* $\bar{\nabla} f = (f_x, f_y)$
* $D_{\bar{u}} f\big|_P = \bar{\nabla} f \cdot \bar{u}$
* eq. of Normal plane:
$$\frac{y-y_0}{x-x_0} = \frac{f_y}{f_x}$$
* eq. of Tangent plane:
$$\frac{y-y_0}{x-x_0} = -\frac{f_x}{f_y}$$
* $df = (\bar{\nabla} f|_{P_0} \cdot \bar{u})\, \underset{\text{distance}}{ds}$

* $dA = r\,dr\,d\theta$
$dv = (r\,dr\,d\theta)\,dz$

* Spherical coordinates

$r = \rho \sin\phi$, $x = r\cos\theta$
$x = \rho\sin\phi\cos\theta$

$z = \rho\cos\phi$, $y = r\sin\theta$
$y = \rho\sin\phi\sin\theta$

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}$
↳ dist. of P from origin

* Jacobian
$$\iint_D f(x,y)\,dA = \iint_R f(g(u,v), h(u,v)) \cdot |J| \cdot dA$$
$\boxed{J \cdot J' = 1}$

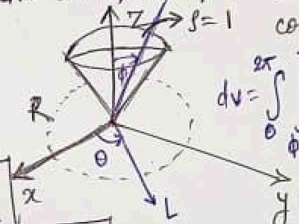
Steps
(i) Sketch region in x-y plane.
(ii) choose substitution (may be given).
(iii) find x and y in terms of u & v.
(iv) Find $J = \partial(x,y)/\partial(u,v)$
(v) Find the eqn. in terms of u, v so as to draw a region in u-v plane with help of step-(i) & (ii).
(vi) Sketch region in u-v plane.
(vii) Find limits for u, v then apply formula & solve.

* Local max if $f_{xx} < 0$, $f_{xx}f_{yy} - f_{xy}^2 > 0$
Local min if $f_{xx} > 0$, $f_{xx}f_{yy} - f_{xy}^2 > 0$
Saddle if $f_{xx}f_{yy} - f_{xy}^2 < 0$
$f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a,b) ⇒ Inconclusive

* $\phi \to$ Angle $\overrightarrow{OP}$ makes with +ve Z-axis $(0 \le \phi \le \pi)$
* $\theta \to$ Angle from cylindrical coordinates
* $dv = \rho^2 \sin\phi \cdot d\rho\, d\phi \cdot d\theta$
*

* $\nabla f = \lambda \cdot \nabla g$. Find λ then find (x,y) corresponding to λ and find f and check for max./min.
$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$ · same procedure

* $\frac{\partial w}{\partial x}$ (w → dep. variable, x → indep. variable)
y = dep. var.
$w = x^2+y^2$
$\frac{\partial w}{\partial x} = 2x + 2y\frac{\partial y}{\partial x}$



cone $\phi = \pi/3$
$$dv = \int_0^{2\pi}\int_{\phi=0}^{\pi/3}\int_{\rho=0}^{1} \rho^2 \sin\phi\, d\rho \cdot d\phi \cdot d\theta$$

Arpit Raj

* $f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) +$
$\frac{1}{2}\left(f_{xx}(x_0,y_0)(x-x_0)^2 + f_{yy}(x_0,y_0)(y-y_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0)\right)$
$+\frac{1}{6}\left(f_{xxx}(x-x_0)^3 + f_{yyy}(y-y_0)^3 + 3f_{xxy}(x-x_0)^2(y-y_0) + 3f_{xyy}(x-x_0)(y-y_0)^2\right)$

Vectors & Geometry

$\bar{u}\cdot\bar{v} = u_1v_1 + u_2v_2$
Angle $= \cos\theta = \frac{\bar{u}\cdot\bar{v}}{|\bar{u}||\bar{v}|}$
↳ 0 when $u \perp v$

* Error in Lin. approx.:
$$|E(x,y)| \le \frac{1}{2} M \left(|x-x_0| + |y-y_0|\right)^2$$
$M = \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\}_{(x,y)}$

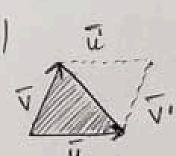
* $|\text{Proj}_{\bar{v}} \bar{u}| = |\bar{u}|\cos\theta$
$= \cancel{|\bar{u}|} \cdot \frac{\bar{u}\cdot\bar{v}}{\cancel{|\bar{u}|}\cdot|\bar{v}|} = \frac{\bar{u}\cdot\bar{v}}{|\bar{v}|}$
$= \frac{(\bar{u}\cdot\bar{v})}{|\bar{v}|} \cdot \frac{\bar{v}}{|\bar{v}|} = \underbrace{\left(\frac{\bar{u}\cdot\bar{v}}{|\bar{v}|^2}\right)}_{\text{scalar}} \underbrace{\bar{v}}_{\text{vector}}$

* $\int u \cdot v = u\int v - \int u' \int v$
* Avg. Value of F over D $= \frac{1}{\text{Vol. of } D} \iiint_D F\,dv$

* $\bar{u} = \underbrace{\text{Proj}_{\bar{v}}\bar{u}}_{\parallel \text{ to } V} + \underbrace{(\bar{u} - \text{Proj}_{\bar{v}}\bar{u})}_{\perp \text{ to } V}$

* $M = \iiint_D \delta(x,y,z)\,dv$ (curve → ds)
* $M_{yz} = \iiint_D x\,\delta\,dv$ (first moments)
* $\bar{x} = M_{yz}/M$ (COM)
* $I_L = \iiint_D r^2 \delta\,dv$ (MoI)
L = x-axis
$I_x = \iiint_D (y^2+z^2)\delta\,dv$
* $R_L = \sqrt{\frac{I_L}{M}}$ (Radius of Gyration)

* Area of ||gm with sides $\bar{u}$ and $\bar{v}$ $= |\bar{u}\times\bar{v}|$
* Area of Δ with sides $\bar{u}, \bar{v} = \frac{|\bar{u}\times\bar{v}|}{2}$



* Scalar/Triple Box Product $= (\bar{u}\times\bar{v})\cdot\bar{w} = (\bar{v}\times\bar{w})\cdot\bar{u}$
$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$
Vol. of box (parallelopiped)

* Eq. of line L passing thru $P_0(x_0, y_0, z_0)$ and $\parallel$ to $\bar{v}$.

$$(x,y,z) = \bar{r}(t) = \bar{r}_0 + t\,\bar{v}$$

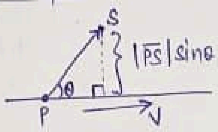
$\bar{r}_0 \to (x_0, y_0, z_0)$, $\bar{v} \to (v_1, v_2, v_3)$

$$\left.\begin{aligned} x &= x_0 + t v_1 \\ y &= y_0 + t v_2 \\ z &= z_0 + t v_3 \end{aligned}\right\} \Rightarrow \frac{x-x_0}{v_1} = \frac{y-y_0}{v_2} = \frac{z-z_0}{v_3}$$

* Dist. of point to line in Space

$$d = |\overrightarrow{PS}| \sin\theta$$

$$d = \frac{|\overrightarrow{PS} \times V|}{|V|}$$



↳ dist. from a point S to a line thru P parallel to V.

* vector $\parallel$ to line of int. of 2 planes

$$\bar{V} = \bar{n}_1 \times \bar{n}_2$$

LOI

$P = (x, y, 0)$ $\quad \bar{V} =$

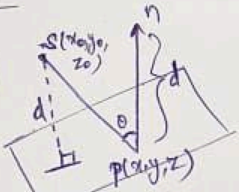
↳ solve the two planes (at $z = 0$).

* Eqn. of plane passing thru $P_0(x_0, y_0, z_0)$ and normal $\bar{n}$

$$\bar{n} \cdot \overline{PP_0} = 0$$

* Dist. of Pt "S" to a plane

$$d = \overline{PS} \cdot \cos\theta = \overline{PS} \cdot \frac{\bar{n}}{\|\bar{n}\|}$$



* Angle b/w 2 planes = Angle b/w 2 normals

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{\big||\bar{n}_1| \cdot |\bar{n}_2|\big|}$$

Line Integrals

* $$\int_C f(x,y,z)\, ds = \int f(x,y,z) \left(\frac{ds}{dt}\right) dt = \int f(x_{(t)}, y_{(t)}, z_{(t)})\, |\bar{v}|\, dt$$

curve "c" $= C_1 \cup C_2 \cup \dots C_n$

$$\int_C f\, ds = \int_{C_1} f ds + \int_{C_2} f ds + \dots + \int_{C_n} f \cdot ds$$

Area of $R = \frac{1}{2} \oint_C x\,dy - y\,dx$

* Work done by force f over a curve

$$W = \int F_{(x,y,z)} \cdot dr = \int F\left(\frac{d\bar{r}}{ds}\right) ds$$

$$\boxed{W = \int_{t=a}^{t=b} F \cdot \underbrace{T}_{\text{unit tang. vector } (V/|V|)} \cdot ds}$$

$$W = \int M\,dx + \int N\,dy + \int P\,dz \qquad F = (M, N, P)$$

* Flow, circulation

$$\text{Flow} = \int_a^b \bar{F} \cdot \bar{T} \cdot ds = \int \bar{F} \cdot d\bar{r}/dt \cdot dt$$

$V = V_x\hat{i} + V_y\hat{j} \Rightarrow n = \frac{+V_y\hat{i} - V_x\hat{j}}{|V|}$

* Flux

$$J = \int_C \bar{F} \cdot \bar{n} \cdot ds \quad \left(\begin{aligned} \bar{n} &= \bar{k} \times \bar{T} \text{ (clockwise)} \\ \bar{n} &= \bar{T} \times \bar{k} \text{ (A.C.W.)} \end{aligned}\right)$$

↳ +ve, net flow across the curve is outward.

* $W = \int_A^B \bar{F} \cdot d\bar{r}$ — ① | If the integral is path independent from A to B, its value is

$$\int_A^B F \cdot dr = f(B) - f(A)$$

① is path indep $\leftrightarrow$ $\bar{F}$ is conservative field.

$$\boxed{\bar{F} = \bar{\nabla} f}$$

↳ potential fn. of F.

* $\bar{F}$ is conservative iff $\qquad \bar{F} = (M, N, P)$

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Exactness

* Green's Theorem $\longrightarrow$ * Area of R $= \frac{1}{2} \oint_C x\,dy - y\,dx$

Form-1: Flux divergence or Normal Thm.

$$\oint \bar{F} \cdot \bar{n} \cdot ds = \iint_R (\bar{\nabla} \cdot \bar{F})\, dA = \iint_R (M_x + N_y)\, dx\, dy$$

outward flux $\longrightarrow \oint_C M\,dy - N\,dx$ $\qquad$ Divergence Integral

Form-2: circulation curl or Tangential form

$$\oint F \cdot T\, ds = \iint \underbrace{(\text{curl}\,\bar{F})}_{(\bar{\nabla} \times \bar{F})} \cdot \hat{k}\, dA = \iint_R (N_x - M_y)\, dx\, dy$$

curl Integral

counter-clockwise circulation $\longrightarrow = \oint_C M\,dx + N\,dy$

$$\boxed{(\text{curl}\,F) \cdot K > 0 \;:\; \text{ACW}}$$

* Surface Area $= \iint_R \frac{|\bar{\nabla} f|}{|\bar{\nabla} f \cdot \underline{p}|} dA$

↳ p: unit vector normal to R

* $A = \iint_{R_{xy}} \sqrt{f_x^2 + f_y^2 + 1}\; dx\, dy \rightarrow$ area of smooth surface $z = f(x,y)$ over region R_{xy} in xy-plane.

* **Length of a Smooth curve**

$$\gamma(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

* **Arc length**: $L = \int_a^b |V|\, dt$

* **Arc length parameter with base point $P(t_0)$**

* $$S(t) = \int_{t_0}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\, d\tau$$

$$= \int_{t_0}^{t} |V(\tau)|\, d\tau$$

* **Speed on a Smooth curve**

$$\frac{ds}{dt} = |V(t)|$$

* **unit Tangent vector T**

$$T = \frac{dr}{ds} = \frac{dr/dt}{ds/dt} = \frac{V}{|V|}$$

* **Curvature**

$$k = \left|\frac{dT}{ds}\right| = \frac{1}{|V|}\left|\frac{dT}{dt}\right|$$

curvature of st. line = 0

curvature of circle of rad. $a = \frac{1}{a}$

* **Principal unit Normal**

$$N = \frac{1}{k}\frac{dT}{ds} = \frac{dT/ds}{|dT/ds|}$$

$$N = \frac{dT/dt}{|dT/dt|}$$

* **Circle of curvature/osculating circle**

at a point P on a curve where $K \neq 0$ is

(i) Tangent to the curve at P.

(ii) has the same curvature that the curve has at 'P'.

(iii) lies towards the concave/inner side of curve.

Rod. of curvature $= \rho = 1/K$

* $$k(t) = \frac{|V \times a|}{|V|^3}$$

* $B = T \times N$. The torsion func. of smooth curve is $\tau = -\frac{dB}{ds} \cdot N$

* **Tangential & Normal comp. of Acc^n.**

$$a = a_T T + a_N N$$

$$\boxed{a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}|V|} \quad \& \quad a_N = k\left(\frac{ds}{dt}\right)^2$$

$$\boxed{a_N = k|V|^2}$$

$$\boxed{a_N = \sqrt{|a|^2 - a_T^2}}$$

* **Vector formula for curvature**

$$k = \frac{|V \times a|}{|V|^3}$$

* $$\tau = \frac{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}{|V \times a|^2} \quad (\text{If } V \times a \neq 0)$$

* $\gamma(t_0)$ lies at any pt. P which lies on the oscillating plane. → (Multiply with B)

T is normal to the normal plane.

N is normal to the rectifying plane.

* $y = f(x) \Rightarrow k = \dfrac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$

* $$\left(\frac{\partial x}{\partial y}\right)_z = -\frac{\partial f/\partial y}{\partial f/\partial z}$$

Arpit

* Integral of g over S $= \iint_R g(x,y,z) \dfrac{|\nabla f|}{|\nabla f \cdot p|}\, dA$

* $\iint_S g \cdot d\sigma$, $d\sigma = \dfrac{|\nabla f|}{|\nabla f \cdot p|}\, dA$

* Surface integral for Flux $= \iint_S F \cdot n\, d\sigma$

If S is a part of a level surface $g(x,y,z) = c$, then n may be taken as: $\boxed{n = \pm \dfrac{\nabla g}{|\nabla g|}}$

* $\gamma(u,v) = f(u,v)\hat{i} + g(u,v)\hat{j} + h(u,v)\hat{k}$

$$\gamma_u = \frac{\partial r}{\partial u} = \frac{\partial f}{\partial u}\hat{i} + \frac{\partial g}{\partial u}\hat{j} + \frac{\partial h}{\partial u}\hat{k}$$

$$\gamma_v = \frac{\partial r}{\partial v} = \frac{\partial f}{\partial v}\hat{i} + \frac{\partial g}{\partial v}\hat{j} + \frac{\partial g}{\partial v}\hat{k}$$

Smooth parametrized surface: $\gamma(u,v)$ is smooth if γ_u and γ_v are continuous and $\gamma_u \times \gamma_v$ is never zero.

* **Area of Smooth Surface**

$$A = \int_c^d \int_a^b |\gamma_u \times \gamma_v|\, du\, dv$$

* Surface Area diff.

$$d\sigma = |r_u \times r_v|\, du\, dv$$

Diff formula for S.A. : $\iint_S d\sigma$

* Integral of G over S ;

$$\iint_S G(x,y,z)\, d\sigma = \int_c^d \int_a^b G(f(u,v), g(u,v), h(u,v))\, |r_u \times r_v|\, du\, dv$$

$$= \iint G\, d\sigma$$

$$n = \frac{r_u \times r_v}{|r_u \times r_v|}$$

* Stoke's Theorem

$$\oint F \cdot dr = \iint_S (\nabla \times F) \cdot n\, d\sigma$$

counter clockwise   curl Integral

$$n = \frac{\nabla f}{|\nabla f|}$$

* curl grad $f = 0$ or $\nabla \times \nabla f = 0$ } holds for any fn. $f(x,y,z)$ whose 2nd partial derivatives are continous.

* Divergence Thm.

$$\iint \bar{F} \cdot \bar{n}\, d\sigma = \iiint_D (\bar{\nabla} \cdot \bar{F})\, dv$$

outward flux   Divergence Integral

* Paramentrization of Sphere

$x^2+y^2+z^2=a^2$

$x = a\sin\phi\cos\theta$

$y = a\sin\phi\sin\theta$

$z = a\cos\phi$

$\boxed{0 \le \phi, \theta \le 2\pi}$

* Hyperboloid (one sheet)

$x^2+y^2-z^2=1$

$x = \cosh u \cos\theta$

$y = \cosh u \sin\theta$

$z = \sinh u$

* Hyperboloid (Two Sheet)

$z^2/c^2 - x^2/a^2 - y^2/b^2 = 1$

$x = a\sinh u\cos\theta$

$y = b\sinh u\sin\theta$

$z = c\cosh u$

* for cone

$z = \sqrt{x^2+y^2}$

$x = r\cos\theta$

$y = r\sin\theta$

$z = r$

* from (0,1) to (1,0)

$r(t) = (0,1) + ((1-0),(0-1))t$

$= (t, 1-t)$

* If $\nabla \times F = 0$, then the circ^n. of F around boundary C of any oriented surface S in the domain of F is zero.

By stoke's: $\oint_C F \cdot dr = \iint_S \nabla \times F \cdot n\, d\sigma = 0$.

* $|z| = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$ $\quad (z = r(\cos\theta + i\sin\theta))$

$\theta = \underbrace{\arg z}_{\text{general arg.} \ni (\theta + 2n\pi)} = \tan^{-1}(y/x)$

$\boxed{-\pi < \operatorname{Arg} z \le \pi}$ → capital A (principal value of argument)

* $|z_1 + z_2| \le |z_1| + |z_2|$

* $Re(z) = \frac{1}{2}(z + \bar{z}) = x$, $Im(z) = \frac{1}{2i}(z - \bar{z}) = y$

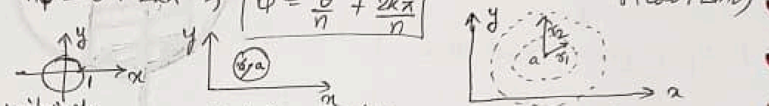
* $z_1 z_2 = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

* $z^n = r^n(\cos n\theta + i\sin n\theta)$, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

* $z = w^n \Rightarrow w = \sqrt[n]{z} \Rightarrow w^n = R^n(\cos n\phi + i\sin n\phi) = z = r(\cos\theta + i\sin\theta)$

$n\phi = \theta + 2k\pi \Rightarrow \boxed{\phi = \frac{\theta}{n} + \frac{2k\pi}{n}}$

*



unit circle   Circle in complex plane   Annulus

$|z-a| = r$: circle of rad $= r$, centre $= a$, $r_1 < |z-a| < r_2$: open Annulus

$|z-a| < r$: disk $r_1 \le |z-a| \le r_2$: closed Annulus

* A fn. is said to be analytic at a pt. if it is diff. in some neighbourhood which is centered around a.

* $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ → $z = z_0 + \Delta z$

* cauchy - Riemann

$\boxed{u_x = v_y}$ $\boxed{u_y = -v_x}$

* $z = r\cos\theta + i\sin\theta$

$f(z) = u(r,\theta) + i \cdot v(r,\theta)$

$\boxed{u_r = \frac{1}{r} v_\theta}$ $\boxed{v_r = \frac{-1}{r} u_\theta}$ → Polar form

* $f(z) = u(x,y) + i\, v(x,y)$

$\nabla^2 u = u_{xx} + u_{yy} = 0$

$\nabla^2 v = v_{xx} + v_{yy} = 0$ } Satisfies ↓ Harmonic

* e^z : Analytic $= \underset{e^x\cos y}{u} + i\underset{e^x\sin y}{v} = e^{re^{i\theta}}$ $\boxed{z = re^{i\theta}}$

* $e^z = e^{(z+2\pi i)}$

* $|e^z| = e^x$. $\arg(e^z) = y \pm 2n\pi$ $(n = 0, 1, 2, \ldots)$

$\operatorname{Arg}(z) = -\pi < y \le \pi$

* Entire fn. : Analytic $\forall$ z.

* $\sin z = \sin x\cosh y + i\cos x\sinh y$ $\boxed{\cos^2 z + \sin^2 z = 1}$

* $\cos z = \cos x\cosh y - i\sin x\sinh y$

* $|\sin z|^2 = \sin^2 x + \sinh^2 y$, $|\cos z|^2 = \cos^2 x + \sinh^2 y$

* $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$

$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$

* $\cosh z = (e^z + e^{-z})/2$, $\sinh z = (e^z - e^{-z})/2$

* $\cosh(iz) = \cos z$, $\sinh(iz) = i\sin z$

$\cos(iz) = \cosh z$, $\sin(iz) = i\sinh z$

$\pi - \alpha$	α
$-(\pi - \alpha)$	$-\alpha$