Quantum Computing - SOC Week 1

Hilbert Space : For a space to be a hilbert space, it must :

1. Be a vector space over $\mathbb{C}$ in the 'mathematical' sense
2. Have a defined inner product
3. Must be 'complete'

State Vectors (in QM)

$|\psi\rangle$: ket : vector in a vector-space : denotes a state independent of basis

Can be represented as a coloumn vector once we choose a basis

Inner Product

$|v\rangle$ & $|w\rangle$ are vectors :- Inner pdt $= \langle v|w\rangle \in \mathbb{C}$

Typically : $\langle v|w\rangle = \int v^*_{(x)}\, w(x)\, t(x)\, dx$ where t is a weight / jacobian

Properties of $\langle v|w\rangle$ are analogous to that of a dot product when dealing with 'normal' vectors.

Projection of $|v\rangle$ on $|w\rangle$ = $P_w(v) = \dfrac{\langle w|v\rangle}{\langle w|w\rangle}|w\rangle$

Outer Product : $\underset{1\times n}{|v\rangle}\underset{m\times 1}{\langle w|} = \underset{n\times m}{M}$: Generates a matrix of order $n \times m$

Outer products are used to build operators $\longrightarrow$ Hamiltonian Construction

$\longrightarrow$ Projective Measurements & Spectral Decomposition

We usually work with a finite n-dim Hilbert space

Any operator can be expressed as a $n\times n$ matrix

Gram-Schmidt Orthonormalisation

Given : $\{|u_1\rangle, |u_2\rangle \cdots |u_n\rangle\}$

$$v_1 = \frac{|u_1\rangle}{\|u_1\|} \quad \& \quad |v_{j+1}\rangle = \frac{|u_{j+1}\rangle - \langle v_j|u_{j+1}\rangle|v_j\rangle}{\| |u_{j+1}\rangle - \langle v_j|u_{j+1}\rangle|v_j\rangle \|}$$

Adjoint of an operator

$\langle\varphi|A\psi\rangle = \langle A^\dagger\varphi|\psi\rangle$ where Adjoint $(A) = A^\dagger = (\bar{A})^T$

A Linear operator A is hermitian if $A = A^\dagger$

Properties of a hermitian operator : ① all eigenvals are real

② Eigenvecs corr. to diff eigenvals are orthogonal

③ diagnolisable

Dual Vector Space

: V^* is dual space of complex vectorspace V

A dual vector $\langle v|$ maps any $|w\rangle \in V$ to a complex number $\langle v|w\rangle \in \mathbb{C}$

$V^* = \{f : V \to \mathbb{C} \mid f \text{ is linear}\}$

$|v\rangle$ is a column vector in $V \Rightarrow \langle v|$ is its hermitian conjugate (row vec in V^*)

$\{|i\rangle\}$ is basis for $V \iff \{\langle i|\}$ is basis for V^*

Tensor Products

$V \to$ vector space of n-dim

$W \to$ vector space of m-dim

$V \otimes W \longrightarrow mn$-dim vector space

Elements of $V \otimes W$ are linear combinations of tensor products of any two vector in V & W

Eg. $V = \{|0\rangle, |1\rangle\} \Rightarrow V \otimes V = \{|0\rangle\otimes|0\rangle, |0\rangle\otimes|1\rangle, |1\rangle\otimes|0\rangle, |1\rangle\otimes|1\rangle\}$

Linear Operators on Tensor Spaces

: A & B are linear operators on V & W

$(A|\psi\rangle)\otimes(B|\omega\rangle) = (A\otimes B)(|\psi\rangle\otimes|\omega\rangle)$ & $\sum_i a_i(A|\psi_i\rangle\otimes B|\omega_i\rangle) = (A\otimes B)\left(\sum_i a_i|\psi_i\rangle\otimes|\omega_i\rangle\right)$

Kronecker Product : $A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ & & & a_{mn}B \end{pmatrix}$

Commutator & Anticommutators

Commutator : $[A, B] = AB - BA$ Note : $[\hat{x}, \hat{p}] = i\hbar$

Anticommutator : $\{A, B\} = AB + BA$

Pauli Matrices : $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$[X, Y] = 2iZ$, $[Y, Z] = 2iX$, $[Z, X] = 2iY$ $\{X, Y\} = \{Y, Z\} = \{Z, X\} = 0$

State Space : $|\psi\rangle \in H$ completely denotes the state of a system

$\langle\psi|\psi\rangle = 1$ & $|\psi\rangle$ is defined only upto a global phase : $|\psi\rangle \sim e^{i\theta}|\psi\rangle$

Evolution of Quantum States :- $t_0 \to |\psi\rangle$ & $t_1 \to |\psi'\rangle \Rightarrow |\psi'\rangle = U|\psi\rangle$ where $UU^\dagger = I$

$\langle\psi'|\psi'\rangle = \langle\psi|\psi\rangle$; In isolated systems, evolution is always linear & is along $\langle\psi|\psi\rangle = 1$

In QC, general state $(|\psi\rangle) = \alpha|0\rangle + \beta|1\rangle$; $|\alpha|^2 + |\beta|^2 = 1$; prob of $|0\rangle = \alpha^2$ & $|1\rangle = \beta^2$

Operators on single qubit states can be denoted by 2x2 matrices

Spectral Decomposition : Express linear operator as sum over its eigenvals & eigenvecs

For a normal operator M : $M = \sum \lambda_i |i\rangle\langle i|$ where λ_i is eigenvals & $\{|i\rangle\}$ forms orthonormal basis

M is diagnol in the basis of its eigenvecs. The eigenvecs form a complete orthonormal set

$f(M) = \sum f(\lambda_i)|i\rangle\langle i|$; Decomposition is unique if all eigenvals are distinct

Polar Decomposition : $\underbrace{A}_{\text{Any linear operator}} = \underbrace{U}_{\text{unitary operator}} \underbrace{P}_{P = \sqrt{A^\dagger A}\text{ \& is positive semidefinite \& uniquely defined}}$

Quantum Measurements : $\{M_m\} \to$ Collection of predefined measurement operators (index 'm' corr. to measurement outcome)

$p(m) = \langle\psi|M^\dagger M|\psi\rangle$ & $|\psi'\rangle = \dfrac{M_m|\psi\rangle}{\sqrt{\langle\psi|M^\dagger M|\psi\rangle}}$; $\sum_m p(m) = 1 \Rightarrow \sum_m M_m^\dagger M_m = 1$

Projective Measurement : $\{P_m\} = \{M_m\}$ where $\{P_m\}$ are orthogonal

Suppose we are measuring an observable $\hat{A}$ with eigenvals $\{m\}$ $\Rightarrow P_m = \sum_{i=1}^{n} |m_i\rangle\langle m_i|$ where m is n-fold degenerate

P_m picks out eigenvecs from $|\psi\rangle$ corr. to eigenvalue m. $P = P^{+} = P^{2}$ $\Rightarrow p(m) = \langle\psi|P_m|\psi\rangle$ & $|\psi'\rangle = \frac{P_m|\psi\rangle}{\sqrt{p(m)}}$

Positive Operator Valued Measurements : $E_m = M^{+}M$ as we are usually interested in prob. of measurment states $p(m) = \langle\psi|E_m|\psi\rangle$