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1. For $Y_1 = \max(X_1, \dots X_n)$.

$$cdf(x) = P(X_1 \leq x, X_2 \leq x, \dots, X_N \leq x)$$

As all X_i are independent random variables

$$= P(X_1 \leq x) . P(X_2 \leq x) . P(X_3 \leq x) \dots P(X_N \leq x)$$

$$= [F_X(x)]^N$$

$$pdf(x) = \frac{d}{dx}(cdf(x))$$

$$= N[F_X(x)]^{N-1} f_X(x).$$

For $Y_2 = \min(X_1, \dots X_N)$

$$cdf(x) = 1 - P(X_1 \geq x, X_2 \geq x \dots X_N \geq x).$$

As all X_i are independent random variables

$$= 1 - P(X_1 \geq x) . P(X_2 \geq x) \dots P(X_N \geq x)$$

$$= 1 - \prod (1 - P(X_i \leq x))$$

$$= 1 - [1 - F_X(x)]^N$$

$$pdf(x) = \frac{d}{dx} cdf(x)$$

$$= -N[1-F_x(x)]^{N-1} \cdot -f_x(x)$$

$$= N f_x(x)[1-F_x(x)]^{N-1}$$

3. $P(X-\mu \geq \tau)$.

Let's take $Y = X - \mu$.

$= P(Y \geq \tau)$.

$= P(Y+b \geq \tau+b) \leq P((Y+b)^2 \geq (\tau+b)^2)$

as the domain of Y increases for which the inequality holds

By Markov's inequality:

$$P((Y+b)^2 \geq (\tau+b)^2) \leq \frac{E((Y+b)^2)}{(\tau+b)^2}$$

Linearity of expectation

$$= \frac{E(Y^2) + 2bE(Y) + \cancel{b^2}E(b^2)}{(\tau+b)^2}$$

$$= \frac{\sigma^2 + 0 + b^2}{(\tau+b)^2}$$

$$= \frac{\sigma^2 + b^2}{(\tau + b)^2}$$

We need to find such a value of b such that the term becomes minimum.

$$f(b) = \frac{\sigma^2 + b^2}{(\tau + b)^2}$$

$$f'(b) = \frac{(\tau+b)^2 (2b) - 2(\sigma^2 + b^2)(\tau + b)}{(\tau+b)^4} = 0$$

$$\cancel{2}b(\tau+b) - \cancel{2}\cancel{(\tau+b)}(\sigma^2+b^2) = 0$$

$$b\tau = \sigma^2$$

$$b = \frac{\sigma^2}{\tau}$$

$$f(b) = \frac{\sigma^2 + \frac{\sigma^4}{\tau^2}}{\left(\tau + \frac{\sigma^2}{\tau}\right)^2}$$

$$= \frac{\sigma^2 \tau^2 + \sigma^4}{(\tau^2 + \sigma^2)^2}$$

$$= \frac{\sigma^2}{\sigma^2 + \tau^2}$$

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For $\tau < 0$

Let $\gamma = -\tau$

$$P(X-\mu<\tau) = P(-Y>\gamma) \leq \frac{\sigma^2}{\gamma^2+\sigma^2}$$

Taking the complement of above set, we get

$$P(X-\mu>\tau) \geqslant 1-\frac{\sigma^2}{\gamma^2+\sigma^2} = 1-\frac{\sigma^2}{\tau^2+\sigma^2}$$

Q2 $X \sim \sum_{i=1}^{K} p_i N(\mu_i, \sigma_i^2)$

Since $X = X_i$ with probability p_i

$$\phi_x(t) = \sum_{i=1}^{K} p_i \phi_{x_i}(t)$$

$$= \sum_{i=1}^{K} p_i e^{(\mu_i t + \sigma_i^2 t^2/2)}$$

$$\phi_x'(0) = E(x) = \frac{\partial}{\partial t}\left[\sum_{i=1}^{K} p_i (e^{\mu_i t + \sigma_i^2 t^2/2})\right]\Bigg|_{t=0}$$

$$\boxed{E(x) = \sum_{i=1}^{K} p_i \mu_i}$$

$$\phi_x''(0) = E(x^2) = \frac{\partial^2}{\partial t^2}\left[\sum_{i=1}^{K} p_i e^{(\mu_i t + \sigma_i^2 t^2/2)}\right]\Bigg|_{t=0}$$

$$= \sum_{i=1}^{K} \frac{\partial}{\partial t}\left[p_i (\mu_i + \sigma_i^2 t) e^{(\mu_i t + \sigma_i^2 t^2/2)}\right]$$

$$= \sum_{i=1}^{K} \left[p_i (\mu_i + \sigma_i^2 t)^2 e^{(\mu_i t + \sigma_i^2 t^2/2)} + p_i \sigma_i^2 e^{(\mu_i t + \sigma_i^2 t^2/2)}\right]_{t=0}$$

$$= \sum_{i=1}^{K} p_i(\mu_i^2 + \sigma_i^2)$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= \sum_{i=1}^{K} p_i (\mu_i^2 + \sigma_i^2) - \left(\sum p_i \mu_i\right)^2$$

$$\phi_Z(t) = E(e^{tX})$$

$$= E(e^{t\sum p_i X_i})$$

as all X_i are independent

$$\cancel{E(e}$$

$$= \prod_{i=1}^{N} E(e^{t p_i X_i})$$

$$= \prod (e^{p_i \mu_i t + p_i^2 \sigma_i^2 t^2/2})$$

$$\phi_Z(t) = e^{\left(\sum_{i=1}^{N} \mu_i p_i\right) t + \sum_{i=1}^{N} (\sigma_i^2 p_i^2) t^2/2}$$

As pdf is unique for a given moment generating function.

$$PDF(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$$

Where:

$$\mu = \sum_{i=1}^{N} \mu_i p_i$$

$$\sigma^2 = \sum_{i=1}^{N} \sigma_i^2 p_i^2$$

as $MGF(Y) \sim N(\mu, \sigma^2)$ whose PDF is the one given above.

Since $\phi_Z(t) = e^{\sum \mu_i p_i t + (\sum \sigma_i^2 p_i^2) t^2/2}$

$\phi_Z'(t)\big|_{t=0} = E(Z) = \sum \mu_i p_i$

$\phi_Z''(t) = \cancel{E(Z^2)} = (\sum \mu_i p_i)^2 e^{\sum \mu_i p_i t + \sum \sigma_i^2 p_i^2 t^2/2}$
$+ (\sum \sigma_i^2 p_i^2) e^{\sum \mu_i p_i t + \sum \sigma_i^2 p_i^2 t^2/2}$

$\phi_Z''(t)\big|_{t=0} = E(Z^2) = (\sum \mu_i p_i)^2 + (\sum \sigma_i^2 p_i)^2$

$Var(Z) = E(Z^2) - [E(Z)]^2$
$= \sum \sigma_i^2 p_i^2$

Date:

Q.4 For $t > 0$

using Markov's inequality $\quad X \geq x \Rightarrow tX \geq tx \; [t>0]$

$$P(X \geq x) = P(e^{tX} \geq e^{tx}) \leq \frac{E(e^{tX})}{e^{tx}} = \phi_X(t)\, e^{-tx}$$

for $t < 0$

$$X \leq x \Rightarrow tX \geq tx$$

$$P(X \leq x) = P(e^{tX} \geq e^{tx}) \leq \frac{E(e^{tX})}{e^{tx}} = \phi_X(t) e^{-tx}$$

$$P(X > (1+\delta)\mu) \leq e^{-t(1+\delta)\mu} \phi_X(t) \quad \text{[By above inequality]}$$

$$= \frac{\phi_X(t)}{e^{(1+\delta)t\mu}} = \frac{E(e^{t\Sigma X_i})}{e^{(1+\delta)t\mu}}$$

$$= \frac{\prod_i E(e^{tX_i})}{e^{(1+\delta)t\mu}} \quad \text{[Since } X_i \text{ are iid]}$$

$$= \frac{\prod (p_i e^t + 1 - p_i)}{e^{(1+\delta)t\mu}}$$

using $1 + x \leq e^x$

$$\leq \frac{\prod e^{(e^t-1)p_i}}{e^{(1+\delta)t\mu}}$$

$$= \frac{e^{\Sigma p_i (e^t - 1)}}{e^{(1+\delta)t\mu}} = \frac{e^{\mu(e^t-1)}}{e^{(1+\delta)t\mu}}$$

Since

$P(X > (1+\delta)\mu) \leq \dfrac{e^{\mu(e^t-1)}}{e^{(1+\delta)t\mu}}$ holds $\forall\, t \geq 0$

It will also hold for $t = t_0$ which minimises R.H.S.

$$\frac{\partial}{\partial t} \frac{e^{\mu(e^t-1)}}{e^{(1+\delta)t\mu}} = 0 \Rightarrow \frac{\partial}{\partial t}\left[\mu(e^t - 1) - (1+\delta)t\mu\right] = 0$$

$\boxed{t = \log(1+\delta)}$ minimises R.H.S. $\Leftarrow$ $\mu e^t - \mu(1+\delta) = 0$

At $t = \log(1+\delta)$

$$P(X > (1+\delta)\mu) \leq \frac{e^{\delta\mu}}{e^{(1+\delta)\mu \log(1+\delta)}}$$

$$\left[\frac{\partial^2}{\partial t^2} \frac{e^{\mu(e^t - 1)}}{e^{(1+\delta)t\mu}} = (+ve) \frac{\partial}{\partial t} \left[\mu e^t - \mu(1+\delta)\right] = (+ve)\, \mu e^t > 0 \right.$$

hence $t = \log(1+\delta)$
is minima$\left.\right]$.

Q.6 When the two images I_1 & I_S' are aligned fully (I_S' is I_2 with shift along x-axis as S) i.e. S=0, then both the images are highly dependent on each other which is reflected in QMI plot but since correlation coefficient uses only mean & variance of intensities (much of the information of PMF not used), we don't see the same behaviour in correlation plot as in QMI plot.

But when I_2 is negative of I_1 then both, QMI & correlation coefficient show the same behaviour in terms of dependence between I_1 & I_2 values.

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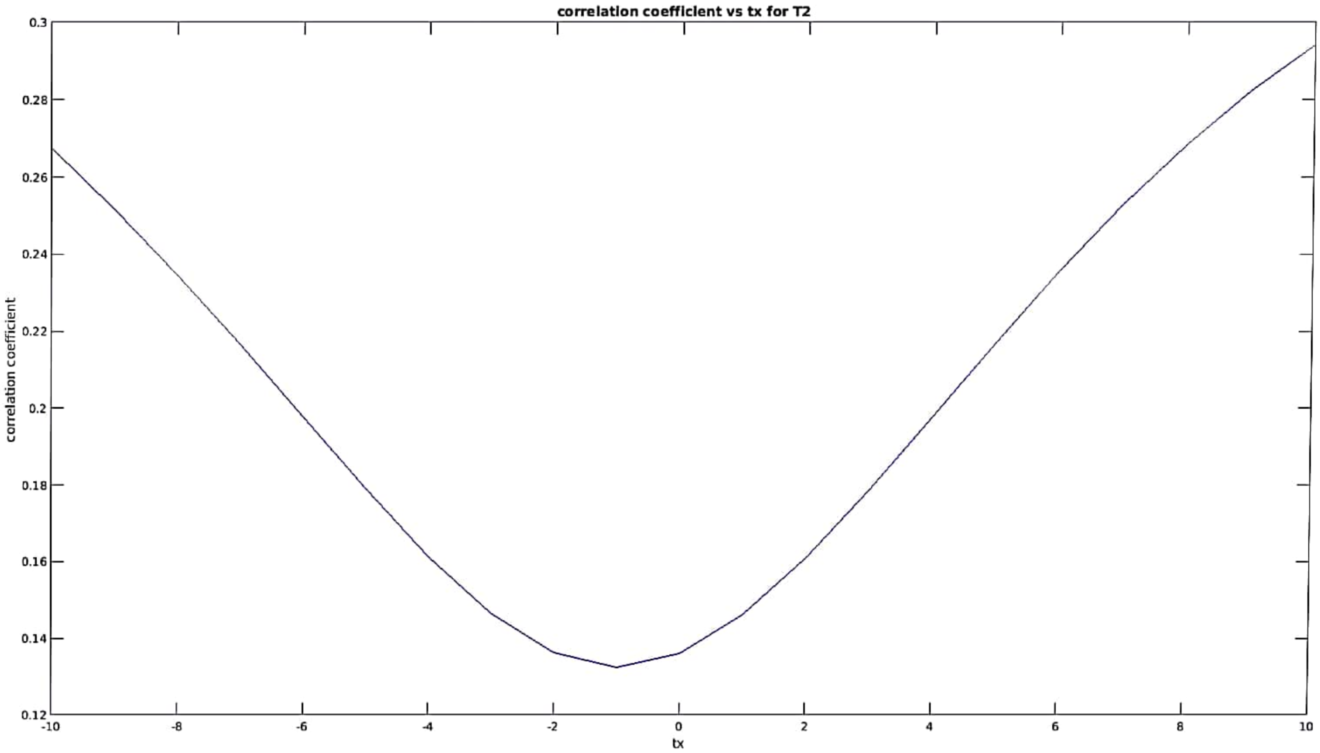
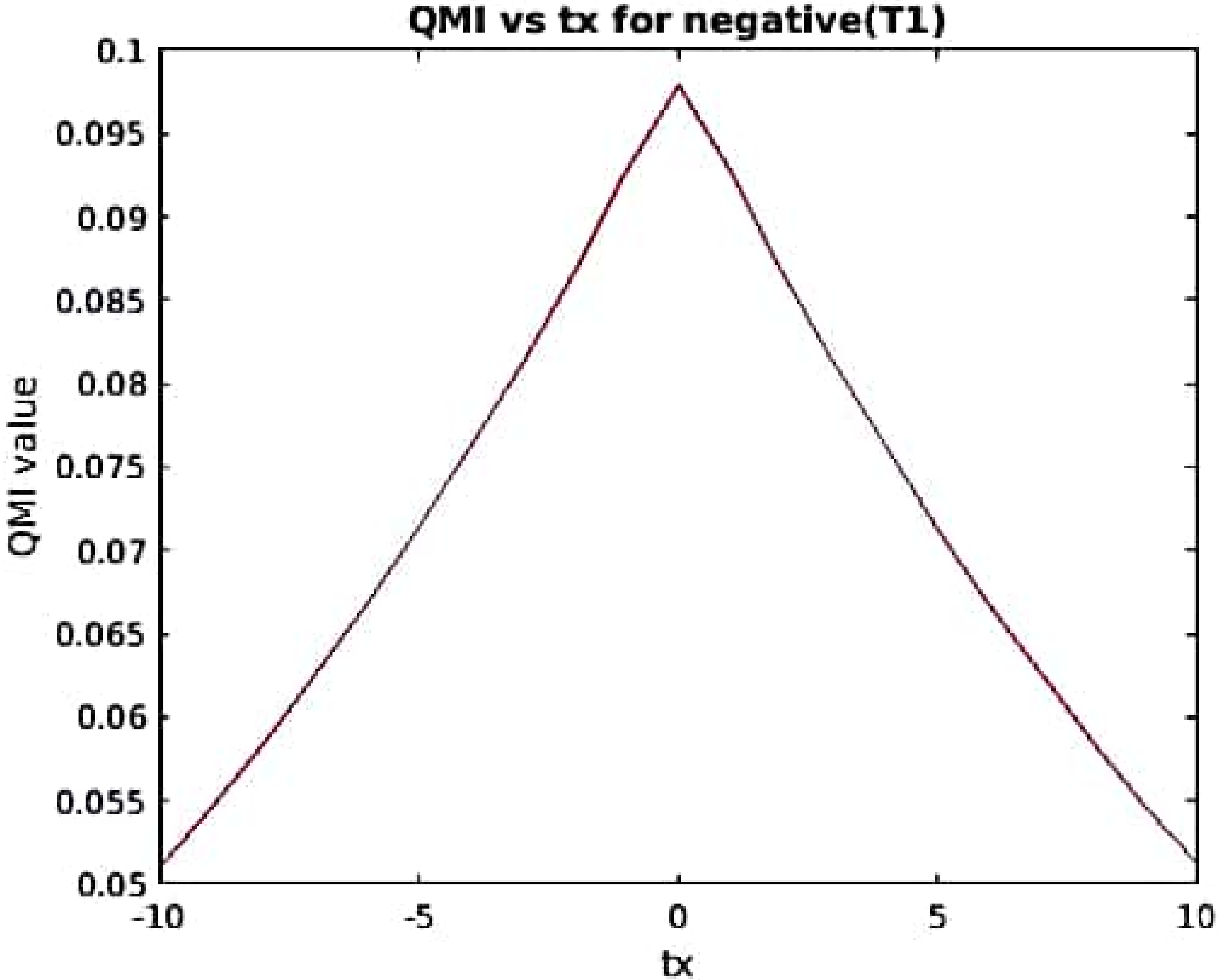
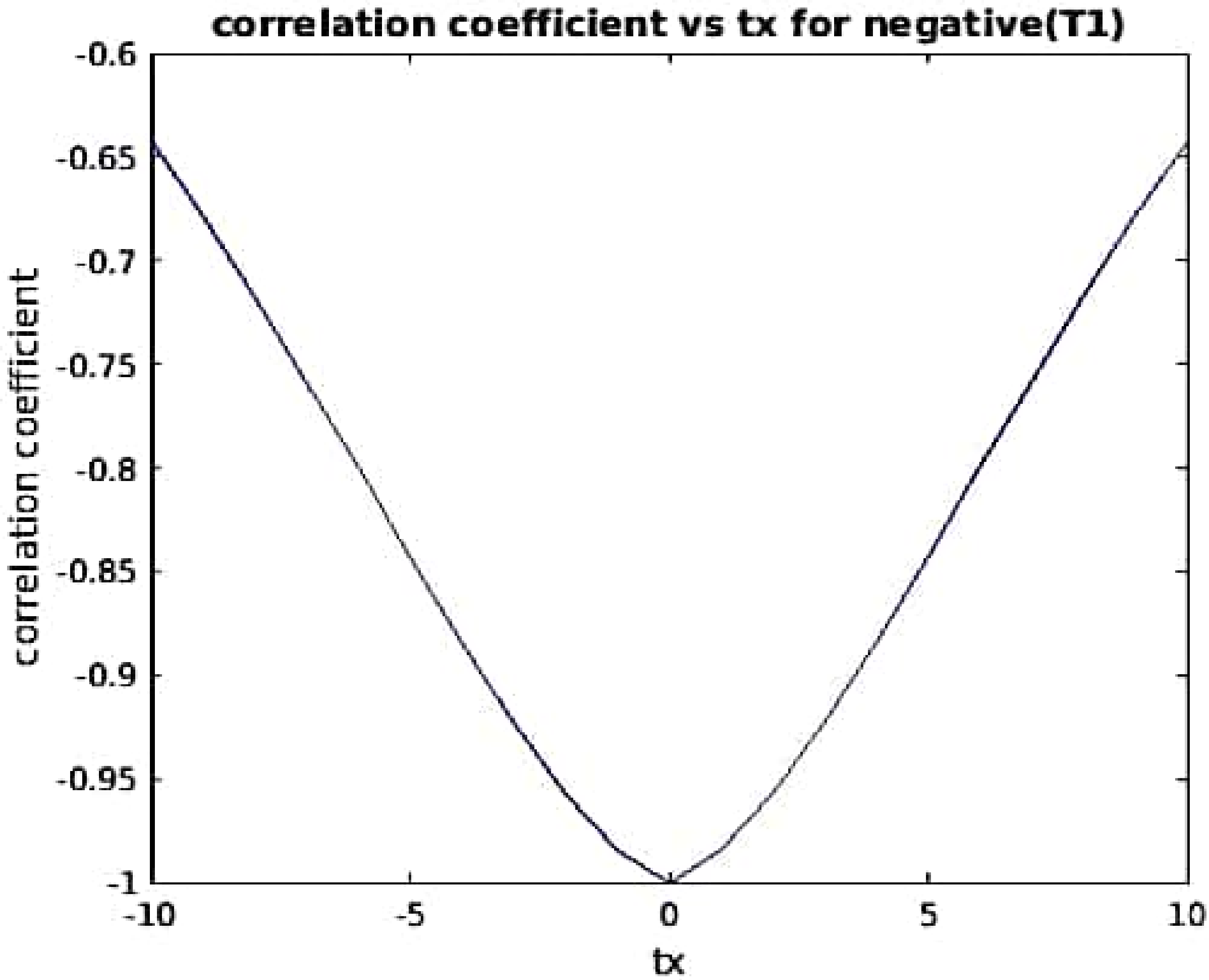
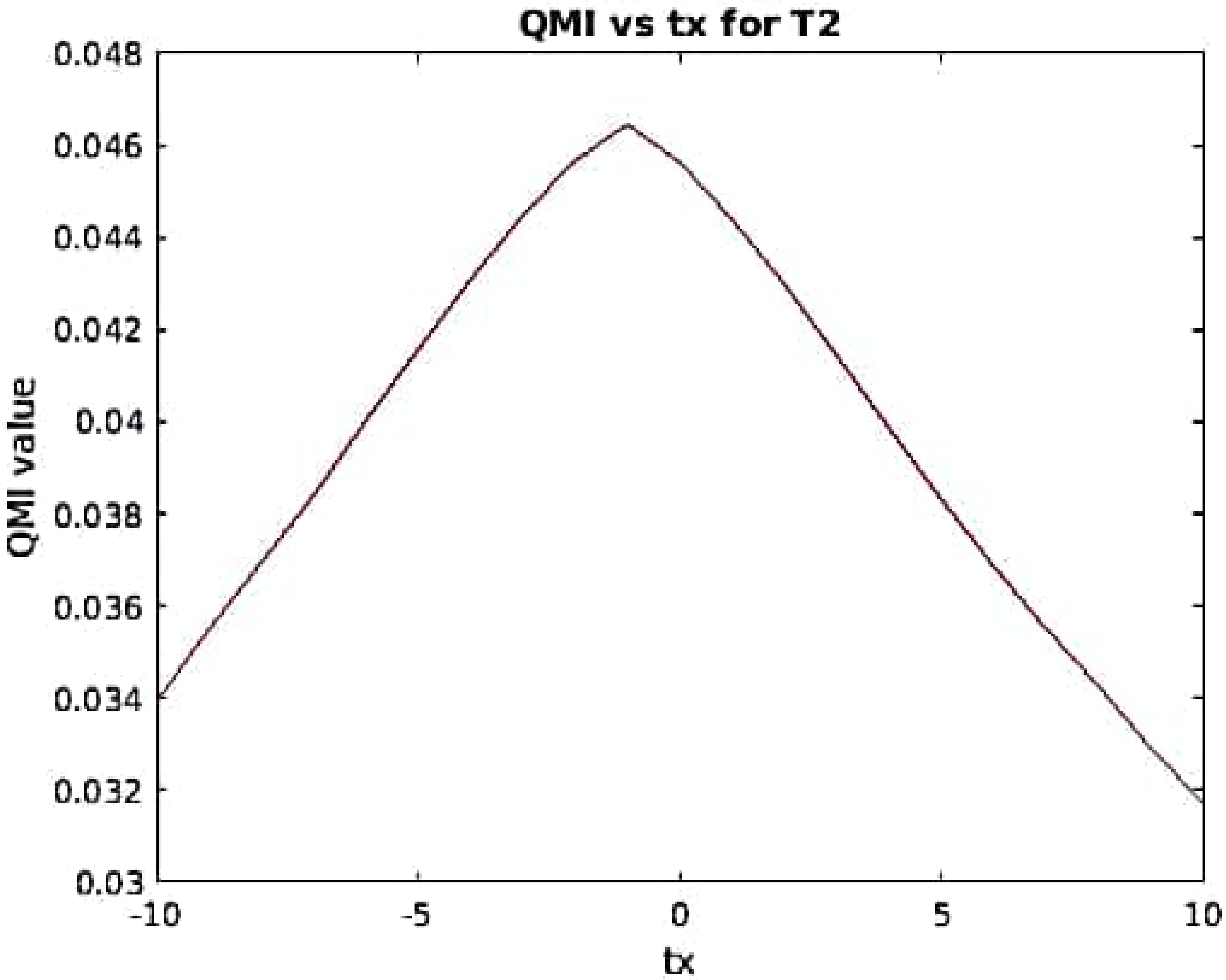


correlation coefficient vs tx for T2
correlation coefficient
tx
0.3
0.28
0.26
0.24
0.22
0.2
0.18
0.16
0.14
0.12
-10
-8
-6
-4
-2
0
2
4
6
8
10


QMI vs tx for negative(T1)
QMI value
tx
0.1
0.095
0.09
0.085
0.08
0.075
0.07
0.065
0.06
0.055
0.05
-10
-5
0
5
10


correlation coefficient vs tx for negative(T1)
correlation coefficient
tx
-0.6
-0.65
-0.7
-0.75
-0.8
-0.85
-0.9
-0.95
-1
-10
-5
0
5
10


QMI vs tx for T2
0.048
0.046
0.044
0.042
0.04
0.038
0.036
0.034
0.032
0.03
QMI value
-10
-5
0
5
10
tx