Solutions Manual to

AN INTRODUCTION TO MATHEMATICAL FINANCE: OPTIONS AND OTHER TOPICS

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1.1 (a) $1 - p_0 - p_1 - p_2 - p_3 = 0.05$ (b) $p_0 + p_1 + p_2 = 0.80$

1.2 $P\{C \cup R\} = P\{C\} + P\{R\} - P\{C \cap R\} = 0.4 + 0.3 - 0.2 = 0.5$

1.3 (a) $\frac{8}{14}\frac{7}{13} = \frac{56}{182}$ (b) $\frac{6}{14}\frac{5}{13} = \frac{30}{182}$ (c) $\frac{6}{14}\frac{8}{13} + \frac{8}{14}\frac{6}{13} = \frac{96}{182}$

1.4 (a) 27/58 (b) 27/35

1.5

1. The probability that their child will develop cystic fibrosis is the probability that the child receives a CF gene from each of his parents, which is 1/4.

2. Given that his sibling died of the disease, each of the parents much have exactly one CF gene. Let A denote the event that he possesses one CF gene and B that he does not have the disease (since he is 30 years old). Then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{2/4}{3/4} = \frac{2}{3}$$

1.6 Let A be the event that they are both aces and B the event they are of different suits. Then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{\frac{4}{52}\frac{3}{51}}{\frac{39}{51}} = \frac{1}{169}$$

1.7

$$\begin{aligned}(a)\ \ P(AB^c) &= P(A) - P(AB) \\ &= P(A) - P(A)P(B) \\ &= P(A)(1 - P(B)) \\ &= P(A)P(B^c)\end{aligned}$$

Part (b) follows from part (a) since from (a) A and B^c are independent, implying from (a) that so are A^c and B^c.

1.8 If the gambler loses both the bets, then $X = -3$. If he wins the first bet, or loses the first bet and wins the second bet, $X = 1$. Therefore,

$$\begin{aligned}P\{X = -3\} &= (\frac{20}{38})^2 = \frac{100}{361} \\ P\{X = 1\} &= \frac{18}{38} + \frac{20}{38}\frac{18}{38} = \frac{261}{361}\end{aligned}$$

1. $P\{X > 0\} = P\{X = 1\} = \frac{261}{361}$

2. $E[X] = 1\frac{261}{361} - 3\frac{100}{361} = \frac{-39}{361}$

1.9

1. $E[X]$ is larger since a bus with more students is more likely to be chosen than a bus with less students.

2.

$$\begin{aligned} E[X] &= \frac{1}{152}(39^2 + 33^2 + 46^2 + 34^2) = \frac{5882}{152} \approx 38.697 \\ E[Y] &= \frac{1}{4}(39 + 33 + 46 + 34) = 38 \end{aligned}$$

1.10 Let N denote the number of sets played. Then it is clear that $P\{N = 2\} = P\{N = 3\} = 1/2$.

1. $E[N] = 2.5$

2. $\text{Var}(N) = \frac{1}{2}(2 - 2.5)^2 + \frac{1}{2}(3 - 2.5)^2 = \frac{1}{4}$

1.11 Let $\mu = E[X]$.

$$\begin{aligned} \text{Var}(X) &= E[(X - \mu)^2] \\ &= E[X^2 - 2\mu X + \mu^2] \\ &= E[X^2] - 2\mu E[X] + \mu^2 \\ &= E[X^2] - \mu^2 \end{aligned}$$

1.12 Let F be her fee if she takes the fixed amount and X when she takes the contingency amount.

$$E[F] = 5,000, \quad SD(F) = 0$$

$$E[X] = 25,000(.3) + 0(.7) = 7,500$$

$$E[X^2] = (25,000)^2(.3) + 0(.7) = 1.875 \times 10^8$$

Therefore,

$$SD(X) = \sqrt{\text{Var}(X)} = \sqrt{1.875 \times 10^8 - (7,500)^2} = \sqrt{1.3125} \times 10^4$$

1.13

$$\begin{aligned} (a)\ E[\bar{X}] &= \frac{1}{n}\sum_{i=1}^{n} E[X_i] \\ &= \frac{1}{n} n\mu = \mu \end{aligned}$$

$$
\begin{aligned}
(b)\ \mathrm{Var}(\bar{X}) &= (\frac{1}{n})^2 \sum_{i=1}^{n} \mathrm{Var}(X_i) \\
&= (\frac{1}{n})^2 n\sigma^2 = \sigma^2/n
\end{aligned}
$$

$$
\begin{aligned}
(c)\ \sum_{i=1}^{n}(X_i - \bar{X})^2 &= \sum_{i=1}^{n}(X_i^2 - 2X_i\bar{X} + \bar{X})^2) \\
&= \sum_{i=1}^{n} X_i^2 - 2\bar{X}\sum_{i=1}^{n} X_i + n\bar{X}^2 \\
&= \sum_{i=1}^{n} X_i^2 - 2\bar{X}n\bar{X} + n\bar{X}^2 \\
&= \sum_{i=1}^{n} X_i^2 - n\bar{X}^2
\end{aligned}
$$

$$
\begin{aligned}
(d)\ E[(n-1)S^2] &= E[\sum_{i=1}^{n} X_i^2] - E[n\bar{X}^2] \\
&= nE[X_1^2] - nE[\bar{X}^2] \\
&= n(\mathrm{Var}(X_1) + E[X_1]^2) - n(\mathrm{Var}(\bar{X}) + E[\bar{X}]^2) \\
&= n\sigma^2 + n\mu^2 - n(\sigma^2/n) - n\mu^2 \\
&= (n-1)\sigma^2
\end{aligned}
$$

1.14

$$
\begin{aligned}
\mathrm{Cov}(X,Y) &= E\left[(X - E[X])(Y - E[Y])\right] \\
&= E\left[XY - XE[Y] - E[X]Y + E[X]E[Y])\right] \\
&= E[XY] - E[Y]E[X] - E[X]E[Y] + E[X]E[Y] \\
&= E[XY] - E[Y]E[X]
\end{aligned}
$$

1.15

$$
\begin{aligned}
(a)\ \mathrm{Cov}(X,Y) &= E\left[(X - E[X])(Y - E[Y])\right] \\
&= E\left[(Y - E[Y])(X - E[X])\right]
\end{aligned}
$$

$$
(b)\ \mathrm{Cov}(X,X) = E[(X - E[X])^2] = \mathrm{Var}(X)
$$

$$
\begin{aligned}
(c)\ \mathrm{Cov}(cX,Y) &= E\left[(cX - E[cX])(Y - E[Y])\right] \\
&= cE\left[(X - E[X])(Y - E[Y])\right] \\
&= c\mathrm{Cov}(X,Y)
\end{aligned}
$$

$$
(d)\ \mathrm{Cov}(c,Y) = E\left[(c - E[c])(Y - E[Y])\right] = 0
$$

1.16

$$\begin{aligned}
\mathrm{Cov}(aU+bV, cU+dV) &= \mathrm{Cov}(aU, cU+dV) + \mathrm{Cov}(bV, cU+dV) \\
&= \mathrm{Cov}(aU, cU) + \mathrm{Cov}(aU, dV) + \mathrm{Cov}(bV, cU) + \mathrm{Cov}(bV, dV) \\
&= ac(1) + ad(0) + bc(0) + bd(1) = ac + bd
\end{aligned}$$

1.17 With $c(i,j) = \mathrm{Cov}(X_i, X_j)$
(a) $c(1,3) + c(1,4) + c(2,3) + c(2,4) = 21$
(b) $2+3+4+4+6+8+6+9+12 = 54$

1.17 Let X_i be the amount it goes up in period i. Then

$$Y = \sum_{i=1}^{3} X_i$$

and

$$\mathrm{Cov}(X_1, Y) = \mathrm{Cov}(X_1, X_1) = \mathrm{Var}(X_1) = 1/4$$

Therefore,

$$\mathrm{Corr}(X, Y) = \frac{1/4}{\sqrt{(1/4)(3/4)}} = 1/\sqrt{3}$$

1.18 No, since for such a pair $\mathrm{Corr}(X, Y) = 2$, and correlations are always between -1 and 1.

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1.20

$$\begin{aligned}
\sum_y h(y)P(Y = y) &= \sum_i \sum_{y:h(y)=h_i} h(y)P(Y = y) \\
&= \sum_i \sum_{y:h(y)=h_i} h_i P(Y = y) \\
&= \sum_i h_i \sum_{y:h(y)=h_i} P(Y = y) \\
&= \sum_i h_i P(h(Y) = h_i)
\end{aligned}$$

1.21 $P(X = i) = F(i) - F(i - 1)$

2.1

1. $P(Z < -.66) = P(Z > .66) = 1 - P(Z < .66) = 1 - \Phi(.66) = 1 - .7454 = .2546$

2.

$$\begin{aligned} P(|Z| < 1.64) &= P(Z < 1.64) - P(Z < -1.64) \\ &= P(Z < 1.64) - [1 - P(Z < 1.64)] \\ &= 2\Phi(1.64) - 1 = 2 \times .9495 - 1 = .8990 \end{aligned}$$

3. $P(|Z| > 2.20) = 2P(Z > 2.20) = 2[1 - P(Z < 2.20)] = 2(1 - .9861) = .0278$

2.2 $x = 2$

2.3

$$P(|Z| > x) = P(Z > x \text{ or } Z < -x) = P(Z > x) + P(Z < -x) = 2P(Z > x)$$

where the last equality comes from the fact that Z is symmetric.

2.4 $a = 2\mu, \quad b = -1.$

$$\mathrm{Cov}(X, Y) = -\mathrm{Var}(X) = -\sigma^2$$

2.5
(a) 127.7 ± 19.2
(b) $127.7 \pm (1.96)(19.2)$
(c) 127.7 ± 57.6

2.6 Let X_1 and X_2 denote the life of the first and the second battery respectively. It is given that X_1 and X_2 are both normal random variables with mean 400 and standard deviation 50. Let Z denote a standard normal random variable.

1. $X_1 + X_2$ is a normal random variable with mean 800 and standard deviation $50\sqrt{2}$.

$$\begin{aligned} P(X_1 + X_2 > 760) &= P\Big(\frac{X_1 + X_2 - 800}{50\sqrt{2}} > \frac{760 - 800}{50\sqrt{2}}\Big) \\ &\approx P(Z > -.5657) = P(Z < .5657) \\ &= \Phi(.5657) \approx 0.7142 \end{aligned}$$

2. $X_2 - X_1$ is a normal random variable with mean 0 and standard deviation $50\sqrt{2}$.

$$\begin{aligned} P(X_2 - X_1 > 25) &= P\Big(\frac{X_2 - X_1}{50\sqrt{2}} > \frac{25}{50\sqrt{2}}\Big) \\ &\approx P(Z > .3536) = 1 - \Phi(.3536) \\ &\approx 1 - .6382 = .3618 \end{aligned}$$

3. $P(|X_1 - X_2| > 25) = 2P(X_2 - X_1 > 25) \approx .7236$

2.7 Let X_i be the time the that it takes to develop the i^{th} print. Then, the time that it takes to develop 100 prints, call it X, can be expressed as

$$X = \sum_{i=1}^{100} X_i$$

It follows from the central limit theorem that X approximately has a normal distribution with mean 1800 and standard deviation 10. Therefore,

$$\begin{aligned} P\{X > 1710\} &= P\{\frac{X - 1800}{10} > \frac{1710 - 1800}{10}\} \\ &= 1 - \Phi(-9) \\ &= \Phi(9) \approx 1 \end{aligned}$$

The probability for part (b) is 0

2.8 Let X_i be the mileage for person i, $i = 1, \ldots, 30$. From central limit theorem, $\sum_{i=1}^{30} X_i$ is approximately a normal random variable with mean 25000×30 and standard deviation $12000 \times \sqrt{30}$.

1. $P\Big(\frac{\sum_{i=1}^{30} X_i}{30} > 25000\Big) = P\Big(\frac{\sum_{i=1}^{30} X_i - 25000\times 30}{12000\times\sqrt{30}} > 0\Big) = \Phi(0) = 0.5$

2.

$$\begin{aligned} &P\Big(23000 < \frac{\sum_{i=1}^{30} X_i}{30} < 27000\Big) \\ &= P\Big(\frac{23000\times 30 - 25000\times 30}{12000\sqrt{30}} < \frac{\sum_{i=1}^{30} X_i - 25000\times 30}{12000\times\sqrt{30}} < \frac{27000\times 30 - 25000\times 30}{12000\times\sqrt{30}}\Big) \\ &= \Phi(5/\sqrt{30}) - \Phi(-5/\sqrt{30}) \\ &= 2\Phi(5/\sqrt{30}) - 1 \approx 2\times 0.8194 - 1 = .6388 \end{aligned}$$

2.9 Let S_i be the price of stock in time period i. Then $S_{i+1} = S_i X_i$ where the random variable X_i is defined as

$$X_i = \begin{cases} u & \text{, with probability } p \\ d & \text{, with probability } 1 - p \end{cases}$$

Then

$$P\Big(\frac{S_{1000}}{S_0} > 1.3\Big) = P\Big(\prod_{i=0}^{999} X_i > 1.3\Big) = P\Big(\sum_{i=0}^{999} \log X_i > \log 1.3\Big)$$

We can use the central limit theorem to approximate $\sum_{i=0}^{999} \log X_i$ with a normal random variable Y with the same mean and variance.

$$\begin{aligned} E[Y] &= 1000\, E[\log X_i] = 1000\,(p \log u + (1-p)\log d) \approx 1.3787 \\ \mathrm{Var}(Y) &= 1000\, \mathrm{Var}(\log X_i) \\ &= 1000\,(p(\log u)^2 + (1-p)(\log d)^2 - .0013787^2) \approx 0.1206 \end{aligned}$$

Therefore

$$\begin{aligned} P\Big(\sum_{i=0}^{999} \log X_i > \log 1.3\Big) &\approx P\Big(Y > \log 1.3\Big) \\ &= P\Big(\frac{Y - 1.3787}{\sqrt{.1206}} > \frac{\log 1.3 - 1.3787}{\sqrt{.1206}}\Big) \\ &\approx P(Z > -3.2146) = \Phi(3.2146) \approx .9993 \end{aligned}$$

where Z stands for a standard normal random variable.

2.10 Let X_i be the movement in period i. Then we can approximate $\sum_{i=1}^{700} X_i$ with a normal random variable Y with the same mean and variance

$$\begin{aligned} E[Y] &= E\Big[\sum_{i=1}^{700} X_i\Big] = 700(-.39 + .41) = 14 \\ \mathrm{Var}(Y) &= \mathrm{Var}\Big(\sum_{i=1}^{700} X_i\Big) = 700(.39 \times 1.02^2 + .20 \times .02^2 + .41 \times .98^2) = 559.72 \end{aligned}$$

Therefore,

$$P\Big(\sum_{i=1}^{700} X_i > 10\Big) \approx P(Y > 10.5) = P\Big(\frac{Y - 14}{\sqrt{559.72}} > \frac{10.5 - 14}{\sqrt{559.72}}\Big) \approx \Phi(.1479) \approx .5588$$

3.1 This follows since $-X(t+y) - (-X(y)) = -(X(t+y) - X(y))$ is the negative of a normal with mean μt and variance $\sigma^2 t$, and so has mean $-\mu t$ and variance $\sigma^2 t$.

3.2 (a) 16; (b) 18

$$\begin{aligned}(c)\ P(X(2) > 20) &= P(\frac{X(2)-16}{\sqrt{18}} > 20 - 16\sqrt{18}) \\ &= P(Z > .9428) \approx .173\end{aligned}$$

$$\begin{aligned}(d)\ P(X(.5) > 10) &= P(\frac{X(.5)-11.5}{\sqrt{14.5}} > \frac{10-11.5}{\sqrt{14.5}}) \\ &= P(Z > -.3939) \\ &= P(Z < .3939) \approx .655\end{aligned}$$

where Z is a standard normal.

3.3 (a) With $2p-1 = .3162$, $E[X(1)] = 10 + 10(.9487)(.3168) = 13.005$
(b) $\text{Var}(X(1)) = 9[1 - .09998] \approx 8.10018$
(c) $P(X(.5) > 10) = (.6581)^5 + 5(.6581)^4(.3419) + 10(.6581)^3(.3419)^2 = .7773$

3.4 (a) With W being normal with mean .1 and variance .04,

$$P(S(1) > S(0)) = P(e^W > 1) = P(W > 0) = P(Z > -.1/.2) = P(Z < .5) = .6915$$

(b) $(.6915)^2$
(c)

$$\begin{aligned}P(S(3) < S(1) > S(0)) &= P(S(1) > S(0))P(S(3) < S(1)|S(1) > S(0)) \\ &= P(S(1) > S(0))P(S(3) < S(1)) \\ &= .6915P(Z < -.2/\sqrt{.08})\end{aligned}$$

3.6 $S(t)/s$ is distributed as e^W, where W is normal with mean μt and variance $t\sigma^2$. Hence,

$$E[S(t)] = sE[e^W] = se^{\mu t + t\sigma^2/2}$$

and

$$E[S^2(t)] = s^2E[e^{2W}] = s^2e^{2\mu t + 2t\sigma^2}$$

Hence,

$$\mathrm{Var}(S(t)) = s^2 e^{2\mu t + 2t\sigma^2} - s^2 e^{2\mu t + t\sigma^2} = s^2 e^{2\mu t + t\sigma^2}(e^{t\sigma^2} - 1)$$

3.7 This follows directly from the formula for $P(T_y \leq t)$ given in the text, upon using that $\lim_{x\to\infty} \bar{\Phi}(x) = 0$ and $\lim_{x\to\infty} \bar{\Phi}(-x) = 1$. Hence, when $\mu < 0$,

$$P(M > y) = P(T_y < \infty) = e^{2y\mu/\sigma^2}, \quad y > 0$$

3.8 Using the representation that $S(t) = se^{X(t)}$, where $X(t), t \geq 0$ is Brownian motion with $X(0) = 0$, gives

$$P(\max_{0\leq v\leq t} S(v) \geq y) = P\left(\max_{0\leq v\leq t} X(v) \geq \log(y/s)\right)$$

Now use Corollary 1, setting $t = \log(y/s)$.

3.9 The desired probability is $P\left(M(t) < \log(1.2)\right)$ where $M(t)$ is the maximum by time t of a Brownian motion having $\mu = .1, \sigma = .3$ and $X(0) = 0$. Now, apply Corollary 1.

4.1

(a) $r_e = (1+0.1/2)^2 - 1 = 0.1025$

(b) $r_e = (1+0.1/4)^4 - 1 \approx 0.1038$

(c) $r_e = e^{0.1} - 1 \approx 0.1052$

4.2 Suppose it takes t years to double, then

$$e^{0.1t} = 2 \Rightarrow t = \frac{\log 2}{0.1} \approx 6.93$$

4.3 Suppose it takes t years to quadruple, then we can solve t from $1.05^t = 4$. We can also use the doulbing rule to approximate t, which gives

$$t \approx \frac{0.7}{0.05} \times 2 = 28$$

If the interest is 4%, then it is approximately $0.7/0.04 \times 2 = 35$ years.

4.4 Using that $e^r \approx 1+r$, when r is small, we see that if $(1+r)^n = 3$ then $e^{nr} \approx 3$. Thus,

$$n \approx \frac{\log(3)}{r} \approx \frac{1.1}{r}$$

4.5 Suppose you need to invest x at the beginning of each of the next 60 months to have a value of $\$100,000$ at the end of 60 months, then

$$100000 = x\sum_{i=1}^{60} 1.005^i = x\frac{1.005(1-1.005^{60})}{1-1.005}$$

Solve to get $x = 1426.15$.

4.6 Let's compute the present value, denoted by S, of this cash flow.

$$S = -1000 - \frac{1200}{1.06} + \frac{800}{1.06^2} + \frac{900}{1.06^3} + \frac{800}{1.06^4} = -30.75$$

Since it is negative, it is not worth investing.

4.7 (15 pts) Let the present value of the first cash flow sequence be S_1 and that of the second cash flow sequence be S_2. Then

$$\begin{aligned} S_1 &= \frac{20}{1+r} + \frac{20}{(1+r)^2} + \frac{20}{(1+r)^3} + \frac{15}{(1+r)^4} + \frac{10}{(1+r)^5} + \frac{5}{(1+r)^6} \\ S_2 &= \frac{10}{1+r} + \frac{10}{(1+r)^2} + \frac{15}{(1+r)^3} + \frac{20}{(1+r)^4} + \frac{20}{(1+r)^5} + \frac{20}{(1+r)^6} \end{aligned}$$

(a) $r = 0.03$, $S_1 = 82.71, S_2 = 84.63$. The second one is preferable.

(b) $r = 0.05$, $S_1 = 78.37, S_2 = 78.60$. The second one is preferable.

(c) $r = 0.1$, $S_1 = 69.01, S_2 = 65.99$. The first one is preferable.

4.8 (15 pts) Let S denote the present value, then

$$S = -10000 + \sum_{i=1}^{10} \frac{500}{(1+r/2)^i} + \frac{10000}{(1+r/2)^{10}}$$

(a) $r = 0.06$, $S = 1706.04$.

(b) $r = 0.10$, $S = 0$.

(c) $r = 0.12$, $S = -736.01$.

4.9 The effective interest rate, call it r, is that value for which

$$160(\frac{1}{1+r} + \frac{1}{(1+r)^2} + \ldots + \frac{1}{(1+r)^{24}} = 3200$$

which reduces to

$$\frac{1 - (\frac{1}{1+r})^2 4}{r} = 20$$

Solution by trial and error shows that $r \approx .015$. That is, the effective interest rate is 1.5 percent per month.

4.11 The cost-flow sequences are as follows

buy at beginning of year 1:	22	7	8	9	10	-4
buy at beginning of year 2:	9	25	7	8	9	-9
buy at beginning of year 3:	9	11	28	7	8	-14
buy at beginning of year 4:	9	11	13	31	7	-19

With the yearly interest rate 10%, the present value of the first cost-flow sequence is

$$22 + \frac{7}{1.1} + \frac{8}{1.1^2} + \frac{9}{1.1^3} + \frac{10}{1.1^4} - \frac{4}{1.1^5} = 46.08$$

Similarly, the prevent values of the other three cost-flow sequences can be determined, and the four present values are

$$46.08,\ 44.08,\ 44.17,\ 46.02$$

Therefore, the company should purchase a new machine one year from now.

4.12 Since the bank charges 2 points, the amount of money we receive for this loan is actually $120,000 \times 0.98 = 117,600$. The interest we need to pay per month is $120,000 \times 0.5\% = 600$. Therefore the cash flow sequence of this loan is

time (mths)	0	1	2	...	35	36
cash flow	117600	-600	-600	...	-600	-120600

Let r be the effective interest rate per month for this loan, then

$$\begin{aligned} 117600 &= \frac{600}{1+r} + \frac{600}{(1+r)^2} + \ldots + \frac{600}{(1+r)^{35}} + \frac{120600}{(1+r)^{36}} \\ &= \frac{600[1-(\frac{1}{1+r})^{35}]}{r} + \frac{120600}{(1+r)^{36}} \end{aligned}$$

We can solve the above to get $r \approx 0.5615\%$.

4.13 The present value of paying the entire amount of \$16,000 now is simply \$16,000, while the present value of paying \$10,000 now and another \$10,000 at the end of ten years is

$$S = 10,000 + 10,000(e^{-r})^{10}$$

Therefore

(a) $r = 0.02$, $S = 18,187.31$, which is not preferable.

(b) $r = 0.05$, $S = 16,065.31$, which is not preferable.

(c) $r = 0.10$, $S = 13,678.79$, which is preferable.

4.14 The cash flow sequence is as follows,

time (yrs)	0	0.5	1	...	4.5	5
cash flow	-1000	30	30	...	30	1030

With a continuously compounded interest 5%, the present value of above is

$$-1000 + \sum_{i=1}^{9} \frac{30}{(e^{0.05/2})^i} + \frac{1030}{(e^{0.05})^5} = 40.94$$

4.15 The present value of a cash flow of 1,000 at the end of 10 years with a continuously compounded interest rate 8% is

$$\frac{1000}{(e^{0.08})^{10}} = 449.33$$

4.16 The rate of return is the effective interest rate which makes the present value of the cash flow streams equal to the initial payment. Therefore

4.15 $(1 + 05/n)^n$ would be the amount on deposit after one year if 1 is initially deposited, the nominal interest rate is 5 percent, and interest is compounded n times in the year. The more times it is compounded the higher this amount should be.

4.16 The amount of interest earned after n days is $100(e^{.06n/365} - 1)$.

4.17 $1000e^{3r} + 2000e^{2r} + 3000e^{r}$

4.18 You would pay the present value of the string of payments:

$$\frac{3}{1.05} + \frac{5}{(1.05)^2} - \frac{6}{(1.05)^3} + \frac{5}{(1.05)^4} = 6.7444$$

4.19 When $20 + \frac{10}{1+r} > \frac{34}{1+r}$. That is, when $r > .2$.

4.20

$$104e^{-r} = 110e^{-2r}$$

giving that $e^r = 110/104$ or $r = \log(110/104) = .0561$

4.21 $Ae^{-rs} + \sum_{n=1}^{\infty} Ae^{-r(s+nt)} = Ae^{-rs} \sum_{n=0}^{\infty} (e^{-rt})^n = \frac{Ae^{-rs}}{1-e^{-rt}}$

4.22 (a) The interest earned by time $t + h$ on the interest earned in $(t, t + h)$ is of smaller order than h.
(b) From (a), we have

$$\frac{D(t+h) - D(t)}{h} \approx rD(t)$$

Letting $h \to 0$, the approximation becomes exact and we obtain that

$$D'(t) = rD(t)$$

(c) Integrating both sides of $\frac{D'(t)}{D(t)} = r$, yields that

$$\log(D(t)) = rt + C$$

or

$$D(t) = Ke^{rt}$$

for some constant K. Evaluating at $t = 0$ gives $D = D(0) = K$.

4.23 By Proposition 4.2.1, the cash flow $100, 140, 131$ is preferable for any positive interest rate.

4.24 (a) $110/(1+r)^2 = 100$ or $r = .0488$
(b) If R is the rate of return, then R is equally likely to be $\sqrt{1.2} - 1$ or 0. Hence, $E[R] = .0477$.

4.25 $1000e^{-.8} = 449.33$

4.26 $100 = 40(1+r)^{-1} + 70(1+r)^{-2}$ yielding that $r = .0498$

4.27 (a) No, it is greater than 10 percent if and only if $\sum_i x_i/(1.1)^i$ is greater than 1.
(b) yes because $\frac{8}{1.11} + \frac{16}{(1.11)^2} + \frac{110}{(1.11)^3} = 100.624 > 100$.

4.28

$$\begin{aligned} P(\frac{X_1}{1.1} + \frac{X_2}{(1.1)^2} > 100) &= P(1.1X_1 + X_2 > 121) \\ &= P(Z > \frac{121 - 126}{\sqrt{55.25}}) \\ &= P(Z < .6727) = .780 \end{aligned}$$

where the preceding used that $1.1X_1 + X_2$ is normal with mean 126 and variance 55.25

5.1 (a) The present value of your net return is $10e^{-.12} - 10 = -1.1308$
(b) -10

5.2 (a) -5; $(b) 2e{-}.03 - 5 = -3.059$

5.3 Because the call option will be exercised, purchasing it costs C at time 0 and then costs K at time 1 with the result being owning the security at time 1. Another investment that yields the security at time 1 is to purchase it at time 0 for its initial price s. By the law of one price

$$s = C + Ke^{-r}$$

giving that $C = s - Ke^{-r}$.

5.4 If $C > S$ an arbitrage is effected by simultaneously selling the call and buying the security.

5.5 Because $P \geq 0$, the put call option parity implies that $Ke^{-rt} \geq S - C$

5.6 Use Exercise 5.5 to obtain $C \geq S - Ke^{-rt} = 30 - 28e^{-.05/3}$

5.7 (a) is not necessarily true (to see this, let K be exceedingly large); (b) is true for if $P > K$ an arbitrage can be effected by selling the put.

5.8 This follows from the put call option parity formula.

5.10 Buying the security, buying the put, and selling the call has an initial cost of $S + P - C$ and no matter what the price at time t yields K at that time. (If $S(t) \leq K$ then the sold call is worthless and we exercise the put to sell the security for K. If $S(t) > K$ then the bought put is worthless but the purchaser of the call will exercise and we will receive K for the security.) A second investment that yields K at time t is to ;put Ke^{-rt} in the bank at time 0. The parity formula now follows from the law of one price.

5.11 (a) K; (b) If P is cost of put then law of one price yields $s + P = Ke^{-rt}$, giving that $P = Ke^{-rt} - s$. (Note that if $Ke^{-rt} < s$ then $se^{rt} > K > S(t)$, showing that selling the security with the intention to purchase it at time t yields an arbitrage.)

5.12 Buying both yields 1 at time t, as would putting e^{-rt} in the bank at time 0. Hence, the law of one price gives $C_1 + C_2 = e^{-rt}$

5.13 Because $25 = S+P-C > Ke^{-rt} = 20e^{-.1/4}$, an arbitrage is effected by selling the security, selling the put, and buying the call. This yields 25 and will cost you 20 at time t.

5.14 If C and P are the costs for the European versions, then $C_a = C$ and $P_a \geq P$. The put call parity formula yields

$$S + P_a - C_a \geq Ke^{-rt}$$

5.15 First note that $P_1 \geq P_2$ for if $P_1 < P_2$ an arbitrage is effected by buying the P_1 put and selling the P_2 put. So assume that $P_1 \geq P_2$. If $K_1 - K_2 < P_1 - P_2$ then an arbitrage is effected by selling the K_1, P_1 put and buying the K_2, P_2 put. This has an immediate return of $P_1 - P_2$. If the sold put is exercised at time t then if you also exercise the bought put you will have to pay $K_1 - K_2$, which is less than your immediate return.

5.16 If it were less than could buy the exercise time t put and sell the exercise time $s < t$ put. If the sold put is ever exercised then you should exercise the bought put at that time. These latter transactions cancel each other and you have the initial difference in prices as your arbitrage.

5.17 (a) True because it is clearly true for an American call option and the European is worth the same amount.
(b) This is only true if the domestic interest rate is at least as large as the foreign rate.
(c) This need not be true since it is sometimes optimal to exercise early and so being forced to continue can be detrimental.

5.18 (a) If the security has a lot of volatility.

5.19 $s - d$

5.20 (a) $(S(t) - K)^+ + (K - S(t))^+ = |S(t) - K|$.
(b) $(S(t) - K_1)^+ - (K_2 - S(t))^+$
(c) $2(S(t) - K)^+ - S(t)$
(d) $S(t) - (S(t) - K)^+$

5.21 If not then an arbitrage would result from buying the one with lower strike and selling the one with higher strike.

5.22 (a) negative

5.23 Suppose you buy the $K = 110$ call and sell the $K = 100$ call. This would give you $20 - C$ at time 0 and would cost at most (if the sold option is exercised then you should exercise the other option) 10 at exercise time t. So there would be an arbitrage if $20 - C \geq 10e^{-rt}$. Hence, $C \geq 20 - 10e^{-rt}$.

5.24 Convexity follows from the analogous result for call options upon using the put call option parity formula.

5.25 yes, to show that having λ put options with strike K_1 and $1 - \lambda$ put options with strike K_2 is better than having one put option with strike $K = \lambda K_1 + (1 - \lambda)K_2$ exercise the put pair at the same time that the single put is exercised, taking $(K^* - s)^+$ as the return from exercising a K^* strike put when the security price is s.

5.26 If s is the price at time t_1 than better than exercising is to sell the security at that time and then exercise and give back the security at time t_2. This follows because exercising at time t_1 gives a time t_1 return of $s - K_1$, whereas the latter policy gives a time t_1 return of $s - K_2e^{-r(t_2 - t_1)}$.

5.27 This follows because

$$S(t) - \max(K, S(t) - A) = S(t) + \min(-K, -S(t) + A) = \min(S(t) - K, A)$$

where we used that $-\max(a, b) = \min(-a, -b)$. Hence,

$$(S(t) - \max(K, S(t) - A))^{+} = \max\{0, \min(S(t) - K, A)\} = \min\{(S(t) - K)^{+}, A)\}$$

5.28 A function is concave if the curve obtained when it is plotted is such that the straight line connecting any two of its points lies below or on the curve.

5.29 An arbitrage is a sure win, so neither is necessarily true.

6.1 We need to see whether we can find a probability vector (p_1, p_2, p_3) for which all bets are fair. In order to have all bets fair, $p_i = 1/(1 + o_i)$. Therefore,

$$p_1 = 1/2 \qquad p_2 = 1/3 \qquad p_3 = 1/6$$

Since the p_i's sum up to 1, (p_1, p_2, p_3) is indeed a probability vector which makes all bets fair. Therefore, no arbitrage is present.

6.2 To rule out the arbitrage opportunity, o_4 must satisfy the equation,

$$\frac{1}{1+2} + \frac{1}{1+3} + \frac{1}{1+4} + \frac{1}{1+o_4} = 1$$

Therefore, $o_4 = 47/13$.

6.3 No arbitrage is present since

$$\frac{1}{1+o_1} + \frac{1}{1+o_2} + \frac{1}{1+o_3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

6.4 If no arbitrage is present, then $(p_1, p_2, p_3) = (1/2, 1/3, 1/6)$ has to be the probability vector which makes all bets fair. Therefore

$$\begin{aligned}
o_{12}(p_1 + p_2) - p_3 = 0 &\;\Rightarrow\; o_{12} = 1/5 \\
o_{23}(p_2 + p_3) - p_1 = 0 &\;\Rightarrow\; o_{23} = 1 \\
o_{13}(p_1 + p_3) - p_2 = 0 &\;\Rightarrow\; o_{13} = 1/2
\end{aligned}$$

6.5 If the outcome is j, then the betting scheme $x_i, i = 1, \ldots, m$, gives me

$$\begin{aligned}
o_j x_j - \sum_{i \neq j} x_i &= o_j x_j - \sum_{i=1}^{m} x_j + x_j = (1 + o_j)x_j - \sum_{i=1}^{m} x_j \\
&= \frac{(1+o_j)(1+o_j)^{-1} - \sum_{i=1}^{m}(1+o_i)^{-1}}{1 - \sum_{i=1}^{m}(1+o_i)^{-1}} = 1
\end{aligned}$$

6.6 From Example 6.1b, $p = (1 + 2r)/3$. The payoff of the put option is 0 if the stock price goes up, and 100 if the stock price goes down. To rule out arbitrage, the expected return of buying one put option under the probability distribution has to be zero. That is,

$$\frac{p0 + (1-p)100}{1+r} - P = 0$$

Therefore

$$P = \left(1 - \frac{1+2r}{3}\right)\frac{100}{1+r} = \frac{200(1-r)}{3(1+r)}$$

The put-call parity says

$$S + P - C = \frac{K}{1+r}$$

In this example, one can check the following indeed holds.

6.7
Since the call option expires in period 2 and the strike price $K = 150$, it is clear that

$$C_{uu} = 250 \qquad C_{ud} = 0 \qquad C_{dd} = 0$$

Let p denote the risk neutral probability that the price of the security goes up, then

$$p = \frac{1+r-d}{u-d} = \frac{1-0.5}{2-0.5} = \frac{1}{3}$$

where we assume $r = 0$. Then we can find C by computing the expected return of the call option in the risk neutral world.

$$\begin{aligned} C_u &= pC_{uu} + (1-p)C_{ud} = \frac{1}{3} \times 250 = \frac{250}{3} \\ C_d &= pC_{ud} + (1-p)C_{dd} = 0 \end{aligned}$$

$$C = p^2 C_{uu} + 2p(1-p)C_{ud} + (1-p)^2 C_{dd} = \frac{1}{3^2} \times 250 = \frac{250}{9}$$

6.8 See Example 8.1a for details.

6.9 We need to find a betting strategy which gives a weak arbitrage if (a) $C = 0$ and (b) $C = 50/3$.

(a) $C = 0$. In this case it is clear that buying one share of stock is a weak arbitrage. At time 0, one does not have to pay out anything. At time 1, the profit is 50 if the stock goes up to 200, and 0 if the stock price is either 100 or 50.

(b) $C = 50/3$. In this case it is not that clear how a weak arbitrage can be established. But since the price of the call option is high, it's intuitive that we want to sell it. So, let's consider a portfolio consisting of selling one share of the call option and buying x share(s) of the stock. Our return depends on the price of stock at time 1, which is tabulated as follows.

stock price at time 1	balance at time 0	value of the portfolio at time 1	profit ($r = 0$)
200	$50/3 - 100x$	$-50 + 200x$	$100x - 100/3$
100	$50/3 - 100x$	$0 + 100x$	$50/3$
50	$50/3 - 100x$	$0 + 50x$	$-50x + 50/3$

From the above table, if we choose $x = 1/3$, then the profit is $50/3$ if the stock price is 100 at time 1, and 0 otherwise, which is a weak arbitrage.

6.3 (a) There is no arbitrage if there are probabilities p_1, p_2, p_3 such that $\sum_i p_i = 1$ and

$$4p_1 + 8p_2 - 10p_3 = 0$$

and

$$6p_1 + 12p_2 - 16p_3 = 0$$

Because the first equation implies that $6p_1 + 12p_2 - 15p_3 = 0$, any solution must have $p_3 = 0$. Consequently, p_1, p_2 would need to satisfy $p_1 + 2p_2 = 0$ which is impossible because the p_i must be nonnegative. Hence, an arbitrage is possible. One such is to let $x_1 = 1.5, x_2 = -1$.
(b) No arbitrage if there are probabilities p_1, p_2, p_3 such that $\sum_i p_i = 1$ and

$$\begin{aligned} 6p_1 - 3p_2 &= 0 \\ -2p_1 + 6p_3 &= 0 \\ 10p_1 + 10p_2 + xp_3 &= 0 \end{aligned}$$

Hence, $p_2 = 2p_1$, and $p_1 = 3p_3$ Using that $\sum_i p_i = 1$, this yields that $p_1 = .3$, $p_2 = .6$, $p_3 = .1$. Therefore, no arbitrage is possible if $x = -90$.

6.10 Let $S(0) = s$ and suppose that $us > K > ds$. If you purchase y shares of the security by borrowing x and investing the remaining $ys - x$ then your payoff at time t is

$$\text{payoff} = \begin{cases} yus - (1+r)x, & \text{if } S(1) = us \\ yds - (1+r)x, & \text{if } S(1) = ds \end{cases}$$

Hence, the payoff from the option is replicated if we choose x, y so that

$$yus - (1+r)x = us - K$$

and

$$yds - (1+r)x = 0$$

Setting $y = \frac{us-K}{(u-d)s}$, $x = \frac{ds(us-K)}{(1+r)(u-d)s}$ does the trick. It now follows, by the law of one price, that the no arbitrage cost of the option is $ys - x = \frac{us-K}{u-d} - \frac{ds(us-K)}{(1+r)(u-d)s}$.

6.11 (a) With $p = \frac{1+r-d}{u-d} = \frac{.3}{.425} = 12/17$, the expected payoff is $25(1-p)^2p^3 = .7606$, so the no arbitrage cost is $.7606(1.1)^{-5} = .4723$
(b) yes
(c) $25(1/2)^5 = 25/32 = .78125$

6.12 No arbitrage is possible if new bet is fair when the up probability is $p = \frac{1+r-d}{u-d} = .7380$. The bet will pay off if at least 2 of the first 3 moves are up moves. Hence, no arbitrage if

$$C = e^{-.15} 100 \left((.7380)^3 + 3(.7380)^2 (.2620) \right)$$

7.1 (a) $.33/\sqrt{225}$, $\quad (b) .33/\sqrt{12}$

7.2 Since the unit of time is one year, $t = 4/12 = 1/3$. The probability that the call option will be exercised at $t = 1/3$ is the probability that the stock price at $t = 1/3$ is greater than the strike price $K = 42$, which is

$$\begin{aligned} P(S(1/3) > 42) &= P\Big(\frac{S(1/3)}{S(0)} > \frac{42}{40}\Big) \\ &= P\Big(\log\frac{S(1/3)}{S(0)} > \log\frac{42}{40}\Big) \\ &= P(X > \log 1.05) \end{aligned}$$

where X is a normal random variable with mean $\mu t = .12/3 = .04$ and standard deviation $\sigma\sqrt{t} = .24/\sqrt{3}$. Therefore, the above probability is equal to

$$P\Big(\frac{X - .04}{.24/\sqrt{3}} > \frac{\log 1.05 - .04}{.24/\sqrt{3}}\Big) = 1 - \Phi\Big(\frac{\log 1.05 - .04}{.24/\sqrt{3}}\Big)$$
$$\approx 1 - \Phi(.0634) \approx 1 - .5253 = .4747$$

7.3 The parameters are

$$t = 1/3 \qquad r = .08 \qquad \sigma = .24 \qquad K = 42 \qquad S = 40$$

so we have that

$$\omega = \frac{.08/3 + .24^2/6 - \log(42/40)}{.24/\sqrt{3}} \approx -.0904$$

Therefore,

$$\begin{aligned} C &\approx 40\Phi(-.0904) - 42e^{-.08/3}\Phi(-.2289) \\ &\approx 40 \times .4639 - 42e^{-.08/3} \times .4094 \\ &\approx 1.8137 \end{aligned}$$

7.4 From the put-call parity, we can derive the no-arbitrage cost to a put option

$$\begin{aligned} P &= C - S(0) + Ke^{-rt} \\ &= S(0)\Phi(\omega) - Ke^{-rt}\Phi(\omega - \sigma\sqrt{t}) - S(0) + K^{-rt} \\ &= S(0)\left(\Phi(\omega) - 1\right) + Ke^{-rt}(1 - \Phi(\omega - \sigma\sqrt{t})) \end{aligned}$$

where ω is defined in equation (7.7) in text (page 87). The parameters are

$$K = 100 \qquad S(0) = 105 \qquad r = .1 \qquad \sigma = .30 \qquad t = 1/2$$

$$\begin{aligned} \omega &\approx .571767 \qquad & \Phi(.571767) \approx .7163 \\ \omega - \sigma\sqrt{t} &\approx .359635 \qquad & \Phi(.359635) \approx .6404 \end{aligned}$$

7.5 (a) With W being a normal random variable with mean .03 and variance .045

$$\begin{aligned} P(S(.5)/S(0) < .9) &= P(W < \log(.9)) \\ &= \Phi(\frac{\log(.9) - .03}{\sqrt{.045}}) \\ &= \Phi(-.638) \\ &= 1 - \Phi(.638) \approx .2617 \end{aligned}$$

(b) Now we use the risk neutral drift $r - \sigma^2/2 = .05 - .045 = .005$. With W being a normal random variable with mean .0025 and variance .045

$$\begin{aligned} P(W < \log(.9)) &= \Phi(\frac{\log(.9) - .0025}{\sqrt{.045}}) \\ &= \Phi(-.508) \approx .3057 \end{aligned}$$

7.6 (a) Use formula in text.
(b) With W being a normal random variable with mean .05/4 and variance .09/4

$$\begin{aligned} P(W < \log(100/95) &= \Phi(\frac{\log(100/95) - .05/4}{\sqrt{.09/4}} \\ &= \Phi(.2586) \approx .602 \end{aligned}$$

(c) The risk neutral drift is $r - \sigma^2/2 = -.005$. With W being a normal random variable with mean $-.0025$ and variance .045 the no arbitrage cost is $50e^{-.04}P(W > \log(105/95))P(W > 0)$.

7.7 With W being a normal random variable with mean $(.06 - (.32)^2/2)/2 = .0044$ and variance $(.32)^2/2 = .0512$ the risk neutral valuation is

$$e^{-.03}100P(W > \log(40/38)) = e^{-.03}100(1 - \Phi(.207)) \approx 40.55$$

7.8 $e^{-.03}100(1 - \Phi(\frac{\log(40/38)}{\sqrt{.0512}}))$

7.9 No, you also need to know the drift of the geometric Brownian motion.

7.10 (a) The risk neutral geometric Brownian motion has drift $r - \sigma^2/2 = .06 - .08 = -.02$. No arbitrage if under the risk neutral geometric Brownian motion. Under this process $W \equiv \log(S(1)/S(0))$ is normal with mean $-.02$ and variance .16. Hence, no arbitrage if

$$10 = e^{-.06}\left(5P(W < \log(.95)) + xP(W > \log(1.1))\right)$$

Now,

$$P(W < \log(.95)) = P(W < -.0513) = \Phi(\frac{-.0313}{.4}) = 1 - \Phi(.07825) \approx .469$$

and

$$P(W > \log(1.1)) = P(W > .0953) = 1 - \Phi(\frac{.1153}{.4}) = 1 - \Phi(.28825) \approx .386$$

Hence,

$$x \approx \frac{10e^{.06} - 5(.469)}{.386} \approx 21.433$$

(b) Using that the actual mean of W is .05 yields that

$$P(S(1) < 95) = P(W < -.0513) = \Phi(\frac{-.1013}{.4}) = 1 - \Phi(.25325) \approx .400$$

7.11 (a) The risk neutral drift is $-.02$. There is no payoff with probability given by

$$P(S(.5) < 42, S(1) < 1.05S(.5)) = P(S(.5) < 42)P(S(1) < 1.05S(.5))$$

Under the risk neutral GBM,

$$P(S(.5) < 42) = \Phi(\frac{\log(42/40) + .01}{\sqrt{.08}} = \Phi(.2078) \approx .590$$

$$P(S(1) < 1.05S(.5)) = \Phi(\frac{\log(1.05) + .02}{.4} = \Phi(.1720) \approx .569$$

Hence, there is no payoff with probability approximately .336, yielding that the expected payoff at time 1 is 66.4. Hence, there is no arbitrage if $C \approx e^{-.06}66.4 \approx 62.53$.
(b) Using the actual drift

$$P(S(.5) < 42) = \Phi(\frac{\log(42/40) - .03}{\sqrt{.08}} = \Phi(.0664) \approx .527$$

$$P(S(1) < 1.05S(.5)) = \Phi(\frac{\log(1.05) - .06}{.4} = \Phi(-.0280) \approx .489$$

Hence, the investment will make money with probability $1 - .258 = .742$.

7.12 (a) The risk neutral drift is 0. Under the risk neutral GBM

$$p \equiv P(S(1)/S(0) > 1 + x) = 1 - \Phi(\frac{\log(1 + x)}{.2})$$

There is no arbitrage if

$$10 = e^{-.02} 100p$$

Thus, there is no arbitrage if $p = .1020$, yielding that

$$\Phi(\frac{\log(1 + x)}{.2}) = .8980$$

Using that $\Phi(1.27) = .8980$ yields that $\log(1 + x) = .254$ or $x = .2892$.
(b) Using the actual drift

$$P(S(1)/S(0) > 1.2892) = 1 - \Phi(\frac{\log(1.2892) - .04}{.2}) = 1 - \Phi(1.070) \approx .157$$

7.13 Lemma 7.5.3 gives the result.

7.14 $S(0)$

7.15 $S(0)$

7.16 The price at time t converges to $S(0)e^{rt}$ as the volatility goes to 0. So the cost should be $e^{-rt}(S(0)e^{rt} - K)^+ = (S(0) - Ke^{-rt})^+$.

7.18 Not necessarily concave nor convex.

8.1 yes

8.2 Geometric Brownian motion with drift $r - \sigma^2/2 - f$ and volatility σ.

8.3 $C(s(1-f)^2, K, t, r, \sigma)$

8.4 Because one should never exercise a call option early when there are no dividends it follows that one should never exercise earlier than t_d or after t_d but before t.

8.5 The payoff form the capped option is the difference between the payoff from at K, t call and the payoff from a $K + B, t$ call. Hence, by the law of one-price its no-arbitrage cost is $C(s, K, t, r, \sigma) - C(s, K + B, t, r, \sigma)$.

8.6 Under the risk neutral Geometric Brownian motion, the expected return from this investment is

$$E[(1+\beta)s + \alpha(S(1) - (1+\beta)s)^+] = (1+\beta)s + \alpha e^r C(s, t, (1+\beta)s, \sigma, r)$$

This bet won't give rise to an arbitrage provided the preceding is equal to se^r. Thus,

$$\alpha = \frac{s(e^r - 1 - \beta)}{e^r C(s, t, (1+\beta)s, \sigma, r)}$$

8.7 Under the risk neutral Geometric Brownian motion, provided that $K > (1+\beta)s$, the expected return from this investment is

$$E[(1+\beta)s + (S(1) - (1+\beta)s)^+ - (S(1) - K)^+] = (1+\beta)s + e^r C(s, t, (1+\beta)s, \sigma, r) - e^r C(s, t, K, \sigma, r)$$

There is no arbitrage provided the preceding is equal to se^r, which will be the case under the stated condition. Because $s(1+\beta)e^{-r} - s < 0$, and $C(s, t, K, \sigma, r)$ is decreasing in K, the condition that K will exceed $s(1+\beta)$ is satisfied.

8.8 With Z being a standard normal random variable

$$\begin{aligned} C(se^{-ft}, t, K, \sigma, r) &= e^{-rt} E[(se^{-ft} e^{\sigma\sqrt{t}Z + (r-\sigma^2/2)t} - K)^+] \\ &= e^{-rt} E[(se^{\sigma\sqrt{t}Z + (r-f-\sigma^2/2)t} - K)^+] \\ &= e^{-ft} e^{-(r-f)t} E[(se^{\sigma\sqrt{t}Z + (r-f-\sigma^2/2)t} - K)^+] \\ &= e^{-ft} C(s, t, K, \sigma, r - f) \end{aligned}$$

8.9 (a) Rather than exercising at time $s < t_1$, and thus paying K_1 at time s, a dominating strategy is to exercise at time t_1 and thus pay K_1 at time t_1.
(b) This follows because $C(x, t - t_1, K, \sigma, r)$ is the value of the call at time t_1 when $S(t_1) = x$.
(c) This follows because $C(y, t - t_1, K, \sigma, r)$ is a strictly increasing function of y.
(d) This follows because the optimal policy is to exercise the option to purchase the call option at time t_1 if and only if $S(t_1) \geq x$.

8.10 (a) Better than exercising at time t_1 is to exercise (no matter what the price) at time t_2, since the only difference is that in the former case you pay the present value amount $K_1 e^{-rt_1}$ whereas in the latter case you pay the present value amount $K_2 e^{-rt_2}$. Hence, if $K_2 e^{-rt_2} < K_1 e^{-rt_1}$ you should never exercise at time t_1.
(b) The time t_1 risk neutral expected return if the option is not exercised at that time is $C(y, t_2 - t_1, K_2, \sigma, r)$ if $S(t_1) = y$. The time t_1 value of the option if it is exercised at time t_1 is $y - K_1$. Hence, if $S(t_1) = y$, one should exercise at time t_1 if

$$y - K_1 > C(y, t_2 - t_1, K_2, \sigma, r)$$

8.12 (a) yes; (b) no; (c) no; (d) yes.

8.15 This option should be exercised whenever the price is at least K. It can be explicitly priced by using the formula derived in Chapter 3 for the maximum by time t of a Brownian motion. It can be approximated by a N period binomial model. Take the same states as used in pricing an American put option, and work backwards to obtain $V_0(0)$. It takes a bit less work than determining the risk neutral cost of an American put option because the optimal strategy for the asset-or-nothing is known in advance.

9.1 With X_i equal to fortune after investment i, we have

$$E[u(X_1)] = 1 - \int_0^\infty e^{-x}e^{-x}dx = 1/2$$

$$E[u(X_2)] = 1 - \int_0^2 e^{-x}\frac{1}{2}dx = 1 - \frac{1}{2}(1 - e^{-2}) = 1/2 + e^{-2}$$

Investment 2 is better.

9.2 $E[X] < 0$ so optimal is $a = 0$.

9.3 The rate of return R is such that $\frac{X}{(1+R)^n} = 1$. Hence, $R = X^{1/n} - 1$. Because $g(x) = x^{1/n}$ is concave in x it follows from Jensen's inequality that

$$E[R] = E[X^{1/n}] - 1 \leq \mu^{1/n} - 1$$

and the result follows because $\mu^{1/n} - 1$ is the rate of return of an investment of 1 that yields μ after n periods.

9.4 The second derivative with respect to α of expected utility is negative, showing the expected utility as a function of α is concave. As, allowing α to range from $-\infty$ to ∞, its minimum is, when $p < 1/1$, obtained when $\alpha < 0$, it follows by concavity that its value when $\alpha = 0$ is greater than its value for $\alpha > 0$.

9.5 y should be chosen to maximize $-.03y - .0025(.04y^2 + .0625(100 - y)^2)$.

9.7 $E[\log(\sum_i \alpha_i w X_i)] = \log(w) + E[\log(\sum_i \alpha_i X_i)]$ and so the optimal fractions do not depend on w.

9.9 With $W = \sum w\alpha_i X_i$, we want to maximize

$$\log(E[W]) - \frac{\text{Var}(W)}{2(E[W])^2} = \log(w) + \log(E[\sum \alpha_i X_i]) - \frac{w^2\text{Var}(\sum \alpha_i X_i)}{2w^2 E[\sum \alpha_i X_i]}$$

Thus, w does not play a role in determining the optimal $\alpha_i, i = 1, \ldots, n$.

9.11 With Z being a standard normal

$$P(W > g) = P\left(Z > \frac{g - E[W]}{\sqrt{\text{Var}(W}}\right)$$

Hence, want to choose so as to minimize $\frac{g-E[W]}{\sqrt{\text{Var}(W)}}$.

9.14 .066 and .076

9.15 $\sum_i \alpha_i \beta_i$

9.16 $a_i = (1 - \beta_i) r_f, \ b_i = \beta_i, \ F = R_m.$

9.17 (a) Yes, because $E[X_1 + X_2] = 2$, we can conclude from Jensen's inequality that a fixed return of 2 is preferable.
(b) $X_1 + X_2$ is preferable to $2X_1$ because they are both normal and $X_1 + X_2$ has the same mean but a smaller variance than does $2X_1$.
(c) No, it depends on the utility function. $3X_1$ has a larger mean but also a larger variance.
(d) Using that $-(X_1 + X_2)$ is normal with mean -2 and variance 2

$$E[1 - e^{-X_1+X_2}] = 1 - e^{-2+1} = 1 - e^{-1}$$

whereas, because $-3X_1$ has mean -3 and variance 9

$$E[1 - e^{-3X_1}] = 1 - e^{-3+4.5} = 1 - e^{1.5}$$

Thus, $X_1 + X_2$ is preferable.

9.18

$$\begin{aligned}
\text{Cov}(X_i, X_j) &= \text{Cov}(\sum_{r=1}^{m} a_{ir} Z_r, \ \sum_{k=1}^{m} a_{jk} Z_k) \\
&= \sum_r \sum_k \text{Cov}(a_{ir} Z_r, \ a_{jk} Z_k)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_r \mathrm{Cov}(a_{ir}Z_r,\ a_{jr}Z_r) + \sum_r \sum_{k \neq r} \mathrm{Cov}(a_{ir}Z_r,\ a_{jk}Z_k) \\
&= \sum_r a_{ir}a_{jr}\mathrm{Cov}(Z_r,\ Z_r) + \sum_r \sum_{k \neq r} a_{ir}a_{jk}\mathrm{Cov}(Z_r,\ Z_k) \\
&= \sum_r a_{ir}a_{jr}
\end{aligned}
$$

where the final equality used that $\mathrm{Cov}(Z_r,\ Z_k)$ is 1 when $k = r$ and is 0 when $k \neq r$.

10.1 Immediate by definition.

10.2 Let $I_j, j \geq 1$, be independent random variables, each equal either to 1 with probability p or to 0 with probability $1 - p$. Then,

$$\sum_{j=1}^{n+1} I_j \geq \sum_{j=1}^{n} I_j$$

which proves the result since $\sum_{j=1}^{k} I_j$ has a binomial distribution with parameters k, p.

10.3 Let $I_j, j \geq 1$, be independent random variables, each equal either to 1 with probability p_1 or to 0 with probability $1 - p_1$; and let $J_j, j \geq 1$, be independent random variables, each equal either to 1 with probability $\frac{p_2}{p_1}$ or to 0 with probability $1 - \frac{p_2}{p_1}$. Assume that the two sequences are independent of each other. Then,

$$\sum_{j=1}^{n} I_j J_j \leq \sum_{j=1}^{n} I_j$$

which proves the result because $\sum_{j=1}^{n} I_j J_j$ has a binomial distribution with parameters n, p_2 (true since $I_j J_j$ is either 1 with probability p_2 or 0 with probability $1 - p_2$) and $\sum_{j=1}^{n} I_j$ has a binomial distribution with parameters n, p_1.

10.4

$$\begin{aligned}
\frac{f_{\mu_1}(x)}{f_{\mu_2}(x)} &= \frac{e^{-(x-\mu_1)^2/2\sigma^2}}{e^{-(x-\mu_2)^2/2\sigma^2}} \\
&= \exp\{\frac{1}{2\sigma^2}\Big((x-\mu_2)^2 - (x-\mu_1)^2\Big)\} \\
&= \exp\{\frac{1}{2\sigma^2}\Big(2(\mu_1-\mu_2)x + \mu_2^2 - \mu_1^2\Big)\}
\end{aligned}$$

and the preceding is increasing in x when $\mu_1 \geq \mu_2$.

10.5

$$\frac{\lambda_1 e^{-\lambda_1 x}}{\lambda_2 e^{-\lambda_2 x}} = \frac{\lambda_1}{\lambda_2} e^{(\lambda_2 - \lambda_1)x}$$

and the preceding is increasing in x when $\lambda_1 \leq \lambda_2$.

10.6
$$\frac{e^{-\lambda_1}\lambda_1^n/n!}{e^{-\lambda_2}\lambda_2^n/n!} = e^{\lambda_2-\lambda_1}(\lambda_1/\lambda_2)^n$$
which increases in n when $\lambda_1 \geq \lambda_2$.

10.7 This is just Jensen's inequality.

10.8 Immediately follows from the hint.

10.9 Let h and g both be increasing and concave. We need to show that $f(x) \equiv h(g(x))$ is also increasing and concave in x. But

$$f'(x) = h'(g(x))g'(x)$$

which is nonnegative since h' and g' are both nonnegative because h and g are increasing. Also,

$$f''(x) = h''(g(x))(g'(x))^2 + h'(g(x))g''(x) \leq 0$$

where the inequality follows because both terms are nonpositive because $h'' \leq 0$ and $h' \geq 0$ and $g'' \leq 0$.

11.1 Optimal is to invest 2 in project 1 and 4 in project 2. The optimal return is $2\log(3) + 2$.

11.4 Suppose under the optimal policy that k is invested in project i and r is invested in project j, where r and k are both positive. Now,

$$f_i(k) - f_i(k-1) \geq f_j(r+1) - f_j(r)$$

for otherwise investing $k-1$ in i and $r+1$ in j would, although investing the same amount, yield a higher return from these two investments. But, by convexity,

$$f_i(k+1) - f_i(k) \geq f_i(k) - f_i(k-1) \quad \text{and} \quad f_j(r+1) - f_j(r) \geq f_j(r) - f_j(r-1)$$

showing that investing $k+1$ in project i and $r-1$ in project j is at least as good as investing k and r. Continuing in this manner shows that investing $k+r$ in project i and 0 in project j s at least as good as investing k and r. Continuing with other projects shows that there is an optimal policy that invests all in a single project.

11.5 (a) Consider a solution that has one value of x equal to $k+i$ and another equal to $k-j$ where i and j are both positive. (Unless all x equal k this will be the case.) We now argue that a better solution is given by leaving all other values unchanged and changing $k+i$ to $k+i-1$ and changing $k-j$ to $k-j+1$. That is, we claim that

$$f(k+i) + f(k-j) \leq f(k+i-1) + f(k-j+1)$$

which is equivalent to

$$f(k+i) - f(k+i-1) \leq f(k-j+1) - f(k-j)$$

which follows because f is concave (and $k-j+1 < k+i$). Hence, we always get a better solution by letting the $x's$ become nearer, and in the limit we get that the maximal occurs when all of them are equal.
(b) When f is convex an analogous argument to the one of part (a) will yield the result. This is also a special case of Exercise 11.4.

11.7 (a) $\sqrt{x}$
(b) With $\beta = 1 + r$,

$$V_2(x) = \max_{y \leq x}\{\sqrt{y} + V_1(\beta(x-y))\} =$$

(c)

$$V_n(x) = \max_{y \leq x}\{\sqrt{y} + V_{n-1}(\beta(x-y))\}$$

(d) Rewriting so that the decision is the fraction of your wealth that you consume

$$\begin{aligned} V_2(x) &= \max_{0\le\alpha\le1}\{\sqrt{\alpha x} + \sqrt{\beta(x-\alpha x)}\} \\ &= \sqrt{x}\max_{0\le\alpha\le1}\{\sqrt{\alpha} + \sqrt{\beta}\sqrt{1-\alpha}\} \end{aligned}$$

Calling the term inside the maximum $f(\alpha)$ and differentiating yields

$$f'(\alpha) = \frac{1}{2}\alpha^{-1/2} - \sqrt{\beta}\,\frac{1}{2}(1-\alpha)^{-1/2}$$

Setting equal to 0 yields that the maximum occurs when $\alpha = \frac{1}{1+\beta}$, with the optimal value being

$$V_2(x) = \sqrt{1+\beta}\,\sqrt{x}$$

Now,

$$\begin{aligned} V_3(x) &= \max_{0\le\alpha\le1}\{\sqrt{\alpha x} + V_2(\beta(1-\alpha)x)\} \\ &= \max_{0\le\alpha\le1}\{\sqrt{\alpha x} + \sqrt{1+\beta}\sqrt{\beta(1-\alpha)x}\} \\ &= \max_{0\le\alpha\le1}\{(\sqrt{\alpha} + \sqrt{1+\beta}\sqrt{\beta(1-\alpha)})\,\sqrt{x}\} \\ &= \sqrt{x}\max_{0\le\alpha\le1}\{\sqrt{\alpha} + \sqrt{\beta(1+\beta)}\sqrt{1-\alpha}\} \end{aligned}$$

The value of α that maximizes is $\alpha = \frac{1}{1+\beta(1+\beta)}$, and

$$V_3(x) = \sqrt{x}\,\sqrt{1+\beta+\beta^2}$$

In general,

$$V_n(x) = \sqrt{x}\,\sqrt{1+\beta+\beta^2+\ldots+\beta^{n-1}}$$

and the optimal fraction to consume witn n periods remaining is $\frac{1}{1+\beta+\ldots+\beta^{n-1}}$

11.8 (a)

$$V(S) = \max_{i\in S}\{R_i(x_i + \sum_{k\notin S} x_k) + V(S-i)\}$$

(b) First solve when S is a one-point set; then when it is a two-point set, and so on.

11.9 In the first investment she should risk $.2x$ if her fortune is x. Her expected utility is $\log(x) + .6\log(1.2) + .4\log(.8) = \log(x) + .0201$. In the second investment, she will risk 0 if the win probability is .4 and $.6x$ if the win probability is .8. Hence, the expected utility of her final fortune for this investment is $\log(x) + .3(.8 \log(1.6) + .2 \log(.4)) = \log(x) + .0578$. Hence, the second investment is preferable.

11.11 To determine the minimal time that node j can be reached, let the decision be the node visited immediately before entering node j. If that node is node i, then if node i is reached at time s then the time at which node j is reached is $s + t_s(i, j)$. Because $s + t_s(i, j)$ is increasing in s we would want to reach node i in minimal time, which proves the equation..

12.1 Letting $q(j) = 1 - p(j)$, then with $V_k(n) = 0$ if $k > n$

$$V_k(n) \max_{1 \le j \le n} \{p(j)V_{k-1}(n-j) + q(j)V_k(n-j)\}$$

Therefore,

$$\begin{aligned}
V_1(1) &= p(1) = .2, \quad a_1(1) = 1 \\
V_1(2) &= \max\{p(1) + q(1)V_1(1),\ p(2)\} = .4, \quad a_1(2) = 2 \\
V_1(3) &= \max\{p(1) + q(1)V_1(2),\ p(2) + q(2)V_1(1),\ p(3)\} = .6, \quad a_1(3) = 3 \\
V_2(2) &= p(1)^2 = .04,\ a_2(2) = 1 \\
V_2(3) &= \max\{p(1)V_1(2) + q(1)V_2(2),\ p(2)V_1(1)\} = .112, \quad a_2(3) = 1 \\
V_2(4) &= \max\{p(1)V_1(3) + q(1)V_2(3),\ p(2)V_1(2) + q(2)V_2(2),\ p(3)V_1(1)\} = .2096, \quad a_2(4) = 1
\end{aligned}$$

The maximal probability is .2096. It is optimal to initially invest 1 and then to invest $a_j(i)$ when you are in position where you still need j machines and you have i remaining units to spend.

12.2 With $V(i,0) = i,\ V(0,i) = 0$,

$$\begin{aligned}
V(1,1) &= \max\{0, 0 + \frac{1}{2}\} = 1/2 \\
V(2,1) &= \max\{0, \frac{1}{3} + \frac{2}{3}V(1,1) + \frac{1}{3}V(2,0)\} = 4/3 \\
V(1,2) &= \max\{0, -\frac{1}{3} + \frac{1}{3}V(0,2) + \frac{2}{3}V(1,1)\} = 0 \\
V(2,2) &= \max\{0, 0 + \frac{1}{2}V(1,2) + \frac{1}{2}V(2,1)\} = 2/3 \\
V(1,3) &\le V(1,2) = 0 \\
V(1,4) &\le V(1,3) = 0 \\
V(2,3) &= \max\{0, -\frac{1}{5} + \frac{2}{5}V(1,3) + \frac{3}{5}V(2,2)\} = 1/5 \\
V(2,4) &= \max\{0, -\frac{2}{6} + \frac{2}{6}V(1,4) + \frac{4}{6}V(2,3)\} = 0 \\
V(3,1) &= \max\{0, \frac{1}{2} + \frac{3}{4}V(2,1) + \frac{1}{4}V(3,0)\} = 9/4 \\
V(3,2) &= \max\{0, \frac{1}{5} + \frac{3}{5}V(2,2) + \frac{2}{5}V(3,1)\} = 3/2 \\
V(3,3) &= \max\{0, 0 + \frac{1}{2}V(2,3) + \frac{1}{2}V(3,2)\} = 17/20 \\
V(3,4) &= \max\{0, -\frac{1}{7} + \frac{3}{7}V(2,4) + \frac{4}{7}V(3,3)\} = 12/35
\end{aligned}$$

12.3 Use mathematical induction to complete the proof. When $E[Y] < 0$, because $\log(x)$ is a concave function, we have by Jensen's inequality that

$$E[\log(\alpha Y + 1 - \alpha)] \leq \log(\alpha E[Y] + 1 - \alpha) \leq \log(1 - \alpha) \leq \log(1) = 0$$

12.4 The optimal policy is to accept any offer i such that

$$i(1 - \beta) \geq \beta(E[(X - i)^+ - c)$$

12.5 (a) Before you can get to k in a row you must have $k - 1$. So the optimal policy when you need k in a row is to get to $k - 1$ in a row at minimal expected cost and then invest some amount x in the next game.
(b) If you invest $x$in the next game after you have reached $k - 1$ in a row at minimal expected cost, then the number of times you will get to $k - 1$ in a row is geometric with parameter $p(x)$. Hence, as the expected cost to get to $k - 1$ in a row is V_{k-1}, your expected cost $\frac{V_{k-1}+x}{p(x)}$.
(c) Once V_n is determined, we find the optimal policy as follows. Let $H(j)$ be the minimal expected additional cost to get to n wins in a row when you currently have j consecutive wins. Then

$$H(j) = \min_x \{x + p(x)H(j + 1) + (1 - p(x))V_n\}, \quad j < n.$$

Starting with $j = n - 1$, then $j = n - 2$, the preceding can be recursively solved for. The x that mininizes the right side of the preceding equation is the optimal amount to invest when you currently have j wins in a row.

12.6 (a) The state is the number of distinct type in the current collection. The action is whether to stop or to collect another coupon.
(b) With $V(j)$ equal to the maximal expected additional return if one currently has j distinct types, the optimality equation is

$$V(j) = \max\{jr, \quad -1 + \frac{j}{n}V(j) + \frac{n-j}{n}V(j+1)\}$$

(c) Theone-stage lookahead policy stops in state j if

$$jr \geq -1 + fracjn\, jr + \frac{n-j}{n}\,(j+1)r$$

That is, it stops in state j if $r\,\frac{n-j}{n} \leq 1$.
(d) It is optimal because, as the state cannot decrease, its set of stopping states is closed.

(e) The state is the subset of types in cone's collection. (d) If S is the subset of types, then the one-stage lookahead policy stops if

$$r \sum_{i \notin S} p_i \leq 1.$$

It is optimal because the set of coupons in one's collection is one continues must always include what is currently in one's collection.

12.7 The one-stage lookahead policy would stop whenever the number of red balls is less than or equal to the number of black balls. It is a very bad policy.

13.1 Assume $u \le r$. Let s be the price of the security at time y, where $y < t$. If, rather than exercising at time $y < t$, we exercised at time t (regardless of the price at time t) then the expected time y return is

$$e^{-r(t-y)}(se^{r(t-y)} - Ke^{ut}) = s - Ke^{ut-r(t-y)} \ge s - Ke^{uy}$$

The result follows because the right hand side is the time y return if we exercise at time y.

13.3 Solution given in text.

13.4

$$\begin{aligned}
\mathrm{Var}(W) &= \mathrm{Cov}(Y + \sum_{i=1}^{n} c_i X_i, \;\; Y + \sum_{j=1}^{n} c_j X_j) \\
&= \mathrm{Cov}(Y, Y) + 2\mathrm{Cov}(Y, \sum_{j=1}^{n} c_j X_j) + \mathrm{Cov}(\sum_{i=1}^{n} c_i X_i, \;\; \sum_{j=1}^{n} c_j X_j) \\
&= \mathrm{Var}(Y) + 2\sum_{i=1}^{n} c_i \mathrm{Cov}(Y, X_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \mathrm{Cov}(X_i, X_j) \\
&= \mathrm{Var}(Y) + 2\sum_{i=1}^{n} c_i \mathrm{Cov}(Y, X_i) + \sum_{i=1}^{n} c_i^2 \mathrm{Cov}(X_i, X_i) + \sum_{i=1}^{n}\sum_{j \ne i} c_i c_j \mathrm{Cov}(X_i, X_j) \\
&= \mathrm{Var}(Y) + 2\sum_{i=1}^{n} c_i \mathrm{Cov}(Y, X_i) + \sum_{i=1}^{n} c_i^2 \mathrm{Var}(X_i)
\end{aligned}$$

where the final equality used the independence of X_i, X_j to conclude that, for $i \ne j$, $\mathrm{Cov}(X_i, X_j) = 0$.
(c) Setting the partial derivative of the preceding with respect to c_i equal to 0 gives the result:

$$2c_i \mathrm{Var}(X_i) + 2\mathrm{Cov}(Y, X_i) = 0$$

13.7 Similar to what is done in Section 11.8 except we now let

$$V_k(i) = \max\{su^i d^{k-i} - K, \quad pV_{k+1}(i+1) + (1-p)W_{k+1}(i)\}$$

13.8 The price is either multiplied by u with probability p or by d with probability $1-p$, and if multiplied by d we need to check if the new price is an end of day price that is below the barrier.

Solutions Manual to

AN INTRODUCTION TO MATHEMATICAL FINANCE: OPTIONS AND OTHER TOPICS

Sheldon M. Ross

1.1 (a) $1 - p_0 - p_1 - p_2 - p_3 = 0.05$ (b) $p_0 + p_1 + p_2 = 0.80$

1.2 $P\{C \cup R\} = P\{C\} + P\{R\} - P\{C \cap R\} = 0.4 + 0.3 - 0.2 = 0.5$

1.3 (a) $\frac{8}{14}\frac{7}{13} = \frac{56}{182}$ (b) $\frac{6}{14}\frac{5}{13} = \frac{30}{182}$ (c) $\frac{6}{14}\frac{8}{13} + \frac{8}{14}\frac{6}{13} = \frac{96}{182}$

1.4 (a) 27/58 (b) 27/35

1.5

1. The probability that their child will develop cystic fibrosis is the probability that the child receives a CF gene from each of his parents, which is 1/4.

2. Given that his sibling died of the disease, each of the parents much have exactly one CF gene. Let A denote the event that he possesses one CF gene and B that he does not have the disease (since he is 30 years old). Then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{2/4}{3/4} = \frac{2}{3}$$

1.6 Let A be the event that they are both aces and B the event they are of different suits. Then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{\frac{4}{52}\frac{3}{51}}{\frac{39}{51}} = \frac{1}{169}$$

1.7

$$\begin{aligned} (a)\ \ P(AB^c) &= P(A) - P(AB) \\ &= P(A) - P(A)P(B) \\ &= P(A)(1 - P(B)) \\ &= P(A)P(B^c) \end{aligned}$$

Part (b) follows from part (a) since from (a) A and B^c are independent, implying from (a) that so are A^c and B^c.

1.8 If the gambler loses both the bets, then $X = -3$. If he wins the first bet, or loses the first bet and wins the second bet, $X = 1$. Therefore,

$$\begin{aligned} P\{X = -3\} &= (\frac{20}{38})^2 = \frac{100}{361} \\ P\{X = 1\} &= \frac{18}{38} + \frac{20}{38}\frac{18}{38} = \frac{261}{361} \end{aligned}$$

1. $P\{X > 0\} = P\{X = 1\} = \frac{261}{361}$

2. $E[X] = 1\frac{261}{361} - 3\frac{100}{361} = \frac{-39}{361}$

1.9

1. $E[X]$ is larger since a bus with more students is more likely to be chosen than a bus with less students.

2.

$$\begin{aligned} E[X] &= \frac{1}{152}(39^2 + 33^2 + 46^2 + 34^2) = \frac{5882}{152} \approx 38.697 \\ E[Y] &= \frac{1}{4}(39 + 33 + 46 + 34) = 38 \end{aligned}$$

1.10 Let N denote the number of sets played. Then it is clear that $P\{N = 2\} = P\{N = 3\} = 1/2$.

1. $E[N] = 2.5$

2. $\mathrm{Var}(N) = \frac{1}{2}(2 - 2.5)^2 + \frac{1}{2}(3 - 2.5)^2 = \frac{1}{4}$

1.11 Let $\mu = E[X]$.

$$\begin{aligned} \mathrm{Var}(X) &= E[(X - \mu)^2] \\ &= E[X^2 - 2\mu X + \mu^2] \\ &= E[X^2] - 2\mu E[X] + \mu^2 \\ &= E[X^2] - \mu^2 \end{aligned}$$

1.12 Let F be her fee if she takes the fixed amount and X when she takes the contingency amount.

$$E[F] = 5,000, \quad SD(F) = 0$$

$$E[X] = 25,000(.3) + 0(.7) = 7,500$$

$$E[X^2] = (25,000)^2(.3) + 0(.7) = 1.875 \times 10^8$$

Therefore,

$$SD(X) = \sqrt{\mathrm{Var}(X)} = \sqrt{1.875 \times 10^8 - (7,500)^2} = \sqrt{1.3125} \times 10^4$$

1.13

$$\begin{aligned} (a)\ E[\bar{X}] &= \frac{1}{n}\sum_{i=1}^{n} E[X_i] \\ &= \frac{1}{n} n\mu = \mu \end{aligned}$$

$$
\begin{aligned}
(b)\ \mathrm{Var}(\bar{X}) &= (\frac{1}{n})^2 \sum_{i=1}^{n} \mathrm{Var}(X_i) \\
&= (\frac{1}{n})^2 n\sigma^2 = \sigma^2/n
\end{aligned}
$$

$$
\begin{aligned}
(c)\ \sum_{i=1}^{n}(X_i - \bar{X})^2 &= \sum_{i=1}^{n}(X_i^2 - 2X_i\bar{X} + \bar{X})^2) \\
&= \sum_{i=1}^{n} X_i^2 - 2\bar{X}\sum_{i=1}^{n} X_i + n\bar{X}^2 \\
&= \sum_{i=1}^{n} X_i^2 - 2\bar{X}n\bar{X} + n\bar{X}^2 \\
&= \sum_{i=1}^{n} X_i^2 - n\bar{X}^2
\end{aligned}
$$

$$
\begin{aligned}
(d)\ E[(n-1)S^2] &= E[\sum_{i=1}^{n} X_i^2] - E[n\bar{X}^2] \\
&= nE[X_1^2] - nE[\bar{X}^2] \\
&= n(\mathrm{Var}(X_1) + E[X_1]^2) - n(\mathrm{Var}(\bar{X}) + E[\bar{X}]^2) \\
&= n\sigma^2 + n\mu^2 - n(\sigma^2/n) - n\mu^2 \\
&= (n-1)\sigma^2
\end{aligned}
$$

1.14

$$
\begin{aligned}
\mathrm{Cov}(X,Y) &= E\left[(X - E[X])(Y - E[Y])\right] \\
&= E\left[XY - XE[Y] - E[X]Y + E[X]E[Y])\right] \\
&= E[XY] - E[Y]E[X] - E[X]E[Y] + E[X]E[Y] \\
&= E[XY] - E[Y]E[X]
\end{aligned}
$$

1.15

$$
\begin{aligned}
(a)\ \mathrm{Cov}(X,Y) &= E\left[(X - E[X])(Y - E[Y])\right] \\
&= E\left[(Y - E[Y])(X - E[X])\right]
\end{aligned}
$$

$$
(b)\ \mathrm{Cov}(X,X) = E[(X - E[X])^2] = \mathrm{Var}(X)
$$

$$
\begin{aligned}
(c)\ \mathrm{Cov}(cX,Y) &= E\left[(cX - E[cX])(Y - E[Y])\right] \\
&= cE\left[(X - E[X])(Y - E[Y])\right] \\
&= c\mathrm{Cov}(X,Y)
\end{aligned}
$$

$$
(d)\ \mathrm{Cov}(c,Y) = E\left[(c - E[c])(Y - E[Y])\right] = 0
$$

1.16

$$\begin{aligned}
\mathrm{Cov}(aU+bV, cU+dV) &= \mathrm{Cov}(aU, cU+dV) + \mathrm{Cov}(bV, cU+dV) \\
&= \mathrm{Cov}(aU, cU) + \mathrm{Cov}(aU, dV) + \mathrm{Cov}(bV, cU) + \mathrm{Cov}(bV, dV) \\
&= ac(1) + ad(0) + bc(0) + bd(1) = ac + bd
\end{aligned}$$

1.17 With $c(i,j) = \mathrm{Cov}(X_i, X_j)$
(a) $c(1,3) + c(1,4) + c(2,3) + c(2,4) = 21$
(b) $2 + 3 + 4 + 4 + 6 + 8 + 6 + 9 + 12 = 54$

1.17 Let X_i be the amount it goes up in period i. Then

$$Y = \sum_{i=1}^{3} X_i$$

and

$$\mathrm{Cov}(X_1, Y) = \mathrm{Cov}(X_1, X_1) = \mathrm{Var}(X_1) = 1/4$$

Therefore,

$$\mathrm{Corr}(X, Y) = \frac{1/4}{\sqrt{(1/4)(3/4)}} = 1/\sqrt{3}$$

1.18 No, since for such a pair $\mathrm{Corr}(X, Y) = 2$, and correlations are always between -1 and 1.

2.1

1. $P(Z < -.66) = P(Z > .66) = 1 - P(Z < .66) = 1 - \Phi(.66) = 1 - .7454 = .2546$

2.

$$\begin{aligned} P(|Z| < 1.64) &= P(Z < 1.64) - P(Z < -1.64) \\ &= P(Z < 1.64) - [1 - P(Z < 1.64)] \\ &= 2\Phi(1.64) - 1 = 2 \times .9495 - 1 = .8990 \end{aligned}$$

3. $P(|Z| > 2.20) = 2P(Z > 2.20) = 2[1 - P(Z < 2.20)] = 2(1 - .9861) = .0278$

2.2 $x = 2$

2.3

$$P(|Z| > x) = P(Z > x \text{ or } Z < -x) = P(Z > x) + P(Z < -x) = 2P(Z > x)$$

where the last equality comes from the fact that Z is symmetric.

2.4 $a = 2\mu, \quad b = -1.$

$$\mathrm{Cov}(X, Y) = -\mathrm{Var}(X) = -\sigma^2$$

2.5

(a) 127.7 ± 19.2
(b) $127.7 \pm (1.96)(19.2)$
(c) 127.7 ± 57.6

2.6 Let X_1 and X_2 denote the life of the first and the second battery respectively. It is given that X_1 and X_2 are both normal random variables with mean 400 and standard deviation 50. Let Z denote a standard normal random variable.

1. $X_1 + X_2$ is a normal random variable with mean 800 and standard deviation $50\sqrt{2}$.

$$\begin{aligned} P(X_1 + X_2 > 760) &= P\Big(\frac{X_1 + X_2 - 800}{50\sqrt{2}} > \frac{760 - 800}{50\sqrt{2}}\Big) \\ &\approx P(Z > -.5657) = P(Z < .5657) \\ &= \Phi(.5657) \approx 0.7142 \end{aligned}$$

2. $X_2 - X_1$ is a normal random variable with mean 0 and standard deviation $50\sqrt{2}$.

$$\begin{aligned} P(X_2 - X_1 > 25) &= P\Big(\frac{X_2 - X_1}{50\sqrt{2}} > \frac{25}{50\sqrt{2}}\Big) \\ &\approx P(Z > .3536) = 1 - \Phi(.3536) \\ &\approx 1 - .6382 = .3618 \end{aligned}$$

3. $P(|X_1 - X_2| > 25) = 2P(X_2 - X_1 > 25) \approx .7236$

2.7 Let X_i be the time the that it takes to develop the i^{th} print. Then, the time that it takes to develop 100 prints, call it X, can be expressed as

$$X = \sum_{i=1}^{100} X_i$$

It follows from the central limit theorem that X approximately has a normal distribution with mean 1800 and standard deviation 10. Therefore,

$$\begin{aligned} P\{X > 1710\} &= P\{\frac{X-1800}{10} > \frac{1710-1800}{10}\} \\ &= 1 - \Phi(-9) \\ &= \Phi(9) \approx 1 \end{aligned}$$

The probability for part (b) is 0

2.8 Let X_i be the mileage for person i, $i = 1, \ldots, 30$. From central limit theorem, $\sum_{i=1}^{30} X_i$ is approximately a normal random variable with mean 25000×30 and standard deviation $12000 \times \sqrt{30}$.

1. $P\Big(\frac{\sum_{i=1}^{30} X_i}{30} > 25000\Big) = P\Big(\frac{\sum_{i=1}^{30} X_i - 25000\times 30}{12000\times\sqrt{30}} > 0\Big) = \Phi(0) = 0.5$

2.

$$\begin{aligned} &P\Big(23000 < \frac{\sum_{i=1}^{30} X_i}{30} < 27000\Big) \\ &= P\Big(\frac{23000\times 30 - 25000\times 30}{12000\sqrt{30}} < \frac{\sum_{i=1}^{30} X_i - 25000\times 30}{12000\times\sqrt{30}} < \frac{27000\times 30 - 25000\times 30}{12000\times\sqrt{30}}\Big) \\ &= \Phi(5/\sqrt{30}) - \Phi(-5/\sqrt{30}) \\ &= 2\Phi(5/\sqrt{30}) - 1 \approx 2\times 0.8194 - 1 = .6388 \end{aligned}$$

2.9 Let S_i be the price of stock in time period i. Then $S_{i+1} = S_i X_i$ where the random variable X_i is defined as

$$X_i = \begin{cases} u & \text{, with probability } p \\ d & \text{, with probability } 1-p \end{cases}$$

Then

$$P\Big(\frac{S_{1000}}{S_0} > 1.3\Big) = P\Big(\prod_{i=0}^{999} X_i > 1.3\Big) = P\Big(\sum_{i=0}^{999} \log X_i > \log 1.3\Big)$$

We can use the central limit theorem to approximate $\sum_{i=0}^{999} \log X_i$ with a normal random variable Y with the same mean and variance.

$$\begin{aligned} E[Y] &= 1000\ E[\log X_i] = 1000\ (p \log u + (1-p)\log d) \approx 1.3787 \\ \mathrm{Var}(Y) &= 1000\ \mathrm{Var}(\log X_i) \\ &= 1000\ (p(\log u)^2 + (1-p)(\log d)^2 - .0013787^2) \approx 0.1206 \end{aligned}$$

Therefore

$$\begin{aligned} P\Big(\sum_{i=0}^{999} \log X_i > \log 1.3\Big) &\approx P\Big(Y > \log 1.3\Big) \\ &= P\Big(\frac{Y - 1.3787}{\sqrt{.1206}} > \frac{\log 1.3 - 1.3787}{\sqrt{.1206}}\Big) \\ &\approx P(Z > -3.2146) = \Phi(3.2146) \approx .9993 \end{aligned}$$

where Z stands for a standard normal random variable.

2.10 Let X_i be the movement in period i. Then we can approximate $\sum_{i=1}^{700} X_i$ with a normal random variable Y with the same mean and variance

$$\begin{aligned} E[Y] &= E\Big[\sum_{i=1}^{700} X_i\Big] = 700(-.39 + .41) = 14 \\ \mathrm{Var}(Y) &= \mathrm{Var}\Big(\sum_{i=1}^{700} X_i\Big) = 700(.39 \times 1.02^2 + .20 \times .02^2 + .41 \times .98^2) = 559.72 \end{aligned}$$

Therefore,

$$P\Big(\sum_{i=1}^{700} X_i > 10\Big) \approx P(Y > 10.5) = P\Big(\frac{Y - 14}{\sqrt{559.72}} > \frac{10.5 - 14}{\sqrt{559.72}}\Big) \approx \Phi(.1479) \approx .5588$$

3.1
(a) $100e^{.1+.2} = 100e^{.3}$
For parts (b) and (c), use that $Y \equiv \log(S(10)/100)$ is normal with mean .1 and variance .4. This gives

$$
\begin{aligned}
(b)\ P\{S(10) > 100\} &= P\{Y > 0\} \\
&= P\{\frac{Y - .1}{\sqrt{.4}} > \frac{-.1}{\sqrt{.4}}\} \\
&= \Phi(\frac{.1}{\sqrt{.4}}) = \Phi(.158) = .5628
\end{aligned}
$$

$$
\begin{aligned}
(b)\ P\{S(10) < 110\} &= P\{Y < \log(1.1)\} \\
&= P\{\frac{Y - .1}{\sqrt{.4}} < \frac{\log(1.1) - .1}{\sqrt{.4}}\} \\
&= \Phi(-.0074) = .4970
\end{aligned}
$$

3.4 Since $S(t)/S(0)$ is distributed as e^X when X is normal with mean μt and variance $t\sigma^2$, it follows that

$$E[S(t)/S(0)] = E[e^X] = e^{\mu t + t\sigma^2/2}$$

3.5 Using the representation in problem 3.4, we obtain

$$E[S^2(t)/S^2(0)] = E[e^{2X}] = e^{2\mu t + 2t\sigma^2}$$

where the preceding used that $2X$ is normal with mean $2\mu t$ and variance $4t\sigma^2$. Therefore,

$$
\begin{aligned}
\mathrm{Var}(S(t)) &= E[S^2(t)] - (E[S(t)])^2 \\
&= s_0^2 e^{2\mu t + 2t\sigma^2} - s_0^2 e^{2\mu t + t\sigma^2} \\
&= s_0^2 e^{2\mu t + t\sigma^2}(e^{t\sigma^2} - 1)
\end{aligned}
$$

4.1

(a) $r_e = (1 + 0.1/2)^2 - 1 = 0.1025$

(b) $r_e = (1 + 0.1/4)^4 - 1 \approx 0.1038$

(c) $r_e = e^{0.1} - 1 \approx 0.1052$

4.2 Suppose it takes t years to double, then

$$e^{0.1t} = 2 \Rightarrow t = \frac{\log 2}{0.1} \approx 6.93$$

4.3 Suppose it takes t years to quadruple, then we can solve t from $1.05^t = 4$. We can also use the doulbing rule to approximate t, which gives

$$t \approx \frac{0.7}{0.05} \times 2 = 28$$

If the interest is 4%, then it is approximately $0.7/0.04 \times 2 = 35$ years.

4.4 Using that $e^r \approx 1+r$, when r is small, we see that if $(1+r)^n = 3$ then $e^{nr} \approx 3$. Thus,

$$n \approx \frac{\log(3)}{r} \approx \frac{1.1}{r}$$

4.5 Suppose you need to invest x at the beginning of each of the next 60 months to have a value of $\$100,000$ at the end of 60 months, then

$$100000 = x \sum_{i=1}^{60} 1.005^i = x \frac{1.005(1 - 1.005^{60})}{1 - 1.005}$$

Solve to get $x = 1426.15$.

4.6 Let's compute the present value, denoted by S, of this cash flow.

$$S = -1000 - \frac{1200}{1.06} + \frac{800}{1.06^2} + \frac{900}{1.06^3} + \frac{800}{1.06^4} = -30.75$$

Since it is negative, it is not worth investing.

4.7 (15 pts) Let the present value of the first cash flow sequence be S_1 and that of the second cash flow sequence be S_2. Then

$$\begin{aligned} S_1 &= \frac{20}{1+r} + \frac{20}{(1+r)^2} + \frac{20}{(1+r)^3} + \frac{15}{(1+r)^4} + \frac{10}{(1+r)^5} + \frac{5}{(1+r)^6} \\ S_2 &= \frac{10}{1+r} + \frac{10}{(1+r)^2} + \frac{15}{(1+r)^3} + \frac{20}{(1+r)^4} + \frac{20}{(1+r)^5} + \frac{20}{(1+r)^6} \end{aligned}$$

(a) $r = 0.03$, $S_1 = 82.71, S_2 = 84.63$. The second one is preferable.

(b) $r = 0.05$, $S_1 = 78.37, S_2 = 78.60$. The second one is preferable.

(c) $r = 0.1$, $S_1 = 69.01, S_2 = 65.99$. The first one is preferable.

4.8 (15 pts) Let S denote the present value, then

$$S = -10000 + \sum_{i=1}^{10} \frac{500}{(1+r/2)^i} + \frac{10000}{(1+r/2)^{10}}$$

(a) $r = 0.06$, $S = 1706.04$.

(b) $r = 0.10$, $S = 0$.

(c) $r = 0.12$, $S = -736.01$.

4.9 The effective interest rate, call it r, is that value for which

$$160(\frac{1}{1+r} + \frac{1}{(1+r)^2} + \ldots + \frac{1}{(1+r)^{24}} = 3200$$

which reduces to

$$\frac{1-(\frac{1}{1+r})^2 4}{r} = 20$$

Solution by trial and error shows that $r \approx .015$. That is, the effective interest rate is 1.5 percent per month.

4.11 The cost-flow sequences are as follows

buy at beginning of year 1:	22	7	8	9	10	-4
buy at beginning of year 2:	9	25	7	8	9	-9
buy at beginning of year 3:	9	11	28	7	8	-14
buy at beginning of year 4:	9	11	13	31	7	-19

With the yearly interest rate 10%, the present value of the first cost-flow sequence is

$$22 + \frac{7}{1.1} + \frac{8}{1.1^2} + \frac{9}{1.1^3} + \frac{10}{1.1^4} - \frac{4}{1.1^5} = 46.08$$

Similarly, the prevent values of the other three cost-flow sequences can be determined, and the four present values are

$$46.08,\ 44.08,\ 44.17,\ 46.02$$

Therefore, the company should purchase a new machine one year from now.

4.12 Since the bank charges 2 points, the amount of money we receive for this loan is actually $120,000 \times 0.98 = 117,600$. The interest we need to pay per month is $120,000 \times 0.5\% = 600$. Therefore the cash flow sequence of this loan is

time (mths)	0	1	2	...	35	36
cash flow	117600	-600	-600	...	-600	-120600

Let r be the effective interest rate per month for this loan, then

$$\begin{aligned} 117600 &= \frac{600}{1+r} + \frac{600}{(1+r)^2} + \ldots + \frac{600}{(1+r)^{35}} + \frac{120600}{(1+r)^{36}} \\ &= \frac{600[1-(\frac{1}{1+r})^{35}]}{r} + \frac{120600}{(1+r)^{36}} \end{aligned}$$

We can solve the above to get $r \approx 0.5615\%$.

4.13 The present value of paying the entire amount of \$16,000 now is simply \$16,000, while the present value of paying \$10,000 now and another \$10,000 at the end of ten years is

$$S = 10,000 + 10,000(e^{-r})^{10}$$

Therefore

(a) $r = 0.02$, $S = 18,187.31$, which is not preferable.

(b) $r = 0.05$, $S = 16,065.31$, which is not preferable.

(c) $r = 0.10$, $S = 13,678.79$, which is preferable.

4.14 The cash flow sequence is as follows,

time (yrs)	0	0.5	1	...	4.5	5
cash flow	-1000	30	30	...	30	1030

With a continuously compounded interest 5%, the present value of above is

$$-1000 + \sum_{i=1}^{9} \frac{30}{(e^{0.05/2})^i} + \frac{1030}{(e^{0.05})^5} = 40.94$$

4.15 The present value of a cash flow of 1,000 at the end of 10 years with a continuously compounded interest rate 8% is

$$\frac{1000}{(e^{0.08})^{10}} = 449.33$$

4.16 The rate of return is the effective interest rate which makes the present value of the cash flow streams equal to the initial payment. Therefore

(a)

$$1000 = \frac{500}{1+r} + \frac{300}{(1+r)^2} \Rightarrow 10(1+r)^2 - 5(1+r) - 3 = 0$$

Take $1+r$ as the variable and use the formula to solve the above, we get

$$1+r = \frac{5 \pm \sqrt{25+120}}{20} = \frac{5 \pm \sqrt{145}}{20}$$

Since $1+r$ can not be negative, so $1+r = \frac{5+\sqrt{145}}{20} \approx 0.852$, or $r \approx -0.148$.

(b) In this case, it is easy to see that $r = 0$.

(c) Similar to part (a),

$$1000 = \frac{500}{1+r} + \frac{700}{(1+r)^2} \Rightarrow r \approx 0.123$$

4.18

$$r_a = \frac{1+r}{1+r_i} - 1 = \frac{1.05}{1.03} - 1 = \frac{2}{103} \approx 1.942\%$$

4.20 Since the interest rates for borrowing and lending are not the same, we can not compute the present value of the whole cash flow stream by summing up the present values of each item as we did before. For example, when we receive \$900 a year from today, we will choose to use that money paying off part of the debt rather than earning interest on it. So, let's start by borrowing \$1000 and examine our balance year by year.

time	account balance
today	-1000
end of year 1	$-1000 \times 1.08 + 900 = -180$
end of year 2	$-180 \times 1.08 + 800 = 605.6$
end of year 3	$605.6 \times 1.05 - 1200 = -564.12$
end of year 4	$-564.12 \times 1.08 + 700 = 90.7504$

This means that if you start with nothing, this investment gives you \$90.7504 four years from today. Therefore,you should invest.

4.21

$$\frac{d}{dt}\bar{r}(t) = \frac{tr(t) - \int_0^t r(s)ds}{t^2}$$

Thus, we must show that

$$tr(t) \geq \int_0^t r(s)ds$$

which follows since

$$\int_0^t r(s)ds \leq \int_0^t r(t)ds = tr(t)$$

4.22 Note that

$$P(\alpha t) = e^{-\alpha t \bar{r}(\alpha t)}, \quad P^{\alpha}(t) = e^{-\alpha t \bar{r}(t)}$$

Hence, $P(\alpha t) \geq P^{\alpha}(t)$ is equivalent to $\bar{r}(\alpha t) \leq \bar{r}(t)$.

5.2 Let s be the price of the stock today and r the one period simple interest rate. Suppose that we buy x shares of call options and y shares of stocks today, then since $K < \min s_i$, the call option will be exercised after one period in all cases. Therefore, the value of our holdings at time 1 is $x(s_i - K) + ys_i$ if the stock price at time 1 is s_i. If we choose $x = -y$, then we have a riskless return of $-xK$. To rule out an arbitrage opportunity, this riskless return should be equal to

$$(1+r)(xc+ys) = (1+r)x(c-s)$$

Therefore

$$-xK = (1+r)x(c-s) \;\Rightarrow\; c = s - \frac{K}{1+r}$$

5.3 Let K denote the strike price of the call and t the expiration date. Also, let $S(t)$ be the price of the stock when the call expires. Then the value of the call at time t is

$$v_c = \max\{0, S(t) - K\}$$

and the value of the stock at time t is

$$v_s = S(t)$$

It is clear that $v_s \geq v_c$ since $K \geq 0$. Therefore one share of stock is preferred to one share of call option, which leads to the conclusion that $C \leq S$.

We can also do this by arguing that an arbitrage opportunity exists if $C > S$. In this case, one can buy one share of stock and sell a call option today, which gives $C - S$ dollars. When the call option expires, the value of the portfolio is

$$v_s - v_c = \min\{S(t), K\} > 0$$

Therefore it is an arbitrage.

5.4 (a) is not true. (b) is true. The easiest way to see this is to use the put call parity and the result from **Exercise 5.3**.

$$P - Ke^{-rt} = C - S \leq 0$$

Therefore

$$P \leq Ke^{-rt} \leq K$$

Also, one can argue that an arbitrage opportunity exists if $P > K$.

5.5 From the put call parity,

$$P - Ke^{-rt} + S = C \geq 0$$

Therefore

$$P \geq Ke^{-rt} - S$$

Also, one can argue that an arbitrage opportunity exists if the inequality doesn't hold.

5.7 Let $P(t)$ denote the price of an American put option having exercise time t. Given $s < t$, we want to show that $P(s) \leq P(t)$.

We prove this by showing that an arbitrage is present if the statement is not true. Suppose $P(s) > P(t)$, then we buy one share of the put option having exercise time t and sell one share of the put option having exercise time s, which gives us $P(s) - P(t)$ dollars today. Consider the strategy that whenever the put option we sold is exercised by the buyer, we exercise our put option at the same time. This strategy guarantees us no cash flow in the time period $(0, s]$. On the other hand, if the put option we sold is not exercised by the buyer, we then still have our put option at time s, whose value is always nonnegative. Either way, it is a sure win situation (remember we receive $P(s) - P(t)$ today), therefore an arbitrage.

5.8 No, it is not valid for European puts. Suppose we buy the put option having exercise time t and sell the put option having exercise time s, where $s < t$. If the sold put option is exercised at time s, we are forced to pay K for one share of stock. At time t, we have a debt of $Ke^{r(t-s)}$ and our put option guarantees us to sell that stock for at least K, which is not enough to pay off the debt. Therefore, we can not guarantee a sure win.

5.9 $s - d$

5.10

(a) If $S(t) > K$, the call option is worth $S(t) - K$ and the put option is worthless. On the other hand if $S(t) < K$, the put option is worth $K - S(t)$ and the call option is worthless. Therefore, the payoff is $|S(t) - K|$.

(b) Consider the following two cases.
$K_1 \geq K_2$, then

$$\text{payoff} = \begin{cases} S(t) - K_1 & , S(t) > K_1 \\ 0 & , K_2 \leq S(t) \leq K_1 \\ S(t) - K_2 & , S(t) < K_2 \end{cases}$$

$K_1 < K_2$, then

$$\text{payoff} = \begin{cases} S(t) - K_1 & , S(t) > K_2 \\ 2S(t) - K_1 - K_2 & , K_1 \leq S(t) \leq K_2 \\ S(t) - K_2 & , S(t) < K_1 \end{cases}$$

5.7 Let $P(t)$ denote the price of an American put option having exercise time t. Given $s < t$, we want to show that $P(s) \leq P(t)$.

We prove this by showing that an arbitrage is present if the statement is not true. Suppose $P(s) > P(t)$, then we buy one share of the put option having exercise time t and sell one share of the put option having exercise time s, which gives us $P(s) - P(t)$

dollars today. Consider the strategy that whenever the put option we sold is exercised by the buyer, we exercise our put option at the same time. This strategy guarantees us no cash flow in the time period $(0, s]$. On the other hand, if the put option we sold is not exercised by the buyer, we then still have our put option at time s, whose value is always nonnegative. Either way, it is a sure win situation (remember we receive $P(s) - P(t)$ today), therefore an arbitrage.

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5.10

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$K_1 \geq K_2$, then

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$K_1 < K_2$, then

$$\text{payoff} = \begin{cases} S(t) - K_1 & , S(t) > K_2 \\ 2S(t) - K_1 - K_2 & , K_1 \leq S(t) \leq K_2 \\ S(t) - K_2 & , S(t) < K_1 \end{cases}$$

6.1 We need to see whether we can find a probability vector (p_1, p_2, p_3) for which all bets are fair. In order to have all bets fair, $p_i = 1/(1+o_i)$. Therefore,

$$p_1 = 1/2 \qquad p_2 = 1/3 \qquad p_3 = 1/6$$

Since the p_i's sum up to 1, (p_1, p_2, p_3) is indeed a probability vector which makes all bets fair. Therefore, no arbitrage is present.

6.2 To rule out the arbitrage opportunity, o_4 must satisfy the equation,

$$\frac{1}{1+2} + \frac{1}{1+3} + \frac{1}{1+4} + \frac{1}{1+o_4} = 1$$

Therefore, $o_4 = 47/13$.

6.3 No arbitrage is present since

$$\frac{1}{1+o_1} + \frac{1}{1+o_2} + \frac{1}{1+o_3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

6.4 If no arbitrage is present, then $(p_1, p_2, p_3) = (1/2, 1/3, 1/6)$ has to be the probability vector which makes all bets fair. Therefore

$$\begin{aligned}
o_{12}(p_1+p_2) - p_3 = 0 \quad &\Rightarrow \quad o_{12} = 1/5 \\
o_{23}(p_2+p_3) - p_1 = 0 \quad &\Rightarrow \quad o_{23} = 1 \\
o_{13}(p_1+p_3) - p_2 = 0 \quad &\Rightarrow \quad o_{13} = 1/2
\end{aligned}$$

6.5 If the outcome is j, then the betting scheme $x_i, i = 1, \ldots, m$, gives me

$$\begin{aligned}
o_j x_j - \sum_{i \neq j} x_i \quad &= \quad o_j x_j - \sum_{i=1}^{m} x_j + x_j = (1+o_j)x_j - \sum_{i=1}^{m} x_j \\
&= \quad \frac{(1+o_j)(1+o_j)^{-1} - \sum_{i=1}^{m}(1+o_i)^{-1}}{1 - \sum_{i=1}^{m}(1+o_i)^{-1}} = 1
\end{aligned}$$

6.6 From Example 6.1b, $p = (1+2r)/3$. The payoff of the put option is 0 if the stock price goes up, and 100 if the stock price goes down. To rule out arbitrage, the expected return of buying one put option under the probability distribution has to be zero. That is,

$$\frac{p0 + (1-p)100}{1+r} - P = 0$$

Therefore

$$P = \left(1 - \frac{1+2r}{3}\right)\frac{100}{1+r} = \frac{200(1-r)}{3(1+r)}$$

The put-call parity says

$$S + P - C = \frac{K}{1+r}$$

In this example, one can check the following indeed holds.

6.7
Since the call option expires in period 2 and the strike price $K = 150$, it is clear that

$$C_{uu} = 250 \qquad C_{ud} = 0 \qquad C_{dd} = 0$$

Let p denote the risk neutral probability that the price of the security goes up, then

$$p = \frac{1+r-d}{u-d} = \frac{1-0.5}{2-0.5} = \frac{1}{3}$$

where we assume $r = 0$. Then we can find C by computing the expected return of the call option in the risk neutral world.

$$\begin{aligned} C_u &= pC_{uu} + (1-p)C_{ud} = \frac{1}{3} \times 250 = \frac{250}{3} \\ C_d &= pC_{ud} + (1-p)C_{dd} = 0 \end{aligned}$$

$$C = p^2 C_{uu} + 2p(1-p)C_{ud} + (1-p)^2 C_{dd} = \frac{1}{3^2} \times 250 = \frac{250}{9}$$

6.8 See Example 8.1a for details.

6.9 We need to find a betting strategy which gives a weak arbitrage if (a) $C = 0$ and (b) $C = 50/3$.

(a) $C = 0$. In this case it is clear that buying one share of stock is a weak arbitrage. At time 0, one does not have to pay out anything. At time 1, the profit is 50 if the stock goes up to 200, and 0 if the stock price is either 100 or 50.

(b) $C = 50/3$. In this case it is not that clear how a weak arbitrage can be established. But since the price of the call option is high, it's intuitive that we want to sell it. So, let's consider a portfolio consisting of selling one share of the call option and buying x share(s) of the stock. Our return depends on the price of stock at time 1, which is tabulated as follows.

stock price at time 1	balance at time 0	value of the portfolio at time 1	profit ($r = 0$)
200	$50/3 - 100x$	$-50 + 200x$	$100x - 100/3$
100	$50/3 - 100x$	$0 + 100x$	$50/3$
50	$50/3 - 100x$	$0 + 50x$	$-50x + 50/3$

From the above table, if we choose $x = 1/3$, then the profit is $50/3$ if the stock price is 100 at time 1, and 0 otherwise, which is a weak arbitrage.

7.1 $(a)\ .33/\sqrt{225}$, $\quad (b).33/\sqrt{12}$

7.2 Since the unit of time is one year, $t = 4/12 = 1/3$. The probability that the call option will be exercised at $t = 1/3$ is the probability that the stock price at $t = 1/3$ is greater than the strike price $K = 42$, which is

$$\begin{aligned} P(S(1/3) > 42) &= P\Big(\frac{S(1/3)}{S(0)} > \frac{42}{40}\Big) \\ &= P\Big(\log\frac{S(1/3)}{S(0)} > \log\frac{42}{40}\Big) \\ &= P(X > \log 1.05) \end{aligned}$$

where X is a normal random variable with mean $\mu t = .12/3 = .04$ and standard deviation $\sigma\sqrt{t} = .24/\sqrt{3}$. Therefore, the above probability is equal to

$$P\Big(\frac{X - .04}{.24/\sqrt{3}} > \frac{\log 1.05 - .04}{.24/\sqrt{3}}\Big) = 1 - \Phi\Big(\frac{\log 1.05 - .04}{.24/\sqrt{3}}\Big)$$
$$\approx 1 - \Phi(.0634) \approx 1 - .5253 = .4747$$

7.3 The parameters are

$$t = 1/3 \qquad r = .08 \qquad \sigma = .24 \qquad K = 42 \qquad S = 40$$

so we have that

$$\omega = \frac{.08/3 + .24^2/6 - \log(42/40)}{.24/\sqrt{3}} \approx -.0904$$

Therefore,

$$\begin{aligned} C &\approx 40\Phi(-.0904) - 42e^{-.08/3}\Phi(-.2289) \\ &\approx 40 \times .4639 - 42e^{-.08/3} \times .4094 \\ &\approx 1.8137 \end{aligned}$$

7.4 From the put-call parity, we can derive the no-arbitrage cost to a put option

$$\begin{aligned} P &= C - S(0) + Ke^{-rt} \\ &= S(0)\Phi(\omega) - Ke^{-rt}\Phi(\omega - \sigma\sqrt{t}) - S(0) + K^{-rt} \\ &= S(0)\left(\Phi(\omega) - 1\right) + Ke^{-rt}(1 - \Phi(\omega - \sigma\sqrt{t})) \end{aligned}$$

where ω is defined in equation (7.7) in text (page 87). The parameters are

$$K = 100 \qquad S(0) = 105 \qquad r = .1 \qquad \sigma = .30 \qquad t = 1/2$$

$$\begin{array}{ll} \omega \approx .571767 & \Phi(.571767) \approx .7163 \\ \omega - \sigma\sqrt{t} \approx .359635 & \Phi(.359635) \approx .6404 \end{array}$$

Therefore,

$$P = 105(.7163 - 1) + 100e^{-.05}(1 - .6404) \approx 4.418$$

7.5 A call option with a strike price equal to zero is equivalent to a stock, since the payoff is $\max\{0, S(t) - K\} = \max\{0, S(t) - 0\} = S(t)$. Therefore, the price of such a call option is equal to the price of the stock $S(0)$.

The same conclusion can be easily verified by the B-S formula by plugging in $K = 0$.

7.6 From the Black-Scholes formula, $\omega \to \infty$ as $t \to \infty$. Therefore, as $t \to \infty$,

$$C = S(0)\Phi(\omega) - Ke^{-rt}\Phi(\omega - \sigma\sqrt{t}) \to S(0) \times 1 - K \times 0 = S(0)$$

7.7 The payoff is F if $S(t) > K$ and 0 otherwise. So, the risk-neutral valuation (or the unique no-arbitrage cost) of such a call is equal to

$$C_a = e^{-rt}F \times P\{S(0)e^W > K\}$$

where W is normal with mean $(r - \sigma^2/2)t$ and variance $\sigma^2 t$ (see page 88 in text). Therefore

$$\begin{aligned} C_a &= e^{-rt}F \times P\{W > \log(K/S(0))\} \\ &= e^{-rt}F \times P\Big(\frac{W - (r - \sigma^2/2)t}{\sigma\sqrt{t}} > \frac{\log(K/S(0)) - (r - \sigma^2/2)t}{\sigma\sqrt{t}}\Big) \\ &= e^{-rt}F \times P\Big(Z < \frac{(r - \sigma^2/2)t - \log(K/S(0))}{\sigma\sqrt{t}}\Big) \\ &= e^{-rt}F\ \Phi(\omega - \sigma\sqrt{t}) \end{aligned}$$

where Z stands for a standard normal random variable and

$$\omega = \frac{(r + \sigma^2/2)t - \log(K/S(0))}{\sigma\sqrt{t}}$$

as defined in equation (7.7) in text (page 87). The parameters

$$F = 100 \qquad K = 40 \qquad S(0) = 38 \qquad \sigma = .32 \qquad r = .06 \qquad t = 1/2$$

So

$$\omega - \sigma\sqrt{t} \approx -.207242 \qquad \Phi(-.207242) \approx .4179$$

Therefore, $C_a \approx e^{-.03} \times 100 \times .4179 = 40.55$.

8.1 If there is no arbitrage, then there exists $\mathbf{p} = (p_{50}, p_{175}, p_{200})$ such that both buying stocks and buying call options are fair. This means we are able to solve $(p_{50}, p_{175}, p_{200})$ from the following linear equations.

$$-C + 25p_{175} + 50p_{200} = 0 \tag{0.1}$$
$$-50p_{50} + 75p_{175} + 100p_{200} = 0 \tag{0.2}$$
$$p_{50} + p_{175} + p_{200} = 1 \tag{0.3}$$
$$0 \leq p_{50}, p_{175}, p_{200} \leq 1 \tag{0.4}$$

By letting $p_{50} = x$, we can solve p_{175} and p_{200} in terms of x from (2) and (3), which gives

$$p_{175} = 4 - 6x \qquad p_{200} = 5x - 3$$

Together with the constraint in (4), the solution from (2)–(4) can be written as

$$(p_{50}, p_{175}, p_{200}) = (x, 4 - 6x, 5x - 3) \qquad 3/5 \leq x \leq 2/3$$

Therefore, if $C = 25p_{175} + 50p_{200} = 100x - 50$ where $3/5 \leq x \leq 2/3$, then we are able to solve the linear equations (1)–(4). In other words, if $10 \leq C \leq 50/3$, there is no arbitrage opportunity.

8.2

(a)

$$u(x) = \log x \qquad u'(x) = 1/x \qquad u''(x) = -1/x^2$$

Therefore $a(x) = 1/x$.

(b)

$$u(x) = 1 - e^{-x} \qquad u'(x) = e^{-x} \qquad u''(x) = -e^{-x}$$

Therefore $a(x) = 1$.

8.3 Using the notation defined in Example 8.2a, let $f(\alpha)$ denote the expected utility of the final fortune, then

$$\begin{aligned} f(\alpha) &= \log x + p\log(1+\alpha) + (1-p)\log(1-\alpha) \\ f'(\alpha) &= \frac{p}{1+\alpha} - \frac{1-p}{1-\alpha} \end{aligned}$$

Since $p < 1/2$, $f'(\alpha) < 0$ for $0 \leq \alpha \leq 1$, the maximum value of $f(\alpha)$ for $0 \leq \alpha \leq 1$ is obtained at $\alpha = 0$.

8.7 Using the notation on page 117,

$$W = w\sum_{i=1}^{n} \alpha_i X_i$$

where w is the initial wealth, α_i is the proportion of the initial wealth invested in security i, and X_i is the return from security i if the initial investment is \$1. If $U(x) = \log x$, then

$$\begin{aligned} E[U(W)] &= E[\log(W)] = E\Big[\log\Big(w\sum_{i=1}^{n}\alpha_i X_i\Big)\Big] \\ &= E\Big[\log w + \log\Big(\sum_{i=1}^{n}\alpha_i X_i\Big)\Big] \\ &= \log w + E\Big[\log\Big(\sum_{i=1}^{n}\alpha_i X_i\Big)\Big] \end{aligned}$$

So, the optimal $\alpha_i, i = 1, \ldots, n$ do not depend on w.

8.8 To show that the second derivative is nondecreasing, we need to show that the third derivative is nonnegative.

(a) For $0 < a < 1$,

$$U'(x) = ax^{a-1} \qquad U''(x) = a(a-1)x^{a-2} \qquad U'''(x) = a(a-1)(a-2)x^{a-3} > 0$$

(b) $U'(x) = be^{-bx}$, $U''(x) = -b^2e^{-bx}$, so $U'''(x) = b^3e^{-bx} > 0$.

(c) $U'(x) = x^{-1}$, $U''(x) = -x^{-2}$, so $U'''(x) = 2x^{-3} > 0$.

8.11 The objective is to maximize the probability that $W > g$, or

$$\begin{aligned} P(W > g) &= P\Big(\frac{W - E[W]}{\sqrt{\mathrm{Var}(W)}} > \frac{g - E[W]}{\sqrt{\mathrm{Var}(W)}}\Big) \\ &= P\Big(Z > \frac{g - E[W]}{\sqrt{\mathrm{Var}(W)}}\Big) \end{aligned}$$

where Z stands for a standard normal random variable. Therefore, to maximize the probability, it is equivalent to minimize

$$\frac{g - E[W]}{\sqrt{\mathrm{Var}(W)}}$$

or to maximize

$$\frac{E[W] - g}{\sqrt{\mathrm{Var}(W)}}$$