi(\mathbf{f}) = \{\varphi(f_1), \ldots, \varphi(f_h)\}$ is also a basis of $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ over $R[\frac{1}{p}]$. From this we can write $\mathbf{f} = \varphi(\mathbf{f})X$ where $X = (x_{ij}) \in \mathrm{Mat}(h, R[\frac{1}{p}])$. For our choice of M and using Theorem 3.9 and Proposition 3.11, we conclude that $x_{ij} \in \frac{1}{p^s}R$ where $1 \leq i,j \leq h$ and s is the height of V. Therefore, we can define

$$\begin{aligned} \psi : R_\varpi^{[u]} \otimes_R M &\longrightarrow \tfrac{1}{p^s}R_\varpi^{[pu]} \otimes_R M \\ \sum_{i=1}^h y_i \otimes f_i = \mathbf{f}\mathbf{y}^\intercal &\longmapsto \mathbf{f}\psi(X\mathbf{y}^\intercal) = \sum_{j=1}^h \Big(\sum_{i=1}^h \psi(y_i x_{ij})\Big) \otimes f_j, \end{aligned} \tag{5.8}$$

where we consider the operator ψ on $R_\varpi^{[u]}$ defined in §2.6. It is easy to show that this map is well-defined, i.e. independent of the choice of the basis for M.

Using the operator ψ on $M_\varpi^{[u]} = R_\varpi^{[u]} \otimes_R M$ as above, we can define the complex

$$\mathrm{Syn}^\psi(M_\varpi^{[u]}, r) := \Big[\mathrm{Fil}^r M_\varpi^{[u]} \otimes \Omega^\bullet_{R_\varpi^{[u]}} \xrightarrow{p^{r+s}\psi - p^{\bullet+s}} M_\varpi^{[pu]} \otimes \Omega^\bullet_{R_\varpi^{[pu]}}\Big],$$

where the operator ψ acts on $\Omega^\bullet_{R_\varpi^{[u]}}$ as in (5.4).

Lemma 5.18. *The commutative diagram*

$$\begin{tikzcd}
\mathrm{Fil}^r M_\varpi^{[u]} \otimes \Omega^\bullet_{R_\varpi^{[u]}} \arrow[r, "p^r - p^\bullet\varphi"] \arrow[d, "id"] & M_\varpi^{[u]} \otimes \Omega^\bullet_{R_\varpi^{[u]}} \arrow[d, "p^s\psi"] \\
\mathrm{Fil}^r M_\varpi^{[u]} \otimes \Omega^\bullet_{R_\varpi^{[u]}} \arrow[r, "p^{r+s}\psi - p^{\bullet+s}"] & M_\varpi^{[pu]} \otimes \Omega^\bullet_{R_\varpi^{[pu]}},
\end{tikzcd}$$

defines a p^{2s}-quasi-isomorphism from $\mathrm{Syn}(M_\varpi^{[u]}, r)$ *to* $\mathrm{Syn}^\psi(M_\varpi^{[u]}, r)$*, where s is the height of V.*

Proof. First, let us look at the cokernel complex. Since the left vertical arrow is identity, we only need to look at the cokernel of the right vertical arrow. Now, by definition we have $\psi(M_\varpi^{[u]}) \subset p^{-s} M_\varpi^{[pu]}$ and in particular, $p^s\psi(M_\varpi^{[u]}) \subset M_\varpi^{[pu]}$. Moreover, note that the operator $\psi : R_\varpi^{[u]} \to R_\varpi^{[pu]}$ is surjective and $p^s M \subset \varphi^*(M)$ (see Theorem 3.9 and Proposition 3.11). Therefore, we have

$$M_\varpi^{[pu]} = R_\varpi^{[pu]} \otimes_R M \subset \psi(R_\varpi^{[u]} \otimes_R \varphi^*(M)) \subset \psi(R_\varpi^{[u]} \otimes_R M) = \psi(M_\varpi^{[u]}).$$

Hence, we get that $p^s\psi(M_\varpi^{[u]})$ is p^s-isomorphic to $M_\varpi^{[pu]}$. In particular, the cokernel complex is killed by p^s.

Next, for the kernel complex, we proceed as follows: Let $M = \oplus_{j=1}^h R f_j$, so that we have $M_\varpi^{[u]} = \oplus_{j=1}^h R_\varpi^{[u]} f_j$. Now we know that $M/\varphi^*(M)$ is killed by p^s, where s is the height of V. So by extending scalars to $R_\varpi^{[u]}$, we obtain a p^s-isomorphism

$$R_\varpi^{[u]} \otimes_R M \simeq \oplus_{j=1}^h R_\varpi^{[u]} \varphi(f_j).$$

Note that an element

$$y = \sum_{j=1}^h y_j \varphi(f_j) \in \Big(\oplus_{j=1}^h R_\varpi^{[u]} \varphi(f_j)\Big)^{\psi=0},$$

if and only if $y_j \in (R_\varpi^{[u]})^{\psi=0}$. Indeed, $\psi(y) = 0$ if and only if $\sum_{j=1}^h \psi(y_j) f_j = 0$. Since f_j are linearly independent over $R[\frac{1}{p}]$, we get that $\psi(y) = 0$ if and only if $\psi(y_j) = 0$ for all $1 \leq j \leq h$. In particular, we have a p^s-isomorphism

$$(M_\varpi^{[u]})^{\psi=0} = (R_\varpi^{[u]} \otimes_R M)^{\psi=0} \simeq \Big(\oplus_{j=1}^h R_\varpi^{[u]} \varphi(f_j)\Big)^{\psi=0} = \oplus_{j=1}^h (R_\varpi^{[u]})^{\psi=0} \varphi(f_j).$$

Next, recall from (5.4) that in the basis of $\Omega^k_{R_\varpi^{[u]}}$, the operator ψ is defined as $\psi\big(\sum_{\mathbf{i} \in I_k} x_{\mathbf{i}} \omega_{\mathbf{i}}\big) = \sum_{\mathbf{i} \in I_k} \psi(x_{\mathbf{i}}) \omega_{\mathbf{i}}$. In particular, we obtain

$$\Big(M \otimes_R \Omega^k_{R_\varpi^{[u]}}\Big)^{\psi=0} = (R_\varpi^{[u]} \otimes_R M)^{\psi=0} \otimes_{\mathbb{Z}} \Omega^k, \tag{5.9}$$

where

$$\Omega^1 = \mathbb{Z}\tfrac{dX_0}{1+X_0} \oplus_{i=1}^d \mathbb{Z}\tfrac{dX_i}{X_i} \quad \text{and} \quad \Omega^k = \textstyle\bigwedge^k \Omega^1.$$

From Lemma 2.23(ii), we have a decomposition $(R_\varpi^{[u]})^{\psi=0} = \oplus_{\alpha \neq 0} R_{\varpi,\alpha}^{[u]} = R_\varpi^{[u]} u_\alpha$, where $u_\alpha = (1+X_0)^{\alpha_0} X_1^{\alpha_1} \cdots X_d^{\alpha_d}$ for $\alpha = (\alpha_0, \ldots, \alpha_d) \in \{0, 1, \ldots, p-1\}^{[0,d]}$. Moreover, from §2.6, we have $\partial_i(u_\alpha) = \alpha_i u_\alpha$ for $0 \leq i \leq d$. In particular, $\partial_i(R_{\varpi,\alpha}^{[u]}) \subset R_{\varpi,\alpha}^{[u]}$.

Now, using the decomposition of $(R_\varpi^{[u]})^{\psi=0}$, we set $M_\alpha = \oplus_{j=1}^h R_{\varpi,\alpha}^{[u]} \varphi(f_j)$ and obtain that $(M_\varpi^{[u]})^{\psi=0}$ is p^s-isomorphic to $\oplus_{\alpha \neq 0} M_\alpha$. From the differentials on $R_{\varpi,\alpha}^{[u]}$ and the connection on $M_{\varpi}^{[u]}$ we obtain an induced connection $\partial : M_\alpha \to M_\alpha \otimes \Omega^1_{R^{[u]}_{\varpi,\alpha}} = M_\alpha \otimes_{\mathbb{Z}} \Omega^1$, which is integrable. The decomposition of $(M_{\varpi}^{[u]})^{\psi=0}$ and (5.9) shows that the kernel complex in the claim is p^s-isomorphic to the direct sum of complexes

$$0 \longrightarrow M_\alpha \longrightarrow M_\alpha \otimes \Omega^1 \longrightarrow M_\alpha \otimes \Omega^2 \longrightarrow \cdots , \tag{5.10}$$

where $\alpha \neq 0$.

We will show that (5.10) is exact for each α. The idea for the rest of the proof is based on [CN17, Lemma 3.4]. Note that since everything is p-adically complete, we only need to show the exactness of (5.10) modulo p. For this we notice that for $y = \sum_{j=1}^{h} y_j \varphi(f_j) \in M_\alpha$, we have

$$\partial\Big(\sum_{j=1}^{h} y_j\varphi(f_j)\Big) = \sum_{j=1}^{h} y_j \partial_M(\varphi(f_j)) + \varphi(f_j)\partial(y_j),$$

where ∂_M denotes the connection on M. Since we modified the definition of Frobenius on differentials in (5.4), we note from Remark 5.6 that we have $\varphi\partial_M = p\partial_M\varphi$. So we obtain that

$$\partial(y) - \sum_{i=1}^{h} \varphi(f_j)\partial(y_j) \in pM_\alpha.$$

Moreover, by Lemma 2.24 we have that $\partial_i(y_j) - \alpha_i y_j \in pR^{[u]}_{\varpi,\alpha}$. So we get that the complex (5.10) has a very simple shape modulo p: if $d = 0$, it is just $M_\alpha \xrightarrow{\alpha_0} M_\alpha$; if $d = 1$, it is the total complex attached to the double complex

$$\begin{tikzcd} M_\alpha \arrow[r, "\alpha_0"] \arrow[d, "\alpha_1"] & M_\alpha \arrow[d, "\alpha_1"] \\ M_\alpha \arrow[r, "\alpha_0"] & M_\alpha, \end{tikzcd}$$

and for general d, it is the total complex attached to a $(d+1)$-dimensional cube with all vertices equal to M_α and arrows in the i-th direction equal to α_i. As one of the α_i is invertible by assumption, this implies that the cohomology of the total complex is 0. This establishes that (5.10) is exact for each α and hence the kernel complex is p^s-acyclic. ∎

Next, we will base change the complex to $R_{\varpi}^{[u,v]}$. As we will compare (ψ, ∂)-complexes, following (5.8) one can define an operator

$$\psi : R_{\varpi}^{[u,v]} \otimes_R M \longrightarrow \tfrac{1}{p^s} R_{\varpi}^{[pu,pv]} \otimes_R M,$$

as a left inverse to φ. Now using $M_{\varpi}^{[u,v]} = R_{\varpi}^{[u,v]} \otimes_R M$, we define the complex

$$\mathrm{Syn}^{\psi}(M_{\varpi}^{[u,v]}, r) := \Big[\mathrm{Fil}^r M_{\varpi}^{[u,v]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[u,v]}} \xrightarrow{p^{r+s}\psi - p^{\bullet+s}} M_{\varpi}^{[pu,v]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[pu,v]}}\Big].$$

We can relate the two (ψ, ∂)-complexes discussed so far,

Lemma 5.19. *Let $u \leq 1 \leq v$. The natural morphism*

$$\mathrm{Syn}^{\psi}(M_{\varpi}^{[u]}, r) \longrightarrow \mathrm{Syn}^{\psi}(M_{\varpi}^{[u,v]}, r),$$

is a p^{2r}-quasi-isomorphism in degrees $k \leq r - s - 1$.

Proof. The map between complexes is induced by the diagram

$$\begin{tikzcd}
\mathrm{Fil}^r M_{\varpi}^{[u]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[u]}} \arrow[r, "p^{r+s}\psi - p^{\bullet+s}"] \arrow[d] & M_{\varpi}^{[pu]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[pu]}} \arrow[d] \\
\mathrm{Fil}^r M_{\varpi}^{[u,v]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[u,v]}} \arrow[r, "p^{r+s}\psi - p^{\bullet+s}"] & M_{\varpi}^{[pu,v]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[pu,v]}},
\end{tikzcd}$$

where the vertical arrows are natural maps induced by the inclusion $R_{\varpi}^{[u]} \subset R_{\varpi}^{[u,v]}$. Therefore, it suffices to show that the mapping fiber

$$\Big[\mathrm{Fil}^r M_{\varpi}^{[u,v]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[u,v]}} / \mathrm{Fil}^r M_{\varpi}^{[u]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[u]}} \xrightarrow{p^{r+s}\psi - p^{\bullet+s}} M_{\varpi}^{[pu,v]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[pu,v]}} / M_{\varpi}^{[pu]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[pu]}}\Big],$$

is p^{2r}-acyclic. By Lemma 5.20, we can ignore the filtration and, working in the basis $\{\omega_{\mathbf{i}},\ \mathbf{i} \in I_k\}$ of Ω^k, it is enough to show that

$$p^{r+s}\psi - p^{k+s} : M_{\varpi}^{[u,v]}/M_{\varpi}^{[u]} \longrightarrow M_{\varpi}^{[pu,v]}/M_{\varpi}^{[pu]},$$

is a p^r-isomorphism for $k \leq r-s-1$. But

$$M_{\varpi}^{[u,v]}/M_{\varpi}^{[u]} \simeq M_{\varpi}^{[pu,v]}/M_{\varpi}^{[pu]},$$

and therefore $1-p^i\psi$ is an endomorphism of this quotient for $i = r-k$. Moreover, for $i \geq s+1$ we get that $1-p^i\psi$ is invertible on $M_{\varpi}^{[u,v]}/M_{\varpi}^{[u]}$ with inverse given as $1+p^{i-s}(p^s\psi)+p^{2(i-s)}(p^s\psi)^2+\cdots$. Therefore $p^{r+s}\psi - p^{k+s} = p^{k+s}(p^{r-k}\psi - 1)$ is a p^{k+s}-isomorphism. Since $k+s \leq r-1$, we obtain that the complex in the claim is p^{2r}-acyclic. ■

Following observation was used above,

Lemma 5.20. *For $u \leq 1 \leq v$, the natural morphism*

$$\mathrm{Fil}^r M_{\varpi}^{[u,v]}/\mathrm{Fil}^r M_{\varpi}^{[u]} \longrightarrow M_{\varpi}^{[u,v]}/M_{\varpi}^{[u]},$$

is a p^r-isomorphism.

Proof. First we recall that

$$\mathrm{Fil}^r M_{\varpi}^{[u,v]} = \text{closure of } \sum_{a+b=r} \mathrm{Fil}^a R_{\varpi}^{[u,v]} \otimes \mathrm{Fil}^b M \subset M_{\varpi}^{[u,v]}.$$

Now the map in the claim is clearly injective. For p^r-surjectivity, let $\{f_1, \ldots, f_h\}$ be an R-basis of M and let $x = \sum_{i=1}^h b_i \otimes f_i \in R_{\varpi}^{[u,v]} \otimes M$. By [CN17, Lemma 3.5], we have a p^r-isomorphism

$$\mathrm{Fil}^r R_{\varpi}^{[u,v]}/\mathrm{Fil}^r R_{\varpi}^{[u]} \longrightarrow R_{\varpi}^{[u,v]}/R_{\varpi}^{[u]},$$

so we can write $p^r b_i = b_{i1} + b_{i2}$, with $b_{i1} \in \mathrm{Fil}^r R_{\varpi}^{[u,v]}$ and $b_{i2} \in R_{\varpi}^{[u]}$. Since $\sum_{i=1}^h b_{i1} \otimes f_i \in \mathrm{Fil}^r M_{\varpi}^{[u,v]}$, we get the desired conclusion. ■

Finally, we can get back to the (φ, ∂)-complex,

Lemma 5.21. *The commutative diagram*

$$\begin{tikzcd}
\mathrm{Fil}^r M_{\varpi}^{[u,v]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[u,v]}} \arrow[r, "p^r - p^{\bullet}\varphi"] \arrow[d, "id"] & M_{\varpi}^{[u,v/p]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[u,v/p]}} \arrow[d, "p^s\psi"] \\
\mathrm{Fil}^r M_{\varpi}^{[u,v]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[u,v]}} \arrow[r, "p^{r+s}\psi - p^{\bullet+s}"] & M_{\varpi}^{[pu,v]} \otimes \Omega^{\bullet}_{R_{\varpi}^{[pu,v]}},
\end{tikzcd}$$

defines a p^{2s}*-quasi-isomorphism from* $\mathrm{Syn}(M_{\varpi}^{[u,v]}, r)$ *to* $\mathrm{Syn}^{\psi}(M_{\varpi}^{[u,v]}, r)$*.*

Proof. We can repeat the arguments in the proof of Lemma 5.18 by replacing $M_{\varpi}^{[u]}$ with $M_{\varpi}^{[u,v]}$ and $R_{\varpi}^{[u]}$ with $R_{\varpi}^{[u,v]}$. We briefly sketch the argument. First, for the cokernel complex, we only need to look at the cokernel of the right vertical arrow. We have $\psi(M_{\varpi}^{[u,v/p]}) \subset p^{-s}M_{\varpi}^{[pu,v]}$, and in particular $p^s\psi(M_{\varpi}^{[u,v/p]}) \subset M_{\varpi}^{[pu,v]}$. Further, the operator $\psi : R_{\varpi}^{[u,v/p]} \to R_{\varpi}^{[pu,v]}$ is surjective and $p^sM \subset \varphi^*(M)$. Therefore, we have

$$M_{\varpi}^{[pu,v]} = R_{\varpi}^{[pu,v]} \otimes_R M \subset \psi(R_{\varpi}^{[u,v/p]} \otimes_R \varphi^*(M)) \subset \psi(R_{\varpi}^{[u,v/p]} \otimes_R M) = \psi(M_{\varpi}^{[u,v/p]})$$

Hence, we get that $p^s\psi(M_{\varpi}^{[u,v/p]})$ is p^s-isomorphic to $M_{\varpi}^{[pu,v]}$. In particular, the cokernel complex is killed by p^s.

Next, we look at the kernel complex. Arguing as in Lemma 5.18, we obtain a p^s-isomorphism

$$(M_{\varpi}^{[u,v]})^{\psi=0} = (R_{\varpi}^{[u,v/p]} \otimes_R M)^{\psi=0} \simeq \Big(\oplus_{j=1}^h R_{\varpi}^{[u,v/p]}\varphi(f_j)\Big)^{\psi=0} = \oplus_{j=1}^h (R_{\varpi}^{[u,v/p]})^{\psi=0}\varphi(f_j).$$

Now using (5.4), we can write

$$\Big(M \otimes_R \Omega^k_{R_{\varpi}^{[u,v/p]}}\Big)^{\psi=0} = (R_{\varpi}^{[u,v/p]} \otimes_R M)^{\psi=0} \otimes_{\mathbb{Z}} \Omega^k, \tag{5.11}$$

where

$$\Omega^1 = \mathbb{Z}\tfrac{dX_0}{1+X_0} \oplus_{i=1}^d \mathbb{Z}\tfrac{dX_i}{X_i} \quad \text{and} \quad \Omega^k = \textstyle\bigwedge^k \Omega^1.$$

From Lemma 2.23(ii), we have a decomposition $(R_{\varpi}^{[u,v/p]})^{\psi=0} = \oplus_{\alpha\neq 0} R_{\varpi,\alpha}^{[u,v/p]} = R_{\varpi}^{[u,v/p]}u_{\alpha}$, where $u_{\alpha} = (1+X_0)^{\alpha_0}X_1^{\alpha_1}\cdots X_d^{\alpha_d}$ for $\alpha = (\alpha_0, \ldots, \alpha_d) \in \{0, 1, \ldots, p-1\}^{[0,d]}$. From §2.6, we have $\partial_i(u_{\alpha}) = \alpha_i u_{\alpha}$ for $0 \leq i \leq d$. In particular, $\partial_i(R_{\varpi,\alpha}^{[u,v/p]}) \subset R_{\varpi,\alpha}^{[u,v/p]}$. So using the decomposition of $(R_{\varpi}^{[u,v/p]})^{\psi=0}$, we set $M_{\alpha} = \oplus_{j=1}^h R_{\varpi,\alpha}^{[u,v/p]}\varphi(f_j)$ and obtain that $(M_{\varpi}^{[u,v]})^{\psi=0}$ is p^s-isomorphic to $\oplus_{\alpha\neq 0}M_{\alpha}$. From the differentials on $R_{\varpi,\alpha}^{[u,v/p]}$ and the connection on $M_{\varpi}^{[u,v]}$ we obtain an induced connection $\partial : M_{\alpha} \to M_{\alpha} \otimes \Omega^1_{R_{\varpi}^{[u,v/p]}} = M_{\alpha} \otimes_{\mathbb{Z}} \Omega^1$, which is integrable. The decomposition of $(M_{\varpi}^{[u,v]})^{\psi=0}$ and (5.11) shows that the kernel complex in the claim is p^s-isomorphic to the direct sum of complexes

$$0 \longrightarrow M_{\alpha} \longrightarrow M_{\alpha} \otimes \Omega^1 \longrightarrow M_{\alpha} \otimes \Omega^2 \longrightarrow \cdots, \tag{5.12}$$

where $\alpha \neq 0$. An analysis similar to Lemma 5.18 shows that the complex (5.12) has a very simple shape modulo p: if $d = 0$, it is just $M_{\alpha} \xrightarrow{\alpha_0} M_{\alpha}$; if $d = 1$, it is the total complex attached to the double complex

$$\begin{tikzcd} M_{\alpha} \arrow[r, "\alpha_0"] \arrow[d, "\alpha_1"] & M_{\alpha} \arrow[d, "\alpha_1"] \\ M_{\alpha} \arrow[r, "\alpha_0"] & M_{\alpha}, \end{tikzcd}$$

and for general d, it is the total complex attached to a $(d+1)$-dimensional cube with all vertices equal to M_{α} and arrows in the i-th direction equal to α_i. As one of the α_i is invertible by assumption, this implies that the cohomology of the total complex is 0. This establishes that (5.12) is exact for each α and hence the kernel complex is p^s-acyclic. ■

5.5. Differential Koszul Complex. In the previous sections we studied syntomic complexes over various base rings with coefficients in M. In this section, we will study differential Koszul complex over the base ring $\mathbf{A}_{R,\varpi}^{[u,v]}$ with coefficients in the Wach module $\mathbf{N}(T)$. As we shall see the differential Koszul complex is very closely related to syntomic complexes. Such a relationship is to be expected, since we have an isomorphism of rings $\iota_{\mathrm{cycl}} : R_{\varpi}^{[u,v]} \xrightarrow{\sim} \mathbf{A}_{R,\varpi}^{[u,v]}$ in §2.7 and there exists a natural comparison between $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ and $\mathbf{N}(V)$ after extension of scalars to $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ on both sides (see Theorem 3.9). Note that from now onwards, we will be working under the assumption that $\frac{p-1}{p} \leq u \leq \frac{v}{p} < 1 < v$, for example, one can take $u = \frac{p-1}{p}$ and $v = p-1$.

The ring $R_{\varpi}^{[u,v]}$ is a p-adically complete $\mathbb{Z}_p$-algebra, equipped with a Frobenius $\varphi : R_{\varpi}^{[u,v]} \to R_{\varpi}^{[u,v/p]}$, lifting the absolute Frobenius on $R_{\varpi}^{[u,v]}/p$. Let $\Omega^1_{\mathbf{A}_{R,\varpi}^{[u,v]}}$ denote the p-adic completion of the module of differentials of $\mathbf{A}_{R,\varpi}^{[u,v]}$ relative to $\mathbb{Z}$. Recall from §2.5 that $\Omega^1_{R_{\varpi}^{[u,v]}}$ has a basis of differentials $\{\frac{dX_0}{1+X_0}, \frac{dX_1}{X_1}, \ldots, \frac{dX_d}{X_d}\}$. So via the identification $\iota_{\mathrm{cycl}} : R_{\varpi}^{[u,v]} \xrightarrow{\sim} \mathbf{A}_{R,\varpi}^{[u,v]}$ we obtain differential operators ∂_i over $\mathbf{A}_{R,\varpi}^{[u,v]}$, for $0 \leq i \leq d$. Moreover, from Definition 2.13 we can endow $\mathbf{A}_{R,\varpi}^{[u,v]}$ with a filtration $\{\mathrm{Fil}^k \mathbf{A}_{R,\varpi}^{[u,v]}\}_{k\in\mathbb{Z}}$ and obtain filtered de Rham complex

$$\mathrm{Fil}^r\Omega^{\bullet}_{\mathbf{A}_{R,\varpi}^{[u,v]}} : \mathrm{Fil}^r\mathbf{A}_{R,\varpi}^{[u,v]} \longrightarrow \mathrm{Fil}^{r-1}\mathbf{A}_{R,\varpi}^{[u,v]} \otimes \Omega^1_{\mathbf{A}_{R,\varpi}^{[u,v]}} \longrightarrow \mathrm{Fil}^{r-2}\mathbf{A}_{R,\varpi}^{[u,v]} \otimes \Omega^2_{\mathbf{A}_{R,\varpi}^{[u,v]}} \longrightarrow \cdots, \quad \text{for } k \in \mathbb{Z}.$$

Further, the differential operators ∂_i can be related to the infinitesimal action of Γ_R by the relation

$$\nabla_i := \log \gamma_i = t\partial_i \ \text{ for } 0 \leq i \leq d,$$

where $\log \gamma_i = \sum_{k\in\mathbb{N}}(-1)^k \frac{(\gamma_i-1)^{k+1}}{k+1}$. We will study similar operators over the $\mathbf{A}_{R,\varpi}^{[u,v]}$-module arising from the Wach module $\mathbf{N}(T)$.

Note that for an indeterminate X we can formally write

$$\begin{aligned}\frac{\log(1+X)}{X} &= 1 + a_1X + a_2X^2 + a_3X^3 + \cdots,\\ \frac{X}{\log(1+X)} &= 1 + b_1X + b_2X^2 + b_3X^3 + \cdots,\end{aligned}$$

where $v_p(a_k) \geq -\frac{k}{p-1}$ for all $k \geq 1$ and therefore, $v_p(b_k) \geq -\frac{k}{p-1}$ for all $k \geq 1$. We have the following claim:

Lemma 5.22. *Let $N_{\varpi}^{[u,v]}(T) = \mathbf{A}_{R,\varpi}^{[u,v]} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$. Then, for $i \in \{0, 1, \ldots, d\}$ the operators*

$$\nabla_i = \log \gamma_i; \quad \frac{\nabla_i}{\gamma_i - 1} = \frac{\log \gamma_i}{\gamma_i - 1}; \quad \text{and} \quad \frac{\gamma_i - 1}{\nabla_i} = \frac{\gamma_i - 1}{\log \gamma_i}.$$

converge as series of operators on $N_{\varpi}^{[u,v]}(T)$.

Proof. For $0 \leq i \leq d$, we have that $\gamma_i - 1$ acts as a twisted derivation, i.e. for $a \in \mathbf{A}_{R,\varpi}^{[u,v]}$ and $x \in \mathbf{N}(T)$, we have

$$(\gamma_i - 1)(ax) = (\gamma_i - 1)a \cdot x + \gamma_i(a)(\gamma_i - 1)x.$$

The action of Γ_R is trivial on $\mathbf{N}(T)/\pi\mathbf{N}(T)$, so we can write $(\gamma_i - 1)x = \pi y$, for some $y \in \mathbf{N}(T)$. Now, using Lemma 2.32 and the preceding discussion, we easily conclude that

$$(\gamma_i - 1)(p^m, \pi_m^{p^m})^k N_{\varpi}^{[u,v]}(T) \subset (p^m, \pi_m^{p^m})^{k+1} N_{\varpi}^{[u,v]}(T).$$

Then similar to (4.2) and (4.3) in the proof of Lemma 4.17, we get that for $k \geq 0$ we have

$$(\gamma_i - 1)^k N_{\varpi}^{[u,v]}(T) \subset (p^m, \pi_m^{p^m})^k N_{\varpi}^{[u,v]}(T).$$

The same estimation of p-adic valuation of coefficients as in the proof Lemma 4.17 helps us in concluding that $\log \gamma_i$ converges as a series of operators on $N_{\varpi}^{[u,v]}(T)$. The claim for the convergence of operators $\frac{\nabla_i}{\gamma_i - 1}$ and $\frac{\gamma_i - 1}{\nabla_i}$ follows in a manner similar to Lemma 4.20. ∎

Note that $N_{\varpi}^{[u,v]}(T)$ is a topological $\mathbf{A}_{R,\varpi}^{[u,v]}$-module equipped with a filtration by $\mathbf{A}_{R,\varpi}^{[u,v]}$-submodules

$$\mathrm{Fil}^k N_{\varpi}^{[u,v]}(T) = \text{closure of } \sum_{i+j=k} \mathrm{Fil}^i \mathbf{A}_{R,\varpi}^{[u,v]} \otimes \mathrm{Fil}^j \mathbf{N}(T) \subset N_{\varpi}^{[u,v]}(T), \ \text{ for } k \in \mathbb{Z}, \tag{5.13}$$

such that $\mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)$ is stable under the action of Γ_R.

Remark 5.23. The results of Lemma 5.22 continue to hold if we replace $\mathbf{N}(T)$ with $\mathbf{N}(T(r))$ for $r \in \mathbb{Z}$, or $\mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)$ for $k \in \mathbb{Z}$, or filtered pieces of $\mathbf{A}_{R,\varpi}^{[u,v]} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T(r))$.

Lemma 5.24. *For the filtered modules and operators ∇_i defined above, we have*

$$\nabla_i(\mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)) \subset \pi \mathrm{Fil}^{k-1} N_{\varpi}^{[u,v]}(T) = t\mathrm{Fil}^{k-1} N_{\varpi}^{[u,v]}(T) \ \textit{ for } \ 0 \leq i \leq d.$$

Proof. Note that the action of Γ_R is trivial on $\mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)/\pi \mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)$ and from this we infer that for $0 \leq i \leq d$, we have

$$\nabla_i(\mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)) \subset \mathrm{Fil}^k N_{\varpi}^{[u,v]}(T) \cap \pi N_{\varpi}^{[u,v]}(T) = \pi \mathrm{Fil}^{k-1} N_{\varpi}^{[u,v]}(T),$$

where the last equality follows from Lemma 3.6. As $\frac{t}{\pi}$ is a unit in $S = \mathbf{A}_{R,\varpi}^{[u,v]}$ (see Lemma 2.30), we can also write $\nabla_i(\mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)) \subset t\mathrm{Fil}^{k-1} N_{\varpi}^{[u,v]}(T)$. ∎

The lemma above enables us to introduce differential operators ∂_i over $N_{\varpi}^{[u,v]}(T)$ by the formula

$$\nabla_i = \log \gamma_i = t\partial_i, \ \text{ for } 0 \leq i \leq d,$$

where the operators ∂_i are well-defined by dividing out the image under the operator ∇_i by t. Recall that via the identification $R_{\varpi}^{[u,v]} \xrightarrow{\sim} \mathbf{A}_{R,\varpi}^{[u,v]}$, we have a basis for $\Omega^1_{\mathbf{A}_{R,\varpi}^{[u,v]}}$ given by $\{\frac{dX_0}{1+X_0}, \frac{dX_1}{X_1}, \ldots, \frac{dX_d}{X_d}\}$. Therefore, by setting $\partial = (\partial_0, \ldots, \partial_d)$ we obtain a connection over $N_{\varpi}^{[u,v]}(T)$

$$\begin{aligned} \partial : N_{\varpi}^{[u,v]}(T) &\longrightarrow N_{\varpi}^{[u,v]}(T) \otimes \Omega^1_{\mathbf{A}_{R,\varpi}^{[u,v]}} \\ ax &\longmapsto a\partial(x) + x \otimes d(a). \end{aligned}$$

Lemma 5.25. *The connection ∂ on $N_{\varpi}^{[u,v]}(T)$ is integrable and satisfies Griffiths transversality with respect to the filtration, i.e.*

$$\partial_i(\mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)) \subset \mathrm{Fil}^{k-1} N_{\varpi}^{[u,v]}(T) \ \textit{ for } \ 0 \leq i \leq d.$$

Proof. Recall that from (4.4) we have $[\nabla_i, \nabla_j] = 0$ for $1 \leq i, j \leq d$, whereas $[\nabla_0, \nabla_i] = p^m \nabla_i$, for $1 \leq i \leq d$. So it follows that over $N_{\varpi}^{[u,v]}(T)$ we have the composition of operators

$$t^2(\partial_i \circ \partial_j - \partial_j \circ \partial_i) = t\partial_i(t\partial_j) - t\partial_j(t\partial_i) = \nabla_i \circ \nabla_j - \nabla_j \circ \nabla_i = 0, \ \text{ for } 1 \leq i, j \leq d.$$

Next, for $1 \leq i \leq d$, we have

$$\begin{aligned} \nabla_0 \circ \nabla_i - \nabla_i \circ \nabla_0 &= t\partial_0 \circ (t\partial_i) - t\partial_i \circ (t\partial_0) \\ &= tp^m \partial_i + t^2 \partial_0 \circ \partial_i - t^2 \partial_i \circ \partial_0 = p^m \nabla_i + t^2(\partial_0 \circ \partial_i - \partial_i \circ \partial_0). \end{aligned}$$

In particular, $\partial_0 \circ \partial_i - \partial_i \circ \partial_0 = 0$. Since $\partial \circ \partial = (\partial_i \circ \partial_j)_{i,j}$ for $0 \leq i \leq j \leq d$ and $N_{\varpi}^{[u,v]}(T)$ is t-torsion free, we conclude that the connection ∂ is integrable. Moreover, it satisfies Griffiths transversailty since $\partial_i(\mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)) = t^{-1}\nabla_i(\mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)) \subset \mathrm{Fil}^{k-1} N_{\varpi}^{[u,v]}(T)$, for $0 \leq i \leq d$. ∎

From the lemma above, we have the filtered de Rham complex for $N_{\varpi}^{[u,v]}(T)$

$$\begin{aligned} \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T) \otimes \Omega^\bullet_{\mathbf{A}_{R,\varpi}^{[u,v]}} : \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T) \longrightarrow \mathrm{Fil}^{r-1} N_{\varpi}^{[u,v]}(T) \otimes \Omega^1_{\mathbf{A}_{R,\varpi}^{[u,v]}} \longrightarrow \\ \longrightarrow \mathrm{Fil}^{r-2} N_{\varpi}^{[u,v]}(T) \otimes \Omega^2_{\mathbf{A}_{R,\varpi}^{[u,v]}} \longrightarrow \cdots. \end{aligned} \tag{5.14}$$

Further, we know that $\Omega^1_{\mathbf{A}_{R,\varpi}^{[u,v]}}$ has a basis $\{\omega_1, \ldots, \omega_d\}$, such that an element of $\Omega^q_{\mathbf{A}_{R,\varpi}^{[u,v]}} = \wedge^q \Omega^1_{\mathbf{A}_{R,\varpi}^{[u,v]}}$ can be uniquely written as $\sum x_{\mathbf{i}}\omega_{\mathbf{i}}$, with $x_{\mathbf{i}} \in \mathbf{A}_{R,\varpi}^{[u,v]}$ and $\omega_{\mathbf{i}} = \omega_{i_1} \wedge \cdots \wedge \omega_{i_q}$ for $\mathbf{i} = (i_1, \ldots, i_q) \in I_q = \{0 \leq i_1 < \cdots < i_q \leq d\}$. In this case, the map involving differential operators becomes

$$(\partial_i) : \big(\mathrm{Fil}^{k-q} N_{\varpi}^{[u,v]}(T)\big)^{I_q} \longrightarrow \big(\mathrm{Fil}^{k-q-1} N_{\varpi}^{[u,v]}(T)\big)^{I_{q+1}}, \ \text{ for } \ 0 \leq i \leq d.$$

Definition 5.26. Define the *∂-Koszul complex* for $\mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)$ as

$$\mathrm{Kos}(\partial_A, \mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)) : \mathrm{Fil}^k N_{\varpi}^{[u,v]}(T) \xrightarrow{(\partial_i)} \big(\mathrm{Fil}^{k-1} N_{\varpi}^{[u,v]}(T)\big)^{I_1} \longrightarrow \big(\mathrm{Fil}^{k-2} N_{\varpi}^{[u,v]}(T)\big)^{I_2} \longrightarrow \cdots.$$

Remark 5.27. (i) By definition, we have an ismorphism of complexes $\mathrm{Fil}^k N_{\varpi}^{[u,v]}(T) \otimes \Omega^\bullet_{\mathbf{A}_{R,\varpi}^{[u,v]}} \simeq \mathrm{Kos}(\partial_A, \mathrm{Fil}^k N_{\varpi}^{[u,v]}(T))$.

(ii) Let $I'_j = \{(i_1, \ldots, i_j)$, such that $1 \leq i_1 < \cdots < i_j \leq d\}$ and let $\partial' = (\partial_1, \ldots, \partial_d)$. We can also set

$$\mathrm{Kos}(\partial'_A, \mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)) : \mathrm{Fil}^k N_{\varpi}^{[u,v]}(T) \xrightarrow{(\partial_i)} \big(\mathrm{Fil}^{k-1} N_{\varpi}^{[u,v]}(T)\big)^{I'_1} \rightarrow \big(\mathrm{Fil}^{k-2} N_{\varpi}^{[u,v]}(T)\big)^{I'_2} \rightarrow \cdots,$$

and therefore we get that

$$\mathrm{Kos}(\partial_A, \mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)) = \big[\mathrm{Kos}(\partial'_A, \mathrm{Fil}^k N_{\varpi}^{[u,v]}(T)) \xrightarrow{\partial_0} \mathrm{Kos}(\partial'_A, \mathrm{Fil}^{k-1} N_{\varpi}^{[u,v]}(T))\big].$$

(iii) The computation carried out in this section remain valid for the ring $\mathbf{A}_{R,\varpi}^{[u,v/p]}$ as well.

5.6. Poincaré Lemma. Recall from §2.8 that given two p-adically complete W-algebras Σ and Λ, and $\iota : \Sigma \rightarrow \Lambda$ a continuous injective morphism of filtered O_F-algebras. Then for $f : \Sigma \otimes \Lambda \rightarrow \Lambda$ the morphism sending $x \otimes y \mapsto \iota(x)y$, we can define the ring $\Sigma\Lambda$ to be the p-adic completion of the PD-envelope of $\Sigma \otimes \Lambda \rightarrow \Lambda$ with respect to Ker f.

Definition 5.28. Let $\star \in \{\mathrm{PD}, [u], [u,v]\}$ and define $E_{R,\varpi}^{\star} = \Sigma\Lambda$ for $\Sigma = R_{\varpi}^{\star}$, $\Lambda = \mathbf{A}_{R,\varpi}^{\star}$, and $\iota = \iota_{\mathrm{cycl}}$ (see §2.7).

Note that we are working under the assumption that $\frac{p-1}{p} \leq u \leq \frac{v}{p} < 1 < v$, for example, one can take $u = \frac{p-1}{p}$ and $v = p-1$. These rings have desirable properties:

Lemma 5.29 ([CN17, Lemma 2.38]). (i) $E_{R,\varpi}^{\mathrm{PD}} \subset E_{R,\varpi}^{[u]} \subset E_{R,\varpi}^{[u,v]}$.

(ii) *The Frobenius φ extends uniquely to continuous morphisms*

$$E_{R,\varpi}^{\mathrm{PD}} \longrightarrow E_{R,\varpi}^{\mathrm{PD}}, \quad E_{R,\varpi}^{[u]} \longrightarrow E_{R,\varpi}^{[u]}, \quad E_{R,\varpi}^{[u,v]} \longrightarrow E_{R,\varpi}^{[u,v/p]}.$$

(iii) *The action of G_R extends uniquely to continuous actions on $E_{R,\varpi}^{\mathrm{PD}}$, $E_{R,\varpi}^{[u]}$, and $E_{R,\varpi}^{[u,v]}$ which commutes with the Frobenius.*

Remark 5.30. (i) In Definition 5.28 if we reverse the roles of Σ and Λ, i.e. if we take $\Sigma = \mathbf{A}_{R,\varpi}^{\bigstar}$, $\Lambda = R_{\varpi}^{\bigstar}$ and $\iota = \iota_{\mathrm{cycl}}^{-1}$, then we get an isomorphism $\Sigma\Lambda \simeq E_{R,\varpi}^{\bigstar}$ with obvious commutativity of the action of Frobenius and the Galois group G_R on each side.

(ii) Let $V_i = \frac{X_i \otimes 1}{1 \otimes \iota(X_i)}$, for $0 \leq i \leq d$. We filter $E_{R,\varpi}^{\bigstar}$ by defining $\mathrm{Fil}^r E_{R,\varpi}^{\bigstar}$ to be the topological closure of the ideal generated by the products of the form $x_1 x_2 \prod (V_i - 1)^{[k_i]}$, with $x_1 \in \mathrm{Fil}^{r_1} R_{\varpi}^{\bigstar}$, $x_2 \in \mathrm{Fil}^{r_2} \mathbf{A}_{R,\varpi}^{\bigstar}$, and $r_1 + r_2 + \sum k_i \geq r$.

From Definition 3.7, we have a p-adically complete ring $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ equipped with a Frobenius and a continuous action of Γ_R. From [Abh21, Remark 4.20], we have an alternative construction of $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ using an embedding $\iota : R \rightarrow \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ defined by sending $X_i \mapsto [X_i^{\flat}]$, for $1 \leq i \leq d$. Identifying R as a subring of R_{ϖ}^{PD}, and extending the embedding ι to $R_{\varpi}^{\mathrm{PD}} \rightarrow \mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ by sending $X_0 \mapsto \pi_m$, we get that the extended embedding is exactly ι_{cycl}. Since the action of the Frobenius and the Galois group G_R over $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ and $E_{R,\varpi}^{\mathrm{PD}}$ can be given by their action on each component of the tensor product, we obtain a Frobenius and Galois-equivariant embedding $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \rightarrowtail E_{R,\varpi}^{\mathrm{PD}}$. Moreover, the filtration on $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ (see Definition 3.8) coincides with the filtration induced from its embedding into $E_{R,\varpi}^{\mathrm{PD}}$. Note that since $R_{\varpi}^{\mathrm{PD}} \subset E_{R,\varpi}^{\mathrm{PD}}$, the key difference between $E_{R,\varpi}^{\mathrm{PD}}$ and $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ is that the former ring contains the indeterminate X_0 and its divided powers, whereas the latter ring does not.

Next, from the natural inclusion $R \rightarrowtail R_{\varpi}^{\mathrm{PD}}$ we know that the differential operator on R is compatible with the differential operator on R_{ϖ}^{PD}. Furthermore, we have an identification $\iota_{\mathrm{cycl}}^{-1} : \mathbf{A}_{R,\varpi}^{\mathrm{PD}} \xrightarrow{\sim} R_{\varpi}^{\mathrm{PD}}$ (see §2.7) as well as differential operators ∂_i for $0 \leq i \leq d$ on $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. Also, over the ring $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$, the operators $\nabla_i = \log \gamma_i$ converge for $0 \leq i \leq d$ (see Lemma 4.17), which are related to the differential operators by the relation $\nabla_i = t\partial_i$. Thus if we denote this differential operator over $\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ as $\partial_A = (\partial_i)_{0 \leq i \leq d}$ and the differential operator over R_{ϖ}^{PD} (as well as over R) as ∂_R, then we see that the induced differential operator $\partial_R \otimes 1 + 1 \otimes \partial_A$ over $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ as well as $E_{R,\varpi}^{\mathrm{PD}}$ are compatible. Note that $E_{R,\varpi}^{\mathrm{PD}}$ is naturally contained in $E_{R,\varpi}^{[u,v]}$ compatible with all the structures. Hence, below we will identify $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ as a subring of $E_{R,\varpi}^{[u,v]}$.

Now we turn to the comparison between M and $\mathbf{N}(T)$ over the ring $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$. From Proposition 3.11, Remark 3.13 and Example 5.5 we have a $p^{n(T,e)}$-isomorphism

$$\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathbf{N}(T), \tag{5.15}$$

compatible with Frobenius, filtration, connection and the action of Γ_R on each side. We can promote the comparison in (5.15), by extension of scalars, over to the ring $E_{R,\varpi}^{[u,v]}$ and obtain a $p^{n(T,e)}$-isomorphism

$$E_{R,\varpi}^{[u,v]} \otimes_R M \longrightarrow E_{R,\varpi}^{[u,v]} \otimes_R \mathbf{N}(T),$$

compatible with Frobenius, filtration, connection and the action of Γ_R on each side. Let $M_{\varpi}^{[u,v]} = R_{\varpi}^{[u,v]} \otimes_R M$, and $N_{\varpi}^{[u,v]}(T) = \mathbf{A}_{R,\varpi}^{[u,v]} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$, then we can rephrase the comparison above as a $p^{n(T,e)}$-isomorphism

$$E_{R,\varpi}^{[u,v]} \otimes_{R_{\varpi}^{[u,v]}} M_{\varpi}^{[u,v]} \simeq E_{R,\varpi}^{[u,v]} \otimes_{\mathbf{A}_{R,\varpi}^{[u,v]}} N_{\varpi}^{[u,v]}(T), \tag{5.16}$$

compatible with Frobenius, filtration, connection, and the action of Γ_R on each side.

Let $R_1 = R_{\varpi}^{[u,v]}$, $R_2 = \mathbf{A}_{R,\varpi}^{[u,v]}$, and $R_3 = E_{R,\varpi}^{[u,v]}$. We set $X_{0,1} = X_0$, $X_{0,2} = \pi_m$ and for $1 \leq i \leq d$, we set $X_{i,1} = X_i$ and $X_{i,2} = [X_i^{\flat}]$. Now for $j = 1, 2$, we set

$$\Omega_j^1 := \mathbb{Z}\tfrac{dX_{0,j}}{1+X_{0,j}} \oplus_{i=1}^d \mathbb{Z}\tfrac{dX_{i,j}}{X_{i,j}},$$

and $\Omega_3^1 := \Omega_1^1 \oplus \Omega_2^1$. For $j = 1, 2, 3$, let $\Omega_i^k = \wedge^k \Omega_j$. Therefore, $\Omega_{R_j}^k = R_j \otimes \Omega_j^k$.

Recall that we have $M_{\varpi}^{[u,v]} = R_{\varpi}^{[u,v]} \otimes_R M$ is a filtered $R_{\varpi}^{[u,v]}$-module equipped with a quasi-nilpotent integrable connection satisfying Griffiths transversality with respect to the filtration as defined above. In other words, for each $r \in \mathbb{N}$, we have a complex

$$\mathrm{Fil}^r M_{\varpi}^{[u,v]} \otimes \Omega_1^\bullet : \mathrm{Fil}^r M_{\varpi}^{[u,v]} \xrightarrow{\partial_{R_1}} \mathrm{Fil}^{r-1} M_{\varpi}^{[u,v]} \otimes \Omega_1^1 \xrightarrow{\partial_{R_1}} \mathrm{Fil}^{r-2} M_{\varpi}^{[u,v]} \otimes \Omega_1^2 \xrightarrow{\partial_{R_1}} \cdots,$$

Next, let $\Delta_1 := E_{R,\varpi}^{[u,v]} \otimes_{R_{\varpi}^{[u,v]}} M_{\varpi}^{[u,v]}$ and define a filtration on Δ_1 using the filtrations on each factor of the tensor product. For $k \in \mathbb{Z}$, we have

$$\partial_{R_3} : \mathrm{Fil}^r E_{R,\varpi}^{[u,v]} \longrightarrow \mathrm{Fil}^{r-1} E_{R,\varpi}^{[u,v]} \otimes_{\mathbb{Z}} \Omega_3^1, \text{ and } \partial_{R_1} : \mathrm{Fil}^r M_{\varpi}^{[u,v]} \longrightarrow \mathrm{Fil}^{r-1} M_{\varpi}^{[u,v]} \otimes_{\mathbb{Z}} \Omega_1^1,$$

therefore we obtain that $\partial_{R_3} : \mathrm{Fil}^r \Delta_1 \to \mathrm{Fil}^{r-1} \Delta_1 \otimes_{\mathbb{Z}} \Omega_3^1$. Hence, we have the filtered de Rham complex

$$\mathrm{Fil}^r \Delta_1 \otimes \Omega_3^\bullet : \mathrm{Fil}^r \Delta_1 \xrightarrow{\partial_{R_3}} \mathrm{Fil}^{r-1} \Delta_1 \otimes \Omega_3^1 \xrightarrow{\partial_{R_3}} \mathrm{Fil}^{r-2} \Delta_1 \otimes \Omega_3^2 \xrightarrow{\partial_{R_3}} \cdots.$$

Lemma 5.31. *The natural map*

$$\mathrm{Fil}^r M_{\varpi}^{[u,v]} \otimes \Omega_1^\bullet \longrightarrow \mathrm{Fil}^r \Delta_1 \otimes \Omega_3^\bullet$$

is a quasi-isomorphism.

Proof. Note that we have assumed $R_1 = R_{\varpi}^{[u,v]}$. Since we have $\mathrm{Fil}^r M_{\varpi}^{[u,v]} = (\mathrm{Fil}^r \Delta_1)^{\partial_{R_2}=0}$, from Lemma 2.37 and Proposition 2.38 we obtain the claim. ■

Next, recall from (5.14) that for $R_2 = \mathbf{A}_{R,\varpi}^{[u,v]}$ and the module $N_{\varpi}^{[u,v]}(T) = \mathbf{A}_{R,\varpi}^{[u,v]} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$, for $r \in \mathbb{Z}$, we have the filtered de Rham complex

$$\mathrm{Fil}^r N_{\varpi}^{[u,v]}(T) \otimes \Omega_2^\bullet : \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T) \longrightarrow \mathrm{Fil}^{r-1} N_{\varpi}^{[u,v]}(T) \otimes \Omega_2^1 \longrightarrow \mathrm{Fil}^{r-2} N_{\varpi}^{[u,v]}(T) \otimes \Omega_2^2 \longrightarrow \cdots.$$

Also, let $\Delta_2 := E_{R,\varpi}^{[u,v]} \otimes_{R_{\varpi}^{[u,v]}} N_{\varpi}^{[u,v]}(T)$ and define a filtration on Δ_2 using the filtrations on each factor of the tensor product. Then similar to the case of Δ_1, we have the de Rham complex

$$\mathrm{Fil}^r \Delta_2 \otimes \Omega_3^\bullet : \mathrm{Fil}^r \Delta_2 \xrightarrow{\partial_{R_3}} \mathrm{Fil}^{r-1} \Delta_2 \otimes \Omega_3^1 \xrightarrow{\partial_{R_3}} \mathrm{Fil}^{r-2} \Delta_2 \otimes \Omega_3^2 \xrightarrow{\partial_{R_3}} \cdots.$$

Now, since $\mathrm{Fil}^r N_{\varpi}^{[u,v]}(T) = (\mathrm{Fil}^r \Delta_2)^{\partial_1=0}$, in a manner similar to Lemma 5.31 one can show that,

Lemma 5.32. *The natural map*

$$\mathrm{Fil}^r N_{\varpi}^{[u,v]}(T) \otimes \Omega_2^\bullet \longrightarrow \mathrm{Fil}^r \Delta_2 \otimes \Omega_3^\bullet,$$

is a quasi-isomorphism.

Remark 5.33. The computations above continue to hold if we replace the ring $R_{\varpi}^{[u,v]}$ (resp. $\mathbf{A}_{R,\varpi}^{[u,v]}$) with the ring $R_{\varpi}^{[u,v/p]}$ (resp. $\mathbf{A}_{R,\varpi}^{[u,v/p]}$).

Definition 5.34. Let $N_{\varpi}^{[u,v]}(T)$ as above such that it admits a Frobenius-semilinear morphism $\varphi : N_{\varpi}^{[u,v]}(T) \to N_{\varpi}^{[u,v/p]}$. Using Definition 5.26 and Remark 5.27, define the (φ, ∂)-complex

$$\mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T)) := \left[\begin{tikzcd} \mathrm{Kos}(\partial'_A, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T)) \arrow[r, "p^r - p^{\bullet}\varphi"] \arrow[d, "\partial_0"] & \mathrm{Kos}(\partial'_A, N_{\varpi}^{[u,v/p]}) \arrow[d, "\partial_0"] \\ \mathrm{Kos}(\partial'_A, \mathrm{Fil}^{r-1} N_{\varpi}^{[u,v]}(T)) \arrow[r, "p^r - p^{\bullet+1}\varphi"] & \mathrm{Kos}(\partial'_A, N_{\varpi}^{[u,v/p]}) \end{tikzcd}\right].$$

Proposition 5.35. *The complexes* $\mathrm{Syn}(M_{\varpi}^{[u,v]}, r)$ *and* $\mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T))$ *are* $p^{2n(T,e)}$*-quasi-isomorphic, where* $n(T,e) \in \mathbb{N}$ *as in Assumption 5.4.*

Proof. Using Lemma 5.31 with $R_1 = R_{\varpi}^{[u,v]}$, $R_3 = E_{R,\varpi}^{[u,v]}$, $\Delta_1 = E_{R,\varpi}^{[u,v]} \otimes_{R_{\varpi}^{[u,v]}} M_{\varpi}^{[u,v]}$, and $\Delta'_1 = E_{R,\varpi}^{[u,v/p]} \otimes_{R_{\varpi}^{[u,v/p]}} M_{\varpi}^{[u,v/p]}$, we have a quasi-isomorphism

$$\mathrm{Syn}(M_{\varpi}^{[u,v]}, r) \simeq \Big[\mathrm{Fil}^r M_{\varpi}^{[u,v]} \otimes \Omega_1^{\bullet} \xrightarrow{p^r - p^{\bullet}\varphi} M_{\varpi}^{[u,v/p]} \otimes \Omega_1^{\bullet}\Big] \simeq \Big[\mathrm{Fil}^r \Delta_1 \otimes \Omega_3^{\bullet} \xrightarrow{p^r - p^{\bullet}\varphi} \Delta'_1 \otimes \Omega_3^{\bullet}\Big].$$

Using Lemma 5.32 with $R_2 = \mathbf{A}_{R,\varpi}^{[u,v]}$, $R_3 = E_{R,\varpi}^{[u,v]}$, $\Delta_2 = E_{R,\varpi}^{[u,v]} \otimes_{\mathbf{A}_{R,\varpi}^{[u,v]}} N_{\varpi}^{[u,v]}(T)$, and $\Delta'_2 = E_{R,\varpi}^{[u,v/p]} \otimes_{\mathbf{A}_{R,\varpi}^{[u,v/p]}} N_{\varpi}^{[u,v/p]}$, we have a quasi-isomorphism

$$\begin{aligned} \mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T)) &\simeq \Big[\mathrm{Fil}^r N_{\varpi}^{[u,v]}(T) \otimes \Omega_2^{\bullet} \xrightarrow{p^r - p^{\bullet}\varphi} \mathrm{Fil}^r N_{\varpi}^{[u,v/p]} \otimes \Omega_2^{\bullet}\Big] \\ &\simeq \Big[\mathrm{Fil}^r \Delta_2 \otimes \Omega_3^{\bullet} \xrightarrow{p^r - p^{\bullet}\varphi} \Delta'_2 \otimes \Omega_3^{\bullet}\Big]. \end{aligned}$$

Note that in the quasi-ismorphism we used Remark 5.27 to identify the complexes $\mathrm{Fil}^r N_{\varpi}^{[u,v]}(T) \otimes \Omega^{\bullet}_{\mathbf{A}_{R,\varpi}^{[u,v]}} \simeq \mathrm{Kos}(\partial_A, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T))$.

Now using (5.16) we have $p^{n(T,e)}$-isomorphisms $\mathrm{Fil}^r \Delta_1 \simeq \mathrm{Fil}^r \Delta_2$ and $\Delta'_1 \simeq \Delta'_2$. Combining this with the isomorphisms above, we obtain a $p^{2n(T,e)}$-quasi-isomorphism

$$\mathrm{Syn}(M_{\varpi}^{[u,v]}, r) \simeq \mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T)).$$

∎

6. Syntomic complex and (φ, Γ)-modules

In this section, we will carry out the second step of the proof of Theorem 5.8, i.e. study complexes computing continuous G_R-cohomology of $T(r)$. To state the main result of this section, we introduce some notations. Recall that we are working under the assumption $\frac{p-1}{p} \leq u \leq \frac{v}{p} < 1 < v$, for example, one can take $u = \frac{p-1}{p}$ and $v = p-1$. Note that we have the finite free $\mathbf{A}_{R,\varpi}^{[u,v]}$-module

$$N_{\varpi}^{[u,v]}(T) = \mathbf{A}_{R,\varpi}^{[u,v]} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T).$$

From (5.13) we have a filtration on $N_{\varpi}^{[u,v]}(T)$ as

$$\mathrm{Fil}^k N_{\varpi}^{[u,v]}(T) = \text{closure of } \sum_{i+j=k} \mathrm{Fil}^i \mathbf{A}_{R,\varpi}^{[u,v]} \otimes_{\mathbf{A}_R^+} \mathrm{Fil}^j \mathbf{N}(T) \subset N_{\varpi}^{[u,v]}(T).$$

These submodules are stable under the action of Γ_S and from Definition 5.34, we have the complex

$$\mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T)) = \left[\begin{array}{ccc} \mathrm{Kos}(\partial'_A, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T)) & \xrightarrow{p^r - p^{\bullet}\varphi} & \mathrm{Kos}(\partial'_A, N_{\varpi}^{[u,v/p]}(T)) \\ \Big\downarrow \partial_0 & & \Big\downarrow \partial_0 \\ \mathrm{Kos}(\partial'_A, \mathrm{Fil}^{r-1} N_{\varpi}^{[u,v]}(T)) & \xrightarrow{p^r - p^{\bullet+1}\varphi} & \mathrm{Kos}(\partial'_A, N_{\varpi}^{[u,v/p]}(T)). \end{array}\right]$$

From the theory of (φ, Γ_S)-modules in §2.4, we have $\mathbf{D}_{R,\varpi}(T(r)) = \mathbf{D}_S(T(r)) = (\mathbf{A} \otimes_{\mathbb{Z}_p} T(r))^{H_S} = \mathbf{A}_S \otimes_{\mathbf{A}_R} \mathbf{D}(T(r)) = \mathbf{A}_{R,\varpi} \otimes_{\mathbf{A}_R} \mathbf{D}(T(r))$. Using Proposition 4.16, we have the complex

$$\mathrm{Kos}(\varphi, \Gamma_S, \mathbf{D}_{R,\varpi}(T(r))) = \left[\begin{array}{ccc} \mathrm{Kos}(\Gamma'_S, \mathbf{D}_{R,\varpi}(T(r))) & \xrightarrow{1-\varphi} & \mathrm{Kos}(\Gamma'_S, \mathbf{D}_{R,\varpi}(T(r))) \\ \Big\downarrow \tau_0 & & \Big\downarrow \tau_0 \\ \mathrm{Kos}^c(\Gamma'_S, \mathbf{D}_{R,\varpi}(T(r))) & \xrightarrow{1-\varphi} & \mathrm{Kos}^c(\Gamma'_S, \mathbf{D}_{R,\varpi}(T(r))) \end{array}\right].$$

By Proposition 4.14 and Theorem 4.4 we see that the Koszul complex defined above computes the continuous G_S-cohomology of $T(r)$, i.e.

$$\mathrm{Kos}(\varphi, \Gamma_S, \mathbf{D}_{R,\varpi}(T(r))) \simeq \mathrm{R}\Gamma_{\mathrm{cont}}(G_S, T(r)).$$

The main result of this section is the comparison between the Koszul complexes introduced above.

Proposition 6.1. *There exists a p^N-quasi-isomorphism*

$$\tau_{\leq r}\mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T)) \simeq \tau_{\leq r}\mathrm{Kos}(\varphi, \Gamma_S, \mathbf{D}_{R,\varpi}(T(r))) \simeq \tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{cont}}(G_S, T(r)),$$

where $N = N(r,s) \in \mathbb{N}$ depending only on the height s of the representation T and r.

6.1. Proof of Theorem 5.8.

Using the results of previous section and Proposition 6.1, we will show Theorem 5.8. Let us recall the statement,

Theorem 6.2. *Let V be a p-adic finite q-height representation of G_R of height s, $T \subset V$ a G_R-stable $\mathbb{Z}_p$-lattice and satisfying Assumption 5.4, and let $r \in \mathbb{Z}$ such that $r \geq s+1$. Then there exists p^N-quasi-isomorphisms*

$$\alpha_r^{\mathcal{L}az} : \tau_{\leq r-s-1}\mathrm{Syn}(S, M, r) \simeq \tau_{\leq r-s-1}\mathrm{R}\Gamma_{\mathrm{cont}}(G_S, T(r)),$$
$$\alpha_{r,n}^{\mathcal{L}az} : \tau_{\leq r-s-1}\mathrm{Syn}(S, M, r)_n \simeq \tau_{\leq r-s-1}\mathrm{R}\Gamma_{\mathrm{cont}}(G_S, T/p^n(r)),$$

where $N = N(T, e, r) \in \mathbb{N}$ depending on the representations T, the absolute ramification index e of K and the twist r.

Proof. We will only prove the first quasi-isomorphism, the second quasi-isomorphism follows by reducing the first one modulo p^n and arguing in exactly the same manner. Note that by combining Proposition 5.15 and Proposition 5.17, we have p^{4r+4s}-quasi-isomorphisms

$$\tau_{\leq r-s-1}\mathrm{Syn}(M_{\varpi}^{\mathrm{PD}}, r) \simeq \tau_{\leq r-s-1}\mathrm{Syn}(N_{\varpi}^{[u]}(T), r) \simeq \tau_{\leq r-s-1}\mathrm{Syn}(M_{\varpi}^{[u,v]}, r).$$

Next, from Proposition 5.35 we have a $p^{2n(T,e)}$-quasi-isomorphism

$$\mathrm{Syn}(M_{\varpi}^{[u,v]}, r) \simeq \mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T)).$$

Finally, thanks to Proposition 6.1, we have a $p^{10r+2s+2}$-quasi-isomorphism (see the proof of the proposition for the explicit constant)

$$\tau_{\leq r}\mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T)) \simeq \tau_{\leq r}\mathrm{Kos}(\varphi, \Gamma_S, \mathbf{D}_{R,\varpi}(T(r))).$$

Combining all these statement gives us the desired conclusion with $N = 2n(T,e) + 14r + 6s + 2$. ■

In the rest of this section, we will prove Proposition 6.1.

6.2. From differential forms to infinitesimal action of Γ_S. Note that we are working under the assumption that $\frac{p-1}{p} \leq u \leq \frac{v}{p} < 1 < v$, for example, one can take $u = \frac{p-1}{p}$ and $v = p-1$. From Definition 4.21 we have the complex $\mathrm{Kos}(\mathrm{Lie}\,\Gamma'_S, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T))$ and we consider a subcomplex, i.e. a complex made of submodules in each degree stable under the differentials of the former complex

$$\begin{aligned}\mathcal{K}(\mathrm{Lie}\,\Gamma'_S, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T)) : \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T) &\xrightarrow{(\nabla_i)} (t\mathrm{Fil}^{r-1} N_{\varpi}^{[u,v]}(T))^{I'_1} \longrightarrow \cdots \\ \cdots \longrightarrow (t^n \mathrm{Fil}^{r-n} N_{\varpi}^{[u,v]}(T))^{I'_n} &\longrightarrow (t^{n+1}\mathrm{Fil}^{r-n-1} N_{\varpi}^{[u,v]}(T))^{I'_{n+1}} \longrightarrow \cdots .\end{aligned}$$

Similarly, we define the complex $\mathcal{K}(\mathrm{Lie}\,\Gamma'_S, t\mathrm{Fil}^{r-1} N_{\varpi}^{[u,v]}(T))$ as a subcomplex of $\mathrm{Kos}(\mathrm{Lie}\,\Gamma'_S, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T))$. Now, consider the map

$$\nabla_0 : \mathcal{K}(\mathrm{Lie}\,\Gamma'_S, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T)) \longrightarrow \mathcal{K}(\mathrm{Lie}\,\Gamma'_S, t\mathrm{Fil}^{r-1} N_{\varpi}^{[u,v]}(T)),$$

defined by the diagram

$$\begin{tikzcd}
\mathrm{Fil}^r N_{\varpi}^{[u,v]}(T) \arrow[r, "(\nabla_i)"] \arrow[d, "\nabla_0"] & (t\mathrm{Fil}^{r-1} N_{\varpi}^{[u,v]}(T))^{I'_1} \arrow[r] \arrow[d, "\nabla_0 - p^m"] & \cdots \arrow[r] & (t^n\mathrm{Fil}^{r-n} N_{\varpi}^{[u,v]}(T))^{I'_n} \arrow[r] \arrow[d, "\nabla_0 - np^m"] & \cdots \\
t\mathrm{Fil}^{r-1} N_{\varpi}^{[u,v]}(T) \arrow[r, "(\nabla_i)"] & (t^2\mathrm{Fil}^{r-2} N_{\varpi}^{[u,v]}(T))^{I'_1} \arrow[r] & \cdots \arrow[r] & (t^{n+1}\mathrm{Fil}^{r-n-1} N_{\varpi}^{[u,v]}(T))^{I'_n} \arrow[r] & \cdots ,
\end{tikzcd}$$

which commutes since $\nabla_0\nabla_i - \nabla_i\nabla_0 = p^m\nabla_i$ for $1 \leq i \leq d$ (see (4.4) and the discussion after Definition 4.21). We write the total complex of the diagram above as $\mathcal{K}(\mathrm{Lie}\,\Gamma_S, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T))$, which is a subcomplex of $\mathrm{Kos}(\mathrm{Lie}\,\Gamma_S, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T))$. In a similar manner, we can define complexes $\mathcal{K}(\mathrm{Lie}\,\Gamma'_S, N_{\varpi}^{[u,v/p]}(T))$ and $\mathcal{K}(\mathrm{Lie}\,\Gamma'_S, tN_{\varpi}^{[u,v/p]}(T))$, and a map ∇_0 from the former to the latter complex. Note that since the filtration on $\mathbf{A}_{R,\varpi}^{[u,v/p]}$ is trivial (see Definition 2.13), therefore $\mathrm{Fil}^k N_{\varpi}^{[u,v/p]}(T) = N_{\varpi}^{[u,v/p]}(T)$ for all $k \in \mathbb{Z}$.

Next, from Definition 5.34 we have the complex $\mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T))$. Since $\nabla_i = t\partial_i$, for $0 \leq i \leq d$, we consider the morphism of complexes $\mathrm{Kos}(\partial'_A, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T)) \to \mathcal{K}(\mathrm{Lie}\,\Gamma'_S, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T))$ given by the diagram

$$\begin{tikzcd}
\mathrm{Fil}^r N_{\varpi}^{[u,v]}(T) \arrow[r, "(\partial_i)"] \arrow[d, "t^0=id"] & (\mathrm{Fil}^{r-1} N_{\varpi}^{[u,v]}(T))^{I_1'} \arrow[r] \arrow[d, "t^1"] & \cdots \arrow[r] & (N_{\varpi}^{[u,v]}(T))^{I_r'} \arrow[r] \arrow[d, "t^r"] & (N_{\varpi}^{[u,v]}(T))^{I_{r+1}'} \arrow[r] \arrow[d, "t^{r+1}"] & \cdots \\
\mathrm{Fil}^r N_{\varpi}^{[u,v]}(T) \arrow[r, "(\nabla_i)"] & (t\mathrm{Fil}^{r-1} N_{\varpi}^{[u,v]}(T))^{I_1'} \arrow[r] & \cdots \arrow[r] & (t^r N_{\varpi}^{[u,v]}(T))^{I_r'} \arrow[r] & (t^{r+1} N_{\varpi}^{[u,v]}(T))^{I_{r+1}'} \arrow[r] & \cdots .
\end{tikzcd}$$

Since the vertical maps are bijective, it is an isomorphism of complexes. Similarly, we can define maps from $\mathrm{Kos}(\partial_A', t\mathrm{Fil}^{r-1} N_{\varpi}^{[u,v]}(T)) \to \mathcal{K}(\mathrm{Lie}\ \Gamma_S', tN_{\varpi}^{[u,v]}(T))$, $\mathrm{Kos}(\partial_A', N_{\varpi}^{[u,v/p]}(T)) \to \mathcal{K}(\mathrm{Lie}\ \Gamma_S', N_{\varpi}^{[u,v/p]}(T))$ and $\mathrm{Kos}(\partial_A', N_{\varpi}^{[u,v/p]}(T)) \to \mathcal{K}(\mathrm{Lie}\ \Gamma_S', tN_{\varpi}^{[u,v/p]}(T))$, which are isomorphisms as well. Since each term of these complexes admit a Frobenius-semilinear morphism $\varphi : t^j \mathrm{Fil}^{r-j} N_{\varpi}^{[u,v]}(T) \to t^j N_{\varpi}^{[u,v/p]}(T)$, we obtain an induced morphism

$$\left[\begin{tikzcd}
\mathrm{Kos}(\partial_A', \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T)) \arrow[r, "p^r - p^\bullet\varphi"] \arrow[d, "\partial_0"] & \mathrm{Kos}(\partial_A', N_{\varpi}^{[u,v/p]}(T)) \arrow[d, "\partial_0"] \\
\mathrm{Kos}(\partial_A', \mathrm{Fil}^{r-1} N_{\varpi}^{[u,v]}(T)) \arrow[r, "p^r - p^{\bullet+1}\varphi"] & \mathrm{Kos}(\partial_A', N_{\varpi}^{[u,v/p]}(T))
\end{tikzcd}\right] \longrightarrow$$

$$\left[\begin{tikzcd}
\mathcal{K}(\mathrm{Lie}\ \Gamma_S', \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T)) \arrow[r, "p^r - \varphi"] \arrow[d, "\nabla_0"] & \mathcal{K}(\mathrm{Lie}\ \Gamma_S', N_{\varpi}^{[u,v/p]}(T)) \arrow[d, "\nabla_0"] \\
\mathcal{K}(\mathrm{Lie}\ \Gamma_S', t\mathrm{Fil}^{r-1} N_{\varpi}^{[u,v]}(T)) \arrow[r, "p^r - \varphi"] & \mathcal{K}(\mathrm{Lie}\ \Gamma_S', tN_{\varpi}^{[u,v/p]}(T))
\end{tikzcd}\right], \tag{6.1}$$

where the source complex in (6.1) above is $\mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T))$. Tautologically, we have that

Lemma 6.3. *The map constructed in* (6.1) *is a quasi-isomorphism of complexes.*

Next, recall that s is the height of V and $r \geq s+1$ is an integer. Let us set $N_{\varpi}^{[u,v]}(T(r)) = \mathbf{A}_{R,\varpi}^{[u,v]} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T(r))$, and we can define a filtration on this module given as

$$\mathrm{Fil}^k N_{\varpi}^{[u,v]}(T(r)) := \text{closure of } \sum_{i+j=k} \mathrm{Fil}^i \mathbf{A}_{R,\varpi}^{[u,v]} \otimes_{\mathbf{A}_R^+} \mathrm{Fil}^j \mathbf{N}(T(r)) \subset N_{\varpi}^{[u,v]}(T(r)), \ \text{ for } k \in \mathbb{Z}.$$

These submodules are stable under the action of Γ_S. Let ϵ^{-r} denote a $\mathbb{Z}_p$-basis of $\mathbb{Z}_p(-r)$, then we have

$$\begin{aligned}
(t^r \otimes \epsilon^{-r})\ \mathrm{Fil}^k N_{\varpi}^{[u,v]}(T(r)) &= \text{closure of } (t^r \otimes \epsilon^{-r}) \sum_{i+j=k} \mathrm{Fil}^i \mathbf{A}_{R,\varpi}^{[u,v]} \otimes_{\mathbf{A}_R^+} \mathrm{Fil}^j \mathbf{N}(T(r)) \\
&= \text{closure of } \tfrac{t^r}{\pi^r} \sum_{i+j=k} \mathrm{Fil}^i \mathbf{A}_{R,\varpi}^{[u,v]} \otimes_{\mathbf{A}_R^+} \mathrm{Fil}^{j+r} \mathbf{N}(T) = \mathrm{Fil}^{r+k} N_{\varpi}^{[u,v]}(T),
\end{aligned} \tag{6.2}$$

where the second equality is the result of observation made in Lemma 3.5, and the third equality comes from the fact that $\frac{t}{\pi}$ is a unit in $\mathbf{A}_{R,\varpi}^{[u,v]}$ (see Lemma 2.30). Moreover, we also have that $(t^r \otimes \epsilon^{-r}) N_{\varpi}^{[u,v/p]}(T(r)) = t^r \pi^{-r} N_{\varpi}^{[u,v/p]}(T) = N_{\varpi}^{[u,v/p]}(T)$.

From Remark 5.23, we have that ∇_i is well-defined over $N_{\varpi}^{[u,v]}(T(r))$, for $0 \leq i \leq d$. Now using Definition 4.21 we have the complex $\mathrm{Kos}(\mathrm{Lie}\ \Gamma_S', \mathrm{Fil}^0 N_{\varpi}^{[u,v]}(T(r)))$, and we consider the subcomplex

$$\begin{aligned}
\mathcal{K}(\mathrm{Lie}\ \Gamma_S', \mathrm{Fil}^0 N_{\varpi}^{[u,v]}(T(r))) : \mathrm{Fil}^0 N_{\varpi}^{[u,v]}(T(r)) &\xrightarrow{(\nabla_i)} (t\mathrm{Fil}^{-1} N_{\varpi}^{[u,v]}(T(r)))^{I_1'} \longrightarrow \cdots \\
&\cdots \longrightarrow (t^q \mathrm{Fil}^{-q} N_{\varpi}^{[u,v]}(T(r)))^{I_q'} \longrightarrow \cdots .
\end{aligned}$$

Similar to above, we can define the complex $\mathcal{K}(\mathrm{Lie}\ \Gamma'_S, t\mathrm{Fil}^{-1}N_{\varpi}^{[u,v]}(T(r)))$ as a subcomplex of $\mathrm{Kos}(\mathrm{Lie}\ \Gamma'_S, \mathrm{Fil}^0 N_{\varpi}^{[u,v]}(T(r)))$, and a map

$$\nabla_0 : \mathcal{K}(\mathrm{Lie}\ \Gamma'_S, \mathrm{Fil}^0 N_{\varpi}^{[u,v]}(T(r))) \longrightarrow \mathcal{K}(\mathrm{Lie}\ \Gamma'_S, t\mathrm{Fil}^{-1}N_{\varpi}^{[u,v]}(T(r))).$$

The total complex of the latter map, written as $\mathcal{K}(\mathrm{Lie}\ \Gamma_S, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T))$, is a subcomplex of $\mathrm{Kos}(\mathrm{Lie}\ \Gamma_S, \mathrm{Fil}^0 N_{\varpi}^{[u,v]}(T(r)))$. Again, in a similar manner, we can define complexes $\mathcal{K}(\mathrm{Lie}\ \Gamma'_S, N_{\varpi}^{[u,v/p]}(T(r)))$ and $\mathcal{K}(\mathrm{Lie}\ \Gamma'_S, tN_{\varpi}^{[u,v/p]}(T(r)))$, and a map ∇_0 from the former to the latter complex.

Consider the morphism $\mathcal{K}(\mathrm{Lie}\ \Gamma'_S, \mathrm{Fil}^0 N^{[u,v]}(T(r))) \to \mathcal{K}(\mathrm{Lie}\ \Gamma'_S, \mathrm{Fil}^r M_{\varpi}^{[u,v]})$ given by the diagram

$$\begin{tikzcd}
\mathrm{Fil}^0 N^{[u,v]}(T(r)) \arrow[r, "(\nabla_i)"] \arrow[d, "t^r\otimes\epsilon^{-r}"] & (t\mathrm{Fil}^{-1}N^{[u,v]}(T(r)))^{I'_1} \arrow[r] \arrow[d, "t^r\otimes\epsilon^{-r}"] & \cdots \arrow[r] & (t^q\mathrm{Fil}^{-q}N^{[u,v]}(T(r)))^{I'_q} \arrow[r] \arrow[d, "t^r\otimes\epsilon^{-r}"] & \cdots \\
\mathrm{Fil}^r M_{\varpi}^{[u,v]} \arrow[r, "(\nabla_i)"] & (t\mathrm{Fil}^{r-1}M_{\varpi}^{[u,v]})^{I'_1} \arrow[r] & \cdots \arrow[r] & (t^q\mathrm{Fil}^{r-q}M_{\varpi}^{[u,v]})^{I'_q} \arrow[r] & \cdots,
\end{tikzcd}$$

which is is bijective in each term and therefore an isomorphism. Considering similar maps between complexes considered above, we obtain a morphism (multiplication by $t^r\otimes\epsilon^{-r}$ on each term)

$$\begin{bmatrix} \mathcal{K}(\mathrm{Lie}\ \Gamma'_S, \mathrm{Fil}^0 N_{\varpi}^{[u,v]}(T(r))) & \xrightarrow{p^r(1-\varphi)} & \mathcal{K}(\mathrm{Lie}\ \Gamma'_S, N_{\varpi}^{[u,v/p]}(T(r))) \\ \downarrow \nabla_0 & & \downarrow \nabla_0 \\ \mathcal{K}(\mathrm{Lie}\ \Gamma'_S, t\mathrm{Fil}^{-1}N_{\varpi}^{[u,v]}(T(r))) & \xrightarrow{p^r(1-\varphi)} & \mathcal{K}(\mathrm{Lie}\ \Gamma'_S, tN_{\varpi}^{[u,v/p]}(T(r))) \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} \mathcal{K}(\mathrm{Lie}\ \Gamma'_S, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T)) & \xrightarrow{p^r-\varphi} & \mathcal{K}(\mathrm{Lie}\ \Gamma'_S, N_{\varpi}^{[u,v/p]}(T)) \\ \downarrow \nabla_0 & & \downarrow \nabla_0 \\ \mathcal{K}(\mathrm{Lie}\ \Gamma'_S, t\mathrm{Fil}^{r-1}N_{\varpi}^{[u,v]}(T)) & \xrightarrow{p^r-\varphi} & \mathcal{K}(\mathrm{Lie}\ \Gamma'_S, tN_{\varpi}^{[u,v/p]}(T)) \end{bmatrix}. \tag{6.3}$$

Again, it is immediate that

Lemma 6.4. *The map constructed in (6.3) is a quasi-isomorphism of complexes.*

In order to proceed from "Lie Γ_S-Koszul complexes" discussed above to "Γ_S-Koszul complexes", we modify the source complex in the map of Lemma 6.4 as follows:

$$\mathcal{K}(\varphi, \mathrm{Lie}\ \Gamma_S, N_{\varpi}^{[u,v]}(T(r))) := \begin{bmatrix} \mathcal{K}(\mathrm{Lie}\ \Gamma'_S, \mathrm{Fil}^0 N_{\varpi}^{[u,v]}(T(r))) & \xrightarrow{1-\varphi} & \mathcal{K}(\mathrm{Lie}\ \Gamma'_S, N_{\varpi}^{[u,v/p]}(T(r))) \\ \nabla_0 \downarrow & & \downarrow \nabla_0 \\ \mathcal{K}(\mathrm{Lie}\ \Gamma'_S, t\mathrm{Fil}^{-1}N_{\varpi}^{[u,v]}(T(r))) & \xrightarrow{1-\varphi} & \mathcal{K}(\mathrm{Lie}\ \Gamma'_S, tN_{\varpi}^{[u,v/p]}(T(r))) \end{bmatrix}.$$

Remark 6.5. The complex $\mathcal{K}(\varphi, \mathrm{Lie}\ \Gamma_S, N_{\varpi}^{[u,v]}(T(r)))$ is p^{4r}-isomorphic to the source complex in the map of Lemma 6.4.

Combining Lemmas 6.3 & 6.4, and Remark 6.5, we get

Proposition 6.6. *There exists a p^{4r}-quasi-isomorphism of complexes*

$$\mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r N_{\varpi}^{[u,v]}(T)) \simeq \mathcal{K}(\varphi, \mathrm{Lie}\ \Gamma_S, N_{\varpi}^{[u,v]}(T(r))).$$

6.3. From infinitesimal action of Γ_S to continuous action of Γ_S. In the previous section, we changed from complexes involving the operators ∂_i to complexes invoving the operators ∇_i. In this section, we will further replace these complexes with complexes involving operators $\gamma_i - 1$. Note that we are working under the assumption that $\frac{p-1}{p} \leq u \leq \frac{v}{p} < 1 < v$, for example, one can take $u = \frac{p-1}{p}$ and $v = p-1$.

Next, we want to construct similar complexes for the action of Γ_S. Note that we have

$$(\gamma_i - 1)\mathrm{Fil}^k N_{\varpi}^{[u,v]}(T(r)) \subset \mathrm{Fil}^k N_{\varpi}^{[u,v]}(T(r)) \cap \pi N_{\varpi}^{[u,v]}(T(r)) = \pi \mathrm{Fil}^{k-1} N_{\varpi}^{[u,v]}(T(r))$$

where the last equality follows from Lemma 3.6. We can define a subcomplex of $\mathrm{Kos}(\Gamma'_S, \mathrm{Fil}^0 N_{\varpi}^{[u,v]}(T(r)))$ as

$$\begin{aligned}\mathcal{K}(\Gamma'_S, \mathrm{Fil}^0 N_{\varpi}^{[u,v]}(T(r))) : \mathrm{Fil}^0 N_{\varpi}^{[u,v]}(T(r)) &\xrightarrow{(\tau_i)} (\pi \mathrm{Fil}^{-1} N_{\varpi}^{[u,v]}(T(r)))^{I'_1} \longrightarrow \\ &\longrightarrow (\pi^2 \mathrm{Fil}^{-2} N_{\varpi}^{[u,v]}(T(r)))^{I'_2} \longrightarrow \cdots .\end{aligned} \tag{6.4}$$

Similarly, we can define the complex $\mathcal{K}^c(\Gamma'_S, \pi\mathrm{Fil}^{-1} N_{\varpi}^{[u,v]}(T(r)))$ as a subcomplex of $\mathrm{Kos}^c(\Gamma'_S, \mathrm{Fil}^0 N_{\varpi}^{[u,v]}(T(r)))$ (see Definition 4.11). Now, consider the map

$$\tau_0 : \mathcal{K}(\Gamma'_S, \mathrm{Fil}^0 N_{\varpi}^{[u,v]}(T(r))) \longrightarrow \mathcal{K}^c(\Gamma'_S, t\mathrm{Fil}^{-1} N_{\varpi}^{[u,v]}(T(r))), \tag{6.5}$$

defined by the commutative diagram

$$\begin{tikzcd}
\mathrm{Fil}^0 N_{\varpi}^{[u,v]}(T(r)) \arrow[r, "(\tau_i)"] \arrow[d, "\tau_0^0"] & (\pi \mathrm{Fil}^{-1} N_{\varpi}^{[u,v]}(T(r)))^{I'_1} \arrow[r] \arrow[d, "\tau_0^1"] & (\pi^2 \mathrm{Fil}^{-2} N_{\varpi}^{[u,v]}(T(r)))^{I'_2} \arrow[r] \arrow[d, "\tau_0^2"] & \cdots \\
\pi \mathrm{Fil}^{-1} N_{\varpi}^{[u,v]}(T(r)) \arrow[r, "(\tau_i)"] & (\pi^2 \mathrm{Fil}^{-2} N_{\varpi}^{[u,v]}(T(r)))^{I'_1} \arrow[r] & (\pi^3 \mathrm{Fil}^{-3} N_{\varpi}^{[u,v]}(T(r)))^{I'_2} \arrow[r] & \cdots ,
\end{tikzcd}$$

where the vertical maps are as in Definitions 4.10 & 4.13. We write the total complex of the diagram above as $\mathcal{K}(\Gamma_S, \mathrm{Fil}^0 N_{\varpi}^{[u,v]}(T(r)))$, which is a subcomplex of $\mathrm{Kos}(\Gamma_S, \mathrm{Fil}^0 N_{\varpi}^{[u,v]}(T(r)))$. In a similar manner, we can define complexes $\mathcal{K}(\Gamma'_S, N_{\varpi}^{[u,v/p]}(T(r)))$ and $\mathcal{K}^c(\Gamma'_S, \pi N_{\varpi}^{[u,v/p]}(T(r)))$ and a map τ_0 from the former to the latter complex.

Next, we consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Fil}^0 N_{\varpi}^{[u,v]}(T(r)) \arrow[r, "(\tau_i)"] \arrow[d, "id"] & (t \mathrm{Fil}^{-1} N_{\varpi}^{[u,v]}(T(r)))^{I'_1} \arrow[r] \arrow[d, "\beta_1"] & (t^2 \mathrm{Fil}^{-2} N_{\varpi}^{[u,v]}(T(r)))^{I'_2} \arrow[r] \arrow[d, "\beta_2"] & \cdots \\
\mathrm{Fil}^0 N_{\varpi}^{[u,v]}(T(r)) \arrow[r, "(\nabla_i)"] & (t \mathrm{Fil}^{-1} N_{\varpi}^{[u,v]}(T(r)))^{I'_1} \arrow[r] & (t^2 \mathrm{Fil}^{-2} N_{\varpi}^{[u,v]}(T(r)))^{I'_2} \arrow[r] & \cdots ,
\end{tikzcd}$$

where $\beta_q : (a_{i_1 \cdots i_q}) \mapsto (\nabla_{i_q} \cdots \nabla_{i_1} \tau_{i_1}^{-1} \cdots \tau_{i_q}^{-1}(a_{i_1 \cdots i_q}))$ for $1 \leq q \leq d$. Notice that since $\frac{t}{\pi}$ is a unit in $\mathbf{A}_{R,\varpi}^{[u,v]}$ (see Lemma 2.30), the top complex in the diagram above is exactly the complex $\mathcal{K}(\Gamma'_S, \mathrm{Fil}^0 N_{\varpi}^{[u,v]}(T(r)))$ from (6.4). This defines a map

$$\beta : \mathcal{K}(\Gamma'_S, \mathrm{Fil}^0 N_{\varpi}^{[u,v]}(T(r))) \longrightarrow \mathcal{K}(\mathrm{Lie}\ \Gamma'_S, \mathrm{Fil}^0 N_{\varpi}^{[u,v]}(T(r))),$$

Similarly, we can consider the commutative diagram

$$\begin{tikzcd}
t \mathrm{Fil}^{-1} N_{\varpi}^{[u,v]}(T(r)) \arrow[r, "(\tau_i^c)"] \arrow[d, "\beta_0^c"] & (t^2 \mathrm{Fil}^{-2} N_{\varpi}^{[u,v]}(T(r)))^{I'_1} \arrow[r] \arrow[d, "\beta_1^c"] & (t^3 \mathrm{Fil}^{-3} N_{\varpi}^{[u,v]}(T(r)))^{I'_2} \arrow[r] \arrow[d, "\beta_2^c"] & \cdots \\
t \mathrm{Fil}^{-1} N_{\varpi}^{[u,v]}(T(r)) \arrow[r, "(\nabla_i)"] & (t^2 \mathrm{Fil}^{-2} N_{\varpi}^{[u,v]}(T(r)))^{I'_1} \arrow[r] & (t^3 \mathrm{Fil}^{-3} N_{\varpi}^{[u,v]}(T(r)))^{I'_2} \arrow[r] & \cdots ,
\end{tikzcd}$$

with $\beta_0^c = \nabla_0\tau_0^{-1}$ and

$$\beta_q^c : (a_{i_1\cdots i_q}) \longmapsto (\nabla_{i_q}\cdots\nabla_{i_1}\nabla_0\tau_0^{-1}\tau_{i_1}^{c,-1}\cdots\tau_{i_q}^{c,-1}(a_{i_1\cdots i_q})) \text{ for } 1 \leq q \leq d.$$

Recall that $c = \chi(\gamma_0) = \exp(p^m)$. Again, this defines a map

$$\beta^c : \mathcal{K}^c(\Gamma'_S, t\mathrm{Fil}^{-1}N_{\varpi}^{[u,v]}(T(r))) \longrightarrow \mathcal{K}^c(\mathrm{Lie}\ \Gamma'_S, t\mathrm{Fil}^{-1}N_{\varpi}^{[u,v]}(T(r))).$$

Remark 6.7. The definition of maps β and β^c continue to hold after base changing each term of the complexes to the ring $\mathbf{A}_{R,\varpi}^{[u,v/p]}$.

Next, for $j \in \mathbb{N}$, we have $t^j\mathrm{Fil}^{-j}N_{\varpi}^{[u,v]}(T(r)) \subset N_{\varpi}^{[u,v]}(T(r))$ and the induced Frobenius gives

$$\varphi(t^j\mathrm{Fil}^{-j}N_{\varpi}^{[u,v]}(T(r))) = \varphi(\pi^{j-r}\mathrm{Fil}^{r-j}N_{\varpi}^{[u,v]}(T)(r)) \subset \pi^{j-r}N_{\varpi}^{[u,v/p]}(T)(r) = t^jN_{\varpi}^{[u,v/p]}(T(r)),$$

where we have used the fact that $\frac{t}{\pi} \in \mathbf{A}_{R,\varpi}^{[u,v]}$ is a unit (see Lemma 2.30). Using the Frobenius morphism and the map between complexes discussed above, we obtain an induced morphism

$$\begin{bmatrix} \mathcal{K}(\Gamma'_S, \mathrm{Fil}^0N_{\varpi}^{[u,v]}(T(r))) \xrightarrow{1-\varphi} \mathcal{K}(\Gamma'_S, N_{\varpi}^{[u,v/p]}(T(r))) \\ \downarrow \tau_0 \qquad\qquad\qquad\qquad \downarrow \tau_0 \\ \mathcal{K}^c(\Gamma'_S, t\mathrm{Fil}^{-1}N_{\varpi}^{[u,v]}(T(r))) \xrightarrow{1-\varphi} \mathcal{K}^c(\Gamma'_S, tN_{\varpi}^{[u,v/p]}(T(r))) \end{bmatrix} \xrightarrow{(\beta,\beta^c)}$$

$$\begin{bmatrix} \mathcal{K}(\mathrm{Lie}\ \Gamma'_S, \mathrm{Fil}^0N_{\varpi}^{[u,v]}(T(r))) \xrightarrow{1-\varphi} \mathcal{K}(\mathrm{Lie}\ \Gamma'_S, N_{\varpi}^{[u,v/p]}(T(r))) \\ \downarrow \nabla_0 \qquad\qquad\qquad\qquad \downarrow \nabla_0 \\ \mathcal{K}(\mathrm{Lie}\ \Gamma'_S, t\mathrm{Fil}^{-1}N_{\varpi}^{[u,v]}(T(r))) \xrightarrow{1-\varphi} \mathcal{K}(\mathrm{Lie}\ \Gamma'_S, tN_{\varpi}^{[u,v/p]}(T(r))) \end{bmatrix}.$$

We denote the complex on left as $\mathcal{K}(\varphi, \Gamma_S, N_{\varpi}^{[u,v]}(T(r)))$ and write the map as

$$\mathscr{L} = (\beta, \beta^c) : \mathcal{K}(\varphi, \Gamma_S, N_{\varpi}^{[u,v]}(T(r))) \longrightarrow \mathcal{K}(\varphi, \mathrm{Lie}\ \Gamma_S, N_{\varpi}^{[u,v]}(T(r))),$$

Proposition 6.8. *The morphism of complexes $\mathscr{L}$ from the construction above is an isomorphism.*

Proof. The proof follows in a manner similar to [CN17, Lemma 4.6]. From the fact that $\nabla_i\tau_i^{-1}$, for $0 \leq i \leq d$, is invertible (see Corollary 5.22) and $[\nabla_i, \nabla_j] = 0$, for $1 \leq i, j \leq d$, we get that the map β above is an isomorphism.

Next, we will show that the map β_q^c, for $1 \leq q \leq d$, is a well-defined isomorphism. For this, we need to show that $\nabla_{i_q}\cdots\nabla_{i_1}\nabla_0\tau_0^{-1}\tau_{i_1}^{c,-1}\cdots\tau_{i_q}^{c,-1}$ are well-defined isomorphisms, for $1 \leq i_1 < \cdots < i_q \leq d$. We can reduce the map to

$$(\nabla_{i_q}/\tau_{i_q})\cdots(\nabla_{i_1}/\tau_{i_1})\tau_{i_q}\cdots\tau_{i_1}\nabla_0\tau_0^{-1}\tau_{i_1}^{c,-1}\cdots\tau_{i_q}^{c,-1},$$

and since ∇_i/τ_i is invertible for $0 \leq i \leq d$, we only need to show that $\tau_{i_q}\cdots\tau_{i_1}\nabla_0\tau_0^{-1}\tau_{i_1}^{c,-1}\cdots\tau_{i_q}^{c,-1}$ is a well-defined isomorphism. Using the proof of Lemma 4.20, we can write

$$\tau_{i_q}\cdots\tau_{i_1}\nabla_0\tau_0^{-1}\tau_{i_1}^{c,-1}\cdots\tau_{i_q}^{c,-1} = \sum_{k\geq 0} a_k\tau_{i_q}\cdots\tau_{i_1}(\gamma_0 - 1)^k\tau_{i_1}^{c,-1}\cdots\tau_{i_q}^{c,-1},$$

where $a_k \in O_F$. Using the fact that $\gamma_0\gamma_i^{a/c} = \gamma_i^a\gamma_0$, we get that

$$(\gamma_i^a - 1)(\gamma_0 - x) = (\gamma_0 - x\delta(\gamma_i^a))(\gamma_i^{a/c} - 1), \text{ where } \delta(\gamma_i^a) := \tfrac{\gamma_i^a - 1}{\gamma_i^{a/c} - 1},$$

which yields

$$(\gamma_i^a - 1)(\gamma_0 - 1)^k = (\gamma_0 - \delta(\gamma_i^a))(\gamma_0 - \delta(\gamma_i^{a/c})) \cdots (\gamma_0 - \delta(\gamma_i^{a/c^{k-1}}))(\gamma_i^{a/c^k} - 1).$$

So we can write

$$\begin{aligned} \tau_{i_q} \cdots \tau_{i_1}(\gamma_0 - 1)^k \tau_{i_1}^{c_i - 1} \cdots \tau_{i_q}^{c_i - 1} &= (\gamma_0 - \delta_k) \cdots (\gamma_0 - \delta_1) \frac{\gamma_{i_q}^{1/c^k} - 1}{\gamma_{i_q}^c - 1} \cdots \frac{\gamma_{i_1}^{1/c^k} - 1}{\gamma_{i_1}^c - 1} \\ &= (\gamma_0 - \delta_k) \cdots (\gamma_0 - \delta_1)\delta_0. \end{aligned} \tag{6.6}$$

Observe that for $0 \leq i \leq d$ and $j \in \mathbb{Z}$, we have

$$\frac{\gamma_i^{1/c^j} - 1}{\gamma_i^{1/c^{j+1}} - 1} = \frac{\gamma_i^{1/c^j} - 1}{\gamma_i - 1} \cdot \frac{\gamma_i - 1}{\gamma_i^{1/c^{j+1}} - 1} \quad \text{and} \quad \frac{\gamma_i^{1/c^k} - 1}{\gamma_i^c - 1} = \frac{\gamma_i^{1/c^k} - 1}{\gamma_i - 1} \cdot \frac{\gamma_i - 1}{\gamma_i^c - 1} \in 1 + (p^m, \gamma_i - 1)\mathbb{Z}_p[[\Gamma_S]].$$

Therefore, in (6.6) we have that $\delta_j \in 1 + (p^m, (\gamma_1 - 1), \ldots, (\gamma_d - 1))$. Writing $(\gamma_0 - \delta_j) = (\gamma_0 - 1) + (1 - \delta_j)$, we conclude that

$$\tau_{i_q} \cdots \tau_{i_1}(\gamma_0 - 1)^k \tau_{i_1}^{c_i - 1} \cdots \tau_{i_q}^{c_i - 1} \in (p^m, \gamma_0 - 1, \ldots, \gamma_d - 1)^k.$$

Now from Lemma 2.32, the fact that $\gamma_i - 1$ acts as a twisted derivation and using the estimate for p-adic valuation of coefficients as in the proof of Lemma 4.20, it follows that the series of operators

$$\sum_{k \geq 0} a_k \tau_{i_q} \cdots \tau_{i_1}(\gamma_0 - 1)^k \tau_{i_1}^{c_i - 1} \cdots \tau_{i_q}^{c_i - 1}$$

converge and therefore $\nabla_{i_q} \cdots \nabla_{i_1} \nabla_0 \tau_0^{-1} \tau_{i_1}^{c_i - 1} \cdots \tau_{i_q}^{c_i - 1}$ is well-defined. The same arguments show that the series of operators $\sum_{k \geq 0} b_k \tau_{i_q}^c \cdots \tau_{i_1}^c (\gamma_0 - 1)^k \tau_{i_1}^{-1} \cdots \tau_{i_q}^{-1}$ converge as an inverse to the previous operator (see Lemma 4.20 for the definition of b_k). This establishes the claim. ■

6.4. Change of annulus of convergence : Part 1. Now that we have changed our original complex to a complex involving operators $\gamma_i - 1$, in this section, we will pass from the ring $\mathbf{A}_{R,\varpi}^{[u,v]}$ to the overconvergent ring $\mathbf{A}_{R,\varpi}^{(0,v]+}$ and also twist our module by r. Note that we are working under the assumption that $\frac{p-1}{p} \leq u \leq \frac{v}{p} < 1 < v$, for example, one can take $u = \frac{p-1}{p}$ and $v = p - 1$.

Let us set $N_\varpi^{(0,v]+}(T(r)) := \mathbf{A}_{R,\varpi}^{(0,v]+} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T(r))$. We can equip this module with a filtration given as

$$\mathrm{Fil}^k N_\varpi^{(0,v]+}(T(r)) := \text{closure of } \sum_{i+j=k} \mathrm{Fil}^i \mathbf{A}_{R,\varpi}^{(0,v]+} \otimes_{\mathbf{A}_R^+} \mathrm{Fil}^j \mathbf{N}(T(r)) \subset N_\varpi^{(0,v]+}(T(r)), \ \text{ for } k \in \mathbb{Z},$$

where we put the filtration on $\mathbf{A}_{R,\varpi}^{(0,v]+}$ by identifying it with the ring $R_\varpi^{(0,v]+}$ via the map ι_{cycl} (see §2.7), and the latter ring has a filtration described in Definition 2.13. These submodules are stable under the action of Γ_S.

Next, we define a subcomplex of $\mathrm{Kos}(\Gamma_S', \mathrm{Fil}^0 N_\varpi^{(0,v]+}(T(r)))$ as

$$\begin{aligned} \mathcal{K}(\Gamma_S', \mathrm{Fil}^0 N_\varpi^{(0,v]+}(T(r))) : \mathrm{Fil}^0 N_\varpi^{(0,v]+}(T(r)) &\xrightarrow{(\tau_i)} (\pi \mathrm{Fil}^{-1} N_\varpi^{(0,v]+}(T(r)))^{I_1'} \longrightarrow \\ &\longrightarrow (\pi^2 \mathrm{Fil}^{-2} N_\varpi^{(0,v]+}(T(r)))^{I_2'} \longrightarrow \cdots. \end{aligned}$$

Similarly, we can define the complex $\mathcal{K}^c(\Gamma_S', \pi \mathrm{Fil}^{-1} N_\varpi^{(0,v]+}(T(r)))$ as a subcomplex of $\mathrm{Kos}^c(\Gamma_S', \mathrm{Fil}^0 N_\varpi^{(0,v]+}(T(r)))$ (see Definition 4.11). Now, consider the map

$$\tau_0 : \mathcal{K}(\Gamma_S', \mathrm{Fil}^0 N_\varpi^{(0,v]+}(T(r))) \longrightarrow \mathcal{K}^c(\Gamma_S', \pi \mathrm{Fil}^{-1} N_\varpi^{(0,v]+}(T(r))),$$

defined by a commutative diagram similar to (6.5) (see also Definitions 4.10 & 4.13)

$$\begin{tikzcd}
\mathrm{Fil}^0 N_{\varpi}^{(0,v]+}(T(r)) \arrow[r, "(\tau_i)"] \arrow[d, "\tau_0^0"] & (\pi\mathrm{Fil}^{-1} N_{\varpi}^{(0,v]+}(T(r)))^{I_1'} \arrow[r] \arrow[d, "\tau_0^1"] & (\pi^2\mathrm{Fil}^{-2} N_{\varpi}^{(0,v]+}(T(r)))^{I_2'} \arrow[r] \arrow[d, "\tau_0^2"] & \cdots \\
\pi\mathrm{Fil}^{-1} N_{\varpi}^{(0,v]+}(T(r)) \arrow[r, "(\tau_i)"] & (\pi^2\mathrm{Fil}^{-2} N_{\varpi}^{(0,v]+}(T(r)))^{I_1'} \arrow[r] & (\pi^3\mathrm{Fil}^{-3} N_{\varpi}^{(0,v]+}(T(r)))^{I_2'} \arrow[r] & \cdots .
\end{tikzcd}$$

We write the total complex of the diagram as $\mathcal{K}(\Gamma_S, \mathrm{Fil}^0 N_{\varpi}^{(0,v]+}(T(r)))$, which is a subcomplex of $\mathrm{Kos}(\Gamma_S, \mathrm{Fil}^0 N_{\varpi}^{(0,v]+}(T(r)))$. In a similar manner, we can define complexes $\mathcal{K}(\Gamma_S', N_{\varpi}^{(0,v/p]+}(T(r)))$ and $\mathcal{K}^c(\Gamma_S', \pi N_{\varpi}^{(0,v/p]+}(T(r)))$ and a map τ_0 from former to the latter complex.

Next, for $j \in \mathbb{N}$, we have $\pi^j \mathrm{Fil}^{-j} N_{\varpi}^{(0,v]+}(T(r)) \subset N_{\varpi}^{(0,v]+}(T(r))$ and the induced Frobenius gives

$$\varphi(\pi^j \mathrm{Fil}^{-j} N_{\varpi}^{(0,v]+}(T(r))) = \varphi(\pi^{j-r}\mathrm{Fil}^{r-j} N_{\varpi}^{(0,v]+}(T)(r)) \subset \pi^{j-r} N_{\varpi}^{(0,v/p]+}(T)(r) = \pi^j N_{\varpi}^{(0,v/p]+}(T(r)).$$

Using the Forbenius morphism and the map between complexes discussed above, we define the complex

$$\mathcal{K}(\varphi, \Gamma_S, N_{\varpi}^{(0,v]+}(T(r))) := \left[\begin{tikzcd}
\mathcal{K}(\Gamma_S', \mathrm{Fil}^0 N_{\varpi}^{(0,v]+}(T(r))) \arrow[r, "1-\varphi"] \arrow[d, "\tau_0"'] & \mathcal{K}(\Gamma_S', N_{\varpi}^{(0,v/p]+}(T(r))) \arrow[d, "\tau_0"] \\
\mathcal{K}^c(\Gamma_S', \pi\mathrm{Fil}^{-1} N_{\varpi}^{(0,v]+}(r)) \arrow[r, "1-\varphi"] & \mathcal{K}^c(\Gamma_S', \pi N_{\varpi}^{(0,v/p]+}(T(r)))
\end{tikzcd}\right].$$

It is obvious that we can compare this to the complex defined in the previous section.

Proposition 6.9. *The natural map*

$$\mathcal{K}(\varphi, \Gamma_S, N_{\varpi}^{(0,v]+}(T(r))) \longrightarrow \mathcal{K}(\varphi, \Gamma_S, N_{\varpi}^{[u,v]}(T(r)))$$

induced by the inclusion $N_{\varpi}^{(0,v]+}(T(r)) \subset N_{\varpi}^{[u,v]}(T(r))$ is a p^{3r}-quasi-isomorphism.

Proof. The map in the claim is injective, so we only need to show that the cokernel complex is killed by p^{3r}. In the cokernel complex, for $k \in \mathbb{Z}$, we have maps

$$1-\varphi : \pi^k \mathrm{Fil}^{-k} N_{\varpi}^{[u,v]}(T(r))/\pi^k \mathrm{Fil}^{-k} N_{\varpi}^{(0,v]+}(T(r)) \longrightarrow \pi^k N_{\varpi}^{[u,v/p]}(T(r))/\pi^k N_{\varpi}^{(0,v/p]+}(T(r)), \tag{6.7}$$

and it is enough to show that these maps are p^{4r}-bijective. Let us define the modules

$$N_{\varpi}^{(0,v]+}(T)(r) := \mathbf{A}_{R,\varpi}^{(0,v]+} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)(r) \quad \text{and} \quad N_{\varpi}^{[u,v]}(T)(r) := \mathbf{A}_{R,\varpi}^{[u,v]} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)(r),$$

equipped with filtrations given by the usual filtration on tensor products. It is also immediately clear that $\pi^k \mathrm{Fil}^{-k} N_{\varpi}^{(0,v]+}(T(r)) = \pi^{k-r}\mathrm{Fil}^{r-k} N_{\varpi}^{(0,v]+}(T)(r)$ and $\pi^k \mathrm{Fil}^{-k} N_{\varpi}^{[u,v]}(T(r)) = \pi^{k-r}\mathrm{Fil}^{r-k} N_{\varpi}^{[u,v]}(T)(r)$, for $k \in \mathbb{Z}$ (see (6.2) for a similar conclusion).

Let $n = r - k$ and we rewrite (6.7) as

$$1-\varphi : \pi^{-n}\mathrm{Fil}^n N_{\varpi}^{[u,v]}(T)(r)/\pi^{-n}\mathrm{Fil}^n N_{\varpi}^{(0,v]+}(T)(r) \longrightarrow \pi^{-n} N_{\varpi}^{[u,v/p]}(T)(r)/\pi^{-n} N_{\varpi}^{(0,v/p]+}(T)(r), \tag{6.8}$$

For $n \leq 0$, the claim follows from Lemma 6.10. For $n > 0$, we begin by showing that the natural map

$$\pi_1^{-n} N_{\varpi}^{[u,v]}(T)(r)/\pi_1^{-n} N_{\varpi}^{(0,v]+}(T)(r) \longrightarrow \pi^{-n}\mathrm{Fil}^n N_{\varpi}^{[u,v]}(T)(r)/\pi^{-n}\mathrm{Fil}^n N_{\varpi}^{(0,v]+}(T)(r), \tag{6.9}$$

is p^n-bijective. Recall that $\xi = \frac{\pi}{\pi_1}$, so we have

$$\begin{aligned}\pi_1^{-n} N_{\varpi}^{[u,v]}(T)(r) &= \pi^{-n}\xi^n N_{\varpi}^{[u,v]}(T)(r) \subset \pi^{-n}\mathrm{Fil}^n N_{\varpi}^{[u,v]}(T)(r),\\ \pi_1^{-n} N_{\varpi}^{[u,v]}(T)(r) \cap \pi^{-n}\mathrm{Fil}^n N_{\varpi}^{(0,v]+}(T)(r) &= \pi_1^{-n} N_{\varpi}^{(0,v]+}(T)(r).\end{aligned}$$

Therefore, we get that (6.9) is injective. Next, we note that from the definitions we can write $\mathbf{A}_{R,\varpi}^{[u,v]} = \mathbf{A}_{R,\varpi}^{[u]} + \mathbf{A}_{R,\varpi}^{(0,v]+}$. So we take $N_{\varpi}^{[u]}(T) := \mathbf{A}_{R,\varpi}^{[u]} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ and $N_{\varpi}^{+}(T) := \mathbf{A}_{R,\varpi}^{+} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ and we endow these modules with filtrations by considering the tensor product of filtrations on each component (note that for simplicity in notation we consider modules without the twist - this is harmless). This reduces (6.9) to the map

$$\pi_1^{-n} N_{\varpi}^{[u]}(T)/\pi_1^{-n} N_{\varpi}^{+}(T) \longrightarrow \pi^{-n}\mathrm{Fil}^n N_{\varpi}^{[u]}(T)/\pi^{-n}\mathrm{Fil}^n N_{\varpi}^{+}(T),$$

and we need to show that for any $x \in \pi^{-n}\mathrm{Fil}^n N_{\varpi}^{[u]}(T)$, there exists $y \in \pi_1^{-n} N_{\varpi}^{[u]}(T)$ such that under the natural map above, y maps to the image of $p^n x$. Let

$$x = \pi^{-n} \sum_{i+j=n} a_i \otimes x_j \in \pi^{-n}\mathrm{Fil}^n N_{\varpi}^{[u]}(T),$$

with $a_i \in \mathrm{Fil}^i \mathbf{A}_{R,\varpi}^{[u]}$ and $x_j \in \mathrm{Fil}^j \mathbf{N}(T)$. From Lemma 2.14, for $i < n$, we can write $a_i = a_{i1} + a_{i2}$, with $a_{i1} \in \mathrm{Fil}^n \mathbf{A}_{R,\varpi}^{[u]}$ and $a_{i2} \in \frac{1}{p^{\lfloor nu \rfloor}} \mathbf{A}_{R,\varpi}^{+}$. However, note that $a_{i2} = a_i - a_{i1} \in \mathrm{Fil}^i \mathbf{A}_{R,\varpi}^{[u]} \cap \frac{1}{p^{\lfloor nu \rfloor}} \mathbf{A}_{R,\varpi}^{+}$, therefore we get that $a_{i2} \in \frac{1}{p^{\lfloor nu \rfloor}} \mathrm{Fil}^i \mathbf{A}_{R,\varpi}^{+}$. Now we set

$$y = \frac{p^n}{\pi^n} \sum_{\substack{i+j=n\\ i<n}} a_{i1} \otimes x_j + \frac{p^n}{\pi^n} \sum_{\substack{i+j=n\\ i\geq n}} a_i \otimes x_j \in \frac{p^n}{\pi^n}\mathrm{Fil}^n \mathbf{A}_{R,\varpi}^{[u]} \otimes \mathbf{N}(T) \subset \pi_1^{-n} \mathbf{A}_{R,\varpi}^{[u]} \otimes \mathbf{N}(T).$$

and we get that $p^n x - y = \pi^{-n} p^n (\sum a_{i2} \otimes x_j) \in \pi^{-n} N_{\varpi}^{+}(T)$ (since $u = \frac{p-1}{p} < 1$). So (6.8) is p^n-isomorphic to the equation

$$1 - \varphi : \pi_1^{-n} N_{\varpi}^{[u,v]}(T)(r)/\pi_1^{-n} N_{\varpi}^{(0,v]+}(T)(r) \longrightarrow \pi^{-n} N_{\varpi}^{[u,v/p]}(T)(r)/\pi^{-n} N_{\varpi}^{(0,v/p]+}(T)(r),$$

Next, recall that we have $v = p - 1$, so it follows from Lemma 2.33 (v) that π divides p in $\mathbf{A}_{R,\varpi}^{(0,v/p]+}$, whereas π_1 divides p in $\mathbf{A}_{R,\varpi}^{(0,v]+}$, therefore (6.8) is p^{2n}-isomorphic to the equation

$$1 - \varphi : N_{\varpi}^{[u,v]}(T)(r)/N_{\varpi}^{(0,v]+}(T)(r) \longrightarrow N_{\varpi}^{[u,v/p]}(T)(r)/N_{\varpi}^{(0,v/p]+}(T)(r).$$

But from Lemma 6.10, we have that this map is bijective (note that Frobenius has no effect on twist). Therefore, we conclude that (6.7) is p^{3n}-bijective. As $n = r - k \leq r$, the cokernel complex of the map in the claim is killed by p^{3r}. This proves the claim. ∎

Following observation was used above,

Lemma 6.10. *The natural map*

$$1 - \varphi : \mathbf{A}_{R,\varpi}^{[u,v]} \otimes \mathbf{N}(T)/\mathbf{A}_{R,\varpi}^{(0,v]+} \otimes \mathbf{N}(T) \longrightarrow \mathbf{A}_{R,\varpi}^{[u,v/p]} \otimes \mathbf{N}(T)/\mathbf{A}_{R,\varpi}^{(0,v/p]+} \otimes \mathbf{N}(T),$$

is bijective.

Proof. We will follow the strategy of the proof of [CN17, Lemma 4.8]. Let us note that the natural map

$$\mathbf{A}_{R,\varpi}^{[u,v]} \otimes \mathbf{N}(T)/\mathbf{A}_{R,\varpi}^{(0,v]+} \otimes \mathbf{N}(T) \longrightarrow \mathbf{A}_{R,\varpi}^{[u,v/p]} \otimes \mathbf{N}(T)/\mathbf{A}_{R,\varpi}^{(0,v/p]+} \otimes \mathbf{N}(T)$$

induced by the inclusion $\mathbf{A}_{R,\varpi}^{[u,v]} \rightarrowtail \mathbf{A}_{R,\varpi}^{[u,v/p]}$ is an isomorphism. Indeed, the map above is injective because the kernel consists of analytic functions that take values in $\mathbf{N}(T)$ and are integral on the annulus $\frac{u}{e} \leq v_p(X_0) \leq \frac{v}{e}$ and which extend to analytic functions taking values in $\mathbf{N}(T)$ and integral on the annulus $0 < v_p(X_0) \leq \frac{v}{pe}$, hence belong to $\mathbf{A}_{R,\varpi}^{(0,v]+} \otimes \mathbf{N}(T)$. It is surjective because we can write $\mathbf{A}_{R,\varpi}^{[u,v/p]} = \mathbf{A}_{R,\varpi}^{[u]} + \mathbf{A}_{R,\varpi}^{(0,v/p]+}$ (clear from the definitions). So, we can consider $(1-\varphi)$ as an endomorphism of the module $Q = \mathbf{A}_{R,\varpi}^{[u,v]} \otimes \mathbf{N}(T)/\mathbf{A}_{R,\varpi}^{(0,v]+} \otimes \mathbf{N}(T)$.

An element $x \in \mathbf{A}_{R,\varpi}^{[u,v]}$ can be written as $x = \sum_{k\in\mathbb{N}} \frac{\pi_m^k}{p^{\lfloor ku/e \rfloor}} x_k$, with $x_k \in \mathbf{A}_{R,\varpi}^{(0,v]+}$ going to 0, p-adically. So,

$$\varphi(x) = \sum_{k\in\mathbb{N}} p^{\lfloor pku/e \rfloor - \lfloor ku/e \rfloor} \big(\tfrac{\varphi(\pi_m)}{\pi_m^p}\big)^k \tfrac{\pi_m^{pk}}{p^{\lfloor pku/e \rfloor}} \varphi(x_k),$$

and since $\lfloor pku/e \rfloor - \lfloor ku/e \rfloor \geq 1$ if $\lfloor ku/e \rfloor \neq 0$, we see that $\varphi(x) \in \mathbf{A}_{R,\varpi}^{(0,v/p]+} + p\mathbf{A}_{R,\varpi}^{[u,v/p]}$. As $\varphi(\mathbf{N}(T)) \subset \mathbf{N}(T)$, we get $\varphi(Q) \subset pQ$. To show the bijectivity of $1-\varphi$, it remains to check that Q does not contain p-divisible elements, which would then imply that $1 + \varphi + \varphi^2 + \cdots$ converges on Q. Let $(f_j)_{j\in J}$ be a collection of elements of $\mathbf{A}_{R,\varpi}^{+}$ whose images form a basis of $\mathbf{A}_{R,\varpi}^{+}/(p,\pi_m)$ over $\kappa = \mathbf{A}_K^{+}/(p,\pi_m)$. Then $(f_j)_{j\in J}$ is a topological basis of $\mathbf{A}_{R,\varpi}^{[u,v]}$ over $\mathbf{A}_K^{[u,v]}$ and of $\mathbf{A}_{R,\varpi}^{(0,v]+}$ over $\mathbf{A}_K^{(0,v]+}$. Writing everything in the basis $\{f_j \otimes e_i, \text{ for } 1 \leq i \leq h, j \in J\}$, where $\{e_i,\ 1 \leq i \leq h\}$ is a basis of $\mathbf{N}(T)$, reduces the question to proving that $\mathbf{A}_K^{[u,v]}/\mathbf{A}_K^{(0,v]+}$ has no p-divisible element. Since all such elements can be written as a power series in $\mathbf{A}_K^{[u]}/\mathbf{A}_K^{+}$, we conclude that there can be no p-divisible elements in this quotient. Hence, we get the desired conclusion. ∎

6.5. Change of annulus of convergence : Part 2. In this section, we will change the ring of coefficients from $\mathbf{A}_{R,\varpi}^{(0,v]+}$ to $\mathbf{A}_{R,\varpi}^{(0,v/p]+}$ by replacing the action of φ with its left inverse ψ in the complexes discussed so far : these steps are required in order to obtain a complex comparable to Koszul complexes computing the Galois cohomology of $T(r)$. Note that we are working under the assumption that $\frac{p-1}{p} \leq u \leq \frac{v}{p} < 1 < v$, for example, one can take $u = \frac{p-1}{p}$ and $v = p-1$.

6.5.1. From (φ, Γ_S)-complex to (ψ, Γ_S)-complex. Recall from Proposition 2.10 that we have a left inverse ψ of the Frobenius such that $\psi(\mathbf{A}) \subset \mathbf{A}$, which induces the operator $\psi : \mathbf{A}^{+} \to \mathbf{A}^{+}$. For the overconvergent rings we can consider the induced operator over $\mathbf{A}^{\dagger}$ and we have that $\psi(\mathbf{A}^{\dagger}) \subset \mathbf{A}^{\dagger}$. This gives us an operator $\psi : \mathbf{A}_{R,\varpi}^{(0,v/p]+} \to \mathbf{A}_{R,\varpi}^{(0,v]+}$. Note that we can also define ψ by identifying $\mathbf{A}_{R,\varpi}^{(0,v/p]+} \simeq R_{\varpi}^{(0,v/p]+}$ via the isomorphism ι_{cycl} in §2.7, and considering the left inverse of the cyclotomic Frobenius over $R_{\varpi}^{(0,v/p]+}$ (see §2.6). Both these definitions coincide since ι_{cycl} commutes with the Frobenius on each side.

From Lemma 3.3 recall that ψ extends to $\mathbf{N}(T(r))$ and $\psi(\mathbf{N}(T(r))) \subset \mathbf{N}(T(r))$. Extending scalars to $\mathbf{A}_{R,\varpi}^{(0,v]+}$ and from the discussion above we obtain the following inclusion of $\mathbf{A}_{R,\varpi}^{(0,v]+}$-modules $\psi(N_{\varpi}^{(0,v]+}(T(r))) \subset \psi(N_{\varpi}^{(0,v/p]+}(T(r))) \subset N_{\varpi}^{(0,v]+}(T(r))$. Moreover, for $0 \leq k \leq r$ we have $\varphi(\mathrm{Fil}^{r-k} N_{\varpi}^{(0,v]+}(T)) \subset q^{r-k} N_{\varpi}^{(0,v/p]+}(T)$. So multiplying the expression by $\varphi(\pi^{k-r})$ and twisting by r, we get that $\varphi(\pi^{k-r}\mathrm{Fil}^{r-k} N_{\varpi}^{(0,v]+}(T)(r)) \subset \pi^{k-r} N_{\varpi}^{(0,v]+}(T)(r)$. In particular, $\pi^k \mathrm{Fil}^{-k} N_{\varpi}^{(0,v]+}(T(r)) \subset \psi(\pi^k N_{\varpi}^{(0,v/p]+}(T(r)))$ and combining it with preceding discussion we get $(\psi-1)(\pi^k \mathrm{Fil}^{-k} N_{\varpi}^{(0,v]+}(T(r))) \subset \psi(\pi^k N_{\varpi}^{(0,v/p]+}(T(r)))$.

Set $\mathcal{K}(\Gamma_S', N_\psi) := \psi(\mathcal{K}(\Gamma_S', N_{\varpi}^{(0,v/p]+}(T(r))))$ and similarly for $\mathcal{K}^c(\Gamma_S', N_\psi)$. In the previous section, we defined $\tau_0 : \mathcal{K}(\Gamma_S', \mathrm{Fil}^0 N_{\varpi}^{(0,v]+}(T(r))) \to \mathcal{K}^c(\Gamma_S', \pi\mathrm{Fil}^{-1} N_{\varpi}^{(0,v]+}(T(r)))$ and since ψ commutes with Γ_S-action we obtain a morphism $\tau_0 : \mathcal{K}(\Gamma_S', N_\psi) \to \mathcal{K}^c(\Gamma_S', N_\psi)$. From preceding discussion note that we have a well defined map between source complexes of τ_0 above given as

$\psi - 1 : \mathcal{K}(\Gamma'_S, \mathrm{Fil}^0 N_{\varpi}^{(0,v]+}(T(r))) \to \mathcal{K}(\Gamma'_S, N_{\psi})$ and similarly for target complexes. Therefore, similar to $\mathcal{K}(\varphi, \Gamma_S, N^{(0,v]+}(T(r)))$ in previous section, define

$$\mathcal{K}(\psi, \Gamma_S, N_{\varpi}^{(0,v]+}(T(r))) := \left[\begin{tikzcd} \mathcal{K}(\Gamma'_S, \mathrm{Fil}^0 N_{\varpi}^{(0,v]+}(T(r))) \arrow[r, "\psi-1"] \arrow[d, "\tau_0"'] & \mathcal{K}(\Gamma'_S, N_{\psi}) \arrow[d, "\tau_0"] \\ \mathcal{K}^c(\Gamma'_S, \pi \mathrm{Fil}^{-1} N_{\varpi}^{(0,v]+}(T(r))) \arrow[r, "\psi-1"] & \mathcal{K}^c(\Gamma'_S, N_{\psi})) \end{tikzcd}\right].$$

Proposition 6.11. *With notations as above, the natural map*

$$\tau_{\leq r}\mathcal{K}(\varphi, \Gamma_S, N_{\varpi}^{(0,v]+}(T(r))) \longrightarrow \tau_{\leq r}\mathcal{K}(\psi, \Gamma_S, N_{\varpi}^{(0,v]+}(T(r))),$$

induced by identity in the first column and ψ in the second column is a p^{r+2}-quasi-isomorphism.

Proof. By definition, note that the map is surjective so we only need to show that the kernel complex is p^{r+2}-acyclic. As the map in claim is identity on first column, the kernel complex can be written as

$$\tau_{\leq r}\big[\mathcal{K}(\Gamma'_S, (N_{\varpi}^{(0,v/p]+}(T(r)))^{\psi=0}) \xrightarrow{\tau_0} \mathcal{K}^c(\Gamma'_S, (\pi N_{\varpi}^{(0,v/p]+}(T(r)))^{\psi=0})\big]. \tag{6.10}$$

Clearly terms of the complex above are $\varphi(\mathbf{A}_{R,\varpi}^{(0,v]+})$-modules. Recall that we have $\frac{p}{\pi} \in \varphi(\mathbf{A}_{R,\varpi}^{(0,v]+})$ (since π_1 divides p in $\mathbf{A}_{R,\varpi}^{(0,v]+}$, see Lemma 2.33 (ii) for $v = p-1$), we obtain that $(\pi^k N_{\varpi}^{(0,v/p]+}(T(r)))^{\psi=0}$ is p^{r-k}-isomorphic to $(N_{\varpi}^{(0,v/p]+}(T)(r))^{\psi=0}$, for $k \leq r$. Using this we see that the complex in (6.10) is p^r-quasi-isomorphic to the complex

$$\tau_{\leq r}\big[\mathrm{Kos}(\Gamma'_S, (N_{\varpi}^{(0,v/p]+}(T)(r))^{\psi=0}) \xrightarrow{\tau_0} \mathrm{Kos}^c(\Gamma'_S, (N_{\varpi}^{(0,v/p]+}(T)(r))^{\psi=0})\big]. \tag{6.11}$$

We will show that the complex in (6.11) is p^2-acyclic, but to prove our claim we need a simpler decription of the $\varphi(\mathbf{A}_{R,\varpi}^{(0,v]+})$-module $(N_{\varpi}^{(0,v/p]+}(T))^{\psi=0}$.

Let us write $\mathbf{N}(T) = \sum_{j=1}^h \mathbf{A}_R^+ e_j$, for a choice of basis. Since the attached (φ, Γ_S)-module $\mathbf{D}_{R,\varpi}(T)$ over $\mathbf{A}_{R,\varpi}$ is étale, we obtain that $\mathbf{D}_{R,\varpi}(T) = \sum_{j=1}^h \mathbf{A}_{R,\varpi}\varphi(e_j)$. Now note that $z = \sum_{j=1}^h z_j\varphi(e_j) \in (\mathbf{D}_{R,\varpi}(T))^{\psi=0} = \big(\sum_{j=1}^h \mathbf{A}_{R,\varpi}\varphi(e_j)\big)^{\psi=0}$, if and only if $z_j \in (\mathbf{A}_{R,\varpi})^{\psi=0}$, for each $1 \leq j \leq h$. Indeed, $\psi(z) = 0$ if and only if $\sum_{j=1}^h \psi(z_j\varphi(e_j)) = \sum_{j=1}^h \psi(z_j)e_j = 0$ and since e_j are linearly independent over $\mathbf{A}_{R,\varpi}$, we get the desired statement.

Next, using Lemma 2.23 (ii), we have a decomposition

$$\mathbf{A}_{R,\varpi}^{\psi=0} = \bigoplus_{\alpha \in \{0,\ldots,p-1\}^{[0,d]}, \alpha \neq 0} \varphi(\mathbf{A}_{R,\varpi})[X^\flat]^\alpha, \quad \text{where } [X^\flat]^\alpha = (1+\pi_m)^{\alpha_0}[X_1^\flat]^{\alpha_0}\cdots[X_d^\flat]^{\alpha_d}.$$

Therefore, we obtain that

$$\begin{aligned}(\mathbf{D}_{R,\varpi}(T))^{\psi=0} = \Big(\sum_{j=1}^h \mathbf{A}_{R,\varpi}\varphi(e_j)\Big)^{\psi=0} &= \bigoplus_{\substack{\alpha \in \{0,\ldots,p-1\}^{[0,d]} \\ \alpha \neq 0}} \sum_{j=1}^h \varphi(\mathbf{A}_{R,\varpi}e_j)[X^\flat]^\alpha \\ &= \bigoplus_{\substack{\alpha \in \{0,\ldots,p-1\}^{[0,d]} \\ \alpha \neq 0}} \varphi(\mathbf{D}_{R,\varpi}(T))[X^\flat]^\alpha.\end{aligned}$$

Now observe that $(N_{\varpi}^{(0,v/p]+}(T))^{\psi=0} = (\mathbf{D}_{R,\varpi}(T))^{\psi=0} \cap N_{\varpi}^{(0,v/p]+}(T)$. Using the decomposition above, we set

$$D[X^\flat]^\alpha := \varphi(\mathbf{D}_{R,\varpi}(T))[X^\flat]^\alpha \cap N_{\varpi}^{(0,v/p]+}(T), \ \text{ for } \alpha \in \{0,\ldots,p-1\} \text{ and } \alpha \neq 0,$$

where we take the intersection inside $(\mathbf{D}_{R,\varpi}(T))^{\psi=0}$. Note that we have $\varphi(\mathbf{A}_{R,\varpi}^{(0,v]+}) \subset \varphi(\mathbf{A}_{R,\varpi}) \cap \mathbf{A}_{R,\varpi}^{(0,v/p]+}$. So we get that the module $D := D[X^\flat]^\alpha[X^\flat]^{-\alpha}$ is a $\varphi(\mathbf{A}_{R,\varpi}^{(0,v]+})$-module contained in $N_\varpi^{(0,v/p]+}(T)$, stable under the action of Γ_S and independent of α. Indeed, for the last part note that for $\alpha \neq \alpha'$, we have $\sum_{i=1}^h \varphi(x_i e_i)[X^\flat]^\alpha \in D[X^\flat]^\alpha$ if and only if $\sum_{i=1}^h \varphi(x_i e_i)[X^\flat]^{\alpha'} \in D[X^\flat]^{\alpha'}$.

Lemma 6.12. *For $v = p-1$, let $x \in \varphi(\mathbf{D}_{R,\varpi}(T))$ such that $\varphi(x) \in N_\varpi^{(0,v/p]+}(T)$ then $x \in N_\varpi^{(0,v]+}$. In paticular, we have $D = \varphi(N_\varpi^{(0,v]+})$.*

Proof. The idea of the proof is motivated by [CN17, Lemma 2.14]. Note that we can write

$$N_\varpi^{(0,v]+}(T) = \sum_{n\in\mathbb{N}} \tfrac{p^n}{\pi_m^{\lfloor ne/v\rfloor}} N_\varpi^+(T).$$

Now if $x \in \varphi(\mathbf{D}_{R,\varpi}(T))$ such that $\varphi(x) \in N_\varpi^{(0,v/p]+}(T)$, then the image $\overline{x}$ of x in $\mathbf{D}_{R,\varpi}(T)/p$ is such that $\varphi(\overline{x}) \in N_\varpi^+(T)/p$. But since $\mathbf{D}_{R,\varpi}(T)/p = N_\varpi^+(T)/p[\frac{1}{\pi_m}]$, we obtain that $\overline{x} \in N_\varpi^+(T)/p$. So we can take $y_0 \in N_\varpi^+(T)$ such that $x - y_0 \in p\mathbf{D}_{R,\varpi}(T)$ and obtain that

$$\varphi(x-y_0) \in \sum_{n\geq 1} \tfrac{p^n}{\pi_m^{\lfloor ne/v\rfloor}} N_\varpi^+(T),$$

Next, if we write $x = y_0 + \frac{p}{\pi_m^{\lfloor e/v\rfloor+1}} x_1$, the image of $\varphi(x_1)$ in $\mathbf{D}_{R,\varpi}(T)/p$ belongs to $\pi_m N_\varpi^+(T)/p$ (since $p(\lfloor e/v\rfloor+1) - \lfloor pe/v\rfloor \geq 1$), hence the image of x_1 belongs to $\pi_m N_\varpi^+(T)/p$ and we can find $y_1 \in N_\varpi^+(T)$ such that $x_1 - \pi_m y_1 \in p\mathbf{D}_{R,\varpi}(T)$. This implies that

$$\varphi(x - y_0 - \tfrac{p}{\pi_m^{\lfloor e/v\rfloor}} y_1) \in \sum_{n\geq 2} \tfrac{p^n}{\pi_m^{\lfloor ne/v\rfloor}} N_\varpi^+(T).$$

Again we can write $x = y_0 + \frac{p}{\pi_m^{\lfloor e/v\rfloor}} y_1 + \frac{p}{\pi_m^{\lfloor 2e/v\rfloor}+1} x_2$ and argue as above to get that $x_2 - X_0 y_2 \in p\mathbf{D}_{R,\varpi}(T)$ with $y_2 \in N_\varpi^+(T)$. Passing to the limit, we obtain that $x = \sum_{n\in\mathbb{N}} \frac{p^n}{\pi_m^{\lfloor ne/v\rfloor}} y_n$ with $y_n \in N_\varpi^+$. This concludes the proof. ■

Remark 6.13. From Lemma 6.12, we have that $D = \varphi(N_\varpi^{(0,v]+})$ and let $i \in \{0,\ldots,d\}$. Moreover, from Lemma 2.32 (i) we have that $(\gamma_i - 1)\mathbf{A}_{R,\varpi}^{(0,v]+} \subset \pi\mathbf{A}_{R,\varpi}^{(0,v]+}$ from Definition 3.2 we know that $(\gamma_i-1)\mathbf{N}(T) \subset \pi\mathbf{N}(T)$. Hence, we conclude that $(\gamma_i - 1)D \subset \varphi(\pi)D$.

Now we return to the complex in (6.11). From the discussion above, we see that the complex in (6.11) is isomorphic to the complex

$$\tau_{\leq r} \bigoplus_{\alpha\in\{0,\ldots,p-1\}^{[0,d]},\alpha\neq 0} \Big[\mathrm{Kos}(\Gamma_S', D(r)[X^\flat]^\alpha) \xrightarrow{\tau_0} \mathrm{Kos}^c(\Gamma_S', D(r)[X^\flat]^\alpha)\Big]. \tag{6.12}$$

Lemma 6.14. *The complex described in (6.12) above is p^2-acyclic.*

Proof. The proof is motivated by the proof of [CN17, Lemma 4.10]. We will treat terms corresponding to each α separately. First, let us assume that $\alpha_k \neq 0$ for some $k \neq 0$. We want to show that both $\mathrm{Kos}(\Gamma_S', D[X^\flat]^\alpha)$ and $\mathrm{Kos}^c(\Gamma_S', D[X^\flat]^\alpha)$ complexes are p-acyclic (the twist has disappeared because the cyclotomic character is trivial on Γ_S'). As the proof is same in both the cases, we only treat the first case. We can write the complex as a double complex

$$\begin{array}{ccccccc}
D[X^\flat]^\alpha & \xrightarrow{(\gamma_i-1)} & D^{I_1''}[X^\flat]^\alpha & \longrightarrow & D^{I_2''}[X^\flat]^\alpha & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \gamma_k-1} & & \downarrow{\scriptstyle \gamma_k-1} & & \downarrow{\scriptstyle \gamma_k-1} & & \\
D[X^\flat]^\alpha & \xrightarrow{(\gamma_i-1)} & D^{I_1''}[X^\flat]^\alpha & \longrightarrow & D^{I_2''}[X^\flat]^\alpha & \longrightarrow & \cdots,
\end{array}$$

where the horizontal maps involve γ_i's with $i \neq k$, $1 \leq i \leq d$. Now, we have

$$(\gamma_k - 1) \cdot (y[X^\flat]^\alpha) = \pi G(y)[X^\flat]^\alpha, \ \text{ for } \ y \in D,$$

where

$$G(y) = (1+\pi)^{\alpha_k}\pi^{-1}(\gamma_k - 1)y + \pi^{-1}((1+\pi)^{\alpha_k} - 1)y,$$

and we have used the fact that

$$\gamma_k([X^\flat]^\alpha) = [\varepsilon]^{\alpha_k}[X^\flat]^\alpha = (1+\pi)^{\alpha_k}[X^\flat]^\alpha.$$

Now, G is π_m-linear and $(\gamma_k - 1)D \subset \varphi(\pi)D$ (see Remark 6.13). Moreover, π divides p in $\varphi(\mathbf{A}_{R,\varpi}^{(0,v]+})$ (since π_1 divides p in $\mathbf{A}_{R,\varpi}^{(0,v]+}$, see Lemma 2.33 (ii) for $v = p-1$), therefore it follows that $\frac{\varphi(\pi)}{\pi^2} \in \varphi(\mathbf{A}_{R,\varpi}^{(0,v]+})$ and modulo π, G is just multiplication by α_k on D. This shows that G is invertible over D, therefore $\gamma_k - 1$ is injective on $D[X^\flat]^\alpha$. Finally, since we have that $\frac{p}{\pi} \in \varphi(\mathbf{A}_{R,\varpi}^{(0,v]+})$, the cokernel of $\gamma_k - 1$ is killed by p.

Next, let $\alpha_k = 0$ for all $k \neq 0$ and $\alpha_0 \neq 0$. To prove that the kernel complex is p-acyclic, we will show that $\tau_0 : \mathrm{Kos} \to \mathrm{Kos}^c$ is injective and the cokernel complex is killed by p. This amounts to showing the same statement for

$$\gamma_0 - \delta_{i_1}\cdots\delta_{i_q} : D[X^\flat]^\alpha(r) \longrightarrow D[X^\flat]^\alpha(r), \ \ \delta_{i_j} = \frac{\gamma_{i_j}^c - 1}{\gamma_{i_j} - 1}. \tag{6.13}$$

We have

$$(\gamma_0 - \delta_{i_1}\cdots\delta_{i_q})(y[X^\flat]^\alpha(r)) = (c^r\gamma_0(y)(1+\pi)^{p^{-m}(c-1)\alpha_0}[X^\flat]^\alpha)(r) - (\delta_{i_1}\cdots\delta_{i_q}(y)[X^\flat]^\alpha)(r).$$

So we are lead to study the map F defined by

$$F = c^r(1+\pi)^z\gamma_0 - \delta_{i_1}\cdots\delta_{i_q}, \ \ z = p^{-m}(c-1)\alpha_0 \in \mathbb{Z}_p^*.$$

Now $c^r - 1$ is divisible by p^m, $(1+\pi)^z = 1 + z\pi \bmod \pi^2$ and $\delta_{i_j} - 1 \in (\gamma_{i_j} - 1)\mathbb{Z}_p[[\gamma_{i_j} - 1]]$. Therefore, we can write $\pi^{-1}F$ in the form $\pi^{-1}F = z + \pi^{-1}F'$, with $F' \in (p^m, \pi^2, \gamma_0 - 1, \ldots, \gamma_d - 1)\mathbb{Z}_p[[\pi, \Gamma_S]]$.

Let $x \in D$ and let $f = \frac{p}{\pi} \in \varphi(\mathbf{A}_R^{(0,v]+})$ (since π_1 divides p in $\mathbf{A}_R^{(0,v]+}$, see Lemma 2.33 2.33 (ii) for $v = p-1$), then we have $\pi^{-1}p^m x = \pi^{m-1}f^m x \in \pi^{m-1}D$. Moreover, we have $(\gamma_j - 1)D \subset \varphi(\pi)D$ for $0 \leq j \leq d$ (see Remark 6.13) and $\frac{\varphi(\pi)}{\pi^2} \in \varphi(\mathbf{A}_{R,\varpi}^{(0,v]+})$ (since π_1 divides p in $\mathbf{A}_{R,\varpi}^{(0,v]+}$, see Lemma 2.33 (ii) for $v = p-1$). Furthermore, $\pi_m^{p^m}$ divides π and p in $\varphi(\mathbf{A}_{R,\varpi}^{(0,v]+})$ (see Lemma 2.33 (ii) for $v = p-1$), so we get that $\pi^{-1}F'(x) \in \pi_m^{p^m}D$. Therefore, $\pi^{-1}F' = 0$ on $\pi_m^a D/\pi_m^{a+b}D$, for all $a \in \mathbb{N}$ and $b = p^m$. Hence, $\pi^{-1}F$ induces multiplication by z on $\pi_m^a D/\pi_m^{a+b}D$ for all $a \in \mathbb{N}$, which implies that it is an isomorphism of D.

From the preceding discussion, we conclude that the map in (6.13) is injective and its image is contained in $\pi D[X^\flat](r)$. But since π divides p in $\varphi(\mathbf{A}_{R,\varpi}^{(0,v]+})$ (see Lemma 2.33 (ii) for $v = p-1$), we obtain that the cokernel of (6.13) is killed by p, as desired. ∎

Combining the analysis for the kernel and cokernel complex, we conclude that the map in the claim of Proposition 6.11 is a p^{r+2}-quasi-isomorphism. ∎

6.5.2. Changing the overconvergence radius. Recall that $m \geq 2$ and let $\ell = p^{m-1}$, then from Proposition 2.26 (i) we have inclusions

$$\psi(\pi_m^{-\ell}\mathbf{A}_{R,\varpi}^{(0,v]+}) \subset \psi(\pi_m^{-\ell}\mathbf{A}_{R,\varpi}^{(0,v/p]+}) \subset \pi_m^{-p^{m-2}}\mathbf{A}_{R,\varpi}^{(0,v]+} \subset \pi_m^{-\ell}\mathbf{A}_{R,\varpi}^{(0,v/p]+}. \tag{6.14}$$

In other words, $\pi_m^{-\ell}\mathbf{A}_{R,\varpi}^{(0,v]+}$ is stable under ψ. Set $D_\varpi^{(0,v]+}(T(r)) := \mathbf{A}_{R,\varpi}^{(0,v]+} \otimes_{\mathbf{A}_R^+} \mathbf{D}^+(T(r))$ and note that it is stable under Γ_S-action. From Lemma 2.23 we have $\psi(\mathbf{A}_{R,\varpi}^{(0,v/p]+}) \subset \mathbf{A}_{R,\varpi}^{(0,v]+}$

and for $v = p-1$, by Lemma 2.33 (iii) $\pi_m^{-p\ell}\pi$ is a unit in $\mathbf{A}_{R,\varpi}^{(0,v/p]+}$. So by combining Lemma 2.25 and Proposition 2.26 (i), we get $\psi(\pi^{-r}\mathbf{A}_{R,\varpi}^{(0,v/p]+}) \subset \pi_1^{-r}\mathbf{A}_{R,\varpi}^{(0,v]+}$ and therefore $\psi(\pi^{-r}D_{\varpi}^{(0,v/p]+}(T(r))) \subset \pi_1^{-r}D_{\varpi}^{(0,v]+}(T(r))$. Since $\psi(\mathbf{N}(T)) \subset \mathbf{D}^+(T)$, using (6.14) we get

$$\psi(N_{\varpi}^{(0,v/p]+}(T(r))) \subset \psi(\pi^{-r}D_{\varpi}^{(0,v/p]+}(T(r))) \subset \pi_1^{-r}D_{\varpi}^{(0,v]+}(T(r)). \tag{6.15}$$

Moreover, for $k \in \mathbb{N}$ with $k \leq r$ we have $\pi^k N_{\varpi}^{(0,v/p]+}(T(r)) \subset \pi^{k-r}D_{\varpi}^{(0,v/p]+}(T(r))$ and also $\psi(\pi^k N_{\varpi}^{(0,v/p]+}(T(r))) \subset \pi_1^{k-r}D_{\varpi}^{(0,v]+}(T(r))$.

Now by replacing v by v/p in §6.4, define a complex $\mathcal{K}(\Gamma'_S, N_{\varpi}^{(0,v/p]+}(T(r)))$ as

$$N_{\varpi}^{(0,v/p]+}(T(r)) \xrightarrow{(\tau_i)} (\pi N_{\varpi}^{(0,v/p]+}(T(r)))^{I'_1} \longrightarrow (\pi^2 N_{\varpi}^{(0,v/p]+}(T(r)))^{I'_2} \longrightarrow \cdots.$$

Similarly, we define a complex $\mathcal{K}^c(\Gamma'_S, N_{\varpi}^{(0,v/p]+}(T(r)))$ and a map τ_0 from former to latter complex. From (6.15) and the inclusion $N_{\varpi}^{(0,v/p]+}(T(r)) \subset \pi^{-r}D_{\varpi}^{(0,v/p]+}(T(r))$, we get $(\psi - 1)(\pi^k N_{\varpi}^{(0,v/p]+}(T(r))) \subset \pi^{-r}D_{\varpi}^{(0,v/p]+}(T(r))$. Define $\mathcal{K}(\psi, \Gamma_S, N_{\varpi}^{(0,v/p]+}(T(r)))$ as

$$\left[\begin{array}{ccc} \mathcal{K}(\Gamma'_S, N_{\varpi}^{(0,v/p]+}(T(r))) & \xrightarrow{\psi-1} & \mathrm{Kos}(\Gamma'_S, \pi^{-r}D_{\varpi}^{(0,v/p]+}(T(r))) \\ \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau_0} \\ \mathcal{K}^c(\Gamma'_S, \pi N_{\varpi}^{(0,v/p]+}(T(r))) & \xrightarrow{\psi-1} & \mathrm{Kos}^c(\Gamma'_S, \pi^{-r}D_{\varpi}^{(0,v/p]+}(T(r))) \end{array}\right].$$

Lemma 6.15. *The natural map*

$$\tau_{\leq r}\mathcal{K}(\psi, \Gamma_S, N_{\varpi}^{(0,v]+}(T(r))) \longrightarrow \tau_{\leq r}\mathcal{K}(\psi, \Gamma_S, N_{\varpi}^{(0,v/p]+}(T(r))),$$

induced by $N_{\varpi}^{(0,v]+}(T(r)) \subset N_{\varpi}^{(0,v/p]+}(T(r))$ *and* $\psi(N_{\varpi}^{(0,v/p]+}(T(r))) \subset \pi^{-r}D_{\varpi}^{(0,v/p]+}(T(r))$ *is a* p^{r+s}*-quasi-isomorphism.*

Proof. As the map is injective we need to show that cokernel complex is killed by p^{r+s}. For $k \in \mathbb{N}$ and $k \leq r$, in the cokernel complex we have maps

$$\psi - 1 : \pi^{k-r}N_{\varpi}^{(0,v/p]+}(T)/\pi^{k-r}\mathrm{Fil}^{r-k}N_{\varpi}^{(0,v]+}(T) \to \pi^{-r}D_{\varpi}^{(0,v/p]+}(T)/\psi(\pi^{k-r}N_{\varpi}^{(0,v/p]+}(T)), \tag{6.16}$$

and to prove the claim it is enough to show that (6.16) is p^{r+s}-bijective (the twist (r) has disappeared since ψ acts trivially on it). We will show the p^{r+s}-surjectivity first. Note that we have $\psi(\pi^{k-r}N_{\varpi}^{(0,v/p]+}(T)) \subset \pi_1^{-r}D_{\varpi}^{(0,v]+}(T)$ so cokernel of the map in (6.16) is given as $\pi^{-r}D_{\varpi}^{(0,v/p]+}(T)/\pi^{k-r}N_{\varpi}^{(0,v/p]+}(T)$. Recall that $\pi^s\mathbf{D}^+(T) \subset \mathbf{N}(T) \subset \mathbf{D}^+(T)$ (see [Abh21, Corollary 4.11]). Extending scalars of the inclusions above to $\mathbf{A}_{R,\varpi}^{(0,v/p]+}$ and dividing by π^r, we get $\pi^{s-r}D_{\varpi}^{(0,v/p]+}(T) \subset \pi^{-r}N_{\varpi}^{(0,v/p]+}(T)$. Therefore, $\pi^{-r}D_{\varpi}^{(0,v/p]+}(T)/\pi^{k-r}N_{\varpi}^{(0,v/p]+}(T)$ is killed by π^{k+s}. Since π divides p in $\mathbf{A}_{R,\varpi}^{(0,v/p]+}$ (see Lemma 2.33 for $v = p-1$), therefore (6.16) is p^{k+s}-surjective (this also shows that truncation in degree $\leq r$ is necessary in order to bound the power of p).

To show injectivity of (6.16), let $x \in N_{\varpi}^{(0,v/p]+}(T)$ such that there is a $y \in N_{\varpi}^{(0,v/p]+}(T)$ satisfying $(\psi - 1)(\pi^{k-r}x) = \psi(\pi^{k-r}y)$, or equivalently $x = \xi^{r-k}\psi(x-y)$. Note that to obtain injectivity of (6.16), it is enough to show that $x \in \mathrm{Fil}^{r-k}N_{\varpi}^{(0,v]+}(T)$. We first observe that

$$x = \xi^{-k}\psi(q^r x - q^r y) \in \xi^{-k}\psi(q^r N_{\varpi}^{(0,v/p]+}(T)) \subset \xi^{-k}N_{\varpi}^{(0,v]+}(T),$$

since $r \geq s+1$. Now since $\mathbf{N}(T)$ is free over $\mathbf{A}_R^+$, inside $N_{\varpi}^{(0,v/p]+}(T)$ and for all $n \in \mathbb{N}$, it is easy to see that

$$\xi^n N_{\varpi}^{(0,v/p]+}(T) \cap N_{\varpi}^{(0,v]+}(T) = \xi^n N_{\varpi}^{(0,v]+}(T).$$

Therefore, inside $\pi^{-r} N_{\varpi}^{(0,v/p]+}(T)$ we get that

$$x \in N_{\varpi}^{(0,v/p]+}(T) \cap \xi^{-k} N_{\varpi}^{(0,v]+}(T) = N_{\varpi}^{(0,v]+}(T).$$

Moreover, $\psi(N_{\varpi}^{(0,v/p]+}(T)) \subset D_{\varpi}^{(0,v]+}(T)$, therefore $x \in \xi^{r-k} D_{\varpi}^{(0,v]+}(T)$. As the filtration on $\mathbf{N}(T)$ is induced from the filtration on $\mathbf{A}_{\inf}(\overline{R}) \otimes_{\mathbb{Z}_p} T$ (see [Abh21, Lemma 4.51]), it easily follows that inside $D_{\varpi}^{(0,v]+}(T)$ we have $\xi^{r-k} D_{\varpi}^{(0,v]+}(T) \cap N_{\varpi}^{(0,v]+}(T) = \mathrm{Fil}^{r-k} N_{\varpi}^{(0,v]+}(T)$. In other words, (6.16) is injective. Putting everything together for $k \leq r$, we conclude that the map in claim is a p^{r+s}-quasi-isomorphism. ■

Note that $\psi(\pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r))) \subset \pi_1^{-r} D_{\varpi}^{(0,v]+}(T(r)) \subset \pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r))$ from (6.15). So using §4, let us define the complex $\mathrm{Kos}(\psi, \Gamma_S, D_{\varpi}^{(0,v/p]+}(T(r)))$ as

$$\begin{bmatrix} \mathrm{Kos}(\Gamma'_S, \pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r))) & \xrightarrow{\psi-1} & \mathrm{Kos}(\Gamma'_S, \pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r))) \\ \downarrow{\scriptstyle\tau_0} & & \downarrow{\scriptstyle\tau_0} \\ \mathrm{Kos}^c(\Gamma'_S, \pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r))) & \xrightarrow{\psi-1} & \mathrm{Kos}^c(\Gamma'_S, \pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r))) \end{bmatrix}.$$

Lemma 6.16. *The natural map*

$$\tau_{\leq r} \mathcal{K}(\psi, \Gamma_S, N_{\varpi}^{(0,v/p]+}(T(r))) \longrightarrow \tau_{\leq r} \mathrm{Kos}(\psi, \Gamma_S, D_{\varpi}^{(0,v/p]+}(T(r))),$$

induced by the inclusion $N_{\varpi}^{(0,v/p]+}(T(r)) \subset \pi_m^{-p\ell r} D_{\varpi}^{(0,v/p]+}(T(r))$, is a p^{r+s}-quasi-isomorphism.

Proof. Since the map is injective it is enough to show that the cokernel complex is killed by p^{r+s}. Note that the cokernel is a complex made up of $\mathbf{A}_{R,\varpi}^{(0,v/p]+}$-modules $\pi_m^{-p\ell r} D_{\varpi}^{(0,v/p]+}(T(r))/\pi^k N_{\varpi}^{(0,v/p]+}(T(r))$, for $k \in \mathbb{N}$ such that $k \leq r$. Recall from [Abh21, Corollary 4.11] that we have $\pi^s \mathbf{D}^+(T)(r) \subset \mathbf{N}(T)(r) = \pi^r \mathbf{N}(T(r)) \subset \mathbf{D}^+(T(r))$. Extending scalars to $\mathbf{A}_{R,\varpi}^{(0,v/p]+}$ in the equation above and dividing by π^r, we obtain natural inclusions

$$\pi^{s-r} D_{\varpi}^{(0,v/p]+}(T(r)) \subset N_{\varpi}^{(0,v/p]+}(T(r)) \subset \pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r)).$$

As $v = p-1$, from Lemma 2.33 (v) we have that π divides p in $\mathbf{A}_{R,\varpi}^{(0,v/p]+}$. Therefore, the module $\pi_m^{-p\ell r} D_{\varpi}^{(0,v/p]+}(T(r))/\pi^k N_{\varpi}^{(0,v/p]+}(T(r)) = \pi^{-r} D_{\varpi}^{(0,v/p]+}(T(r))/\pi^k N_{\varpi}^{(0,v/p]+}(T(r))$ is killed by p^{k+s}. Hence, the cokernel complex (for the truncated complex) is p^{r+s}-acyclic, which proves the claim. ■

6.6. Change of disk of convergence.

Finally, we are in a position to relate our complexes to the Koszul complex computing continuous G_R-cohomology of $T(r)$. Recall that in §2.4.5, we defined an operator $\psi : \mathbf{D}_{R,\varpi}(T(r)) \to \mathbf{D}_{R,\varpi}(T(r))$, as the left inverse of φ. Using this operator, we can define the complex

$$\mathrm{Kos}(\psi, \Gamma_S, \mathbf{D}_{R,\varpi}(T(r))) := \begin{bmatrix} \mathrm{Kos}(\Gamma'_S, \mathbf{D}_{R,\varpi}(T(r))) & \xrightarrow{\psi-1} & \mathrm{Kos}(\Gamma'_S, \mathbf{D}_{R,\varpi}(T(r))) \\ \downarrow{\scriptstyle\tau_0} & & \downarrow{\scriptstyle\tau_0} \\ \mathrm{Kos}^c(\Gamma'_S, \mathbf{D}_{R,\varpi}(T(r))) & \xrightarrow{\psi-1} & \mathrm{Kos}^c(\Gamma'_S, \mathbf{D}_{R,\varpi}(T(r))) \end{bmatrix}.$$

This complex is related to the one from the previous section:

Lemma 6.17. *The natural map*

$$\mathrm{Kos}(\psi, \Gamma_S, D_{\varpi}^{(0,v/p]+}(T(r))) \longrightarrow \mathrm{Kos}(\psi, \Gamma_S, \mathbf{D}_{R,\varpi}(T(r))),$$

induced by the inclusion $\pi_m^{-p\ell r} D_{\varpi}^{(0,v/p]+}(T(r)) \subset \mathbf{D}_{R,\varpi}(T(r))$, is a quasi-isomorphism.

Proof. The map on complexes is injective, so we examine the cokernel complex. Write $\mathbf{D}_{R,\varpi}(T(r)) = D_{\varpi}^{(0,v/p]+}(T(r))[\frac{1}{\pi_m}]^{\wedge}$, where $^{\wedge}$ denotes the p-adic completion. By Lemma 2.23, $\psi(\mathbf{A}_{R,\varpi}^{(0,v/p]+}) \subset \mathbf{A}_{R,\varpi}^{(0,v]+} \subset \mathbf{A}_{R,\varpi}^{(0,v/p]+}$ and for $\ell = p^{m-1}$ by Lemma 2.33 (iii), $\pi_m^{-p\ell}\pi$ is a unit in $\mathbf{A}_{R,\varpi}^{(0,v/p]+}$. So for $k \geq 1$ we get $\psi(\pi_m^{-p^k\ell r}\mathbf{A}_{R,\varpi}^{(0,v/p]+}) \subset \pi_m^{-p^{k-1}\ell r}\mathbf{A}_{R,\varpi}^{(0,v/p]+}$ (Lemma 2.25 and Proposition 2.26 (i)). Moreover, we have $\psi(D_{\varpi}^{(0,v/p]+}(T(r))) \subset D_{\varpi}^{(0,v/p]+}(T(r))$. Coupling this with the observation above, we get $\psi(\pi_m^{-p^k\ell r}D_{\varpi}^{(0,v/p]+}(T(r))) \subset \pi_m^{-p^{k-1}\ell r}D_{\varpi}^{(0,v/p]+}(T(r))$. Therefore, the map

$$\psi : \mathbf{D}_{R,\varpi}(T(r))/\pi^{-r}D_{\varpi}^{(0,v/p]+}(T(r)) \longrightarrow \mathbf{D}_{R,\varpi}(T(r))/\pi^{-r}D_{\varpi}^{(0,v/p]+}(T(r))$$

is (pointwise) topologically nilpotent and $1-\psi$ is bijective over this quotient of modules. Therefore, the following complexes are acyclic

$$\begin{aligned}
&[\mathrm{Kos}(\Gamma'_S, \mathbf{D}_{R,\varpi}(T(r))/\pi^{-r}D_{\varpi}^{(0,v/p]+}(T(r))) \xrightarrow{\psi-1} \mathrm{Kos}(\Gamma'_S, \mathbf{D}_{R,\varpi}(T(r))/\pi^{-r}D_{\varpi}^{(0,v/p]+}(T(r)))],\\
&[\mathrm{Kos}^c(\Gamma'_S, \mathbf{D}_{R,\varpi}(T(r))/\pi^{-r}D_{\varpi}^{(0,v/p]+}(T(r))) \xrightarrow{\psi-1} \mathrm{Kos}^c(\Gamma'_S, \mathbf{D}_{R,\varpi}(T(r))/\pi^{-r}D_{\varpi}^{(0,v/p]+}(T(r)))].
\end{aligned}$$

Hence, the cokernel complex of the map in the claim is acyclic. ∎

Next, recall that we have the complex

$$\mathrm{Kos}(\varphi, \Gamma_S, \mathbf{D}_{R,\varpi}(T(r))) = \left[\begin{array}{ccc}
\mathrm{Kos}(\Gamma'_S, \mathbf{D}_{R,\varpi}(T(r))) & \xrightarrow{1-\varphi} & \mathrm{Kos}(\Gamma'_S, \mathbf{D}_{R,\varpi}(T(r))) \\
\downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau_0} \\
\mathrm{Kos}^c(\Gamma'_S, \mathbf{D}_{R,\varpi}(T(r))) & \xrightarrow{1-\varphi} & \mathrm{Kos}^c(\Gamma'_S, \mathbf{D}_{R,\varpi}(T(r)))
\end{array}\right].$$

Proposition 6.18. *With notations as above, the natural map*

$$\mathrm{Kos}(\varphi, \Gamma_S, \mathbf{D}_{R,\varpi}(T(r))) \longrightarrow \mathrm{Kos}(\psi, \Gamma_S, \mathbf{D}_{R,\varpi}(T(r))),$$

induced by identity on the first column and ψ on the second column is a quasi-isomorphism.

Proof. We will examine the kernel and cokernel of the map above. Notice that the map ψ is surjective on $\mathbf{D}_{R,\varpi}(T(r))$, so the cokernel complex is 0. For the kernel complex, we need to show that the complex

$$[\mathrm{Kos}(\Gamma'_S, \mathbf{D}_{R,\varpi}(T(r))^{\psi=0}) \xrightarrow{\tau_0} \mathrm{Kos}(\Gamma'_S, \mathbf{D}_{R,\varpi}(T(r))^{\psi=0})],$$

is acyclic. For this, we will analyze the module $(\mathbf{D}_{R,\varpi}(T(r)))^{\psi=0}$. Let us write $\mathbf{N}(T) = \sum_{j=1}^h \mathbf{A}_R^+ e_j$ for a choice of $\mathbf{A}_R^+$-basis. Since $\mathbf{D}(T(r)) \simeq \mathbf{D}(T)(r) \simeq \mathbf{A}_R \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)(r)$, we obtain that $\{e_1 \otimes \epsilon^{\otimes r}, \ldots, e_h \otimes \epsilon^{\otimes r}\}$ is an $\mathbf{A}_R$-basis of $\mathbf{D}(T(r))$, where $\epsilon^{\otimes r}$ is a basis of $\mathbb{Z}_p(r)$. Further, since $\mathbf{D}(T(r))$ is étale and $\mathbf{D}_{R,\varpi}(T(r)) = \mathbf{A}_{R,\varpi} \otimes_{\mathbf{A}_R} \mathbf{D}(T(r))$, we obtain a decomposition

$$\mathbf{D}_{R,\varpi}(T(r)) \simeq \sum_{j=1}^h \mathbf{A}_{R,\varpi}\varphi(e_j) \otimes \epsilon^{\otimes r}.$$

Using this decomposition, note that we can write

$$z = \sum_{j=1}^h z_j\varphi(e_j) \in \Big(\sum_{j=1}^h \mathbf{A}_{R,\varpi}\varphi(e_j)\Big)^{\psi=0} = (\mathbf{D}_{R,\varpi}(T))^{\psi=0}$$

if and only if $z_j \in \mathbf{A}_{R,\varpi}^{\psi=0}$ for each $1 \leq j \leq h$. Indeed, $\psi(z) = 0$ if and only if $\sum_{j=1}^h \psi(z_j\varphi(e_j)) = \sum_{j=1}^h \psi(z_j)e_j = 0$. As e_j are linearly independent over $\mathbf{A}_{R,\varpi}$, we get the desired conclusion.

Next, according to Proposition 2.26, we have a decomposition

$$\mathbf{A}_{R,\varpi}^{\psi=0} \simeq \bigoplus_{\alpha\in\{0,\dots,p-1\}^{[0,d]},\alpha\neq 0} \varphi(\mathbf{A}_{R,\varpi})[X^\flat]^\alpha, \quad \text{where } [X^\flat]^\alpha = (1+\pi_m)^{\alpha_0}[X_1^\flat]^{\alpha_0}\cdots[X_d^\flat]^{\alpha_d}.$$

Therefore, we obtain that

$$(\mathbf{D}_{R,\varpi}(T(r)))^{\psi=0} \simeq (\mathbf{D}_{R,\varpi}(T))^{\psi=0}(r) \simeq \Big(\sum_{i=1}^h \mathbf{A}_{R,\varpi}e_j\Big)^{\psi=0}(r) \simeq \bigoplus_{\substack{\alpha\in\{0,\dots,p-1\}^{[0,d]},\alpha\neq 0\\ j\in\{1,\dots,h\}}} \varphi(\mathbf{A}_{R,\varpi}e_j)(r)[X^\flat]^\alpha,$$

We have $\mathbf{D}_{R,\varpi}(T) = \sum_{j=1}^h \mathbf{A}_{R,\varpi}e_j$ and we see that the kernel complex of the map in the claim is isomorphic to the complex

$$\bigoplus_{\alpha\in\{0,\dots,p-1\}^{[0,d]},\alpha\neq 0} \Big[\operatorname{Kos}(\Gamma_S', \varphi(\mathbf{D}_{R,\varpi}(T))(r)[X^\flat]^\alpha) \xrightarrow{\tau_0} \operatorname{Kos}^c(\Gamma_S', \varphi(\mathbf{D}_{R,\varpi}(T))(r)[X^\flat]^\alpha)\Big]. \tag{6.17}$$

Lemma 6.19. *The complex described in* (6.17) *is acyclic.*

Proof. The proof is motivated by [CN17, Lemma 4.10, Remark 4.11] and essentially similar to Lemma 6.14. We will treat terms corresponding to each α separately. First, let us assume that $\alpha_k \neq 0$ for some $k \neq 0$. We want to show that both $\operatorname{Kos}(\Gamma_S', \varphi(\mathbf{D}_{R,\varpi}(T))[X^\flat]^\alpha)$ and $\operatorname{Kos}^c(\Gamma_S', \varphi(\mathbf{D}_{R,\varpi}(T))[X^\flat]^\alpha)$ complexes are acyclic (the twist has disappeared because the cyclotomic character is trivial on Γ_S'). As the proof is same in both the cases, we only treat the first case. We can write the complex as a double complex

$$\begin{tikzcd}
\varphi(\mathbf{D}_{R,\varpi}(T))[X^\flat]^\alpha \arrow[r, "(\gamma_i-1)"] \arrow[d, "\gamma_k-1"] & \varphi(\mathbf{D}_{R,\varpi}(T))^{I_1''}[X^\flat]^\alpha \arrow[r] \arrow[d, "\gamma_k-1"] & \varphi(\mathbf{D}_{R,\varpi}(T))^{I_2''}[X^\flat]^\alpha \arrow[r] \arrow[d, "\gamma_k-1"] & \cdots \\
\varphi(\mathbf{D}_{R,\varpi}(T))[X^\flat]^\alpha \arrow[r, "(\gamma_i-1)"] & \varphi(\mathbf{D}_{R,\varpi}(T))^{I_1''}[X^\flat]^\alpha \arrow[r] & \varphi(\mathbf{D}_{R,\varpi}(T))^{I_2''}[X^\flat]^\alpha \arrow[r] & \cdots,
\end{tikzcd}$$

where the first horizontal maps involve γ_i's with $i \neq k$, $1 \leq i \leq d$. Since $\mathbf{D}_{R,\varpi}(T)$ is p-adically complete, it enough to show that $\gamma_k - 1$ is bijective on $\varphi(\mathbf{D}_{R,\varpi}(T))[X^\flat]^\alpha$ modulo p. Indeed, this follows from inductively applying five lemma to following exact sequences, for $n \in \mathbb{N}_{\geq 1}$,

$$\begin{tikzcd}
0 \arrow[r] & p^n\varphi(\mathbf{D}_{R,\varpi}(T))[X^\flat]^\alpha/p^{n+1} \arrow[r] \arrow[d, "\gamma_k-1"] & \varphi(\mathbf{D}_{R,\varpi}(T))[X^\flat]^\alpha/p^{n+1} \arrow[r] \arrow[d, "\gamma_k-1"] & \varphi(\mathbf{D}_{R,\varpi}(T))[X^\flat]^\alpha/p^n \arrow[r] \arrow[d, "\gamma_k-1"] & 0 \\
0 \arrow[r] & p^n\varphi(\mathbf{D}_{R,\varpi}(T))[X^\flat]^\alpha/p^{n+1} \arrow[r] & \varphi(\mathbf{D}_{R,\varpi}(T))[X^\flat]^\alpha/p^{n+1} \arrow[r] & \varphi(\mathbf{D}_{R,\varpi}(T))[X^\flat]^\alpha/p^n \arrow[r] & 0.
\end{tikzcd}$$

So below, we will work modulo p, however with slight abuse, we will hide this from the notation.

Note that we have

$$(\gamma_k - 1)\cdot(\varphi(y)[X^\flat]^\alpha) = \varphi(\pi_1(G(y)))[X^\flat]^\alpha,$$

where

$$G(y) = \tfrac{(1+\pi_1)^{\alpha_k}(\gamma_k-1)y}{\pi_1} + \tfrac{((1+\pi_1)^{\alpha_k}-1)y}{\pi_1}, \quad \text{for } y \in \mathbf{D}_{R,\varpi}(T).$$

Also, note that $\mathbf{E}_{R,\varpi} = \mathbf{E}_{R,\varpi}^+[\frac{1}{\pi_m}]$, and setting $\overline{N}_\varpi = N_\varpi^+(T)/p = \sum_{i=1}^h \mathbf{E}_{R,\varpi}^+ e_i$, we obtain that $\mathbf{D}_{R,\varpi}(T)/p = \overline{N}_\varpi[\frac{1}{\pi_m}]$. Now, G is π_m-linear, $(\gamma_k - 1)\mathbf{N}(T) \subset \pi\mathbf{N}(T)$ (see Definition 3.2), and γ_k fixes π_m. Therefore, G is just multiplication by α_k on $\pi_m^a\overline{N}_\varpi/\pi_m^{a+b}\overline{N}_\varpi$ for $a \in \mathbb{Z}$ and $b = p^{m-1}$. Looking at the following diagram and applying five lemma for $a \in \mathbb{Z}$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_m^{a+b}\overline{N}_\varpi/\pi_m^{a+2b}\overline{N}_\varpi & \longrightarrow & \pi_m^{a}\overline{N}_\varpi/\pi_m^{a+2b}\overline{N}_\varpi & \longrightarrow & \pi_m^{a}\overline{N}_\varpi/\pi_m^{a+b}\overline{N}_\varpi & \longrightarrow & 0 \\ & & \big\downarrow G & & \big\downarrow G & & \big\downarrow G & & \\ 0 & \longrightarrow & \pi_m^{a+b}\overline{N}_\varpi/\pi_m^{a+2b}\overline{N}_\varpi & \longrightarrow & \pi_m^{a}\overline{N}_\varpi/\pi_m^{a+2b}\overline{N}_\varpi & \longrightarrow & \pi_m^{a}\overline{N}_\varpi/\pi_m^{a+b}\overline{N}_\varpi & \longrightarrow & 0, \end{array}$$

we obtain that, G is bijective over $\mathbf{D}_{R,\varpi}(T)/p$. Finally, since π_1 is invertible in $\mathbf{E}_{R,\varpi}$, we obtain that $\gamma_k - 1$ is bijective over $\varphi(\mathbf{D}_{R,\varpi}(T))[X^\flat]^\alpha$ modulo p, as desired.

Next, let $\alpha_k = 0$ for all $k \neq 0$ and $\alpha_0 \neq 0$. To show that the complex in the claim is acyclic, we will show that the map $\tau_0 : \mathrm{Kos} \to \mathrm{Kos}^c$ is bijective. This amounts to showing that the following map

$$\gamma_0 - \delta_{i_1}\cdots\delta_{i_q} : \varphi(\mathbf{D}_{R,\varpi}(T))[X^\flat]^\alpha(r) \longrightarrow \varphi(\mathbf{D}_{R,\varpi}(T))[X^\flat]^\alpha(r), \quad \delta_{i_j} = \tfrac{\gamma_{i_j}^c - 1}{\gamma_{i_j}-1},$$

is bijective. Again, arguing as in the previous part, we see that it is enough to show this statement modulo p. We have

$$(\gamma_0 - \delta_{i_1}\cdots\delta_{i_q})(\varphi(y)[X^\flat]^\alpha(r)) = (c^r\varphi(\gamma_0(y))(1+\pi)^{p^{-m}(c-1)\alpha_0}[X^\flat]^\alpha)(r) - (\varphi(\delta_{i_1}\cdots\delta_{i_q}(y))[X^\flat]^\alpha)(r).$$

So we are lead to study the map F defined by

$$F = c^r(1+\pi_1)^z\gamma_0 - \delta_{i_1}\cdots\delta_{i_q}, \quad z = p^{-m}(c-1)\alpha_0 \in \mathbb{Z}_p^*.$$

Now $c^r - 1$ is divisible by p^m, $(1+\pi_1)^z = 1 + z\pi_1 \bmod \pi_1^2$ and $\delta_{i_j} - 1 \in (\gamma_{i_j} - 1)\mathbb{Z}_p[[\gamma_{i_j} - 1]]$. Therefore, we can write $\pi_1^{-1}F$ in the form $\pi_1^{-1}F = z + \pi_1^{-1}F'$, with $F' \in (p^m, \pi_1^2, \gamma_0 - 1, \ldots, \gamma_d - 1)\mathbb{Z}_p[[\pi_1, \Gamma_S]]$. It follows from Lemma 2.31, Lemma 2.33 and Definition 3.2, that for $b = p^m$ we have that $\pi_1^{-1}F' = 0$ on $\pi_m^a\overline{N}_\varpi/\pi_m^{a+b}\overline{N}_\varpi$, for all $a \in \mathbb{Z}$. Hence, $\pi_1^{-1}F$ induces multiplication by z on $\pi_m^a\overline{N}_\varpi/\pi_m^{a+N}\overline{N}_\varpi$ for all $a \in \mathbb{Z}$, which implies that it is an isomorphism of $\mathbf{D}_{R,\varpi}(T)$ modulo p. This allows us to conclude since π_1 is invertible in $\mathbf{A}_{R,\varpi}$. ■

Combining the analysis for the kernel and cokernel complex, we conclude that the map in the claim of Proposition 6.18 is a quasi-isomorphism. ■

Proof of Proposition 6.1. Recall that s is the height of V (see Definition 3.2). From Lemmas 6.3 & 6.4 and Remark 6.5, we have a p^{4r}-quasi-isomorphism

$$\mathrm{Kos}(\varphi, \partial_A, \mathrm{Fil}^r N_\varpi^{[u,v]}(T)) \simeq \mathcal{K}(\varphi, \mathrm{Lie}\,\Gamma_S, N_\varpi^{[u,v]}(T(r))).$$

Changing from infinitesimal action of Γ_S to the continuous action of Γ_S is an isomorphism of complexes by Proposition 6.8,

$$\mathcal{K}(\varphi, \mathrm{Lie}\,\Gamma_S, N_\varpi^{[u,v]}(T(r))) \simeq \mathcal{K}(\varphi, \Gamma_S, N_\varpi^{[u,v]}(T(r))).$$

Further, from Proposition 6.9 we have a p^{3r}-quasi-isomorphism

$$\mathcal{K}(\varphi, \Gamma_S, N_\varpi^{[u,v]}(T(r))) \simeq \mathcal{K}(\varphi, \Gamma_S, N_\varpi^{(0,v]+}(T(r))).$$

Next, from Proposition 6.11 and Lemmas 6.15 & 6.16, we have $p^{3r+2s+2}$-quasi-isomorphisms

$$\begin{aligned}\tau_{\leq r}\mathcal{K}(\varphi, \Gamma_S, N_\varpi^{(0,v]+}(T(r))) &\simeq \tau_{\leq r}\mathcal{K}(\psi, \Gamma_S, N_\varpi^{(0,v]+}(T(r))) \\ &\simeq \tau_{\leq r}\mathcal{K}(\psi, \Gamma_S, N_\varpi^{(0,v/p]+}(T(r))) \simeq \tau_{\leq r}\mathrm{Kos}(\psi, \Gamma_S, D_\varpi^{(0,v/p]+}(T(r))).\end{aligned}$$

Finally, From Lemma 6.17 and Proposition 6.18 we obtain quasi-isomorphisms

$$\mathrm{Kos}(\psi, \Gamma_S, D_\varpi^{(0,v/p]+}(T(r))) \simeq \mathrm{Kos}(\psi, \Gamma_S, \mathbf{D}_{R,\varpi}(T(r))) \simeq \mathrm{Kos}(\varphi, \Gamma_S, \mathbf{D}_{R,\varpi}(T(r))).$$

Combining these statements we get the claim with $N = 10r + 2s + 2$. ■

6.7. Comparison with Fontaine-Messing period map. The aim of this section is to show that the comparison map from $\mathrm{Syn}(S, M, r)$ to $\mathrm{R}\Gamma_{\mathrm{cont}}(G_S, (T(r)))$ coincides with the period map of Fontaine-Messing. We will follow the strategy of Colmez-Nizioł (see [CN17, §4.7]).

Let us begin by defining a certain period ring (see §2.4 for similar definitions). Note that since $S = R[\varpi]$, we have $\overline{S} = \overline{R} \subset \overline{\mathrm{Fr}\, R}$ and $S_\infty = R_\infty \subset \mathrm{Fr}\, R$. Moreover, we are working under the assumption that $\frac{p-1}{p} \leq u \leq \frac{v}{p} < 1 < v$, for example, one can take $u = \frac{p-1}{p}$ and $v = p-1$. Let $\alpha, \beta \in O_{\mathbb{C}_p}^\flat$ such that $v^\flat(\alpha) = 1/v$ and $v^\flat(\beta) = 1/u$. We take $\mathbf{A}_{S_\infty}^{[u,v]} :=$ p-adic completion of $\mathbf{A}_{\mathrm{inf}}(\widehat{S}_\infty^\flat)\big[\frac{p}{[\alpha]}, \frac{[\beta]}{p}\big]$.

Definition 6.20. Following Definition 2.34, define the following rings:

(i) $E_{\overline{S}}^{\mathrm{PD}} = \Sigma\Lambda$ for $\Sigma = R_\varpi^{\mathrm{PD}}$, $\Lambda = \mathbf{A}_{\mathrm{cris}}(\overline{S})$, and $\iota = \iota_{\mathrm{cycl}}$ (see §2.7).

(ii) $E_{\overline{S}}^{[u,v]} = \Sigma\Lambda$ for $\Sigma = R_\varpi^{[u,v]}$, $\Lambda = \mathbf{A}_{\overline{S}}^{[u,v]}$, and $\iota = \iota_{\mathrm{cycl}}$.

(iii) $E_{S_\infty}^{[u,v]} = \Sigma\Lambda$ for $\Sigma = R_\varpi^{[u,v]}$, $\Lambda = \mathbf{A}_{S_\infty}^{[u,v]}$, and $\iota = \iota_{\mathrm{cycl}}$.

These rings have desirable properties:

Lemma 6.21 ([CN17, Lemma 2.38])**.** (i) *$E_{\overline{S}}^{\mathrm{PD}} \subset E_{\overline{S}}^{[u,v]}$ and $E_{S_\infty}^{[u,v]} \subset E_{\overline{S}}^{[u,v]}$.*

(ii) *The Frobenius φ extends uniquely to continuous morphisms*

$$E_{\overline{S}}^{\mathrm{PD}} \longrightarrow E_{\overline{S}}^{\mathrm{PD}}, \qquad E_{S_\infty}^{[u,v]} \longrightarrow E_{S_\infty}^{[u,v/p]} \qquad E_{\overline{S}}^{[u,v]} \longrightarrow E_{\overline{S}}^{[u,v/p]}.$$

(iii) *The action of G_S extends uniquely to continuous actions on $E_{\overline{S}}^{\mathrm{PD}}$,$E_{S_\infty}^{[u,v]}$ and $E_{\overline{S}}^{[u,v]}$ which commutes with the Frobenius.*

The diagram in (2.7) extends to the following diagram

$$\begin{tikzcd}
& \mathrm{Spec}\,(E_{\overline{S},n}^{\mathrm{PD}}) \arrow[dr] \arrow[dd] & \\
\mathrm{Spec}\,(\overline{S}_n) \arrow[ur, tail] \arrow[rr, tail] \arrow[dd] & & \mathrm{Spec}\,(\mathbf{A}_{\mathrm{cris}}(\overline{S})_n \otimes R_\varpi^+) \arrow[dd] \\
& \mathrm{Spec}\,(R_{\varpi,n}^{\mathrm{PD}}) \arrow[dr] \arrow[dd] & \\
\mathrm{Spec}\,(R[\varpi]_n) \arrow[ur, tail] \arrow[rr, tail] \arrow[dd] & & \mathrm{Spec}\,(R_{\varpi,n}^+) \arrow[dd] \\
& \mathrm{Spec}\,(r_{\varpi,n}^{\mathrm{PD}}) \arrow[dr] & \\
\mathrm{Spec}\,(O_{K,n}) \arrow[ur, tail] \arrow[rr, tail] & & \mathrm{Spec}\,(r_{\varpi,n}^+),
\end{tikzcd}$$

where the horizontal maps are given by $X_0 \mapsto \varpi$ on algebras.

Let V be a positive finite q-height representation of G_R as in Assumption 5.4. From the definition of Wach modules we have that $\mathbf{A}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T) \subset \mathbf{A}^+ \otimes_{\mathbb{Z}_p} T$. Now we have $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R$

$M \subset \mathcal{O}\mathbf{A}^{\mathrm{PD}}_{R,\varpi} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ compatible with Frobenius, filtration connection and the action of Γ_R. Therefore, by extension of scalars we obtain an injective map

$$\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}) \otimes_R M \longrightarrow \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Z}_p} T, \tag{6.18}$$

compatible with Frobenius, filtration, connection and the action of G_R.

Remark 6.22. Recall that we have p^r-exact sequence from (2.2)

$$0 \longrightarrow \mathbb{Z}_p(r)' \longrightarrow \mathrm{Fil}^r \mathbf{A}_{\mathrm{cris}}(\overline{R}) \xrightarrow{p^r - \varphi} \mathbf{A}_{\mathrm{cris}}(\overline{R}) \longrightarrow 0.$$

Tensoring the exact sequence above with T, we obtain a p^r-exact sequence

$$0 \longrightarrow T(r)' \longrightarrow \mathrm{Fil}^r \mathbf{A}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Z}_p} T \xrightarrow{p^r - \varphi} \mathbf{A}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Z}_p} T \longrightarrow 0.$$

Using the rings discussed above, we will introduce the local Fontaine-Messing period map. We set $\Omega_{E^{\mathrm{PD}}_{\overline{S},n}} := E^{\mathrm{PD}}_{\overline{S},n} \otimes_{R^+_{\varpi,n}} \Omega_{R^+_{\varpi,n}}$, $\Delta^{\mathrm{PD}} = E^{\mathrm{PD}}_{\overline{S}} \otimes_R M$ and $\Delta^{\mathrm{PD}}_n = \Delta^{\mathrm{PD}}/p^n$ equipped with natural filtration, Frobenius, integrable connection ∂ and the action of G_S. Note that from (6.18) we have an injective map

$$(\Delta^{\mathrm{PD}})^{\partial=0} = (E^{\mathrm{PD}}_{\overline{S}} \otimes_R M)^{\partial=0} \longrightarrow (E^{\mathrm{PD}}_{\overline{S}} \otimes_{\mathbb{Z}_p} T)^{\partial=0} = \mathbf{A}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Z}_p} T. \tag{6.19}$$

For $r \in \mathbb{Z}$, we have filtered de Rham complex

$$\mathrm{Fil}^r \mathcal{D}^\bullet_{\overline{S},M,n} : \mathrm{Fil}^r \Delta^{\mathrm{PD}}_n \to \mathrm{Fil}^{r-1} \Delta^{\mathrm{PD}}_n \otimes_{R^+_{\varpi,n}} \Omega^1_{R^+_{\varpi,n}} \to \mathrm{Fil}^{r-2} \Delta^{\mathrm{PD}}_n \otimes_{R^+_{\varpi,n}} \Omega^2_{R^+_{\varpi,n}} \to \cdots.$$

Using filtered Poincaré Lemma 2.37 and the discussion above, we get a natural map

$$\mathrm{Fil}^r \mathcal{D}^\bullet_{\overline{S},M,n} \xleftarrow{\sim} (\mathrm{Fil}^r \Delta^{\mathrm{PD}}_n)^{\partial=0} \longrightarrow \mathrm{Fil}^r \mathbf{A}_{\mathrm{cris}}(\overline{R})_n \otimes_{\mathbb{Z}_p} T. \tag{6.20}$$

Notation. For a G_S-module D, let $C(G_S, D)$ denote the complex of continuous cochains of G_S with values in D.

Define the syntomic complex with coefficients in M as

$$\mathrm{Syn}(\overline{S}, M, r)_n := [\mathrm{Fil}^r \mathcal{D}^\bullet_{\overline{S},M,n} \xrightarrow{p^r - p^\bullet \varphi} \mathcal{D}^\bullet_{\overline{S},M,n}]. \tag{6.21}$$

We define the Fontaine-Messing period map

$$\tilde{\alpha}^{\mathrm{FM}}_{r,n,S} : \mathrm{Syn}(S, M, r)_n \longrightarrow C(G_S, T/p^n(r)') \tag{6.22}$$

as the composition

$$\begin{aligned} \mathrm{Syn}(S, M, r)_n = [\mathrm{Fil}^r \mathcal{D}^\bullet_{S,M,n} \xrightarrow{p^r - p^\bullet \varphi} \mathcal{D}^\bullet_{S,M,n}] &\longrightarrow C(G_S, [\mathrm{Fil}^r \mathcal{D}^\bullet_{\overline{S},M,n} \xrightarrow{p^r - p^\bullet \varphi} \mathcal{D}^\bullet_{\overline{S},M,n}]) \longrightarrow \\ &\longrightarrow C(G_S, [\mathrm{Fil}^r \mathbf{A}_{\mathrm{cris}}(\overline{R})_n \otimes T \xrightarrow{p^r - \varphi} \mathbf{A}_{\mathrm{cris}}(\overline{R})_n \otimes T]) \xleftarrow{\sim} C(G_S, T/p^n(r)'). \end{aligned}$$

Here the second right arrow is injective from (6.20) (a consequence of filtered Poincaré Lemma 2.37) and the only left arrow is a p^r-quasi-isomorphism as noted in Remark 6.22 (a consequence of the fundamental exact sequence (2.2)).

Remark 6.23. The definition of Fontaine-Messing period map in (6.22) can also be given for R: one uses the ring $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$ instead of $E^{\mathrm{PD}}_{\overline{S}}$ and obtains $\Delta^{\mathrm{PD}} = \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}) \otimes_R M$. The injective map in (6.20) gets replaced by an injective map $\mathrm{Fil}^r \mathcal{D}^\bullet_{\overline{R},M,n} \xrightarrow{\sim} \mathrm{Fil}^r \mathbf{A}_{\mathrm{cris}}(\overline{R})_n \otimes T$ (where the latter complex is the filtered de Rham complex similar to mod p^n of the complex $\mathrm{Fil}^r \mathcal{D}^\bullet_{\overline{R},M}$ in (5.5)). The definition of $\mathrm{Syn}(\overline{R}, M, r)_n$ follows naturally and since the fundamental exact sequence is G_R-equivariant, one obtains

$$\tilde{\alpha}^{\mathrm{FM}}_{r,n,R} : \mathrm{Syn}(R, M, r)_n \longrightarrow C(G_R, T/p^n(r)').$$

Theorem 6.24. *The map $\tilde{\alpha}^{\mathrm{FM}}_{r,n,S}$ in (6.22) is $p^{N(T,e,r)}$-equal to $\alpha^{\mathcal{L}az}_{r,n}$ from Theorem 5.8.*

Proof. The p-power equality of the two maps follows from the diagram below (where we only show the p-adic version to simplify notations). The objects and morphisms are described after the diagram. Note that we have $\mathrm{K}_{\partial,\varphi}(\mathrm{F}^r M^{\mathrm{PD}}_{\varpi}) = \mathrm{Syn}(S, M, r)$ and the map $\tilde{\alpha}^{\mathrm{FM}}_{r,S}$ is obtained by composing the arrows in the top row (note that $C_G(T(r))$ is p^r-isomorphic to $C_G(T(r)')$. Furthermore, the map $\alpha^{\mathcal{L}\mathrm{az}}_{r}$ is obtained by composing the maps in the lower boundary. The isomorphisms in the diagram indicate a p-power quais-isomorphism between complexes. Finally, a simple diagram chase gives us the claim. ∎

$$\begin{tikzcd}
\mathrm{K}_{\partial,\varphi}(\mathrm{F}^r M^{\mathrm{PD}}_{\varpi}) \arrow[r] \arrow[d, "\tau_{\leq r}", "\wr"'] & C_G(\mathrm{K}_{\partial,\varphi}(\mathrm{F}^r \Delta^{\mathrm{PD}})) \arrow[d] & C_G(\mathrm{K}_{\varphi}(\mathrm{F}^r \Delta^{\mathrm{PD},\partial})) \arrow[l, "\mathrm{PL}"', "\sim"] \arrow[r] \arrow[d] & C_G(\mathrm{K}_{\varphi}(\mathrm{F}^r T A_{\mathrm{cris}})) \\
\mathrm{K}_{\partial,\varphi}(\mathrm{F}^r M^{[u,v]}_{\varpi}) \arrow[r] \arrow[d, "\mathrm{PL}", "\wr"'] & C_G(\mathrm{K}_{\partial,\varphi}(\mathrm{F}^r \Delta^{[u,v]})) & C_G(\mathrm{K}_{\varphi}(\mathrm{F}^r \Delta^{[u,v],\partial})) \arrow[l, "\mathrm{PL}"', "\sim"] \arrow[d] & C_G(T(r)) \arrow[u, "\mathrm{FES}"', "\wr"] \arrow[dl, bend right, "\sim"', "\mathrm{FES}"] \\
\mathrm{K}_{\partial,\varphi,\partial_A}(\mathrm{F}^r \Delta^{[u,v]}_{\varpi}) \arrow[ur, bend right] & & C_G(\mathrm{K}_{\varphi}(\mathrm{F}^r T A^{[u,v]})) & C_G(\mathrm{K}_{\varphi}(T A_{\overline{R}}(r))) \arrow[u, "\mathrm{AS}"', "\wr"] \\
\mathrm{K}_{\varphi,\partial_A}(\mathrm{F}^r N^{[u,v]}_{\varpi}) \arrow[u, "\mathrm{PL}"', "\wr"] \arrow[d, "t^{\bullet}", "\tau_{\leq r}\wr"'] & & & C_{\Gamma}(\mathrm{K}_{\varphi}(D_{R_{\infty}}(r))) \arrow[u, "\wr"] \\
\mathcal{K}_{\varphi,\mathrm{Lie}\,\Gamma}(\mathrm{F}^r N^{[u,v]}_{\varpi}) & \mathcal{K}_{\varphi,\Gamma}(\mathrm{F}^r N^{[u,v]}_{\varpi}) \arrow[l, "\sim", "\mathcal{L}\mathrm{az}"'] \arrow[ur, bend right] & & C_{\Gamma}(\mathrm{K}_{\varphi}(D_{\varpi}(r))) \arrow[u, "\wr"] \\
\mathcal{K}_{\varphi,\mathrm{Lie}\,\Gamma}(N^{[u,v]}_{\varpi}(r)) \arrow[u, "t^r"', "\wr"] & \mathcal{K}_{\varphi,\Gamma}(N^{[u,v]}_{\varpi}(r)) \arrow[l, "\sim", "\mathcal{L}\mathrm{az}"'] \arrow[u, "t^r"] & \mathcal{K}_{\varphi,\Gamma}(N^{(0,v]+}_{\varpi}(r)) \arrow[l, "\sim", "\mathrm{can}"'] \arrow[r, "\sim"] & \mathrm{K}_{\varphi,\Gamma}(D_{\varpi}(r)). \arrow[u, "\wr"]
\end{tikzcd}$$

In the diagram,

- $\Delta^{\mathrm{PD}} = E^{\mathrm{PD}}_{\overline{S}} \otimes_R M$, $\Delta^{\mathrm{PD},\partial} = (\Delta^{\mathrm{PD}})^{\partial=0}$, $TA_{\mathrm{cris}} = \mathbf{A}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Z}_p} T$, $\Delta^{[u,v]} = E^{[u,v]}_{\overline{S}} \otimes_R M$, $\Delta^{[u,v],\partial} = (\Delta^{[u,v]})^{\partial=0}$, $TA^{[u,v]} = \mathbf{A}^{[u,v]}_{\overline{R}} \otimes_{\mathbb{Z}_p} T$, $\Delta^{[u,v]}_{\varpi} = E^{[u,v]}_{R,\varpi} \otimes_R M$ (see Definition 5.28 for $E^{[u,v]}_{R,\varpi}$), $TA_{\overline{R}}(r) = \mathbf{A}_{\overline{R}} \otimes_{\mathbb{Z}_p} T(r)$, $D_{\varpi}(r) = \mathbf{D}_{R,\varpi}(T(r))$, $D_{R_{\infty}}(r) = \mathbf{A}_{S_{\infty}} \otimes_{\mathbf{A}_{R,\varpi}} D_{\varpi}(r)$ and $N^{\star}_{\varpi}(r) = N^{\star}_{\varpi}(T(r))$.
- Moreover, $G = G_S$, $\Gamma = \Gamma_S$ with C_G and C_{Γ} denoting the complex of continuous cochains of G and Γ, respectively.
- The letter "K" denotes the Koszul complex with subscripts: ∂ denotes the operators $((1+X_0)\frac{\partial}{\partial X_0}, \ldots, X_d\frac{\partial}{\partial X_d})$, Γ denotes the operators $(\gamma_0 - 1, \ldots, \gamma_d - 1)$ for our choice of topological generators of Γ, Lie Γ denotes the operators $(\nabla_0, \ldots, \nabla_d)$ with $\nabla_i = \log \gamma_i$ and ∂_A denotes $((1+X_0)\frac{\partial}{\partial T}, X_1\frac{\partial}{\partial X_1}, \ldots, X_d\frac{\partial}{\partial X_d})$ as operators on $\mathbf{A}^{[u,v]}_R$ and $E^{[u,v]}_R$ via the isomorphism $\iota_{\mathrm{cycl}} : \mathbf{A}^{[u,v]}_R \xrightarrow{\sim} R^{[u,v]}_{\varpi}$.
- The letter "$\mathcal{K}$" denotes a subcomplex of the Koszul complex as considered in §6.2, §6.3, §6.4 and §6.5.

Next, let us describe the maps between rows:

- FES denotes a map originating from fundamental exact sequences in (2.2) and (2.6).
- AS denotes a map coming from the Artin-Schreier theory in (2.5).
- PL denotes maps originating from filtered Poincaré Lemma of §2.8.

- Going from the first row to the second row is induced by the inclusion $R_{\varpi}^{\mathrm{PD}} \subset R_{\varpi}^{[u,v]}$ and the leftmost slanted vertical map from third to second row is induced by the inclusion $E_{R,\varpi}^{[u,v]} \subset E_{\overline{S}}^{[u,v]}$.
- The vertical map from second to third row is induced similar to (6.19).
- The rightmost vertical map from the fourth to third row is the inflation map from Γ_R to G_R, using the inclusion $R_\infty \subset \overline{R}$ (one could use almost étale descent to obtain the quasi-isomorphism) and the rightmost vertical map from the fifth to fourth row uses the inclusion $R \subset R_\infty$ (the quasi-isomorphism is obtained by decompletion techniques).
- The leftmost vertical arrow from fourth to fifth row is given by multplication by suitable powers of t as in Lemma 6.3 and the rightmost vertical arrow from sixth to fifth row is the comparison between complex computing continuous cohomology of Γ_R and Koszul complex as in §4.2.
- The inclusions $\mathbf{A}_{R,\varpi}^+ \subset \mathbf{A}_{\inf}(\overline{R}) \subset \mathbf{A}_{\overline{R}}^{[u,v]}$ and $\mathbf{A}_{\inf}(\overline{R}) \otimes_{\mathbf{A}_R^+} \mathbf{N}(T) \subset \mathbf{A}_{\inf}(\overline{R}) \otimes_{\mathbb{Z}_p} T$ induce the slanted vertical arrow from fifth to third row.

Finally, let us describe the maps between columns,

- Top two maps between first and second column are induced by the inclusion $R_{\varpi}^{\mathrm{PD}} \subset E_{\overline{S}}^{\mathrm{PD}}$ and $R_{\varpi}^{[u,v]} \subset E_{\overline{S}}^{[u,v]}$.
- The bottom two maps $\mathcal{L}\mathrm{az}$ between first and second column are Lazard isomorphisms discussed in §6.2.
- The bottom map from third to second column is induced by the canonical inclusion $\mathbf{A}_{R,\varpi}^{(0,v]+} \subset \mathbf{A}_{R,\varpi}^{[u,v]}$.
- The horizontal map from third to fourth column is induced similar to (6.19).
- The bottom horizontal map from fifth to fourth column is obtained by the inclusion $\mathbf{A}_{R,\varpi}^{(0,v]+} \subset \mathbf{A}_{R,\varpi}$ (see §6.5 & §6.6).

Corollary 6.25. *The map $\tilde{\alpha}_{r,n,R}^{\mathrm{FM}}$ in Remark 6.23 is a $p^{N(p,r,s)}$-quasi-isomorphism.*

Proof. Let $m = 2$, i.e. $K = F(\zeta_{p^2} - 1)$. Then, over $S = O_K \otimes_{O_F} R$ we know that the local Fontaine-Messing period map $\tilde{\alpha}_{r,n,S}^{\mathrm{FM}}$ is p^N-isomorphic to the Lazard map $\alpha_{r,n}^{\mathcal{L}\mathrm{az}}$ from Theorem 6.24 and the map $\alpha_{r,n}^{\mathcal{L}\mathrm{az}}$ is a p^N-quasi-isomorphism for $N = N(p,r,s) \in \mathbb{N}$ by Theorem 5.8 and Example 5.5 (ii). To descend, note that the period map is $G = \mathrm{Gal}(F(\zeta_p)/F)$-equivariant, i.e. the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Syn}(R,M,r)_n \arrow[r, "\tilde{\alpha}_{r,n,R}^{\mathrm{FM}}"] \arrow[d] & C(G_R, T/p^n(r)') \arrow[d, "\wr"] \\
\mathrm{R}\Gamma(G, \mathrm{Syn}(S,M,r)_n) \arrow[r, "\tilde{\alpha}_{r,n,S}^{\mathrm{FM}}"] & \mathrm{R}\Gamma(G, C(G_S, T/p^n(r)')).
\end{tikzcd}$$

The right vertical map is a quasi-isomorphism. To conclude, we apply the Galois descent argument in Lemma 6.26 (for $e = p(p-1)$) to the left vertical arrow. ■

6.8. Galois descent. In this section we will describe a Galois descent for syntomic cohomology with coefficients. The result will be used to prove Corollary 5.12 and Theorem 8.8. Let $e = [K : F] = p^{m-1}(p-1)$, $G = \mathrm{Gal}(K/F)$ and $S = O_K \otimes_{O_F} R$. For the sake of convenience with notations, we will use the formulation of crystalline and syntomic complexes from §7.2. We will view the R-module M in Assumption 5.4 as an object in $\mathrm{CR}(R/O_F, \mathrm{Fil}, \varphi)$, i.e. a filtered crystal equipped with Frobenius (see Remark 7.9 and Definition 7.10).

Lemma 6.26. *The natural map*

$$\mathrm{R\Gamma}_{\mathrm{syn}}(R, M, r) \longrightarrow \mathrm{R\Gamma}(G, \mathrm{R\Gamma}_{\mathrm{syn}}(S, M, r)), \tag{6.23}$$

is a p^{4r+3e}-quasi-isomorphism.

Proof. We will closely follow the proof of [CN17, Lemma 5.9]. Recall from (5.3) that we have the filtered de Rham complex

$$\mathrm{Fil}^r \mathscr{D}^\bullet_{S,M} : \mathrm{Fil}^r M^{\mathrm{PD}}_\varpi \longrightarrow \mathrm{Fil}^{r-1} M^{\mathrm{PD}}_\varpi \otimes_{R^{\mathrm{PD}}_\varpi} \Omega^1_{R^{\mathrm{PD}}_\varpi} \longrightarrow \cdots,$$

and $\mathrm{R\Gamma}_{\mathrm{cris}}(S, \mathrm{Fil}^r M) \simeq \mathrm{Fil}^r \mathscr{D}^\bullet_{S,M}$. Furthermore, we have

$$\mathrm{Syn}(S, M, r) = [\mathrm{Fil}^r \mathscr{D}^\bullet_{S,M} \xrightarrow{p^r - \varphi} \mathscr{D}^\bullet_{S,M}] \simeq [\mathrm{R\Gamma}_{\mathrm{cris}}(S, \mathrm{Fil}^r M) \xrightarrow{p^r - \varphi} \mathrm{R\Gamma}_{\mathrm{cris}}(S, M)] = \mathrm{R\Gamma}_{\mathrm{syn}}(S, M, r).$$

From Remark 7.12 we have

$$\mathrm{R\Gamma}_{\mathrm{syn}}(S, M, r) = [\mathrm{R\Gamma}_{\mathrm{cris}}(S, M)^{\varphi = p^r} \xrightarrow{\mathrm{can}} \mathrm{R\Gamma}_{\mathrm{cris}}(S, M)/\mathrm{Fil}^r],$$

where we write $\mathrm{R\Gamma}_{\mathrm{cris}}(S, M)^{\varphi = p^r} = [\mathrm{R\Gamma}_{\mathrm{cris}}(S, M) \xrightarrow{p^r - \varphi} \mathrm{R\Gamma}_{\mathrm{cris}}(S, M)]$ and we view $\mathrm{R\Gamma}_{\mathrm{cris}}(S, \mathrm{Fil}^r M)$ as a subcomplex of $\mathrm{R\Gamma}_{\mathrm{cris}}(S, M)$ via the identification $\mathrm{R\Gamma}_{\mathrm{cris}}(S, \mathrm{Fil}^r M) \simeq \mathrm{Fil}^r \mathscr{D}^\bullet_{S,M}$. One can write similar statements for $\mathrm{R\Gamma}_{\mathrm{syn}}(R, M, r)$. We need to show that we have p-power quasi-isomorphisms

$$\begin{aligned} \mathrm{R\Gamma}_{\mathrm{cris}}(R, M)^{\varphi = p^r} &\xrightarrow{\sim} \mathrm{R\Gamma}(G, \mathrm{R\Gamma}_{\mathrm{cris}}(S, M)^{\varphi = p^r}), \\ \mathrm{R\Gamma}_{\mathrm{cris}}(R, M)/\mathrm{Fil}^r &\xrightarrow{\sim} \mathrm{R\Gamma}(G, \mathrm{R\Gamma}_{\mathrm{cris}}(S, M)/\mathrm{Fil}^r). \end{aligned}$$

For the first map, let $R_\kappa = R \otimes_{O_F} \kappa$ and consider the following diagram of formal schemes

$$\begin{tikzcd} & \mathrm{Spf}\, S \arrow[d] \\ \mathrm{Spec}\, R_\kappa \arrow[ur, tail, "i_S"] \arrow[r, tail, "i_R"] \arrow[d] & \mathrm{Spf}\, R \arrow[d] \\ \mathrm{Spec}\, \kappa \arrow[r, tail] & \mathrm{Spf}\, O_F. \end{tikzcd}$$

It gives us a commutative diagram

$$\begin{tikzcd} \mathrm{R\Gamma}_{\mathrm{cris}}(R, M) \arrow[r, "i_R^*", "\sim"'] \arrow[d] & \mathrm{R\Gamma}_{\mathrm{cris}}(R_\kappa, M) \arrow[d] \\ \mathrm{R\Gamma}(G, \mathrm{R\Gamma}_{\mathrm{cris}}(S, M)) \arrow[r, "i_S^*"] & \mathrm{R\Gamma}(G, \mathrm{R\Gamma}_{\mathrm{cris}}(R_\kappa, M)). \end{tikzcd}$$

The top arrow is a quasi-isomorphism and the right vertical arrow is an e-quasi-isomorphism. So we are left to show that

$$\mathrm{R}\Gamma_{\mathrm{cris}}(R,M)^{\varphi=p^r} \longrightarrow \mathrm{R}\Gamma(G,\mathrm{R}\Gamma_{\mathrm{cris}}(R,M)^{\varphi=p^r})$$

is a p-power quasi-isomorphism. Now let $n\in\mathbb{N}$ such that $p^n\geq e$ and consider the following factorization

$$\varphi^n : \mathrm{R}\Gamma_{\mathrm{cris}}(S,M)^{\varphi=p^r} \xrightarrow{i_S^*} \mathrm{R}\Gamma_{\mathrm{cris}}(R_\kappa,M)^{\varphi=p^r} \xrightarrow{j_n} \mathrm{R}\Gamma_{\mathrm{cris}}(S,M)^{\varphi=p^r},$$

where the maps i_S^* and j_n are obvious by using the complex $\mathscr{E}^\bullet_{R,M}$ in Remark 5.10 to describe the crystalline cohomology complex of R with coefficients in M. We also have the following factorization

$$\varphi^n : \mathrm{R}\Gamma_{\mathrm{cris}}(R_\kappa,M)^{\varphi=p^r} \xrightarrow{j_n} \mathrm{R}\Gamma_{\mathrm{cris}}(S,M)^{\varphi=p^r} \xrightarrow{i_S^*} \mathrm{R}\Gamma_{\mathrm{cris}}(S,M)^{\varphi=p^r},$$

where now we use the complex $\mathscr{D}^\bullet_{R,M}$ in (5.5) to describe the crystalline cohomology complex of R with coefficients in M. The map φ^n is a p^{2rn}-quasi-isomorphism on $\mathrm{R}\Gamma_{\mathrm{cris}}(R,M)^{\varphi=p^r}$ and $\mathrm{R}\Gamma_{\mathrm{cris}}(S,M)^{\varphi=p^r}$. Therefore, i_S^* and j_n as above are p^{4rn}-quasi-isomorphisms.

Finally, we need to show that the map

$$\mathrm{R}\Gamma_{\mathrm{cris}}(R,M)/\mathrm{Fil}^r \longrightarrow \mathrm{R}\Gamma(G,\mathrm{R}\Gamma_{\mathrm{cris}}(S,M)/\mathrm{Fil}^r)$$

is a p-power isomorphism. Note that we have $\mathrm{R}\Gamma_{\mathrm{cris}}(R,M)\simeq \mathrm{R}\Gamma_{\mathrm{dR}}((R,M)/O_F)$ and by writing down the complexes explicitly, one obtains a p^r-quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{cris}}(S,M)/\mathrm{Fil}^r \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}((S,M)/O_K)/\mathrm{Fil}^r.$$

This reduces us to showing that the map

$$\mathrm{R}\Gamma_{\mathrm{dR}}((R,M)/O_F)/\mathrm{Fil}^r \longrightarrow \mathrm{R}\Gamma(G,\mathrm{R}\Gamma_{\mathrm{dR}}((S,M)/O_K)/\mathrm{Fil}^r),$$

is a p-power isomorphism. But since we have $\Omega^\bullet_{S/O_K} = \Omega^\bullet_{R/O_F}\otimes_{O_F} O_K$, we conclude that the map above is an e-quasi-isomorphism. Putting everything together we see that (6.23) is a p^{4r+3e}-quasi-isomorphism. ∎

7. Crystals and syntomic cohomology

7.1. Filtered F-crystals. In this section, first we will recall some general facts on finite locally free filtered F-crystals from [Ber74; BO78]. Then we will consider consider crystals on the special fiber of a (p-adic formal) scheme defined over a discrete valuation ring of mixed chracteristic.

7.1.1. Generalities on crystals. Let $(\Sigma, \mathcal{J}_\Sigma, \gamma_\Sigma)$ be a PD-scheme on which p is locally nilpotent (see [Ber74, Définition 1.9.6]), and let $\mathfrak{X}$ be a Σ-scheme such that the PD-structure γ_Σ extends to $\mathfrak{X}$. Let $\mathrm{CRIS}(\mathfrak{X}/\Sigma)$ denote the big crystalline site of $\mathfrak{X}$ over Σ with the underlying topology being the étale topology, and let $(\mathfrak{X}/\Sigma)_{\mathrm{CRIS}}$ be the PD-ringed topos equipped with the PD-ring $(\mathcal{O}_{\mathfrak{X}/\Sigma}, \mathcal{J}_{\mathfrak{X}/\Sigma})$ (see [Ber74, Définitions 1.9.1, 1.9.3]). By [Ber74, §III.4.1.2] the category of $\mathcal{O}_{\mathfrak{X}/\Sigma}$-modules is equivalent to the category of data $(\mathcal{F}_\mathfrak{T}, \tau_f)$ consisting of an $\mathcal{O}_\mathfrak{T}$-module $\mathcal{F}_\mathfrak{T}$ on $\mathfrak{T}_{\text{ét}} = (\mathfrak{T}_{\text{ét}}, (\mathcal{O}_\mathfrak{T}, \mathcal{J}_\mathfrak{T}, \gamma_\mathfrak{T}))$ for each object $\mathfrak{T}$ of $\mathrm{CRIS}(\mathfrak{X}/\Sigma)$ and a morphism of $\mathcal{O}_{\mathfrak{T}'}$-modules $\tau_f : f^*(\mathcal{F}_\mathfrak{T}) \to \mathcal{F}_{\mathfrak{T}'}$ on $\mathfrak{T}'_{\text{ét}}$ for each morphism $f : \mathfrak{T}' \to \mathfrak{T}$ of $\mathrm{CRIS}(\mathfrak{X}/\Sigma)$ satisfying $\tau_{id} = id$ and the cocycle condition for composition of f's, and being an isomorphism if f is étale and $\mathcal{J}_{\mathfrak{T}'} = f^*\mathcal{J}_\mathfrak{T}$.

Definition 7.1. An $\mathcal{O}_{\mathfrak{X}/\Sigma}$-module $\mathcal{F}$ is said to be *crystal* if for every f and the corresponding data $(\mathcal{F}_\mathfrak{T}, \tau_f)$ as above, $\tau_f : f^*(\mathcal{F}_\mathfrak{T}) \xrightarrow{\sim} \mathcal{F}_{\mathfrak{T}'}$ is an isomorphism on $\mathfrak{T}'_{\text{ét}}$. Further, a crystal $\mathcal{F}$ is said to be *quasi-coherent* (resp. *coherent*, resp. *finite locally free*) if for every object $\mathfrak{T}$ of $\mathrm{CRIS}(\mathfrak{X}/\Sigma)$ the $\mathcal{O}_\mathfrak{T}$-module $\mathcal{F}_\mathfrak{T}$ is quasi-coherent (resp. coherent, resp. locally free of finite type). We will denote by $\mathrm{CR}(\mathfrak{X}/\Sigma)$ the category of finite locally free crystals on $\mathrm{CRIS}(\mathfrak{X}/\Sigma)$.

Remark 7.2. In the definition above, we consider the big crystalline with étale topology. One can consider other topologies as well, for example, Zariski or syntomic. Crystals on these different sites are comparable (see [BBM82, §1.1.18, §1.1.19] and [Bau92, Corollary 1.15, Proposition 1.17]). However, unless otherwise stated, in the rest of the text we will work with the setting described above.

Next, we will introduce filtered crystals. We equip $\mathcal{O}_{\mathfrak{X}/\Sigma}$ with a filtration given as $\mathrm{Fil}^r\mathcal{O}_{\mathfrak{X}/\Sigma} = \mathcal{J}_{\mathfrak{X}/\Sigma}^{[r]}$ for $r > 0$ and $\mathcal{O}_{\mathfrak{X}/\Sigma}$ for $r \leq 0$. By [Ber74, §III.4.1.2] and [Tsu20, Lemma 14], we see that the category of filtered $\mathcal{O}_{\mathfrak{X}/\Sigma}$-modules is equivalent to the category of data $(\mathcal{F}_\mathfrak{T}, \tau_f)$ consisting of a filtered module $\mathcal{F}_\mathfrak{T}$ on $\mathfrak{T}_{\text{ét}}$ for each object $\mathfrak{T}$ of $\mathrm{CRIS}(\mathfrak{X}/\Sigma)$ and a morphism of filtered modules $\tau_f : f^*(\mathcal{F}_\mathfrak{T}) \to \mathcal{F}_{\mathfrak{T}'}$ on $\mathfrak{T}'_{\text{ét}}$ for each morphism $f : \mathfrak{T}' \to \mathfrak{T}$ of $\mathrm{CRIS}(\mathfrak{X}/\Sigma)$ satisfying analogous conditions as above.

Definition 7.3. A filtered $\mathcal{O}_{\mathfrak{X}/\Sigma}$-module $\mathcal{F}$ is said to be *a filtered crystal* if for every f and the corresponding data $(\mathcal{F}_\mathfrak{T}, \tau_f)$ as above, $\tau_f : f^*(\mathcal{F}_\mathfrak{T}) \xrightarrow{\sim} \mathcal{F}_{\mathfrak{T}'}$ is a filtered isomorphism on $T'_{\text{ét}}$, i.e. $\mathrm{Fil}^r\mathcal{F}_{\mathfrak{T}'} = \sum_{s\in\mathbb{Z}} \mathrm{Fil}^s\mathcal{O}_{\mathfrak{T}'} \cdot \mathrm{Im}(f^{-1}(\mathrm{Fil}^{r-s}\mathcal{F}_\mathfrak{T}) \to \mathcal{F}_{\mathfrak{T}'})$ for all $r \in \mathbb{Z}$. Further, a filtered crystal $(\mathcal{F}, \mathrm{Fil}^\bullet\mathcal{F})$ is said to be *finite locally free* if the underlying crystal is locally free of finite type and for every object $\mathfrak{T}$ of $\mathrm{CRIS}(\mathfrak{X}/\Sigma)$ the $\mathcal{O}_\mathfrak{T}$-modules $\mathrm{Fil}^r\mathcal{F}_\mathfrak{T}$ are quasi-coherent for all $r \in \mathbb{Z}$. We will denote by $\mathrm{CR}(\mathfrak{X}/\Sigma, \mathrm{Fil})$ the full subcategory of finite locally free filtered crystals on $\mathrm{CRIS}(\mathfrak{X}/\Sigma)$.

Let us now assume that we are given a closed immersion $\iota : \mathfrak{X} \rightarrowtail \mathfrak{Y}$ where $\mathfrak{Y}/\Sigma$ is smooth. Let $\mathfrak{D}$ denote the γ_Σ-compatible PD-envelope of the immersion $\mathfrak{X} \rightarrowtail \mathfrak{Y} \times_\Sigma \mathfrak{Y}$ induced by ι. Consider the category of $\mathcal{O}_\mathfrak{D}$-modules $\mathcal{M}$ on the PD-scheme $\mathfrak{D}$ equipped with a quasi-nilpotent integrable connection $\partial : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_\mathfrak{Y}} \Omega_{\mathfrak{Y}/\Sigma}$, and a morphism between two such objects being ∂-compatible morphism of $\mathcal{O}_\mathfrak{D}$-modules.

Definition 7.4. We will denote by $\mathrm{MIC}(\mathfrak{X} \rightarrowtail \mathfrak{Y}/\Sigma)$ the category of finite locally free $\mathcal{O}_\mathfrak{D}$-modules.

Now we will consider $\mathcal{O}_{\mathfrak{D}}$-modules equipped with a filtration. Consider the category of filtered $\mathcal{O}_{\mathfrak{D}}$-modules $\mathcal{M}$ on the PD-scheme $\mathfrak{D}$ equipped with a quasi-nilpotent integrable connection $\partial : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \Omega_{\mathfrak{Y}/\Sigma}$ satisfying Griffiths transversality with respect to the filtration, i.e. $\partial(\mathrm{Fil}^r \mathcal{M}) \subset \mathrm{Fil}^{r-1}\mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{Y}}} \Omega_{\mathfrak{Y}/\Sigma}$ for every $r \in \mathbb{Z}$. A morphism between two such objects is a morphism of underlying $\mathcal{O}_{\mathfrak{D}}$-modules compatible with ∂ and filtration.

Definition 7.5. A filtered $\mathcal{O}_D$-module $(\mathcal{M}, \mathrm{Fil}^{\bullet}\mathcal{M})$ is said to be *finite locally free* if $\mathcal{M}$ is locally free of finite type and $\mathrm{Fil}^r\mathcal{M}$ are quasi-coherent for all $r \in \mathbb{Z}$. We will denote by $\mathrm{MIC}(\mathfrak{X} \rightarrowtail \mathfrak{Y}/\Sigma)$ the full subcategory of finite locally free filtered $\mathcal{O}_{\mathfrak{D}}$-modules.

By [Ber74, Chapitre IV, Théorèm 1.6.5], we have a natural equivalence of categories

$$\mathrm{CR}(\mathfrak{X}/\Sigma) \xrightarrow{\sim} \mathrm{MIC}(\mathfrak{X} \rightarrowtail \mathfrak{Y}/\Sigma), \tag{7.1}$$

which restricts to an equivalence of categories (see [Tsu20, Theorem 17])

$$\mathrm{CR}(\mathfrak{X}/\Sigma, \mathrm{Fil}) \xrightarrow{\sim} \mathrm{MIC}(\mathfrak{X} \rightarrowtail \mathfrak{Y}/\Sigma, \mathrm{Fil}). \tag{7.2}$$

7.1.2. Our setup. Let κ be a finite field of characteristic p, $O_F = W(\kappa)$ the ring of p-typical Witt vectors with coefficients in κ and $F = \mathrm{Fr}\, O_F$. Furthermore, let K be a finite extension of F with O_K its ring of integers and ϖ a uniformizer and such that $K \cap F^{\mathrm{ur}} = F$.

Notation. In this section we will use same letters $\mathfrak{X}$ to denote schemes as well as (p-adic) formal schemes. As definitions are same in both cases, it is easier to define them at the same time to avoid repetition.

Let $\mathfrak{X}$ be a (p-adic formal) scheme over O_K with X as its (rigid) generic fiber and $\mathfrak{X}_\kappa$ its special fiber. Set $\Sigma = \mathrm{Spec}\, O_F$ (resp. $\Sigma = \mathrm{Spf}\, O_F$), for $n \in \mathbb{N}$, let $\mathfrak{X}_n = \mathfrak{X} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n$ and $\Sigma_n = \mathrm{Spec}\,(O_F/p^n)$. Consider the big crystalline site $\mathrm{CRIS}(\mathfrak{X}_n/\Sigma_n)$ with the PD-ideal $(p(O_F/p^n), [\,])$. By Definition 7.1 we can define the category $\mathrm{CR}(\mathfrak{X}_n/\Sigma_n)$ of finite locally free crystals on $\mathrm{CRIS}(\mathfrak{X}_n/\Sigma_n)$. Furthermore, the homomorphisms $\mathfrak{X}_n \to \mathfrak{X}_{n+1}$ and $\Sigma_n \to \Sigma_{n+1}$ induce the pullback functor $i^*_{n,n+1} : \mathrm{CR}(\mathfrak{X}_{n+1}/\Sigma_{n+1}) \to \mathrm{CR}(\mathfrak{X}_n/\Sigma_n)$. In a similar manner, one can define the crystalline site $\mathrm{CRIS}(\mathfrak{X}_1/\Sigma_n)$, the category of finite locally free crystals $\mathrm{CR}(\mathfrak{X}_1/\Sigma_n)$ and the natural pullback functor $i^*_n : \mathrm{CR}(\mathfrak{X}_n/\Sigma_n) \to \mathrm{CR}(\mathfrak{X}_1/\Sigma_n)$, which is an equivalence by [Ber74, Chapitre IV, Théorèm 1.4.1]. So, we consider the following category of crystals:

Definition 7.6. A finite locally free crystal on $\mathrm{CRIS}(\mathfrak{X}/\Sigma)$ is the data $\mathcal{F} = (\mathcal{F}_n)_{n \geq 1}$ where $\mathcal{F}_n$ is an object of $\mathrm{CR}(\mathfrak{X}_n/\Sigma_n)$ and we have isomorphisms $i^*_{n,n+1}(\mathcal{F}_{n+1}) \xrightarrow{\sim} \mathcal{F}_n$. The morphism between two crystals $\mathcal{F}$ and $\mathcal{G}$ on $\mathrm{CRIS}(\mathfrak{X}/\Sigma)$ is a collection of morphisms $\mathcal{F}_n \to \mathcal{G}_n$ for each $n \geq 1$ compatible with the pullback isomorphism. We denote this category by $\mathrm{CR}(\mathfrak{X}/\Sigma)$. A finite locally free crystal on $\mathrm{CRIS}(\mathfrak{X}_1/\Sigma)$ is defined similarly using $\mathrm{CR}(\mathfrak{X}_1/\Sigma_n)$. The pullback functor $i^* : \mathrm{CR}(\mathfrak{X}/\Sigma) \to \mathrm{CR}(\mathfrak{X}_1/\Sigma)$ induces an equivalence of categories.

Remark 7.7. Let $R = p$-adic completion of an étale algebra over $O_F[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$, in particular, R is formally smooth over O_F. Let $\mathrm{MIC}(R, \partial)$ denote the category of finite projective R-modules equipped with an integrable connection. Further, let $\mathrm{MIC}_{\mathrm{conv}}(R) \subset \mathrm{MIC}(R)$ denote the full subcategory of modules whose connection is p-adically quasi-nilpotent. Let $\mathfrak{X} = \mathrm{Spf}\, R$, then from [Ber74, Chapitre IV, Théorèm 1.6.5] and [MT20, Lemma 1.9] we obtain an equivalence of categories $\mathrm{CR}(\mathfrak{X}/\Sigma) \xrightarrow{\sim} \mathrm{MIC}_{\mathrm{conv}}(R)$ obtained by taking the inverse limit of the evaluation $\mathcal{F}_n$ on the objects $\mathfrak{X}_n \xrightarrow{id} \mathfrak{X}_n$ of $\mathrm{CRIS}(\mathfrak{X}_n/\Sigma_n)$ equipped with a natural integrable connection.

Next, we will consider finite locally free crystals on $\mathrm{CRIS}(\mathfrak{X}/\Sigma)$ equipped with a filtration. By Definition 7.3 we have the category $\mathrm{CR}(\mathfrak{X}_n/\Sigma_n, \mathrm{Fil})$ of finite locally free filtered crystals on $\mathrm{CRIS}(\mathfrak{X}_n/\Sigma_n)$. Furthermore, we have the natural pullback functor $i^*_{n,n+1} : \mathrm{CR}(\mathfrak{X}_{n+1}/\Sigma_{n+1}, \mathrm{Fil}) \to \mathrm{CR}(\mathfrak{X}_n/\Sigma_n, \mathrm{Fil})$. So, we consider the following category of crystals:

Definition 7.8. A finite locally free filtered crystal on $\mathrm{CRIS}(\mathfrak{X}/\Sigma)$ is the data $(\mathcal{F}_n)_{n\geq 1}$ where $\mathcal{F}_n$ is an object of $\mathrm{CR}(\mathfrak{X}_n/\Sigma_n, \mathrm{Fil})$ and we have filtered isomorphisms $i^*_{n,n+1}(\mathcal{F}_{n+1}) \xrightarrow{\sim} \mathcal{F}_n$. The morphisms between filtered crystals is defined in an obvious way and we denote this category as $\mathrm{CR}(\mathfrak{X}/\Sigma, \mathrm{Fil})$.

Remark 7.9. In the setting of Remark 7.7, the equivalence of categories restricts to an equivalence $\mathrm{CR}(\mathfrak{X}/\Sigma, \mathrm{Fil}) \xrightarrow{\sim} \mathrm{MIC}_{\mathrm{conv}}(R, \mathrm{Fil})$, where the target is the full subcategory of $\mathrm{MIC}(R, \mathrm{Fil})$ consisting of objects equipped with p-adically quasi-nilpotent integrable connection.

Finally, we will consider crystals equipped with a Frobenius structure. The Frobenius endomorphism of O_F and the absolute Frobenius on $\mathfrak{X}_1$ induce Frobenius pullbacks $F^*_{\mathfrak{X}_1} : \mathrm{CR}(\mathfrak{X}_1/\Sigma_n) \to \mathrm{CR}(\mathfrak{X}_1/\Sigma_n)$ and $F^*_{\mathfrak{X}_1} : \mathrm{CR}(\mathfrak{X}_1/\Sigma) \to \mathrm{CR}(\mathfrak{X}_1/\Sigma)$. Also, recall that we have the natural pullback functor $i^* : \mathrm{CR}(\mathfrak{X}/\Sigma) \to \mathrm{CR}(\mathfrak{X}_1/\Sigma)$.

Definition 7.10. A *Frobenius structure* on a finite locally free crystal $\mathcal{F}$ on $\mathrm{CRIS}(\mathfrak{X}/\Sigma)$ is a morphism $\varphi_{\mathcal{F}} : F^*_{\mathfrak{X}_1} i^*\mathcal{F} \to i^*\mathcal{F}$ such that it becomes an isomorphism in the isogeny category $\mathrm{CR}(\mathfrak{X}/\Sigma)_{\mathbb{Q}}$. A morphism between two crystals with Frobenius structure is taken to be a morphism in $\mathrm{CR}(\mathfrak{X}/\Sigma)$ compatible with respective Frobenius structures. We denote the category of finite locally free crystals (resp. filtered crystals) equipped with Frobenius structure as $\mathrm{CR}(\mathfrak{X}/\Sigma, \varphi)$ (resp. $\mathrm{CR}(\mathfrak{X}/\Sigma, \mathrm{Fil}, \varphi)$).

7.2. Syntomic complex. In this section we will study syntomic cohomology with coefficients in a finite locally free filtered F-crystal.

Notation. In this section again we will use letters (e.g. $\mathfrak{X}, \mathfrak{U}, Z$ etc.) to denote schemes as well as (p-adic) formal schemes instead of calligraphic notations for the latter.

7.2.1. Via étale site. Let $\mathfrak{X}$ be a smooth (p-adic formal) scheme over O_K, let $\Sigma = \mathrm{Spec}\, O_F$ (resp. $\Sigma = \mathrm{Spf}\, O_F$) and let $\mathcal{F}$ be an object of $\mathrm{CR}(\mathfrak{X}/\Sigma, \mathrm{Fil}, \varphi)$, i.e. a finite locally free filtered crystal on $\mathrm{CRIS}(\mathfrak{X}/\Sigma)$ equipped with a Frobenius structure. Further, let $u_{\mathfrak{X}_n/\Sigma_n} : (\mathfrak{X}_n/\Sigma_n)_{\mathrm{cris}} \to \mathfrak{X}_{n,\text{ét}}$ denote the projection from crystalline topos to étale topos. In the following, we regard sheaves on $\mathfrak{X}_{n,\text{ét}}$ as sheaves on $\mathfrak{X}_{\kappa,\text{ét}}$. For $r \geq 0$ we have filtered crystalline cohomology complexes of $\mathcal{F}$

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{cris}}(\mathfrak{X}, \mathrm{Fil}^r\mathcal{F})_n &:= \mathrm{R}\Gamma(\mathfrak{X}_{n,\text{ét}}, \mathrm{R}u_{\mathfrak{X}_n/\Sigma_n *}\mathrm{Fil}^r\mathcal{F}_n), \\ \mathrm{R}\Gamma_{\mathrm{cris}}(\mathfrak{X}, \mathrm{Fil}^r\mathcal{F}) &:= \mathrm{holim}_n\, \mathrm{R}\Gamma_{\mathrm{cris}}(\mathfrak{X}, \mathrm{Fil}^r\mathcal{F})_n. \end{aligned}$$

Definition 7.11. Define the mod p^n and completed syntomic complex with coefficients in $\mathcal{F}$ as

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{X}, \mathcal{F}, r)_n &:= [\mathrm{R}\Gamma_{\mathrm{cris}}(\mathfrak{X}, \mathrm{Fil}^r\mathcal{F})_n \xrightarrow{p^r-\varphi} \mathrm{R}\Gamma_{\mathrm{cris}}(\mathfrak{X}, \mathcal{F})_n], \\ \mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{X}, \mathcal{F}, r) &:= \mathrm{holim}_n\, \mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{X}, \mathcal{F}, r)_n. \end{aligned}$$

The mapping fibers are taken in the ∞-derived category of abelian groups.

Remark 7.12. We have $\mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{X}, \mathcal{F}, r)_n \simeq \mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{X}, \mathcal{F}, r) \otimes^L_{\mathbb{Z}_p} \mathbb{Z}/p^n$ and $\mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{X}, \mathcal{F}, r)_n \simeq [\mathrm{R}\Gamma_{\mathrm{cris}}(\mathfrak{X}, \mathcal{F})_n \xrightarrow{(p^r-\varphi, \mathrm{can})} \mathrm{R}\Gamma_{\mathrm{cris}}(\mathfrak{X}, \mathcal{F})_n \oplus \mathrm{R}\Gamma_{\mathrm{cris}}(\mathfrak{X}, \mathcal{F}/\mathrm{Fil}^r\mathcal{F})_n]$ in $D^+(\mathfrak{X}_{\kappa,\text{ét}}, \mathbb{Z}/p^n)$.

The definitions above sheafify:

Definition 7.13. Let $\mathfrak{X}$ be a smooth (p-adic formal) scheme over O_K and $\mathcal{F} \in \mathrm{CR}(\mathfrak{X}/\Sigma, \mathrm{Fil}, \varphi)$. Define

$$\begin{aligned} \mathscr{F}_{n,\text{ét},\mathfrak{X}} &: \text{étale sheafification of } (\mathfrak{U} \to \mathfrak{X}) \mapsto \mathrm{R}\Gamma_{\mathrm{cris}}(\mathfrak{U}, \mathcal{F})_n, \\ \mathrm{Fil}^r\mathscr{F}_{n,\text{ét},\mathfrak{X}} &: \text{étale sheafification of } (\mathfrak{U} \to \mathfrak{X}) \mapsto \mathrm{R}\Gamma_{\mathrm{cris}}(\mathfrak{U}, \mathrm{Fil}^r\mathcal{F})_n, \end{aligned}$$

where $\mathfrak{U} \to \mathfrak{X}$ is any étale map. Similarly, we define

$$\mathscr{S}_{n,\text{ét}}(\mathcal{F}, r)_{\mathfrak{X}} : \text{étale sheafification of } (\mathfrak{U} \to \mathfrak{X}) \mapsto \mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{U}, \mathcal{F}, r)_n.$$

Lemma 7.14. *In the setting described, we have*

$$\mathscr{S}_{n,\text{ét}}(\mathcal{F},r)_{\mathfrak{X}} = [\mathrm{Fil}^r\mathscr{F}_{n,\text{ét},\mathfrak{X}} \xrightarrow{p^r-\varphi} \mathscr{F}_{n,\text{ét},\mathfrak{X}}],$$
$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{X},\mathcal{F},r)_n = \mathrm{R}\Gamma(\mathfrak{X}_{\kappa,\text{ét}},\mathscr{S}_{n,\text{ét}}(\mathcal{F},r)_{\mathfrak{X}}).$$

Remark 7.15. We describe a formulation of syntomic cohomology using hypercoverings. The advantage of this definition is that it lets us reduce to local computations. Let $\mathfrak{X}$ be a smooth (p-adic formal) scheme over O_K. Let $\mathfrak{U}^\bullet$ denote an étale hyper-covering of $\mathfrak{X}$ and a morphism of simplicial (formal) schemes $i^\bullet : \mathfrak{U}^\bullet \to \mathfrak{Z}^\bullet$, with the property that for each $s \in \mathbb{N}$, i^s is an immersion of (formal) schemes and $\mathfrak{Z}^s$ is smooth over O_F in such a manner that there exists a compatible system of liftings of Frobenius $F_{\mathfrak{Z}^\bullet} := \{F_{\mathfrak{Z}^\bullet_n} : \mathfrak{Z}^\bullet_n \to \mathfrak{Z}^\bullet_n\}$. Also, set $\mathfrak{U}^\bullet_\kappa := \mathfrak{U}^\bullet \otimes_{O_F} \kappa$.

For a fixed hypercovering $\mathfrak{U}^\bullet \to \mathfrak{X}$, let $(\mathfrak{U}^\bullet)^\sim_{\text{ét}}$ denote the topos whose object is a system which associates to each integer $s \geq 0$ a sheaf $\mathcal{F}^s$ on $\mathfrak{U}^s_{\text{ét}}$, and to each non-decreasing map $a : \{0,\dots,s\} \to \{0,\dots,t\}$ a morphism $\rho_a : \underline{a}^{-1}(\mathcal{F}^s) \to \mathcal{F}^t$ where $\underline{a} : \mathfrak{U}^s \to \mathfrak{U}^t$ corresponds to a, satisfying $\rho_{id} = id$ and $\rho_{ab} = \rho_a \circ \underline{a}^{-1}(\rho_b)$. The morphism of toposes $\theta : (\mathfrak{U}^\bullet)^\sim_{\text{ét}} \to \mathfrak{X}^\sim_{\text{ét}}$ satisfies $\mathcal{F} \xrightarrow{\sim} \mathrm{R}\theta_*(\theta^*\mathcal{F})$ for $\mathcal{F}$ a torsion abelian sheaf on $\mathfrak{X}_{\text{ét}}$ (see [AGV71, §V.7] and [Con03]). In other words, the hypercovering $\mathfrak{U}^\bullet \to \mathfrak{X}$ satisfies cohomological descent. Next, given a sheaf $\mathcal{F}^\bullet$ on $(\mathfrak{U}^\bullet)^\sim_{\text{ét}}$, we define the global sections functor on $\mathfrak{U}^\bullet$ as $\Gamma(\mathfrak{U}^\bullet_{\text{ét}},\mathcal{F}^\bullet) = \mathrm{Ker}\,(\Gamma(\mathfrak{U}^0_{\text{ét}},\mathcal{F}^0) \to \Gamma(\mathfrak{U}^1_{\text{ét}},\mathcal{F}^1))$ which satisfies $\Gamma(\mathfrak{U}^\bullet_{\text{ét}},\mathcal{F}^\bullet) = \Gamma(\mathfrak{X}_{\text{ét}},\theta_*\mathcal{F}^\bullet)$ (see [Con03, Definition 6.10]). This functor is left exact and we write $\mathrm{R}\Gamma(\mathfrak{U}^\bullet_{\text{ét}},\mathcal{F}^\bullet)$ for the resulting total right derived functor. Similarly, one can define $\mathrm{R}\Gamma(\mathfrak{U}^\bullet_{\kappa,\text{ét}},\mathcal{F}^\bullet)$ using $\theta_\kappa : (\mathfrak{U}^\bullet_{\kappa,\text{ét}})^\sim \to \mathfrak{X}^\sim_{\kappa,\text{ét}}$.

Now let $\mathcal{F} \in \mathrm{CR}(\mathfrak{X}/\Sigma,\mathrm{Fil},\varphi)$ and $(\mathfrak{U}^\bullet,\mathfrak{Z}^\bullet,F_{\mathfrak{Z}^\bullet})$ where $\mathfrak{U}^\bullet \to \mathfrak{X}$ is a hypercovering and $\mathfrak{Z}^\bullet$ and $F_{\mathfrak{Z}^\bullet}$ are chosen as above. For each $s \in \mathbb{N}$ let D^s_n denote the divided-power envelope of the injection $i^s_n : \mathfrak{U}^s_n \rightarrowtail \mathfrak{Z}^s_n$. Then for each $r \in \mathbb{Z}$, we have filtered crystalline cohomology complexes

$$\mathrm{Fil}^r\mathscr{C}^\bullet_{\mathfrak{U}^s_n,\mathfrak{Z}^s_n}(\mathcal{F}) : \mathrm{Fil}^r\mathcal{F}_{D^s_n} \xrightarrow{\partial} \mathrm{Fil}^{r-1}\mathcal{F}_{D^s_n} \otimes_{\mathcal{O}_{\mathfrak{Z}^s_n}} \Omega^1_{\mathfrak{Z}^s_n/\Sigma_n} \xrightarrow{\partial} \mathrm{Fil}^{r-2}\mathcal{F}_{D^s_n} \otimes_{\mathcal{O}_{\mathfrak{Z}^s_n}} \Omega^2_{\mathfrak{Z}^s_n/\Sigma_n} \xrightarrow{\partial} \cdots.$$

Define the mod p^n syntomic complex on $\mathfrak{U}^\bullet_{\kappa,\text{ét}}$ with coefficients in $\mathcal{F}$ as

$$\mathscr{S}_n(\mathcal{F},r)_{(\mathfrak{U}^s,\mathfrak{Z}^s)} := [\mathrm{Fil}^r\mathscr{C}^\bullet_{\mathfrak{U}^s_n,\mathfrak{Z}^s_n}(\mathcal{F})_n \xrightarrow{p^r-p^\bullet\varphi} \mathscr{C}^\bullet_{\mathfrak{U}^s_n,\mathfrak{Z}^s_n}(\mathcal{F})_n],$$

where φ denotes the morphism induced by $F_{\mathfrak{Z}_n}$. Finally, we take $\mathrm{R}\Gamma_{\mathrm{syn}}((\mathfrak{U}^\bullet,\mathfrak{Z}^\bullet,F_{\mathfrak{Z}^\bullet}),\mathcal{F},r)_n$ to be the right derived functor of the global sections functor for the complex of sheaves $\mathscr{S}_n$.

The complex $\mathrm{R}\Gamma_{\mathrm{syn}}((\mathfrak{U}^\bullet,\mathfrak{Z}^\bullet,F_{\mathfrak{Z}^\bullet}),\mathcal{F},r)_n$ is very precisely related to the complex $\mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{X},\mathcal{F},r)_n$ from Definition 7.13. Let $\mathrm{HC}(\mathfrak{X})$ denote the category of triples $(\mathfrak{U}^\bullet,\mathfrak{Z}^\bullet,F_{\mathfrak{Z}^\bullet})$ where $\mathfrak{U}^\bullet \to \mathfrak{X}$ is a hypercovering and $\mathfrak{Z}^\bullet$ and $F_{\mathfrak{Z}^\bullet}$ are defined as above. A morphism $(\mathfrak{U}^\bullet,\mathfrak{Z}^\bullet,F_{\mathfrak{Z}^\bullet}) \to (\mathfrak{V}^\bullet,\mathfrak{Y}^\bullet,F_{\mathfrak{Y}^\bullet})$ is given by a pair of morphisms $(f : \mathfrak{U}^\bullet \to \mathfrak{V}^\bullet, \tilde{f} : \mathfrak{Z}^\bullet \to \mathfrak{Y}^\bullet)$ such that for all $s \in \mathbb{N}$, the diagram

$$\begin{tikzcd} \mathfrak{Z}^s \arrow[r, "\tilde{f}^s"] & \mathfrak{Y}^s \\ \mathfrak{U}^s \arrow[u] \arrow[r, "f^s"] & \mathfrak{V}^s \arrow[u] \end{tikzcd}$$

commutes and we have $F_{\mathfrak{Y}^s_n} \circ \tilde{f}^s = \tilde{f}^s \circ F_{\mathfrak{Z}^s_n}$ for all $n \in \mathbb{N}$. Consider the category of hypercoverings $\mathrm{HC}(\mathfrak{X},\mathcal{F})$ as our index category for the diagram

$$\mathrm{R}\Gamma_{\mathrm{syn}}(-,\mathcal{F},r)_n : \mathrm{HC}(\mathfrak{X},\mathcal{F}) \longrightarrow Ab,$$

where *Ab* is the category of abelian groups. This diagram is directed and we obtain a quasi-isomorphism (see [Sta22, Theorem 01H0])

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{X},\mathcal{F},r)_n \xrightarrow{\sim} \mathrm{colim}_{\mathrm{HC}(\mathfrak{X},\mathcal{F})}\mathrm{R}\Gamma_{\mathrm{syn}}((\mathfrak{U}^\bullet,\mathfrak{Z}^\bullet,F^\bullet_{\mathfrak{Z}}),\mathcal{F},r)_n.$$

7.2.2. Via syntomic site. One can also define syntomic cohomology using the syntomic site. Let $\mathfrak{X}$ be a smooth (p-adic formal) scheme over O_K, let $\Sigma = \operatorname{Spec} O_F$ and $\mathcal{F}$ an object of $\mathrm{CR}(\mathfrak{X}/\Sigma, \mathrm{Fil}, \varphi)$. We will denote by $\mathfrak{X}_{\mathrm{syn}}$ the small syntomic site of $\mathfrak{X}$, i.e. the category of syntomic (p-adic formal) $\mathfrak{X}$-schemes such that morphisms between objects is syntomic as well. We define

$$\mathcal{F}_n(\mathfrak{X}) := H^0_{\mathrm{cris}}(\mathfrak{X}, \mathcal{F})_n,$$
$$\mathrm{Fil}^r \mathcal{F}_n(\mathfrak{X}) := H^0_{\mathrm{cris}}(\mathfrak{X}, \mathrm{Fil}^r \mathcal{F})_n.$$

The presheaves $\mathcal{F}_n$ and $\mathrm{Fil}^r \mathcal{F}_n$ are sheaves on $\mathfrak{X}_{n,\mathrm{syn}}$ (see [BBM82, §1.1.18, §1.1.19]), flat as $\mathbb{Z}/p^n$-module and $\mathrm{Fil}^r \mathcal{F}_{n+1} \otimes_{\mathbb{Z}/p^{n+1}} \mathbb{Z}/p^n \simeq \mathrm{Fil}^r \mathcal{F}_n$. Moreover, we have a canonical isomorphism (see [Bau92, Corollary 1.15, Proposition 1.17])

$$\mathrm{R}\Gamma(\mathfrak{X}_{n,\mathrm{syn}}, \mathrm{Fil}^r \mathcal{F}_n) \simeq \mathrm{R}\Gamma_{\mathrm{cris}}(\mathfrak{X}, \mathrm{Fil}^r \mathcal{F})_n,$$

compatible with Frobenius.

Definition 7.16. Define the syntomic Tate twist on $\mathfrak{X}_{n,\mathrm{syn}}$ with coeffcients in $\mathcal{F}$ as

$$\mathscr{S}_{n,\mathrm{syn}}(\mathcal{F}, r)_{\mathfrak{X}} := \left[\mathrm{Fil}^r \mathcal{F}_n \xrightarrow{p^r - \varphi} \mathcal{F}_n\right].$$

Similar to above, we can define syntomic complex with coefficients in $\mathcal{F}$ and by abuse of notations denote them as $\mathscr{S}_{n,\mathrm{syn}}(\mathcal{F}, r)_{\mathfrak{X}}$ on $\mathfrak{X}_{m,\mathrm{syn}}$ for all $m \geq n$. Moreover, we have the natural map $i : \mathfrak{X}_{m,\mathrm{syn}} \to \mathfrak{X}_{\mathrm{syn}}$, and i_* is exact. So we get that $\mathrm{R}\Gamma(\mathfrak{X}_{m,\mathrm{syn}}, \mathscr{S}_{n,\mathrm{syn}}(\mathcal{F}, r)_{\mathfrak{X}}) = \mathrm{R}\Gamma(\mathfrak{X}_{\mathrm{syn}}, i_* \mathscr{S}_{n,\mathrm{syn}}(\mathcal{F}, r)_{\mathfrak{X}})$. Furthermore, we have the natural projection $\varepsilon : \mathfrak{X}_{m,\mathrm{syn}} \to \mathfrak{X}_{m,\text{ét}}$ and we set

$$\mathscr{S}'_{n,\text{ét}}(\mathcal{F}, r)_{\mathfrak{X}} = \mathrm{R}\varepsilon_* \mathscr{S}_{n,\mathrm{syn}}(\mathcal{F}, r)_{\mathfrak{X}}.$$

Proposition 7.17. *Let $\mathfrak{X}$ be a smooth (p-adic formal) scheme over O_K and $\mathcal{F}$ an object of $\mathrm{CR}(\mathfrak{X}/\Sigma, \mathrm{Fil}, \varphi)$, i.e. locally finite free filtered crystal equipped with a Frobenius structure. Then we have canonical isomorphism of complexes $\mathscr{S}_{n,\text{ét}}(\mathcal{F}, r)_{\mathfrak{X}} \simeq \mathscr{S}'_{n,\text{ét}}(\mathcal{F}, r)_{\mathfrak{X}}$.*

Remark 7.18. In the rest of this the text we will denote the mod p^n (resp. completed) syntomic complex with coefficients in $\mathcal{F}$ as $\mathscr{S}_n(\mathcal{F}, r)_{\mathfrak{X}}$ (resp. $\mathscr{S}(\mathcal{F}, r)_{\mathfrak{X}}$).

8. p-ADIC NEARBY CYCLES

We finally come to global applications of computations done in the previous sections.

8.1. Fontaine-Laffaille modules. In this section we will consider global Fontaine-Laffaille modules introduced by Faltings in [Fal89, §II]. These objects will be obtained by gluing together local data which we recall below from §3.3. Let R denote the p-adic completion of an étale algebra over $O_F[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ for some $d \in \mathbb{N}$ and such that R has non-empty geometrically integral special fiber (see §2.1 for details). Let $s \in \mathbb{N}$ such that $s \leq p-2$.

Definition 8.1. Define the category of *free relative Fontaine-Laffaille* modules of level $[0, s]$, denoted by $\mathrm{MF}_{[0,s],\,\mathrm{free}}(R, \Phi, \partial)$, as follows:
An object with weights in the interval $[0, s]$ is a quadruple $(M, \mathrm{Fil}^\bullet M, \partial, \Phi)$ such that,

(i) M is a free R-module of finite rank.

(ii) M is equipped with a decreasing filtration $\{\mathrm{Fil}^k M\}_{k \in \mathbb{Z}}$ by finite R-submodules with $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{s+1} M = 0$ such that $\mathrm{gr}^k_{\mathrm{Fil}} M$ is a finite free R-module for every $k \in \mathbb{Z}$.

(iii) The connection $\partial : M \to M \otimes_R \Omega^1_R$ is quasi-nilpotent and integrable, and satisfies Griffiths transversality with respect to the filtration, i.e. $\partial(\mathrm{Fil}^k M) \subset \mathrm{Fil}^{k-1} M \otimes_R \Omega^1_R$ for $k \in \mathbb{Z}$.

(iv) Let $(\varphi^*(M), \varphi^*(\partial))$ denote the pullback of (M, ∂) by $\varphi : R \to R$, and equip it with a decreasing filtration $\mathrm{Fil}^k_p(\varphi^*(M)) = \sum_{i \in \mathbb{N}} p^{[i]} \varphi^*(\mathrm{Fil}^{k-i} M)$ for $k \in \mathbb{Z}$. We suppose that there is an R-linear morphism $\Phi : \varphi^*(M) \to M$ such that Φ is compatible with connections, $\Phi(\mathrm{Fil}^k_p(\varphi^*(M))) \subset p^k M$ for $0 \leq k \leq s$, and $\sum_{k=0}^{s} p^{-k} \Phi(\mathrm{Fil}^k_p(\varphi^*(M))) = M$. We denote the composition $M \to \varphi^*(M) \xrightarrow{\Phi} M$ by φ.

A morphism between two objects of the category $\mathrm{MF}_{[0,s],\,\mathrm{free}}(R, \Phi, \partial)$ is a continuous R-linear map compatible with the homomorphism Φ and the connection ∂ on each side.

Remark 8.2. Note that we fixed a lifting φ on R of the absolute Frobenius on R/p. However, for a different lift of Frobenius φ on R the categories $\mathrm{MF}_{[0,s],\,\mathrm{free}}(R, \Phi, \partial)$ and $\mathrm{MF}_{[0,s],\,\mathrm{free}}(R, \Phi', \partial)$ are naturally equivalent satisfying a cocycle condition (see [Fal89, Theorem 2.3] and [Tsu20, Remark 33]). In particular, there is a well-defined isomorphism $\alpha_{\varphi,\varphi'} : \varphi^* M \xrightarrow{\sim} \varphi'^* M$ compatible with connection on each side.

Let us now globalize the construction above. Let $\mathfrak{X}$ be a smooth (p-adic formal) scheme defined over O_F. We consider a covering $\{\mathfrak{U}_i\}_{i \in I}$ of $\mathfrak{X}$ with $\mathfrak{U}_i = \mathrm{Spec}\, A_i$ (resp. $\mathfrak{U}_i = \mathrm{Spf}\, A_i$) such that the p-adic completions $\widehat{A}_i$ satisfy Assumption 2.1 for each $i \in I$. We fix lifts of Frobenius modulo p as $\varphi_i : \widehat{A}_i \to \widehat{A}_i$.

Definition 8.3. Define $\mathrm{MF}_{[0,s],\,\mathrm{free}}(\mathfrak{X}, \Phi, \partial)$ as the category of finite locally free filtered $\mathcal{O}_{\mathfrak{X}}$-modules $\mathcal{M}$ equipped with a p-adically quasi-nilpotent integrable connection satisfying Griffiths transverality with respect to the filtration and such that there exists a covering $\{\mathfrak{U}_i\}_{i \in I}$ of $\mathfrak{X}$ as above with $\mathcal{M}_{\mathfrak{U}_i} \in \mathrm{MF}_{[0,s],\,\mathrm{free}}(\widehat{A}_i, \Phi, \partial)$ for all $i \in I$ and on $\mathfrak{U}_{ij}$ the two structures glue well under $\alpha_{\varphi_i,\varphi_j}$.

Remark 8.4. Let $\Sigma = \mathrm{Spec}\, O_F$ (resp. $\Sigma = \mathrm{Spf}\, O_F$), then the category $\mathrm{MF}_{[0,s],\,\mathrm{free}}(\mathfrak{X}, \Phi, \partial)$ is a full subcategory of $\mathrm{MIC}(\mathfrak{X}/\Sigma, \mathrm{Fil}, \varphi)$ described in Definition 7.10.

Remark 8.5. By [Fal89, Theorem 2.6*], the functor T_{cris} associates to any object of $\mathrm{MF}_{[0,s],\,\mathrm{free}}(\mathfrak{X}, \Phi, \partial)$ a compatible system of étale sheaves on $\mathrm{Sp}(\widehat{A}_i[\frac{1}{p}])$. These can be expressed in terms of certain finite étale coverings of $\mathfrak{X}$. Extending these by normalization to $\mathrm{Spec}\,(\widehat{A}_i)$, the results glue to give a finite covering of the formal O_F-scheme $\mathfrak{X}'$ associated to $\mathfrak{X}$. For $\mathfrak{X}$ a formal scheme $\mathfrak{X} = \mathfrak{X}'$ and this gives us an étale sheaf on the generic fiber X of $\mathfrak{X}$, or if $\mathfrak{X}$ is a scheme this covering is algebraic and we obtain an étale sheaf on $X = \mathfrak{X} \otimes_{O_F} F$. The étale $\mathbb{Z}_p$-local system on the generic fiber associated to $\mathcal{M}$ will be denoted as $\mathbb{L}$.

Notation. For $\mathfrak{X}$ a (p-adic formal) scheme over O_F, we will denote its (rigid) generic fiber as X and its special fiber as $\mathfrak{X}_\kappa$.

8.2. Fontaine-Messing period map. Let $\Sigma = \operatorname{Spec} O_F$ (resp. $\Sigma = \operatorname{Spf} O_F$) and K a finite extension of F such that $K \cap F^{\mathrm{ur}} = F$. In this section, we will recall the classical definition of Fontaine-Messing period map for (p-adic formal) schemes.

8.2.1. The case of schemes. Let $\mathfrak{X}$ be a smooth scheme over O_F with $i : \mathfrak{X}_{\kappa,\text{ét}} \to \mathfrak{X}_{\text{ét}}$ the map of sites from its special fiber and $j : X_{\text{ét}} \to \mathfrak{X}_{\text{ét}}$ the map of sites from its generic fiber. Let $\mathcal{M} \in \mathrm{MF}_{[0,s],\,\mathrm{free}}(\mathfrak{X}, \Phi, \partial)$ and $\mathbb{L}$ the associated $\mathbb{Z}_p$-local system on the generic fiber of $\mathfrak{X}$. In this section we will construct the Fontaine-Messing period map from syntomic complex with coefficients in $\mathcal{M}$ to the complex of p-adic nearby cycles with coefficients in $\mathbb{L}$.

From [Abh21, §5.3] and (7.2), we know that the $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ corresponds to a finite locally free filtered crystal in $\mathrm{CR}(\mathfrak{X}/\Sigma, \mathrm{Fil}, \varphi)$ equipped with Frobenius structure and (by abuse of notations) we will denote this crystal again by $\mathcal{M}$. Now recall from §7.2 that we have mod p^n syntomic complex with coeffecients in $\mathcal{M}$ denoted as $\mathscr{S}_n(\mathcal{M}, r)_{\mathfrak{X}}$.

We will follow the construction in [Tsu96, §5] and [Tsu99, §3.1]. Let us first describe the local version of Fontaine-Messing period map, i.e. let $\mathfrak{X}$ be an affine smooth scheme over O_F. Let $\mathfrak{Y} = \mathfrak{X} \otimes_{O_F} O_K$ and choose an embedding $\mathfrak{Y} \rightarrowtail \mathfrak{Z}$ such that $\mathfrak{Z}$ is an affine smooth scheme over O_F. Then $\mathfrak{Y}$ can be covered by affine étale $\mathfrak{Y}$-schemes $\mathfrak{U} = \operatorname{Spec} A$ with $A = O_K \otimes_{O_F} B$ and B an étale algebra over $O_F[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ such that its p-adic completion $\widehat{B}$ satisfies Assumption 2.1.

Remark 8.6. (i) For A as above, let A^h denote the p-adic henselization of A and $G_{A^h} = \mathrm{Gal}(\overline{A^h}[\frac{1}{p}]/A^h[\frac{1}{p}])$ where $\overline{A^h}$ denotes the union of finite A^h-subalgebras $S \subset \mathrm{Fr}\, \overline{A^h}$, such that $S[\frac{1}{p}]$ is étale over $A^h[\frac{1}{p}]$. Then, by Elkik's approximation theorem [Elk73, Corollary p. 579], we have a natural isomorphism of Galois groups $G_{A^h} \simeq G_{\widehat{A}}$. Therefore, we can regard discrete $G_{\widehat{A}}$-modules as locally constant sheaves on the étale site of the generic fiber $U^h = \mathfrak{U}^h \otimes_{O_K} K$, where $\mathfrak{U}^h = \operatorname{Spec} A^h$.

(ii) We can consider the henselian version of the fundamental exact sequence in (2.2) and in Remark 6.22 which can be obtained by replacing $\overline{\widehat{A}}$ by $\overline{A^h}$ and $G_{\widehat{A}}$ with G_{A^h}. In particular, similar to (6.21) one obtains a syntomic complex $\mathrm{Syn}(\overline{A^h}, \mathcal{M}_{\mathfrak{U}}, r)$ of discrete G_{A^h}-modules. We will denote this complex by $\overline{\mathscr{S}}_n(\mathcal{M}, r)_{\mathfrak{U}}$.

(iii) By (i) the complex of G_{A^h}-modules $\overline{\mathscr{S}}_n(\mathcal{M}, r)_{\mathfrak{U}}$ can be regarded as a complex of locally constant sheaves on $U^h_{\text{ét}}$ and we obtain a morphism

$$\Gamma(\mathfrak{U}, i_*\mathscr{S}_n(\mathcal{M}, r)_{\mathfrak{Y}}) \longrightarrow \Gamma(U^h, \overline{\mathscr{S}}_n(\mathcal{M}, r)_{\mathfrak{U}}),$$

and a natural map

$$\mathrm{R}\Gamma(G_{\widehat{A}}, T_{\mathrm{cris}}(\mathcal{M}_{\mathfrak{U}})/p^n(r)) \longrightarrow \mathrm{R}\Gamma_{\text{ét}}(U^h, \mathbb{L}/p^n(r)_U). \tag{8.1}$$

Now we take a sufficiently large algebraically closed field Ω of characteristic 0. Let C^* denote the Godement resolution with respect to all Ω-rational points. Then we have the following morphisms of complexes

$$\begin{aligned}
\Gamma(\mathfrak{U}, i_*\mathscr{S}_n(\mathcal{M},r)_{\mathfrak{Y}}) &\longrightarrow \Gamma(U^h, \overline{\mathscr{S}}_n(\mathcal{M},r)_{\mathfrak{U}}) \\
&\longrightarrow \Gamma(U^h, \mathrm{Tot}^{\oplus} C^*(\overline{\mathscr{S}}_n(\mathcal{M},r)_{\mathfrak{U}})) \\
&\xrightarrow{(1)} \Gamma(U^h, C^*(\mathbb{L}/p^n(r)'_{U^h})) \\
&\xleftarrow{(2)} \Gamma(U^h, C^*(\mathbb{L}/p^n(r)'_{Y})|U^h) \\
&\longrightarrow \Gamma(\mathfrak{U}^h, i_{\mathfrak{U}^h *} i^*_{\mathfrak{U}^h} j_{\mathfrak{U}^h *}(C^*(\mathbb{L}/p^n(r)'_Y)|U^h)) \\
&\xleftarrow{(3)} \Gamma(\mathfrak{U}, i_{\mathfrak{U}*} i^*_{\mathfrak{U}} j_{\mathfrak{U}*}(C^*(\mathbb{L}/p^n(r)'_Y)|U)) \\
&\xleftarrow{\simeq} \Gamma(\mathfrak{U}, i_* i^* j_* C^*(\mathbb{L}/p^n(r)'_Y)).
\end{aligned} \tag{8.2}$$

Here we set $U^h = \mathfrak{U}^h \otimes_{O_K} K$ where $\mathfrak{U}^h$ denotes the p-adic henselization of $\mathfrak{U}$. Moreover, the morphisms $i_{\mathfrak{U}}, j_{\mathfrak{U}}$ and $i_{\mathfrak{U}^h}, j_{\mathfrak{U}^h}$ are defined by the commutative diagram

$$\begin{tikzcd}
U^h \arrow[r, "j_{\mathfrak{U}^h}"] \arrow[d] & \mathfrak{U}^h \arrow[d] & \mathfrak{U}^h \otimes_{O_F} \kappa \arrow[l, "i_{\mathfrak{U}^h}"'] \arrow[d, "\simeq"] \\
U \arrow[r, "j_{\mathfrak{U}}"'] & \mathfrak{U} & \mathfrak{U} \otimes_{O_F} \kappa. \arrow[l, "i_{\mathfrak{U}}"]
\end{tikzcd}$$

In (8.2), "$\mathrm{Tot}^{\oplus}$" denotes the associated simple complex of a double complex, $\mathbb{L}(r)' = \frac{1}{p^{a(r)}}\mathbb{L}$ where $a(r)$ is determined by the equation $r = (p-1)a(r) + b(r)$ with $0 \le b(r) < p-1$.

Now let us describe the non-obvious (labeled) morphisms. The morphism (1) is determined by the Poincaré Lemma 2.37, fundamental exact sequence (see (2.2), Remarks 8.6 (ii), 6.22 and (6.20)) in combination with (8.1). Furthermore, since $\mathfrak{U}^h$ is a filtered inverse limit of affine étale $\mathfrak{U}$-schemes, the morphism (2) is a quasi-isomorphism. Moreover, since $\mathcal{O}_{\mathfrak{U},\overline{x}} \simeq \mathcal{O}_{\mathfrak{U}^h,\overline{x}}$ for any geometric point $\overline{x}$ on the special fibre, the morphism (3) is an isomorphism. Finally, the functoriality with respect to $\mathfrak{U}$, of the complex $\overline{\mathscr{S}}_n(\mathcal{M},r)_{\mathfrak{U}}$ and the morphisms of complexes discussed above follows similar to [Tsu99, p. 321] (also see [Tsu99, §1.4]).

Next, let $F_n(r)_{\mathfrak{Y},3}$ denote the complex of étale sheaves on $\mathfrak{Y}$ associated to the complex of presheaves

$$\mathfrak{U} \mapsto \begin{cases} \Gamma(U^h, \mathrm{Tot}^{\oplus} C^*(\overline{\mathscr{S}}_n(\mathcal{M},r)_{\mathfrak{U}})) & \text{if } \mathfrak{U} \otimes_{O_K} \kappa \neq \emptyset, \\ 0 & \text{if } \mathfrak{U} \otimes_{O_K} \kappa = \emptyset, \end{cases}$$

where $\mathfrak{U} = \operatorname{Spec} A$ is an affine étale $\mathfrak{Y}$-scheme such that $A = O_K \otimes_{O_F} B$ and B is the p-adic comlpetion $\widehat{A}$ satisfies Assumption 2.1 or $\mathfrak{U}$ is an étale X-scheme. Similarly, define $G_n(r)_{\mathfrak{Y},3}$ to be the complex of étale sheaves on $\mathfrak{Y}$ by modifying the complex of presheaves above as

$$\mathfrak{U} \mapsto \begin{cases} \Gamma(U^h, C^*(\mathbb{L}/p^n(r)'_U)|X^h) & \text{if } \mathfrak{U} \otimes_{O_K} \kappa \neq \emptyset, \\ 0 & \text{if } \mathfrak{U} \otimes_{O_K} \kappa = \emptyset. \end{cases}$$

Then we have a sequence of morphisms of complexes on $\mathfrak{Y}_{\text{ét}}$

$$i_*\mathscr{S}_n(\mathcal{M},r)_{\mathfrak{Y}} \longrightarrow F_n(r)_{\mathfrak{Y},3} \xrightarrow{(1)} G_n(r)_{\mathfrak{Y},3} \longrightarrow i_* i^* j_* C^*(\mathbb{L}/p^n(r)'_Y), \tag{8.3}$$

in which (1) is determined by the Poincaré Lemma 2.37, fundamental exact sequence (see (2.2), Remarks 8.6 (ii) and 6.22 and (6.20)) in combination with (8.1). Thus by composing the maps we obtain a natural morphism

$$\mathscr{S}_n(\mathcal{M},r)_{\mathfrak{Y}} \longrightarrow i^* \mathrm{R} j_* \mathbb{L}/p^n(r)'_Y,$$

in $D^+(\mathfrak{Y}_{\kappa,\text{ét}}, \mathbb{Z}/p^n)$.

Finally, we will globalize this construction. Let $\mathfrak{X}$ be a proper and smooth scheme over O_F and let $\mathfrak{Y} = \mathfrak{X} \otimes_{O_F} O_K$. Take $\mathfrak{U}^\bullet \to \mathfrak{Y}$, $\mathfrak{U}^\bullet \rightarrowtail \mathfrak{Z}^\bullet$ and $F_{\mathfrak{Z}^\bullet} = \{F_{\mathfrak{Z}_n^\bullet} : \mathfrak{Z}_n^\bullet \to \mathfrak{Z}_n^\bullet\}$ as in Remark 7.15. Furthermore, assume that $\mathfrak{U}^s$ and $\mathfrak{Z}^s$ are affine schemes for each $s \geq 0$. Let $F_n(r)_{\mathfrak{U}^\bullet,\mathfrak{Z}^\bullet}$, $G_n(r)_{\mathfrak{U}^\bullet,\mathfrak{Z}^\bullet}$, and $H_n(r)_{\mathfrak{U}^\bullet,\mathfrak{Z}^\bullet}$ denote the complexes of sheaves on $(\mathfrak{U}^\bullet)^\sim_{\text{ét}}$ which give complexes $F_n(r)_{\mathfrak{U}^s,\mathfrak{Z}^s}$, $G_n(r)_{\mathfrak{U}^s,\mathfrak{Z}^s}$, and $i^s_* i^{s*} j^s_* C^*(\mathbb{Z}/p^n\mathbb{Z}(r))'_{\mathfrak{U}^s_K}$ respectively on $\mathfrak{U}^s_{\text{ét}}$ for each $s \geq 0$. Here i^s (resp. j^s) denotes the morphism of sites i from étale site of the special fiber (resp. j from étale site of the generic fiber) to $\mathfrak{U}^s_{\text{ét}}$. Then from (8.3) we obtain morphisms of complexes on $(\mathfrak{U}^\bullet)^\sim_{\text{ét}}$,

$$
\begin{aligned}
i^\bullet_* \mathscr{S}_n(\mathcal{M}, r)_{\mathfrak{U}^\bullet} &\longrightarrow F_n(r)_{\mathfrak{U}^\bullet,\mathfrak{Z}^\bullet} \\
&\xrightarrow{(1)} G_n(r)_{\mathfrak{U}^\bullet,\mathfrak{Z}^\bullet} \\
&\longrightarrow H_n(r)_{\mathfrak{U}^\bullet,\mathfrak{Z}^\bullet} \\
&\xleftarrow{(2)} \theta^* i_* i^* j_* C^*(\mathbb{L}/p^n(r)'_Y),
\end{aligned}
$$

where $\theta : (\mathfrak{U}^\bullet)^\sim_{\text{ét}} \to \mathfrak{Y}^\sim_{\text{ét}}$ denotes the canonical morphism of toposes and (1) is determined by the Poincaré Lemma 2.37, fundamental exact sequence (see (2.2), Remarks 8.6 (ii) and 6.22 and (6.20)) in combination with (8.1) and (2) is a quasi-isomorphism. Taking $\mathrm{R}\theta_*$ and taking the colimit over the category of hypercoverings $\mathrm{HC}(\mathfrak{Y}, \mathcal{F})$ (see Remark 7.15) we obtain a morphism

$$
i_* \mathscr{S}_n(\mathcal{M}, r)_{\mathfrak{Y}} \longrightarrow \mathrm{R}\theta_* \theta^* i_* i^* \mathrm{R}j_* \mathbb{L}/p^n(r)'_Y \simeq i_* i^* \mathrm{R}j_* \mathbb{L}/p^n(r)'_Y,
$$

in $D^+(\mathfrak{Y}_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$ and hence a natural map

$$
\alpha^{\mathrm{FM}}_{r,n,\mathfrak{Y}} : \mathscr{S}_n(\mathcal{M}, r)_{\mathfrak{Y}} \longrightarrow i^* \mathrm{R}j_* \mathbb{L}/p^n(r)'_Y. \tag{8.4}
$$

8.2.2. The case of formal schemes. The construction of Fontaine-Messing period map in the case of formal schemes largely follows the same procedure as in the case of schemes with certain key differences which we will point out below. Let $\mathfrak{X}$ be a smooth p-adic formal scheme over O_F and $\mathfrak{Y} = \mathfrak{X} \otimes_{O_F} O_K$. In this case, an affine étale formal scheme $\mathfrak{U} \to \mathfrak{Y}$ can be covered by affine formal schemes $\mathfrak{U} = \operatorname{Spf} S$ with $S = O_K \otimes_{O_F} R$ and R satisfying Assumption 2.1. Next, for such local models, we need to consider the completed version of the Fontaine-Messing period map described in (8.2). Finally, to obtain the global version, one proceeds in exactly the same manner as in the case of schemes (with hypercovering $(\mathfrak{U}^\bullet, \mathfrak{Z}^\bullet, F_{\mathfrak{Z}^\bullet})$ where each $\mathfrak{U}^s$ is of the form described above).

Remark 8.7. We note that in the local cyclotomic case, i.e. $K = F(\zeta_{p^m})$ for $m \in \mathbb{N}$, the map described in (8.2) coincides with composition of the map $\tilde{\alpha}^{\mathrm{FM}}_{r,n,S}$ described in §6.7 with the quasi-isomorphism $C(G_S, T/p^n(r)') \xrightarrow{\sim} \mathrm{R}\Gamma_{\text{ét}}(U, \mathbb{L}/p^n(r)')$ obtained by applying $K(\pi, 1)$-Lemma for p-coefficients (see [Sch13, Theorem 4.9] and [CN17, §5.4.1]).

8.3. A global result.

The aim of this section is to prove the following result:

Theorem 8.8. *Let $\mathfrak{X}$ be a smooth (p-adic formal) scheme over O_F, $\mathcal{M} \in \mathrm{MF}_{[0,s],\,\mathrm{free}}(\mathfrak{X}, \Phi, \partial)$ a Fontaine-Laffaille module of level $[0, s]$ for $0 \leq s \leq p-2$ and let $\mathbb{L}$ be the associated $\mathbb{Z}_p$-local system on the (rigid) generic fiber X of $\mathfrak{X}$. Then for $0 \leq k \leq r-s-1$ the Fontaine-Messing period map*

$$
\alpha^{\mathrm{FM}}_{r,n,\mathfrak{X}} : \mathcal{H}^k(\mathscr{S}_n(\mathcal{M}, r)_{\mathfrak{X}}) \longrightarrow i^* \mathrm{R}^k j_* \mathbb{L}/p^n(r)'_X, \tag{8.5}
$$

is a p^N-isomorphism for an integer $N = N(p, r, s)$, which depends on p, r and s but not on $\mathfrak{X}$ or n.

Proof for schemes. By the definition of Fontaine-Messing period map in §8.2, we see that it is enough to show the p-power quasi-isomorphism locally (provided the power of p does not depend on the local model). Let A be an O_F-algebra such that its p-adic completion $\widehat{A}$ satisfies Assumption 2.1, $\mathfrak{U} = \operatorname{Spec} A$ and $M := \mathcal{M}_{\mathfrak{U}}$. We have

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{U}, \mathcal{M}_{\mathfrak{U}}, r)_n = \mathrm{Syn}(\widehat{A}, M, r)_n, \qquad \mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{U}, \mathcal{M}_{\mathfrak{U}}, r) = \mathrm{Syn}(\widehat{A}, M, r).$$

The Fontaine-Messing period map

$$\alpha^{\mathrm{FM}}_{r,n,\mathfrak{U}} : \mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{U}, \mathcal{M}_{\mathfrak{U}}, r)_n \longrightarrow \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(U^h, \mathbb{L}/p^n(r)'_{U^h}),$$

is the same as the composition of the henselian version of the map $\tilde{\alpha}^{\mathrm{FM}}_{r,n}$ (see Remarks 6.23 and 8.7 for the completed version) with the natural map $C(G_{A^h}, T/p^n(r)') \to \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(U^h, \mathbb{L}/p^n(r)'_{U^h})$ as in (8.1). The henselian version of the map $\tilde{\alpha}^{\mathrm{FM}}_{r,n}$ is obtained by replacing $\overline{\widehat{A}}$ by $\overline{A^h}$ and $G_{\widehat{A}}$ with G_{A^h}. We set $\mathrm{Syn}(A, M, r) := \mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{U}, \mathcal{M}_{\mathfrak{U}}, r)$.

Let $k \leq r - s - 1$, then we need to show that the map

$$\alpha^{\mathrm{FM}}_{r,n,A} : H^k(\mathrm{Syn}(A, M, r)_n) \xrightarrow{\tilde{\alpha}^{\mathrm{FM}}_{r,n}} H^k(G_{A^h}, T/p^n(r)') \longrightarrow H^k(U^h_{\mathrm{\acute{e}t}}, \mathbb{L}/p^n(r)'_{U^h}), \tag{8.6}$$

is an isomorphism (up to some power of p). To show (8.6), we will pass to the p-adic completion of A. Let $\mathcal{U} := \mathrm{Sp}(\widehat{A}[\frac{1}{p}])$ and consider the following commutative diagram:

$$\begin{tikzcd}
H^k(\mathrm{Syn}(A, M, r)_n) \arrow[r, "\tilde{\alpha}^{\mathrm{FM}}_{r,n,A}"] \arrow[d, equal] & H^k(G_{A^h}, T/p^n(r)') \arrow[r] \arrow[d, "\wr"] & H^k(U^h_{\mathrm{\acute{e}t}}, \mathbb{L}/p^n(r)'_{U^h}) \arrow[d, "\wr"] \\
H^k(\mathrm{Syn}(\widehat{A}, M, r)_n) \arrow[r, "\tilde{\alpha}^{\mathrm{FM}}_{r,n,\widehat{A}}", "\sim"'] & H^k(G_{\widehat{A}}, T/p^n(r)') \arrow[r, "\sim"] & H^k(\mathcal{U}_{\mathrm{\acute{e}t}}, \mathbb{L}/p^n(r)'_{\mathcal{U}}).
\end{tikzcd}$$

The middle vertical arrow is an isomorphism because the two Galois groups are equal by Elkik's approximation theorem [Elk73, Corollary p. 579] (see Remark 8.6 (i)). The right vertical arrow is an isomorphism due to Gabber [Gab94, Theorem 1]. The left horizontal arrow in the bottom row is a p^N-isomorphism for $N = N(p, r, s) \in \mathbb{N}$ as shown in the case of formal schemes below (for $R = \widehat{A}$). The right horizontal arrow in the bottom row is an isomorphism by a $K(\pi, 1)$-Lemma due to Scholze [Sch13, Theorem 4.9]. ■

Proof for formal schemes. By the definition of Fontaine-Messing period map in §8.2, we see that it is enough to show the p-power quasi-isomorphism locally (provided the power of p does not depend on the local model). Let R be an O_F-algebra satisfying Assumption 2.1, $\mathfrak{U} = \operatorname{Spf} R$ and $M := \mathcal{M}_{\mathfrak{U}}$. We have that the Fontaine-Messing period map

$$\alpha^{\mathrm{FM}}_{r,n,R} : H^k(\mathrm{Syn}(R, M, r)_n) \longrightarrow H^k(G_R, T/p^n(r)') \xrightarrow{\sim} H^k(U_{\mathrm{\acute{e}t}}, \mathbb{L}/p^n(r)'_U),$$

is the same as the composition of the map $\tilde{\alpha}^{\mathrm{FM}}_{r,n,R}$ (see Remarks 6.23 and 8.7) with the natural isomorphism $H^k(G_R, T/p^n(r)') \xrightarrow{\sim} H^k(U_{\mathrm{\acute{e}t}}, \mathbb{L}/p^n(r)'_U)$ by a $K(\pi, 1)$-Lemma due to Scholze [Sch13, Theorem 4.9].

Finally, to show the isomorphism in degrees $0 \leq k \leq r - s - 1$ we use Corollary 6.25 with Example 5.5 (iii) for Fontaine-Laffaille modules. To compute $N = N(p, r, s) \in \mathbb{N}$, we combine the constants obtained in the proof of Theorem 5.8, Corollary 6.25 (i.e. Lemma 6.26 for $e = p(p-1)$) and Example 5.5 (iii) and get that $N = 40r + 14s + 3p(p-1) + 4$. In particular, N does not depend on n or the local model $\mathfrak{U}$. This allows us to conclude the theorem. ■

References

[Abh21] Abhinandan. "Crystalline representations and Wach modules in the relative case". In: arXiv:2103.17097 (Mar. 2021). To appear in Annales de l'Institut Fourier.

[And06] Fabrizio Andreatta. "Generalized ring of norms and generalized (ϕ, Γ)-modules". In: *Ann. Sci. École Norm. Sup. (4)* 39.4 (2006), pp. 599–647. ISSN: 0012-9593.

[AB08] Fabrizio Andreatta and Olivier Brinon. "Surconvergence des représentations p-adiques: le cas relatif". In: *Astérisque* 319 (2008). Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (ϕ, Γ)-modules, pp. 39–116. ISSN: 0303-1179.

[AI08] Fabrizio Andreatta and Adrian Iovita. "Global applications of relative (φ, Γ)-modules. I". In: *Astérisque* 319 (2008). Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (φ, Γ)-modules, pp. 339–420. ISSN: 0303-1179.

[AI13] Fabrizio Andreatta and Adrian Iovita. "Comparison isomorphisms for smooth formal schemes". In: *J. Inst. Math. Jussieu* 12.1 (2013), pp. 77–151. ISSN: 1474-7480.

[AGV71] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier. *Theorie de Topos et Cohomologie Etale des Schemas I, II, III.* Vol. 269, 270, 305. Lecture Notes in Mathematics. Springer, 1971.

[Bau92] Werner Bauer. "On the conjecture of Birch and Swinnerton-Dyer for abelian varieties over function fields in characteristic $p > 0$". In: *Invent. Math.* 108.2 (1992), pp. 263–287. ISSN: 0020-9910.

[Bei12] Alexander Beilinson. "p-adic periods and derived de Rham cohomology". In: *J. Amer. Math. Soc.* 25.3 (2012), pp. 715–738. ISSN: 0894-0347.

[Bei13] Alexander Beilinson. "On the crystalline period map". In: *Camb. J. Math.* 1.1 (2013), pp. 1–51. ISSN: 2168-0930.

[Ber02] Laurent Berger. "Représentations p-adiques et équations différentielles". In: *Invent. Math.* 148.2 (2002), pp. 219–284. ISSN: 0020-9910.

[Ber04] Laurent Berger. "Limites de représentations cristallines". In: *Compos. Math.* 140.6 (2004), pp. 1473–1498. ISSN: 0010-437X.

[Ber74] Pierre Berthelot. *Cohomologie cristalline des schémas de caractéristique $p > 0$.* Lecture Notes in Mathematics, Vol. 407. Springer-Verlag, Berlin-New York, 1974, p. 604.

[BBM82] Pierre Berthelot, Lawrence Breen, and William Messing. *Théorie de Dieudonné cristalline. II.* Vol. 930. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1982, pp. x+261. ISBN: 3-540-11556-0.

[BO78] Pierre Berthelot and Arthur Ogus. *Notes on crystalline cohomology.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978, pp. vi+243. ISBN: 0-691-08218-9.

[BMS18] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. "Integral p-adic Hodge theory". In: *Publ. Math. Inst. Hautes Études Sci.* 128 (2018), pp. 219–397. ISSN: 0073-8301.

[BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. "Topological Hochschild homology and integral p-adic Hodge theory". In: *Publ. Math. Inst. Hautes Études Sci.* 129 (2019), pp. 199–310. ISSN: 0073-8301.

[Bri08] Olivier Brinon. "Représentations p-adiques cristallines et de de Rham dans le cas relatif". In: *Mém. Soc. Math. Fr. (N.S.)* 112 (2008), pp. vi+159. ISSN: 0249-633X.

[CC98] Frédéric Cherbonnier and Pierre Colmez. “Représentations p-adiques surconvergentes”. In: *Invent. Math.* 133.3 (1998), pp. 581–611. ISSN: 0020-9910.

[Col99] Pierre Colmez. “Représentations cristallines et représentations de hauteur finie”. In: *J. Reine Angew. Math.* 514 (1999), pp. 119–143. ISSN: 0075-4102.

[CN17] Pierre Colmez and Wiesława Nizioł. “Syntomic complexes and p-adic nearby cycles”. In: *Invent. Math.* 208.1 (2017), pp. 1–108. ISSN: 0020-9910.

[Con03] Brian Conrad. *Cohomological Descent.* Available at: `https://math.stanford.edu/~conrad/papers/hypercover.pdf`. 2003.

[DLMS22] Heng Du, Tong Liu, Yong Suk Moon, and Koji Shimizu. “Completed prismatic F-crystals and crystalline $\mathbf{Z}_p$-local systems”. In: arXiv:2203.03444 (Mar. 2022).

[Elk73] Renée Elkik. “Solutions d’équations à coefficients dans un anneau hensélien”. In: *Ann. Sci. École Norm. Sup. (4)* 6 (1973), 553–603 (1974). ISSN: 0012-9593.

[Fal88] Gerd Faltings. “p-adic Hodge theory”. In: *J. Amer. Math. Soc.* 1.1 (1988), pp. 255–299. ISSN: 0894-0347.

[Fal89] Gerd Faltings. “Crystalline cohomology and p-adic Galois-representations”. In: *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988).* Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 25–80.

[Fal02] Gerd Faltings. “Almost étale extensions”. In: *Astérisque* 279 (2002). Cohomologies p-adiques et applications arithmétiques, II, pp. 185–270. ISSN: 0303-1179.

[Fon79] Jean-Marc Fontaine. “Modules galoisiens, modules filtrés et anneaux de Barsotti-Tate”. In: *Journées de Géométrie Algébrique de Rennes. (Rennes, 1978), Vol. III.* Vol. 65. Astérisque. Soc. Math. France, Paris, 1979, pp. 3–80.

[Fon82] Jean-Marc Fontaine. “Sur certains types de représentations p-adiques du groupe de Galois d’un corps local; construction d’un anneau de Barsotti-Tate”. In: *Ann. of Math. (2)* 115.3 (1982), pp. 529–577. ISSN: 0003-486X.

[Fon90] Jean-Marc Fontaine. “Représentations p-adiques des corps locaux. I”. In: *The Grothendieck Festschrift, Vol. II.* Vol. 87. Progr. Math. Birkhäuser Boston, Boston, MA, 1990, pp. 249–309.

[Fon94a] Jean-Marc Fontaine. “Le corps des périodes p-adiques”. In: *Astérisque* 223 (1994). With an appendix by Pierre Colmez, Périodes p-adiques (Bures-sur-Yvette, 1988), pp. 59–111. ISSN: 0303-1179.

[Fon94b] Jean-Marc Fontaine. “Représentations p-adiques semi-stables”. In: *Astérisque* 223 (1994). With an appendix by Pierre Colmez, Périodes p-adiques (Bures-sur-Yvette, 1988), pp. 113–184. ISSN: 0303-1179.

[FL82] Jean-Marc Fontaine and Guy Laffaille. “Construction de représentations p-adiques”. In: *Ann. Sci. École Norm. Sup. (4)* 15.4 (1982), 547–608 (1983). ISSN: 0012-9593.

[FM87] Jean-Marc Fontaine and William Messing. “p-adic periods and p-adic étale cohomology”. In: *Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985).* Vol. 67. Contemp. Math. Amer. Math. Soc., Providence, RI, 1987, pp. 179–207.

[FW79a] Jean-Marc Fontaine and Jean-Pierre Wintenberger. “Extensions algébrique et corps des normes des extensions APF des corps locaux”. In: *C. R. Acad. Sci. Paris Sér. A-B* 288.8 (1979), A441–A444. ISSN: 0151-0509.

[FW79b] Jean-Marc Fontaine and Jean-Pierre Wintenberger. “Le “corps des normes” de certaines extensions algébriques de corps locaux”. In: *C. R. Acad. Sci. Paris Sér. A-B* 288.6 (1979), A367–A370. ISSN: 0151-0509.

[Gab94] Ofer Gabber. “Affine analog of the proper base change theorem”. In: *Israel J. Math.* 87.1-3 (1994), pp. 325–335. ISSN: 0021-2172.

[Gil21] Sally Gilles. “Morphismes de périodes et cohomologie syntomique”. In: arXiv:2101.04987 (Jan. 2021).

[Gro63] Alexander Grothendieck. *Revêtements étales et groupe fondamental. Fasc. I: Exposés 1 à 5.* Troisième édition, corrigée, Séminaire de Géométrie Algébrique, 1960/61. Institut des Hautes Études Scientifiques, Paris, 1963, iv+143 pp. (not consecutively paged) (loose errata).

[GR22] Haoyang Guo and Emanuel Reinecke. “A prismatic approach to crystalline local systems”. In: arXiv:2203.09490 (Mar. 2022).

[Her98] Laurent Herr. “Sur la cohomologie galoisienne des corps p-adiques”. In: *Bull. Soc. Math. France* 126.4 (1998), pp. 563–600. ISSN: 0037-9484.

[Kat87] Kazuya Kato. “On p-adic vanishing cycles (application of ideas of Fontaine-Messing)”. In: *Algebraic geometry, Sendai, 1985.* Vol. 10. Adv. Stud. Pure Math. North-Holland, Amsterdam, 1987, pp. 207–251.

[Kat94] Kazuya Kato. “Semi-stable reduction and p-adic étale cohomology”. In: *Astérisque* 223 (1994). Périodes p-adiques (Bures-sur-Yvette, 1988), pp. 269–293. ISSN: 0303-1179.

[KM92] Kazuya Kato and William Messing. “Syntomic cohomology and p-adic étale cohomology”. In: *Tohoku Math. J. (2)* 44.1 (1992), pp. 1–9. ISSN: 0040-8735.

[Kur87] Masato Kurihara. “A note on p-adic étale cohomology”. In: *Proc. Japan Acad. Ser. A Math. Sci.* 63.7 (1987), pp. 275–278. ISSN: 0386-2194.

[Laz65] Michel Lazard. “Groupes analytiques p-adiques”. In: *Inst. Hautes Études Sci. Publ. Math.* 26 (1965), pp. 389–603. ISSN: 0073-8301.

[Mor08] Kazuma Morita. “Galois cohomology of a p-adic field via (Φ,Γ)-modules in the imperfect residue field case”. In: *J. Math. Sci. Univ. Tokyo* 15.2 (2008), pp. 219–241. ISSN: 1340-5705.

[MT20] Matthew Morrow and Takeshi Tsuji. “Generalised representations as q-connections in integral p-adic Hodge theory”. In: arXiv:2010.04059 (Oct. 2020).

[Niz98] Wiesława Nizioł. “Crystalline conjecture via K-theory”. In: *Ann. Sci. École Norm. Sup. (4)* 31.5 (1998), pp. 659–681. ISSN: 0012-9593.

[Sch13] Peter Scholze. “p-adic Hodge theory for rigid-analytic varieties”. In: *Forum Math. Pi* 1 (2013), e1, 77.

[Sta22] The Stacks project authors. *The Stacks project.* Available at: `https://stacks.math.columbia.edu`. 2022.

[Tsu96] Takeshi Tsuji. “Syntomic complexes and p-adic vanishing cycles”. In: *J. Reine Angew. Math.* 472 (1996), pp. 69–138. ISSN: 0075-4102.

[Tsu99] Takeshi Tsuji. “p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case”. In: *Invent. Math.* 137.2 (1999), pp. 233–411. ISSN: 0020-9910.

[Tsu20] Takeshi Tsuji. “Crystalline $\mathbb{Z}_p$-representations and A_{inf}-Representations with Frobenius”. In: *Proceeedings in Simons Symposium: p-adic Hodge theory.* Simons symposia (2020), pp. 161–319.

[Tyc88] Andrzej Tyc. “Differential basis, p-basis, and smoothness in characteristic $p > 0$”. In: *Proc. Amer. Math. Soc.* 103.2 (1988), pp. 389–394. ISSN: 0002-9939.

[Wac96] Nathalie Wach. “Représentations p-adiques potentiellement cristallines”. In: *Bull. Soc. Math. France* 124.3 (1996), pp. 375–400. ISSN: 0037-9484.

[Wac97] Nathalie Wach. "Représentations cristallines de torsion". In: *Compositio Math.* 108.2 (1997), pp. 185–240. ISSN: 0010-437X.

[Win83] Jean-Pierre Wintenberger. "Le corps des normes de certaines extensions infinies de corps locaux; applications". In: *Ann. Sci. École Norm. Sup. (4)* 16.1 (1983), pp. 59–89. ISSN: 0012-9593.

[YY14] Go Yamashita and Seidai Yasuda. "p-adic étale cohomology and crystalline cohomology for open varieties with semistable reduction". In: *preprint* (2014).

Abhinandan
University of Tokyo, Japan
E-mail : abhi@ms.u-tokyo.ac.jp