Complex Variables & Linear Algebra (BMAT201L)

Module 1
Analytic Functions

Dr. T. Phaneendra

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1 Differentiability and Analyticity

Let $S \subset \mathbb{C}$ be a path-connected open set and $z_0 \in S$. A mapping $f : S \to \mathbb{C}$ is said to be *differentiable* at z_0, if $f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$, provided this limit exists, and $f'(z_0)$ is called the *derivative* of f at z_0.

Limit Test for Nonexistence of the derivative

If $L = \frac{f(z)-f(z_0)}{z-z_0}$ is different along different paths of approach as $z \to z_0$, then the derivative $f'(z_0)$ of f does not exist at z_0.

Example 1.1. Consider $f(z) = f(x+iy) = x$ for all $z \in S$. Let

$$L = \frac{f(z)-f(0)}{z-0} = \frac{x}{x+iy}.$$

Along the real axis, $y = 0$ and $x \to 0$ as $z \to 0$. Thus $L = x/x = 1$, while along the imaginary axis, $x = 0$ and $y \to 0$ as $z \to 0$. Thus $L = x/iy = 0$. That is, L is different along different paths of approach as $z \to 0$. Therefore, $f'(0)$ does not exist, and f is not differentiable at $z = 0$.

Cauchy-Riemann equations

Theorem 1.1. Consider $f(z) = u(z) + iv(z)$ and $z_0 = (x_0, y_0)$. If $f'(z_0)$ exists, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ at } (x_0, y_0). \tag{1.1}$$

Conditions (1.1) are called the *Cauchy-Riemann* equations.

Remark 1.1. If the Cauchy-Riemann equations are not satisfied at a point, f cannot be differentiable at that point. Thus the conditions (1.1) are *necessary* for a function $f(z)$ to be differentiable at a point z_0.

Example 1.2. The complex function $f(z) = x - iy$ is nowhere differentiable.

Exercise 1.1. Show that $f(z) = \mathrm{Re}(z)$ is everywhere differentiable as a real variable function, but nowhere differentiable as a complex variable function.

Remark 1.2. Cauchy-Riemann equations (1.1) are not sufficient for the differentiability of f at a point z_0. That is, a complex function f may satisfy the Cauchy-Riemann equations at a point, without being differentiable at that point.

Example 1.3. Consider

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z} = \frac{\bar{z}^3}{|z|^2} = \left(\frac{x^3-3xy^2}{x^2+y^2}\right) + i\left(\frac{y^3-3x^2y}{x^2+y^2}\right) & \text{if } z \neq 0 \\ 0, & \text{if } z = 0. \end{cases}$$

Then

$$\frac{\partial u}{\partial x}(0,0)=\lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x-0}=\lim_{x\to 0}\frac{1}{x}\left(\frac{x^3-0}{x^2+0}\right)=1.$$

Similarly, $\frac{\partial u}{\partial y}(0,0)=0$, $\frac{\partial v}{\partial x}=0$ and $\frac{\partial v}{\partial y}(0,0)=1$. Thus the Cauchy-Riemann equations hold good at the origin. Write

$$L=\frac{f(z)-f(0)}{z-0}=\frac{\bar{z}^2}{z^2}.$$

Along the line $y=mx$: $z=(1+im)x$ and $L=\frac{(1-im)^2}{(1+im)^2}$. Thus L is different for different m-values, that is L is different along different linear paths $y=mx$ of approach as $z\to 0$. Hence, f is not differentiable at 0.

Exercise 1.2. Show that each of the following functions $f(z)$ satisfies the Cauchy Riemann equations at the origin, but $f'(0)$ does not exist:

$$(a)\quad f(z)=\begin{cases}\dfrac{xy^2(x+\mathrm{i}y)}{x^2+y^4}, & z\neq 0\\ 0, & z=0\end{cases}$$

$$(b)\quad f(z)=\begin{cases}\dfrac{x^2y^5(x+\mathrm{i}y)}{x^4+y^{10}}, & z\neq 0\\ 0, & z=0\end{cases}$$

$$(c)\quad f(z)=\begin{cases}\dfrac{x^3y(y-\mathrm{i}x)}{x^6+y^2}, & z\neq 0\\ 0, & z=0\end{cases}$$

$$(d)\quad f(z)=\begin{cases}\dfrac{x^3-y^3}{x^2+y^2}+\mathrm{i}\left(\dfrac{x^3+y^3}{x^2+y^2}\right), & z\neq 0\\ 0, & z=0\end{cases}$$

$$(e)\quad f(z)=\begin{cases}\dfrac{xy}{x^2+y^2}, & z\neq 0\\ 0, & z=0\end{cases}$$

$$(f)\quad f(z)=\begin{cases}\dfrac{x^3y^2}{(x^2+y^2)^2}, & z\neq 0\\ 0, & z=0\end{cases}$$

Theorem 1.2. Consider $f(z)=u(z)+iv(z)$ and $z_0=(x_0,y_0)\in\mathbb{C}$. Suppose that

(a) the partial derivatives $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous at (x_0,y_0), and

(b) satisfy the Cauchy-Riemann equations: $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$.

Then f will be differentiable at z_0, and its *derivative* is given by

$$f'(z_0)=\frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0). \tag{1.2}$$

Cauchy-Riemann Equations in Polar Form

If $f(z) = u(r,\theta) + iv(r,\theta)$ is differentiable at $z = re^{i\theta} \neq 0$, then

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}, \text{ and} \tag{1.3}$$

$$f'(z) = (\cos\theta - i\sin\theta)\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{\cos\theta - i\sin\theta}{r}\left(\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right) \tag{1.4}$$

Example 1.4. Using the polar Cauchy-Riemann equations, let us find the derivative of $f(z) = z^n$, where $n \geq 2$ at $z \neq 0$.

Solution. Let $z = re^{i\theta} \neq 0$. Then $r \neq 0$, and

$$f(z) = z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta) = u + iv$$

so that $u = r^n \cos n\theta$ and $v = r^n \sin n\theta$, and

$$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta, \frac{\partial u}{\partial \theta} = -nr^n \sin n\theta$$
$$\frac{\partial v}{\partial r} = nr^{n-1}\sin n\theta, \frac{\partial v}{\partial \theta} = nr^n \cos n\theta.$$

We find that

$$f'(z) = \frac{\cos\theta - i\sin\theta}{r}\left(\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right)$$

$$\begin{aligned}
&= \frac{\cos\theta - i\sin\theta}{r}\left(nr^n \cos n\theta + inr^n \sin n\theta\right)\\
&= nr^{n-1}\left[(\cos n\theta\cos\theta + \sin n\theta\sin\theta) + i(\sin n\theta\cos\theta - \cos n\theta\sin\theta)\right]\\
&= nr^{n-1}\left[\cos(n-1)\theta + i\sin(n-1)\theta\right]\\
&= nr^{n-1}e^{i(n-1)\theta} = nz^{n-1}, z \neq 0.
\end{aligned}$$

Analytic Function

A function f is said to be *analytic* at a point $z_0 \in S$, if it is differentiable on some neighborhood of z_0. A function $w = f(z)$ is *analytic* on S, if and only if it is analytic at every point of S.

Remark 1.3. Everywhere differentiable functions are analytic everywhere on $\mathbb{C}$. Conversely, everywhere analytic functions are everywhere differentiable. A function which is analytic throughout the complex plane $\mathbb{C}$ is called an *entire* function.

Example 1.5. The exponential mapping e^z is entire

Solution. We have

$$e^z = e^{x+iy} = e^x.e^{iy} = e^x(\cos y + \mathrm{i}\sin y) = u + iv, \tag{1.5}$$

so that $u = e^x \cos y$ and $v = e^x \sin y$. Then $\frac{\partial u}{\partial x} = e^x \cos y$, $\frac{\partial u}{\partial y} = -e^x \sin y$, $\frac{\partial v}{\partial x} = e^x \sin y$, $\frac{\partial v}{\partial y} = e^x \cos y$. Thus at every point $z = (x, y)$ the Cauchy-Reimann equations (1.1) hold good. Also, these first order partial derivatives are known to be continuous in the entire complex plane. Thus, by Theorem 1.2, $f = u + iv$ is analytic throughout the complex plane and hence entire.

Example 1.6. The trigonometric sine $f(z) = \sin z$ is analytic in the entire complex plane

Solution. We have

$$\begin{aligned}\sin z &= \sin(x+iy) = \sin x \cos(iy) + \cos x \sin(iy)\\ &= \sin x \cosh y + i \cos x \sinh y = u + iv\end{aligned}$$

so that $u = \sin x \cosh y$ and $v = \cos x \sinh y$. Then $\frac{\partial u}{\partial x} = \cos x \cosh y$, $\frac{\partial u}{\partial y} = \sin x \sinh y$, $\frac{\partial v}{\partial x} = -\sin x \sinh y$, $\frac{\partial v}{\partial y} = \cos x \cosh y$. Thus at every point $z = (x, y)$ the Cauchy-Reimann equations (1.1) hold good. Also, these first order partial derivatives are known to be continuous in the entire complex plane. Thus, by Theorem 1.2, $f = u + iv$ is analytic throughout the complex plane. Also,

$$\begin{aligned}f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \cos x \cosh y + i[-\sin x \sinh y]\\ &= \cos x \cos(iy) - \sin x \sin(iy) = \cos(x+iy) = \cos z\end{aligned}$$

Example 1.7. The trigonometric cosine $f(z) = \cos z$ is an entire function

Solution. We have

$$\begin{aligned}\cos z &= \cos(x+iy) = \cos x \cos(iy) - \sin x \sin(iy)\\ &= \cos x \cosh y - i \sin x \sinh y = u + iv.\end{aligned}$$

Then proceed as in the previous lines.

Example 1.8. Identify the real and imaginary parts of $f(z) = e^{z^2}$, and show that f is everywhere differentiable

Let $z = x + iy$. Then $z^2 = (x+iy)^2 = x^2 - y^2 + i.2xy$, and

$$\begin{aligned}e^{z^2} &= e^{x^2-y^2+i.2xy} = e^{x^2-y^2}.e^{i.2xy}\\ &= e^{x^2-y^2}(\cos 2xy + i \sin 2xy) = u + iv\end{aligned}$$

so that $u = e^{x^2-y^2}\cos 2xy$ and $v = e^{x^2-y^2}\sin 2xy$ are real and imaginary parts of f. Since f is a composition of the entire functions e^z and z^2, it will also be analytic, and by the chain rule of differentiation, we have

$$f'(z) = \tfrac{d}{dz}\left(e^{z^2}\right) = e^{z^2}(2z) = 2ze^{z^2}$$

Example 1.9. Identify the real and imaginary parts of $f(z) = e^{e^z}$, and hence show that f is everywhere differentiable

We see that $e^{e^z} = e^{e^x\cos y}[\cos(e^x\sin y) + \mathrm{i}\sin(e^x\sin y)] = u + iv$, so that $u = e^{e^x\cos y}\cos(e^x\sin y)$ and $v = e^{e^x\cos y}\sin(e^x\sin y)$, and $f'(z) = e^z e^{e^z}$

Example 1.10. Show that $f(z) = e^{1/z}$ is analytic at all $z \neq 0$, and hence find its derivative

For $z \neq 0$, $e^{1/z} = e^{x/(x^2+y^2)}\left[\cos\left(\frac{y}{x^2+y^2}\right) - \mathrm{i}\sin\left(\frac{y}{x^2+y^2}\right)\right] = u + iv$ so that $u = e^{x/(x^2+y^2)}\cos\left(\frac{y}{x^2+y^2}\right)$ and $v = -e^{x/(x^2+y^2)}\sin\left(\frac{y}{x^2+y^2}\right)$

Example 1.11. The principal branch

$$F(z) = \log\sqrt{x^2+y^2} + i\tan^{-1}(y/x) \tag{1.6}$$

of the complex logarithm $f(z) = \log z$, is analytic at all points $z = x + iy \in \mathbb{C}$ with

$$\sqrt{x^2+y^2} > 0 \text{ and } -\pi < \tan^{-1}(y/x) < \pi. \tag{1.7}$$

Solution. Let $z = re^{i\theta}$ with $r > 0$ and $-\pi < \theta < \pi$. Then

$$F(z) = \log\sqrt{x^2+y^2} + i\tan^{-1}(y/x) = \log r + i\theta = u(r,\theta) + iv(r,\theta),$$

so that real and imaginary parts of F are $u(r,\theta) = \log r$ and $v(r,\theta) = \theta$ respectively. We find that

$$\tfrac{\partial u}{\partial r} = \tfrac{1}{r}, \tfrac{\partial u}{\partial \theta} = 0, \tfrac{\partial v}{\partial r} = 0, \tfrac{\partial v}{\partial \theta} = 1.$$

Thus u and v satisfy the Cauchy-Reimann equations (1.3). Since u, v and their first partial derivatives are continuous at all points $z = x + iy \in \mathbb{C}$ in the domain (1.7), it follows that F is analytic in this domain. Also, the derivative of F is given by

$$f'(z) = (\cos\theta - i\sin\theta)\left(\tfrac{\partial u}{\partial r} + i\tfrac{\partial v}{\partial r}\right) = e^{-i\theta}\left(\tfrac{1}{r} + i.0\right) = 1/re^{i\theta} = 1/z$$

Exercise 1.3. Identify the real and imaginary parts of $f(z) = (z^2-2)e^{-z}$, and show that f is entire. Then find $f'(z)$.

Exercise 1.4. Explain why $f(z) = 2z^2 - 3 - ze^z + e^{-z}$ and $g(z) = 3z^2 + 5z - 6i$ are entire

Exercise 1.5. Explain why $f(z) = x + \sin x \cosh y + i(y + \cos x \sinh y)$ is analytic at every point of the complex plane

Exercise 1.6. Show that each of the following functions $f(z)$ is entire, and find the derivative at each point z of the complex plane:

(a) $z^3 = x^3 - 3xy^2 + \mathrm{i}(3x^2y - y^3)$

(b) $4x^2 + 5x - 4y^2 + 9 + \mathrm{i}(8xy + 5y - 1)$

Exercise 1.7. Find the choice of the constants (*mentioned in brackets*) such that each of the following functions is entire:

(a) $f(z) = x + ay + \mathrm{i}(bx + cy), \quad (a, b, c)$

(b) $f(z) = (ax + y) + \mathrm{i}(3y - bx), \quad (a, b)$

(c) $f(z) = e^x[\cos ay + \mathrm{i}\sin(y + b)] + c, \quad (a, b, c)$

(d) $f(z) = x^2 + axy + by^2 + \mathrm{i}(cx^2 + dxy + y^2), \quad (a, b, c, d)$

(e) $f(z) = (3x - y + 5) + \mathrm{i}(ax + b - 3), \quad (a, b)$

Exercise 1.8. Show that each of the following functions $f(z)$ is analytic in an appropriate domain, and find its derivative in the domain:

(a) $\frac{x-1}{(x-1)^2+y^2} - \mathrm{i}\frac{y}{(x-1)^2+y^2}$

(b) $\frac{x^3+xy^2+x}{x^2+y^2} + \mathrm{i}\frac{x^2y+y^3-y}{x^2+y^2}$

Remark 1.4. Analyticity at a point implies the differentiability at a point, but the converse is not true. In fact, there are functions which are differentiable at a point but are nowhere analytic

Example 1.12. Consider $f(z) = |z|^2 = x^2 + y^2$ for all $z = (x, y) \in \mathbb{C}$. Then $u_x \equiv \frac{\partial u}{\partial x} = 2x, u_y \equiv \frac{\partial u}{\partial y} = 2y, v_x \equiv \frac{\partial v}{\partial x} = 0, v_y \equiv \frac{\partial v}{\partial y} = 0$, which are continuous functions and $u_x = v_y, v_x = -u_y$ only if $x = 0 = y$. That is, the Cauchy-Riemann equations are satisfied at $z = (0, 0)$ only. Thus, by Theorem 1.2, f is differentiable only at $z = 0$. Since the Cauchy-Riemann equations are not satisfied at $z \neq (0, 0)$, there exists no neighborhood of $(0, 0)$, in which f is differentiable, except at $(0, 0)$. That is, f is not analytic at 0.

Since the Cauchy-Riemann equations are not satisfied at the points $z \neq 0$, f is not differentiable and hence not analytic at $z = 0$. Thus f is nowhere analytic.

Example 1.13. If $f(z) = x^2y^2$ for all $z = (x,y) \in \mathbb{C}$, we see that

$$u_x \equiv \frac{\partial u}{\partial x} = 2xy^2, u_y \equiv \frac{\partial u}{\partial y} = 2x^2y, v_x \equiv \frac{\partial v}{\partial x} = 0, v_y \equiv \frac{\partial v}{\partial y} = 0,$$

which are continuous functions and

$$u_x = v_y, v_x = -u_y \quad \text{only if} \quad x = 0 \text{ or } y = 0.$$

That is, the Cauchy-Riemann equations are satisfied at all points on the coordinate axes. Then by Theorem 1.2, f will be differentiable on the coordinate axes. But there is no neighborhood of any of the points on the axes, on which f is differentiable. Hence it is nowhere analytic.

Exercise 1.9. Show that $f(z) = e^{i\bar{z}}$ and $g(z) = 2x + ixy^2$ are *nowhere differentiable and hence nowhere analytic*

Exercise 1.10 (HOT). Show that

$$f(z) = \cos x - i \sinh y$$
$$g(z) = (\bar{z} + 1)^3 - 3\bar{z}$$

are nowhere analytic

Exercise 1.11. Show that each of the following functions is differentiable only at the origin but nowhere analytic:

(a) $z^2\bar{z} = z\bar{z}^2$, $z\mathrm{Im}z$

(b) $|z|^2 = x^2 + y^2$, $x^2 - ixy$

Exercise 1.12. Show that each of the following functions is nowhere analytic but is differentiable along the indicated curve(s)

(a) $f(z) = x^2 + y^2 + i2xy$, the real axis

(b) $f(z) = 3x^2y^2 - i6x^2y^2$, the coordinate axes

(c) $f(z) = x^3 + 3xy^2 - x + i(y^3 + 3x^2 - y)$, the coordinate axes

Exercise 1.13 (HOT). If $f(z) = \begin{cases} e^{-1/z^4}, & z \neq 0 \\ 0, & z = 0, \end{cases}$ show that the Cauchy-Riemann equations are satisfied at every point of the complex plane but f is analytic only at all $z \neq 0$.

Exercise 1.14 (HOT). Prove or disprove the following statements:

(a) Everywhere differentiable functions are analytic everywhere

(b) If $f = u + iv$ satisfies the Cauchy-Riemann equations at every point of the complex plane, then it will be differentiable at at least one point of the complex plane

(c) Nowhere analytic function is nowhere differentiable

(d) If f is differentiable at a point of the complex plane, then it will be analytic at that point

(e) If $f = u + iv$ is analytic, then $f = v + iu$ is analytic

Exercise 1.15. If $f = u + iv$ is differentiable at a point, compute the Jacobian $J\left(\frac{u,v}{x,y}\right)$ in terms of the derivative $f'(z)$

Exercise 1.16 (HOT). If $f(z)$ is differentiable at a point z_0, show that

$$\left(\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}}\right) + \left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right) = \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0 \text{ at } z_0$$

Exercise 1.17 (HOT). Show that $f(z) = \sin(\bar{z})$ is nowhere differentiable and hence nowhere analytic

2 Applications

Theorem 2.1. If $f(z)$ is an analytic with constant modulus (that is, $|f(z)| = constant$) on a domain S, then $f(z)$ will also be constant on S.

Theorem 2.2. If $f(z)$ is an analytic with $f'(z) = 0$ on a domain S, then $f(z)$ will be constant on S.

Harmonic function: A function $\psi(x,y)$ which has continuous partial derivatives of second order on a domain S is said to be *harmonic*, if it satisfies the Laplace's equation $\nabla^2\psi \equiv \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$ on S.

Theorem 2.3. Let $f(z) = u(x,y) + iv(x,y)$ be analytic on a domain S. Then the real and imaginary parts $u \equiv u(x,y)$ and $v \equiv v(x,y)$ will be harmonic on S, that is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ and } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \text{ on } S.$$

Remark 2.1. If $f = u + iv$ is analytic, we say that the imaginary part v is a harmonic conjugate of the real part u of f. Note that v is a harmonic conjugate of u if and only if u is a harmonic conjugate of $-v$. This follows from the observation that $if = i(u + iv) = -v + iu$ is analytic whenever $f = u + iv$ is analytic. This property is referred to as the *antisymmetry* of harmonic conjugates.

Finding a Harmonic Conjugate

Let f be an analytic function with the real part u. Then u is harmonic. The imaginary part v of f is the conjugate harmonic of u, and is determined by integrating the total derivative

$$dv = \left(\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial v}{\partial y}\right)dy = -\left(\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy.$$

Thus the short-cut formula is

$$v \equiv v(x,y) = -\int_{y-\text{constant}} \left(\frac{\partial u}{\partial y}\right)dx + \int_{x-\text{free}} \left(\frac{\partial u}{\partial x}\right)dy + \text{constant} \tag{2.1}$$

Exercise 2.1. Given a harmonic function $u \equiv u(x,y)$, determine its harmonic conjugate $v \equiv v(x,y)$ and the analytic function $f(z) = u + iv$:

(a) $\cos x \cosh y$

(b) $y + e^x \cos y$, $e^{2xy}\sin(x^2 - y^2)$, $e^{x^2-y^2}\cos(2xy)$

(c) $x^2 - y^2$, $4xy^3 - 4x^3y + x$

(d) $\frac{x^2-y^2}{(x^2+y^2)^2}$, $x^2 - y^2 + \frac{x}{x^2+y^2}$

(e) $\frac{\sin 2x}{\cos 2x + \cosh 2y}$, $e^x(x\cos y - y\sin y)$

Remark 2.2. Let f be an analytic function with the imaginary part v. Then v is harmonic. The real part u of f is determined by the short-cut formula

$$u \equiv u(x,y) = \int_{y-\text{constant}} \left(\frac{\partial v}{\partial y}\right)dx - \int_{x-\text{free}} \left(\frac{\partial v}{\partial x}\right)dy + \text{constant} \tag{2.2}$$

Milne-Thomson method: Given the harmonic function $u(x,y)$, to find the corresponding analytic function f whose real part is u, first we write

$$f'(z) == \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}.$$

We replace x by z and y by 0 in this and then integrate w. r. t. z.

Exercise 2.2. Without finding the real parts, find the analytic functions whose imaginary parta are $e^x(x\sin y + y\cos y)$ and $\frac{x-y}{x^2+y^2}$, respectively.

Exercise 2.3 (HOT). Determine the analytic function $f(z) = u + iv$ where u and v satisfy the following relations:

(a) $u - v = \frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$

(b) $u + v = \frac{x-y}{x^2+4xy+y^2}$

Theorem 2.4 (Analytic Function and Orthogonal Curves)**.** Suppose that $f(z) = u(x,y) + iv(x,y)$ is analytic at a point P. Then the curves $u(x,y) = a$ and $v(x,y) = b$ are orthogonal at P.

Flow Problems: Consider the irrotational motion of a frictionless, incompressible fluid moving in planes parallel to the xy-plane (that is, planar flow). Suppose that $\mathbf{F}(x,y) = f_1\mathbf{i} + f_2\mathbf{j}$ is the two dimensional velocity field of a fluid particle. Then $\nabla \times \mathbf{F} = 0$ so that $\mathbf{F} = \text{grad } \phi$ for some scalar potential $\phi(x,y)$. Since the flow is incompressible, $\nabla.\mathbf{F} = 0$ or $\nabla.\nabla\phi = \nabla^2\phi = 0$, that is ϕ is harmonic. The scalar potential ϕ is called a *velocity potential*. It follows that there exists a harmonic conjugate $\psi(x,y)$ of ϕ such that $f(z) = \phi(x,y) + i\psi(x,y)$ is analytic, and the harmonic conjugate ψ is called the *stream function*. Since f is analytic, the level curves $\phi(x,v) = c_1$ and $\psi(x,y) = c_2$ are orthogonal. The level curves corresponding to the velocity potential ϕ are known as *equipotential* curves, while the level curves corresponding to the stream function are called the *stream lines*. Stream lines represent the actual paths along which the fluid particles will move. The flow pattern is fully represented by the analytic function $f(z) = \phi(x,y) + i\psi(x,y)$, called the *complex potential*. In electrostatics and gravitational fields, the curves $\phi = c_1$ and $\psi = c_2$ are equipotential lines and lines of force. In steady state heat flow problems, the curves $\phi = c_1$ and $\psi = c_2$ are isotherms and lines of flow or flux lines

Exercise 2.4 (HOT). You should be little clever in giving the conclusions for the following problems:

1. If $f(z)$ is an entire function, is $\overline{f(z)}$ analytic at at least one point of the complex plane? If both $f(z)$ and $\overline{f(z)}$ are analytic, what can you say about f?
2. If u and v are harmonic functions, can $f = u + iv$ be analytic? Justify!
3. If $f = u + iv$ is analytic where u is a harmonic conjugate of v, what can you say about f?
4. If $f = u + iv$ is analytic, prove that $U = e^u \cos v$ and $V = e^x \sin v$ are harmonic conjugates of each other.
5. If $u(x,y)$ is a harmonic function and $v(x,y)$ is its harmonic conjugate, show that $\phi(x,y) = u(x,y)v(x,y)$ and $\psi(x,y) = u^2 - v^2$ are harmonic.