Part 1 A) For a general bivariate Normal distribution

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \Rightarrow \Sigma^{-1} = \frac{1}{|\Sigma|} \begin{pmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{pmatrix}$$

where $|\Sigma| = \sigma_1^2 \sigma_2^2 - (\sigma_{12})^2$

$$p(x_1, x_2) = \frac{1}{(2\pi)^{d/2} (|\Sigma|)^{1/2}} \exp\left\{-\frac{1}{2}(x-u)^T (\Sigma^{-1}) (x-u)\right\}$$

Let $Q(x_1, x_2) = (x-u)^T \Sigma^{-1} (x-u)$

$$\Rightarrow \frac{1}{|\Sigma|}\left([x_1 - u_1 \ , \ x_2 - u_2] \begin{bmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{bmatrix} \begin{bmatrix} x_1 - u_1 \\ x_2 - u_2 \end{bmatrix} \right) = Q(x_1, x_2)$$

$$\Rightarrow \frac{1}{|\Sigma|}\left(\begin{bmatrix} x_1-\mu_1 , & x_2-\mu_2 \end{bmatrix} \begin{bmatrix} \sigma_2^2 (x_1-\mu_1) - \sigma_{12}(x_2-\mu_2) \\ -\sigma_{12}(x_1-\mu_1) + \sigma_1^2 (x_2-\mu_2) \end{bmatrix} \right.$$

$$\Rightarrow \frac{1}{\sigma_1^2\sigma_2^2 - (\sigma_{12})^2}\left(\sigma_2^2 (x_1-\mu_1)^2 - 2\sigma_{12}(x_1-\mu_1)(x_2-\mu_2) + \sigma_1^2 (x_2-\mu_2)^2 \right)$$

$$\Rightarrow \frac{1}{1-\left(\frac{\sigma_{12}}{\sigma_1\sigma_2}\right)^2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\left(\frac{\sigma_{12}}{\sigma_1^2\sigma_2^2}\right)(x_1-\mu_1)(x_2-\mu_2) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 \right]$$

$$\rho = \frac{\sigma_{12}}{\sigma_1\sigma_2}$$

$$\therefore Q(x_1, x_2) = \frac{1}{1-\rho^2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho(x_1-\mu_1)(x_2-\mu_2) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 \right]$$

$\therefore \cancel{p(x_1, x_2) = \frac{1}{2\pi}}$

$$|\Sigma| = \sigma_1^2 \sigma_2^2 - (\sigma_{12})^2$$
$$= \sigma_1^2 \sigma_2^2 (1 - \rho^2)$$

$$\therefore p(x_1, x_2) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{(1-\rho^2)}} \left[-\frac{1}{2} Q(x_1, x_2) \right]$$

$$\therefore p(x_1 | x_2) = \frac{p(x_1, x_2)}{p(x_2)}$$

$$\Rightarrow \frac{\exp\left\{ \frac{-1}{(1-\rho^2)2} \left[\left(\frac{x_1 - u_1}{\sigma_1}\right)^2 + \left(\frac{x_2 - u_2}{\sigma_2}\right)^2 - 2\left(\frac{x_1 - u_1}{\sigma_1}\right)\left(\frac{x_2 - u_2}{\sigma_2}\right) \right] \right\} \Big/ 2\pi \sigma_1 \sigma_2 \sqrt{1-\rho}}{\exp\left\{ -\frac{1}{2} \left(\frac{x_2 - u_2}{\sigma_2}\right)^2 \right\} \Big/ \sigma_2 \sqrt{2\pi}}$$

$$\Rightarrow \frac{\exp\left\{ -\frac{1}{2(1-\rho^2)} \left[\left(\frac{x_1-u_1}{\sigma_1}\right)^2 + \left(\frac{x_2-u_2}{\sigma_2}\right)^2 - \frac{2\rho(x_1-u_1)(x_2-u_2)}{\sigma_1 \sigma_2} \right] + \left(\frac{x_2-u_2}{\sigma_2}\right)^2 \right\}}{\sigma_1\sqrt{1-\rho^2}\ \sqrt{2\pi}}$$

$$\Rightarrow \frac{\exp\left\{ -\frac{1}{2\sigma_1^2(1-\rho^2)} \left[(x_1-u_1)^2 + \rho^2\frac{\sigma_1^2}{\sigma_2^2}(x_2-u_2)^2 - 2\rho\frac{\sigma_1}{\sigma_2}(x_1-u_1)(x_2-u_2) \right] \right\}}{\sqrt{2\pi}\ \sigma_1\sqrt{1-\rho^2}}$$

$$\Rightarrow \frac{1}{\sigma_1\sqrt{1-\rho^2}\ \sqrt{2\pi}} \exp\left\{ -\frac{1}{2\sigma_1^2(1-\rho^2)} \left[x_1 - u_1 - \rho\frac{\sigma_1}{\sigma_2}(x_2-u_2) \right]^2 \right\}$$

$$\therefore\ p(x_1|x_2) = \frac{1}{\sigma_1\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left\{ -\frac{1}{2\sigma_1^2(1-\rho^2)} \left[x_1 - u_1 - \rho\frac{\sigma_1}{\sigma_2}(x_2 - u_2) \right]^2 \right\}$$

Similarly, $$p(x_2|x_1) = \frac{1}{\sigma_2\sqrt{1-\rho^2}\sqrt{2\pi}} \exp\left\{ -\frac{1}{2\sigma_2^2(1-\rho^2)} \left[x_2 - u_2 - \rho\frac{\sigma_2}{\sigma_1}(x_1 - u_1) \right]^2 \right\}$$

$$\therefore\ u = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\quad \Sigma = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \Rightarrow \sigma_1 = 1,\ \sigma_2 = 1,\ \rho = a$$

$$u_1 = 1,\ u_2 = 2$$

$$\therefore\ \boxed{\begin{aligned} p(x_1|x_2) &= \frac{1}{\sqrt{1-a^2}\sqrt{2\pi}} \exp\left\{ -\frac{1}{2(1-a^2)} \left[x_1 - 1 - a(x_2 - 2) \right]^2 \right\} \\ p(x_2|x_1) &= \frac{1}{\sqrt{1-a^2}\sqrt{2\pi}} \exp\left\{ \frac{-1}{2(1-a^2)} \left[x_2 - 2 - a(x_1 - 1) \right]^2 \right\} \end{aligned}}$$