Quiz 2 Solutions (RUBRIC)

A.1) $\phi(r) = \frac{k}{r^4}$ (No step marking)

$\vec{E} = -\vec{\nabla}\phi$

$\Rightarrow \vec{E} = -\frac{\partial \phi}{\partial r}\hat{r}$

$\vec{E} = \frac{4k}{r^5}\hat{r}$ → It can be written as $E = \frac{4k}{r^5}$

but $\vec{E} = \frac{4k}{r^5}$ (one side without vector sign will result in -1).

Energy Density $= \frac{1}{2}\varepsilon_0 E^2$ 3

$= \frac{1}{2}\varepsilon_0\left(\frac{4k}{r^5}\right)^2$

$= \frac{8\varepsilon_0 k^2}{r^{10}}$

A.2) $\because \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(r)$

and $\int f(r)\delta^3(r-a)\,dV = f(a)$, if $r=a$ is in V and is 0 otherwise.

(a) $\therefore \int_{-1}^{+1}(r^2+2)\,\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)dV = \int_{-1}^{+1}(r^2+2)\,4\pi\delta^3(r)\,dV$

No step marking 1.5

$= 4\pi\int_{-1}^{+1}(r^2+2)\delta^3(r)\,dV$

$= 4\pi(0+2) = 8\pi$

(b) $\int_1^{\infty}(r^2+2)\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)dV = \int_1^{\infty}(r^2+2)\,4\pi\delta^3(r)\,dV$

No step marking 1.5

$= 4\pi\int_1^{\infty}(r^2+2)\delta^3(r)\,dV = 0$

A.3) (a) (iii) $\int(\vec{\nabla}\cdot\vec{E})\,dv = \oint_S \vec{E}\cdot d\vec{S}$ 1

(b) (i) $E_{t1} = E_{t2}$ & (iii) $\varepsilon_1 E_{n1} = \varepsilon_2 E_{n2}$ 1

Award marks only when both options are written 0 otherwise.

A.4) (a) Inside $E = 0$ $(r < R)$

$\therefore$ Energy density $= \frac{1}{2}\varepsilon_0 E^2 = 0$

1 mark

(b) Outside, $E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$ $(r > R)$

$\therefore$ Energy density $= \frac{1}{2}\varepsilon_0 E^2$

$$= \frac{1}{2}\varepsilon_0\left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)^2$$

$$= \frac{Q^2}{32\pi^2\varepsilon_0 r^4}$$

1 mark

(b) Total Energy

No part marking

$$= \frac{Q^2}{32\pi^2\varepsilon_0}\int_{r=R}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{1}{r^4}\, r^2\sin\theta\, d\theta\, d\phi$$

$$= \frac{Q^2}{32\pi^2\varepsilon_0}\int_R^{\infty}\frac{1}{r^2}\int_0^{\pi}\sin\theta\, d\theta\int_0^{2\pi}d\phi$$

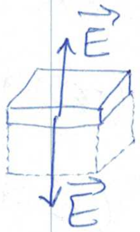
$$= \frac{Q^2}{32\pi^2\varepsilon_0}\left[-\frac{1}{r}\right]_R^{\infty}\left[-\cos\theta\right]_0^{\pi}\left[\phi\right]_0^{2\pi}$$

$$= \frac{Q^2}{32\pi^2\varepsilon_0}\left[\frac{1}{r}\right]_{\infty}^{R}\left[-(-1)+1\right]2\pi$$

3 mark

$$= \frac{Q^2}{\cancel{32}_{8}\,\cancel{\pi^2}\varepsilon_0}\frac{1}{R}\times \cancel{4\pi} = \frac{Q^2}{8\pi\varepsilon_0 R}$$

A.5)



$$EA + EA = \frac{Q_{enc}}{\varepsilon_0}$$

$$\Rightarrow 2EA = \frac{\sigma A}{\varepsilon_0}$$

3

No part marking

$$\Rightarrow \vec{E} = \frac{\sigma}{2\varepsilon_0} \Rightarrow \vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$$

A.6)

In presence of an external field $\vec{E}$, the nucleus will shift slightly to ~~the right~~, creating a dipole moment

$$\boxed{\vec{P} = q\vec{d}}$$

The shift stop increasing when the restoring electric force from the displaced cloud balance the external force on the nucleus.

For a uniformly charged sphere of radius a and total charge $-q$. The electric field inside at a distance r from centre is

$$E_{cloud}(r) = \frac{1}{4\pi\varepsilon_0}\frac{-q\,(r)}{a^3}$$ (valid for $r<a$) (using Gauss's law)

↑ Award ② 'if correct till here'

~~At equilibrium~~

So, if the nucleus is displaced by $\vec{d}$, it feels a restoring force from this field,

$$\vec{F}_{restoring} = q\cdot\vec{E}_{cloud}(d)$$

$$= q\cdot\frac{1}{4\pi\varepsilon_0}\frac{-q\vec{d}}{a^3} = \frac{-q^2}{4\pi\varepsilon_0 a^3}\vec{d}$$

In the external field, the nucleus also experience a force:

$$\vec{F}_{ext} = q\cdot\vec{E}$$

At equilibrium, ~~the~~

$$\vec{F}_{restoring} + \vec{F}_{ext} = 0$$

$$\Rightarrow \frac{-q^2}{4\pi\varepsilon_0 a^3}\vec{d} + q\vec{E} = 0$$

$$\Rightarrow \cancel{q}\vec{E} = \frac{q^{\cancel{2}}}{4\pi\varepsilon_0 a^3}\vec{d}$$

~~No part marking~~

$$\Rightarrow \vec{d} = (4\pi\varepsilon_0 a^3)\frac{\vec{E}}{q}$$ ← Award ~~2.5~~ 3 if correct till here.

Now, $\because \vec{p} = q\vec{d}$

Substituting $\vec{d}$, $\vec{p} = (4\pi\varepsilon_0 a^3)\vec{E}$

Also, it is given that $\vec{p} = \alpha\vec{E}$

where α is polarizability.

Comparing both eqn(s) for $\vec{p}$, we get

$$\alpha = 4\pi\varepsilon_0 a^3$$ ④